Advanced algorithms
Linear Programming Notes

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1 Disposition

1. What is a linear program?
2. Standard and slack forms
3. Pivoting
4. Simplex algorithm. Pivot until we're done.
5. Simplex terminates. A slack form is determined by B (or N). Finite amount. Cycling can be avoided
6. Dual and primal. Weak duality
7. Intuition that optimal solutions will be equal.
8. Initial slack form.

2 Summary

A linear program is where we want to maximize/minimize some linear function:

$$\text{maximize} \sum_{j=1}^{n} c_j x_j$$

subject to a set of linear constraints

$$\text{subject to} \sum_{j=1}^{m} a_{ij} x_j \leq b_i \quad i = 1..m$$

We can also have $\geq$ or $=$.

3 Standard and slack forms

A linear program in standard form requires that all variables be positive and all linear constraints be "less than or equal". That is:

$$\text{max.} \sum_{j=1}^{n} c_j x_j$$

$$\text{st.} \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1..m \tag{1}$$

$$x_j \geq 0 \quad \text{for } j = 1..n \tag{2}$$

We can write these as matrices: $A = (a_{ij})$ an $m \times n$ matrix. $b = (b_i)$, $c = (c_j)$ and $x = (x_j)$. respectively m-vector and two n-vectors. Then we want to maximize $c^T x$ subject to $Ax \leq b$ and $x \geq 0$.

We call a solution $\overline{x}$ feasible if it satisfies all constraints. An optimal solution is a solution $\overline{x}$ which has maximum objective value $c^T \overline{x}$ over all feasible solutions. Some linear programs may be unbounded.

To convert a linear program into standard form do the following:

1. To convert minimization to maximization, just multiply c by -1.
2. For each variable x_j without nonnegativity constraint. Replace it with $x'_j - x''_j$. We can set $x'_j, x''_j \geq 0$
3. Equality constraints can be replaced by two inequality constraints.
4. $\geq$ constraints can be replaced by $\leq$ by multiplying with -1.

3.1 Slack form

In slack form we want to replace all inequality constraints with equality constraints, so we introduce variables

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

$$x_{n+i} \geq 0$$

If these two equations hold then clearly the inequalities from the standard form holds - and vice versa. We call this variable a slack variable because it indicates how much slack there is between the left and right-hand side.

Example:

$$z = 2x_1 - 3x_2 + 3x_3 \tag{3}$$

$$x_4 = 7 - x_1 - x_2 + x_3 \tag{4}$$

$$x_5 = -7 + x_1 + x_2 - x_3 \tag{5}$$

$$x_6 = 4 - x_1 + 2x_2 - 2x_3 \tag{6}$$

We call the variables on the left-hand side of the equations for the basic variables and denote their indices by the set B. The variables on the right-hand side are called nobasic variables, denoted N. This gives:

$$z = v + \sum_{j \in N} c_j x_j \tag{7}$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B \tag{8}$$

In the example we have:

$$B = \{1, 2, 3\}$$
$$N = \{4, 5, 6\}$$
$$A = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & -2 \end{pmatrix}$$
$$b = \begin{pmatrix} 7 & -7 & 4 \end{pmatrix}^T$$
$$c = \begin{pmatrix} 2 & -3 & 3 \end{pmatrix}^T$$
$$v = 0$$

4 SIMPLEX

The idea is to rewrite the program every step until we have rewritten it in a way such that the optimal solution is easy to obtain.

Simplex starts by converting the linear program into slack form. There are a couple of concepts to simplex.

Pivoting From one slack form we pick an "entering variable". This is picked at random from the nonbasic variables. It must have $c_j > 0$. Let this variable be $x_e, e \in N \wedge c_e > 0$. As we increase this variable the values of $x_j, j \in B$ might either increase, decrease or stay the same. If no x_j decreases it is clear that we can increase x_e infinitely and the objective function is unbounded.

We cannot allow any x_j to become negative, as the solution won't be feasible then. We therefore pick the $i \in B$ that minimizes b_i/a_{ie}. We call this the leaving variable l. We now look at the constraint

$$x_l = b_l - \sum_{j \in N} aljx_j$$

and we exchange the roles of x_l and x_e which gives us:

$$x_e = b_l/a_{le} - \sum_{j \in N + \{l\} - \{e\}} a_{lj}/a_{le} x_j$$

We then replace x_e in all other $x_i, i \in B$ with this new expression.

Finally we change the objection value $v = v + c_e b_e = v + b_l/ale$

In order to show that simplex returns either “unbounded” or a feasible solution we use the following invariant:

1. The slack form is equivalent to the initial slack form
2. All $b_i \geq 0$
3. The basic solution is feasible

Initialization:
1. They are the same
2. The assumption is that the initial is feasible, so this must be true.
3. Is true because of (2).

Maintenance: We assume that there is a bounded variable x_e

1. Can be shown with simple linear algebra.
2. Clearly $b_e = b_l/a_{le}$ is positive. We also know that $b_i/a_{ie} \geq b_l/a_{le}$. And that the new $b_i = b_i - a_{ie}b_e$. This clearly gives $b_i \geq 0$ (because $a_{le} > 0$).
3. (2) causes this to be true because all $x_j, j \in N$ are 0 and all $b_i, i \in B$ are ≥ 0.

The case where a variable is unbounded causes the algorithm to terminate.

Termination: If the algorithm terminates because there is no positive coef. in the objective function we have from maintenance that the solution is feasible.

If we find an unbounded variable, we want to show that

$$\overline{x}_i = \begin{cases} \infty & \text{if } i = e \\ 0 & \text{if } i \in N - \{e\} \\ b_i - \sum_{j \in N} a_{ij}\overline{x}_j & \text{if } i \in B \end{cases}$$

We know that all $b_i \geq 0$. We also know that all x_i in N are nonnegative. We need to see that $x_i = b_i - a_{ie}x_e$ is nonnegative. Because $a_{ie} \leq 0$ (Other wise it would be bounding for x_e) this is okay.

This gives an objective value of ∞.

We now need to show that Simplex terminates, and that the returned solution is optimal. We also need to show how to find an initial solution.

4.1 Termination

There is a problem. No pivoting will decrease the objective value, but it might not increase. See example:

$$z = 8 + x_3 - x_4$$

$$x_1 = 8 - x_2 - x_4$$

$$x_5 = x_2 - x_3$$

We have to pick x_3 as x_e and x_5 as x_l which gives no increase in pivot value. This also means that SIMPLEX can cycle.

First let's show that N and B uniquely define the slack form. We use that if we have

$$\sum_{j \in I} \alpha_j x_j = \gamma + \sum_{j \in I} \beta_j x_j$$

for any setting of x_j's. Then $\gamma = 0$ and $\alpha_j = \beta_j$. This is easy to show (set $x_j = 0$ then $\gamma = 0$ and so on...)

For any standard form (A, b, c) and a set of basic variables B the slack form is uniquely defined.

Proof. Assume two different slack forms have the same basic/nonbasic variables.

Because all slack forms are equivalent we can subtract the two forms from each other, giving:

$$0 = (b_i - b'_i) - \sum_{j \in N} (a_{ij} - a'_{ij}) x_j \quad \text{for } i \in B$$

or

$$\sum_{j \in N} a_{ij} x_j = (b_i - b'_i) + \sum_{j \in N} a'_{ij} x_j$$

We know that this makes the form equivalent. Same goes for the objective function. □

This means that SIMPLEX will always terminate unless it cycles. Because we can only chose m basic variables out of the $n+m$ variables in $n+mCm$ ways simplex will return in that amount of iterations unless it cycles. We can make it not cycle by always picking the entering variable with lowest index (bland's rule).

5 Duality

We want to show that simplex returns an *optimal* solution.

Given a primal linear program in standard form we create the dual linear program:

$$\text{min.} \sum_{i=1}^{m} b_i y_i$$

$$\text{s.t.} \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad \text{for } j = 1..n$$

$$y_i \geq 0 \quad \text{for } i = 1..n$$

If we have feasible solutions $\overline{x}, \overline{y}$ we have:

$$\sum_{j=1}^{n} c_j \overline{x}_j \leq \sum_{i=1}^{m} b_i \overline{y}_i$$

Proof.

$$\sum_{j=1}^{n} c_j \overline{x}_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \overline{y}_i \right) \overline{x}_j \tag{9}$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \overline{x}_j \right) \overline{y}_i \tag{10}$$

$$\leq \sum_{i=1}^{m} b_i \overline{y}_i \tag{11}$$

□

From this it is clear, that if the objective value of the primal and dual are the same, then they are optimal.

If we have a slack form returned by simplex of the primal:

$$z = v' + \sum_{j \in N} c'_j x_j$$

$$x_i = b'_i - \sum_{j \in N} a'_{ij} x_j \quad \text{for } i \in B$$

we can produce an optimal dual solution by setting

$$\overline{y}_i = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N \\ 0 & \text{otherwise} \end{cases}$$

If SIMPLEX returns $\overline{x}$ and the last slack form N, B, c. Let $\overline{y}$ be defined as stated above. Then the $\overline{x}$ is optimal for the primal and $\overline{y}$ is optimal for the dual.

Because all $c'_j \leq 0, j \in N$ in the last slack form, if we set $c'_j = 0, j \in B$. We can write:

$$z = v' + \sum_{j \in N} c'_j x_j \tag{12}$$

$$= v' + \sum_{j=1}^{m+n} c'_j x_j \tag{13}$$

Because all $x_j = 0, j \in N$ in the basic solution we have that $z = v'$.

Because of the equality of slack forms we have

$$\sum_{j=1}^{n} c_j x_j = v' + \sum_{j=1}^{n+m} c'_j x_j$$

(Note that $c'_j = 0, j \in B$). We therefor have:

$$\sum_{j=1}^{n} c_j \overline{x}_j = v' + \sum_{j=1}^{n+m} c'_j \overline{x}_j \tag{14}$$

$$= v' + \sum_{j=1}^{n} c'_j \overline{x}_j + \sum_{i=1}^{m} c'_{n+i} \overline{x}_{n_i} \tag{15}$$

$$\tag{16}$$

We can replace c'_{n+i} with $-\overline{y}_i$. And $\overline{x}_{n+i}$ are the slack variables, so these can be replaced with their formula in the original slack:

$$= v' + \sum_{j=1}^{n} c'_j \overline{x}_j + \sum_{i=1}^{m} (-y_i) \left(b_i - \sum_{j=1}^{n} a_{ij} \overline{x}_j \right)$$

Regrouping gives us

$$= v' + \sum_{j=1}^{n} c'_j \overline{x}_j - \sum_{i=1}^{m} b_i \overline{y}_i + \sum_{j=1}^{n} \sum_{i=1}^{m} (a_{ij} \overline{y}_i) \overline{x}_j$$

We then get

$$\sum_{j=1}^{n} c_j \overline{x}_j = \left(v' - \sum_{i=1}^{m} b_i \overline{y}_i \right) + \sum_{j=1}^{n} \left(c'_j + \sum_{i=1}^{m} a_{ij} \overline{y}_i \right) \overline{x}_j$$

This is of the form we proved earlier, so we have

$$v' - \sum_{i=1}^{m} b_i \overline{y}_i = 0$$

$$c'_j + \sum_{i=1}^{m} a_{ij} \overline{y}_i = c_j \quad \text{for } j = 1..n$$

The first of these two give that $v' = \sum_{i=1}^{m} b_i \overline{y}_i$, so the objective values of the primal and dual are the same. We now only to show, that $\overline{y}$ is feasible.

Because $c'_j \leq 0$ (or simplex would continue). From the equations above we have

$$c_j = c'_j + \sum_{i=1}^{m} a_{ij} \overline{y}_i \leq \sum_{i=1}^{m} a_{ij} \overline{y_i}$$

We therefore have that the feasible solution returned by SIMPLEX is optimal.

6 Finding the initial slack form

The idea is to create a new linear program L_{aux}:

$$\text{maximize } -x_0$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij}x_j - x_0 \leq b_i$$
$$x_j \geq 0$$

This has objective value 0 if and only if L is feasible.

Proof. $\Rightarrow$: If $\overline{X}$ is feasible solution to L, then we can set $x_0 = 0$ and all the constraints are satisfied. This will give $z = 0$, but this is obviously optimal.

$\Leftarrow$: If $x_0 = 0$ then the remaining values make up a solution to L. $\square$

The idea is, that we can perform one pivot with x_0 as the leaving variable to get a feasible solution to L_{aux}. Then we can use SIMPLEX to get the optimal solution. If it's objective value is 0, we're good.

Proof. If we perform a pivot with 0 entering and i such that b_i is minimal leaving, then the base solution is feasible. This is true becaus x_0 has the same coefficient in alle the constraints, so picking the minimum b_i is enough.

Simplex does the rest. $\square$