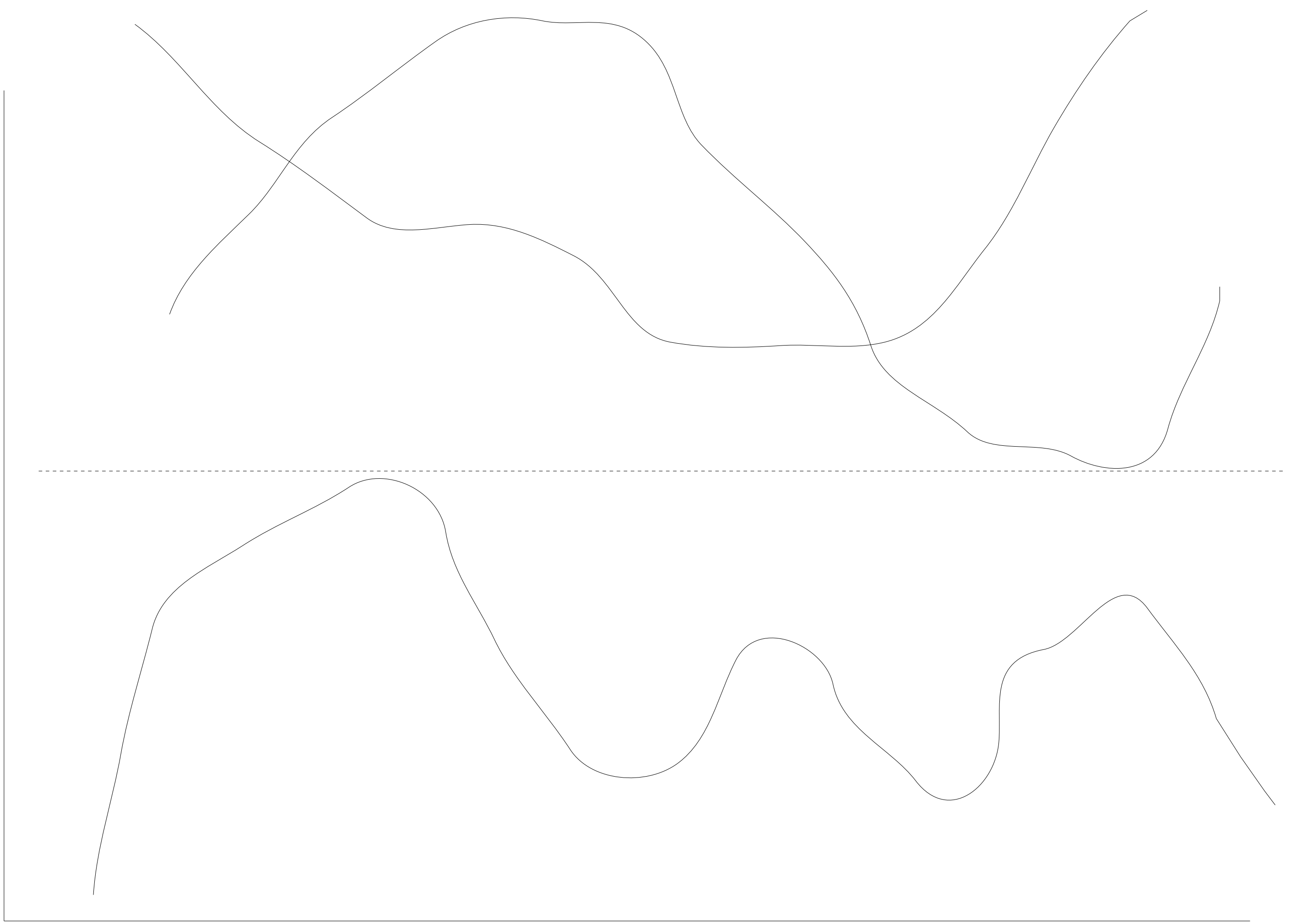
Duality



Upper Bounds on Maximization LP

$$\begin{array}{lccccccccl}
max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

- Multiply second constraint by 5/3:

$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

Upper Bounds on Maximization LP

$$
\begin{array}{llllllllll}
max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

- Add the second constraint to the third constraint:

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq 4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

Upper Bounds on Maximization LP

$$
\begin{array}{lcccccccccc}
max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

- Construct a linear combination of the constraints using nonnegative multipliers y_1, y_2, and y_3:

$$y_1(x_1 - x_2 - x_3 + 3x_4) + y_2(5x_1 + x_2 + 3x_3 + 8x_4) + y_3(-x_1 + 2x_2 + 3x_3 - 5x_4) \leq y_1 + 55y_2 + 3y_3$$

$$(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 + (3y_1 + 8y_2 - 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3$$

- Left-hand side will be an upper bound for the LP if the coefficients at each x_j are at least as big as the corresponding coefficients in the objective function

$$y_1 + 5y_2 - y_3 \geq 4 \qquad -y_1 + y_2 + 2y_3 \geq 1 \qquad 3y_1 + 8y_2 - 5y_3 \geq 3 \qquad -y_1 + 3y_2 + 3y_3 \geq 5$$

- Any set of nonnegative multipliers y_i satisfying these inequalities also satisfies $4x_1 + x_2 + 5x_3 + 3x_4 \leq y_1 + 55y_2 + 3y_3$
- Good upper bound: minimize right-hand side s.t. constraints.

Good Upper Bound

$$
\begin{array}{lccccccc}
\min & y_1 & + & 55y_2 & + & 3y_3 & & \\
s.t. & y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\
 & -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\
 & -y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\
 & 3y_1 & + & 8y_2 & - & 5y_3 & \geq & 3 \\
 & y_1, & & y_2, & & y_3 & \geq & 0
\end{array}
$$

Duality

- The identification of a dual problem is almost always coupled with the discovery of a polynomial-time algorithm.
- Duality provides a proof that a solution is optimal.

LP in Standard Form and Its Dual

$$
\begin{array}{llllllll}
max & 3x_1 & + & x_2 & + & 2x_3 & & \\
s.t. & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

$$
\begin{array}{llllllll}
min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}
$$

LP in Standard Form and Its Dual

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad && for \quad i=1,2,\dots,m \\ & x_j \ge 0 && for \quad j=1,2,\dots,n \end{aligned}$$

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} b_i y_i \\ s.t. \quad & \sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad && for \quad j=1,2,\dots,n \\ & y_i \ge 0 && for \quad i=1,2,\dots,m \end{aligned}$$

Weak Duality

- ***x****: feasible solution to the primal LP.
- ***y****: feasible solution to the dual LP.
- Claim $\sum_{j=1}^{n} c_j x_j^* \le \sum_{i=1}^{m} b_i y_i^*$
- Proof:

$$
\begin{aligned}
\sum_{j=1}^{n} c_j x_j^* &\le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i^*\right) x_j^* && \text{from the dual} \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j^*\right) y_i^* \\
&\le \sum_{i=1}^{m} b_i y_i^* && \text{from the primal}
\end{aligned}
$$

Importance of Weak Duality

- ***x****: feasible solution to the primal LP.
- ***y****: feasible solution to the dual LP.
- If

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

then ***x**** and ***y**** are optimal solutions to the primal and to the dual LPs, respectively.

Final Feasible Basic Solution and Corresponding Dual Solution

$$\begin{array}{lllllllll}
max & z & = & 28 & - & x_3/6 & - & x_5/6 & - & 2x_6/3 \\
s.t. & x_4 & = & 18 & - & x_3/2 & + & x_5/2 & + & 0x_6 \\
 & x_2 & = & 4 & - & 8x_3/3 & - & 2x_5/3 & + & x_6/3 \\
 & x_1 & = & 8 & + & x_3/6 & + & x_5/6 & - & x_6/3
\end{array}$$

Basic variables: x_1=8, x_2=4, x_4=18

Objective value z = 28

$$y_i = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N \\ 0 & \text{otherwise} \end{cases}$$

y_1 = 0 (since x_4 is basic), y_2 = 1/6, y_3 = 2/3

$$\begin{array}{llllllll}
min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}$$

Feasible Solution to the Dual

$$\begin{array}{lccccccc}
min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}$$

$y_1 = 0$ (since x_4 is basic)
$y_2 = 1/6$
$y_3 = 2/3$

Objective value: 30 x 0 + 24 x 1/6 + 36 x 2/3 = 28

1 x 0 + 2 x 1/6 + 4 x 2/3 = 3
1 x 0 + 2 x 1/6 + 1 x 2/3 = 1
3 x 0 + 5 x 1/6 + 2 x 2/3 = 13/6

Duality Theorem

- Suppose that SIMPLEX terminates with a feasible basic solution $\boldsymbol{x^*} = (x_1^*, x_2^*, \ldots, x_n^*)$ with:
  - N and B denoting the nonbasic and basic variables for the final slack form
  - c' denoting the coefficients of the objective function in the final slack form.
- Let $\boldsymbol{y^*} = (y_1^*, y_2^*, \ldots, y_m^*)$ be defined by

$$y_i^* = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N \\ 0 & \text{otherwise} \end{cases}$$

- Then $\boldsymbol{x^*}$ is an optimal solution to the primal LP, $\boldsymbol{y^*}$ is an optimal solution to the dual LP and

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Proof of Duality Theorem

- We have to show that
  - y^* is feasible solution for the dual, and
  - $\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$

Proof of Duality Theorem

- Objective function of the final slack form of the primal is:

$$v^* + \sum_{j \in N} c^*_j x_j = v^* + \sum_{j \in N} c^*_j x_j + \sum_{j \in B} 0\, x_j = v^* + \sum_{j=1}^{n+m} c^*_j x_j$$

- Optimal value of the primal objective function: $v^* = \sum_{j=1}^{n} c_j x^*_j$

$$\sum_{j=1}^{n} c_j x_j = v^* + \sum_{j=1}^{n} c^*_j x_j + \sum_{j=n+1}^{n+m} c^*_j x_j = v^* + \sum_{j=1}^{n} c^*_j x_j - \sum_{i=1}^{m} y^*_i (b_i - \sum_{j=1}^{n} a_{ij} x_j)$$

$$= \ (v^* - \sum_{i=1}^{m} b_i y^*_i) + \sum_{j=1}^{n} (c^*_j + \sum_{i=1}^{m} a_{ij} y^*_i) x_j$$

- This must hold for every choice of x_1, x_2, ..., x_n. Hence

$v^* = \sum_{i=1}^{m} b_i y^*_i$ and $c_j = c^*_j + \sum_{i=1}^{m} a_{ij} y^*_i, \forall\, j = 1, 2, \ldots, n$

- Since $c^*_j \le 0, \forall\, k = 1, 2, \ldots, n+m$, we get

$\sum_{i=1}^{m} a_{ij} y^*_i \ge c_j, \forall\, j = 1, 2, \ldots, n$ and $y^*_i \ge 0, \forall\, i = 1, 2, \ldots, m$

Primal Dual Combinations

		Dual		
		Optimal	**Infeasible**	**Unbounded**
	Optimal	*Possible*	*Impossible*	*Impossible*
Primal	**Infeasible**	*Impossible*	*Possible*	*Possible*
	Unbounded	*Impossible*	*Possible*	*Impossible*

Both Primal and Dual Infeasible

$$
\begin{array}{llcllcl}
max & 2\mathrm{x}_1 & - & x_2 & & \\
s.t. & x_1 & - & x_2 & \leq & 1 \\
 & -x_1 & + & x_2 & \leq & -2 \\
 & x_1, & & x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{llcllcl}
min & y_1 & - & 2\mathrm{y}_2 & & \\
s.t. & y_1 & - & y_2 & \geq & 2 \\
 & -y_1 & + & y_2 & \geq & -1 \\
 & y_1, & & y_2 & \geq & 0
\end{array}
$$

Practical Implications

- If $m >> n$ then the number of constraint in the dual will be much smaller than in the primal.
- Number of pivots in SIMPLEX is usually less than $1.5m$ and only rarely is higher than $3m$.
- Number of pivots increases very slowly with n.
- Solving dual will in such cases be more efficient.