SIMPLEX

- SIMPLEX starts with a slack form corresponding to some feasible basic solution and iterates:
  - Select a nonbasic variable x_e with $c_e > 0$. If no such x_e exists, SIMPLEX terminates. We will show that the feasible basic solution is optimal.
  - Select a basic variable x_l that most severely limits the nonbasic variable x_e. Ties are broken arbitrarily. We will show that LP is unbounded if no such x_l exists.
  - Pivot.

SIMPLEX – Open Issues

- How to decide that LP is feasible?
- What to do if the initial basic solution is infeasible?
- How to select entering and leaving variables?
- How to decide that LP is unbounded?
- Does SIMPLEX terminate?
- Does it terminate with an optimal solution?

Termination

- SIMPLEX computes a feasible basic solution during each iteration.
- When does SIMPLEX terminate?
  - When all coefficients in the objective function are negative.
  - When it becomes obvious that LP is unbounded.

SIMPLEX: LP is Unbounded

- Can x_2 be increased without violating feasibility? By how much?

$$
\begin{aligned}
z &= 111/4 + x_2/16 - x_5/8 - 11x_6/16 \\
x_4 &= 69/4 + 3x_2/16 + 5x_5/8 - x_6/16 \\
x_3 &= 3/2 + 3x_2/8 - x_5/4 + x_6/8 \\
x_1 &= 33/4 + x_2/16 + x_5/8 - 5x_6/16
\end{aligned}
$$

- If x_2 is increased then x_4 also increases.
- If x_2 is increased then x_3 also increases.
- If x_2 is increased then x_1 also increases.
- No constraint is binding; LP is unbounded.

Termination

- The number of basic solutions is finite: Number of basic variables is *m.* They are selected from among *m*+*n* variables. This can be done in

$$\binom{m+n}{m}=\frac{(n+m)!}{n!m!}$$

ways

- Each basic solution has exactly one objective value. If the objective value increases at each iteration, we will eventually end up with a solution where the coefficients of the objective function are all negative (or we will realize that the LP is unbounded).
- Is it possible that the objective value does not change?

Degeneracy

$$
\begin{aligned}
z &= 0 + x_1 + x_2 + x_3 \\
x_4 &= 8 - x_1 - x_2 \\
x_5 &= x_2 - x_3
\end{aligned}
$$

$$
\begin{aligned}
z &= 8 + x_3 - x_4 \\
x_1 &= 8 - x_2 - x_4 \\
x_5 &= x_2 - x_3
\end{aligned}
$$

$$
\begin{aligned}
z &= 8 + x_2 - x_4 - x_5 \\
x_1 &= 8 - x_2 - x_4 \\
x_3 &= x_2 - x_5
\end{aligned}
$$

Degeneracy

$$
\begin{aligned}
z &= 8 + x_2 - x_4 - x_5 \\
x_1 &= 8 - x_2 - x_4 \\
x_3 &= x_2 - x_5
\end{aligned}
$$

$$
\begin{aligned}
z &= 16 - x_1 - 2x_4 - x_5 \\
x_2 &= 8 - x_1 - x_4 \\
x_3 &= 8 - x_1 - x_4 - x_5
\end{aligned}
$$

Cycling

- Is it possible to get the same basis more than once? Then SIMPLEX has a problem.
- It is in fact possible. This happens even if we use specific rules for selecting entering and leaving variables at each iteration.
- The entering variable will always be a nonbasic variable with the largest positive coefficient in the z-row.
- If two or more basic variables compete for leaving the basis, then the candidate with the smallest subscript is made to leave.

Cycling - Example

$$
\begin{aligned}
z &= 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
x_5 &= 0 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 53x_2 + 41x_3 - 204x_4 - 20x_5 \\
x_1 &= 0 + 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
x_6 &= 0 - 4x_2 - 2x_3 + 8x_4 + x_5 \\
x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 14.5x_3 - 98x_4 - 6.75x_5 - 13.25x_6 \\
x_1 &= 0 - 0.5x_3 + 4x_4 + 0.75x_5 - 2.75x_6 \\
x_2 &= 0 - 0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6 \\
x_7 &= 1 + 0.5x_3 - 4x_4 - 0.75x_5 - 13.25x_6
\end{aligned}
$$

Cycling - Example

$$
\begin{aligned}
z &= 0 + 18x_4 + 15x_5 - 93x_6 - 29x_1\\
x_2 &= 0 - 2x_4 - 0.5x_5 + 2.5x_6 + x_1\\
x_3 &= 0 + 8x_4 + 1.5x_5 - 5.5x_6 - 2x_1\\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 10.5x_5 - 70.5x_6 - 20x_1 - 9x_2\\
x_3 &= 0 - 0.5x_5 + 4.5x_6 + 2x_1 - 4x_2\\
x_4 &= 0 - 0.25x_5 + 1.25x_6 + 0.5x_1 - 0.5x_2\\
x_7 &= 1 - x_1
\end{aligned}
$$

Cycling - Example

$$
\begin{aligned}
z &= 0 + 24x_6 + 22x_1 - 93x_2 - 21x_3 \\
x_4 &= 0 - x_6 - 0.5x_1 + 1.5x_2 + 0.5x_3 \\
x_5 &= 0 + 9x_6 + 4x_1 - 8x_2 - 2x_3 \\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
x_5 &= 0 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1
\end{aligned}
$$

Cycling

- **Claim**: If SIMPLEX fails to terminate then it cycles.
- **Proof:** Suppose that SIMPLEX does not cycle but it fails to terminate. So it must generate infinite number of different slack forms.
- However, the number of different bases is finite. If we can show that a slack form for a given basis is unique then we have a contradiction.

$$
\begin{aligned}
z &= v + \sum_{j \notin B} c_j x_j \\
x_i &= b_i - \sum_{j \notin B} a_{ij} x_j \quad \text{for} \quad i \in B
\end{aligned}
\qquad\qquad
\begin{aligned}
z &= v^* + \sum_{j \notin B} c_j^* x_j \\
x_i &= b_i^* - \sum_{j \notin B} a_{ij}^* x_j \quad \text{for} \quad i \in B
\end{aligned}
$$

–

$$\sum_{j \notin B} c_j^* x_j = (v - v^*) + \sum_{j \notin B} c_j x_j$$

$$\sum_{j \notin B} a_{ij} x_j = (b_i - b_i^*) + \sum_{j \notin B} a_{ij}^* x_j, \; \forall i \in B$$

Now use Lemma 29.3 on p. 876

Avoiding Cycling

- Perturb input slightly so that it is impossible to have two basic solutions with the same objective value.
- Always choose the entering and leaving variables with the smallest indicies.

Perturbation Method

$$1 \gg e_1 \gg e_2 \gg \ldots \gg e_{m-1} \gg e_m > 0$$

$$\begin{array}{ccccccccccc}
z & = & 0 & + & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4 \\
x_5 & = & e_1 & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\
x_6 & = & e_2 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\
x_7 & = & 1+e_3 & - & x_1 & & & & & &
\end{array}$$

$$\begin{array}{ccccccccccccc}
z & = & 20e_2 & - & 27x_2 & + & x_3 & - & 44x_4 & - & 20x_6 \\
x_5 & = & e_1-e_2 & + & 4x_2 & + & 2x_3 & - & 8x_4 & + & x_6 \\
x_1 & = & 2e_2 & + & 3x_2 & + & x_3 & - & 2x_4 & - & 2x_6 \\
x_7 & = & 1-2e_2+e_3 & - & 3x_2 & - & x_3 & + & 2x_4 & + & 2x_6
\end{array}$$

$$\begin{array}{ccccccccccc}
z & = & 1+18e_2+e_3 & - & 30x_2 & - & 42x_4 & - & 18x_6 & - & x_7 \\
x_5 & = & 2+e_1-5e_2+2e_3 & - & 2x_2 & - & 4x_4 & + & 5x_6 & - & 2x_7 \\
x_1 & = & 1+e_3 & & & & & & & - & x_7 \\
x_3 & = & 1-2e_2+e_3 & - & 3x_2 & + & 2x_4 & + & 2x_6 & - & x_7
\end{array}$$

Small Indicies

$$\begin{aligned}
z &= 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
x_5 &= 0 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1
\end{aligned}$$

$$\begin{aligned}
z &= 0 + 53x_2 + 41x_3 - 204x_4 - 20x_5 \\
x_1 &= 0 + 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
x_6 &= 0 - 4x_2 - 2x_3 + 8x_4 + x_5 \\
x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5
\end{aligned}$$

$$\begin{aligned}
z &= 0 + 14.5x_3 - 98x_4 - 6.75x_5 - 13.25x_6 \\
x_1 &= 0 - 0.5x_3 + 4x_4 + 0.75x_5 - 2.75x_6 \\
x_2 &= 0 - 0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6 \\
x_7 &= 1 + 0.5x_3 - 4x_4 - 0.75x_5 - 13.25x_6
\end{aligned}$$

Small Indicies

$$
\begin{array}{lclclclclcl}
z & = & 0 & + & 18x_4 & + & 15x_5 & - & 93x_6 & - & 29x_1 \\
x_2 & = & 0 & - & 2x_4 & - & 0.5x_5 & + & 2.5x_6 & + & x_1 \\
x_3 & = & 0 & + & 8x_4 & + & 1.5x_5 & - & 5.5x_6 & - & 2x_1 \\
x_7 & = & 1 & & & & & & & - & x_1
\end{array}
$$

$$
\begin{array}{lclclclclcl}
z & = & 0 & + & 10.5x_5 & - & 70.5x_6 & - & 20x_1 & - & 9x_2 \\
x_3 & = & 0 & - & 0.5x_5 & + & 4.5x_6 & + & 2x_1 & - & 4x_2 \\
x_4 & = & 0 & - & 0.25x_5 & + & 1.25x_6 & + & 0.5x_1 & - & 0.5x_2 \\
x_7 & = & 1 & & & & & - & x_1 & &
\end{array}
$$

Small Indicies

$$\begin{array}{rcllllll}
z & = & 0 & + 24x_6 & + 22x_1 & - 93x_2 & - 21x_3 \\
x_5 & = & 0 & + 9x_6 & + 4x_1 & - 8x_2 & - 2x_3 \\
x_4 & = & 0 & - x_6 & - 0.5x_1 & + 1.5x_2 & + 0.5x_3 \\
x_7 & = & 1 & & - x_1 & &
\end{array}$$

$$\begin{array}{rcllllll}
z & = & 0 & - 20x_6 & - 27x_2 & + x_3 & - 44x_4 \\
x_5 & = & 0 & + x_6 & + 4x_2 & + 2x_3 & - 8x_4 \\
x_1 & = & 0 & - 2x_6 & + 3x_2 & + x_3 & - 2x_4 \\
x_7 & = & 1 & + 2x_6 & - 3x_2 & - x_3 & + 2x_4
\end{array}$$

$$\begin{array}{rcllllll}
z & = & 1 & - 18x_6 & - 30x_2 & - 42x_4 & - x_7 \\
x_5 & = & 2 & + 5x_6 & - 2x_2 & - 4x_4 & - 2x_7 \\
x_1 & = & 1 & & & & - x_7 \\
x_3 & = & 1 & + 2x_6 & - 3x_2 & + 2x_4 & - x_7
\end{array}$$

Infeasible First Basic Solution

$$\begin{array}{lcccccc} max & 2x_1 & - & x_2 & & \\ s.t. & 2x_1 & - & x_2 & \leq & 2 \\ & x_1 & - & 5x_2 & \leq & -4 \end{array}$$

$$\begin{array}{ccccccc} z & = & 0 & + & 2x_1 & - & x_2 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 \end{array}$$

- x_1 and x_2 is set to 0 in the first basic solution.
- This solution is infeasible since x_4 = -4.

Auxiliary LP

- We will define a related auxiliary LP.
- This auxiliary LP is feasible and bounded.
- Optimal value of this auxiliary LP will indicate if the LP is feasible.
- If LP is feasible, then the slack form of this auxiliary LP will yield a feasible basic solution to the LP (and the corresponding slack form).

Auxiliary Linear Program

- L: LP in standard form:

$$
\begin{array}{llcccl}
max & \sum_{j=1}^{n} c_j x_j & & & & \\
s.t. & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & for & i=1,2,\dots,m \\
 & x_j & \geq & 0 & for & j=1,2,\dots,n
\end{array}
$$

- L_{aux}: Auxiliary LP:

$$
\begin{array}{llcccl}
max & -x_0 & & & & \\
s.t. & \sum_{j=1}^{n} a_{ij} x_j - x_0 & \leq & b_i & for & i=1,2,\dots,m \\
 & x_j & \geq & 0 & for & j=0,1,2,\dots,n
\end{array}
$$

- L_{aux} is bounded and feasible.

INITIALIZE_SIMPLEX

- Let l be the index of the minimum b_l.
- If $b_l \geq 0$, let N={1,2,...,n}, B={n+1,n+2,...,n+m} and return (N,B,A,b,c,0).
- Otherwise form L_{aux} in its slack form.
- (N,B,A,b,c,v) $\leftarrow$ PIVOT(N,B,A,b,c,v,l,0). This basic solution is feasible for L_{aux}.
- Keep pivoting until SIMPLEX terminates. If the returned objective value is 0, then use the returned slack form (with x_0 removed) and restored objective function as a starting point for original LP problem. If the returned objective value is negative, then the original LP problem is infeasible.

INITIALIZE_SIMPLEX - Example

L:

$$\begin{aligned} \max \quad & 2x_1 - x_2 \\ s.t. \quad & 2x_1 - x_2 \leq 2 \\ & x_1 - 5x_2 \leq -4 \end{aligned}$$

L_{aux}

$$\begin{aligned} \max \quad & -x_0 \\ s.t. \quad & 2x_1 - x_2 - x_0 \leq 2 \\ & x_1 - 5x_2 - x_0 \leq -4 \end{aligned}$$

- If the value of the optimal solution to L_{aux} is 0, then if we disregard x_0, we get a feasible solution to L. It is basic (at most 2 variables are positive, others are 0).
- If the value of the optimal solution to L_{aux} is negative, then at least one of the constraints of L is never satisfied. L is infeasible.

INITIALIZE_SIMPLEX - Example

$$
\begin{array}{ccccccccc}
z & = & 0 & & & & & - & x_0 \\
x_3 & = & 2 & - & 2\mathrm{x}_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5\mathrm{x}_2 & + & x_0
\end{array}
$$

$$
\begin{array}{ccccccccc}
z & = & & & & & & - & (x_4 + 4 + x_1 - 5\mathrm{x}_2) \\
x_3 & = & 2 & - & 2\mathrm{x}_1 & + & x_2 & + & (x_4 + 4 + x_1 - 5\mathrm{x}_2) \\
x_0 & = & 4 & + & x_1 & - & 5\mathrm{x}_2 & + & x_4
\end{array}
$$

$$
\begin{array}{ccccccccc}
z & = & -4 & - & x_1 & + & 5\mathrm{x}_2 & - & x_4 \\
x_3 & = & 6 & - & x_1 & - & 4\mathrm{x}_2 & + & x_4 \\
x_0 & = & 4 & + & x_1 & - & 5\mathrm{x}_2 & + & x_4
\end{array}
$$

feasible!!

INITIALIZE_SIMPLEX - Example

$$\begin{array}{ccccccccc} z & = & -4 & - & x_1 & + & 5\mathrm{x}_2 & - & x_4 \\ x_3 & = & 6 & - & x_1 & - & 4\mathrm{x}_2 & + & x_4 \\ x_0 & = & 4 & + & x_1 & - & 5\mathrm{x}_2 & + & x_4 \end{array}$$

$$\begin{array}{ccccccccc} z & = & -4 & - & x_1 & + & 5\left(4/5+x_1/5+x_4/5-x_0/5\right) & - & x_4 \\ x_3 & = & 6 & - & x_1 & - & 4\left(4/5+x_1/5+x_4/5-x_0/5\right) & + & x_4 \\ x_2 & = & 4/5 & + & x_1/5 & - & x_0/5 & + & x_4/5 \end{array}$$

INITIALIZE_SIMPLEX - Example

$$
\begin{aligned}
z &= 0 && && - && x_0 \\
x_3 &= 14/5 &&- 9x_1/5 &&+ 4x_0/5 &&+ x_4/5 \\
x_2 &= 4/5 &&+ x_1/5 &&- x_0/5 &&+ x_4/5
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 &&+ 2x_1 &&- \left(4/5 + x_1/5 - x_0/5 + x_4/5\right) \\
x_3 &= 14/5 &&- 9x_1/5 &&+ 4x_0/5 &&+ x_4/5 \\
x_2 &= 4/5 &&+ x_1/5 &&- x_0/5 &&+ x_4/5
\end{aligned}
$$

$$
\begin{aligned}
z &= -4/5 &&+ 9x_1/4 &&- x_4/5 \\
x_3 &= 14/5 &&- 9x_1/5 &&+ x_4/5 \\
x_2 &= 4/5 &&+ x_1/5 &&+ x_4/5
\end{aligned}
$$

L is feasible $\Leftrightarrow$ Optimal objective value of L_{aux} is 0

$\Rightarrow$ Let $(s_1,s_2,...,s_n)$ be a feasible solution for L.

Let s_0=0. Then $(s_0,s_1,s_2,...,s_n)$ is a feasible solution for L_{aux} with the objective value 0.

Since $x_0 \geq 0$ in L_{aux} and the objective is to maximize $-x_0$ (or minimize x_0), $(s_0,s_1,s_2,...,s_n)$ is an optimal solution for L_{aux} with the objective value 0.

$\Leftarrow$ Let $(s_0,s_1,s_2,...,s_n)$ be the optimal solution for L_{aux} with the objective value 0.

Then s_0=0 and $(s_1,s_2,...,s_n)$ is a feasible solution for L.

If L is infeasible then L_{aux} has a negative optimal solution

$\Rightarrow$ Suppose that L is infeasible.

- Optimal objective value of L_{aux} is not 0.
- L_{aux} is bounded: Let

$$s_0 = | min_{i=1}^{m} \{ b_i \} |$$

(s_0, 0, 0,..., 0) is a feasible solution for L_{aux} with negative objective value.

If L is feasible then L_{aux} returns a basic solution for L

- We create L_{aux} only if the first basic solution to L is infeasible. This is the case when some *b*-values in L are negative. The same *b*-values reappear in L_{aux} and therefore its first basic solution is infeasible.
- Assume that $b_l < 0$ is the smallest *b*-value in L. We have to show that after the first pivoting (where x_0 enters the basis and x_l leaves the basis, all *b*-values become nonnegative.
- This is a straightforward algebraic manipulation, see Lemma 29.12.
- Optimal solution to L_{aux} has objective value 0. Since the first and the last slack form of L_{aux} are equivalent, the value of x_0 must be 0. When x_0 is removed, a slack form feasible for L is obtained.

SIMPLEX – Open Issues

- How to decide that LP is feasible? **SOLVED**
- What to do if the initial basic solution is infeasible? **SOLVED**
- How to select entering and leaving variables? **SOLVED**
- How to decide that LP is unbounded? **SOLVED**
- Does SIMPLEX terminate? **SOLVED**
- Does it terminate with an optimal solution?