Maximum Flow

Flow networks, flow, and max flow problem.

Residual networks, Ford-Fulkerson algorithm.

Cuts and max-flow min-cut theorem.

Edmonds-Karp algorithm.

Matching in bipartite graphs – application of max flow.

Flow Networks

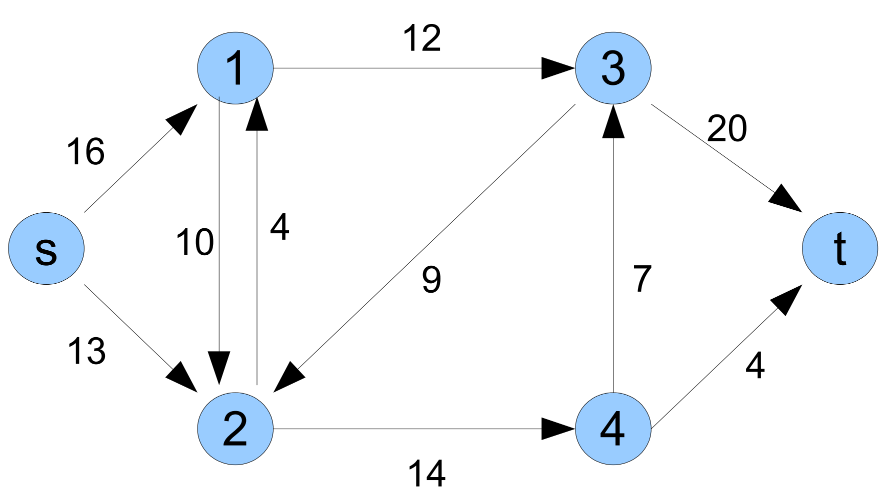
Directed graph $G = (V,E)$.

Edge capacities $c(u,v) \geq 0$ for all $u, v \in V$.

Source vertex s.

Sink vertex t.

Every vertex is reachable from s. Every vertex can reach t.



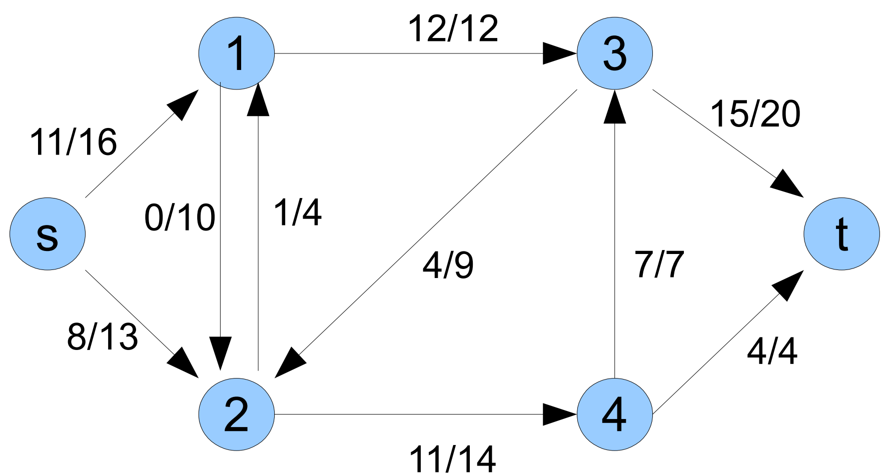
Flow

Flow: Real-valued function f: $V \times V \rightarrow R$ satisfying:

Capacity constraint: $0 \leq f(u,v) \leq c(u,v)$ for all $u, v \in V$

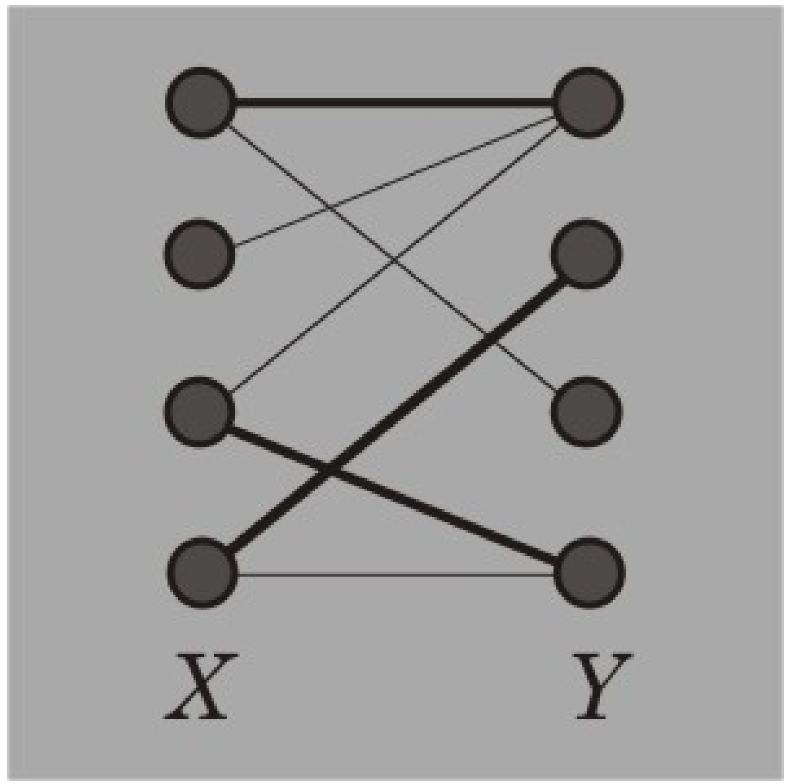
Flow conservation: $\sum_{v \in V} f(u,v) = \sum_{v \in V} f(v,u)$ for all $u \in V - \{ s,t \}$

Flow value: $$|f| = \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s)$$



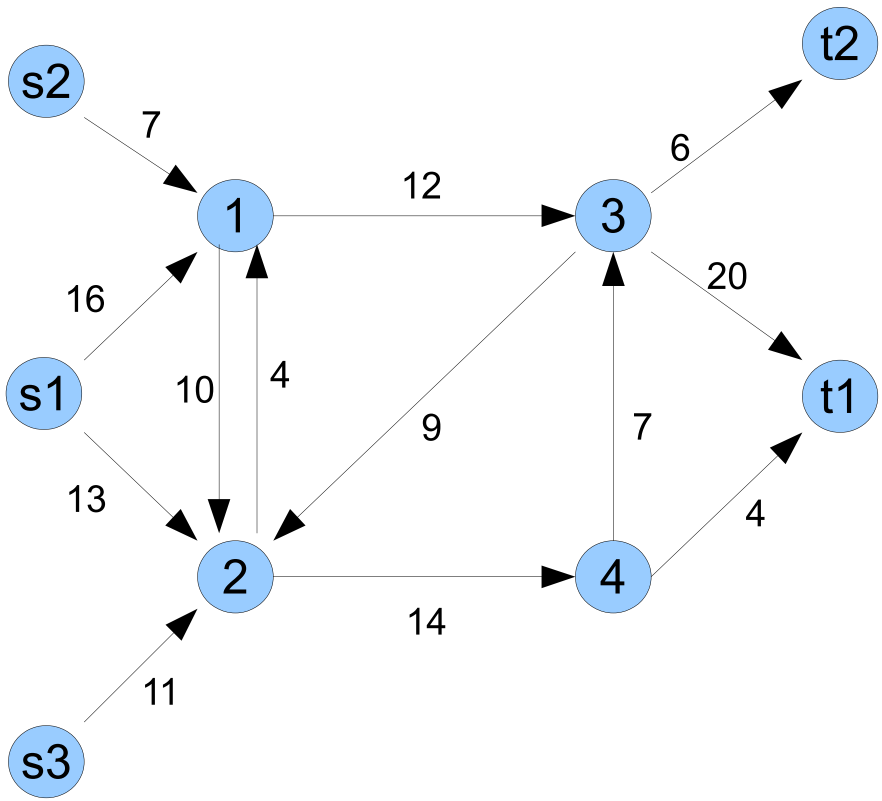
Problem: Maximize flow value subject to the constraints

Assigning Workers to Machines



High Speed Railroad Network

Multiple Sorces and/or Sinks



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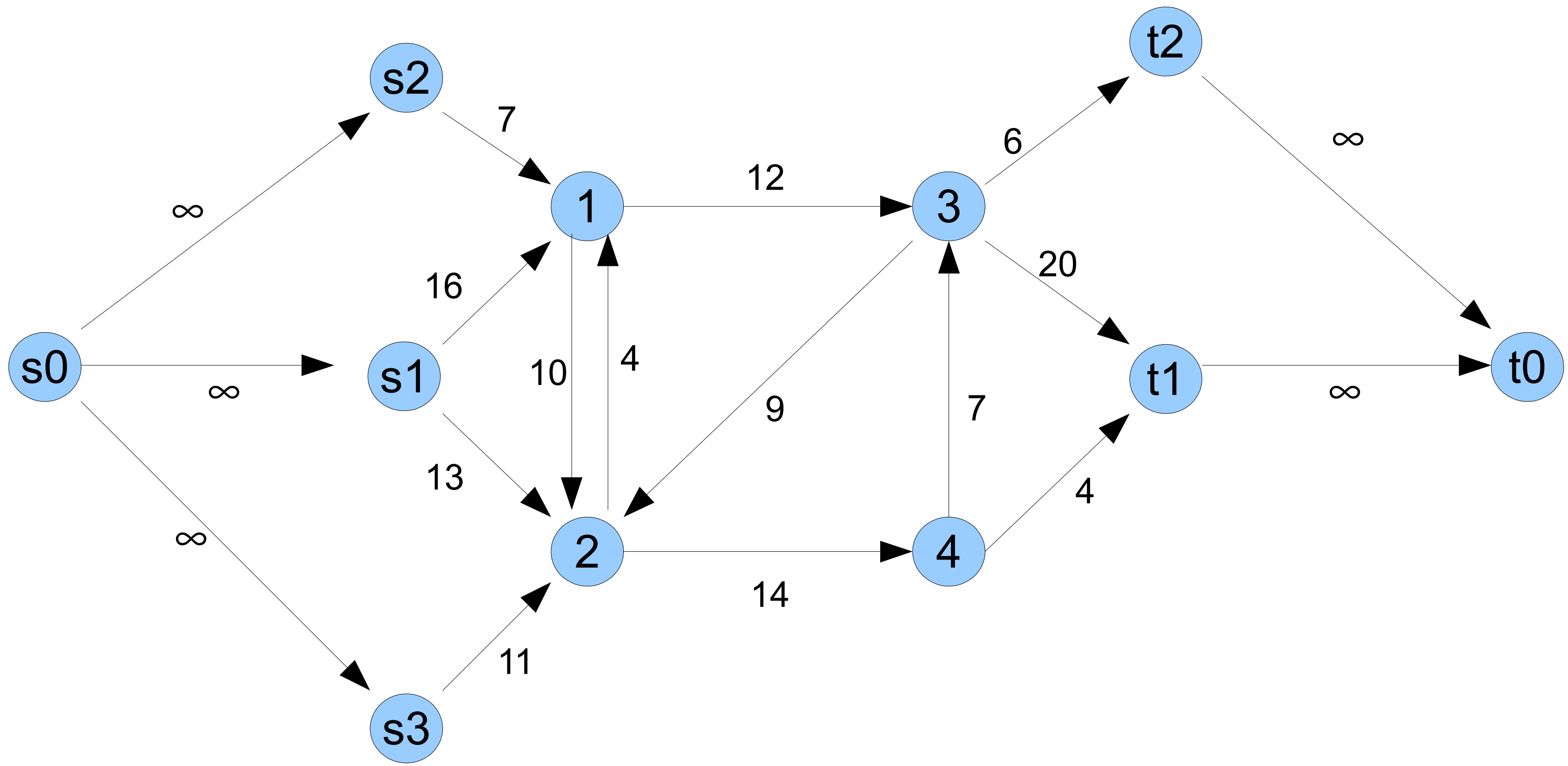
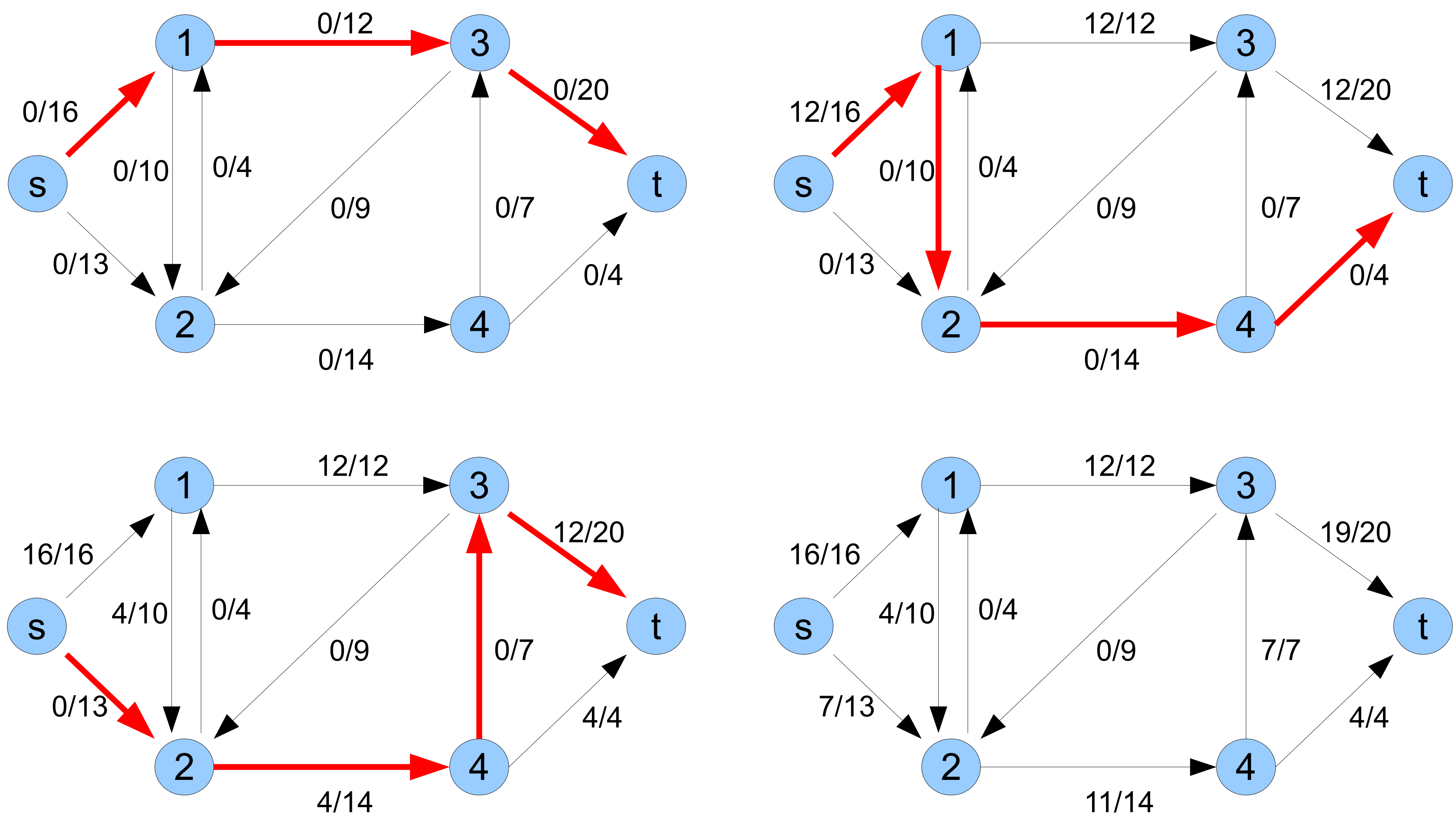


Ford-Fulkerson Method

initialize flow *f* to 0-flow.

while there is a flow augmenting path do augment *f*.



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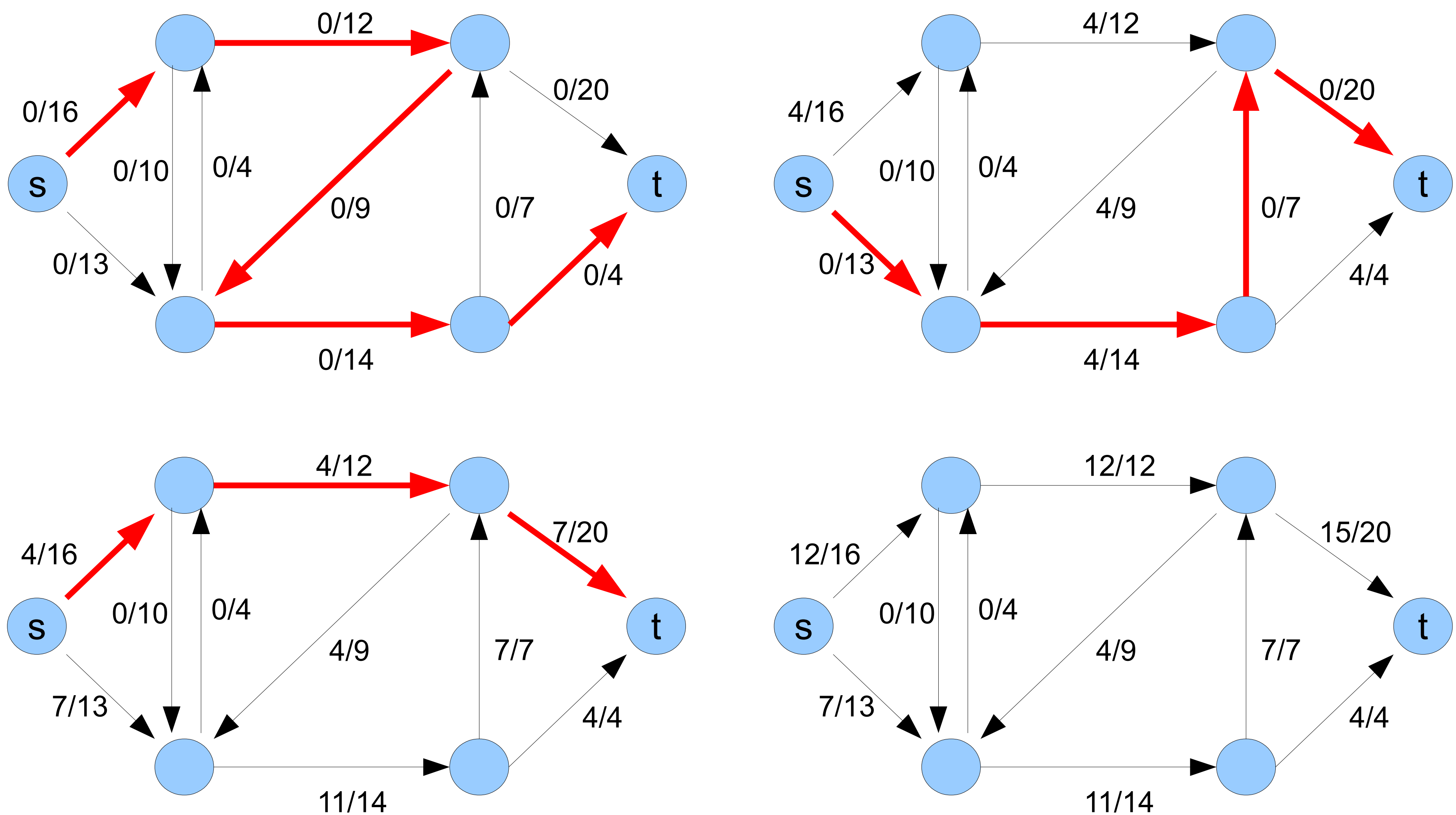
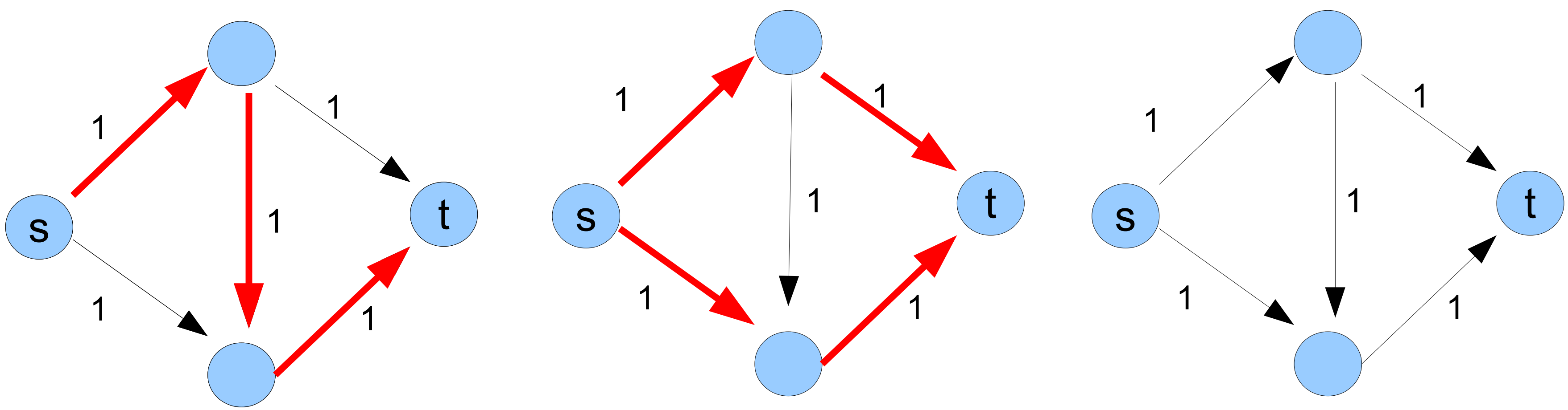
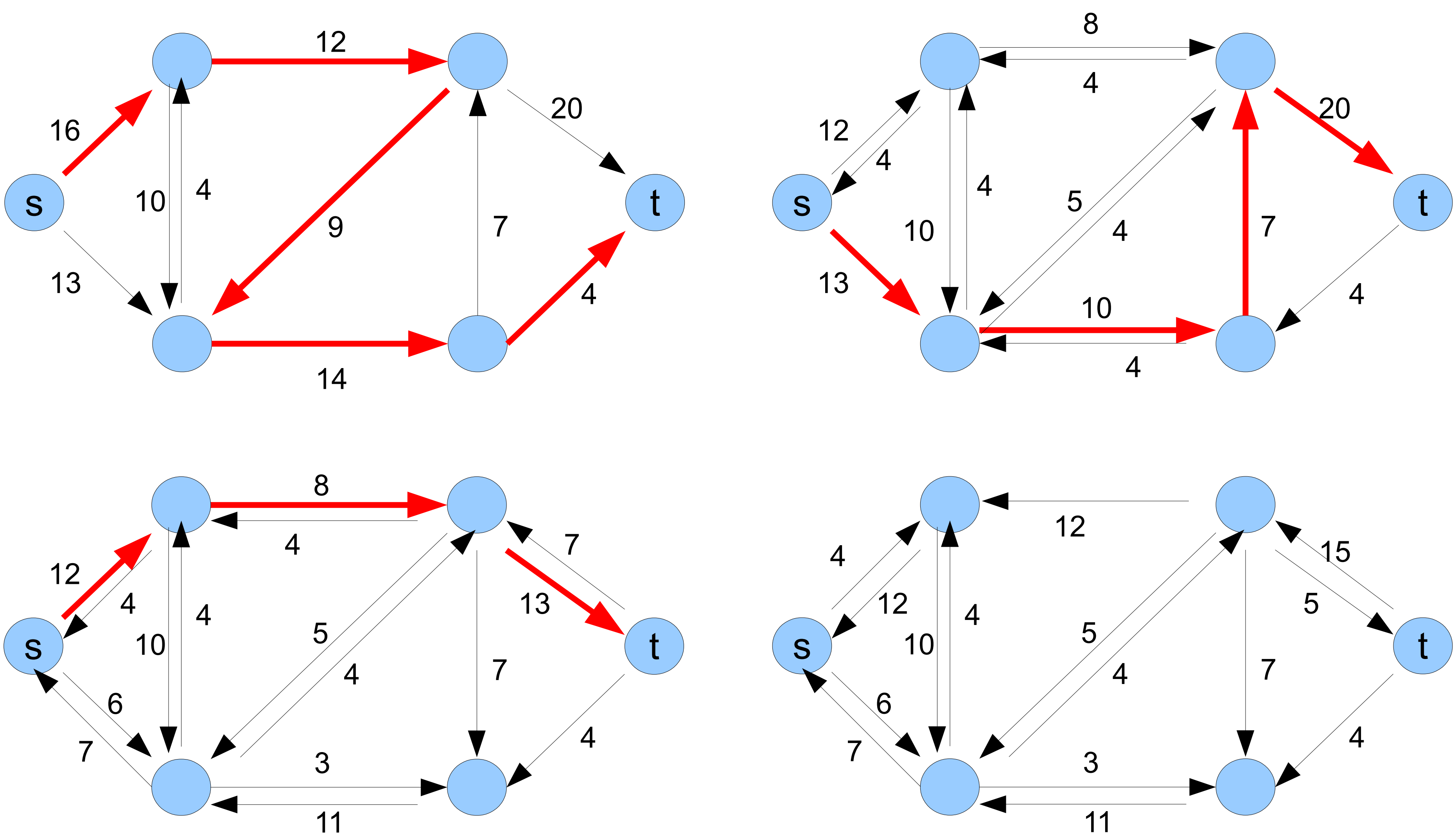


Something is Wrong!



Residual Networks



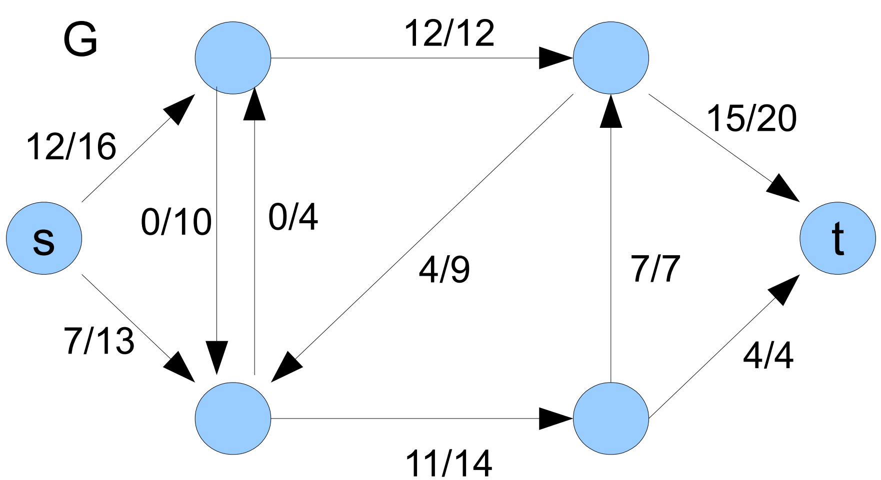
Flows in residual and original networks

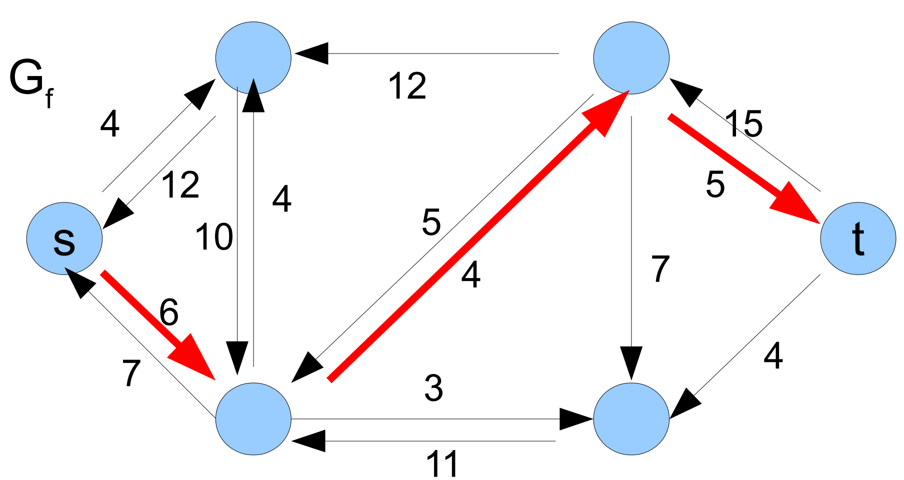
Let f be a flow in network $G = (V,E)$

Let f' be a flow in the residual network G_f of G induced by the flow f.

Let $(f + f')(u,v) = f(u,v) + f'(u,v) – f'(v,u)$ for all (u,v) in E.

$f + f'$ is a flow in G with value $| f+f' | = | f | + | f' |$.





$f + f'$ is a flow

- Capacity constrant: Prove that $(f + f')(u,v) \leq \ c(u,v)$
- Conservation constraint: Prove that

$$\sum_{v \in V} (f + f')(u, v) = \sum_{v \in V} (f + f')(v, u)$$

- Flow value: Prove that $| f + f' | = | f | + | f' |$
- These follow directly from the definitions, see Cormen et al., Lemma 26.1

Flow Between Subset

- Let $G = (V,E)$ be a flow network.
- Let X and Y be two subsets of V.
- Let f be a flow in G.
- Define

$$f(X,Y)=\sum_{x\in X}\sum_{y\in Y} f(x,y)-\sum_{x\in X}\sum_{y\in Y} f(y,x)$$

Cut of Flow Network

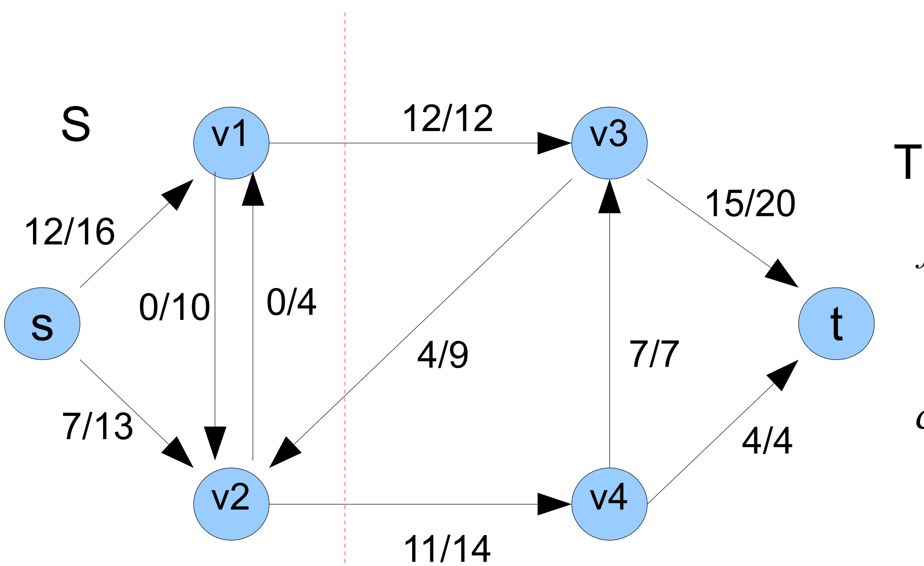
Let $G=(V,E)$, $S \subseteq V$, $T=V-S$, $s \in S$, $t \in T$.

Partition (S,T) of V is then called a **cut**.

Net flow across the cut (S,T) for a flow f is defined to be $f(S,T)$.

Capacity $c(S,T)$ of the cut (S,T) is defined to be $\sum_{u \in S, v \in T} c(u,v)$

Minimum cut of a flow network is a cut of minimum capacity (over all cuts of the network).



$$f(S,T)=f(v_1,v_3)+f(v_2,v_4)-f(v_3,v_2)=12+11-4=19$$

$$c(S,T)=c(v_1,v_3)+c(v_2,v_4)=12+14=26$$