Introduction

- Linear Programming by Example
- Geometric Interpratation
- Linear Programming – Brief History
- Standard and Slack Formulations
- SIMPLEX by Example

Diet Problem (after Chvatal)

- Every day Polly needs:
  - 2000 kcal,
  - 55g protein,
  - 800mg calcium.
- She will get other stuff (e.g., iron and vitamins) by taking pills. Not that this could not be included in the model – we just want to keep it simple.
- She wants a diet that will meet the requirements while being neither expensive nor boring.

Value and Price per Serving

Food	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving
Oatmeal	110	4	2	3
Chicken	205	32	12	24
Eggs	160	13	54	13
Whole milk	160	8	285	9
Cherry pie	420	4	22	20
Pork with beans	260	14	80	19

10 portions of pork with beans would cover her needs!
And would cost only 190. But ...

Limits to What Polly Can Stomach

- Oatmeal: at most 4 servings a day.
- Chicken: at most 3 servings a day.
- Eggs: at most 2 servings a day.
- Milk: at most 8 servings a day.
- Cherry pie: at most 2 servings a day.
- Pork with beans: at most 2 servings a day.

8 servings of milk and 2 servings of cherry pie would meet her needs. Boring but she could stomach it. Especially since it would cost 112. Can she find a less expensive diet?

Variables

- X_1: number of oatmeal servings.
- X_2: number of chicken servings.
- X_3: number of eggs servings.
- X_4: number of milk servings.
- X_5: number of cherry pie servings.
- X_6: number of pork and pie servings.

Linear Constraints

$$
\begin{array}{ccccccccccccc}
x_1 & & & & & & & & & & & \le & 4 \\
 & & x_2 & & & & & & & & & \le & 3 \\
 & & & & x_3 & & & & & & & \le & 2 \\
 & & & & & & x_4 & & & & & \le & 8 \\
 & & & & & & & & x_5 & & & \le & 2 \\
 & & & & & & & & & & x_6 & \le & 2 \\
110x_1 & + & 205x_2 & + & 160x_3 & + & 160x_4 & + & 420x_5 & + & 260x_6 & \ge & 2000 \\
4x_1 & + & 32x_2 & + & 13x_3 & + & 8x_4 & + & 4x_5 & + & 14x_6 & \ge & 55 \\
2x_1 & + & 12x_2 & + & 54x_3 & + & 285x_4 & + & 22x_5 & + & 80x_6 & \ge & 800 \\
x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \ge & 0
\end{array}
$$

Linear Programming Problem

$$
\begin{array}{lcccccccccccccc}
\min & 3x_1 & + & 24x_2 & + & 13x_3 & + & 9x_4 & + & 20x_5 & + & 19x_6 & & \\
s.t. & x_1 & & & & & & & & & & & \leq & 4 \\
 & & & x_2 & & & & & & & & & \leq & 3 \\
 & & & & & x_3 & & & & & & & \leq & 2 \\
 & & & & & & & x_4 & & & & & \leq & 8 \\
 & & & & & & & & & x_5 & & & \leq & 2 \\
 & & & & & & & & & & & x_6 & \leq & 2 \\
 & 110x_1 & + & 205x_2 & + & 160x_3 & + & 160x_4 & + & 420x_5 & + & 260x_6 & \geq & 2000 \\
 & 4x_1 & + & 32x_2 & + & 13x_3 & + & 8x_4 & + & 4x_5 & + & 14x_6 & \geq & 55 \\
 & 2x_1 & + & 12x_2 & + & 54x_3 & + & 285x_4 & + & 22x_5 & + & 80x_6 & \geq & 800 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}
$$

http://www-neos.mcs.anl.gov/CaseStudies/dietpy/WebForms/

Linear Objective Function

$$\begin{array}{ll} \mathit{min} & 3\mathrm{x}_1 + 24\mathrm{x}_2 + 13\mathrm{x}_3 + 9\mathrm{x}_4 + 20\mathrm{x}_5 + 19\mathrm{x}_6 \\ s.t. & \text{linear constraints} \end{array}$$

- The value of the **objective function** for a particular set of values for x_1, x_2, x_3, x_4, x_5, x_6 is called its **objective value**.
- If a particular set of values for x_1, x_2, x_3, x_4, x_5, x_6, satisfies all constraints, it is said to be a **feasible solution**.
- The set of all feasible solutions is called the **feasible region**. It can be shown to be convex.
- A feasible solution that has the minimum (or maximum) objective value is called an **optimal solution**.

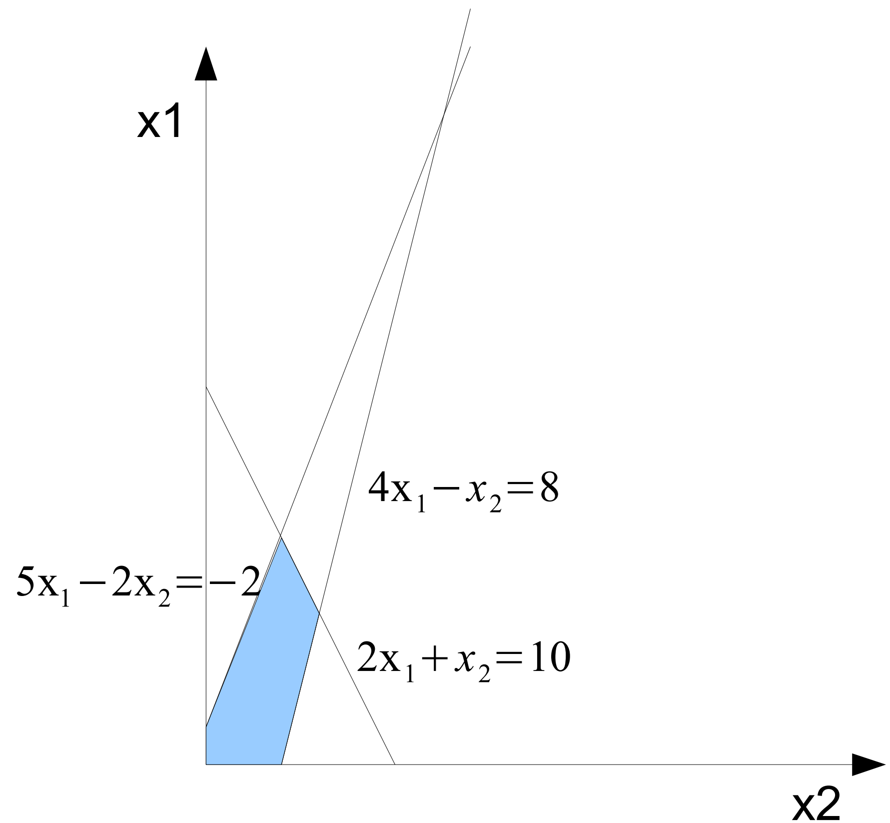
General LP Problem

$$
\begin{aligned}
& min \quad \sum_{j=1}^{n} c_j x_j \\
& s.t. \quad m \text{ linear constraints}
\end{aligned}
$$

- Minimization or maximization of a **linear** objective function with *n* real-valued variables.
- An optimal solution must satisfy *m* **linear** constraints (inequalities or equalities).
- Strict inequalities are not allowed.
- ”programming” in ”linear programming” does not refer to any code. It was chosen before computer programming was born.

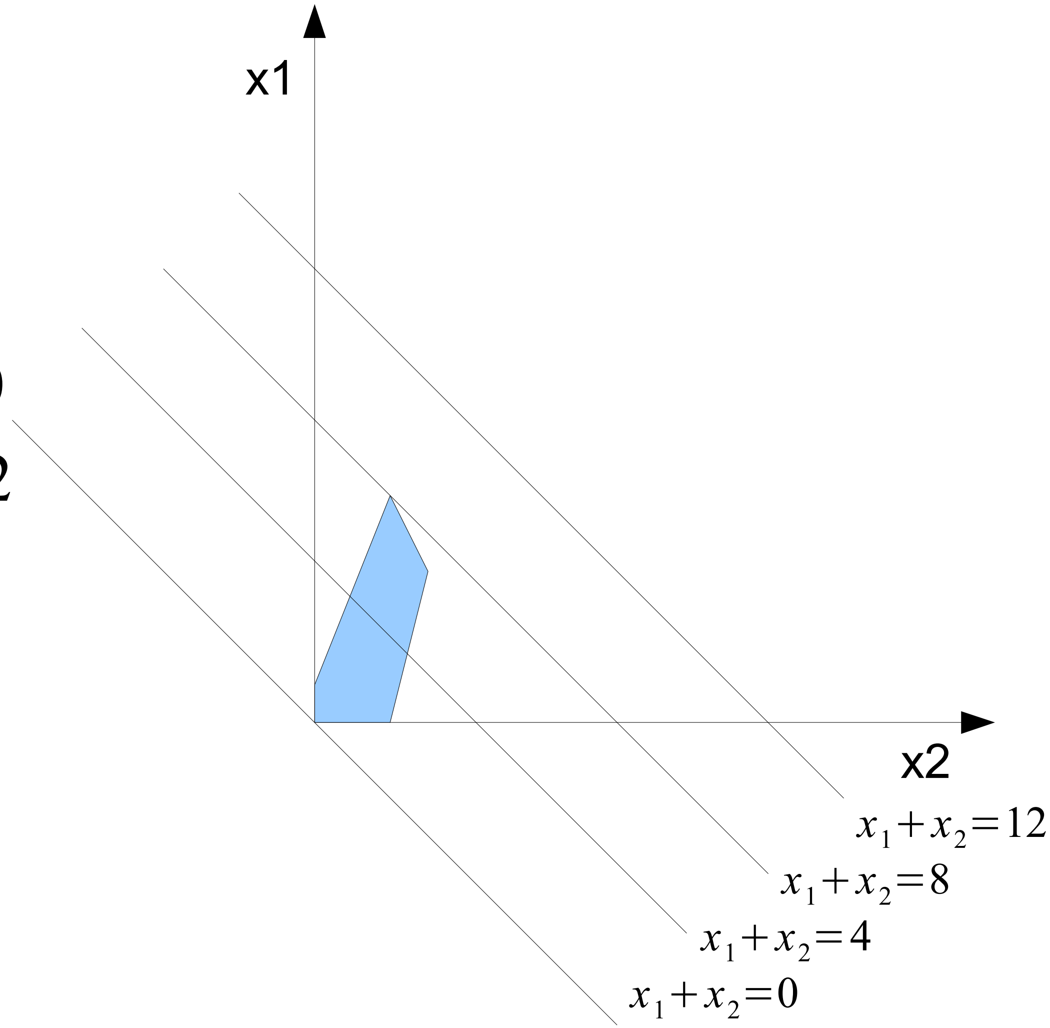
Geometric Interpretation

$$\begin{array}{llllll} max & x_1 & + & x_2 & & \\ s.t. & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$



Geometric Interpretation

$$
\begin{array}{llclcl}
\max & x_1 & + & x_2 & & \\
s.t. & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1, & & x_2 & \geq & 0
\end{array}
$$



Special Cases of LP

- LP may have no feasible solution (in case of conflicting constraints).
- LP may have feasible solutions but no optimal solution (in case of unboundedness).
- LP may have more than one optimal solution.

Geometric Interpretation in R^d

- d variables.
- Each constraint defines a half-space in R^d. The set of feasible solutions is the intersection of these half-spaces, called **simplex**. It is convex. Can be unbounded or empty.
- The set of points in which the objective function has the same value z is a **hyperplane**.
- The value of the objective function increases or decreases as the hyperplane is translated.
- If the set of feasible solutions is bounded and not empty, then there is an optimal solution in an extreme vertex of the simplex.

General Idea Behind SIMPLEX Algorithm

- SIMPLEX starts with a feasible solution corresponding to some vertex of the simplex. We will show how to find such a vertex (or decide that the feasible region is empty).
- SIMPLEX keeps ”jumping” from a vertex of the simplex to a new vertex if the new vertex offers a feasible solution that is better (or at least not worse). We will show how SIMPLEX ”jumps”. We will show how to avoid ”cycling” when SIMPLEX jumps through feasible solutions with the same objective value.
- When no more ”jumps” are possible, we will show that SIMPLEX is in an optimal vertex (or the LP is unbounded).
- We will show that the number of jumps is finite (at most equal to the number of simplex vertices).

History of LP

- L.V. Kantorovich pointed out in 1939 the importance of restricted classes of LPs.
- T.C. Koopmans realized in 1947 the importance of LP for the analysis of classical economic theories.
- G.B. Dantzig designed in 1947 the simplex method to solve LP for U.S. Air Force. Not a polynomial algorithm!
- Many applications followed over the years. Work of Kantorovich was rediscovered in 1950's.
- In 1975 Kantorovich and Koopmans got the Nobel prize.
- L.G. Khachian discovered first polynomial algorithm in 1979. Terribly slow.
- N. Karmarkar discovered second polynomial algorithm in 1984. Practical.

Applications of LP

- Scheduling problems: airline wishes to schedule its flight crews on all flights while using as few crew members as possible.
- Manufacturing problems: How much of each type of product should be produced subject to technical and financial constraints.
- Location problems: Locating drills to maximize the amount of oil that will be extracted under given budget constraints.
- Many network and graph problems can be formulated as LP; dimensionig telecommunication and distribution networks.
- Allocation of financial assets to maximize profit or minimize risk.
- Integer linear programming problems.

General LP Problem

$$\begin{aligned} &min \quad \sum\nolimits_{j=1}^{n} c_j x_j \\ &s.t. \quad m \,\text{linear constraints} \end{aligned}$$

- Minimization or maximization of a **linear** objective function with *n* real-valued variables.
- An optimal solution must satisfy *m* **linear** constraints (inequalities or equalities).
- Strict inequalities are not allowed.
- ”programming” in ”linear programming” does not refer to any code. It was chosen before computer programming was born.

Equivalent LPs

- Two max LPs *L* and *L*' are **equivalent**
  - if for each feasible solution ***x*** to *L* with the objective value z, there is a corresponding feasible solution ***x'*** to *L*' with the same objective value *z*,
  - if for each feasible solution ***x'*** to *L'* with the objective value z, there is a corresponding feasible solution ***x*** to *L* with the same objective value *z*.
- Similarly for two min LPs.
- A min LP *L* and a max LP *L*' are **equivalent**
  - if for each feasible solution ***x*** to *L* with the objective value *z*, there is a corresponding feasible solution ***x'*** to *L*' with the objective value -*z,*
  - if for each feasible solution ***x'*** to *L'* with the objective value *z*, there is a corresponding feasible solution ***x*** to *L* with the objective value -*z*.

LP in Standard Form

- Maximization of a linear function.
- *n* non-negative real-valued variables.
- *m* linear inequalities ("less than or equal to").

$$
\begin{array}{lll}
max & \sum_{j=1}^{n} c_j x_j & \\
s.t. & \sum_{j=1}^{n} a_{ij} x_j \le b_i & for \quad i = 1, 2, \ldots, m \\
 & x_j \ge 0 & for \quad j = 1, 2, \ldots, n
\end{array}
$$

Converting LP into Standard Form

$$
\begin{array}{lrcrcl}
min & -2x_1 & + & 3x_2 & & \\
s.t. & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & & & x_1 & \geq & 0
\end{array}
$$

- Minimization LP is converted to an equivalent maximization problem by negating the coefficients of the objective function.

$$
\begin{array}{lrcrcl}
max & 2x_1 & - & 3x_2 & & \\
s.t. & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & & & x_1 & \geq & 0
\end{array}
$$

Converting LP into Standard Form

$$
\begin{array}{lccccc}
max & 2x_1 & - & 3x_2 & & \\
s.t. & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & x_1 & & & \geq & 0
\end{array}
$$

- Every variable x_j without the non-negativity constraint is replaced by two non-negative variables x'_j and x''_j and each occurrence of x_j is replaced by $x'_j - x''_j$.

$$
\begin{array}{lccccccc}
max & 2x_1 & - & 3x'_2 & + & 3x''_2 & & \\
s.t. & x_1 & + & x'_2 & - & x''_2 & = & 7 \\
 & x_1 & - & 2x'_2 & + & 2x''_2 & \leq & 4 \\
 & x_1, & & x'_2, & & x''_2 & \geq & 0
\end{array}
$$

Converting LP into Standard Form

$$
\begin{array}{llllllll}
max & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
s.t. & x_1 & + & x_2' & - & x_2'' & = & 7 \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & x_1, & & x_2', & & x_2'' & \geq & 0
\end{array}
$$

- Each equality constraint is replaced by a pair of ”opposite” inequality constraints.

$$
\begin{array}{llllllll}
max & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
s.t. & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
 & x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & x_1, & & x_2', & & x_2'' & \geq & 0
\end{array}
$$

Converting LP into Standard Form

$$
\begin{array}{llllllll}
max & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
s.t. & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
 & x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & x_1, & & x_2', & & x_2'' & \geq & 0
\end{array}
$$

- Inequalities are ”turned around” by multiplying both sides by -1.

$$
\begin{array}{llllllll}
max & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
s.t. & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
 & -x_1 & - & x_2' & + & x_2'' & \leq & -7 \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & x_1, & & x_2', & & x_2'' & \geq & 0
\end{array}
$$

Converting LP into a Standard Form

$$\begin{array}{lccccccc}
max & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
s.t. & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
 & -x_1 & - & x_2' & + & x_2'' & \leq & -7 \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & x_1, & & x_2', & & x_2'' & \geq & 0
\end{array}$$

- Renaming the variables

$$\begin{array}{lccccccc}
max & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
s.t. & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}$$

LP in Standard Form

- *n* variables, *m* constraints

$$
\begin{array}{llcccl}
max & \sum_{j=1}^{n} c_j x_j & & & & \\
s.t. & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & for & i = 1, 2, \ldots, m \\
 & x_j & \geq & 0 & for & j = 1, 2, \ldots, n
\end{array}
$$

Slack Variables

- Consider one of the constraints, for example

$$2x_1 + 3x_2 + x_3 \leq 5$$

- For every feasible solution x_1, x_2, x_3 , the value of the left-hand side is at most the value of the right-hand side.
- Often there can be a **slack** between these two values.
- Denote the slack by x_4.
- By requiring that $x_4 \geq 0$, we can replace the inequality by the equality

$$2x_1 + 3x_2 + x_3 + x_4 = 5$$

Slack Variables

$$\begin{aligned} &max && \sum_{j=1}^{n} c_j x_j \\ &s.t. && \sum_{j=1}^{n} a_{ij} x_j \le b_i && for \quad i=1,2,\dots,m \\ & && x_j \ge 0 && for \quad j=1,2,\dots,n \end{aligned}$$

$$\begin{aligned} &max && \sum_{j=1}^{n} c_j x_j \\ &s.t. && \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i && for \quad i=1,2,\dots,m \\ & && x_j \ge 0 && for \quad j=1,2,\dots,n+m \end{aligned}$$

LP in Slack Form

$$\begin{array}{llll} max & \sum_{j=1}^{n} c_j x_j & & \\ s.t. & \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} & = b_i & for \quad i=1,2,\dots,m \\ & x_j & \geq 0 & for \quad j=1,2,\dots,n+m \end{array}$$

$$\begin{array}{llll} max & z & = \sum_{j=1}^{n} c_j x_j & \\ s.t. & x_{n+i} & = b_i - \sum_{j=1}^{n} a_{ij} x_j & for \quad i=1,2,\dots,m \\ & x_j & \geq 0 & for \quad j=1,2,\dots,n+m \end{array}$$

$$\begin{array}{lll} z & = 0 + \sum_{j=1}^{n} c_j x_j & \\ x_{n+i} & = b_i - \sum_{j=1}^{n} a_{ij} x_j & for \quad i=1,2,\dots,m \end{array}$$

Standard to Slack Form - Example

$$
\begin{array}{lccccccc}
max & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
s.t. & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4
\end{array}
$$

$$
\begin{array}{lccccccccccccc}
max & 2x_1 & - & 3x_2 & + & 3x_3 & & & & & & & & \\
s.t. & x_1 & + & x_2 & - & x_3 & + & x_4 & & & & & = & 7 \\
 & -x_1 & - & x_2 & + & x_3 & & & + & x_5 & & & = & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & & & & & + & x_6 & = & 4
\end{array}
$$

$$
\begin{array}{ccccccccc}
z & = & 0 & + & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Basic Solutions

- Any solution of LP in standard form yields a solution of LP in the corresponding slack form (with the same objective value) and vice versa.
- Setting right-hand side variables of the slack form to 0 yields a **basic solution.**
- Left-hand side variables are called **basic**. Right-hand side variables are called **nonbasic**.
- The basic variables are said to constitute a **basis.**
- Note that a basic solution does not need to be feasible.

SIMPLEX - Example

- LP problem in standard form:

$$
\begin{array}{lccccccc}
max & 3x_1 & + & x_2 & + & 2x_3 & & \\
s.t. & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

SIMPLEX – Example Continued

- LP in slack form:

$$
\begin{array}{rcrcrcrcr}
z & = & 0 & + & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

- Set all **nonbasic** variables (right-hand side) to 0.
- Compute values of **basic** variables: x_4=30, x_5=24, x_6=36.
- Compute the objective value z (= 0).
- This gives the feasible basic solution (0,0,0,30,24,36).
- It is feasible; not always the case – we were lucky.

SIMPLEX: 1. Pivoting

- Can x_1 be increased without violating feasibility?

$$
\begin{array}{ccccccccc}
z & = & 0 & + & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

- If x_1 is increased to 1, then x_4=29, x_5=22, x_6=32 while z=3. (1,0,0,29,22,32) is a feasible solution.
- If x_1 is increased to 2, then x_4=28, x_5=20, x_6=28 while z=6. (2,0,0,28,20,28) is a feasible solution.
- If x_1 is increased to 3, then x_4=27, x_5=18, x_6=24 while z=9. (3,0,0,27,18,24) is a feasible solution.

SIMPLEX: 1. Pivoting

- Can x_1 be increased without violating feasibility? By how much?

$$
\begin{array}{lclclclcl}
z & = & 0 & + & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

- If x_1 is increased beyond 30 then x_4 becomes negative.
- If x_1 is increased beyond 12 then x_5 becomes negative.
- If x_1 is increased beyond 9 then x_6 becomes negative.
- Constraint defining x_6 is **binding**.

SIMPLEX: 1. Pivoting

- So x_1 can be increased to 9 without losing feasibility. The feasible solution is (9,0,0,21,6,0) and z = 27.
- We will now rewrite the slack form to an equivalent slack form with x_1, x_4, x_5 as basic variables and with (9,0,0,21,6,0) being its feasible basic solution.
- This rewriting is called **pivoting.**
- Binding constraint defining x_6 is rewritten so that it has x_1 on its left-hand side.
- All occurences of x_1 in other constraints and in the objective function are replaced by the right-hand side of the binding constraint.

SIMPLEX: 1. Pivoting

$$
\begin{array}{lclclclclcl}
z & = & 0 & + & 3\left(9-x_2/4-x_3/2-x_6/4\right) & + & x_2 & + & 2\mathrm{x}_3 \\
x_4 & = & 30 & - & \left(9-x_2/4-x_3/2-x_6/4\right) & - & x_2 & - & 3\mathrm{x}_3 \\
x_5 & = & 24 & - & 2\left(9-x_2/4-x_3/2-x_6/4\right) & - & 2\mathrm{x}_2 & - & 5\mathrm{x}_3 \\
x_1 & = & 9 & - & x_2/4 & - & x_3/2 & - & x_6/4
\end{array}
$$

$$
\begin{array}{lclclclcl}
z & = & 27 & + & x_2/4 & + & x_3/2 & - & 3\mathrm{x}_6/4 \\
x_4 & = & 21 & - & 3\mathrm{x}_2/4 & - & 5\mathrm{x}_3/2 & + & x_6/4 \\
x_5 & = & 6 & - & 3\mathrm{x}_2/2 & - & 4\mathrm{x}_3 & + & x_6/2 \\
x_1 & = & 9 & - & x_2/4 & - & x_3/2 & - & x_6/4
\end{array}
$$

New basic variables: $x_1=9$, $x_4=21$, $x_5=6$

New objective value z = 27

New feasible basic solution: (9,0,0,21,6,0)

SIMPLEX: 2. Pivoting

- Can x_3 be increased without violating feasibility? By how much?

$$
\begin{array}{ccccccccc}
z & = & 27 & + & x_2/4 & + & x_3/2 & - & 3x_6/4 \\
x_4 & = & 21 & - & 3x_2/4 & - & 5x_3/2 & + & x_6/4 \\
x_5 & = & 6 & - & 3x_2/2 & - & 4x_3 & + & x_6/2 \\
x_1 & = & 9 & - & x_2/4 & - & x_3/2 & - & x_6/4
\end{array}
$$

- If x_3 is increased beyond 42/5 then $x_4 < 0$.
- If x_3 is increased beyond 3/2 then $x_5 < 0$.
- If x_3 is increased beyond 18 then $x_1 < 0$.
- Constraint defining x_5 is binding.

SIMPLEX: 2. Pivoting

$$
\begin{array}{ccccccccc}
z & = & 27 & + & x_2/4 & + & \frac{1}{2}(3/2-3\mathrm{x}_2/8+x_6/8-x_5/4) & - & 3\mathrm{x}_6/4 \\
x_4 & = & 21 & - & 3\mathrm{x}_2/4 & - & \frac{5}{2}(3/2-3\mathrm{x}_2/8+x_6/8-x_5/4) & + & x_6/4 \\
x_3 & = & 3/2 & - & 3\mathrm{x}_2/8 & - & x_5/4 & + & x_6/8 \\
x_1 & = & 9 & - & x_2/4 & - & \frac{1}{2}(3/2-3\mathrm{x}_2/8+x_6/8-x_5/4) & - & x_6/4
\end{array}
$$

$$
\begin{array}{ccccccccc}
z & = & 111/4 & + & x_2/16 & - & x_5/8 & - & 11\mathrm{x}_6/16 \\
x_4 & = & 69/4 & + & 3\mathrm{x}_2/16 & + & 5\mathrm{x}_5/8 & - & x_6/16 \\
x_3 & = & 3/2 & - & 3\mathrm{x}_2/8 & - & x_5/4 & + & x_6/8 \\
x_1 & = & 33/4 & - & x_2/16 & + & x_5/8 & - & 5\mathrm{x}_6/16
\end{array}
$$

New basic variables: x_1=33/4, x_3=3/2, x_4=69/4

New objective value z = 27.75

New feasible basic solution: (33/4, 0, 3/2, 69/4, 0, 0)

SIMPLEX: 3. Pivoting

- Can x_2 be increased without violating feasibility? By how much?

$$
\begin{array}{ccccccccc}
z & = & 111/4 & + & x_2/16 & - & x_5/8 & - & 11x_6/16 \\
x_4 & = & 69/4 & + & 3x_2/16 & + & 5x_5/8 & - & x_6/16 \\
x_3 & = & 3/2 & - & 3x_2/8 & - & x_5/4 & + & x_6/8 \\
x_1 & = & 33/4 & - & x_2/16 & + & x_5/8 & - & 5x_6/16
\end{array}
$$

- If x_2 is increased then x_4 also increases.
- If x_2 is increased beyond 4 then $x_3 < 0$.
- If x_2 is increased beyond 132 then $x_1 < 0$.
- Constraint defining x_3 is binding.

SIMPLEX: 3.Pivoting

$$
\begin{array}{lllllllll}
z & = & 111/4 & + & \frac{1}{16}(4 - 8x_3/3 - 2x_5/3 + x_6/3) & - & x_5/8 & - & 11x_6/16 \\
x_4 & = & 69/4 & + & \frac{1}{16}(4 - 8x_3/3 - 2x_5/3 + x_6/3) & + & 5x_5/8 & - & x_6/16 \\
x_2 & = & 4 & - & 8x_3/3 & - & 2x_5/3 & + & x_6/3 \\
x_1 & = & 33/4 & - & \frac{1}{16}(4 - 8x_3/3 - 2x_5/3 + x_6/3) & + & x_5/8 & - & 5x_6/16
\end{array}
$$

$$
\begin{array}{lllllllll}
z & = & 28 & - & x_3/6 & - & x_5/6 & - & 2x_6/3 \\
x_4 & = & 18 & - & x_3/2 & + & x_5/2 & + & 0x_6 \\
x_2 & = & 4 & - & 8x_3/3 & - & 2x_5/3 & + & x_6/3 \\
x_1 & = & 8 & + & x_3/6 & + & x_5/6 & - & x_6/3
\end{array}
$$

New basic variables: x_1=8, x_2=4, x_4=18

New objective value z = 28

New feasible basic solution: (8, 4, 0, 18, 0, 0) is optimal

Pivoting in General

- PIVOT(*N*,*B*,*A*,*b*,*c*,*v*,*l*,*e*)
  - Compute the coefficients of the bounding constraint so that the **entering** basic variable x_e is expressed as a linear combination of the other variables.

$$b_e = b_l / a_{le} \qquad a_{ej} = a_{lj} / a_{le}, \forall j \in N \setminus e \qquad a_{el} = 1 / a_{le}$$

  - Compute the coefficients of the remaining constraints and the objective function (by substituting x_e by the right-hand side of the rewritten binding equation).

$$b_i = b_i - a_{ie} b_e, \forall i \in B \setminus l \qquad a_{ij} = a_{ij} - a_{ie} a_{ej}, \forall j \in N \setminus e \qquad a_{il} = -a_{ie} a_{el}$$

$$v = v + c_e b_e \qquad c_j = c_j - c_e a_{ej}, \forall j \in N \setminus e \qquad c_l = -c_e a_{el}$$

  - Compute new sets of basic and nonbasic variables (remove x_e from *N* and add it to *B*, remove x_l from *B* and add it to *N*).