



Faculty of Science



Loop Parallelism I

Cosmin E. Oancea

`cosmin.oancea@diku.dk`

Department of Computer Science (DIKU)
University of Copenhagen

September 2019 PMPH Lecture Slides



Course Organization

W	HARDWARE		SOFTWARE	LAB/CUDA
1	Trends Vector Machine	←	List HOM (Map-Reduce)	Intro & Simple Map Programming
2	In Order Processor	→ ←	VLIW Instr Scheduling	Scan & Reduce
3	Cache Coherence		Loop Parallelism I	Sparse Vect Matrix Mult
4	Interconnection Networks		Case Studies & Optimizations	Transpose & Matrix Matrix Mult
5	Memory Consistency		Optimising Locality	Sorting & Profiling & Mem Optimizations
6	OoO, Spec Processor		Thread-Level Speculation	Project Work

Three narrative threads: the path to complex & good design:

- **Design Space** tradeoffs, constraints, common case, trends.
- **Reasoning**: from simple to complex, **Applying Concepts**.



Motivation

- + So far we reasoned about how to parallelize a known algorithm
- + using a clean, functional approach, e.g., flattening,
- + which provides work and depth guarantees,
 - but does **NOT** account for locality of reference.

Why do we have to look at imperative loops?



Motivation

- + So far we reasoned about how to parallelize a known algorithm
- + using a clean, functional approach, e.g., flattening,
- + which provides work and depth guarantees,
 - but does **NOT** account for locality of reference.

Why do we have to look at imperative loops?

- A lot of legacy sequential imperative code, C++/Java/Fortran.
- Need to parallelize the implementation of unknown algorithm,
- Need to optimize parallelism, e.g., locality of reference requires subscript analysis.



- 1 Direction-Vector Analysis
- 2 Block Tiling: Matrix Multiplication Case Study
- 3 Coalesced Accesses: Matrix Transposition Case Study



Problem Statement

Three Loop Examples

DO i = 1, N	DO i = 2, N	DO i = 2, N
DO j = 1, N	DO j = 2, N	DO j = 1, N
A[j,i] = A[j,i] ..	A[j,i] = A[j-1,i-1]...	A[i,j] = A[i-1,j+1]...
ENDDO	B[j,i] = B[j-1,i]...	ENDDO
ENDDO	ENDDO ENDDO	ENDDO

Iterations are ordered *lexicographically*, i.e., in the order they occur in the sequential execution, e.g., $\vec{k} = (i=2, j=4) < \vec{l} = (i=3, j=3)$.

- Which of the three loop nests is amenable to parallelization?
- Loop interchange is one of the most simple and useful code transformations, e.g., used to enhance locality of reference, parallel-loop granularity, and even to “create” parallelism.
- In which loop nest is it safe to interchange the loops?



Definition of a Dependency

Load-Store Classification of Dependencies

True Dependency (RAW)

S1 X = ..

S2 .. = X

Anti Dependency (WAR)

S1 .. = X

S2 X = ..

Output dependency (WAW)

S1 X = ...

S2 X = ...

Def. Loop Dependence: There is a dependence from statement S_1 to S_2 in a loop nest *iff* \exists iterations \vec{k}, \vec{l} such that:



Definition of a Dependency

Load-Store Classification of Dependencies

True Dependency (RAW)		Anti Dependency (WAR)		Output dependency (WAW)	
S1	X = ..	S1	.. = X	S1	X = ...
S2	.. = X	S2	X = ..	S2	X = ...

Def. Loop Dependence: There is a dependence from statement $S1$ to $S2$ in a loop nest *iff* \exists iterations \vec{k}, \vec{l} such that:

1. $\vec{k} < \vec{l}$ or $\vec{k} = \vec{l}$ and \exists an execution path from statement $S1$ to statement $S2$ **such that:**
2. $S1$ accesses memory location M on iteration \vec{k} , and
3. $S2$ accesses memory location M on iteration \vec{l} , and
4. one of these accesses is a write.

We say that $S1$ is the source and $S2$ is the sink of the dependence, because $S1$ executes before $S2$ in the sequential program execution. Dependence depicted with an arrow pointing from source to sink.



Definition of a Dependency

Load-Store Classification of Dependencies

True Dependency (RAW)	Anti Dependency (WAR)	Output dependency (WAW)
S1 X = ..	S1 .. = X	S1 X = ...
S2 .. = X	S2 X = ..	S2 X = ...

Def. Loop Dependence: There is a dependence from statement $S1$ to $S2$ in a loop nest *iff* \exists iterations \vec{k}, \vec{l} such that:

1. $\vec{k} < \vec{l}$ or $\vec{k} = \vec{l}$ and \exists an execution path from statement $S1$ to statement $S2$ **such that:**
2. $S1$ accesses memory location M on iteration \vec{k} , and
3. $S2$ accesses memory location M on iteration \vec{l} , and
4. one of these accesses is a write.

We say that $S1$ is the source and $S2$ is the sink of the dependence, because $S1$ executes before $S2$ in the sequential program execution. Dependence depicted with an arrow pointing from source to sink.

We are most interested in cross iteration dependencies, i.e., $\vec{k} < \vec{l}$. Intra iteration dependencies, i.e., $\vec{k} = \vec{l}$ are analysed for ILP.



Loop-Nest Dependencies

Lexicographic ordering, e.g., $\vec{k}=(i=2,j=4) < \vec{l}=(i=3,j=3)$.

Three Loop Examples

```
DO i = 1, N
  DO j = 1, N
    A[j,i] = A[j,i] ..
  ENDDO
ENDDO
```

```
DO i = 2, N
  DO j = 2, N
    A[j,i] = A[j-1,i-1]...
    B[j,i] = B[j-1,i]...
  ENDDO ENDDO
```

```
DO i = 2, N
  DO j = 1, N
    A[i,j] = A[i-1,j+1]...
  ENDDO
ENDDO
```



Loop-Nest Dependencies

Lexicographic ordering, e.g., $\vec{k} = (i=2, j=4) < \vec{l} = (i=3, j=3)$.

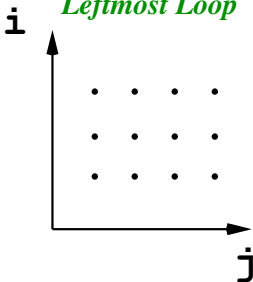
Three Loop Examples

```
DO i = 1, N
  DO j = 1, N
    A[j,i] = A[j,i] ..
  ENDDO
ENDDO
```

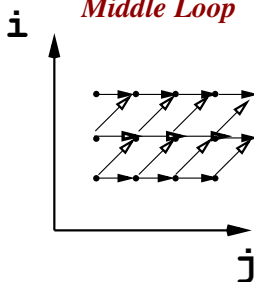
```
DO i = 2, N
  DO j = 2, N
    A[j,i] = A[j-1,i-1]...
    B[j,i] = B[j-1,i]...
  ENDDO
ENDDO
```

```
DO i = 2, N
  DO j = 1, N
    A[i,j] = A[i-1,j+1]...
  ENDDO
ENDDO
```

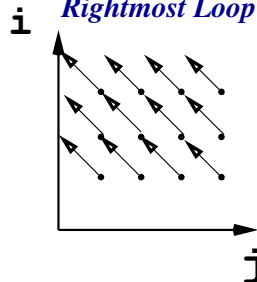
Leftmost Loop



Middle Loop



Rightmost Loop



How can I summarize this information?



Aggregate Dependencies via Direction Vectors

Write the Direction Vectors for Each Loop:

DO i = 1, N	DO i = 2, N	DO i = 2, N
DO j = 1, N	DO j = 2, N	DO j = 1, N
S1 A[j,i]=A[j,i]..	S1 A[j,i]=A[j-1,i]...	S1 A[i,j]=A[i-1,j+1]...
ENDDO	S2 B[j,i]=B[j-1,i-1]...	ENDDO
ENDDO	ENDDO ENDDO	ENDDO

Dependencies depicted via an edge *from* the stmt that executes first in the loop nest, i.e., *the source*, to the one that executes later, *the sink*.

Def. Dependence Direction: Assume \exists a dependence from $S1$ in iteration \vec{k} to $S2$ in \vec{l} ($\vec{k} \leq \vec{l}$). *Dependence-direction vector* $\vec{D}(\vec{k}, \vec{l})$:

1. $\vec{D}(\vec{k}, \vec{l})_m = "<"$ if $\vec{k}_m < \vec{l}_m$,
2. $\vec{D}(\vec{k}, \vec{l})_m = "="$ if $\vec{k}_m = \vec{l}_m$,
3. $\vec{D}(\vec{k}, \vec{l})_m = ">"$ if $\vec{k}_m > \vec{l}_m$.

If the source is a write and the sink a read then RAW dependency,
if the source is a read then WAR, if both are writes then WAW.



How to Compute the Direction Vectors?

- For any two statements $S1$ and $S2$ that may access the same array A (and one of the accesses is a write),
- in two symbolic iterations $I^1 \equiv (i_1^1, \dots, i_m^1)$ and $I^2 = (i_1^2, \dots, i_m^2)$ (such that $I^1 < I^2$)
- on indices $A[e_1^1, \dots, e_n^1]$ and $A[e_1^2, \dots, e_n^2]$, respectively,
- then *the direction vectors may be derived* from the equations
$$\begin{cases} e_1^1 = e_1^2 \\ \dots \\ e_n^1 = e_n^2 \end{cases}$$
(The system of equations models the definition of a dependency: both accesses need to refer to the same memory location!)



Parallelism and Loop Interchange

Direction Vectors/Matrix for Three Loops

```
DO i = 1, N
  DO j = 1, N
S1    A[j,i]=A[j,i]..
      ENDDO
  ENDDO
```



Parallelism and Loop Interchange

Direction Vectors/Matrix for Three Loops

```

DO i = 1, N
  DO j = 1, N
S1    A[j,i]=A[j,i]..
      ENDDO
  ENDDO
For S1→S1:
  (j1,i1)=(j2,i2)
  i1 = i2 & j1 = j2

```

```

DO i = 2, N
  DO j = 2, N
S1    A[j,i]=A[j-1,i]...
S2    B[j,i]=B[j-1,i-1]...
      ENDDO
  ENDDO

```

Direction matrix:

S1→S1: [=,=]



Parallelism and Loop Interchange

Direction Vectors/Matrix for Three Loops

```

DO i = 1, N
  DO j = 1, N
S1    A[j,i]=A[j,i]..
      ENDDO
  ENDDO

```

```

For S1→S1:
  (j1,i1)=(j2,i2)
  i1 = i2 & j1 = j2

```

Direction matrix:

S1→S1: [=,=]

```

DO i = 2, N
  DO j = 2, N
S1    A[j,i]=A[j-1,i]...
S2    B[j,i]=B[j-1,i-1]...
      ENDDO
  ENDDO

```

```

S1→S1: (j1,i1)=(j2-1,i2)
        i1 = i2 & j1 < j2
S2→S2: (j1,i1)=(j2-1,i2-1)
        i1 < i2 & j1 < j2

```

S1→S1: [=,<]

S2→S2: [<,<]

```

DO i = 2, N
  DO j = 1, N
S1    A[i,j]=A[i-1,j+1]...
      ENDDO
  ENDDO

```

For S1→S1:



Parallelism and Loop Interchange

Direction Vectors/Matrix for Three Loops

<pre> DO i = 1, N DO j = 1, N S1 A[j,i]=A[j,i].. ENDDO ENDDO For S1→S1: (j1,i1)=(j2,i2) i1 = i2 & j1 = j2 Direction matrix: S1→S1: [=,=]</pre>	<pre> DO i = 2, N DO j = 2, N S1 A[j,i]=A[j-1,i]... S2 B[j,i]=B[j-1,i-1]... ENDDO ENDDO For S1→S1: (j1,i1)=(j2-1,i2) i1 = i2 & j1 < j2 S2→S2: (j1,i1)=(j2-1,i2-1) i1 < i2 & j1 < j2 Direction matrix: S1→S1: [=,<] S2→S2: [<,<]</pre>	<pre> DO i = 2, N DO j = 1, N S1 A[i,j]=A[i-1,j+1]... ENDDO ENDDO For S1→S1: (i1,j1) = (i2-1,j2+1) i1 < i2 & j1 > j2 Direction matrix: S1→S1: [<,>]</pre>
---	---	--

Th. Parallelism: A loop in a loop nest is parallel *iff* all its directions are either = or there exists an outer loop whose corresp. direction is <.

A direction vector cannot have > as the first non-= symbol, as that would mean that I depend on something in the future.



Parallelism and Loop Interchange

Direction Vectors/Matrix for Three Loops

<pre> DO i = 1, N DO j = 1, N S1 A[j,i]=A[j,i].. ENDDO ENDDO For S1→S1: (j1,i1)=(j2,i2) i1 = i2 & j1 = j2 Direction matrix: S1→S1: [=,=]</pre>	<pre> DO i = 2, N DO j = 2, N S1 A[j,i]=A[j-1,i]... S2 B[j,i]=B[j-1,i-1]... ENDDO ENDDO S1→S1: (j1,i1)=(j2-1,i2) i1 = i2 & j1 < j2 S2→S2: (j1,i1)=(j2-1,i2-1) i1 < i2 & j1 < j2 Direction matrix: S1→S1: [=,<] S2→S2: [<,<]</pre>	<pre> DO i = 2, N DO j = 1, N S1 A[i,j]=A[i-1,j+1]... ENDDO ENDDO For S1→S1: (i1,j1) = (i1-1,j2+1) i1 < i2 & j1 > j2 Direction matrix: S1→S1: [<,>]</pre>
---	---	--

Th. Loop Interchange: A column permutation of the loops in a loop nest is legal *iff* permuting the direction matrix in the same way *does NOT* result in a > direction as the leftmost non-= direction in a row.



Parallelism and Loop Interchange

Direction Vectors/Matrix for Three Loops

DO i = 1, N DO j = 1, N S1 A[j,i]=A[j,i].. ENDDO ENDDO	DO i = 2, N DO j = 2, N S1 A[j,i]=A[j-1,i]... S2 B[j,i]=B[j-1,i-1]... ENDDO ENDDO	DO i = 2, N DO j = 1, N S1 A[i,j]=A[i-1,j+1]... ENDDO ENDDO
For S1→S1: j1 = j2 i1 = i2 (i2,j2)-(i1,j1)= [=,=]	For S1→S1: j1 = j2-1 i1 = i2 (i2,j2)-(i1,j1)=[=,<] For S2→S2: j1 = j2-1 i1 = i2-1 (i2,j2)-(i1,j1)=[<,<]	For S1→S1: i1 = i2-1 j1 = j2+1 (i2,j2)-(i1,j1)=[<,>]

Interchange is safe for the first and second nests, but not for the third!

e.g., $\begin{bmatrix} =, < \\ <, < \end{bmatrix} \rightarrow \begin{bmatrix} <, = \\ <, < \end{bmatrix}$ (for the second loop nest)



Parallelism and Loop Interchange

Direction Vectors/Matrix for Three Loops

DO i = 1, N	DO i = 2, N	DO i = 2, N
DO j = 1, N	DO j = 2, N	DO j = 1, N
S1 A[j,i]=A[j,i]..	S1 A[j,i]=A[j-1,i]...	S1 A[i,j]=A[i-1,j+1]...
ENDDO	S2 B[j,i]=B[j-1,i-1]...	ENDDO
ENDDO	ENDDO ENDDO	ENDDO
For S1→S1: j1 = j2	For S1→S1: j1 = j2-1	For S1→S1: i1 = i2-1
i1 = i2	i1 = i2	j1 = j2+1
(i2,j2)-(i1,j1)=	(i2,j2)-(i1,j1)=[=,<]	(i2,j2)-(i1,j1)=[<,>]
[=,=]	For S2→S2: j1 = j2-1	
	i1 = i2-1	
	(i2,j2)-(i1,j1)=[<,<]	

Interchange is safe for the first and second nests, but not for the third!

e.g., [=,<] → [<,<] (for the second loop nest)

[<,<] [<,<]

After interchange, loop j of the second loop nest is parallel.

Corollary: A parallel loop can be always interchanged inwards.



Dependency Graph and Loop Distribution

Def. Dependency Graph: edges from the source of the dependency, i.e., early iteration, to the sink, i.e., later iteration.

Th. Loop Distribution: Statements that are in a dependence cycle remain in one (sequential) loop. The others are distributed to separate loops in graph order; if no cycle then parallel loops.

Vectorization Example: Remember Vector Machines?

```
DO i = 3, N
S1  A[i] = B[i-2] ...
S2  B[i] = B[i-1] ...
ENDDO

For S2→S1: i1 = i2-2, [<]
For S2→S2: i1 = i2-1, [<]
```



Dependency Graph and Loop Distribution

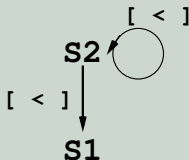
Def. Dependency Graph: edges from the source of the dependency, i.e., early iteration, to the sink, i.e., later iteration.

Th. Loop Distribution: Statements that are in a dependence cycle remain in one (sequential) loop. The others are distributed to separate loops in graph order; if no cycle then parallel loops.

Vectorization Example: Remember Vector Machines?

```
DO i = 3, N
S1  A[i] = B[i-2] ...
S2  B[i] = B[i-1] ...
ENDDO
```

```
For S2→S1: i1 = i2-2, [<]
For S2→S2: i1 = i2-1, [<]
```



```
DO i = 3, N
S2  B[i] = B[i-1] ...
ENDDO
```

```
DOALL i = 3, N
S1  A[i] = B[i-2] ...
ENDDOALL
```

Corollary: It is always legal to distribute a parallel loop;
but requires array expansion for local variables or if output
dependencies are present.



Loop Distribution May Require Array Expansion

```
float tmp;  
for(i=2; i<N; i++) {  
    tmp = 2*B[i-2];  
    A[i] = tmp;  
    B[i] = tmp+B[i-1]  
}
```

```
float tmp[N];  
for(int i=2; i<N; i++) {  
    tmp[i] = 2*B[i-2];  
    B[i] = tmp[i]+B[i-1];  
}  
  
forall(int i=2; i<N; i++) {  
    A[i] = tmp[i];  
}
```

No matter where `tmp` is declared (inside or outside the loop) it needs to be expanded into an array in order to do loop distribution.

If `tmp` is declared outside the loop then requires **privatization**,



Loop Distribution May Require Array Expansion

```
float tmp;  
for(i=2; i<N; i++) {  
    tmp = 2*B[i-2];  
    A[i] = tmp;  
    B[i] = tmp+B[i-1]  
}
```

```
float tmp[N];  
for(int i=2; i<N; i++) {  
    tmp[i] = 2*B[i-2];  
    B[i] = tmp[i]+B[i-1];  
}  
  
forall(int i=2; i<N; i++) {  
    A[i] = tmp[i];  
}
```

No matter where `tmp` is declared (inside or outside the loop) it needs to be expanded into an array in order to do loop distribution.

If `tmp` is declared outside the loop then requires **privatization**, because it actually causes frequent WAW dependencies. However its value is written before being used within the same iteration. Hence it is semantically equivalent to a locally declared variable, which will remove the output (WAW) dependency.

Distribution requires array expansion of the scalar `tmp`.



False Dependencies (WAR/WAW)

- **Cross-Iteration Anti Dependencies (WAR)** correspond to a read from the array as it was before the loop \Rightarrow can be eliminated by reading from a copy of the array.
- **Cross-Iteration WAW Dependencies (WAW):**
If they correspond to the case in which every **read** from a scalar or array location is covered by a **previous same-iteration write** \Rightarrow can be eliminated **privatization (renaming)**, which semantically moves the declaration of the variable (scalar or array) inside the loop.
- Direction-vectors reasoning is limited to relatively simple loop nests, e.g., difficult to reason about privatization in such a way.



Example: Eliminating WAR Dependencies

Anti Dependency (WAR) and Output Dependency (WAW)

```
float tmp = A[1];  
for (i=0; i<N-1; i++)  
S1  A[i] = A[i+1];  
    A[N-1] = tmp;  
  
//S1→S1: i1+1=i2, [<] WAR
```



Example: Eliminating WAR Dependencies

Anti Dependency (WAR) and Output Dependency (WAW)

```

float tmp = A[1];
for (i=0; i<N-1; i++)
S1  A[i] = A[i+1];
    A[N-1] = tmp;

//S1→S1: i1+1=i2, [<] WAR

// Solution: copy A into A'
// and use A' for the reads!
float Acopy[N];
#pragma omp parallel for
    for(i=0; i<N; i++) {
        Acopy[i] = A[i];
    }
    tmp = A[1];
#pragma omp parallel for private(i)
    for (i=0; i<N-1; i++) {
        A[i] = Acopy[i+1];
    }
    A[N-1] = tmp;

```

For OpenMP code, run the g++ compiler with the -fopenmp flag. You may play with the degree of parallelism by setting the OMP_NUM_THREADS environment variable, e.g., to run on 4 cores use:

```
$ export OMP_NUM_THREADS=4.
```



Example: Eliminating WAW Dependencies

Anti Dependency (WAR) and Output Dependency (WAW)

```
int A[M];
for(i=0; i<N; i++){
    for(int j=0, j<M; j++)
        A[j] = (4*i+4*j) % M;
    for(int k=0; k<N; k++)
        X[i,k] = X[i,k-1] *
            A[A[(2*i+k)%M]%M];
}
```



Example: Eliminating WAW Dependencies

Anti Dependency (WAR) and Output Dependency (WAW)

// The write to A[j] causes multiple WAWs,

```
int A[M];
for(i=0; i<N; i++){
    for(int j=0, j<M; j++)
        A[j] = (4*i+4*j) % M;
    for(int k=0; k<N; k++)
        X[i,k] = X[i,k-1] *
                A[A[(2*i+k)%M]%M];
}
```



Example: Eliminating WAW Dependencies

Anti Dependency (WAR) and Output Dependency (WAW)

```

int A[M];
for(i=0; i<N; i++){
    for(int j=0, j<M; j++)
        A[j] = (4*i+4*j) % M;
    for(int k=0; k<N; k++)
        X[i,k] = X[i,k-1] *
            A[A[(2*i+k)%M]%M];
}

// The write to A[j] causes multiple WAWs,
// but A is fully written in the inner loop
#pragma omp parallel{
    int A[M];
    #pragma omp for
    for(int i=0; i<N; i++){
        for(int j=0, j<M; j++)
            A[j] = (4*i+4*j) % M;
        for(int k=0; k<N; k++)
            X[i,k]=X[i,k-1] *
                A[A[(2*i+k)%M]%M];
    }
}

```

The declaration of `int A[M]` inside the OMP parallel region, will create an array A for every thread (and visible/accessible inside the thread context). Alternatively you may expand A with a new dimension of number-of-processor size, and use explicit indexing.



Reduction is Typically Easy To Recognize

If all the statements in which a scalar variable x appears are of the form $x \oplus = \text{exp}$, where x does not appear in exp and \oplus is associative then the cross-iteration RAWs on x can be resolved by:

- privatizing x initialized with the neutral element,
- computing the per-processor partial values of x ,
- reducing the x s across processors and with the initial value.

```
// compilation requires g++ -fopenmp ...
float x = 6.0;
#pragma omp parallel for reduction(+:x) private(i,j)
for(i=1; i<N; i++) {
    for(j=1; j<N; j++) {
        if ( A[i,j] >= 2.0 )    x += 2*A[i,j-1];
        else if( A[i,j] > 0.0 ) x += A[i-1,j+1];
    }
    if ( i % (j+1) == 3 ) x += A[i,i];
}
```



Scan and Segmented Scan Are Difficult!

Compilers cannot recognize and parallelize even simple scans:

- they raise a cross-iteration true dependency (RAW),
- they appear in a multitude of forms,
- hence they are difficult to analyze.

// What kind of scans are these?

1. `A[0] = B[0];`
 `for(i=1; i<N; i++) {`
 `A[i] = A[i-1] + B[i];`
 `}`
2. `acc = 0;`
 `for(i=0; i<N; i++){`
 `acc = acc ^ i;`
 `A[i] = acc;`
 `}`
3. `for(j=0; j<M; j++)`
 `A[0,j] = B[0,j];`
 `for(i=1; i<N; i++) {`
 `for(j=0; j<M; j++)`
 `A[i,j] = A[i-1,j] + B[i,j];`
 `}`



Scan and Segmented Scan Are Difficult!

Compilers cannot recognize and parallelize even simple scans:

- they raise a cross-iteration true dependency (RAW),
- they appear in a multitude of forms,
- hence they are difficult to analyze.

// What kind of scans are these?

```
1. A[0] = B[0];
   for(i=1; i<N; i++) {
       A[i] = A[i-1] + B[i];
   }
```

```
2. acc = 0;
   for(i=0; i<N; i++){
       acc = acc ^ i;
       A[i] = acc;
   }
```

```
3. for(j=0; j<M; j++)
    A[0,j] = B[0,j];
   for(i=1; i<N; i++) {
       for(j=0; j<M; j++)
           A[i,j] = A[i-1,j] + B[i,j];
   }
```

```
1. let A = scan (+) 0 B
```

```
2. let A = scan (^) 0 (iota N)
```

```
3. let A = scan (\ a b -> map2 (+) a b)
      (replicate M 0.0) B
```

≡

```
let A = transpose <|
    map (scanInc (+) 0.0) <|
    transpose B
```



- 1 Direction-Vector Analysis
- 2 **Block Tiling: Matrix Multiplication Case Study**
- 3 Coalesced Accesses: Matrix Transposition Case Study



Matrix Multiplication: Loop Strip Mining

```
DOALL i = 1, M, 1    // Parallel
  DOALL j = 1, N, 1  // Parallel
    float tmp = 0.0
    DO k = 1, U, 1 // Reduction
      tmp += A[i,k]*B[k,j]
    ENDDO
    C[i,j] = tmp;
  ENDDO
ENDDO
```

←Matrix Multiplication. Matrices:

- input A has M rows and U columns
- input B has U rows and N columns
- result C has M rows and N columns

Loops of indices i and j are parallel
(can be proved by direction vectors).

Accesses to A and B invariant to loops i and j \Rightarrow Block Tiling to optimize locality of reference!



Matrix Multiplication: Loop Strip Mining

```
DOALL i = 1, M, 1    // Parallel
  DOALL j = 1, N, 1  // Parallel
    float tmp = 0.0
    DO k = 1, U, 1 // Reduction
      tmp += A[i,k]*B[k,j]
    ENDDO
    C[i,j] = tmp;
  ENDDO
ENDDO
```

← Matrix Multiplication. Matrices:

- input A has M rows and U columns
- input B has U rows and N columns
- result C has M rows and N columns

Loops of indices i and j are parallel
(can be proved by direction vectors).

Accesses to A and B invariant to loops i and j \Rightarrow Block Tiling to optimize locality of reference!

First step: Strip Mining, always safe since the transformed loop executes the same instructions in the same order as the original loop:

```
DO i = 1, N, 1  // stride 1
  loop_body(i)
ENDDO
```

```
DO ii = 1, N, T    // stride T
  DO i = ii, MIN(ii+T-1,N), 1
    loop_body(i)
  ENDDO
ENDDO
```



Matrix Multiplication: Loop Interchange

After strip mining all loops with a tile of size T:

```
DOALL ii = 1, M, T
  DOALL i = ii, MIN(ii+T-1,M), 1      // loop
    DOALL jj = 1, N, T                // interchange. Why Safe?
      DOALL j = jj, MIN(jj+T-1,N), 1
        float tmp = 0.0
        DO kk = 1, U, T
          DO k = kk, MIN(kk+T-1,U), 1
            tmp += A[i,k]*B[k,j]
          ENDDO ENDDO
        C[i,j] = tmp;
      ENDDO ENDDO ENDDO ENDDO
```



Matrix Multiplication: Loop Interchange

After strip mining all loops with a tile of size T:

```
DOALL ii = 1, M, T
  DOALL i = ii, MIN(ii+T-1,M), 1    // loop
    DOALL jj = 1, N, T              // interchange. Why Safe?
      DOALL j = jj, MIN(jj+T-1,N), 1
        float tmp = 0.0
        DO kk = 1, U, T
          DO k = kk, MIN(kk+T-1,U), 1
            tmp += A[i,k]*B[k,j]
          ENDDO ENDDO
        C[i,j] = tmp;
      ENDDO ENDDO ENDDO ENDDO
```

The second step is to apply loop interchange between the loops of indices *i* and *jj*. This is safe because loop *i* is parallel, hence it can always be interchanged inwards!



Matrix Multiplication: Summarizing Read Subscripts

After loop interchange we have a grid shape, as in CUDA:

```
DOALL ii = 1, M, T           // grid.y
  DOALL jj = 1, N, T         // grid.x
    DOALL i = ii, MIN(ii+T-1,M), 1 // block.y
      DOALL j = jj, MIN(jj+T-1,N), 1 // block.x
        float tmp = 0.0
        DO kk = 1, U, T
          DO k = kk, MIN(kk+T-1,U), 1
            tmp += A[i,k]*B[k,j]
          ENDDO
        ENDDO
      ENDDO
    C[i,j] = tmp;
  ENDDO ENDDO ENDDO ENDDO
```

The third step is to summarize the subscripts of A and B read inside the loop of index k, for fixed ii, jj and kk (x:y denotes [x..y]):



Matrix Multiplication: Summarizing Read Subscripts

After loop interchange we have a grid shape, as in CUDA:

```
DOALL ii = 1, M, T           // grid.y
  DOALL jj = 1, N, T         // grid.x
    DOALL i = ii, MIN(ii+T-1,M), 1 // block.y
      DOALL j = jj, MIN(jj+T-1,N), 1 // block.x
        float tmp = 0.0
        DO kk = 1, U, T
          DO k = kk, MIN(kk+T-1,U), 1
            tmp += A[i,k]*B[k,j]
          ENDDO
        ENDDO
      ENDDO
    ENDDO
  ENDDO
ENDDO ENDDO ENDDO ENDDO
```

The third step is to summarize the subscripts of A and B read inside the loop of index k, for fixed ii, jj and kk (x:y denotes [x...y]):

- A subscripts [ii : MIN(ii+T-1,M), kk : MIN(kk+T-1,U)]
- B subscripts [kk : MIN(kk+T-1,U), jj : MIN(jj+T-1,N)]
- Summaries have size at most T^2 & independent on i, j, and k \Rightarrow CUDA-block threads cooperatively copy-in data to shared mem!



Block Tiled Matrix Multiplication CUDA Kernel

Shared memory padded with zeros to remove the branch from loop k!

```
DOALL ii = 1, M, T    // grid.y
  DOALL jj = 1, N, T // grid.x
    DOALL i = ii, MIN(ii+T-1,M), 1
      DOALL j = jj, MIN(jj+T-1,N), 1
        float tmp = 0.0
        DO kk = 1, U, T
          //we would like to copy
          //to shared memory here
          //& use it inside loop k
          DO k = kk, MIN(kk+T-1,U), 1
            tmp += A[i,k]*B[k,j]
          ENDDO
        ENDDO
      ENDDO
    ENDDO ENDDO ENDDO ENDDO
```



Block Tiled Matrix Multiplication CUDA Kernel

Shared memory padded with zeros to remove the branch from loop k!

```

DOALL ii = 1, M, T    // grid.y
  DOALL jj = 1, N, T  // grid.x
    DOALL i = ii, MIN(ii+T-1,M), 1
      DOALL j = jj, MIN(jj+T-1,N), 1
        float tmp = 0.0
        DO kk = 1, U, T
          //we would like to copy
          //to shared memory here
          //& use it inside loop k
          DO k = kk, MIN(kk+T-1,U), 1
            tmp += A[i,k]*B[k,j]
          ENDDO
        ENDDO
      ENDDO
    ENDDO
  ENDDO
ENDDO ENDDO ENDDO ENDDO

```

```

__global__ void matMultTiledKer( ... ) {
  __shared__ T Ash[T][T], Bsh[T][T];
  int ii = blockIdx.y * T; //blockDim.x==T
  int jj = blockIdx.x * T; //blockDim.y==T
  int tidy = threadIdx.y, i = tidy+ii;
  int tidx = threadIdx.x, j = tidx+jj;
  float tmp = 0.0;

  for(int kk=0; kk<U; kk+=T) {
    Ash[tidy][tidx] = (i<M && kk+tidx<U) ?
                      A[i*U + (kk+tidx)] : 0.0;
    Bsh[tidy][tidx] = (j<N && kk_tidx<U) ?
                      B[(kk+tidy)*N + j] : 0.0;
    __syncthreads();
    for(int k=0; k<T; k++) {
      tmp += Ash[tidy][k] * Bsh[k][tidx]
    } __syncthreads();
  } if (i<M && j<N) C[i*N + j] = tmp;
}

```

A global memory access amortized by (T-1) shared memory accesses.



Measuring GFlops Performance

Sequential matrix multiplication $\sim 2 \times M \times N \times U$ floating point operations. What is the GFlops performance of our implementation?

```
// CPU code
int dimy = ceil( ((float)M) / T );
int dimx = ceil( ((float)N) / T );
dim3 block(T,T,1), grid(dimx,dimy,1);

unsigned long int elapsed;
struct timeval t_start,t_end,t_diff;
gettimeofday(&t_start, NULL);

// ignoring generic shared mem problems
matMultTiledKer<T><<<grid, block>>>
    (d_A, d_B, d_C, U, M, N);

gettimeofday(&t_end, NULL);
timeval_subtract(&t_diff,
                &t_end,&t_start);
elapsed=(t_diff.tv_sec*1e6 +
        t_diff.tv_usec);
double flops = 2.0 * M * N * U;
double gigaFlops=(flops*1.0e-3f) /
    elapsed;

template <int T> // KERNEL
__global__ void matMultTiledKer( ... ) {
    __shared__ float Ash[T][T], Bsh[T][T];
    int ii = blockIdx.y * T; //blockDim.x==T
    int jj = blockIdx.x * T; //blockDim.y==T
    int tidy = threadIdx.y, i = tidy+ii;
    int tidx = threadIdx.x, j = tidx+jj;
    float tmp = 0.0;

    for(int kk=0; kk<U; kk+=T) {
        Ash[tidy][tidx] = (i<M && kk+tidx<U) ?
            A[i*U + (kk+tidx)] : 0.0;
        Bsh[tidy][tidx] = (j<N && kk_tidy<U) ?
            B[(kk+tidy)*N + j] : 0.0;
        __syncthreads();
        for(int k=0; k<T; k++) {
            tmp += Ash[tidy][k] * Bsh[k][tidx];
        } __syncthreads();
    } if (i<M && j<N) C[i*N + j] = tmp;
}
```



- 1 Direction-Vector Analysis
- 2 Block Tiling: Matrix Multiplication Case Study
- 3 Coalesced Accesses: Matrix Transposition Case Study



Matrix Transposition: Motivation

```

// Non-Coalesced Memory Access
// Transposition to coalesce it ⇒
DOALL i = 0 to N-1 // parallel
  float accum = 0
  DO j = 0, 63 // sequential
    float tmpA = A[i, j]
    accum = accum*accum + tmpA*tmpA
    B[i,j] = accum
  ENDDO
ENDDO

A' = transpose(A)
DOALL i = 0 to N-1 // parallel
  float accum = 0
  DO j = 0, 63 // sequential
    float tmpA = A'[j, i]
    accum = accum*accum + tmpA*tmpA
    B'[j, i] = accum
  ENDDO
B = transpose(B')

```

The transformed program performs about three-times more accesses to global memory than the original.

But exhibits only coalesced accesses!

Coalesced Access: a (half) warp accesses in a SIMD instruction consecutive memory (word) locations.



Transposition: Strip Mining, Interchange & Kernel

```
//Both loops are parallel
//Strip mining & interchange⇒
for(i = 0; i < rowsA; i++) {
    for(j = 0; j < colsA; j++) {
        trA[j*rowsA+i] = A[i*colsA+j];
    } }

for(ii=0; ii<rowsA; ii+=T) {
    for(jj=0; jj<colsA; jj+=T) {
        for(i=ii; i<min(ii+T,rowsA); i++) {
            for(j=jj; j<min(jj+T,colsA); j++) {
                trA[j*rowsA+i] = A[i*colsA+j];
            } } } } }
```



Transposition: Strip Mining, Interchange & Kernel

```
//Both loops are parallel
//Strip mining & interchange⇒
for(i = 0; i < rowsA; i++) {
    for(j = 0; j < colsA; j++) {
        trA[j*rowsA+i] = A[i*colsA+j];
    } }
```

```
for(ii=0; ii<rowsA; ii+=T) {
    for(jj=0; jj<colsA; jj+=T) {
        for(i=ii; i<min(ii+T,rowsA); i++) {
            for(j=jj; j<min(jj+T,colsA); j++) {
                trA[j*rowsA+i] = A[i*colsA+j];
            } } } }
```

```
__global__ void matTranspose(
    float* A, float* trA,
    int rowsA, int colsA ) {
    __shared__ float tile[T][T+1];
    int tidx = threadIdx.x;
    int tidy = threadIdx.y;
    int j = blockIdx.x*T + tidx;
    int i = blockIdx.y*T + tidy;
    if( j < colsA && i < rowsA )
        tile[tidy][tidx] = A[i*colsA+j];
    __syncthreads();
    i = blockIdx.y*T + threadIdx.x;
    j = blockIdx.x*T + threadIdx.y;
    if( j < colsA && i < rowsA )
        trA[j*rowsA+i] = tile[tidx][tidy];
}
```

- Trick is to write the element of the symmetric thread in the same block.
- What is the problem?



Transposition: Strip Mining, Interchange & Kernel

```
//Both loops are parallel
//Strip mining & interchange⇒
for(i = 0; i < rowsA; i++) {
    for(j = 0; j < colsA; j++) {
        trA[j*rowsA+i] = A[i*colsA+j];
    } }
```

```
for(ii=0; ii<rowsA; ii+=T) {
    for(jj=0; jj<colsA; jj+=T) {
        for(i=ii; i<min(ii+T,rowsA); i++) {
            for(j=jj; j<min(jj+T,colsA); j++) {
                trA[j*rowsA+i] = A[i*colsA+j];
            } } } }
```

```
__global__ void matTranspose(
    float* A, float* trA,
    int rowsA, int colsA ) {
    __shared__ float tile[T][T+1];
    int tidx = threadIdx.x;
    int tidy = threadIdx.y;
    int j = blockIdx.x*T + tidx;
    int i = blockIdx.y*T + tidy;
    if( j < colsA && i < rowsA )
        tile[tidy][tidx] = A[i*colsA+j];
    __syncthreads();
    i = blockIdx.y*T + threadIdx.x;
    j = blockIdx.x*T + threadIdx.y;
    if( j < colsA && i < rowsA )
        trA[j*rowsA+i] = tile[tidx][tidy];
}
```

- Trick is to write the element of the symmetric thread in the same block.
- What is the problem?
- Number of shared memory banks typically 16 or 32.
- T is also either 16 or 32 ⇒
- 16 consecutive threads will read the same memory bank at the same time.
- Solution: `tile[T][T+1]`;



Lessons Learned So Far

- Tiled transposition is about $2\times$ faster than the naive version,
- but the motivating example runs much faster than that when transposition coalesces accesses to arrays A and B. **Why?**



Lessons Learned So Far

- Tiled transposition is about $2\times$ faster than the naive version,
- but the motivating example runs much faster than that when transposition coalesces accesses to arrays A and B. **Why?**
- **Better to eliminate rather than hide latency.** Impact of hardware multi-threading limited by the amount of available resources!



Constant (Read-Only) Memory in CUDA

- 64KB of `__constant__` memory on device (global/slow), cached in each multiprocessor, e.g., 8KB (fast).
- May reduce the required memory bandwidth:
 - if found in cache, then no extra traffic,
 - if a (half) warp accesses the same location and misses in cache \Rightarrow only one request is sent and the result is broadcast back to all,
 - **serialized** accesses if a warp of threads read different locations!
 - latency can range from one to hundreds of cycles.
- Best Use: when an entire block accesses the same location in the same SIMD instruction: even on a miss, the first warp brings the data in cache @ minimal traffic, the rest find it in cache.

```

// C in __constant__ memory: Good!
DO i = 1, N, 1 // grid
  DO j = 1, M, 1 // block(s)
    A[i,j] = A[i,j] % C[i]
  ENDDO
ENDDO

// C in __constant__ memory: Bad!
DO i = 1, N, 1 // grid
  DO j = 1, M, 1 // block(s)
    A[i,j] = A[i,j] % C[j]
  ENDDO
// Either global memory or loop interchange

```

