Problem 1:

Solution: step 1: A set G with a binary operation $*$ is a group if it satisfies:

1. closure: For all $a, b \in G$, $a*b \in G$.
2. Associatively: $(a*b)*c = a*(b*c)$
3. Identity Element: There exists $e \in G$ such that $a*e = e*a = a$ for all $a \in G$.
4. Inverse element: For each $a \in G$, there exists $a^{-1} \in G$ such that $a + a^{-1} = e$

Additionally, if $a*b = b*a$ for all $a, b \in G$, then the group is abelian.

step 2: Take the set of odd integers

Let
$$O = \{\ldots, -3, -1, 1, 3, 5 \ldots\}$$
with binary operation $+$ (usual addition)

step 3: verify group axioms

1. closure:

odd + odd = Even

Example: $3+5 = 8$ (not odd)

Thus, clousure fails.

2. **Associatively:**

Addition of integer is associative:
$(a+b)+c = a+(b+c)$.

3. **Identity element:**

The additive identity in $(\mathbb{Z},+)$ is 0. But 0 is not odd, so O has no identity element.

× Fails.

4. **Inverse element:**

For an odd integer a, its inverse under addition is $-a$.
Example: if $3 \in O$, inverse is -3, which is also odd.
This works.

step 4: **conclusion:**

since closure and identity fail, the set of odd integers with + is not a group. Therefore, it cannot be an abelian group either.

Final answer:

The set of odd integers under addition is not an abelian group because:

(i) it is not closed (odd + odd = even $\notin O$)

(ii) it does not contain the identity element (since 0 is not odd).

Problem 2

6. statement: Let G be finite and let p be the smallest prime dividing $|G|$. Any subgroup of index p in G is normal.

Answer: True. Let H have index p. The action of G on cosets gives $\varphi : G \to S_p$. By minimality of p the image $\varphi(G)$ must have order either 1 or p. transitivity forces order p. But a subgroup of S_p of order p fixes a coset, so the kernel of the action equals H. Hence H is normal.

7. Answer: False

explanation: if a and b commute then $(ab)^6 = a^6 b^6$. From $b^2 = a^4$ we get $b^6 = (b^2)^3 = a^{12}$, so, $(ab)^6 = a^{18}$.
There is no reason $a^{18} = e$ in general.
Counterexample: in the infinite cyclic group $\langle g \rangle$ take $a = g$, $b = g^2$. Then $a^4 = g^4 = b^2$. they commute, but $(ab)^6 = g^{18} \neq e$. The claim needs extra hypotheses (e.g. finite orders forcing $a^{18} = e$) to hold.

8. Answer: False

correct version: The general true statement is $x^{n!} \in H$ for all $x \in G$. Reason: The Permutation action of G on the n cosets gives $\varphi : G \longrightarrow S_n$; the order of $\varphi(x)$ divides $n!$, so $x^{n!} \in \ker \varphi = \bigcap_{g \in G} g H g^{-1} \subseteq H$
The exponent n is not sufficient in general.

9. Answer: True

why: Let P be the (unique) subgroup of order p^k for each $k \leq n$ (and $p^n \| |G|$). then G has a normal sylow p-subgroup. Thus P is normal and is the slowly p-subgroup.

10. True.

why: A subgroup of order p^n is a sylow p-subgroup. By sylow theorems, the number n_p of such subgroups divides m and satisfies $n_p = 1 \equiv 1$. since $p \nmid m$, the only divisor of m congruent to 1 (mod p) is 1. so. $n_p = 1$. uniqueness implies normality.