ASSIGNEMNT-1 PROBABILITY

Katherapaka Nikhil*

Question 35. Suppose that $(\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3})$ has $N_3(\mu, \mathbf{\Sigma})$ distrubution with $\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\Sigma = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Given that $\phi(-0.5)$=0.3085, where $\phi(.)$ denotes the cumulative distribution function of a standard normal random variable, $P\left((X_1 - 2X_2 + 2X_3)^2 < \frac{7}{2}\right)$ equals to

Solution:

Let $\mathbf{Y} = \mathbf{X_1} - \mathbf{2X_2} + \mathbf{2X_3}$ and $\mathbf{a}$=$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

$$\mu_Y = \mathbf{a}^T \mu \tag{1}$$

$$= \begin{bmatrix} 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

$$= 0 \tag{3}$$

$$\sigma_Y^2 = \mathbf{a}^T \mathbf{\Sigma} \mathbf{a} \tag{4}$$

$$= \begin{bmatrix} 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \tag{5}$$

$$= 1 \tag{6}$$

$$\Pr\left(Y^2 < \frac{7}{2}\right) = \Pr\left(-\sqrt{\frac{7}{2}} < Y < \sqrt{\frac{7}{2}}\right) \tag{7}$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \tag{8}$$

$$\Phi\left(\sqrt{\tfrac{7}{2}}\right) = \int_{-\infty}^{\sqrt{\frac{7}{2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \tag{9}$$

$$= \int_{-\infty}^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} + \int_{-\frac{1}{2}}^{\sqrt{\frac{7}{2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \tag{10}$$

$$= \Phi(-0.5) + \int_{-\frac{1}{2}}^{\sqrt{\frac{7}{2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \tag{11}$$

$$= 0.3085 + 0.66082 \tag{12}$$

$$= 0.96932 \tag{13}$$

$$\Phi\left(-\sqrt{\tfrac{7}{2}}\right) = \int_{-\infty}^{-\sqrt{\frac{7}{2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \tag{14}$$

$$= 0.03068 \tag{15}$$

$$\Pr\left(\left(X_1 - 2X_2 + 2X_3\right)^2 < \frac{7}{2}\right) = \Phi\left(\sqrt{\tfrac{7}{2}}\right) - \Phi\left(-\sqrt{\tfrac{7}{2}}\right) \tag{16}$$

$$= 0.96932 - 0.03068 \tag{17}$$

$$= 0.93864 \tag{18}$$

1 Theory

To calculate the probability $P((X_1 - 2X_2 + 2X_3)^2 < \frac{7}{2})$, where X_1, X_2, and X_3 are random variables with mean vector $\mu = [0, 0, 0]$ and covariance matrix Σ given by:

$$\Sigma = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and the coefficients for the linear combination $\mathbf{a} = [1, -2, 2]$, we can follow these steps:

1) Calculate Mean and Variance of $\mathbf{Y}$=$\mathbf{X_1} - \mathbf{2X_2} + \mathbf{2X_3}$
   The mean of Y, denoted as μ_Y, is calculated as follows:

$$\mu_Y = \sum_{i=1}^{3} a_i \mu_i \tag{19}$$

$$= 1 \times 0 + (-2) \times 0 + 2 \times 0 \tag{20}$$

$$= 0 \tag{21}$$

   The variance of Y, denoted as σ_Y^2, is calculated using the formula:

$$\sigma_Y^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i \Sigma_{ij} a_j \tag{22}$$

$$= 1 \tag{23}$$

2) Calculate Lower and Upper Bounds for Standard Normal Distribution
   Using the variance σ_Y^2, calculate the lower and

upper bounds for the standard normal distribution corresponding to $\frac{7}{2}$:

$$\text{Lower Bound} = -\sqrt{\frac{7}{2}} \tag{24}$$

$$\text{Upper Bound} = \sqrt{\frac{7}{2}} \tag{25}$$

3) Calculate Cumulative Probabilities Using Standard Normal CDF
Calculate the cumulative probabilities for the lower and upper bounds using the error function (`erf`) and the standard normal cumulative distribution function (CDF) formula:

$$\text{Lower Probability} = \frac{1}{2}\left(1 + \text{erf}\left(\frac{\text{Lower Bound}}{\sqrt{2}}\right)\right) \tag{26}$$

$$\text{Upper Probability} = \frac{1}{2}\left(1 + \text{erf}\left(\frac{\text{Upper Bound}}{\sqrt{2}}\right)\right) \tag{27}$$

4) Calculate Required Probability
Calculate the required probability by subtracting the lower probability from the upper probability:

$$\text{Required Probability} = \text{Upper Probability} - \text{Lower Probability}$$

2 Simulation

The given C code calculates and prints the required probability using the calculated lower and upper probabilities.

```
#include <stdio.h>
#include <math.h>

int main() {
    // Given mean vector and covariance matrix
    double mu[3] = {0, 0, 0};
    double Sigma[3][3] = {{2, 2, 1},
                          {2, 5, 1},
                          {1, 1, 1}};

    // Coefficients for the linear combination
    double a[3] = {1, -2, 2};

    // Calculate mean and variance of $Y = X_1 - 2X_2 + 2X_3$
    double mu_Y = 0;
    double sigma_Y_squared = 0;
    for (int i = 0; i < 3; ++i) {
        mu_Y += a[i] * mu[i];
        for (int j = 0; j < 3; ++j) {
            sigma_Y_squared += a[i] * Sigma[i][j
        }
    }

    // Calculate the probability using standard
    double lower_bound = -sqrt(sigma_Y_squared);
    double upper_bound = sqrt(sigma_Y_squared);

    // Calculate cumulative probabilities using
    double lower_prob = 0.5 * (1 + erf(lower_bou
    double upper_prob = 0.5 * (1 + erf(upper_bou

    // Calculate the required probability
    double required_prob = upper_prob - lower_pr

    printf("P((X_1 - 2X_2 + 2X_3)^2 < 7/2) = %f\

    return 0;
}
```