Leibniz's Product Rule for Higher derivatives

Rule: If $u(x)$ and $v(x)$ are two n-times differentiable functions of x then

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n}v + \binom{n}{1}\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \binom{n}{2}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2 v}{dx^2} + \cdots\cdots + \binom{n}{n-1}\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + u\frac{d^n v}{dx^n}$$

Or,

$$(uv)^{(n)} = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \binom{n}{2}u^{(n-2)}v^{(2)} + \cdots\cdots + \binom{n}{n-1}u^{(1)}v^{(n-1)} + uv^{(n)}$$

Where,

$$u^{(1)} = \frac{du}{dx}, \qquad u^{(2)} = \frac{d^2u}{dx^2}, \cdots\ \cdots, u^{(n)} = \frac{d^n u}{dx^n}$$

and

$$\binom{n}{r} = \frac{n(n-1)(n-2)(n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!}$$

Example Find the 3rd derivative of $y = x^3 \sin x$ or find d^3y/dx^3

Solution:

Let us consider,

$$u = x^3 \text{ and}$$

$$v = \sin x$$

Applying Leibniz's theorem we obtain,

$$\frac{d^3}{dx^3}(x^3 \sin x) = \frac{d^3(x^3)}{dx^3}(\sin x) + \binom{3}{1}\frac{d^2(x^3)}{dx^2}\frac{d(\sin x)}{dx} + \binom{3}{2}\frac{d(x^3)}{dx}\frac{d^2(\sin x)}{dx^2} + (x^3)\frac{d^3(\sin x)}{dx^3} \tag{1}$$

Now,

$$\begin{array}{l|l} \dfrac{d(\sin x)}{dx} = \cos x & \dfrac{d(x^3)}{dx} = 3x^2 \\ \dfrac{d^2(\sin x)}{dx^2} = -\sin x & \dfrac{d^2(x^3)}{dx^2} = 6x \\ \dfrac{d^3(\sin x)}{dx^3} = -\cos x & \dfrac{d^3(x^3)}{dx^3} = 6 \end{array}$$

Putting these values into the equation (1), we get,

$$\frac{d^3}{dx^3}(x^3 \sin x) = 6 \sin x + 3 \times 6x \times (\cos x) + 3 \times 3x^2 \times (-\sin x) + x^3(-\cos x)$$

$$\boxed{\therefore \frac{d^3}{dx^3}(x^3 \sin x) = -x^3 \cos x - 9x^2 \sin x + 18x \cos x + 6 \sin x}$$

Problem Find the 4th derivative of $y = x^4 \cos 2x$