14.4

(1) The portion of the cylinder $y^2+z^2=9$ that is above the rectangle

$R = \{(x,y) : 0 \le x \le 2, -3 \le y \le 3\}$

$y^2+z^2=9$

$z^2 = 9-y^2$

$z = \sqrt{9-y^2}$

$\frac{dz}{dx} = 0$

$\frac{dz}{dy} = \frac{1}{2\sqrt{9-y^2}}(-2y)$

$= \frac{-2y}{2\sqrt{9-y^2}}$

$= \frac{-y}{2\sqrt{9-y^2}}$

$\therefore \sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1}$

$= \sqrt{0^2 + \left(\frac{-y}{2\sqrt{9-y^2}}\right)^2 + 1}$

$= \sqrt{\frac{y^2}{2(9-y^2)} + 1}$

$$= \sqrt{\frac{y^2 + 18 - y^2}{2(9-y^2)}}$$

$$= \sqrt{\frac{9}{9-y^2}}$$

$$= \frac{3}{\sqrt{9-y^2}}$$

$$\int_{-3}^{3}\int_{0}^{2} \frac{3}{\sqrt{9-y^2}}\, dx\, dy$$

$$= \int_{-3}^{3} \frac{3 \times 2}{\sqrt{9-y^2}}\, dy$$

$$= 6\int_{-3}^{3} \frac{1}{\sqrt{9-y^2}}\, dy$$

$$= 6 \cancel{\int_{-3}^{3}} \left[\sin^{-1} y/3\right]_{-3}^{3}$$

$$= 6\left[\sin^{-1}1 + \sin^{-1}1\right]$$

$$= 12\sin^{-1}1 = 12\pi$$

Ans.

(2) The portion of the plane

$2x + 2y + z = 8$ in the first octant

$2x + 2y + z = 8$

$z = 8 - 2x - 2y$

$\frac{dz}{dx} = -2 \qquad \frac{dz}{dy} = -2$

$\sqrt{(-2)^2 + (-2)^2 + 1} = \sqrt{4+4+1} = 3$

$\int_0^4 \int_0^{4-y} 3 \, dx \, dy$

$2x + 2y = 8$

$x + y = 4$

$x = 4 - y$

$y = 4 - x$

$y = 4$

$= 3 \int_0^4 4 - y \, dy$

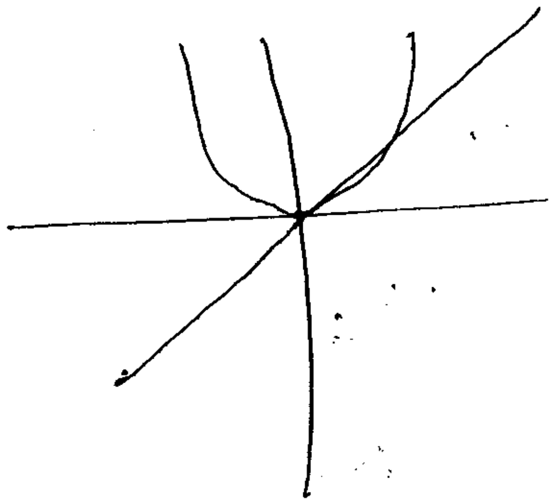
$= 3 \left[4y - y^2/2 \right]_0^4$

$= 3 \left[4 \times 4 - 16/2 \right]$

$= 3 [16 - 8]$

$= 24$

③ The portion of the cone $z^2 = 4x^2 + 4y^2$ that is above the region in the first quadrant bounded by the line $y = x$ and parabola $y = x^2$



$x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$

$x = 1$

$z^2 = 4x^2 + 4y^2$

$z = \sqrt{4x^2 + 4y^2}$

$\frac{dz}{dx} = \frac{1}{2\sqrt{4x^2+4y^2}} 8x = \frac{4x}{2\sqrt{x^2+y^2}} = \frac{2x}{\sqrt{x^2+y^2}}$

$\frac{dz}{dy} = \frac{1}{2\sqrt{4x^2+4y^2}} 8y = \frac{8y}{2 \times 2\sqrt{x^2+y^2}} = \frac{2y}{\sqrt{x^2+y^2}}$

$\sqrt{\frac{4x^2}{x^2+y^2} + \frac{4y^2}{x^2+y^2} + 1} = \sqrt{\frac{4x^2+4y^2+x^2+y^2}{x^2+y^2}}$

$= \sqrt{\frac{5(x^2+y^2)}{x^2+y^2}} = \sqrt{5}$

$$\int_0^1 \int_{x^2}^{x} \sqrt{5}\, dy\, dx$$

$$= \sqrt{5} \int_0^1 (x - x^2)\, dx$$

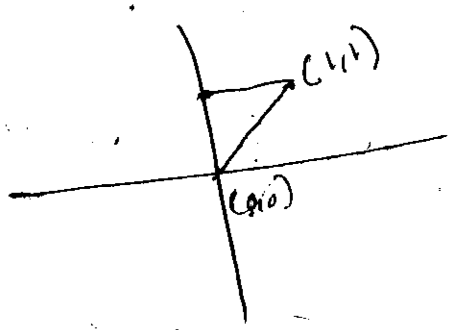
$$= \sqrt{5} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= \sqrt{5}\left(\tfrac{1}{2} - \tfrac{1}{3} \right)$$

$$= \sqrt{5} \left(\frac{3-2}{6} \right) = \sqrt{5}/6$$

Ans.

(4) The portion of the surface $z = 2x + y^2$ that is above the triangular region with vertices $(0,0)$, $(0,1)$ and $(1,1)$;



$$\frac{x-0}{0-1} = \frac{y-0}{0-1}$$

$x = y$

$y = x$

$y = 1$

$$\frac{dz}{dx} = 2 \qquad \frac{dz}{dy} = 2y$$

$$\sqrt{4+4y^2+1} = \sqrt{4y^2+5}$$

$$\int_0^1 \int_x^1 \sqrt{4y^2+5}\ dy\ dx$$

$$\int_0^1 \left[\frac{1}{2\sqrt{4y^2+5}}\ 8y \right]$$

$2x+y^2=0$

$x=y$

$y=1$

$x=1$

$y=1$

$r\sin\theta=1$

$r=\frac{1}{\sin\theta}$

$=\cos\theta$

$x=y$

$r\cos\theta = r\sin\theta$

$r\cos\theta - r\sin\theta = 0$

$r(\cos\theta - \sin\theta) = 0$

$r = \frac{0}{\cos\theta}$

$$\int_0^1 \int_0^y \sqrt{4y^2+5}\ dx\ dy$$

$$\int_0^1 y\sqrt{4y^2+5}\ dy$$

$$\frac{1}{8}\int_0^1 \sqrt{p}\ dp = \frac{1}{8}\times\frac{2}{3}\left[(4y^2+5)^{3/2}\right]_0^1$$

$$= \frac{1}{12}\times(27-5^{3/2})$$ Ans.

(5) The portion of the cone $z=\sqrt{x^2+y^2}$ that lies inside the cylinder $x^2+y^2=2x$

$$\frac{dz}{dx} = \frac{1}{2\sqrt{x^2+y^2}}\,2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{dz}{dy} = \frac{1}{2\sqrt{x^2+y^2}}\,2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}$$

$$=\sqrt{\frac{x^2+y^2+x^2+y^2}{x^2+y^2}}$$

$$=\sqrt{2}$$

$$\int_0^{2\pi}\int_0^{2\cos\theta}\sqrt{2}\,r\,dr\,d\theta$$

$x^2+y^2=2x$

$r^2=2r\cos\theta$

$r=2\cos\theta$

$x^2-2x+1+y^2=1$

$(x-1)^2+y^2=1$

$$=\frac{1}{\sqrt{2}}\int_{\cancel{0}\,-\pi/2}^{\cancel{2}\pi/2} 4\cos^2\theta\,d\theta$$

$$=\frac{4}{\sqrt{2}}\times\frac{1}{2}\int_{\cancel{0}\,-\pi/2}^{\cancel{2}\pi/2} 1+\cos 2\theta\,d\theta$$

$$= \frac{4}{2\sqrt{2}} \left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{4}{2\sqrt{2}} \left(\frac{2\pi}{2} + 0 \right)$$

$$= \sqrt{2}\,\pi$$

$$= 2\sqrt{2}\pi$$

$$\frac{\sqrt{2}}{\sqrt{2}\,2} \cdot \times 4 \times \frac{1}{2}$$

$$\frac{\sqrt{2} \times 4}{4}$$

(6) The portion of paraboloid $z = 1 - x^2 - y^2$ that is above xy plane.

$$\frac{dz}{dx} = -2x \qquad \frac{dz}{dy} = -2y$$

$$\sqrt{4x^2 + 4y^2 + 1} = \sqrt{4(x^2+y^2)+1}$$

$$= \sqrt{4r^2+1}$$

$$\therefore 0 = 1 - x^2 - y^2$$

$$x^2 = 1 - y^2 \qquad \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 1$$

$$x = \sqrt{1-y^2} \qquad r^2(1) = 1$$

$$r^2 = 1$$

$$r = \pm 1$$

$$\int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\; r\, dr\, d\theta$$

$4r^2+1=p$

$8r = \frac{dp}{dr}$

$8r\,dr = dp$

$$= \frac{1}{8}\int_0^{2\pi}\int_0^1 p^{1/2}\, dp\, d\theta$$

$$= \frac{1}{8}\times\frac{2}{3}\int_0^{2\pi}\left[p^{3/2}\right]_0^1 \, d\theta$$

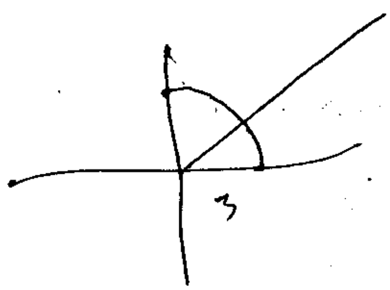
$$= \frac{1}{12}\int_0^{2\pi} \cancel{5-1 d\theta}\; (\sqrt{5})^3-(1)^3$$

$$= \frac{1}{3}\cancel{(2\pi)} \qquad \frac{5\sqrt{5}-1}{\cancel{12}\,6}\times\cancel{2}\pi$$

$$= 2\cancel{\pi/3}$$

(7) the portion of the surface $z=xy$ that is above the sector in the first quadrant bounded by the lines $y=x/\sqrt{3}$, $y=0$, $x^2+y^2=9$

$r\sin\theta = \frac{r\cos\theta}{\sqrt{3}}$



$$\int_0^{\pi/6}\int_0^{3} \sqrt{r^2+1}\; r\,dr\,d\theta$$

$$= \frac{2}{3}\cdot\frac{1}{2}\int_0^{\pi/6} \left[(r^2+1)^{3/2}\right]_0^3 d\theta$$

$$= \frac{1}{3}\int_0^{\pi/6} \left(10^{3/2} - 1\right) d\theta$$

$$= \frac{10\sqrt{10}-1}{3}\;\pi/6$$

$$= \frac{(10\sqrt{10}-1)\pi}{18}$$

$$x^2 + \frac{x^2}{3} = 9$$

$$\frac{3x^2+x^2}{3} = 9$$

$$4x^2 = 27$$

$$x^2 = \frac{27}{4} \qquad x = \frac{3\sqrt{3}}{2}$$

$$\sqrt{x^2+y^2+1}$$

$$= \sqrt{r^2+1}$$

$$r^2\cos^2\theta = \frac{27}{4}$$

$$r^2 = \frac{27}{4}\sec^2\theta$$

$$r = \frac{\sqrt{27}}{2}\sec\theta$$

$$y = \frac{1}{\sqrt{3}}\;\frac{\sqrt{27}}{2}$$

$$= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}\times 3}{2}$$

$$= 3/2$$

$$\therefore \theta = \tan^{-1}\left(\frac{3}{2} \times \frac{2}{3\sqrt{3}}\right)$$

$$= \tan^{-1}(1/\sqrt{3})$$

$$= 30^\circ = \pi/6$$

(8) The portion of the paraboloid $2z = x^2 + y^2$ that is inside the cylinder $x^2 + y^2 = 8$

$2z = x^2 + y^2$

$z = \frac{x^2}{2} + \frac{y^2}{2}$

$\frac{dz}{dx} = x \qquad \frac{dz}{dy} = y$

$\sqrt{x^2 + y^2 + 1}$

$\sqrt{r^2 + 1}$

$x^2 + y^2 = 8$

$r^2 = (2\sqrt{2})^2$

$r = 2\sqrt{2}$

$$\int_0^{2\pi} \int_0^{2\sqrt{2}} \sqrt{r^2+1} \; r \, dr \, d\theta$$

$$= \frac{1}{2} \cdot \frac{2}{3} \int_0^{2\pi} \left[(r^2 + 1)^{3/2} \right]_0^{2\sqrt{2}} d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \left[27 - 1 \right] d\theta$$

$$= \frac{26}{3} \, 2\pi$$

(9) **

The portion of the sphere $x^2+y^2+z^2=16$ between the planes $z=1$ and $z=2$

$$z^2 = 16-x^2-y^2$$

$$z = \sqrt{16-x^2-y^2}$$

$$\frac{dz}{dx} = \frac{1}{2\sqrt{16-x^2-y^2}}(-2x) \qquad \frac{dz}{dy} = \frac{1}{2\sqrt{16-x^2-y^2}}(-2y$$

$$\sqrt{\frac{x^2}{16-x^2-y^2}+\frac{y^2}{16-x^2-y^2}+1}$$

$$= \sqrt{\frac{x^2+y^2+16-x^2-y^2}{16-x^2-y^2}}$$

$$= \sqrt{\frac{16}{16-x^2-y^2}}$$

$$= \frac{4}{\sqrt{16-x^2-y^2}}$$

$z=1$

$16 = x^2+y^2$

$r^2 = 15$

$r = \sqrt{15}$

~~$z=0$~~

$z=2$

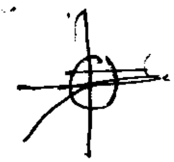
$x^2+y^2=12$

$r=\sqrt{12}$

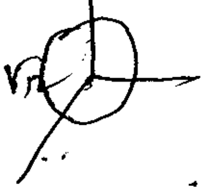
$$\int_0^{2\pi}\int_{\sqrt{12}}^{\sqrt{15}} \frac{4r}{\sqrt{16-r^2}}\, dr\, d\theta \qquad -\tfrac{1}{2}+1 \qquad \frac{-1+2}{2}$$

$$= \frac{4}{2}\int_0^{2\pi} \frac{1}{\sqrt{p}}\, dp\, d\theta \qquad \tfrac{1}{2}$$

$$= 2\times 2\int_0^{2\pi} p^{1/2}\, d\theta$$



$$= 4\int_0^{2\pi} \left[(16-r^2)^{1/2}\right]_{\sqrt{12}}^{\sqrt{15}}$$



$$= -4\int_0^{2\pi} d\theta$$

$$= -8\pi$$