Fourier Series
And
Application

1 ⓐ

$$f(x) = \begin{cases} -x, & -4 \le x \le 0 \\ x, & 0 \le x \le 4 \end{cases}$$

period $= 8$

$\Rightarrow$ $2L = 8$

$\therefore L = 4$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx$$

$$= \frac{1}{4} \int_{-4}^{4} f(x) \cos \frac{n\pi x}{4} \, dx$$

$$= \frac{1}{4} \left[\int_{-4}^{0} -x \cos \frac{n\pi x}{4} \, dx + \int_{0}^{4} x \cos \frac{n\pi x}{4} \, dx \right]$$

$$= -\frac{1}{4} \int_{-4}^{0} x \cos \frac{n\pi x}{4} \, dx + \frac{1}{4} \int_{0}^{4} x \cos \frac{n\pi x}{4} \, dx$$

$$\therefore \int x \cos n\pi x/4 \, dx$$

$$= x \int \cos \frac{n\pi x}{4} \, dx - \int \left\{ \frac{d}{dx}(x) \int \cos n\pi x/4 \, dx \right\} dx$$

$$= x \times \frac{4}{n\pi} \sin \frac{n\pi x}{4} - \int \frac{4}{n\pi} \sin \frac{n\pi x}{4} \, dx$$

$$= \frac{4x}{n\pi} \sin \frac{n\pi x}{4} - \frac{4}{n\pi} \times \left(- \frac{4}{n\pi} \cos \frac{n\pi x}{4} \right)$$

$$= \frac{4x}{n\pi} \sin \frac{n\pi x}{4} + \frac{16}{n^2\pi^2} \cos \frac{n\pi x}{4}$$

$$\therefore \int_{-4}^{0} x \cos n\pi x/4 \, dx = \left[\frac{4x}{n\pi} \sin \frac{n\pi x}{4} + \frac{16}{n^2\pi^2} \cos \frac{n\pi x}{4} \right]_{-4}^{0}$$

$$= \left[\frac{4 \times 0}{n\pi} \sin \frac{n\pi \times 0}{4} + \frac{16}{n^2\pi^2} \cos \frac{n\pi \times 0}{4} - \left(\frac{4(-4)}{n\pi} \sin \frac{n\pi(-4)}{4} + \frac{16}{n^2\pi^2} \cos \frac{n\pi(-4)}{4} \right) \right]$$

$$= \left[0 + \frac{16}{n^2\pi^2} - \left(\frac{-16}{n\pi} (-\sin n\pi) + \frac{16}{n^2\pi^2} \cos n\pi \right) \right]$$

$$= \left[0 + \frac{16}{n^2\pi^2} - \left(-\frac{16}{n\pi} \times 0 + \frac{16}{n^2\pi^2} \cos n\pi \right) \right.$$

$$= \frac{16}{n^2\pi^2} - \frac{16}{n^2\pi^2}\cos n\pi$$

$$= \frac{16}{n^2\pi^2}(1 - \cos n\pi)$$

$$\int_0^4 x\cos\frac{n\pi}{4}\,dx = \left[\frac{4x}{n\pi}\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_0^4$$

$$= \left(\frac{4\times 4}{n\pi}\sin\frac{n\pi\times 4}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi\times 4}{4}\right) -$$

$$\left(0 + \frac{16}{n^2\pi^2}\cos 0\right)$$

$$= \frac{16}{n\pi}\sin n\pi + \frac{16}{n^2\pi^2}\cos n\pi - \frac{16}{n^2\pi^2}$$

$$= \frac{16}{n\pi}\times 0 + \frac{16}{n^2\pi^2}\cos n\pi - \frac{16}{n^2\pi^2}$$

$$= \frac{16}{n^2\pi^2}(\cos n\pi - 1)$$

$$\therefore a_n = -\frac{1}{4} \times \frac{16}{n^2\pi^2}(1-\cos n\pi) + \frac{1}{4} \times \frac{16}{n^2\pi^2}(\cos n\pi - 1)$$

$$= -\frac{4}{n^2\pi^2}(1-\cos n\pi) + \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1) + \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \frac{2 \times 4}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \frac{8}{n^2\pi^2}(\cos n\pi - 1)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{4}\int_{-4}^{4} f(x)\,dx$$

$$= \frac{1}{4}\left[\int_{-4}^{0} -x\,dx + \int_{0}^{4} x\,dx\right]$$

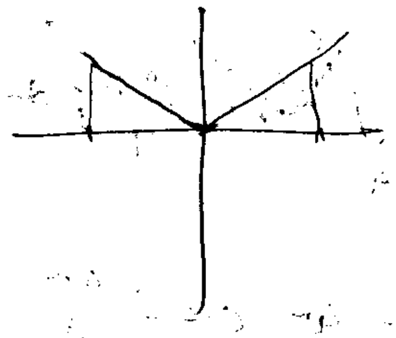
$$= \frac{1}{4}\left[-\left[\frac{x^2}{2}\right]_{-4}^{0} + \left[\frac{x^2}{2}\right]_{0}^{4}\right]$$

$$= \frac{1}{4}\left[-\left(0 - \frac{16}{2}\right) + \left(\frac{16}{2} - 0\right)\right]$$

$$= \frac{1}{4}\left[-0 + 8 + 8 - 0\right] = \frac{1}{4} \times 16$$

$$= 4$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$



$$= \frac{1}{4}\int_{-4}^{4} f(x)\sin\frac{n\pi x}{4}\,dx$$

$$= \frac{1}{4}\left[\int_{-4}^{0} -x\sin\frac{n\pi x}{4} + \int_{0}^{4} x\sin\frac{n\pi x}{4}\,dx\right]$$

$$= -\frac{1}{4}\int_{-4}^{0} x\sin\frac{n\pi x}{4} + \int_{0}^{4} x\sin\frac{n\pi x}{4}\,dx$$

$$\therefore \int x\sin\frac{n\pi x}{4}\,dx$$

$$= x\int \sin\frac{n\pi x}{4}\,dx - \int\left(\frac{d}{dx}(x)\int \sin\frac{n\pi x}{4}\,dx\right)dx$$

$$= x\times\frac{4}{n\pi}\left(-\cos\frac{n\pi x}{4}\right) - \int \frac{4}{n\pi}\cos\frac{n\pi x}{4}\,dx$$

$$= -\frac{4x}{n\pi}\cos\frac{n\pi x}{4} + \frac{4}{n\pi}\times\frac{4}{n\pi}\sin\frac{n\pi x}{4}$$

$$= -\frac{4x}{n\pi}\cos\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}$$

$$\int_{-4}^{0} x \sin \frac{n\pi x}{4}\, dx$$

$$= \left[-\frac{4x}{n\pi} \cos \frac{n\pi x}{4} + \frac{16}{n^2\pi^2} \sin \frac{n\pi x}{4} \right]_{-4}^{0}$$

$$= \left(-\frac{4\times 0}{n\pi} \cos \frac{n\pi \times 0}{4} + \frac{16}{n^2\pi^2} \sin \frac{n\pi \times 0}{4} \right)$$

$$- \left(-\frac{4(-4)}{n\pi} \cos \frac{n\pi(-4)}{4} + \frac{16}{n^2\pi^2} \sin \frac{n\pi(-4)}{4} \right)$$

$$= (0 + 0) - \left(\frac{16}{n\pi} \times \cos n\pi + \frac{16}{n^2\pi^2} \sin(-n\pi) \right)$$

$$= (0+0) - \left(\frac{16}{n\pi} \cos n\pi + 0 \right)$$

$$= -\frac{16}{n\pi} \cos n\pi$$

Again,

$$\int_{0}^{4} x \sin \frac{n\pi x}{4}\, dx$$

$$= \left[-\frac{4x}{n\pi} \cos \frac{n\pi x}{4} + \frac{16}{n^2\pi^2} \sin \frac{n\pi x}{4} \right]_{0}^{4}$$

$$= \left(-\frac{4\times 4}{n\pi}\cos\frac{n\pi\times 4}{n\pi} + \frac{16}{n^2\pi^2}\sin n\pi\right)$$

$$-\left(0+0\right)$$

$$= \left(-\frac{16}{n\pi}\cos n\pi + 0\right) - 0$$

$$= -\frac{16}{n\pi}\cos n\pi$$

$$\therefore b_n = -\frac{1}{4}\left(-\frac{16}{n\pi}\cos n\pi\right) + \frac{1}{4}\left(-\frac{16}{n\pi}\cos n\pi\right)$$

$$= +\frac{4}{n\pi}\cos n\pi - \frac{4}{n\pi}\cos n\pi$$

$$= 0$$

$$\therefore f(x) = \frac{4}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{2} + b_n\sin\frac{n\pi x}{L}\right)$$

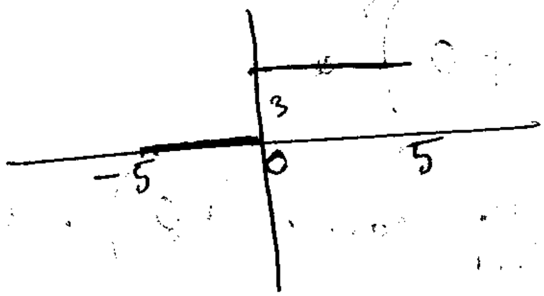
$$= 2 + \sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{4}$$

Ans.

(b)

$$f(x) = \begin{cases} 0, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases}$$

period $= 10$



$\Rightarrow$ $2L = 10$

$L = 5$

$$\therefore\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{5}\left[\int_{-5}^{0} 0 \times \cos\frac{n\pi x}{5}\,dx + \int_{0}^{5} 3\cos\frac{n\pi x}{5}\,dx\right]$$

$$= \frac{1}{5}\left[0 + 3\int_{0}^{5}\cos\frac{n\pi x}{5}\,dx\right]$$

$$= 3/5 \times \frac{5}{n\pi}\cancel{\int_0^5}\left[\sin\frac{n\pi x}{5}\right]_0^5$$

$$= \frac{3}{n\pi}(\sin n\pi - \sin 0)$$

$$= 0$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx$$

$$= \frac{1}{5}\int_{-5}^{5} f(x)\, dx$$

$$= \frac{1}{5}\int_{-5}^{0} 0\, dx + \frac{1}{5}\int_{0}^{5} 3\, dx$$

$$= 0 + \frac{1}{5} \times 3\, [x]_0^5$$

$$= \frac{1}{5} \times 3 \times 5$$

$$= 3$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{5}\int_{-5}^{5} f(x) \sin\frac{n\pi x}{5}\, dx$$

$$= \frac{1}{5}\int_{-5}^{0} 0 \sin\frac{n\pi x}{5}\, dx + \frac{1}{5}\int_{0}^{5} 3 \sin\frac{n\pi x}{5}\, dx$$

$$= \frac{1}{5} \times 3 \times \frac{5}{n\pi} \left[-\cos\frac{n\pi x}{5}\right]_0^5$$

$$= \frac{-3}{n\pi}(\cos n\pi - 1)$$

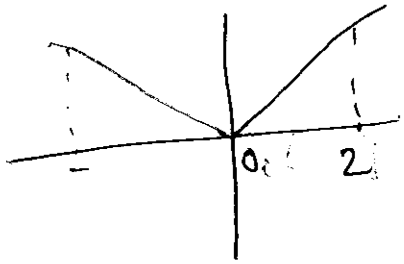
$$= \frac{3(1-\cos n\pi)}{n\pi}$$

$\therefore f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{3(1-\cos n\pi)}{n\pi} \sin \frac{n\pi x}{2}$

$\boxed{2}$

$f(x) = x, \quad 0<x<2$ in a half-range $\quad f(-x) = -x$

(i) sine series



$\Rightarrow$ $L = 2$

$a_n = 0$

$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$

$= \frac{2}{2} \int_0^2 x \sin \frac{n\pi x}{2} \, dx$

$= \int_0^2 x \sin \frac{n\pi x}{2} \, dx$

$\therefore \int x \sin \frac{n\pi x}{2} \, dx$

$= x \int \sin \frac{n\pi x}{2} - \int \left(\frac{d}{dx}(x) \int \sin \frac{n\pi x}{2} \, dx \right) dx$

$= -x \cdot \frac{2}{n\pi} \cos \frac{n\pi x}{2} + \frac{2}{n\pi} \int \cos \frac{n\pi x}{2} \, dx$

$= -\frac{2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{2}{n\pi} \frac{2}{n\pi} \sin \frac{n\pi x}{2}$

$= -\frac{2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2}$

$$\int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx = \left[-\frac{2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= \left(-\frac{2\times 2}{n\pi}\cos\frac{n\pi 2}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi 2}{2}\right) -$$

$$(0+0)$$

$$= -\frac{4}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin n\pi$$

$$= -\frac{4}{n\pi}\cos n\pi \qquad [\sin n\pi = 0]$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

$$= \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx$$

$$= \int_0^2 x\cos\frac{n\pi x}{2}dx$$

$$= \int_0^2 x\,dx \qquad [n=0]$$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{4}{2} - 0 = 2$$

$$f(x) = 2/2 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + \right)$$

$$= 2/2 + \sum_{n=1}^{\infty}\left(-\frac{4}{n\pi}\right)\cos n\pi$$

(11) Cosine Series

$b_n = 0 \qquad L = 2$

$$a_n = 2/L \int_0^L f(x) \cos\frac{n\pi x}{L}\, dx$$

$$= 2/2 \int_0^2 x \cos\frac{n\pi x}{2}\, dx$$

$$= \int_0^2 x \cos\frac{n\pi x}{2}\, dx$$

$$\int x\cos\frac{n\pi x}{2}\, dx = x\int \cos\frac{n\pi x}{2} - \int \frac{d}{dx}(x) \int \cos\frac{n\pi x}{2}\, dx$$

$$= x\frac{2}{n\pi}\sin\frac{n\pi x}{2} - \int \frac{2}{n\pi}\sin\frac{n\pi x}{2}\, dx$$

$$\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}$$

$$\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_0^2$$

$$= \left(\frac{4}{n\pi}\sin n\pi + \frac{4}{n^2\pi^2}\cos n\pi\right) - \left(0 + \frac{4}{n^2\pi^2}\right)$$

$$= \left(0 + \frac{4}{n^2\pi^2}\cdot\cos n\pi\right) - \frac{4}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

$$a_0 = \frac{2}{L}\int_0^2 x\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{2}\int_0^2 x\,dx$$

$$= 1\left[\frac{x^2}{2}\right]_0^2$$

$$= 2$$

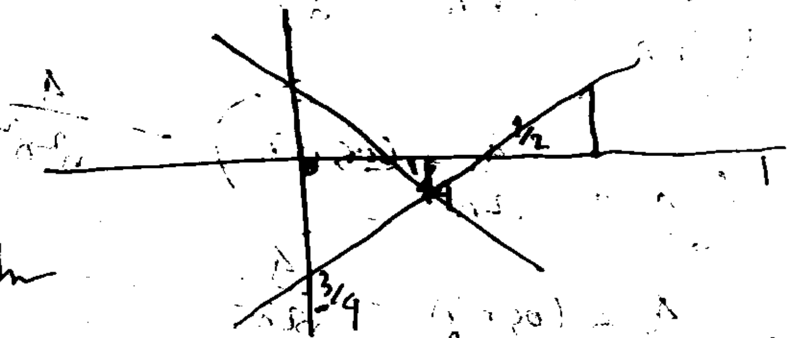
$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}$$ b.

(3) Expand $f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1 \end{cases}$ in sine series

$\Rightarrow$

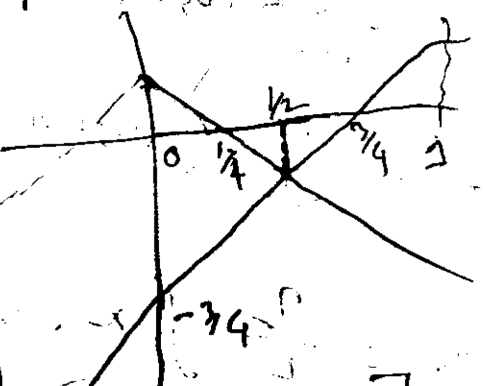
$2L = 1$

$L = \frac{1}{2}$



$$a_0 = \frac{1}{L}\int_0^1 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{1/2}\left[\int_0^{1/2} \frac{1}{4} - x + \int_{1/2}^{1} x - \frac{3}{4}\right]$$



$$= 2\left[\left(\frac{1}{4}x - \frac{x^2}{2}\right)_0^{1/2} + \left(\frac{x^2}{2} - \frac{3}{4}x\right)_{1/2}^{1}\right]$$

$$= 2\left[\left(\frac{1}{4}\times\frac{1}{2} - \frac{\frac{1}{4}}{2} - 0\right) + \left(\frac{1}{2} - \frac{3}{4} - \frac{1}{8} + \frac{3}{4}\times\frac{1}{2}\right)\right]$$

$$= 2\left[\left(\frac{1}{8} - \frac{1}{8}\right) + \left(\frac{1}{2} - \frac{3}{4} - \frac{1}{8} + \frac{3}{8}\right)\right]$$

$$= 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin \cancel{\cos}\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{1/2}\left[\int_{\cancel{-1/2}\ 0}^{1/2}\left(\frac{1}{4}-x\right)\sin\frac{n\pi x}{1/2}\,dx + \int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin\frac{n\pi x}{1/2}\,dx\right]$$

$$= 2\left[\frac{1}{4}\int_0^{1/2}\sin\frac{n\pi x}{1/2}\,dx - \int_0^{1/2} x\sin\frac{n\pi x}{1/2}\,dx + \int_{1/2}^{1} x\sin\frac{n\pi x}{1/2}\,dx - \int_{1/2}^{1}\frac{3}{4}\sin\frac{n\pi x}{1/2}\,dx\right]$$

$$= 2\left[\frac{1}{4}\int_0^{1/2}\sin 2n\pi x\,dx - \int_0^{1/2} x\sin 2n\pi x\,dx + \int_{1/2}^{1} x\sin 2n\pi x\,dx - \frac{3}{4}\int_{1/2}^{1}\sin 2n\pi x\,dx\right]$$

$$= 2\left[-\frac{1}{4}\times\frac{1}{2n\pi}\left(\cos 2n\pi x\right)_0^{1/2} - \int_0^{1/2} x\sin 2n\pi x\,dx + \int_{1/2}^{1} x\sin 2\pi n x\,dx + \frac{3}{4}\,\frac{1}{2n\pi}\left(\cos 2n\pi x\right)_{1/2}^{1}\right]$$

$\int x \sin 2\pi n x \, dx$

$= x \int \sin 2\pi n x \, dx - \int \frac{d}{dx}(x) \int \sin 2\pi n x \, dx \, dx$

$= -x \frac{1}{2\pi n} \cos 2\pi n x + \frac{1}{2\pi n} \int \cos 2\pi n x \, dx$

$= -\frac{x}{2\pi n} \cos 2\pi n x + \frac{1}{2n\pi} \cdot \frac{1}{2n\pi}$

$= -\frac{x}{2\pi n} \cos 2\pi n x + \frac{1}{4n^2\pi^2} \sin$

$\therefore \int_0^{1/2} x \sin 2\pi n x \, dx$

$= \left(-\frac{\frac{1}{2}}{2\pi n} \cos \frac{2\pi n}{2} + \frac{1}{4n^2\pi^2} \sin \right.$

$- (0 + 0)$

$= -\frac{1}{4n\pi} \cos n\pi$

$\int_{1/2}^{1} x \sin 2\pi n x \, dx = \left(-\frac{1}{2\pi n} \cos 2\pi \right)$

$2n\pi \frac{1}{2})$

$- \left(-\frac{1}{4n\pi} \cos 2 \right.$

$= -\frac{1}{2\pi n} \cos 2n \quad n\pi$

$$= -\frac{1}{2n\pi} + 0 - \frac{1}{4n\pi}\cos n\pi$$

$$= -\frac{1}{2n\pi} - \frac{1}{4n\pi}\cos n\pi$$

$$\therefore b_n = 2\left[-\frac{1}{4}\times\frac{1}{2n\pi}\left(\cos 2n\pi\tfrac{1}{2} - \cos 0\right)\right.$$

$$+ \frac{1}{4n\pi}\cos n\pi - \frac{1}{2n\pi} - \frac{1}{4n\pi}\cos n\pi$$

$$\left. + \frac{3}{8n\pi}\left(\cos 2n\pi - \cos n\pi\right)\right]$$

$$= 2\left[-\frac{1}{8n\pi}(\cos n\pi - 1) - \frac{1}{2n\pi} + \frac{3}{8n\pi}(1-\cos n\pi)\right]$$

$$= -2/8n\pi\,(\cos n\pi - 1) - \frac{2}{2n\pi} + \frac{3\times 2}{8n\pi}(1-\cos n\pi)$$

$$= -\frac{1}{4n\pi}(\cos n\pi - 1) - \frac{1}{n\pi} + \frac{3}{4}n\pi\,(1-\cos n\pi)$$

$$= \frac{1}{4n\pi}(1-\cos n\pi) + \tfrac{3}{4}n\pi\,(1-\cos n\pi) - \frac{1}{n\pi}$$

$$= (1-\cos n\pi)\left(\frac{1}{4n\pi} + \frac{3}{4}n\pi\right) - \frac{1}{n\pi}$$

$$= (1-\cos n\pi)\left(\frac{1+3n^2\pi^2}{4n\pi}\right) - \frac{1}{n\pi}$$

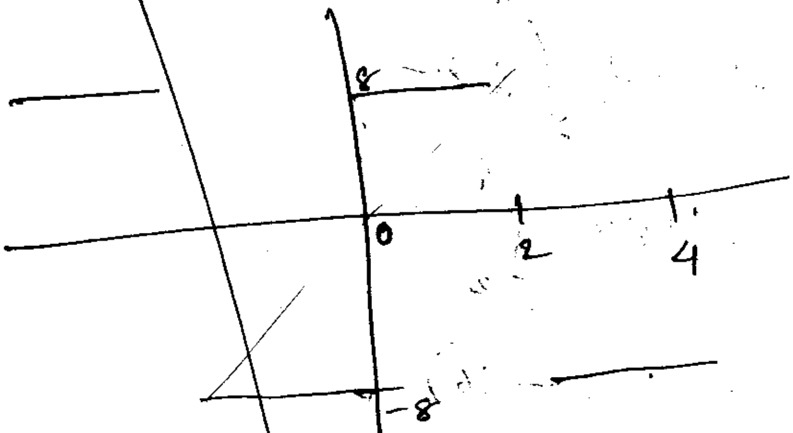
$$f(x) = \frac{a_0}{2} + \sum_{n}^{\infty}\left(a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right)$$

$$= \frac{0}{2} + \sum_{n}^{\infty}\left(0 + \left\{(1-\cos n\pi)\left(\frac{1+3n^2\pi^2}{4n\pi}\right) - \frac{1}{n\pi}\right\}\right.$$

$$= \sum_{n}^{\infty}\left\{(1-\cos n\pi)\left(\frac{1+3n^2\pi^2}{4n\pi}\right) - \frac{1}{n\pi}\right.$$

$\boxed{4}$

(a) $f(x) = \begin{cases} 8, & 0 < x < 2 \\ -8, & 2 < x < 4 \end{cases}$ period 4 $\quad 2L = 4$, $L = 2$



This is an odd function

$\therefore a_n = 0$

$b_n = \frac{1}{L} \int_{0}^{4} f(x) \sin \frac{n\pi x}{2} \, dx$

$= \frac{1}{2} \int_0^4 f(x) \sin \frac{n\pi x}{2} \, dx$

$= \frac{1}{2} \int_0^2 8 \sin \frac{n\pi x}{2} \, dx + \frac{1}{2} \int_2^4 -8 \sin \frac{n\pi x}{2} \, dx$

$= -\frac{1}{2} \times 8 \, \frac{2}{n\pi} \left[\cos \frac{n\pi x}{2} \right]_0^2 + \frac{1}{2} \times 8 \, \frac{2}{n\pi} \left[\cos \frac{n\pi x}{2} \right]_2^4$

$= -\frac{8}{n\pi} \left(\cos n\pi - 1 \right) + \frac{8}{n\pi} \left(\cos \frac{n\pi 4}{2} - \cos \frac{n\pi 2}{2} \right)$

$= \frac{8}{n\pi} \left(1 - \cos n\pi \right) + \frac{8}{n\pi} \left(1 - \cos n\pi \right)$