CSE 221: Algorithms

Heapsort

Mumit Khan

Computer Science and Engineering
BRAC University

References

1. T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms, Second Edition*. The MIT Press, September 2001.

Last modified: May 14, 2010

Contents

Contents

Heapsort

- $O(n \lg n)$ in the worst case – like *merge sort*.

Heapsort

- $O(n \lg n)$ in the worst case – like *merge sort*.
- Sorts *in place* – like *insertion sort*.

Heapsort

- $O(n \lg n)$ in the worst case – like *merge sort*.
- Sorts *in place* – like *insertion sort*.
- Combines the best of both algorithms.

Heapsort

- $O(n \lg n)$ in the worst case – like *merge sort*.
- Sorts *in place* – like *insertion sort*.
- Combines the best of both algorithms.
- Uses a data structure called the heap, which is also extensively used in other applications.

Contents

Heap data structure

- A data structure that provides worst-case $O(1)$ time access to the largest (max heap) or smallest (min heap) element.

Heap data structure

- A data structure that provides worst-case $O(1)$ time access to the largest (max heap) or smallest (min heap) element.
- A data structure that provides worst-case $\Theta(\lg n)$ time extract the largest (max heap) or smallest (min heap) element.

Heap data structure

- A data structure that provides worst-case $O(1)$ time access to the largest (max heap) or smallest (min heap) element.
- A data structure that provides worst-case $\Theta(\lg n)$ time extract the largest (max heap) or smallest (min heap) element.
- Priority queue is a prototypical application, where the keys are retrieved by priority.

Heap data structure

- A data structure that provides worst-case $O(1)$ time access to the largest (max heap) or smallest (min heap) element.
- A data structure that provides worst-case $\Theta(\lg n)$ time extract the largest (max heap) or smallest (min heap) element.
- Priority queue is a prototypical application, where the keys are retrieved by priority.
- Heapsort is an another application, where the keys can be sorted by repeatedly extracting the largest from the heap.

Heap data structure

- A data structure that provides worst-case $O(1)$ time access to the largest (max heap) or smallest (min heap) element.
- A data structure that provides worst-case $\Theta(\lg n)$ time extract the largest (max heap) or smallest (min heap) element.
- Priority queue is a prototypical application, where the keys are retrieved by priority.
- Heapsort is an another application, where the keys can be sorted by repeatedly extracting the largest from the heap.

Max vs. Min Heap

Unless explicitly stated as max heap or min heap, heap means max heap in this course.

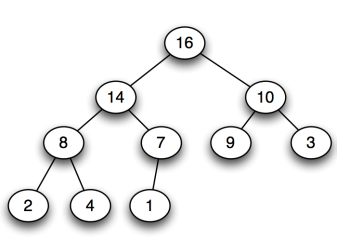
Heap-ordered tree

Definition

A binary tree is heap-ordered if:

1. the value at a node is $\geq$ the value at each of its children.

Example of (max) heap



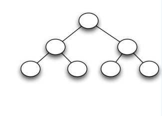
Heap-ordered tree

Definition

A binary tree is heap-ordered if:

1. the value at a node is $\geq$ the value at each of its children.
2. the tree is almost-complete.

Example of complete tree (or *not*)



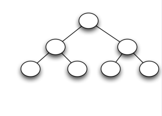
Heap-ordered tree

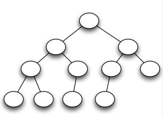
Definition

A binary tree is heap-ordered if:

1. the value at a node is $\geq$ the value at each of its children.
2. the tree is almost-complete.

Example of complete tree (or *not*)





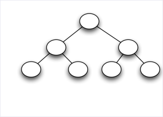
Heap-ordered tree

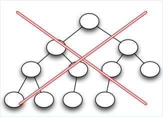
Definition

A binary tree is heap-ordered if:

1. the value at a node is $\geq$ the value at each of its children.
2. the tree is almost-complete.

Example of complete tree (or *not*)





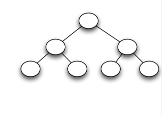
Heap-ordered tree

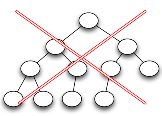
Definition

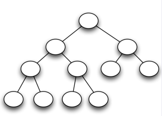
A binary tree is heap-ordered if:

1. the value at a node is $\geq$ the value at each of its children.
2. the tree is almost-complete.

Example of complete tree (or *not*)







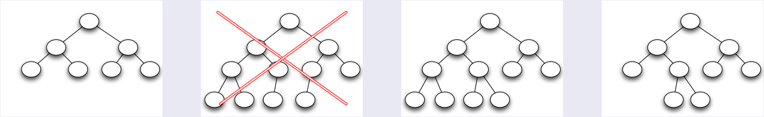
Heap-ordered tree

Definition

A binary tree is heap-ordered if:

1. the value at a node is $\geq$ the value at each of its children.
2. the tree is almost-complete.

Example of complete tree (or *not*)



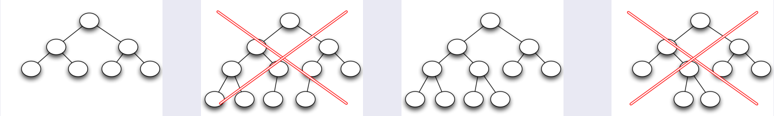
Heap-ordered tree

Definition

A binary tree is heap-ordered if:

1. the value at a node is $\geq$ the value at each of its children.
2. the tree is almost-complete.

Example of complete tree (or *not*)



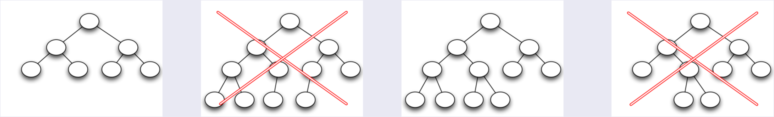
Heap-ordered tree

Definition

A binary tree is heap-ordered if:

1. the value at a node is $\geq$ the value at each of its children.
2. the tree is almost-complete. Height of tree is $\Theta(\lg n)$.

Example of complete tree (or *not*)



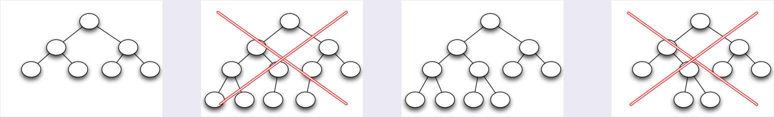
Heap-ordered tree

Definition

A binary tree is heap-ordered if:

1. the value at a node is $\geq$ the value at each of its children.
2. the tree is almost-complete. Height of tree is $\Theta(\lg n)$. Why?

Example of complete tree (or *not*)



Height of a heap-ordered tree

- Height h of a tree is the maximum distance of any leaf node to the root.

Height of a heap-ordered tree

- Height h of a tree is the maximum distance of any leaf node to the root.
- A heap of height h has the most number of elements if the tree is complete, so n equals the sum of nodes at each level.

$$
\begin{aligned}
n &\leq 2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^h \\
&= \frac{2^{h+1} - 1}{2 - 1} \\
&= 2^{h+1} - 1.
\end{aligned}
$$

Height of a heap-ordered tree

- Height h of a tree is the maximum distance of any leaf node to the root.
- A heap of height h has the most number of elements if the tree is complete, so n equals the sum of nodes at each level.

$$\begin{aligned} n &\leq 2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^h \\ &= \frac{2^{h+1} - 1}{2 - 1} \\ &= 2^{h+1} - 1. \end{aligned}$$

- It has the least number of elements if the lowest level has a single element and all higher levels are complete, so $n \geq 2^h - 1 + 1 = 2^h$.

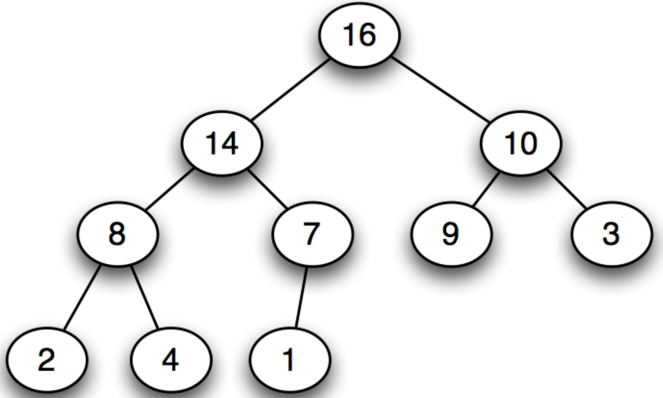
Height of a heap-ordered tree

- Height h of a tree is the maximum distance of any leaf node to the root.
- A heap of height h has the most number of elements if the tree is complete, so n equals the sum of nodes at each level.

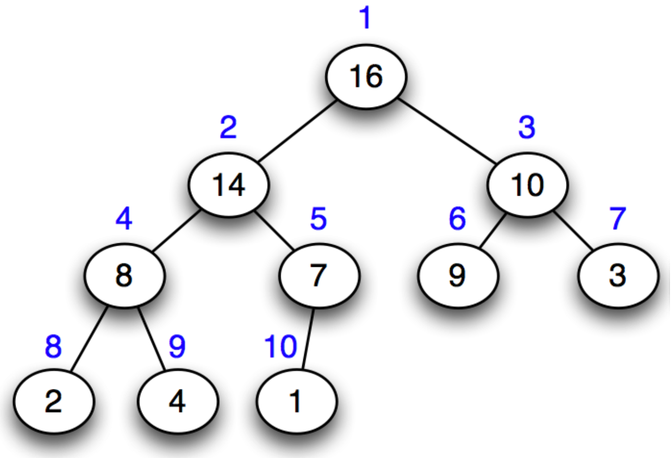
$$\begin{aligned} n &\leq 2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^h \\ &= \frac{2^{h+1} - 1}{2 - 1} \\ &= 2^{h+1} - 1. \end{aligned}$$

- It has the least number of elements if the lowest level has a single element and all higher levels are complete, so $n \geq 2^h - 1 + 1 = 2^h$.
- $2^h \leq n \leq 2^{h+1} - 1 < 2^{h+1} \quad \Rightarrow \quad h \leq \lg n < h + 1$. Since h is an integer, $h = \lfloor \lg n \rfloor = \Theta(\lg n)$.

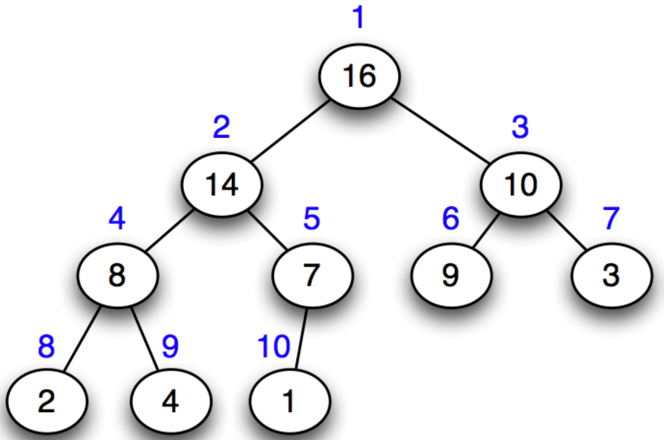
Heap – array representation of heap-ordered tree



Heap – array representation of heap-ordered tree

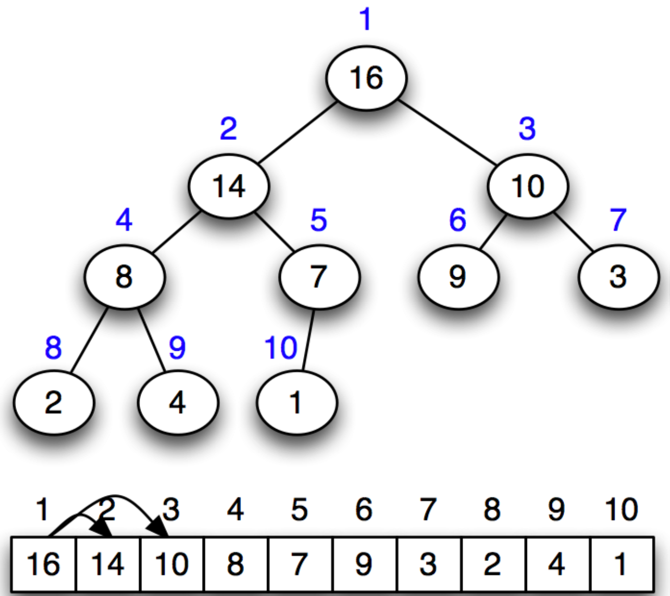


Heap – array representation of heap-ordered tree

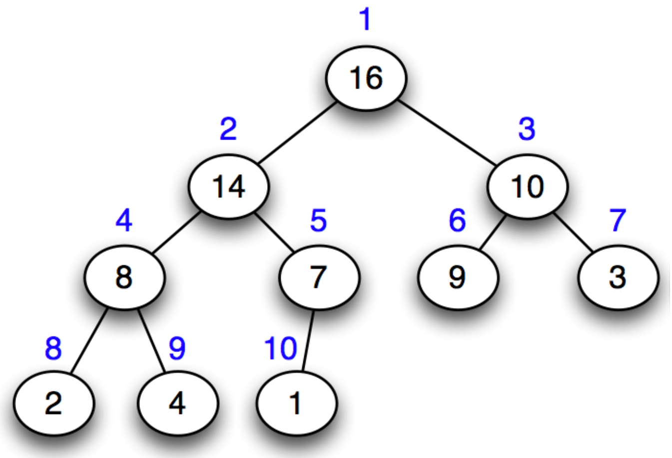


1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

Heap – array representation of heap-ordered tree

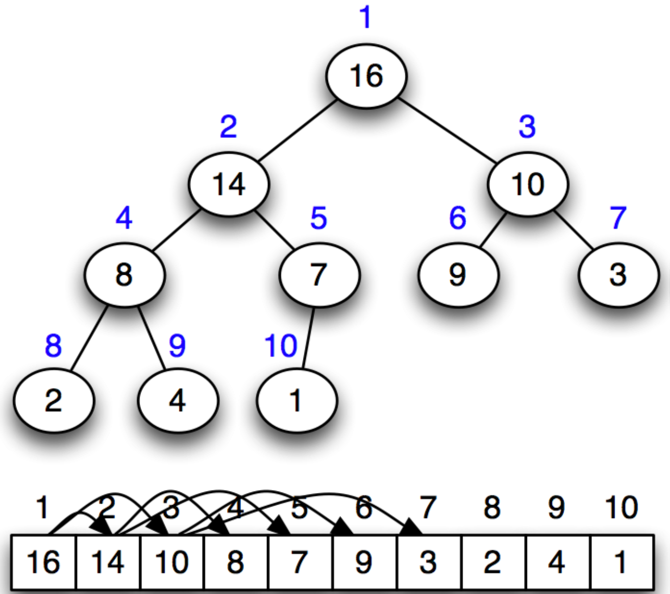


Heap – array representation of heap-ordered tree

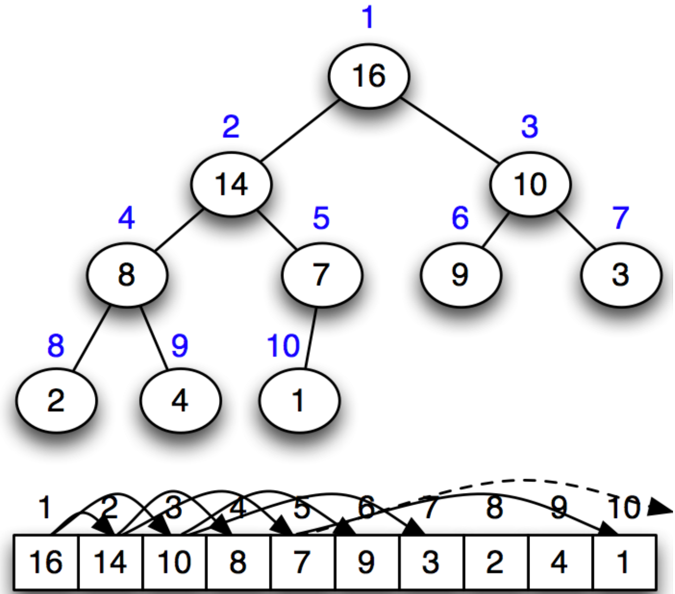


1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

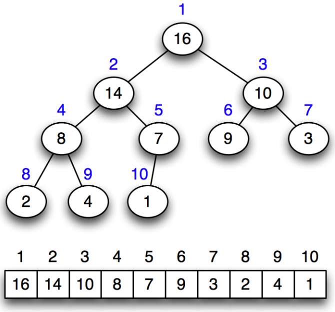
Heap – array representation of heap-ordered tree



Heap – array representation of heap-ordered tree



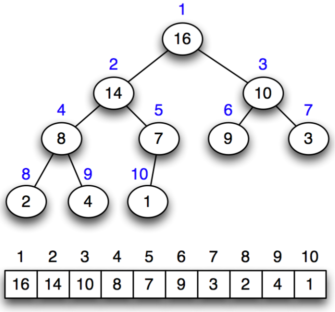
Heap – accessing parent and children



MAXIMUM(A)

return $A[1]$

Heap – accessing parent and children



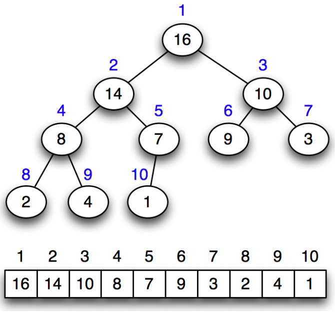
MAXIMUM(A)

return $A[1]$

PARENT(i)

return $\lfloor i/2 \rfloor$

Heap – accessing parent and children



MAXIMUM(A)

return $A[1]$

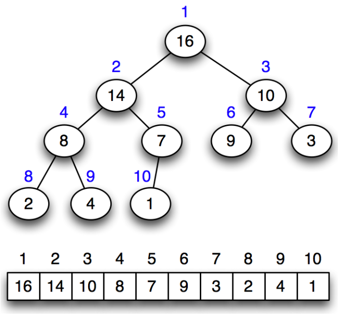
PARENT(i)

return $\lfloor i/2 \rfloor$

Question

What if PARENT$(i) < 1$?

Heap – accessing parent and children



MAXIMUM(A)

return $A[1]$

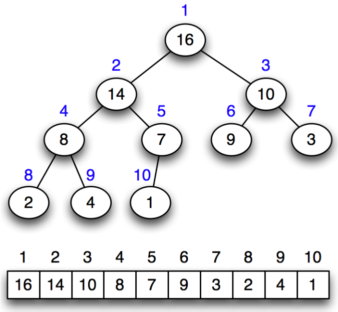
PARENT(i)

return $\lfloor i/2 \rfloor$

LEFT(i)

return $2i$

Heap – accessing parent and children



MAXIMUM(A)

return $A[1]$

PARENT(i)

return $\lfloor i/2 \rfloor$

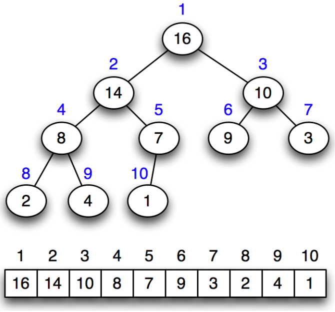
LEFT(i)

return $2i$

Question

What if LEFT$(i) > n$?

Heap – accessing parent and children



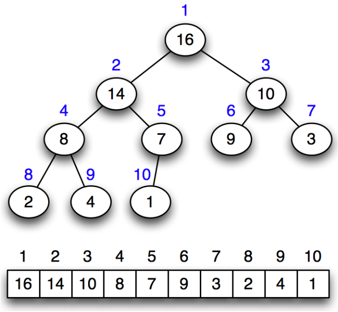
MAXIMUM(A)
  return $A[1]$

PARENT(i)
  return $\lfloor i/2 \rfloor$

LEFT(i)
  return $2i$

RIGHT(i)
  return $2i + 1$

Heap – accessing parent and children



$\textsc{Maximum}(A)$

  return $A[1]$

$\textsc{Parent}(i)$

  return $\lfloor i/2 \rfloor$

$\textsc{Left}(i)$

  return $2i$

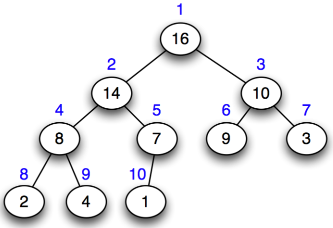
$\textsc{Right}(i)$

  return $2i + 1$

Question

What if $\textsc{Right}(i) > n$?

Heap – accessing parent and children



1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

MAXIMUM(A)

  return $A[1]$

PARENT(i)

  return $\lfloor i/2 \rfloor$

LEFT(i)

  return $2i$

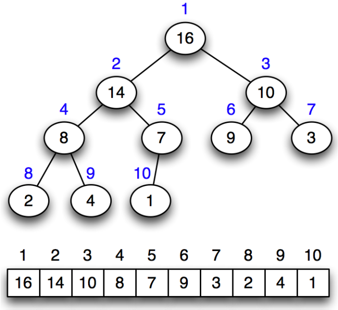
RIGHT(i)

  return $2i + 1$

Lemma

*All nodes $i > \lfloor length[A]/2 \rfloor$ (or equivalently, $i > \lfloor heap\text{-}size[A]/2 \rfloor$) are leaf nodes. *Check Note*

Heap – accessing parent and children



$\text{MAXIMUM}(A)$

return $A[1]$

$\text{PARENT}(i)$

return $\lfloor i/2 \rfloor$

$\text{LEFT}(i)$

return $2i$

$\text{RIGHT}(i)$

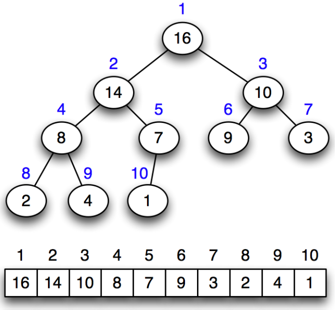
return $2i + 1$

Definition (Heap property)

Heap property: For every node i other than the root,

$$A[\text{PARENT}(i)] \geq A[i].$$

Heap – accessing parent and children



MAXIMUM(A)

return $A[1]$

PARENT(i)

return $\lfloor i/2 \rfloor$

LEFT(i)

return $2i$

RIGHT(i)

return $2i + 1$

Question

Why do we insist that a heap-ordered tree be a complete binary tree? (Hint: draw the array representation of a tree that is not complete and see the *gaps*).

Contents

Operations on heap

1. MAX-HEAPIFY(A, i) – Ensure the heap property of A starting at node i. Also known as "sink" operation since it sinks the lighter elements down the tree.

Operations on heap

1. MAX-HEAPIFY(A, i) – Ensure the heap property of A starting at node i. Also known as "sink" operation since it sinks the lighter elements down the tree.
2. MAX-HEAP-INSERT(A, key) – Insert key in the heap A, maintaining A's heap property.

Operations on heap

1. MAX-HEAPIFY(A, i) – Ensure the heap property of A starting at node i. Also known as "sink" operation since it sinks the lighter elements down the tree.
2. MAX-HEAP-INSERT(A, key) – Insert key in the heap A, maintaining A's heap property.
3. BUILD-MAX-HEAP(A) – Build a max heap given an array A.

Operations on heap

1. MAX-HEAPIFY(A, i) – Ensure the heap property of A starting at node i. Also known as "sink" operation since it sinks the lighter elements down the tree.
2. MAX-HEAP-INSERT(A, key) – Insert key in the heap A, maintaining A's heap property.
3. BUILD-MAX-HEAP(A) – Build a max heap given an array A.
4. HEAPSORT(A) – Sort the elements in array A using the heap operations.

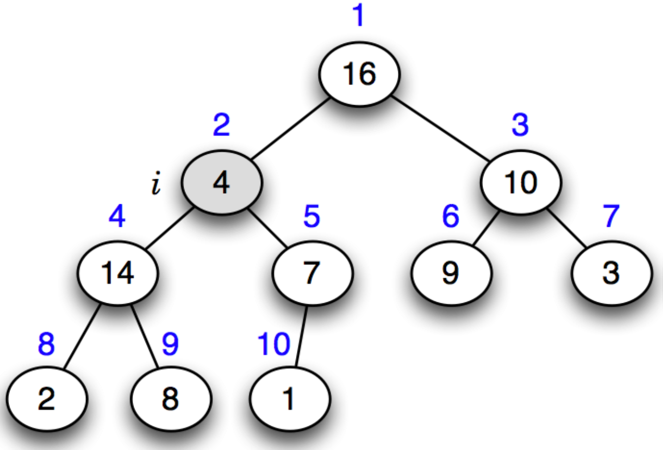
Operations on heap

1. MAX-HEAPIFY(A, i) – Ensure the heap property of A starting at node i. Also known as "sink" operation since it sinks the lighter elements down the tree.
2. MAX-HEAP-INSERT(A, key) – Insert key in the heap A, maintaining A's heap property.
3. BUILD-MAX-HEAP(A) – Build a max heap given an array A.
4. HEAPSORT(A) – Sort the elements in array A using the heap operations.
5. HEAP-INCREASE-KEY(A, i, key) – Increase the value of element at node i to key, and ensure the heap property of A by moving larger elements upwards. Also known as "swim" operation as it moves larger elements upwards.

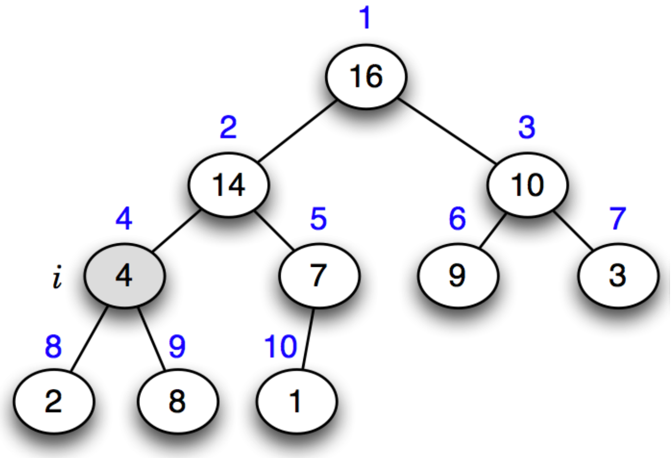
Operations on heap

1. MAX-HEAPIFY(A, i) – Ensure the heap property of A starting at node i. Also known as "sink" operation since it sinks the lighter elements down the tree.
2. MAX-HEAP-INSERT(A, key) – Insert key in the heap A, maintaining A's heap property.
3. BUILD-MAX-HEAP(A) – Build a max heap given an array A.
4. HEAPSORT(A) – Sort the elements in array A using the heap operations.
5. HEAP-INCREASE-KEY(A, i, key) – Increase the value of element at node i to key, and ensure the heap property of A by moving larger elements upwards. Also known as "swim" operation as it moves larger elements upwards.
6. HEAP-EXTRACT-MAX(A) – Extract the largest element from heap A.

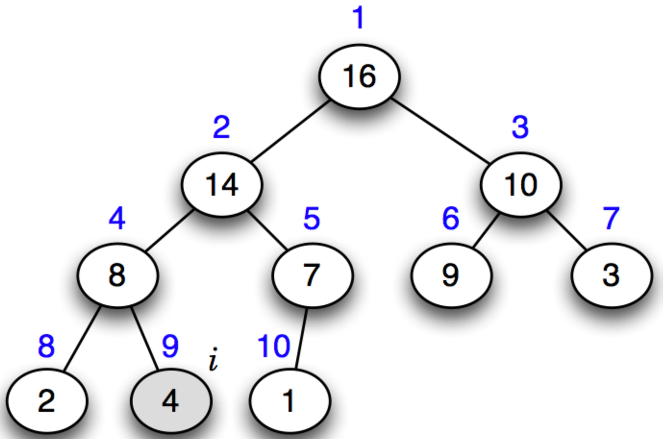
Example of MAX-HEAPIFY ("sink") operation



Example of MAX-HEAPIFY ("sink") operation



Example of MAX-HEAPIFY ("sink") operation



MAX-HEAPIFY algorithm

MAX-HEAPIFY(A, i)

```
l ← left(i)
r ← right(i)
if l ≤ heap-size[A] and A[l] > A[i]
    then largest ← l
    else largest ← i
if r ≤ heap-size[A] and A[r] > A[largest]
    then largest ← r
if largest ≠ i
    then exchange A[i] ↔ A[largest]
         MAX-HEAPIFY(A, largest)
```

MAX-HEAPIFY algorithm

MAX-HEAPIFY(A, i)

$l \leftarrow left(i)$
$r \leftarrow right(i)$
if $l \leq heap\text{-}size[A]$ and $A[l] > A[i]$
  then $largest \leftarrow l$
  else $largest \leftarrow i$
if $r \leq heap\text{-}size[A]$ and $A[r] > A[largest]$
  then $largest \leftarrow r$
if $largest \neq i$
  then exchange $A[i] \leftrightarrow A[largest]$
    MAX-HEAPIFY$(A, largest)$

Analysis – first way

Since the children's subtrees each have at most size of $2n/3$ (when the last row is exactly half full), we have

$$T(n) \leq T(2n/3) + \Theta(1).$$

According to case 2 of the Master theorem, $T(n) = O(\lg n)$.

MAX-HEAPIFY algorithm

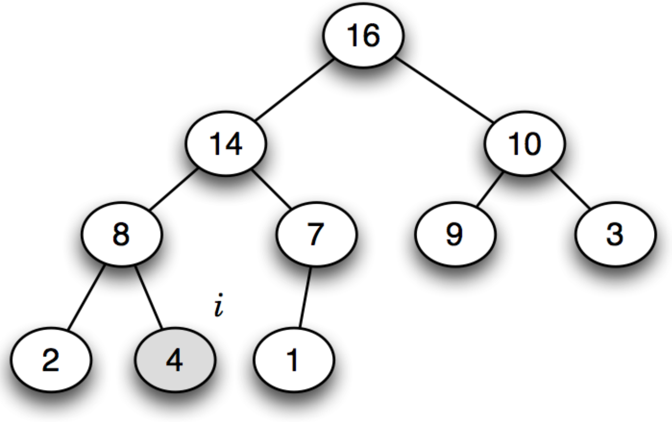
MAX-HEAPIFY(A, i)

$l \leftarrow left(i)$
$r \leftarrow right(i)$
if $l \leq heap\text{-}size[A]$ and $A[l] > A[i]$
  then $largest \leftarrow l$
  else $largest \leftarrow i$
if $r \leq heap\text{-}size[A]$ and $A[r] > A[largest]$
  then $largest \leftarrow r$
if $largest \neq i$
  then exchange $A[i] \leftrightarrow A[largest]$
    MAX-HEAPIFY$(A, largest)$

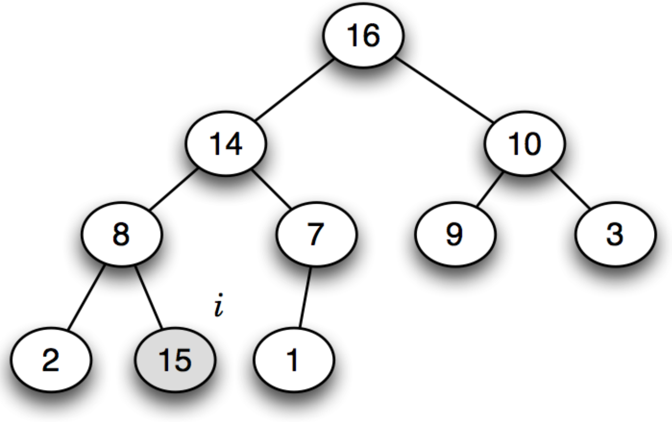
Analysis – second way

The running time of MAX-HEAPIFY on a node of height h is $T(n) = O(h) = O(\lg n)$.

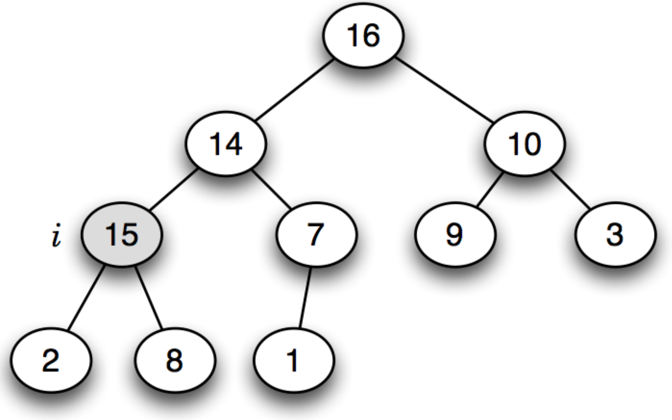
Example of HEAP-INCREASE-KEY ("swim") operation



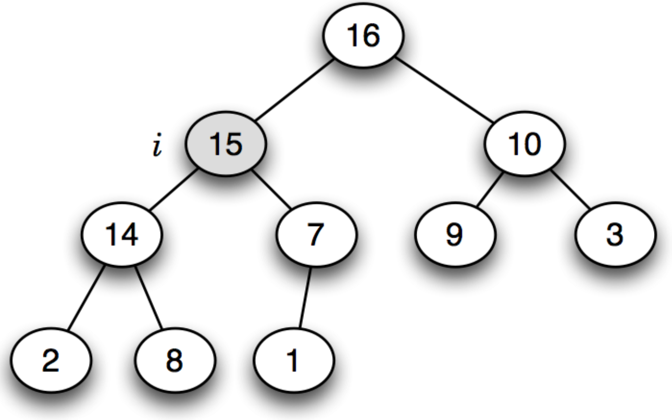
Example of HEAP-INCREASE-KEY ("swim") operation



Example of HEAP-INCREASE-KEY (“swim”) operation



Example of HEAP-INCREASE-KEY ("swim") operation



HEAP-INCREASE-KEY algorithm

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
   then error "new key is smaller than current key"
A[i] ← key
while i > 1 and A[PARENT(i)] < A[i]
      do exchange A[i] ↔ A[parent(i)]
         i ← PARENT(i)
```

HEAP-INCREASE-KEY algorithm

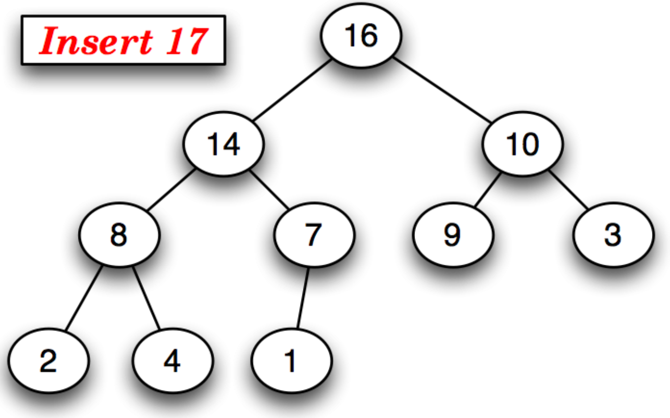
HEAP-INCREASE-KEY(A, i, key)

if $key < A[i]$
  then error "new key is smaller than current key"
$A[i] \leftarrow key$
while $i > 1$ and $A[\text{PARENT}(i)] < A[i]$
    do exchange $A[i] \leftrightarrow A[parent(i)]$
      $i \leftarrow \text{PARENT}(i)$

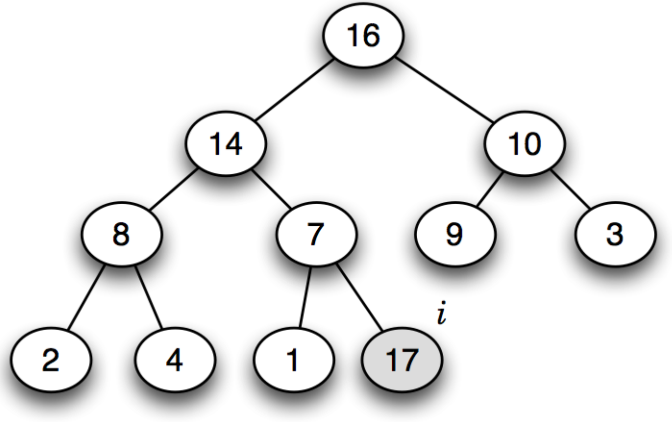
Analysis

A node may move all the way from a leaf node to the root because of increased value, so $T(n) = O(h) = O(\lg n)$.

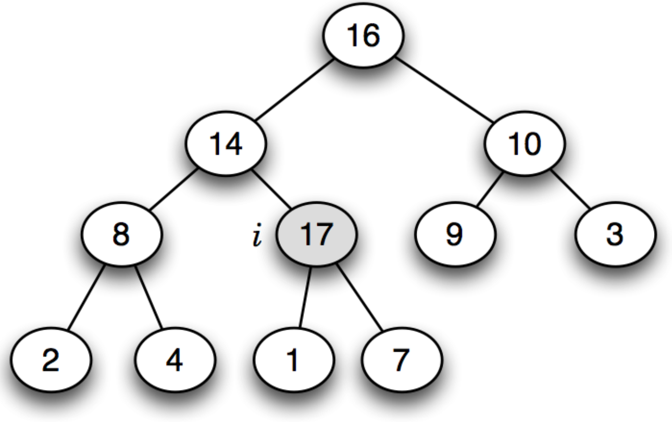
Example of MAX-HEAP-INSERT operation



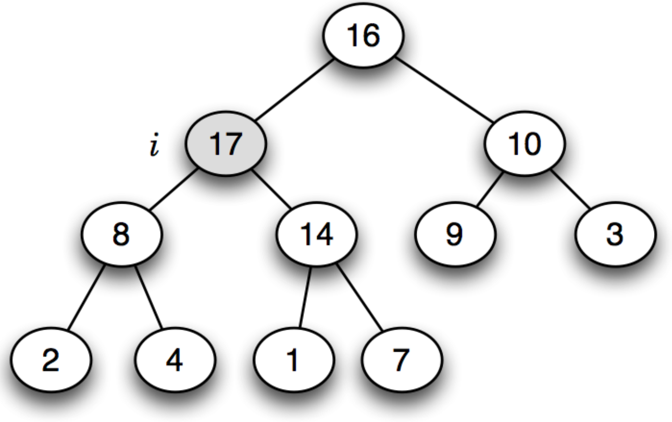
Example of MAX-HEAP-INSERT operation



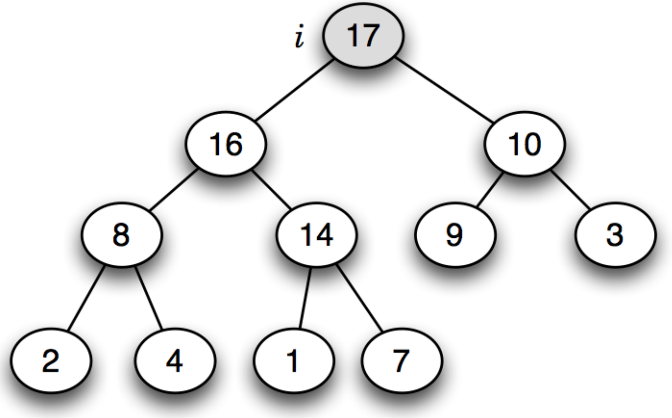
Example of MAX-HEAP-INSERT operation



Example of MAX-HEAP-INSERT operation



Example of MAX-HEAP-INSERT operation



MAX-HEAP-INSERT algorithm

$\textsc{Max-Heap-Insert}(A, key)$

$heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
$A[heap\text{-}size[A]] \leftarrow key$
$i \leftarrow heap\text{-}size[A]$
while $i > 1$ and $A[\textsc{Parent}(i)] < A[i]$
    do exchange $A[i] \leftrightarrow A[parent(i)]$
        $i \leftarrow A[\textsc{Parent}(i)]$

MAX-HEAP-INSERT algorithm

MAX-HEAP-INSERT(A, key)

$heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
$A[heap\text{-}size[A]] \leftarrow key$
$i \leftarrow heap\text{-}size[A]$
while $i > 1$ and $A[\text{PARENT}(i)] < A[i]$
  do exchange $A[i] \leftrightarrow A[parent(i)]$
    $i \leftarrow A[\text{PARENT}(i)]$

Can also be done using HEAP-INCREASE-KEY.

MAX-HEAP-INSERT(A, key)

$heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
$A[heap\text{-}size[A]] \leftarrow -\infty$
HEAP-INCREASE-KEY$(A, heap\text{-}size[A], key)$

MAX-HEAP-INSERT algorithm

MAX-HEAP-INSERT(A, key)

$heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
$A[heap\text{-}size[A]] \leftarrow key$
$i \leftarrow heap\text{-}size[A]$
while $i > 1$ and $A[\text{PARENT}(i)] < A[i]$
  do exchange $A[i] \leftrightarrow A[parent(i)]$
    $i \leftarrow A[\text{PARENT}(i)]$

Can also be done using HEAP-INCREASE-KEY.

MAX-HEAP-INSERT(A, key)

$heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
$A[heap\text{-}size[A]] \leftarrow -\infty$
HEAP-INCREASE-KEY$(A, heap\text{-}size[A], key)$

Analysis

$T(n) = O(h) = O(\lg n)$.

Simple BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP'(A)

$heap\text{-}size[A] \leftarrow 1$
for $i \leftarrow 2$ **to** $length[A]$
    do MAX-HEAP-INSERT($A, A[i]$)

Simple BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP'(A)

```
1   heap-size[A] ← 1
2   for i ← 2 to length[A]
3         do MAX-HEAP-INSERT(A, A[i])
```

Analysis

There are $n-1$ calls to MAX-HEAP-INSERT, each taking $O(\lg n)$ time, so $T(n) = O(n \lg n)$.

Simple BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP'(A)

1 $\textit{heap-size}[A] \leftarrow 1$
2 **for** $i \leftarrow 2$ **to** $\textit{length}[A]$
3 **do** MAX-HEAP-INSERT$(A, A[i])$

Analysis

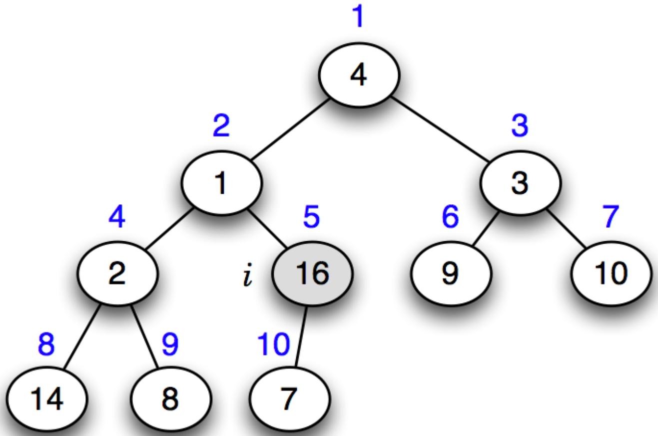
There are $n-1$ calls to MAX-HEAP-INSERT, each taking $O(\lg n)$ time, so $T(n) = O(n \lg n)$.

Better way?

A better way is to build up the heap from the smaller trees. See next.

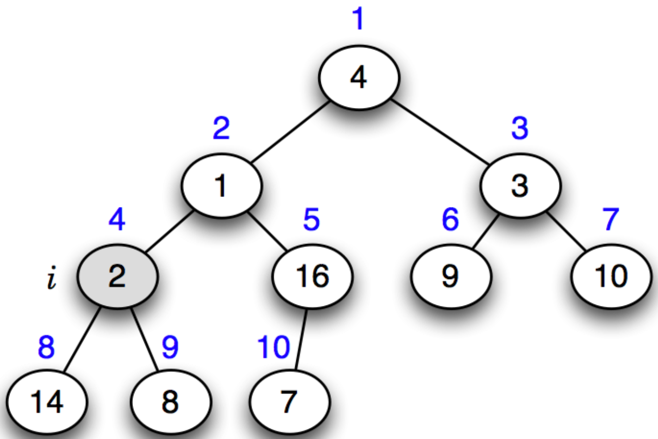
Example of BUILD-MAX-HEAP (“heapify”) operation

1	2	3	4	5	6	7	8	9	10
4	1	3	2	16	9	10	14	8	7



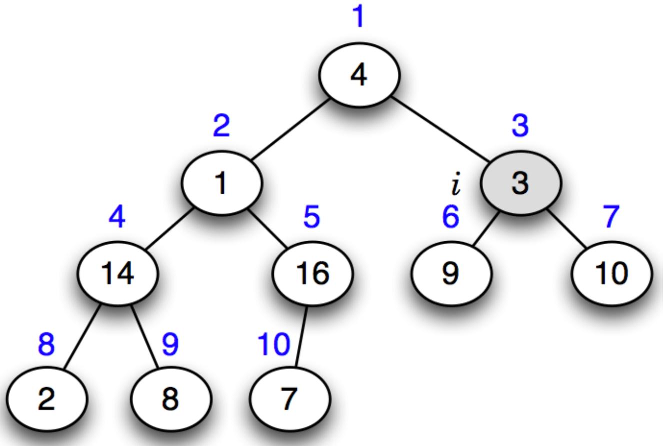
Example of BUILD-MAX-HEAP ("heapify") operation

1	2	3	4	5	6	7	8	9	10
4	1	3	2	16	9	10	14	8	7



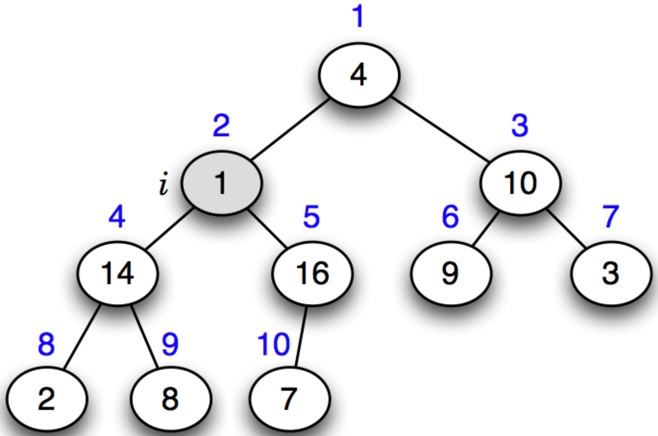
Example of BUILD-MAX-HEAP (“heapify”) operation

1	2	3	4	5	6	7	8	9	10
4	1	3	2	16	9	10	14	8	7



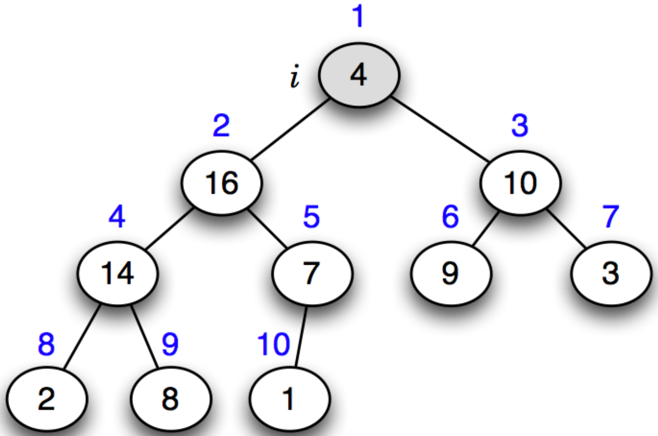
Example of BUILD-MAX-HEAP ("heapify") operation

1	2	3	4	5	6	7	8	9	10
4	1	3	2	16	9	10	14	8	7



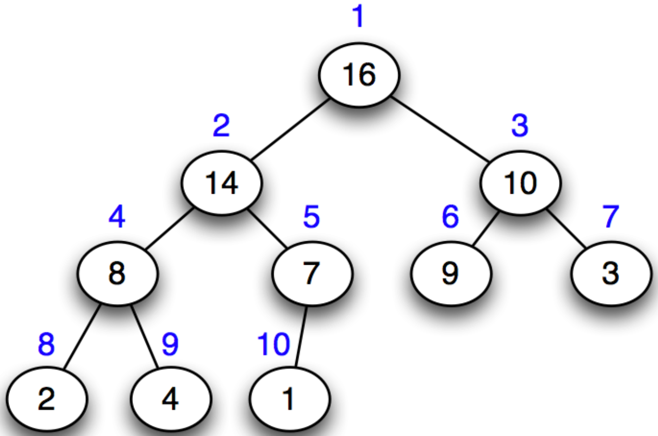
Example of BUILD-MAX-HEAP (“heapify”) operation

1	2	3	4	5	6	7	8	9	10
4	1	3	2	16	9	10	14	8	7



Example of BUILD-MAX-HEAP ("heapify") operation

1	2	3	4	5	6	7	8	9	10
4	1	3	2	16	9	10	14	8	7



BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP(A)

```
heap-size[A]] ← length[A]
for i ← ⌊length[A]/2⌋ downto 1
        do MAX-HEAPIFY(A, i)
```

BUILD-MAX-HEAP algorithm

BUILD-MAX-HEAP(A)

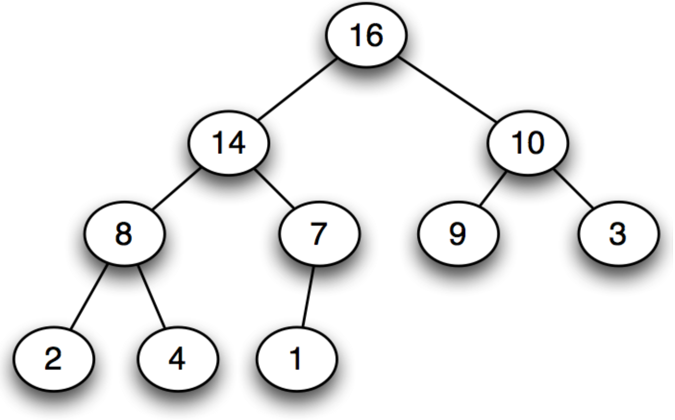
1 *heap-size*$[A]] \leftarrow$ *length*$[A]$
2 **for** $i \leftarrow \lfloor length[A]/2 \rfloor$ **downto** 1
3 **do** MAX-HEAPIFY(A, i)

Analysis

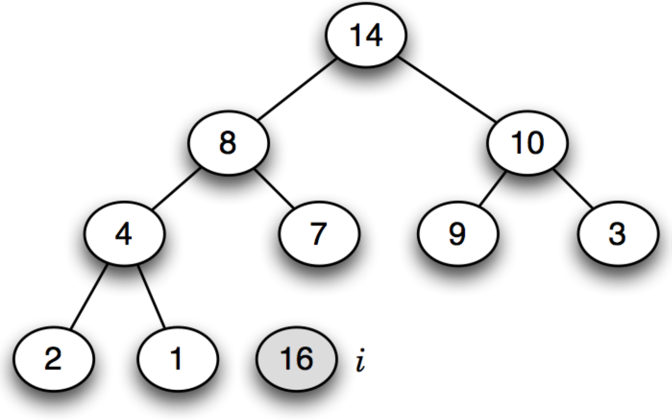
$T(n) = O(n)$ (see textbook for details)

Contents

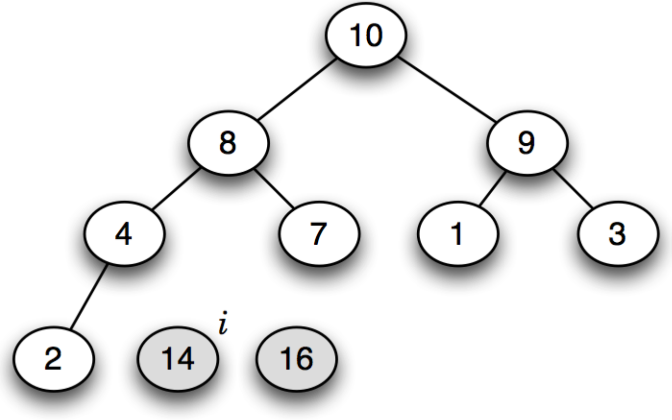
Heap use - heapsort



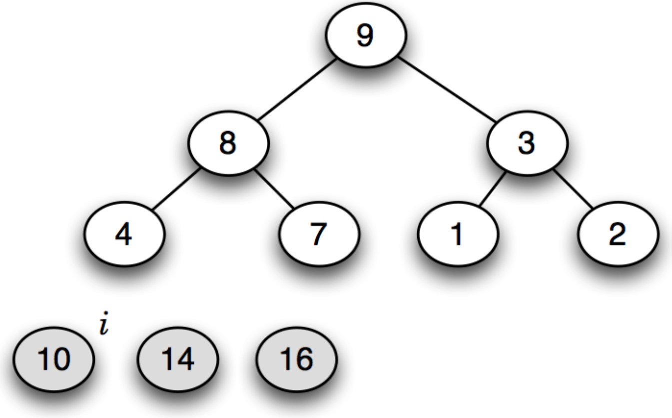
Heap use - heapsort



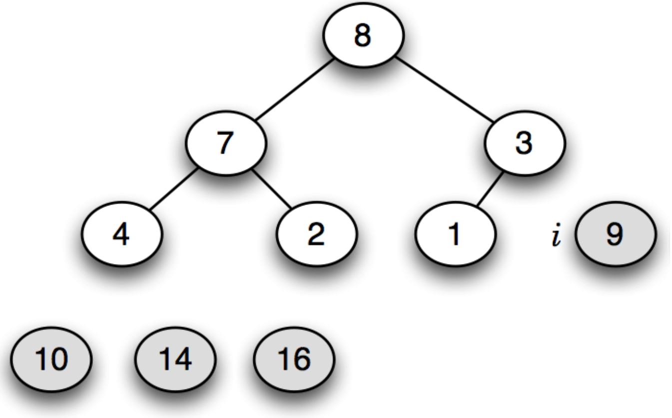
Heap use - heapsort



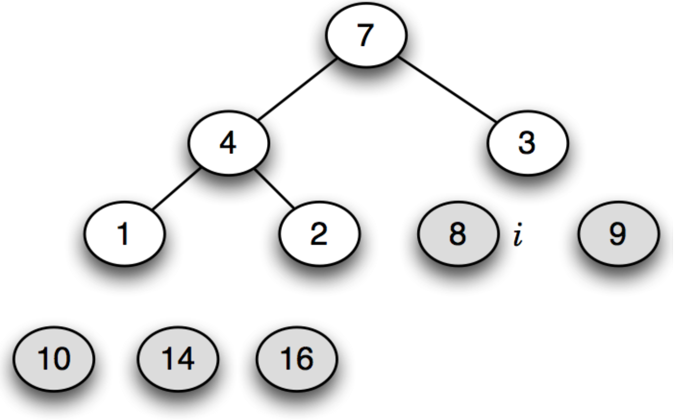
Heap use - heapsort



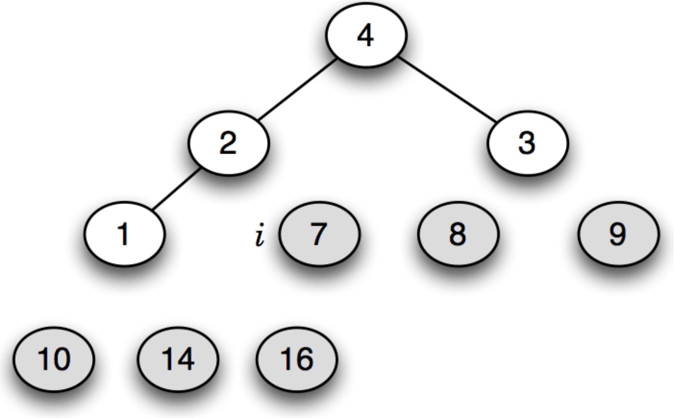
Heap use - heapsort



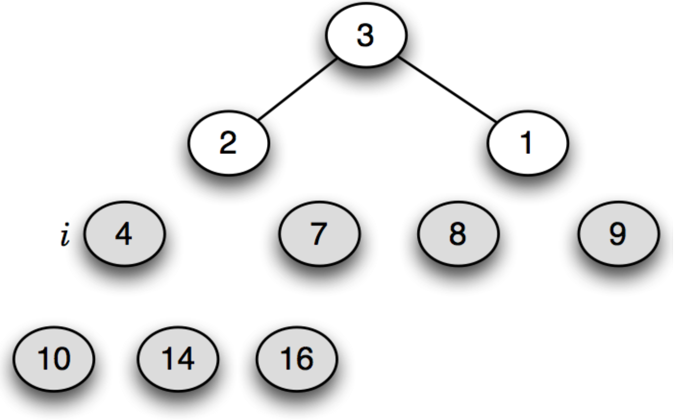
Heap use - heapsort



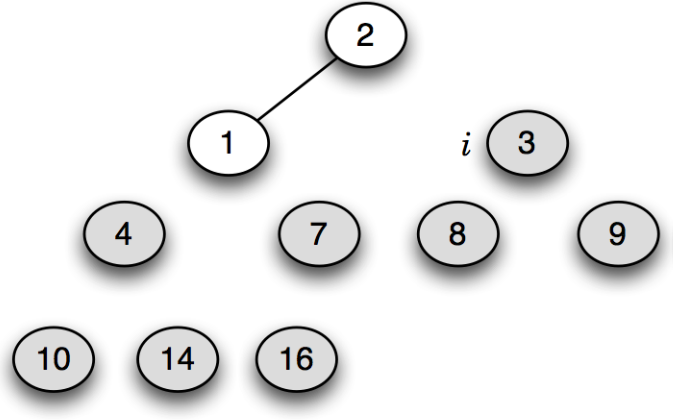
Heap use - heapsort



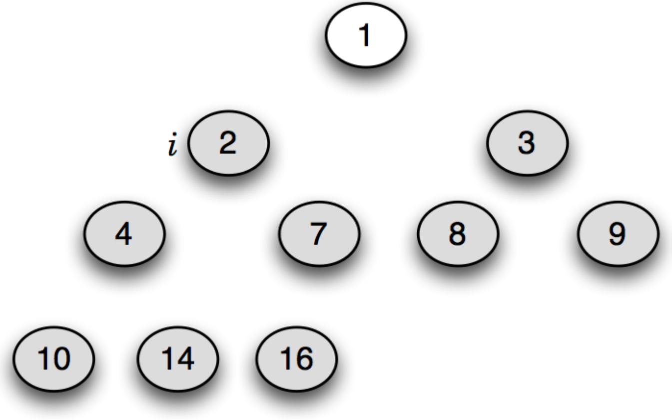
Heap use - heapsort



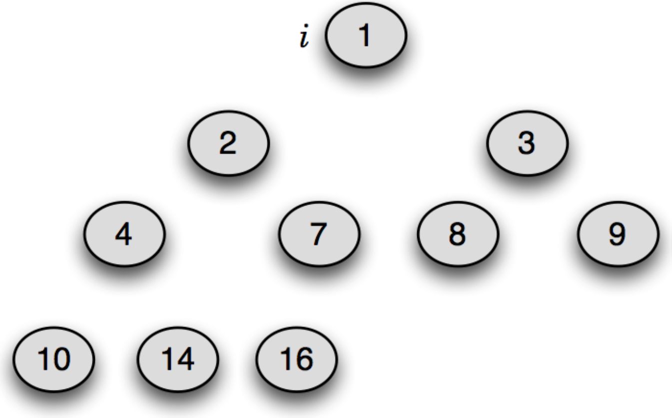
Heap use - heapsort



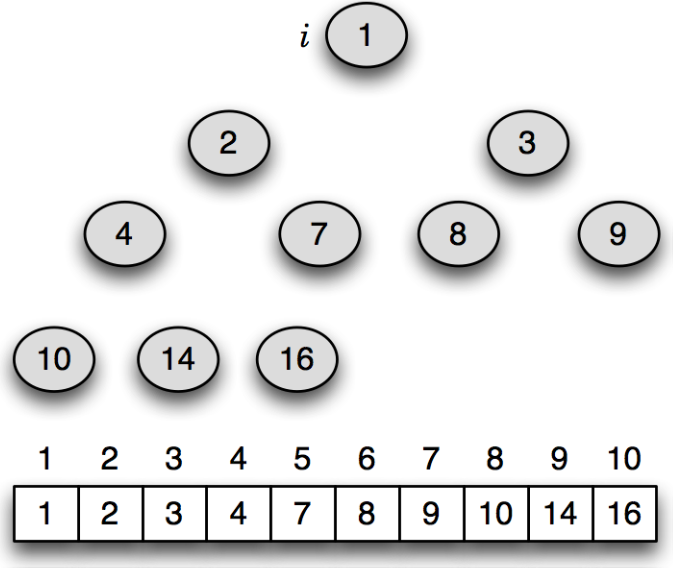
Heap use - heapsort



Heap use - heapsort



Heap use - heapsort



HEAPSORT algorithm

HEAPSORT(A)

	cost	*times*
BUILD-MAX-HEAP(A)	$\Theta(n)$	1
for $i \leftarrow$ *length*$[A]$ **downto** 2	$\Theta(1)$	n
do exchange $A[1] \leftrightarrow A[i]$	$\Theta(1)$	$n-1$
heap-size$[A] \leftarrow$ *heap-size*$[A] - 1$	$\Theta(1)$	$n-1$
MAX-HEAPIFY$(A, 1)$	$\Theta(\lg n)$	$n-1$

HEAPSORT algorithm

HEAPSORT(A)

	cost	*times*
BUILD-MAX-HEAP(A)	$\Theta(n)$	1
for $i \leftarrow \mathit{length}[A]$ **downto** 2	$\Theta(1)$	n
do exchange $A[1] \leftrightarrow A[i]$	$\Theta(1)$	$n-1$
$\mathit{heap\text{-}size}[A] \leftarrow \mathit{heap\text{-}size}[A] - 1$	$\Theta(1)$	$n-1$
MAX-HEAPIFY$(A, 1)$	$\Theta(\lg n)$	$n-1$

HEAPSORT algorithm

HEAPSORT(A)

	cost	*times*
BUILD-MAX-HEAP(A)	$\Theta(n)$	1
for $i \leftarrow$ *length*$[A]$ **downto** 2	$\Theta(1)$	n
do exchange $A[1] \leftrightarrow A[i]$	$\Theta(1)$	$n-1$
heap-size$[A] \leftarrow$ *heap-size*$[A] - 1$	$\Theta(1)$	$n-1$
MAX-HEAPIFY$(A, 1)$	$\Theta(\lg n)$	$n-1$

HEAPSORT algorithm

HEAPSORT(A)

	cost	*times*
BUILD-MAX-HEAP(A)	$\Theta(n)$	1
for $i \leftarrow \mathit{length}[A]$ **downto** 2	$\Theta(1)$	n
do exchange $A[1] \leftrightarrow A[i]$	$\Theta(1)$	$n-1$
$\mathit{heap\text{-}size}[A] \leftarrow \mathit{heap\text{-}size}[A] - 1$	$\Theta(1)$	$n-1$
MAX-HEAPIFY$(A, 1)$	$\Theta(\lg n)$	$n-1$

HEAPSORT algorithm

HEAPSORT(A)

	cost	*times*
BUILD-MAX-HEAP(A)	$\Theta(n)$	1
for $i \leftarrow length[A]$ **downto** 2	$\Theta(1)$	n
do exchange $A[1] \leftrightarrow A[i]$	$\Theta(1)$	$n-1$
$heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$	$\Theta(1)$	$n-1$
MAX-HEAPIFY$(A, 1)$	$\Theta(\lg n)$	$n-1$

HEAPSORT algorithm

HEAPSORT(A)

	cost	*times*
BUILD-MAX-HEAP(A)	$\Theta(n)$	1
for $i \leftarrow length[A]$ **downto** 2	$\Theta(1)$	n
do exchange $A[1] \leftrightarrow A[i]$	$\Theta(1)$	$n-1$
$heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$	$\Theta(1)$	$n-1$
MAX-HEAPIFY($A, 1$)	$\Theta(\lg n)$	$n-1$

HEAPSORT algorithm

HEAPSORT(A)

	cost	*times*
BUILD-MAX-HEAP(A)	$\Theta(n)$	1
for $i \leftarrow$ *length*$[A]$ **downto** 2	$\Theta(1)$	n
do exchange $A[1] \leftrightarrow A[i]$	$\Theta(1)$	$n-1$
heap-size$[A] \leftarrow$ *heap-size*$[A] - 1$	$\Theta(1)$	$n-1$
MAX-HEAPIFY($A, 1$)	$\Theta(\lg n)$	$n-1$

Worst-case analysis

$T(n) = \Theta(n \lg n)$

Contents

Heap use - priority queue

Definition (Priority Queue)

A priority queue is a data structure for maintaining a set S of elements, each with an associated value called a *key*. A max-priority queue supports the following operations.

1. $\text{INSERT}(S, x)$ – inserts the element x into the set S. This operation could be written as $S \leftarrow S \cup \{x\}$.
2. $\text{MAXIMUM}(S)$ – returns the element of S with the largest key.
3. $\text{EXTRACT-MAX}(S)$ – removes and returns the element of S with the largest key.
4. $\text{INCREASE-KEY}(S, x, k)$ – increases the value of element x's key to k. Assume $k \geq x$'s current value.

Heap use - priority queue

Definition (Priority Queue)

A priority queue is a data structure for maintaining a set S of elements, each with an associated value called a *key*. A max-priority queue supports the following operations.

1. $\text{INSERT}(S, x)$ – inserts the element x into the set S. This operation could be written as $S \leftarrow S \cup \{x\}$.
2. $\text{MAXIMUM}(S)$ – returns the element of S with the largest key.
3. $\text{EXTRACT-MAX}(S)$ – removes and returns the element of S with the largest key.
4. $\text{INCREASE-KEY}(S, x, k)$ – increases the value of element x's key to k. Assume $k \geq x$'s current value.

- Used in many scheduling applications where jobs or tasks are scheduled according to priority.

Heap use - priority queue

Definition (Priority Queue)

A priority queue is a data structure for maintaining a set S of elements, each with an associated value called a *key*. A max-priority queue supports the following operations.

1. $\mathrm{INSERT}(S, x)$ – inserts the element x into the set S. This operation could be written as $S \leftarrow S \cup \{x\}$.
2. $\mathrm{MAXIMUM}(S)$ – returns the element of S with the largest key.
3. $\mathrm{EXTRACT\text{-}MAX}(S)$ – removes and returns the element of S with the largest key.
4. $\mathrm{INCREASE\text{-}KEY}(S, x, k)$ – increases the value of element x's key to k. Assume $k \geq x$'s current value.

- Used in many scheduling applications where jobs or tasks are scheduled according to priority.
- A FIFO queue is a priority queue where the priority is inversely proportional to time of arrival.

Heap use - priority queue

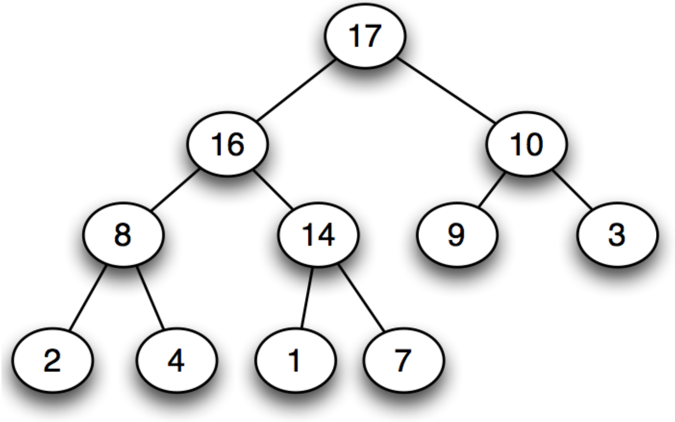
Definition (Priority Queue)

A priority queue is a data structure for maintaining a set S of elements, each with an associated value called a *key*. A max-priority queue supports the following operations.

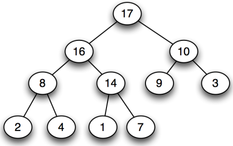
1. $\mathrm{INSERT}(S, x)$ – inserts the element x into the set S. This operation could be written as $S \leftarrow S \cup \{x\}$.
2. $\mathrm{MAXIMUM}(S)$ – returns the element of S with the largest key.
3. $\mathrm{EXTRACT\text{-}MAX}(S)$ – removes and returns the element of S with the largest key.
4. $\mathrm{INCREASE\text{-}KEY}(S, x, k)$ – increases the value of element x's key to k. Assume $k \geq x$'s current value.

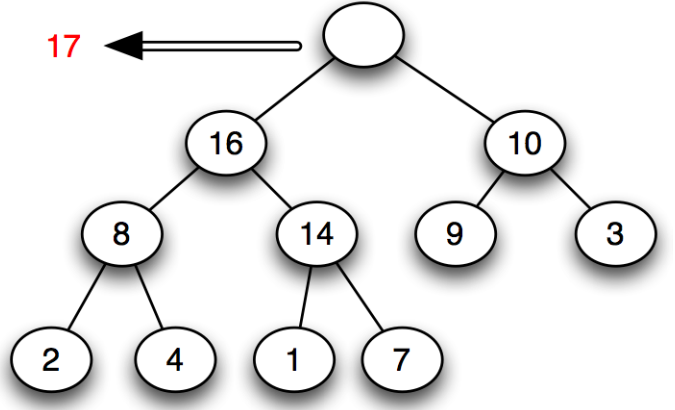
- Used in many scheduling applications where jobs or tasks are scheduled according to priority.
- A FIFO queue is a priority queue where the priority is inversely proportional to time of arrival.
- A LIFO stack is a priority queue where the priority is proportional to time of arrival.

Example of HEAP-EXTRACT-MAX operation

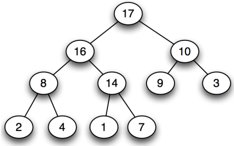


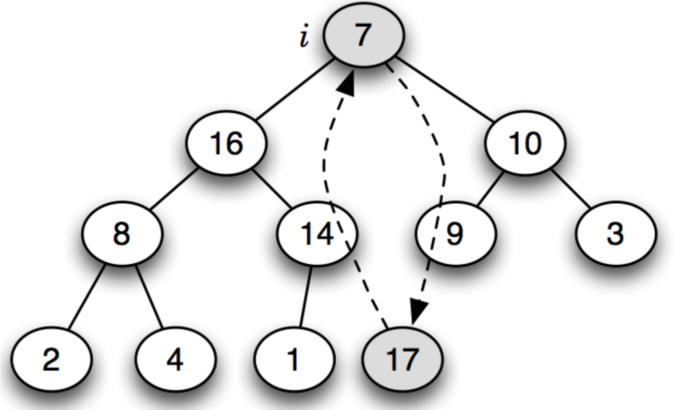
Example of HEAP-EXTRACT-MAX operation



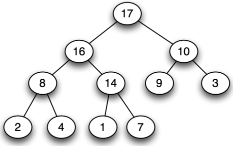


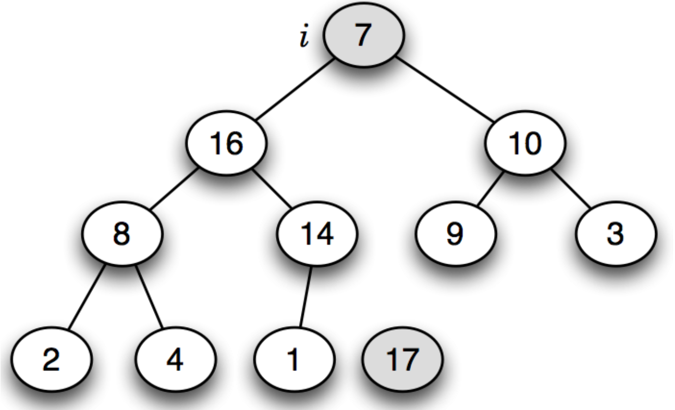
Example of HEAP-EXTRACT-MAX operation



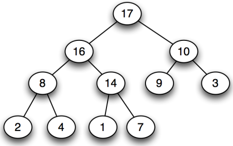


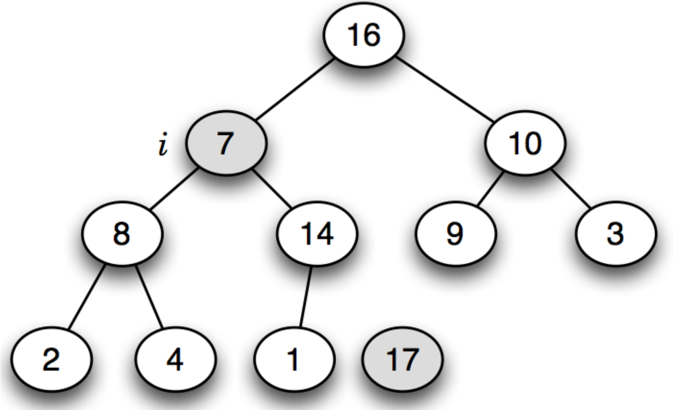
Example of HEAP-EXTRACT-MAX operation



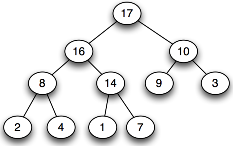


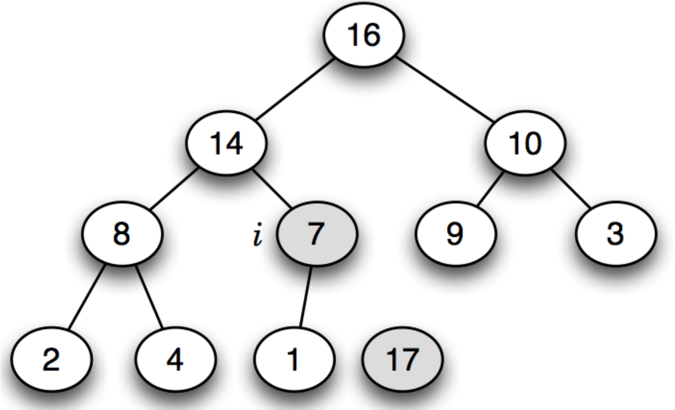
Example of HEAP-EXTRACT-MAX operation



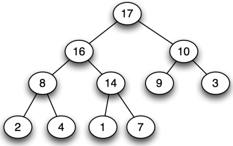


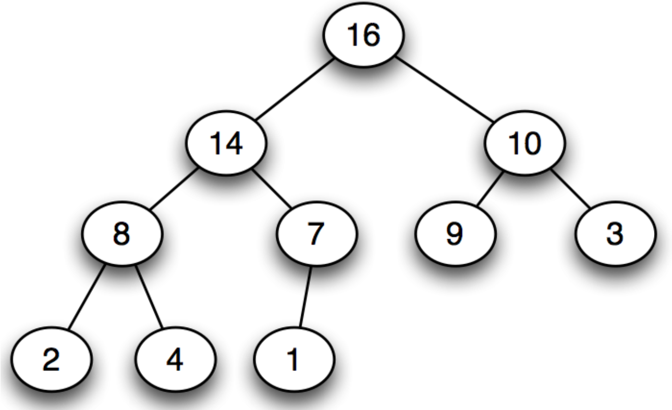
Example of HEAP-EXTRACT-MAX operation





Example of HEAP-EXTRACT-MAX operation





HEAP-EXTRACT-MAX algorithm

EXTRACT-MAX(A)

	cost	*times*
if $heap\text{-}size[A] < 1$	$\Theta(1)$	1
then error "heap underflow"	$\Theta(1)$	1
$max \leftarrow A[1]$	$\Theta(1)$	1
$A[1] \leftarrow A[heap\text{-}size[A]]$	$\Theta(1)$	1
$heap\text{-}size[A]] \leftarrow heap\text{-}size[A] - 1$	$\Theta(1)$	1
MAX-HEAPIFY($A, 1$)	$\Theta(\lg n)$	1

HEAP-EXTRACT-MAX algorithm

EXTRACT-MAX(A)

	cost	times
if *heap-size*$[A] < 1$	$\Theta(1)$	1
then error "heap underflow"	$\Theta(1)$	1
$max \leftarrow A[1]$	$\Theta(1)$	1
$A[1] \leftarrow A[\textit{heap-size}[A]]$	$\Theta(1)$	1
$\textit{heap-size}[A]] \leftarrow \textit{heap-size}[A] - 1$	$\Theta(1)$	1
MAX-HEAPIFY$(A, 1)$	$\Theta(\lg n)$	1

HEAP-EXTRACT-MAX algorithm

EXTRACT-MAX(A)

	cost	*times*
if $\textit{heap-size}[A] < 1$	$\Theta(1)$	1
then error “heap underflow”	$\Theta(1)$	1
$\textit{max} \leftarrow A[1]$	$\Theta(1)$	1
$A[1] \leftarrow A[\textit{heap-size}[A]]$	$\Theta(1)$	1
$\textit{heap-size}[A]] \leftarrow \textit{heap-size}[A] - 1$	$\Theta(1)$	1
MAX-HEAPIFY$(A, 1)$	$\Theta(\lg n)$	1

HEAP-EXTRACT-MAX algorithm

EXTRACT-MAX(A)

	cost	*times*
if *heap-size*$[A] < 1$	$\Theta(1)$	1
then error "heap underflow"	$\Theta(1)$	1
$max \leftarrow A[1]$	$\Theta(1)$	1
$A[1] \leftarrow A[heap\text{-}size[A]]$	$\Theta(1)$	1
$heap\text{-}size[A]] \leftarrow heap\text{-}size[A] - 1$	$\Theta(1)$	1
MAX-HEAPIFY$(A, 1)$	$\Theta(\lg n)$	1

HEAP-EXTRACT-MAX algorithm

EXTRACT-MAX(A)

	cost	*times*
if $\textit{heap-size}[A] < 1$	$\Theta(1)$	1
then error "heap underflow"	$\Theta(1)$	1
$\textit{max} \leftarrow A[1]$	$\Theta(1)$	1
$A[1] \leftarrow A[\textit{heap-size}[A]]$	$\Theta(1)$	1
$\textit{heap-size}[A]] \leftarrow \textit{heap-size}[A] - 1$	$\Theta(1)$	1
MAX-HEAPIFY$(A, 1)$	$\Theta(\lg n)$	1

HEAP-EXTRACT-MAX algorithm

EXTRACT-MAX(A)

	cost	*times*
if $\mathit{heap\text{-}size}[A] < 1$	$\Theta(1)$	1
then error "heap underflow"	$\Theta(1)$	1
$\mathit{max} \leftarrow A[1]$	$\Theta(1)$	1
$A[1] \leftarrow A[\mathit{heap\text{-}size}[A]]$	$\Theta(1)$	1
$\mathit{heap\text{-}size}[A]] \leftarrow \mathit{heap\text{-}size}[A] - 1$	$\Theta(1)$	1
MAX-HEAPIFY$(A, 1)$	$\Theta(\lg n)$	1

HEAP-EXTRACT-MAX algorithm

EXTRACT-MAX(A)

	cost	*times*
if $\textit{heap-size}[A] < 1$	$\Theta(1)$	1
then error "heap underflow"	$\Theta(1)$	1
$\textit{max} \leftarrow A[1]$	$\Theta(1)$	1
$A[1] \leftarrow A[\textit{heap-size}[A]]$	$\Theta(1)$	1
$\textit{heap-size}[A]] \leftarrow \textit{heap-size}[A] - 1$	$\Theta(1)$	1
MAX-HEAPIFY$(A, 1)$	$\Theta(\lg n)$	1

Worst-case analysis

$T(n) = \Theta(\lg n)$

Conclusion

- Heap plays a very important role in many algorithms, either used directly or as part of a priority queue implementation.
- If the size of a queue is known in advance, then an array representation (using a fixed size array) provides compact storage coupled with fast operations.
- Even if the size of the heap is not known in advance, "intelligent" resizing can still provide good benefits.
- Heapsort is a natural application of Heap with two very important properties – $\Theta(n \lg n)$ complexity, and in-place sorting.