Changes of Variable (14.7)

$$\partial(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v} \end{vmatrix}.$$

ex

$$\iint_R \frac{x-y}{x+y}\, dA$$

where R is the region enclosed by $x-y=0$

$x-y=1$ $x+y=1$ $x+y=3$

$U = x+y$ $\quad U=1 \quad V=0$

$V = x-y$ $\quad U=3 \quad V=1$

$U+V = 2x\cancel{+2y}$ $\quad U-V = 2y$

$x = \frac{U+V}{2}$ $\quad y = \frac{U-V}{2}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{2} \times \frac{1}{2} - \frac{1}{2} \times \frac{1}{2}$$

$$= -\frac{1}{4} - \frac{1}{4}$$

$$= -\frac{1+1}{4} = -\frac{2}{4} = -\frac{1}{2}$$

$$\iint_R \frac{x-y}{x+y}\, dA = \int_0^1 \int_1^3 \frac{v}{u} \left|-\tfrac{1}{2}\right| du\, dv$$

$$= \frac{1}{2}\int_0^1 v \Big[\ln u\Big]_1^3 dv$$

$$= \frac{1}{2}\int_0^1 v\,(\ln 3 - \ln 1)\, dv$$

$$= \frac{1}{2}\ln 3 \int_0^1 v\, dv$$

$$= \frac{\ln 3}{4}$$

(1) $x = u + 4v$

$y = 3u - 5v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 4 \\ 3 & -5 \end{vmatrix}$$

$$= -5 - 12$$

$$= -17$$

(2) $x = u + 2v^2$

$y = 2u^2 - v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 4v \\ 4u & -1 \end{vmatrix}$$

$$= -1 - 16uv$$

(3) $x = \sin u + \cos v$

$y = -\cos u + \sin v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos u & -\sin v \\ \sin u & \cos v \end{vmatrix}$$

$$= \cos u \cos v + \sin u \sin v$$

$$= \cos(u - v)$$

(4)

$$x = \frac{2u}{u^2+v^2}, \quad y = -\frac{2v}{u^2+v^2}$$

$$= \cancel{\frac{2u}{u^2}} + \cancel{\frac{2u}{v^2}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{4u^2 - 2(u^2+v^2)}{(u^2+v^2)^2} & -\frac{4uv}{(u^2+v^2)^2} \\ \frac{4uv}{(u^2+v^2)^2} & \frac{2(v^2-u^2)}{(u^2+v^2)^2} \end{vmatrix}$$

(21) Use the transformation

$u = x - 2y$

$v = 2x + y$ to find

$$\iint_R \frac{x-2y}{2x+y}\, dA$$

where R is the rectangular region enclosed by the lines $v\begin{cases} x-2y=1, \\ x-2y=4 \end{cases}$

~~x-2y~~

$$u\begin{cases} 2x+y=1 \\ 2x+y=3 \end{cases}$$

$x - 2y = u$
$2x + y = v$
$5x = u + 2v \quad x = \frac{u + 2v}{5}$

$y = \frac{-2u - v}{5}$

$x = u + 2y$

$y = v - 2x \qquad x = \frac{u + 2v}{5}$

$= \frac{u}{5} + \frac{2v}{5}$

$2u - v = -5y$

$5y = v - 2u$

$y = \frac{v - 2u}{5}$

$= \frac{v}{5} - \frac{2u}{5}$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{vmatrix}$$

$= \frac{1}{5} \times \frac{1}{5} - \frac{2}{5} \times \frac{1}{5}$

$= \frac{1}{25} + \frac{4}{25}$

$= 1/5$

$= -\frac{2}{25} - \frac{2}{25}$

$= -\frac{2+2}{25} = -\frac{4}{25}$

$$\frac{1}{5} \int_1^3 \int_1^4 \frac{u}{v} \, du \, dv$$

$$= \frac{1}{10} \int_1^3 \frac{1}{v} [u^2]_1^4 \, dv = \frac{15}{10} \int_1^3 \frac{1}{v} \, dv$$

$$= \frac{15}{10} \ln 3$$

$$= 3/2 \ln 3$$

(22) Use the transformation

$u = x + y$

$v = x - y$

to find $\iint_R (x-y) e^{x^2-y^2} \, dA$ $\quad (x+y)(x-y)$ $\quad v e^{uv}$

over the rectangular region R enclosed by the lines $x + y = 0,$

$x + y = 1$

$x - y = 1$

$x - y = 4$ $\quad x =$

$u + v = 2x$. $\quad u - v = 2y$

$x = \frac{u+v}{2}$ $\quad y = \frac{u-v}{2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{4} - \frac{1}{4}$$

$$= -\frac{2}{4} = -\frac{1}{2}$$

$$\frac{1}{2}\int_1^4\int_0^1 v e^{uv}\,du\,dv$$

$$= \frac{1}{2}\int_1^4 \frac{v}{v}\left[e^{uv}\right]_0^1$$

$$= \frac{1}{2}\int_1^4 e^{v}-1\,dv$$

$$= \frac{1}{2}\left[e^{v}-v\right]_1^4 = \frac{1}{2}\left[e^4-4-e^1+1\right]$$

$$= \frac{1}{2}\left(e^4-e-3\right)$$

(23) Use the transformation $u = \frac{1}{2}(x+y)$

$v = \frac{1}{2}(x-y)$

to find $\iint_R \sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)\,dA$

over the triangular region R with vertices $(0,0)$ $(2,0)$ $(1,1)$

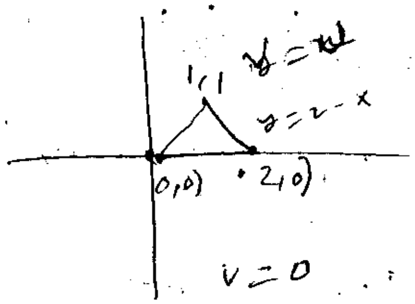
$u+v = \frac{1}{2}(2x)$

$= x$

$x = u+v$

$u-v = \frac{1}{2}(2y)$

$y = u-v$



$v = 0$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -1 - 1$$

$$= -2$$

$y = x$

$u + v = u - v$

$2v = 0$

$v = 0$

$y = 0$

$u - v = 0$

$u = v$

$y = 2 - x$

$u - v = 2 - u - v$

$2u = 2$

$u = 1$

$u = 0 \quad u = 1$

$v = 0 \quad v = u$

$$2\int_0^1 \int_0^u \sin u \cos v \, dv\, du$$

$$= 2\int_0^1 \sin u \left(\sin u\right) du$$

$$= 2\int_0^1 \sin^2 u \, du$$

$$= 2\int_0^1 \frac{1}{2} - \frac{1}{2}\cos 2u$$

$$= 2\left[\frac{1}{2}u - \frac{1}{4}\sin 2u\right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{4}\sin 2\right)$$

$$= (1 - \frac{1}{2}\sin 2$$

Ans.

(29)

use the transformation $u = \frac{y}{x}$

$v = xy$

to find

$$\iint_R xy^3 \, dA$$

over the region R in the first quadrant

enclosed by

$$y = x \qquad \frac{y}{x} = 1$$
$$y = 3x \quad \Rightarrow \quad \frac{y}{x} = 3$$
$$xy = 1 \qquad xy = 1$$
$$xy = 4 \qquad xy = 4$$

~~$u + v = \frac{y}{x} + xy$~~

$$y = ux \qquad\qquad x = y/u$$

$$v = x^2 u \qquad\qquad v = y \frac{y}{u}$$

$$x^2 = v/u \qquad\qquad y^2 = uv$$

$$x = \sqrt{v/u} \qquad\qquad y = \sqrt{uv}$$

$$= v^{1/2} u^{-1/2} \qquad\qquad = u^{1/2} v^{1/2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v^{1/2}\, u^{-3/2} & \frac{u^{-1/2}}{2\sqrt{v}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix}$$

$$= -\frac{1}{2}\frac{\cancel{\sqrt{v}}}{(\sqrt{u})^3} \times \frac{\sqrt{u}}{\cancel{2\sqrt{v}}} - \frac{1}{2\sqrt{u}\,\cancel{\sqrt{v}}}\,\frac{\cancel{\sqrt{v}}}{2\sqrt{u}}$$

$$= -\frac{1}{2u} - \frac{1}{4u}$$

$$= -\frac{2\cdot 1+1}{4u} \quad = \cancel{\frac{1}{4u}}$$

$$= -\frac{1}{2u}$$

$$\frac{1}{2}\int_1^4 \int_1^3 \frac{u v^2}{u}\, du\, dv$$

$$= \frac{1}{2}\int_1^4 \int_1^3 v^2\, du\, dv$$

$$= \frac{1}{2} \times 2 \int_1^4 v^2\, dv = \frac{1}{3}\left[v^3\right]_1^4$$

$$= \frac{1}{3} \times 63 = \frac{63}{3}$$

$$= 21$$

(35)

$$\iint_R \frac{y-4x}{y+4x}\, dA$$ where R is the region

enclosed by lines

$y = 4x$
$y = 4x+2$
$y = 2-4x$
$y = 5-4x$

$\Rightarrow$

$y - 4x = 0$
$y - 4x = 2$
$y + 4x = 2$
$y + 4x = 5$

$u = y - 4x$
$v = y + 4x$
$u + v = 2y$
$y = \frac{u+v}{2}$

$u - v = -8x$
$x = \frac{v-u}{8}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{8} & \frac{1}{8} \end{vmatrix} = \frac{1}{2}\times\frac{1}{8} + \frac{1}{2}\times\frac{1}{8}$$

$$= \frac{1}{8}$$

$$\frac{1}{8}\int_2^5\int_0^2 \frac{u}{v}\,du\,dv$$

$$= \frac{1}{2\times 8}\times 4\int_2^5 \frac{1}{v}\,dv$$

$$= \frac{1}{4}\left(\ln 5 - \ln 2\right)$$

$$= \frac{\ln 5/2}{4}$$

(36) $\iint_R (x^2-y^2)\,dA$, where R is the rectangular region enclosed by the lines $y = ~~ax~~$ ~~y~~

$y=-x$
$y=1-x$
$y=x$
$y=x+2$

$\Rightarrow$

$y+x=0$
$y+x=1$
$y-x=0$
$y-x=2$

Let
$u = y+x$
$v = y-x$

$u+v = 2y$
$y = \frac{u+v}{2}$

$u-v = 2x$
$x = \frac{u-v}{2}$

$$\frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2}$$

$$-\frac{1}{2}\int_0^2\int_0^1 u\,v\;du\,dv = -\frac{1}{2}\int_0^2 v\,\frac{u^2}{2}\,dv.$$

$$= -\frac{1}{4}\int_0^2 v\,dv = -\frac{1}{4}\times\frac{1}{2}\left[4-0\right]$$

$$= -\frac{1}{8}\times 4 = -\frac{1}{2}$$

(37) $\iint_R \frac{\sin(x-y)}{\cos(x+y)}\, dA$ where R is

the triangular region enclosed by

the lines $y = 0$

$y = x$

$x + y = \pi/4$

$v = u$

$u + v = v - u$

$2u = 0$

$u = 0$

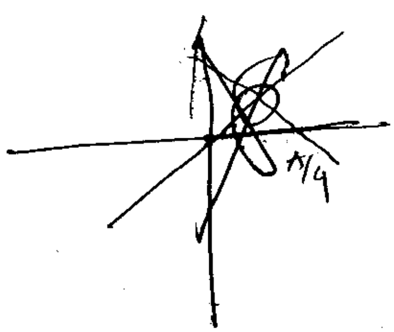
$y = \frac{\pi}{4} - \frac{u+v}{2}$

$= \frac{\pi - 2u - 2v}{4}$

$2v - \cancel{2u} = \pi - \cancel{2u} - 2v$

$4v = \pi$

$v = \pi/4$



$y = x$

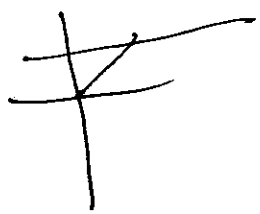
$y = \pi/4 - x$

$x = \pi/4 - x$

$2x = \pi/4$

$x = \pi/8$

let



$u = x - y$

$v = x + y$

$x = \frac{u+v}{2}$ $\qquad$ $y = \frac{v-u}{2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = 1/2$$

$$\frac{1}{2}\int_0^{\pi/4}\int_0^{v} \frac{\sin u}{\cos v}\, du\, dv$$

$$= -\frac{1}{2}\int_0^{\pi/4} \frac{1}{\cos v}\left[\cos u\right]_0^{v} \, dv$$

$$= -\frac{1}{2}\int \frac{1}{\cos v}(\cos v - 1)\, dv$$

$$= -\frac{1}{2}\int_0^{\pi/4} 1 - \frac{1}{\cos v}\, dv$$

$$= -\frac{1}{2}\left[v - \ln|\sec v + \tan v|\right]_0^{\pi/4}$$

$$= -\frac{1}{2}\left[\pi/4 - \ln \sec \pi/4 + \tan \pi/4\right]$$

$$= -1/2\,(\pi/4 - \ln \sec \pi/4 + 1)$$

$$= 1/2\,(\ln(\sqrt{2}+1) - \pi/4)$$

Am.

$$= -\frac{1}{2}\int_0^{\pi/4} \frac{1}{\cos v}\Big[\cos u\Big]_0^{v}\, dv$$

$$= -\frac{1}{2}\int \frac{1}{\cos v}(\cos v - 1)\, dv$$

$$= -\frac{1}{2}\int_0^{\pi/4} 1 - \frac{1}{\cos v}\, dv$$

$$= -\frac{1}{2}\Big[v - \ln|\sec v + \tan v|\Big]_0^{\pi/4}$$

$$= -\frac{1}{2}\Big[\pi/4 - \ln \sec \pi/4 + \tan \pi/4\Big]$$

$$= -1/2\,(\pi/4 - \ln \sec \pi/4 + 1)$$

$$= 1/2\,(\ln(\sqrt{2}+1) - \pi/4)$$

Ans.