Find the nth derivative of the following functions:

1. $y = x^n$ **2.** $y = (a + bx)^n$ **3.** $y = \ln(a + bx)$

4. $y = \dfrac{1}{x + a}$ **5.** $y = e^{ax}$ **6.** $y = \sin(ax + b)$

7. $y = \cos(ax + b)$ **8.** $y = \sin^3 x$ **9.** $y = e^{ax}\sin(bx + c)$

10. If $y = e^{ax}\sin bx$ then show that, $y_2 - 2ay_1 + (a^2 + b^2)y = 0$

11. If $y = e^x \sin x$ then show that, $y_4 + 4y = 0$

Example Find the nth derivative of the following function

$$y = x^n$$

where n is natural number.

Solution

Differentiating once we obtain,

$$\frac{dy}{dx} = nx^{n-1}$$

Similarly differentiating,

$$\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3}$$

$$\frac{d^ry}{dx^r} = n(n-1)(n-2)\cdots\cdots\big(n-(r-1)\big)x^{n-r} = n(n-1)(n-2)\cdots\cdots(n-r+1)x^{n-r}$$

Now considering n is a positive integer, then we can write,

$$\frac{d^ry}{dx^r} = \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)(n-r)!}{(n-r)!}x^{n-r} = \frac{n!}{(n-r)!}x^{n-r}$$

$$\boxed{\frac{d^ry}{dx^r} = \frac{n!}{(n-r)!}x^{n-r}}$$

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Now the nth derivative gives,

$$\frac{d^n y}{dx^n} = \frac{n!}{(n-n)!}x^{n-n} = \frac{n!}{0!}x^0 = \frac{n!}{1}1 = n!$$

Example Find the nth derivative of the following function

$$y = \ln(a + bx).$$

Solution

Differentiating once we obtain,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{b}{(a+bx)} & &= \frac{b}{(a+bx)} & &= \frac{b}{(a+bx)}\\
\frac{d^2y}{dx^2} &= -\frac{b^2}{(a+bx)^2} & &= (-1)\frac{1!\,b^2}{(a+bx)^2} & &= (-1)\frac{1!\,b^2}{(a+bx)^2}\\
\frac{d^3y}{dx^3} &= \frac{2\times b^3}{(a+bx)^3} & &= (-1)^2\frac{1!\times 2\times b^3}{(a+bx)^3} & &= (-1)^2\frac{2!\,b^3}{(a+bx)^3}\\
\frac{d^4y}{dx^4} &= -\frac{2\times 3\times b^4}{(a+bx)^4} & &= (-1)^3\frac{2!\times 3\times b^4}{(a+bx)^4} & &= (-1)^3\frac{3!\,b^4}{(a+bx)^4}
\end{aligned}$$

Similarly, the nth derivative gives,

$$\boxed{\frac{d^n y}{dx^n} = (-1)^{n-1}\frac{n!\,b^n}{(a+bx)^n}.}$$

Example Find the nth derivative of the following function

$$y = \sin(a + bx).$$

Solution

Differentiating we obtain,

$$\begin{aligned}
\frac{dy}{dx} &= b\cos(a+bx) & &= b\sin\left(\frac{\pi}{2} + a + bx\right)\\
\frac{d^2y}{dx^2} &= -b^2\sin(a+bx) & &= b^2\sin\left(2\frac{\pi}{2} + a + bx\right)\\
\frac{d^3y}{dx^3} &= -b^3\cos(a+bx) & &= b^3\sin\left(3\frac{\pi}{2} + a + bx\right)
\end{aligned}$$

$$\frac{d^4y}{dx^4} = b^4 \sin(a + bx) \qquad = b^4 \sin\left(4\frac{\pi}{2} + a + bx\right)$$

Similarly, the nth derivative gives,

$$\boxed{\frac{d^ny}{dx^n} = b^n \sin\left(n\frac{\pi}{2} + a + bx\right).}$$

Problem Show that the nth derivative of the following function $y = \cos(a + bx)$ is

$$\frac{d^ny}{dx^n} = b^n \cos\left(n\frac{\pi}{2} + a + bx\right).$$

Example Find the d^ny/dx^n of the following:

$$y = \sin^2 x.$$

Solution

Using the trigonometry identity,

$$2\sin^2 x = 1 - \cos 2x$$

we obtain

$$y = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x$$

Now differentiating n times, we get,

$$\frac{d^ny}{dx^n} = \frac{d^n}{dx^n}\left(\frac{1}{2}\right) - \frac{1}{2}\frac{d^n}{dx^n}(\cos 2x)$$

$$\frac{d^ny}{dx^n} = -\frac{1}{2}2^n \cos\left(n\frac{\pi}{2} + 2x\right) = -2^{n-1}\cos\left(\frac{n\pi}{2} + 2x\right)$$

$$\boxed{\frac{d^ny}{dx^n} = -2^{n-1}\cos\left(\frac{n\pi}{2} + 2x\right)}$$

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Problem Find the $d^n y/dx^n$ of the following:

$$y = \cos^2 x.$$

Ans.

$$\frac{d^n y}{dx^n} = 2^{n-1}\cos\left(\frac{n\pi}{2} + 2x\right).$$

Problem Find the $d^n y/dx^n$ of the following:

$$y = \sin^3 x.$$

Hint: Using the trigonometric identity

$$\sin 3x = 3\sin x - 4\sin^3 x$$

We can write,

$$y = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

Ans.

$$\frac{d^n y}{dx^n} = \frac{3}{4}\sin\left(\frac{n\pi}{2} + x\right) - \frac{3^n}{4}\sin\left(\frac{n\pi}{2} + 3x\right).$$

Problem Find the $d^n y/dx^n$ of the following:

$$y = e^{ax}\sin(bx).$$

Solution

Differentiating once we obtain,

$$\frac{dy}{dx} = ae^{ax}\sin(bx) + be^{ax}\cos(bx)$$

Let us consider,

$$a = r\cos\theta, \qquad b = r\sin\theta$$

$$r = \sqrt{a^2 + b^2}, \qquad \tan\theta = \frac{b}{a}$$

$$\frac{dy}{dx} = re^{ax}\sin(bx)\cos\theta + re^{ax}\cos(bx)\sin\theta = re^{ax}[\sin(bx)\cos\theta + \cos(bx)\sin\theta]$$

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$$\frac{dy}{dx} = re^{ax}\sin(bx+\theta)$$

Now differentiating once again, we obtain,

$$\frac{d^2y}{dx^2} = are^{ax}\sin(bx+\theta) + bre^{ax}\cos(bx+\theta)$$

$$= r^2e^{ax}\sin(bx+\theta)\cos\theta + r^2e^{ax}\cos(bx+\theta)\sin\theta$$

$$= r^2e^{ax}[\sin(bx+\theta)\cos\theta + \cos(bx+\theta)\sin\theta]$$

$$= r^2e^{ax}\sin(bx+\theta+\theta)$$

$$\frac{d^2y}{dx^2} = r^2e^{ax}\sin(bx+2\theta)$$

In a similar fashion we can obtain,

$$\frac{d^ny}{dx^n} = r^ne^{ax}\sin(bx+n\theta) = \left(\sqrt{a^2+b^2}\right)^n e^{ax}\sin\left(bx+n\tan^{-1}\frac{b}{a}\right)$$

$$\boxed{\frac{d^ny}{dx^n} = \left(\sqrt{a^2+b^2}\right)^n e^{ax}\sin\left(bx+n\tan^{-1}\frac{b}{a}\right).}$$

Problem Find the d^ny/dx^n of the following:

$$y = e^{ax}\cos(bx).$$

Ans.

$$\frac{d^ny}{dx^n} = \left(\sqrt{a^2+b^2}\right)^n e^{ax}\cos\left(bx-n\tan^{-1}\frac{b}{a}\right).$$

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LEIBNIZ'S FORMULA

Rule: If $u(x)$ and $v(x)$ are two n-times differentiable functions of x then

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n} v + \binom{n}{1}\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \binom{n}{2}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2 v}{dx^2} + \cdots\cdots + \binom{n}{n-1}\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + u\frac{d^n v}{dx^n}$$

Or,

$$(uv)^{(n)} = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \binom{n}{2}u^{(n-2)}v^{(2)} + \cdots\cdots + \binom{n}{n-1}u^{(1)}v^{(n-1)} + uv^{(n)}$$

$$\boxed{(uv)^{(n)} = \sum_{i=0}^{n}\binom{n}{i}u^{(n-i)}v^{(i)}}$$

where,

$$u^{(1)} = \frac{du}{dx}, \qquad u^{(2)} = \frac{d^2u}{dx^2}, \cdots\cdots, u^{(n)} = \frac{d^n u}{dx^n}$$

and

$$\binom{n}{r} = \frac{n(n-1)(n-2)(n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!}$$

Example Find the 3rd derivative of $y = x^3 \sin x$ or find d^3y/dx^3.

Solution

Let us consider,

$$u = x^3, \qquad v = \sin x$$

Applying Leibniz's theorem we obtain,

$$\frac{d^3}{dx^3}(x^3 \sin x)$$

$$= \frac{d^3(x^3)}{dx^3}(\sin x) + \binom{3}{1}\frac{d^2(x^3)}{dx^2}\frac{d(\sin x)}{dx} + \binom{3}{2}\frac{d(x^3)}{dx}\frac{d^2(\sin x)}{dx^2} + (x^3)\frac{d^3(\sin x)}{dx^3} \qquad (1)$$

Now,

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$$\frac{d(\sin x)}{dx} = \cos x$$
$$\frac{d^2(\sin x)}{dx^2} = -\sin x$$
$$\frac{d^3(\sin x)}{dx^3} = -\cos x$$

$$\frac{d(x^3)}{dx} = 3x^2$$
$$\frac{d^2(x^3)}{dx^2} = 6x$$
$$\frac{d^3(x^3)}{dx^3} = 6$$

Putting these values into the equation (1), we get,

$$\frac{d^3}{dx^3}(x^3 \sin x) = 6 \sin x + 3 \times 6x \times (\cos x) + 3 \times 3x^2 \times (-\sin x) + x^3(-\cos x)$$

$$\boxed{\therefore \frac{d^3}{dx^3}(x^3 \sin x) = -x^3 \cos x - 9x^2 \sin x + 18x \cos x + 6 \sin x}$$

Problem Find the 4th derivative of $y = x^4 \cos 2x$.

Example Find the 4th derivative of $y = e^{3x} \cos 2x$ or find d^4y/dx^4.

Solution

Let us consider,

$$u = e^{3x}, \qquad v = \cos 2x$$

Applying Leibniz's theorem we obtain,

$$(e^{3x} \cos 2x)^{(4)}$$
$$= (e^{3x})^{(4)} \cos 2x + \binom{4}{1}(e^{3x})^{(4)}(\cos 2x)^{(1)} + \binom{4}{2}(e^{3x})^{(2)}(\cos 2x)^{(2)}$$
$$+ \binom{4}{3}(e^{3x})^{(1)}(\cos 2x)^{(3)} + \binom{4}{4}(e^{3x})(\cos 2x)^{(4)}$$

Now,

$(e^{3x})^{(1)} = 3e^{3x}$	$(e^{3x})^{(2)} = 9e^{3x}$	$(e^{3x})^{(3)} = 27e^{3x}$	$(e^{3x})^{(4)} = 81e^{3x}$
$(\cos 2x)^{(1)}$ $= -2 \sin 2x$	$(\cos 2x)^{(2)}$ $= -4 \cos 2x$	$(\cos 2x)^{(3)}$ $= 8 \sin 2x$	$(\cos 2x)^{(4)}$ $= 16 \cos 2x$

Putting these values into the equation (1), we get,

$(e^{3x}\cos 2x)^{(4)}$

$$= 81e^{3x}\times\cos 2x + 4\times 27e^{3x}\times(-2\sin 2x) + 6\times 9e^{3x}\times(-4\cos 2x) + 4 \times 3e^{3x}\times 8\sin 2x + e^{3x}\times 16\cos 2x$$

$$= 81e^{3x}\cos 2x - 216e^{3x}\sin 2x - 216e^{3x}\cos 2x + 96e^{3x}\sin 2x + 16e^{3x}\cos 2x$$

Problem Find the 4th derivative of $y = e^{-3x}\sin 2x$ or find d^4y/dx^4.

Example If $y = \tan^{-1} x$, then show that,

$$(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0.$$

Solution

Differentiating the given function with respect to x, we get,

$$y_1 = \frac{1}{1+x^2} \Rightarrow y_1(1+x^2) = 1$$

Differentiating once more gives,

$$y_2(1+x^2) + 2xy_1 = 0$$

Now differentiating the above equation n times by using Leibniz's theorem, we obtain,

$$[y_2(1+x^2)]_n + [2xy_1]_n = 0$$

$$\left[y_{n+2}(1+x^2) + \binom{n}{1}y_{n-1+2}(1+x^2)_1 + \binom{n}{2}y_{n-2+2}(1+x^2)_2\right] + 2\left[y_{n+1}x + \binom{n}{1}y_{n-1+1}(x)_1\right] = 0$$

$$(1+x^2)y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}y_n 2 + 2(xy_{n+1} + ny_n) = 0$$

$$(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$$

Example If $y = a\cos(\ln x) + b\sin(\ln x)$ then show that,

$$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$$

Solution

Differentiating the given function with respect to x, we get,

$$y_1 = -a\sin(\ln x)\frac{1}{x} + b\cos(\ln x)\frac{1}{x}$$

$$xy_1 = -a\sin(\ln x) + b\cos(\ln x)$$

Differentiating once more gives,

$$xy_2 + y_1 = -a\cos(\ln x)\frac{1}{x} + b\sin(\ln x)\frac{1}{x}$$

$$x^2y_2 + xy_1 = -a\cos(\ln x) + b\sin(\ln x) = -y$$

$$x^2y_2 + xy_1 + y = 0$$

Now differentiating the above equation n times by using Leibniz's theorem, we obtain,

$$[y_2x^2]_n + [xy_1]_n + [y]_n = 0$$

$$\left[y_{n+2}x^2 + \binom{n}{1}y_{n-1+2}(x^2)_1 + \binom{n}{2}y_{n-2+2}(x^2)_2\right] + \left[y_{n+1}x + \binom{n}{1}y_{n-1+1}(x)_1\right] + y_n = 0$$

$$x^2y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}y_n 2 + xy_{n+1} + ny_n + y_n = 0$$

$$x^2y_{n+2} + (2nx + x)y_{n+1} + (n(n-1) + n + 1)y_n = 0$$

$$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$$

Example If $y\sqrt{1-x^2} = \sin^{-1}x$ then show that,

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0.$$

Solution

Squaring the given expression, we get,

$$y^2(1-x^2) = (\sin^{-1}x)^2$$

Differentiating with respect to x,

$$2yy_1(1-x^2) - 2xy^2 = 2(\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$$

$$2yy_1(1-x^2) - 2xy^2 = 2y$$

$$y_1(1-x^2) - xy = 1 \tag{1}$$

Differentiating once more gives,

$$y_2(1-x^2)-2xy_1-(xy_1+y)=0$$

$$y_2(1-x^2)-3xy_1-y=0$$

Now differentiating the above equation n times by using Leibniz's theorem, we obtain,

$$[y_2(1-x^2)]_n-[3xy_1]_n-[y]_n=0$$

$$\left[y_{n+2}(1-x^2)+\binom{n}{1}y_{n-1+2}(1-x^2)_1+\binom{n}{2}y_{n-2+2}(1-x^2)_2\right]$$
$$-3\left[y_{n+1}x+\binom{n}{1}y_{n-1+1}(x)_1\right]-y_n=0$$

$$(1-x^2)y_{n+2}-2nxy_{n+1}-\frac{n(n-1)}{2}y_n2-3xy_{n+1}-3ny_n-y_n=0$$

$$(1-x^2)y_{n+2}-(2n+3)xy_{n+1}-(n+1)^2y_n=0$$

Or, by substituting $n-1$ instead of n we can also prove that,

$$(1-x^2)y_{n+1}-(2n+1)xy_n-n^2y_{n-1}=0$$

Or, differentiating eq. (1) n times we can show the above.

Example If $y=e^{\tan^{-1}x}$ then show that,

$$(1+x^2)y_{n+2}+(2nx+2x-1)y_{n+1}+n(n+1)y_n=0.$$

Solution

Differentiating the given function with respect to x, we get,

$$y_1=e^{\tan^{-1}x}\frac{1}{(1+x^2)}$$

$$y_1(1+x^2)=e^{\tan^{-1}x}=y$$

Differentiating once more gives,

$$(1+x^2)y_2+2xy_1=y_1$$

$$(1+x^2)y_2+(2x-1)y_1=0$$

Now differentiating the above equation n times by using Leibniz's theorem, we obtain,

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$[y_2(1+x^2)]_n + [(2x-1)y_1]_n = 0$

$$\left[y_{n+2}(1+x^2) + \binom{n}{1} y_{n-1+2}(1+x^2)_1 + \binom{n}{2} y_{n-2+2}(1+x^2)_2\right] + \left[y_{n+1}(2x-1) + \binom{n}{1} y_{n-1+1}(2x-1)_1\right] = 0$$

$$\left[y_{n+2}(1+x^2) + 2nxy_{n+1} + \frac{n(n-1)}{2} y_n 2\right] + [y_{n+1}(2x-1) + 2ny_n] = 0$$

$$(1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + n(n+1)y_n = 0.$$