MAXIMA AND MINIMA

Fermat's theorem

If f has a relative extremum at a, then either $f'(a) = 0$ or $f'(a)$ does not exist.

Example Find the intervals on which the following function is

a) Increasing and Decreasing or Constant
b) Concave up and Concave down

Also find the point of inflection, where the concavity changes.

$$f(x) = 4x^3 - 12x + 2$$

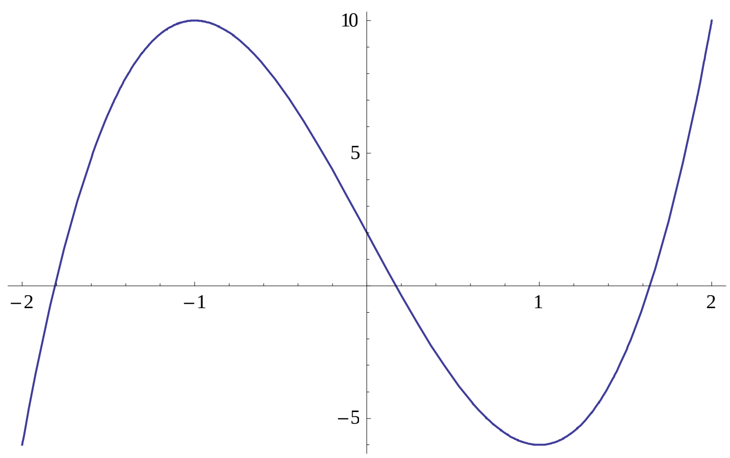
Solution: a)

The first derivative of the given function is,

$$f'(x) = 12x^2 - 12 = 12(x-1)(x+1) \tag{1}$$

So the derivative becomes zero at $x = -1$ and $x = 1$

Interval	$12(x-1)(x+1)$	Sign of $f'(x)$	f
$x < -1$	$(+)(-)(-)$	$+$	Increasing
$-1 < x < 1$	$(+)(-)(+)$	$-$	Decreasing
$x > 1$	$(+)(+)(+)$	$+$	Increasing



b) The second derivative of the function is given by,

$$f''(x) = 24x \tag{2}$$

So the second derivative becomes zero at $x = 0$

Interval	$24x$	Sign of $f''(x)$	f	Conclusion
$x < 0$	$(+)(-)$	$-$	Concave down	f has a point of
$x > 0$	$(+)(+)$	$+$	Concave up	Inflection at $x = 0$

As the second derivative changes sign left and right to the point $x = 0$, so the function has a point of inflection thereat.

Example Use

a) *First derivative test*
b) *Second derivative test*

to find the relative/local extrema (maxima or minima) values of the following function

$$f(x) = 4x^3 - 12x + 2$$

Solution: a)

The first derivative of the given function is,

$$f'(x) = 12x^2 - 12 = 12(x-1)(x+1) \tag{1}$$

The derivative becomes zero at $x = -1$ and $x = 1$ so the critical points of f are,

$$x = -1, \qquad 1$$

Interval	$12(x-1)(x+1)$	Sign of $f'(x)$	f	Conclusion
$x < -1$	$(+)(-)(-)$	$+$	Increasing	Relative maxima at
$-1 < x < 1$	$(+)(-)(+)$	$-$	Decreasing	$x = -1$
$-1 < x < 1$	$(+)(-)(+)$	$-$	Decreasing	Relative minima at
$1 < x$	$(+)(+)(+)$	$+$	Increasing	$x = 1$

b)

The second derivative of the given function is given by,

$$f''(x) = 24x \tag{2}$$

Now, for $x = -1$

$$f''(-1) = -24 < 0$$

Since $f''(-1) < 0$, f has a local maximum value at $x = -1$ and the value is

$$f(-1) = 4(-1)^3 - 12 \times (-1) + 2 = -4 + 12 + 2 = 10$$

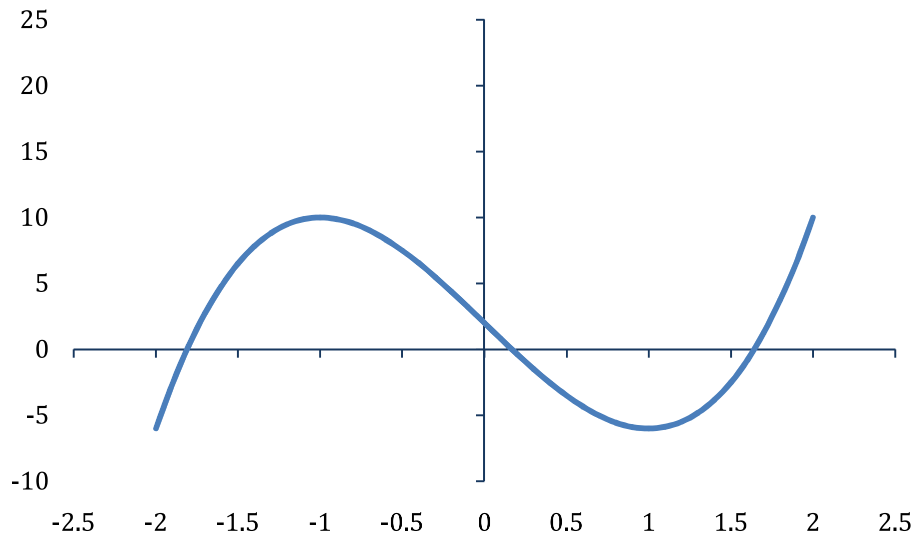
Similarly, for $x = 1$

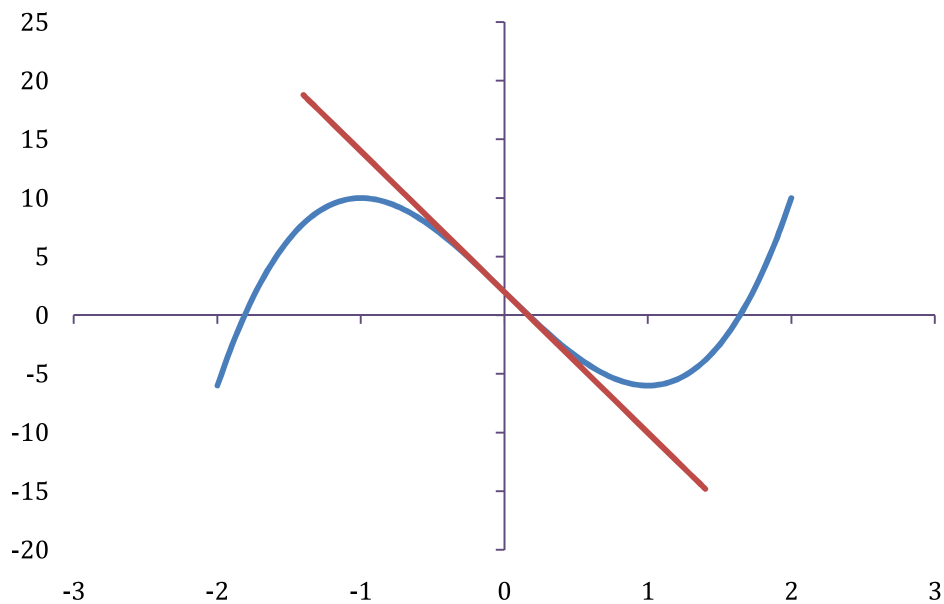
$$f''(1) = 24 > 0$$

Since, $f''(1) > 0$, f has a local minimum value at $x = 1$ and the value is

$$f(1) = 4(1)^3 - 12 \times (1) + 2 = 4 - 12 + 2 = -6$$

Ans. Local maximum value 10 at $x = -1$, local minimum value -6 at $x = 1$.





Problem Use

a) *First derivative test*
b) *Second derivative test*

to find the relative/local extrema (maxima or minima) values of the following function

$$f(x) = 2x^3 + 3x^2 - 36x + 5$$

Example Find the intervals on which the following function is increasing, decreasing, concave up, concave down, relative extrema, and point of inflection (if any).

$$f(x) = 4x^3$$

Solution: The first and the second derivatives of the given function are,

$$f'(x) = 12x^2 \tag{1}$$

$$f''(x) = 24x \tag{2}$$

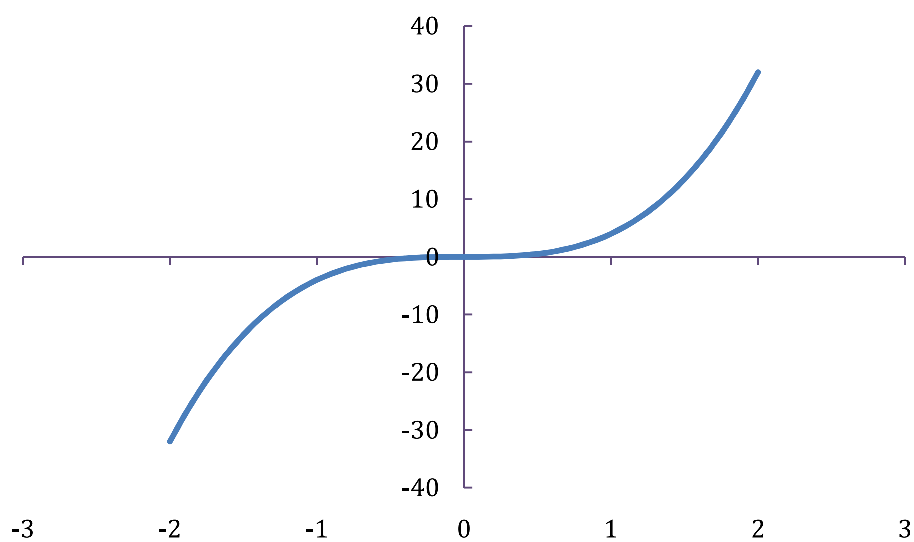
The critical points of f are given by,

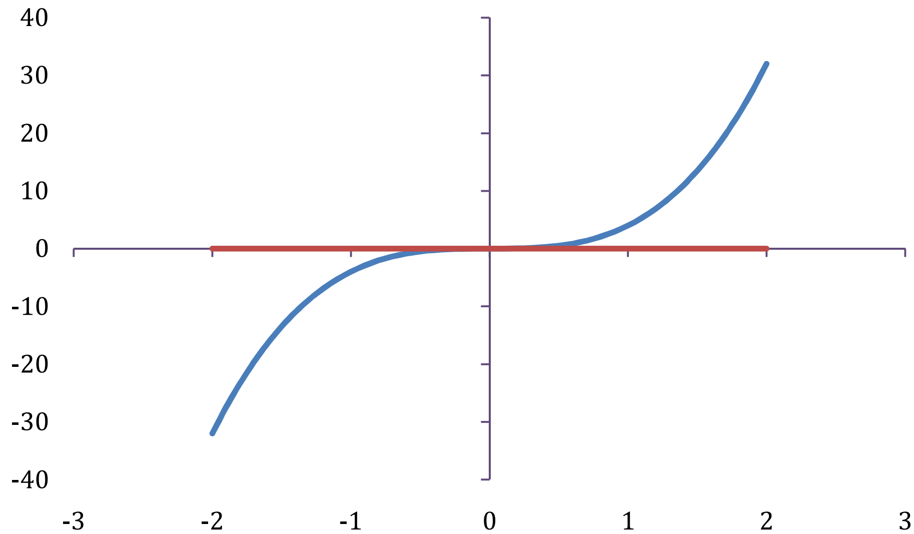
$$f'(x) = 0 \Rightarrow 12x^2 = 0 \Rightarrow x = 0$$

Interval	$12x^2$	Sign of $f'(x)$	f
$x < 0$	$(+)(+)$	$+$	Increasing
$x > 0$	$(+)(+)$	$+$	Increasing

As the first derivative does not change sign left and right to the point $x = 0$, so the function has no relative extrema at the critical point.

Interval	$24x$	Sign of $f''(x)$	f	Conclusion
$x < 0$	$(+)(-)$	$+$	Concave down	f has a point of
$x > 0$	$(+)(+)$	$+$	Concave up	Inflection at $x = 0$





Problem Find the extreme (maximum or minimum) values and the point of inflection (if any) of the function

$$f(x) = x^3 - 2x^2 + x + 1$$

Example Find the intervals on which the following function is increasing, decreasing, concave up, concave down, relative extrema, and point of inflection (if any).

$$f(x) = -5x^4$$

Solution: The first and the second derivatives of the given function are,

$$f'(x) = -20x^3 \quad (1)$$

$$f''(x) = -60x^2 \quad (2)$$

The critical points of f are given by,

$$f'(x) = 0 \Rightarrow -20x^3 = 0 \Rightarrow x = 0$$

Interval	$-20x^3$	Sign of $f'(x)$	f
$x < 0$	$(-)(-)$	$+$	Increasing
$x > 0$	$(-)(+)$	$-$	Increasing

As the first derivative does not change sign left and right to the point $x = 0$, so the function has no relative extrema at the critical point.

Interval	$24x$	Sign of $f''(x)$	f	Conclusion
$x < 0$	$(+)(-)$	$+$	Concave down	f has a point of
$x > 0$	$(+)(+)$	$+$	Concave up	Inflection at $x = 0$

As the second derivative changes sign left and right to the point $x = 0$, so the function has a point of inflection at the critical point.

Example Find the intervals on which the following function is increasing, decreasing, concave up, concave down, relative extrema, and point of inflection (if any).

$$f(x) = 3x^5 - 5x^3 + 8$$

Solution: The first and the second derivatives of the given function are,

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x-1)(x+1) \tag{1}$$

$$f''(x) = 60x^3 - 30x = 30x(2x^2 - 1) = 30x(\sqrt{2}x - 1)(\sqrt{2}x + 1) \tag{2}$$

The critical points of f are given by,

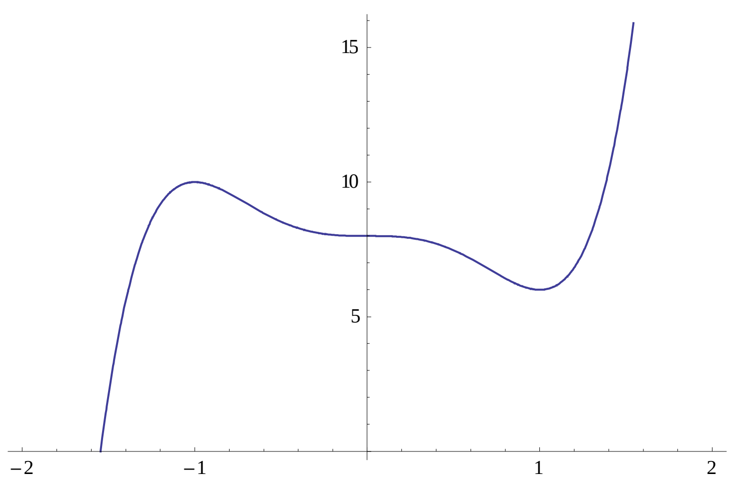
$$f'(x) = 0 \Rightarrow 15x^2(x-1)(x+1) = 0 \Rightarrow x = -1, \quad 0, \quad 1$$

Interval	$15x^2(x-1)(x+1)$	Sign of $f'(x)$	f	Conclusion	Conclusion
$x < -1$	$(+)(+)(-)(-)$	$+$	Increasing	f has a relative	f has a relative
$-1 < x < 0$	$(+)(+)(-)(+)$	$-$	Decreasing	maxima at $x = -1$	maxima at $x = -1$
$-1 < x < 0$	$(+)(+)(-)(+)$	$-$	Decreasing	f has no relative	f has no relative
$0 < x < 1$	$(+)(+)(-)(+)$	$-$	Decreasing	extrema at $x = 0$	extrema at $x = 0$
$0 < x < 1$	$(+)(+)(-)(+)$	$-$	Decreasing	f has a relative	f has a relative
$x > 1$	$(+)(+)(+)(+)$	$+$	Increasing	minima at $x = 1$	minima at $x = 1$

So the second derivative becomes zero at $x = -\sqrt{2}/2$, 0, $\sqrt{2}/2$, and as $x = 0$ is only the critical point it is required to examine the change of sign of $f''(x)$ left and right to $x = 0$.

Interval	$30x(\sqrt{2}x - 1)(\sqrt{2}x + 1)$	Sign of $f''(x)$	f	Conclusion
$-\sqrt{2}/2 < x < 0$	$(+)(-)(+)(+)$	$-$	Concave down	f has a point of
$0 < x < \sqrt{2}/2$	$(+)(+)(+)(+)$	$+$	Concave up	Inflection at $x = 0$

As the second derivative changes sign left and right to the point $x = 0$, so the function has a point of inflection at the critical point.



therefore the point is $(0, 8)$

f has a local minimum value at $x = 1$ and the value is

$$f(1) = 3(1)^5 - 5(1)^3 + 8 = 3 - 5 + 8 = 6$$

For, $x = -\sqrt{2}/2$

$$f\left(-\frac{\sqrt{2}}{2}\right) = 3\left(-\frac{\sqrt{2}}{2}\right)^5 - 5\left(-\frac{\sqrt{2}}{2}\right)^3 + 8$$

$$= -3\left(\frac{4\sqrt{2}}{32}\right) + 5\left(\frac{2\sqrt{2}}{8}\right) + 8 = -3\left(\frac{\sqrt{2}}{8}\right) + 5\frac{\sqrt{2}}{4} + 8 = 8 + 7\frac{\sqrt{2}}{4}$$

Similarly, for $x = \sqrt{2}/2$

$$f\left(\frac{\sqrt{2}}{2}\right) = 8 - 7\frac{\sqrt{2}}{4}$$

Ans. f has local maximum value 10 at $x = -1$, local minimum value 6 at $x = 1$ and a points of inflection $(0, 8)$, $\left(-\sqrt{2}/2\,, 8 + 7\sqrt{2}/4\right)$, $\left(\sqrt{2}/2\,, 8 - 7\sqrt{2}/4\right)$.

Example Find the intervals on which the following function is increasing, decreasing, concave up, concave down, relative extrema, and point of inflection (if any).

$$f(x) = xe^{-x}$$

Solution: The first and the second derivatives of the given function are,

$$f'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x} \tag{1}$$

$$f''(x) = -e^{-x} - (1 - x)e^{-x} = (x - 2)e^{-x} \tag{2}$$

The critical points of f are given by,

$$f'(x) = 0 \Rightarrow (1 - x)e^{-x} = 0 \Rightarrow x = 1$$

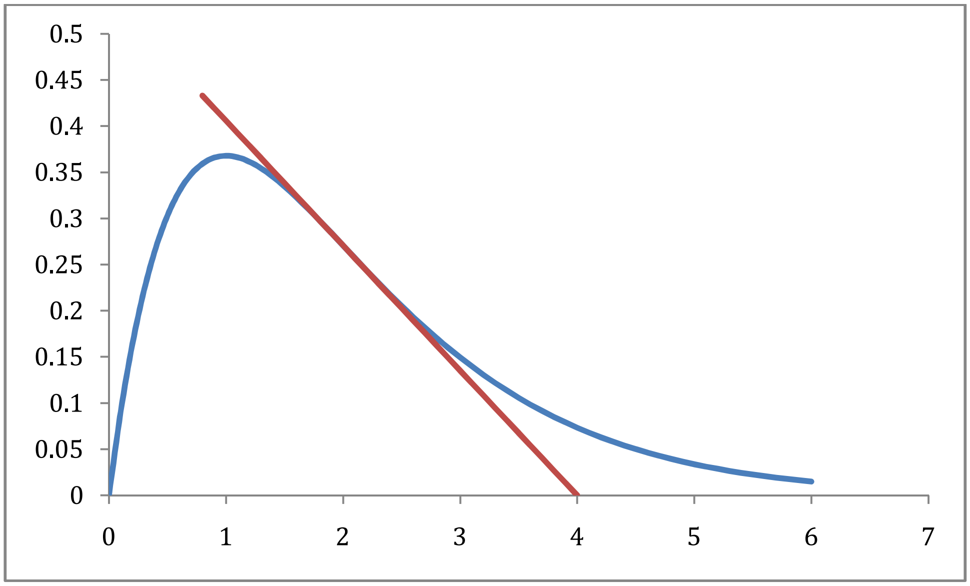
Interval	$(1 - x)e^{-x}$	Sign of $f'(x)$	f	Conclusion	$f(1)$
$x < 1$	$(+)(+)$	$+$	Increasing	f has a relative	e^{-1}
$x > 1$	$(-)(+)$	$-$	Decreasing	maximum at $x = 1$	

$(1,\ e^{-1})$ is the relative maximum point.

So the second derivative becomes zero at $x = 2$ and it is required to examine the change of sign of $f''(x)$ left and right to $x = 2$.

Interval	$(x-2)e^{-x}$	Sign of $f''(x)$	f	Conclusion	$f(2)$
$x<2$	$(-)(+)$	$-$	Concave down	f has a point of	$2e^{-2}$
$x>2$	$(+)(+)$	$+$	Concave up	Inflection at $x=2$	

As the second derivative changes sign left and right to $x=2$, so the function has a point of inflection at the critical number and the point is $(2, 2e^{-2})$.



Ans. f has local maximum value e^{-1} at $x=1$, and a point of inflection $(2, 2e^{-2})$.

Example Using *Second derivative test* to find the maximum, minimum values and the point of inflection of the following function

$$f(x) = x^2e^{-x}$$

Solution: Given that,

$$f(x) = x^2e^{-x} \tag{1}$$

Now differentiating (1) w. r. t. x we get,

$$f'(x) = 2xe^{-x} - x^2e^{-x}$$

$$f'(x) = xe^{-x}(2-x) \tag{2}$$

Let,

$$f'(x) = 0 \Rightarrow xe^{-x}(2-x) = 0 \Rightarrow x(2-x) = 0 \qquad e^{-x} \neq 0$$

Solving the above equation, we get,

$$x = 0, 2$$

Again differentiating (2) w. r. t. x we get,

$$f''(x) = e^{-x}(2-x) - xe^{-x}(2-x) - xe^{-x} = e^{-x}(2-x-2x+x^2-1)$$

$$f''(x) = e^{-x}(1-3x+x^2) \tag{3}$$

Let,

$$f''(x) = 0 \Rightarrow x^2 - 3x + 1 = 0 \qquad [e^{-x} \neq 0]$$

Solving the above equation, we get,

$$x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}, \qquad \frac{3-\sqrt{5}}{2}$$

Now, for $x = 0$

$$f''(0) = 1$$

Since $f''(0) > 0$, f has a local minimum value at $x = 0$ and the value is

$$f(0) = 0$$

Again, for $x = 2$

$$f''(2) = -e^{-2}$$

Since $f''(0) < 0$, f has a local maximum value at $x = 2$ and the value is

$$f(2) = 4e^{-2}$$

Points of Inflection

Substituting $x = (3-\sqrt{5})/2$ into the given function

$$f\left(\frac{3-\sqrt{5}}{2}\right) = \left(\frac{3-\sqrt{5}}{2}\right)^2 \exp\left(-\frac{3-\sqrt{5}}{2}\right) \approx 0.099578$$

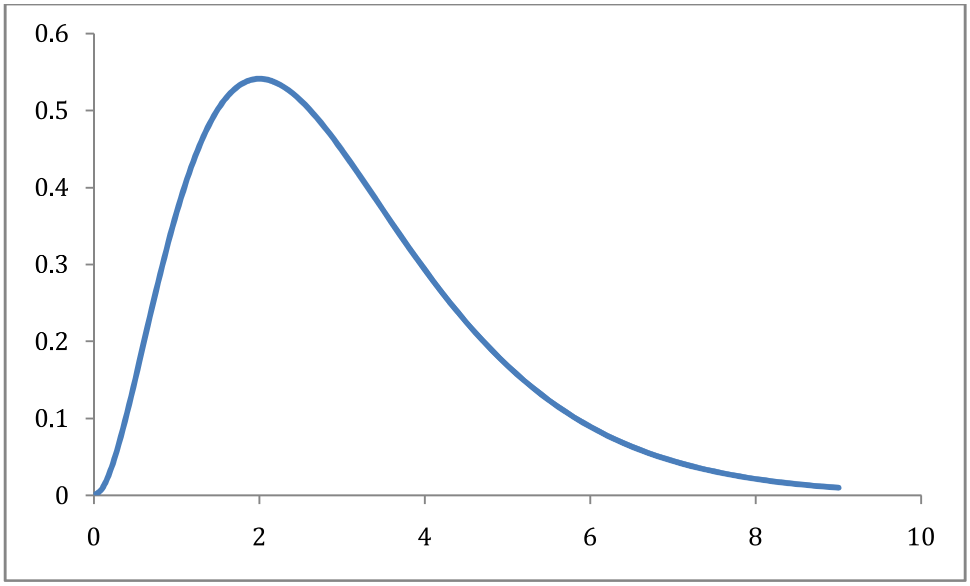
$$= (7-\sqrt{5}) \exp\left(-\frac{3-\sqrt{5}}{2}\right)$$

Similarly, for $x = (3+\sqrt{5})/2$

$$f\left(\frac{3+\sqrt{5}}{2}\right) = \left(\frac{3+\sqrt{5}}{2}\right)^2 \exp\left(-\frac{3+\sqrt{5}}{2}\right) \approx 0.49998$$

$$= (7+\sqrt{5}) \exp\left(-\frac{3+\sqrt{5}}{2}\right)$$

Therefore the point is $(2, 2e^{-2})$



Ans. The function f has local maximum value $4e^{-2}$ at $x = 2$, and a points of inflection

$$\left(\frac{3-\sqrt{5}}{2}, (7-\sqrt{5})\exp\left(-\frac{3-\sqrt{5}}{2}\right)\right), \left(\frac{3+\sqrt{5}}{2}, (7+\sqrt{5})\exp\left(-\frac{3+\sqrt{5}}{2}\right)\right)$$

Problem Find the extreme (maximum or minimum) values and the point of inflection (if any) of the function

1. $\boldsymbol{f(x) = 12x^5 - 45x^4 + 40x^3 + 6}$ **2.** $\boldsymbol{f(x) = x^4 - 2x^3 - 3x^2 + 4x + 4}$ **3.** $\boldsymbol{f(x) = x^3 + 2x^2 - 4x - 8}$
4. $\boldsymbol{f(x) = x^3 - 6x^2 + 9x + 6}$ **5.** $\boldsymbol{f(x) = xe^{-2x},\quad xe^{-\frac{x}{2}},\quad x^2e^{-2x},\quad x^3e^{-\frac{x}{2}}}$ **6.** $\boldsymbol{f(x) = x^3 + \frac{48}{x}}$

Problem Find the minimum distance from the point (4, 2) to the parabola $\boldsymbol{y^2 = 8x}$.

☹ **Air Pollution** The level of ozone, an invisible gas that irritates and impairs breathing, which was present in the atmosphere on a certain day in May in the city of Riverside is approximated by

$$A(t) = 1.0974t^3 - 0.0915t^4 \qquad\qquad 0 \le t \le 11$$

where $A(t)$ is measured in pollutant standard index (PSI) and t is measured in hours, with corresponding to 7 A.M. Use the *Second Derivative Test* to show that the function has a relative maximum at approximately. Interpret your results.