62. The side of a square is measured with a possible percentage error of $\pm 1\%$. Use differentials to estimate the percentage error in the area.

63. The side of a cube is measured with a possible percentage error of $\pm 2\%$. Use differentials to estimate the percentage error in the volume.

64. The volume of a sphere is to be computed from a measured value of its radius. Estimate the maximum permissible percentage error in the measurement if the percentage error in the volume must be kept within $\pm 3\%$. ($V = \frac{4}{3}\pi r^3$ is the volume of a sphere of radius r.)

65. The area of a circle is to be computed from a measured value of its diameter. Estimate the maximum permissible percentage error in the measurement if the percentage error in the area must be kept within $\pm 1\%$.

66. A steel cube with 1-inch sides is coated with 0.01 inch of copper. Use differentials to estimate the volume of copper in the coating. [*Hint:* Let ΔV be the change in the volume of the cube.]

67. A metal rod 15 cm long and 5 cm in diameter is to be covered (except for the ends) with insulation that is 0.1 cm thick. Use differentials to estimate the volume of insulation. [*Hint:* Let ΔV be the change in volume of the rod.]

68. The time required for one complete oscillation of a pendulum is called its ***period***. If L is the length of the pendulum and the oscillation is small, then the period is given by $P = 2\pi\sqrt{L/g}$, where g is the constant acceleration due to gravity. Use differentials to show that the percentage error in P is approximately half the percentage error in L.

69. If the temperature T of a metal rod of length L is changed by an amount ΔT, then the length will change by the amount $\Delta L = \alpha L \Delta T$, where α is called the ***coefficient of linear expansion***. For moderate changes in temperature α is taken as constant.

(a) Suppose that a rod 40 cm long at 20°C is found to be 40.006 cm long when the temperature is raised to 30°C. Find α.

(b) If an aluminum pole is 180 cm long at 15°C, how long is the pole if the temperature is raised to 40°C? [Take $\alpha = 2.3 \times 10^{-5}/°\text{C}$.]

70. If the temperature T of a solid or liquid of volume V is changed by an amount ΔT, then the volume will change by the amount $\Delta V = \beta V \Delta T$, where β is called the ***coefficient of volume expansion***. For moderate changes in temperature β is taken as constant. Suppose that a tank truck loads 4000 gallons of ethyl alcohol at a temperature of 35°C and delivers its load sometime later at a temperature of 15°C. Using $\beta = 7.5 \times 10^{-4}/°\text{C}$ for ethyl alcohol, find the number of gallons delivered.

71. **Writing** Explain why the local linear approximation of a function value is equivalent to the use of a differential to approximate a change in the function.

72. **Writing** The local linear approximation

$$\sin x \approx x$$

is known as the *small angle approximation* and has both practical and theoretical applications. Do some research on some of these applications, and write a short report on the results of your investigations.

✔ QUICK CHECK ANSWERS 3.5

1. tangent; $f(x)$; x_0 **2.** $y = 1 + (-4)(x - 2)$ or $y = -4x + 9$ **3.** $dy = -0.4$, $\Delta y = -0.41$ **4.** within $\pm 1\%$

3.6 L'HÔPITAL'S RULE; INDETERMINATE FORMS

In this section we will discuss a general method for using derivatives to find limits. This method will enable us to establish limits with certainty that earlier in the text we were only able to conjecture using numerical or graphical evidence. The method that we will discuss in this section is an extremely powerful tool that is used internally by many computer programs to calculate limits of various types.

INDETERMINATE FORMS OF TYPE 0/0

Recall that a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)} \tag{1}$$

in which $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ is called an ***indeterminate form of type 0/0***. Some examples encountered earlier in the text are

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2, \quad \lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

The first limit was obtained algebraically by factoring the numerator and canceling the common factor of $x-1$, and the second two limits were obtained using geometric methods. However, there are many indeterminate forms for which neither algebraic nor geometric methods will produce the limit, so we need to develop a more general method.

To motivate such a method, suppose that (1) is an indeterminate form of type $0/0$ in which f' and g' are continuous at $x=a$ and $g'(a)\neq 0$. Since f and g can be closely approximated by their local linear approximations near a, it is reasonable to expect that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)} \tag{2}$$

Since we are assuming that f' and g' are continuous at $x=a$, we have

$$\lim_{x\to a}f'(x)=f'(a) \quad \text{and} \quad \lim_{x\to a}g'(x)=g'(a)$$

and since the differentiability of f and g at $x=a$ implies the continuity of f and g at $x=a$, we have

$$f(a)=\lim_{x\to a}f(x)=0 \quad \text{and} \quad g(a)=\lim_{x\to a}g(x)=0$$

Thus, we can rewrite (2) as

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(a)(x-a)}{g'(a)(x-a)}=\lim_{x\to a}\frac{f'(a)}{g'(a)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{3}$$

This result, called ***L'Hôpital's rule***, converts the given indeterminate form into a limit involving derivatives that is often easier to evaluate.

Although we motivated (3) by assuming that f and g have continuous derivatives at $x=a$ and that $g'(a)\neq 0$, the result is true under less stringent conditions and is also valid for one-sided limits and limits at $+\infty$ and $-\infty$. The proof of the following precise statement of L'Hôpital's rule is omitted.

3.6.1 THEOREM (*L'Hôpital's Rule for Form* $0/0$) *Suppose that f and g are differentiable functions on an open interval containing $x=a$, except possibly at $x=a$, and that*

$$\lim_{x\to a}f(x)=0 \quad \textit{and} \quad \lim_{x\to a}g(x)=0$$

If $\lim_{x\to a}[f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x\to a^-$, $x\to a^+$, $x\to -\infty$, or as $x\to +\infty$.

WARNING

Note that in L'Hôpital's rule the numerator and denominator are differentiated individually. This is *not* the same as differentiating $f(x)/g(x)$.

In the examples that follow we will apply L'Hôpital's rule using the following three-step process:

Applying L'Hôpital's Rule

Step 1. Check that the limit of $f(x)/g(x)$ is an indeterminate form of type $0/0$.

Step 2. Differentiate f and g separately.

Step 3. Find the limit of $f'(x)/g'(x)$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $f(x)/g(x)$.

▶ Example 1 Find the limit

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

using L'Hôpital's rule, and check the result by factoring.

Solution. The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{\dfrac{d}{dx}[x^2-4]}{\dfrac{d}{dx}[x-2]}=\lim_{x\to 2}\frac{2x}{1}=4$$

This agrees with the computation

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4 \blacktriangleleft$$

The limit in Example 1 can be interpreted as the limit form of a certain derivative. Use that derivative to evaluate the limit.

▶ Example 2 In each part confirm that the limit is an indeterminate form of type 0/0, and evaluate it using L'Hôpital's rule.

(a) $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{x}$ (b) $\displaystyle\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}$ (c) $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x^3}$

(d) $\displaystyle\lim_{x\to 0^-}\frac{\tan x}{x^2}$ (e) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (f) $\displaystyle\lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}$

Solution (a). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\frac{\dfrac{d}{dx}[\sin 2x]}{\dfrac{d}{dx}[x]}=\lim_{x\to 0}\frac{2\cos 2x}{1}=2$$

Observe that this result agrees with that obtained by substitution in Example 4(b) of Section 1.6.

WARNING

Applying L'Hôpital's rule to limits that are not indeterminate forms can produce incorrect results. For example, the computation

$$\lim_{x\to 0}\frac{x+6}{x+2}=\lim_{x\to 0}\frac{\dfrac{d}{dx}[x+6]}{\dfrac{d}{dx}[x+2]}=\lim_{x\to 0}\frac{1}{1}=1$$

is ***not valid***, since the limit is not an indeterminate form. The correct result is

$$\lim_{x\to 0}\frac{x+6}{x+2}=\frac{0+6}{0+2}=3$$

Solution (b). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}=\lim_{x\to \pi/2}\frac{\dfrac{d}{dx}[1-\sin x]}{\dfrac{d}{dx}[\cos x]}=\lim_{x\to \pi/2}\frac{-\cos x}{-\sin x}=\frac{0}{-1}=0$$

Guillaume François Antoine de L'Hôpital (1661–1704) French mathematician. L'Hôpital, born to parents of the French high nobility, held the title of Marquis de Sainte-Mesme Comte d'Autrement. He showed mathematical talent quite early and at age 15 solved a difficult problem about cycloids posed by Pascal. As a young man he served briefly as a cavalry officer, but resigned because of nearsightedness. In his own time he gained fame as the author of the first textbook ever published on differential calculus, *L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (1696). L'Hôpital's rule appeared for the first time in that book. Actually, L'Hôpital's rule and most of the material in the calculus text were due to John Bernoulli, who was L'Hôpital's teacher. L'Hôpital dropped his plans for a book on integral calculus when Leibniz informed him that he intended to write such a text. L'Hôpital was apparently generous and personable, and his many contacts with major mathematicians provided the vehicle for disseminating major discoveries in calculus throughout Europe.

[Image: http://en.wikipedia.org/wiki/File:Guillaume_de_l%27H%C3%B4pital.jpg]

Solution (c). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{e^x-1}{x^3}=\lim_{x\to 0}\frac{\frac{d}{dx}[e^x-1]}{\frac{d}{dx}[x^3]}=\lim_{x\to 0}\frac{e^x}{3x^2}=+\infty$$

Solution (d). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^-}\frac{\tan x}{x^2}=\lim_{x\to 0^-}\frac{\sec^2 x}{2x}=-\infty$$

Solution (e). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}$$

Since the new limit is another indeterminate form of type 0/0, we apply L'Hôpital's rule again:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

Solution (f). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}=\lim_{x\to +\infty}\frac{-\frac{4}{3}x^{-7/3}}{(-1/x^2)\cos(1/x)}=\lim_{x\to +\infty}\frac{\frac{4}{3}x^{-1/3}}{\cos(1/x)}=\frac{0}{1}=0 \blacktriangleleft$$

INDETERMINATE FORMS OF TYPE ∞/∞

When we want to indicate that the limit (or a one-sided limit) of a function is $+\infty$ or $-\infty$ without being specific about the sign, we will say that the limit is ∞. For example,

$$\lim_{x\to a^+} f(x)=\infty \quad \text{means} \quad \lim_{x\to a^+} f(x)=+\infty \quad \text{or} \quad \lim_{x\to a^+} f(x)=-\infty$$

$$\lim_{x\to +\infty} f(x)=\infty \quad \text{means} \quad \lim_{x\to +\infty} f(x)=+\infty \quad \text{or} \quad \lim_{x\to +\infty} f(x)=-\infty$$

$$\lim_{x\to a} f(x)=\infty \quad \text{means} \quad \lim_{x\to a^+} f(x)=\pm\infty \quad \text{and} \quad \lim_{x\to a^-} f(x)=\pm\infty$$

The limit of a ratio, $f(x)/g(x)$, in which the numerator has limit ∞ and the denominator has limit ∞ is called an ***indeterminate form of type ∞/∞***. The following version of L'Hôpital's rule, which we state without proof, can often be used to evaluate limits of this type.

3.6.2 THEOREM (*L'Hôpital's Rule for Form ∞/∞*) *Suppose that f and g are differentiable functions on an open interval containing $x=a$, except possibly at $x=a$, and that*

$$\lim_{x\to a} f(x)=\infty \quad \textit{and} \quad \lim_{x\to a} g(x)=\infty$$

If $\lim_{x\to a}[f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x\to a^-$, $x\to a^+$, $x\to -\infty$, or as $x\to +\infty$.

▶ **Example 3** In each part confirm that the limit is an indeterminate form of type ∞/∞ and apply L'Hôpital's rule.

$$\text{(a)}\ \lim_{x\to+\infty}\frac{x}{e^x} \qquad \text{(b)}\ \lim_{x\to 0^+}\frac{\ln x}{\csc x}$$

Solution (a). The numerator and denominator both have a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to+\infty}\frac{x}{e^x} = \lim_{x\to+\infty}\frac{1}{e^x} = 0$$

Solution (b). The numerator has a limit of $-\infty$ and the denominator has a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^+}\frac{\ln x}{\csc x} = \lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x} \tag{4}$$

This last limit is again an indeterminate form of type ∞/∞. Moreover, any additional applications of L'Hôpital's rule will yield powers of $1/x$ in the numerator and expressions involving $\csc x$ and $\cot x$ in the denominator; thus, repeated application of L'Hôpital's rule simply produces new indeterminate forms. We must try something else. The last limit in (4) can be rewritten as

$$\lim_{x\to 0^+}\left(-\frac{\sin x}{x}\tan x\right) = -\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\lim_{x\to 0^+}\tan x = -(1)(0) = 0$$

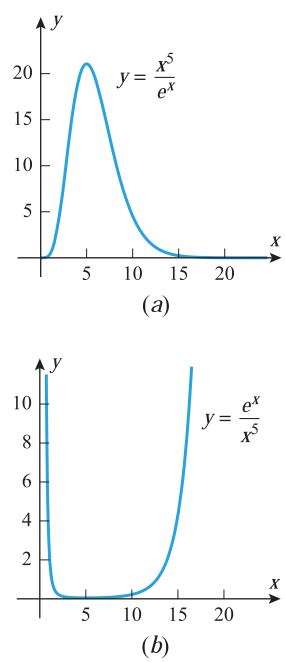
Thus,

$$\lim_{x\to 0^+}\frac{\ln x}{\csc x} = 0 \blacktriangleleft$$

ANALYZING THE GROWTH OF EXPONENTIAL FUNCTIONS USING L'HÔPITAL'S RULE

If n is any positive integer, then $x^n \to +\infty$ as $x\to+\infty$. Such integer powers of x are sometimes used as "measuring sticks" to describe how rapidly other functions grow. For example, we know that $e^x\to+\infty$ as $x\to+\infty$ and that the growth of e^x is very rapid (Table 0.5.5); however, the growth of x^n is also rapid when n is a high power, so it is reasonable to ask whether high powers of x grow more or less rapidly than e^x. One way to investigate this is to examine the behavior of the ratio x^n/e^x as $x\to+\infty$. For example, Figure 3.6.1a shows the graph of $y = x^5/e^x$. This graph suggests that $x^5/e^x\to 0$ as $x\to+\infty$, and this implies that the growth of the function e^x is sufficiently rapid that its values eventually overtake those of x^5 and force the ratio toward zero. Stated informally, "e^x eventually grows more rapidly than x^5." The same conclusion could have been reached by putting e^x on top and examining the behavior of e^x/x^5 as $x\to+\infty$ (Figure 3.6.1b). In this case the values of e^x eventually overtake those of x^5 and force the ratio toward $+\infty$. More generally, we can use L'Hôpital's rule to show that e^x *eventually grows more rapidly than any positive integer power of* x, that is,

$$\boxed{\lim_{x\to+\infty}\frac{x^n}{e^x} = 0} \quad\text{and}\quad \boxed{\lim_{x\to+\infty}\frac{e^x}{x^n} = +\infty} \tag{5–6}$$



▲ **Figure 3.6.1**

Both limits are indeterminate forms of type ∞/∞ that can be evaluated using L'Hôpital's rule. For example, to establish (5), we will need to apply L'Hôpital's rule n times. For this purpose, observe that successive differentiations of x^n reduce the exponent by 1 each time, thus producing a constant for the nth derivative. For example, the successive derivatives of x^3 are $3x^2$, $6x$, and 6. In general, the nth derivative of x^n is $n(n-1)(n-2)\cdots 1 = n!$ (verify).* Thus, applying L'Hôpital's rule n times to (5) yields

$$\lim_{x\to+\infty} \frac{x^n}{e^x} = \lim_{x\to+\infty} \frac{n!}{e^x} = 0$$

Limit (6) can be established similarly.

INDETERMINATE FORMS OF TYPE $0 \cdot \infty$

Thus far we have discussed indeterminate forms of type $0/0$ and ∞/∞. However, these are not the only possibilities; in general, the limit of an expression that has one of the forms

$$\frac{f(x)}{g(x)}, \quad f(x)\cdot g(x), \quad f(x)^{g(x)}, \quad f(x)-g(x), \quad f(x)+g(x)$$

is called an *indeterminate form* if the limits of $f(x)$ and $g(x)$ individually exert conflicting influences on the limit of the entire expression. For example, the limit

$$\lim_{x\to 0^+} x\ln x$$

is an ***indeterminate form of type*** $\mathbf{0\cdot\infty}$ because the limit of the first factor is 0, the limit of the second factor is $-\infty$, and these two limits exert conflicting influences on the product. On the other hand, the limit

$$\lim_{x\to+\infty} [\sqrt{x}(1-x^2)]$$

is not an indeterminate form because the first factor has a limit of $+\infty$, the second factor has a limit of $-\infty$, and these influences work together to produce a limit of $-\infty$ for the product.

Indeterminate forms of type $0\cdot\infty$ can sometimes be evaluated by rewriting the product as a ratio, and then applying L'Hôpital's rule for indeterminate forms of type $0/0$ or ∞/∞.

WARNING

It is tempting to argue that an indeterminate form of type $0\cdot\infty$ has value 0 since "zero times anything is zero." However, this is fallacious since $0\cdot\infty$ is not a product of numbers, but rather a statement about limits. For example, here are two indeterminate forms of type $0\cdot\infty$ whose limits are *not* zero:

$$\lim_{x\to 0}\left(x\cdot\frac{1}{x}\right) = \lim_{x\to 0} 1 = 1$$

$$\lim_{x\to 0^+}\left(\sqrt{x}\cdot\frac{1}{x}\right) = \lim_{x\to 0^+}\left(\frac{1}{\sqrt{x}}\right) = +\infty$$

Example 4 Evaluate

$$\text{(a)}\ \lim_{x\to 0^+} x\ln x \qquad \text{(b)}\ \lim_{x\to\pi/4} (1-\tan x)\sec 2x$$

Solution (a). The factor x has a limit of 0 and the factor $\ln x$ has a limit of $-\infty$, so the stated problem is an indeterminate form of type $0\cdot\infty$. There are two possible approaches: we can rewrite the limit as

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} \quad \text{or} \quad \lim_{x\to 0^+} \frac{x}{1/\ln x}$$

the first being an indeterminate form of type ∞/∞ and the second an indeterminate form of type $0/0$. However, the first form is the preferred initial choice because the derivative of $1/x$ is less complicated than the derivative of $1/\ln x$. That choice yields

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

Solution (b). The stated problem is an indeterminate form of type $0\cdot\infty$. We will convert it to an indeterminate form of type $0/0$:

$$\lim_{x\to\pi/4} (1-\tan x)\sec 2x = \lim_{x\to\pi/4} \frac{1-\tan x}{1/\sec 2x} = \lim_{x\to\pi/4} \frac{1-\tan x}{\cos 2x}$$

$$= \lim_{x\to\pi/4} \frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1 \blacktriangleleft$$

*Recall that for $n \ge 1$ the expression $n!$, read ***n-factorial***, denotes the product of the first n positive integers.

INDETERMINATE FORMS OF TYPE $\infty - \infty$

A limit problem that leads to one of the expressions

$$(+\infty) - (+\infty), \quad (-\infty) - (-\infty),$$
$$(+\infty) + (-\infty), \quad (-\infty) + (+\infty)$$

is called an ***indeterminate form of type*** $\boldsymbol{\infty - \infty}$. Such limits are indeterminate because the two terms exert conflicting influences on the expression: one pushes it in the positive direction and the other pushes it in the negative direction. However, limit problems that lead to one of the expressions

$$(+\infty) + (+\infty), \quad (+\infty) - (-\infty),$$
$$(-\infty) + (-\infty), \quad (-\infty) - (+\infty)$$

are not indeterminate, since the two terms work together (those on the top produce a limit of $+\infty$ and those on the bottom produce a limit of $-\infty$).

Indeterminate forms of type $\infty - \infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $0/0$ or ∞/∞.

▶ **Example 5** Evaluate $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

Solution. Both terms have a limit of $+\infty$, so the stated problem is an indeterminate form of type $\infty - \infty$. Combining the two terms yields

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+}\frac{\sin x - x}{x\sin x}$$

which is an indeterminate form of type $0/0$. Applying L'Hôpital's rule twice yields

$$\begin{aligned}\lim_{x\to 0^+}\frac{\sin x - x}{x\sin x} &= \lim_{x\to 0^+}\frac{\cos x - 1}{\sin x + x\cos x}\\ &= \lim_{x\to 0^+}\frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0\end{aligned}$$ ◀

INDETERMINATE FORMS OF TYPE 0^0, ∞^0, 1^∞

Limits of the form

$$\lim f(x)^{g(x)}$$

can give rise to ***indeterminate forms of the types*** $\boldsymbol{0^0}$, $\boldsymbol{\infty^0}$, ***and*** $\boldsymbol{1^\infty}$. (The interpretations of these symbols should be clear.) For example, the limit

$$\lim_{x\to 0^+}(1+x)^{1/x}$$

whose value we know to be e [see Formula (1) of Section 3.2] is an indeterminate form of type 1^∞. It is indeterminate because the expressions $1+x$ and $1/x$ exert two conflicting influences: the first approaches 1, which drives the expression toward 1, and the second approaches $+\infty$, which drives the expression toward $+\infty$.

Indeterminate forms of types 0^0, ∞^0, and 1^∞ can sometimes be evaluated by first introducing a dependent variable

$$y = f(x)^{g(x)}$$

and then computing the limit of $\ln y$. Since

$$\ln y = \ln[f(x)^{g(x)}] = g(x)\cdot\ln[f(x)]$$

the limit of $\ln y$ will be an indeterminate form of type $0\cdot\infty$ (verify), which can be evaluated by methods we have already studied. Once the limit of $\ln y$ is known, it is a straightforward matter to determine the limit of $y = f(x)^{g(x)}$, as we will illustrate in the next example.

Example 6 Find $\lim_{x \to 0} (1 + \sin x)^{1/x}$.

Solution. As discussed above, we begin by introducing a dependent variable

$$y = (1 + \sin x)^{1/x}$$

and taking the natural logarithm of both sides:

$$\ln y = \ln(1 + \sin x)^{1/x} = \frac{1}{x} \ln(1 + \sin x) = \frac{\ln(1 + \sin x)}{x}$$

Thus,

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1 + \sin x)}{x}$$

which is an indeterminate form of type $0/0$, so by L'Hôpital's rule

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1 + \sin x)}{x} = \lim_{x \to 0} \frac{(\cos x)/(1 + \sin x)}{1} = 1$$

Since we have shown that $\ln y \to 1$ as $x \to 0$, the continuity of the exponential function implies that $e^{\ln y} \to e^1$ as $x \to 0$, and this implies that $y \to e$ as $x \to 0$. Thus,

$$\lim_{x \to 0} (1 + \sin x)^{1/x} = e$$ ◀

✔ QUICK CHECK EXERCISES 3.6 (See page 228 for answers.)

1. In each part, does L'Hôpital's rule apply to the given limit?
   (a) $\lim_{x \to 1} \frac{2x - 2}{x^3 + x - 2}$ (b) $\lim_{x \to 0} \frac{\cos x}{x}$
   (c) $\lim_{x \to 0} \frac{e^{2x} - 1}{\tan x}$
2. Evaluate each of the limits in Quick Check Exercise 1.
3. Using L'Hôpital's rule, $\lim_{x \to +\infty} \frac{e^x}{500x^2} =$ ________.

EXERCISE SET 3.6 Graphing Utility C CAS

1–2 Evaluate the given limit without using L'Hôpital's rule, and then check that your answer is correct using L'Hôpital's rule. ■

1. (a) $\lim_{x \to 2} \frac{x^2 - 4}{x^2 + 2x - 8}$ (b) $\lim_{x \to +\infty} \frac{2x - 5}{3x + 7}$
2. (a) $\lim_{x \to 0} \frac{\sin x}{\tan x}$ (b) $\lim_{x \to 1} \frac{x^2 - 1}{x^3 - 1}$

3–6 True–False Determine whether the statement is true or false. Explain your answer. ■

3. L'Hôpital's rule does not apply to $\lim_{x \to -\infty} \frac{\ln x}{x}$.
4. For any polynomial $p(x)$, $\lim_{x \to +\infty} \frac{p(x)}{e^x} = 0$.
5. If n is chosen sufficiently large, then $\lim_{x \to +\infty} \frac{(\ln x)^n}{x} = +\infty$.
6. $\lim_{x \to 0^+} (\sin x)^{1/x} = 0$

7–45 Find the limits. ■

7. $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$
8. $\lim_{x \to 0} \frac{\sin 2x}{\sin 5x}$
9. $\lim_{\theta \to 0} \frac{\tan \theta}{\theta}$
10. $\lim_{t \to 0} \frac{te^t}{1 - e^t}$
11. $\lim_{x \to \pi^+} \frac{\sin x}{x - \pi}$
12. $\lim_{x \to 0^+} \frac{\sin x}{x^2}$
13. $\lim_{x \to +\infty} \frac{\ln x}{x}$
14. $\lim_{x \to +\infty} \frac{e^{3x}}{x^2}$
15. $\lim_{x \to 0^+} \frac{\cot x}{\ln x}$
16. $\lim_{x \to 0^+} \frac{1 - \ln x}{e^{1/x}}$
17. $\lim_{x \to +\infty} \frac{x^{100}}{e^x}$
18. $\lim_{x \to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)}$
19. $\lim_{x \to 0} \frac{\sin^{-1} 2x}{x}$
20. $\lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3}$
21. $\lim_{x \to +\infty} xe^{-x}$
22. $\lim_{x \to \pi^-} (x - \pi) \tan \frac{1}{2}x$
23. $\lim_{x \to +\infty} x \sin \frac{\pi}{x}$
24. $\lim_{x \to 0^+} \tan x \ln x$
25. $\lim_{x \to \pi/2^-} \sec 3x \cos 5x$
26. $\lim_{x \to \pi} (x - \pi) \cot x$
27. $\lim_{x \to +\infty} (1 - 3/x)^x$
28. $\lim_{x \to 0} (1 + 2x)^{-3/x}$

29. $\lim_{x\to 0}(e^x + x)^{1/x}$

30. $\lim_{x\to +\infty}(1 + a/x)^{bx}$

31. $\lim_{x\to 1}(2 - x)^{\tan[(\pi/2)x]}$

32. $\lim_{x\to +\infty}[\cos(2/x)]^{x^2}$

33. $\lim_{x\to 0}(\csc x - 1/x)$

34. $\lim_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{\cos 3x}{x^2}\right)$

35. $\lim_{x\to +\infty}(\sqrt{x^2 + x} - x)$

36. $\lim_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$

37. $\lim_{x\to +\infty}[x - \ln(x^2 + 1)]$

38. $\lim_{x\to +\infty}[\ln x - \ln(1 + x)]$

39. $\lim_{x\to 0^+} x^{\sin x}$

40. $\lim_{x\to 0^+}(e^{2x} - 1)^x$

41. $\lim_{x\to 0^+}\left[-\dfrac{1}{\ln x}\right]^x$

42. $\lim_{x\to +\infty} x^{1/x}$

43. $\lim_{x\to +\infty}(\ln x)^{1/x}$

44. $\lim_{x\to 0^+}(-\ln x)^x$

45. $\lim_{x\to \pi/2^-}(\tan x)^{(\pi/2)-x}$

46. Show that for any positive integer n

(a) $\lim_{x\to +\infty}\dfrac{\ln x}{x^n} = 0$ (b) $\lim_{x\to +\infty}\dfrac{x^n}{\ln x} = +\infty.$

FOCUS ON CONCEPTS

47. (a) Find the error in the following calculation:

$$\lim_{x\to 1}\frac{x^3 - x^2 + x - 1}{x^3 - x^2} = \lim_{x\to 1}\frac{3x^2 - 2x + 1}{3x^2 - 2x} = \lim_{x\to 1}\frac{6x - 2}{6x - 2} = 1$$

(b) Find the correct limit.

48. (a) Find the error in the following calculation:

$$\lim_{x\to 2}\frac{e^{3x^2-12x+12}}{x^4 - 16} = \lim_{x\to 2}\frac{(6x - 12)e^{3x^2-12x+12}}{4x^3} = 0$$

(b) Find the correct limit.

49–52 Make a conjecture about the limit by graphing the function involved with a graphing utility; then check your conjecture using L'Hôpital's rule.

49. $\lim_{x\to +\infty}\dfrac{\ln(\ln x)}{\sqrt{x}}$

50. $\lim_{x\to 0^+} x^x$

51. $\lim_{x\to 0^+}(\sin x)^{3/\ln x}$

52. $\lim_{x\to (\pi/2)^-}\dfrac{4\tan x}{1 + \sec x}$

53–56 Make a conjecture about the equations of horizontal asymptotes, if any, by graphing the equation with a graphing utility; then check your answer using L'Hôpital's rule.

53. $y = \ln x - e^x$

54. $y = x - \ln(1 + 2e^x)$

55. $y = (\ln x)^{1/x}$

56. $y = \left(\dfrac{x + 1}{x + 2}\right)^x$

57. Limits of the type

$$0/\infty,\quad \infty/0,\quad 0^\infty,\quad \infty\cdot\infty,\quad +\infty + (+\infty),$$
$$+\infty - (-\infty),\quad -\infty + (-\infty),\quad -\infty - (+\infty)$$

are *not* indeterminate forms. Find the following limits by inspection.

(a) $\lim_{x\to 0^+}\dfrac{x}{\ln x}$ (b) $\lim_{x\to +\infty}\dfrac{x^3}{e^{-x}}$

(c) $\lim_{x\to (\pi/2)^-}(\cos x)^{\tan x}$ (d) $\lim_{x\to 0^+}(\ln x)\cot x$

(e) $\lim_{x\to 0^+}\left(\dfrac{1}{x} - \ln x\right)$ (f) $\lim_{x\to -\infty}(x + x^3)$

58. There is a myth that circulates among beginning calculus students which states that all indeterminate forms of types 0^0, ∞^0, and 1^∞ have value 1 because "anything to the zero power is 1" and "1 to any power is 1." The fallacy is that 0^0, ∞^0, and 1^∞ are not powers of numbers, but rather descriptions of limits. The following examples, which were suggested by Prof. Jack Staib of Drexel University, show that such indeterminate forms can have any positive real value:

(a) $\lim_{x\to 0^+}[x^{(\ln a)/(1+\ln x)}] = a$ (form 0^0)

(b) $\lim_{x\to +\infty}[x^{(\ln a)/(1+\ln x)}] = a$ (form ∞^0)

(c) $\lim_{x\to 0}[(x + 1)^{(\ln a)/x}] = a$ (form 1^∞).

Verify these results.

59–62 Verify that L'Hôpital's rule is of no help in finding the limit; then find the limit, if it exists, by some other method.

59. $\lim_{x\to +\infty}\dfrac{x + \sin 2x}{x}$

60. $\lim_{x\to +\infty}\dfrac{2x - \sin x}{3x + \sin x}$

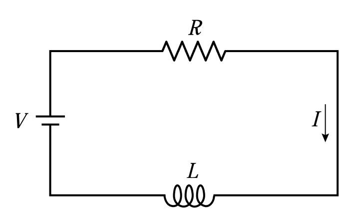
61. $\lim_{x\to +\infty}\dfrac{x(2 + \sin 2x)}{x + 1}$

62. $\lim_{x\to +\infty}\dfrac{x(2 + \sin x)}{x^2 + 1}$

63. The accompanying schematic diagram represents an electrical circuit consisting of an electromotive force that produces a voltage V, a resistor with resistance R, and an inductor with inductance L. It is shown in electrical circuit theory that if the voltage is first applied at time $t = 0$, then the current I flowing through the circuit at time t is given by

$$I = \frac{V}{R}(1 - e^{-Rt/L})$$

What is the effect on the current at a fixed time t if the resistance approaches 0 (i.e., $R\to 0^+$)?



◀ **Figure Ex-63**

64. (a) Show that $\lim_{x\to \pi/2}(\pi/2 - x)\tan x = 1$.

(b) Show that

$$\lim_{x\to \pi/2}\left(\frac{1}{\pi/2 - x} - \tan x\right) = 0$$

(c) It follows from part (b) that the approximation

$$\tan x \approx \frac{1}{\pi/2 - x}$$