BRAC UNIVERSITY

Inspiring Excellence

Department of Mathematics and Natural Sciences
Final Examination
Semester: Fall 2015
Course Title: Linear Algebra and Fourier Analysis
Course No.: MAT216

Time: 3 hours
Total Marks: 50

Date: December 13, 2015

Note: Question 1 is compulsory. Answer any TWO from Part A, any TWO from Part B, and any ONE from Part C.

1. Answer all of the following:
   (a) "A homogeneous linear system with fewer equations than the number of variables is always consistent and has infinitely many solutions."- explain your reasoning. [1]
   (b) What do you mean by nonsingularity of a matrix? Relate the concept of nonsingularity to the solution of a system of linear equations. [1]
   (c) Formulate the volume of a sphere with centre at the origin and radius r using double integral. [1]
   (d) Write the geometrical significance of $\iint\limits_R dA$. [1]
   (e) Sketch the odd extension of the function $f(x) = x^2$, $0 < x < 2$ and find its period. [1]

Part A

2. (a) Define eigenvalue and eigenvector. [2]
   (b) Find the eigenvalues of the matrix: [3]

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix}.$$

   (c) Find the eigenvectors of the matrix A and also find the matrix P, if it exists, that diagonalizes A. [4]

3. (a) State the elementary row operations. [2]

   (b) Find the rank of the following matrix: [3]

$$A = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{pmatrix}.$$

   (c) Write $\begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}$ as a linear combination of vectors in the set S. [4]

$$S = \left\{ \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \right\}.$$

   Are the vectors in S linearly independent?

4. (a) Define basis and dimension of a vector space with example. [2]

   (b) Solve the following system of equations: [3]

$$\begin{aligned} x_1 - 2x_3 + 3x_4 &= 1 \\ x_1 + x_2 - 3x_3 + 4x_4 &= 0 \\ x_1 + 2x_2 - 4x_3 + 5x_4 &= -1. \end{aligned}$$

   (c) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation defined by: [4]

$$T(x_1, x_2, x_3) = (x_1 + 4x_2 + 5x_3, 2x_1 + x_2 + 3x_3, -x_1 + 3x_2 + 2x_3, x_1 + 11x_2 + 12x_3).$$

   Find the standard matrix for this transformation. Also find the basis and dimension for $\ker(T)$.

Part B

5. (a) Write the transformation formulas from three dimensional spherical polar coordinates to Cartesian coordinates. [2]

   (b) Use spherical coordinates to evaluate: [3]

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2 + y^2 + z^2}\, dz dy dx.$$

   (c) Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \leq 1$ by the planes $y = x$ and $x = 0$. Evaluate $\iiint\limits_G z\, dv$. [4]

6. (a) Define gradient of a scalar field, and curl and divergence of a vector field. [2]

   (b) Calculate $\nabla \times \vec{F}$, where $\vec{F} = 2xy \cos z \hat{i} + (x^2 \cos z - 3y^2 z)\hat{j} + (-x^2 y \sin z - y^3)\hat{k}$. [3]

   (c) Use the transformation $u = x - y$ and $v = x + y$ to find [4]

$$\iint\limits_R \frac{e^{x-y}}{x+y}\, dA$$

   over the rectangular region R enclosed by the lines $y = x$, $y = 5 + x$, $y = 2 - x$, and $y = 4 - x$.

7. (a) Express the double integral $\iint_R 4xy^3\, dA$, where R is bounded by the curves $y = x$ and $y = \sqrt{x}$, as iterated integrals and evaluate. [2]

   (b) Use double integral to find the volume of a right circular cylinder of unit height whose base is $x^2 + y^2 = 4$. [3]

   (c) Let $\vec{F}(x, y) = 2xy^3\hat{\imath} + (1 + 3x^2y^2)\hat{\jmath}$ is a force field. Show that $\vec{F}$ is a conservative force field. Also find the potential function ϕ, such that, $\nabla\phi = \vec{F}$. [4]

8. (a) State Green's Theorem. [2]

   (b) Use Green's Theorem to evaluate $\oint_C (x^3 - y)dx + (x + y^3)dy$, where C is the path formed by $y = x^2$ and $y = x$, oriented counterclockwise. [3]

   (c) Evaluate $\displaystyle\int_1^2 \int_z^2 \int_0^{\sqrt{3}y} \frac{y}{x^2 + y^2}\, dxdydz.$ [4]

Part C

9. (a) Define periodic, even, and odd functions with examples. Graph the following triangular-wave function: [4]

$$f(x) = \begin{cases} -x & -\pi \le x < 0 \\ x & 0 \le x \le \pi. \end{cases}$$

   (b) Expand the function in question 9(a) in Fourier series, assuming that the f is periodic outside the interval $[-\pi, \pi]$. [5]

10. (a) Define Fourier series for a periodic function f defined over $(-L, L)$ with period $2L$. Define half-range Fourier sine and cosine series. Show that an odd periodic function can have no cosine terms (or constant term) in its Fourier expansion. [4]

   (b) Expand the following function in a Fourier sine series. [5]

$$f(x) = \begin{cases} x & 0 < x < 3 \\ 6 - x & 3 < x < 6. \end{cases}$$

   Also discuss the convergence of the Fourier series of the function f.