CSE 221: Algorithms

Quicksort

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References

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Why do we want to study Quicksort?

One of the most widely used, and extensively studied, sorting algorithms.

Divide and conquer

Quicksort an *n*-element array:

1. *Divide* Partition the array into subarrays around a *pivot* x such that the elements in lower subarray $\leq x \leq$ elements in the upper subarray.

2. *Conquer* Recursively sort the two subarrays.
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Key

Linear-time partitioning algorithm.

Quicksort in action

1	2	3	4	5	6	7	8
6	10	13	5	8	3	2	11

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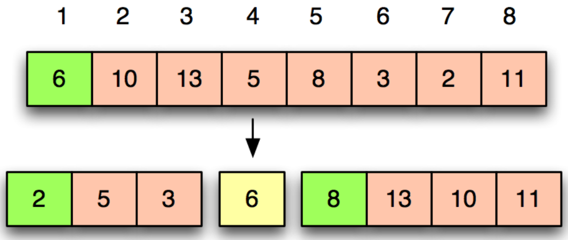

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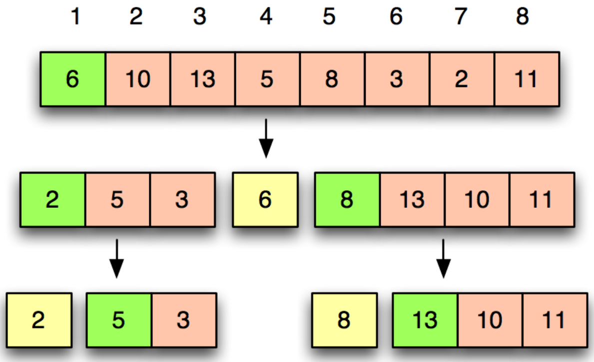

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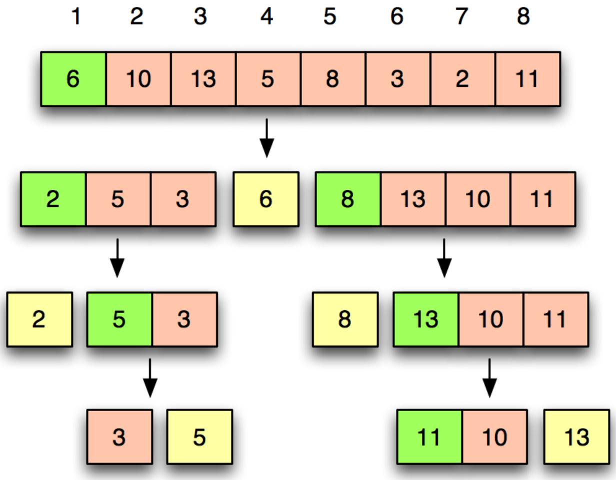

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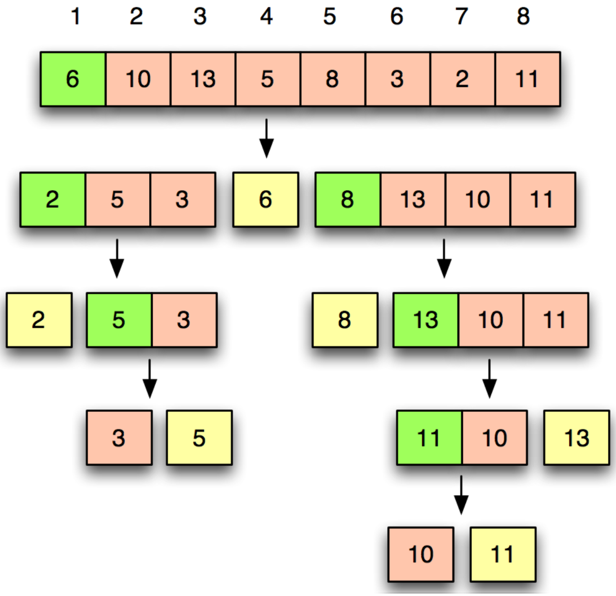

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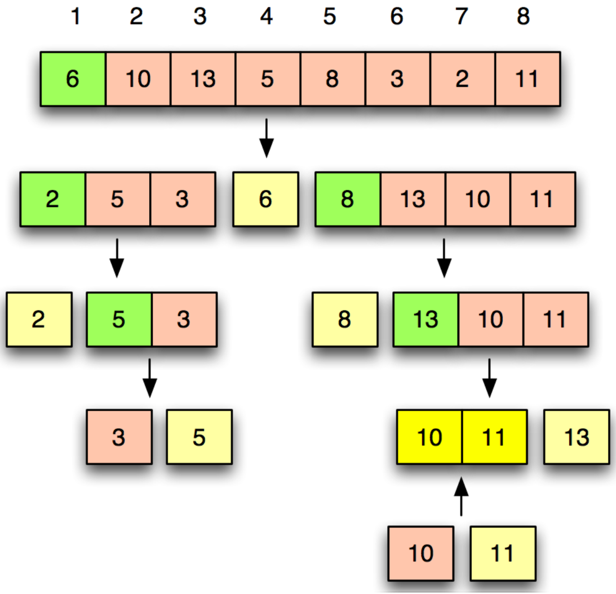

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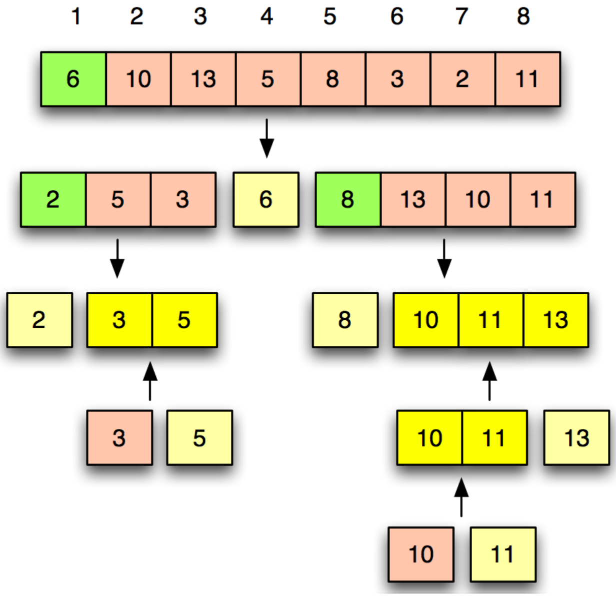

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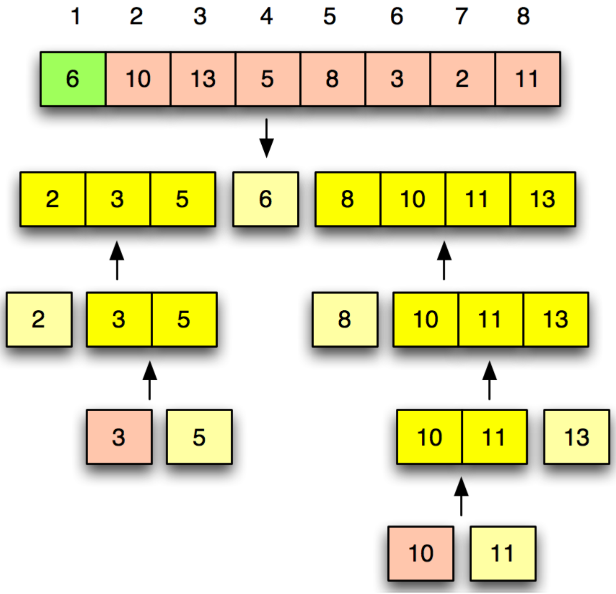

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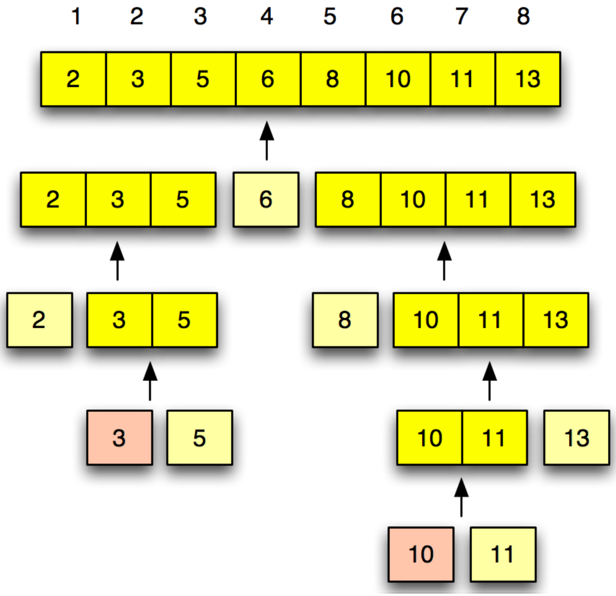

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Partitioning algorithm

Algorithm

$\textsc{Partition}(A, p, q) \rhd A[p \mathinner{..} q]$

$x \leftarrow A[p]$ $\qquad\qquad \rhd$ pivot $= A[p]$

$i \leftarrow p$

for $j \leftarrow p + 1$ **to** q

  do if $A[j] \leq x$

    then $i \leftarrow i + 1$

      exchange $A[i] \leftrightarrow A[j]$

exchange $A[p] \leftrightarrow A[i]$

return i

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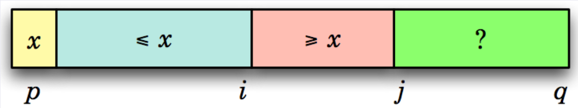
Invariant



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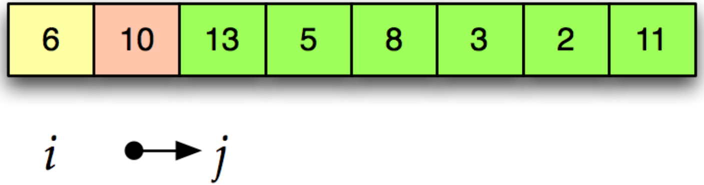

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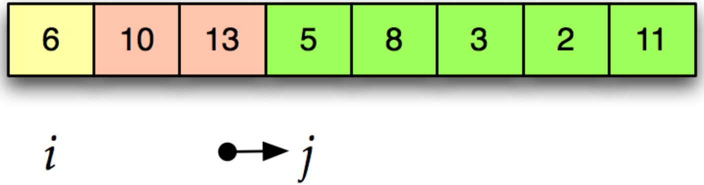

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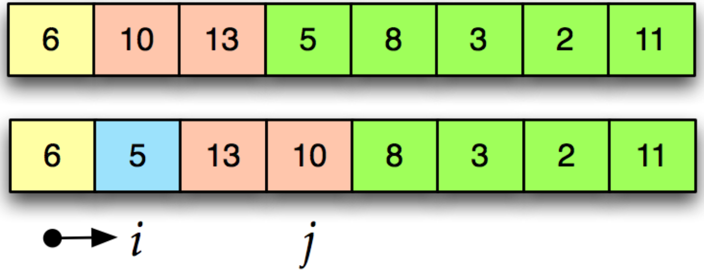

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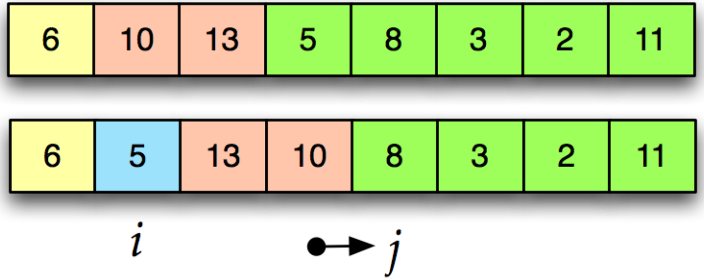

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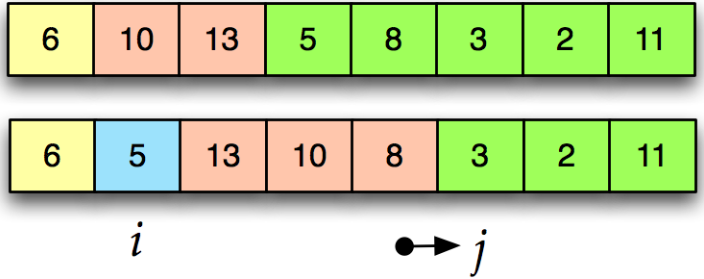

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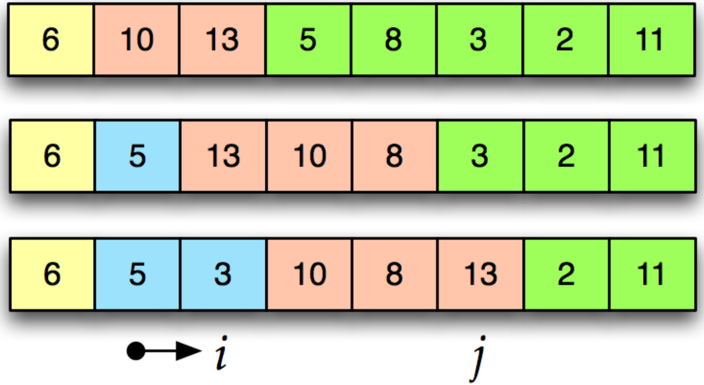

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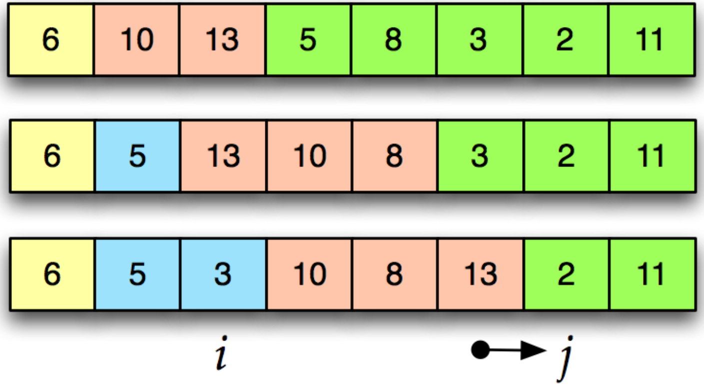

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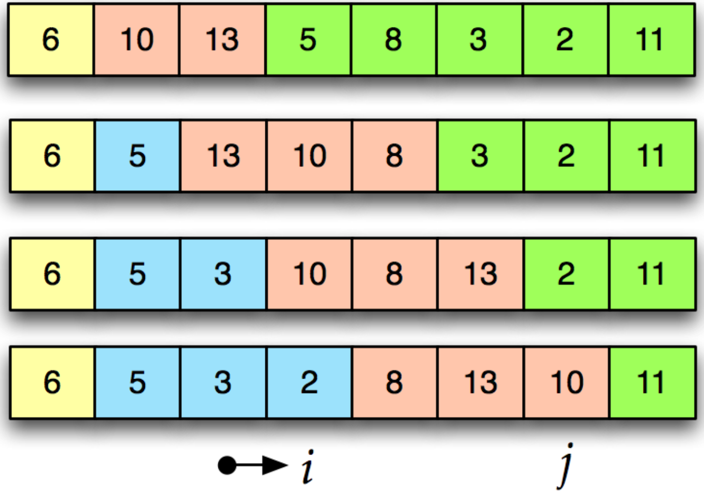

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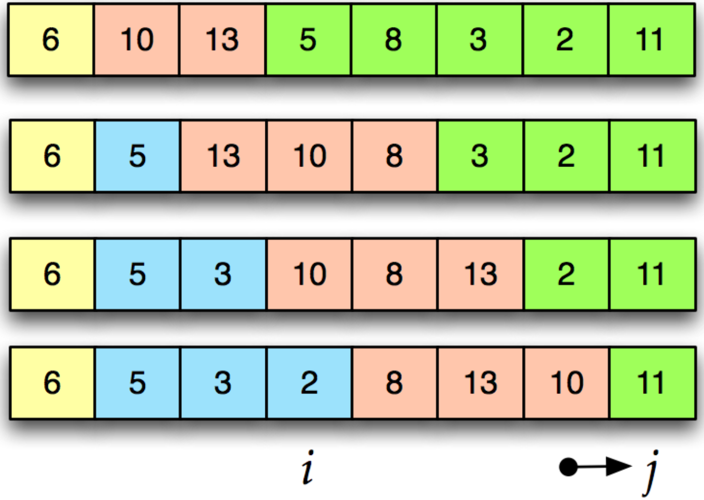

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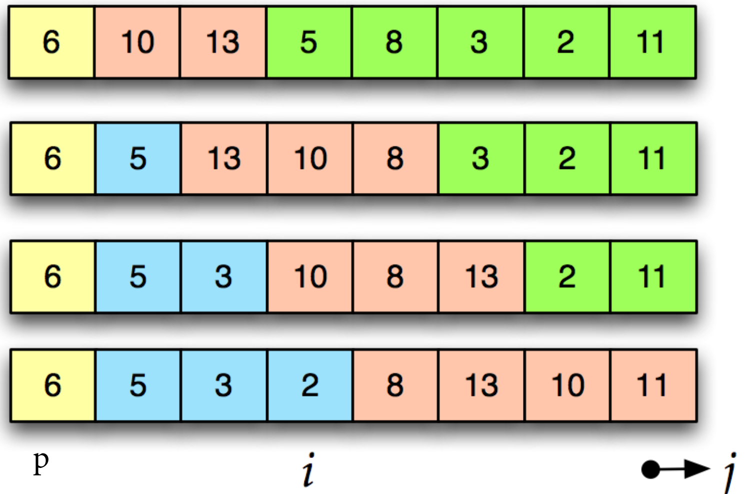

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Quicksort algorithm

Algorithm

QUICKSORT$(A, p, r) \rhd A[p\,..\,r]$

if $p < r$

  then $q \leftarrow$ PARTITION(A, p, r)

    QUICKSORT$(A, p, q - 1)$

    QUICKSORT$(A, q + 1, r)$

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Initial call

$\textsc{Quicksort}(A, 1, n)$

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Worst-case analysis

(Note: the worst-case running time for partitioning is $\Theta(n)$.)

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \boxed{\Theta(n^2)}
\end{aligned}
$$

Worst-case recursion tree

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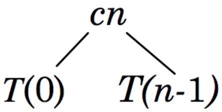

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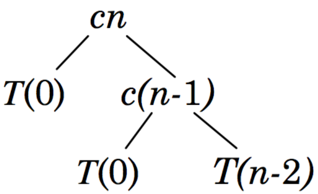

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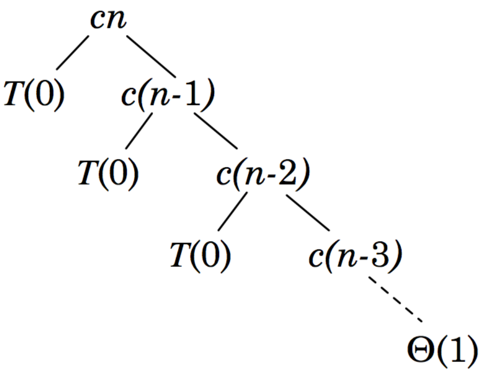

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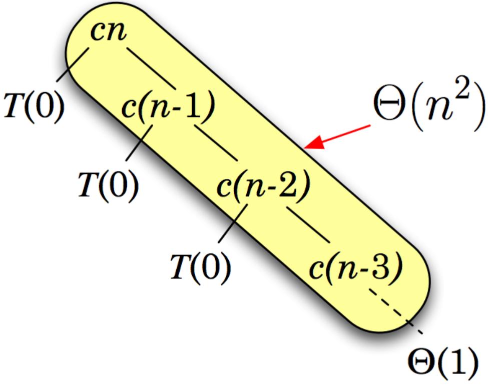

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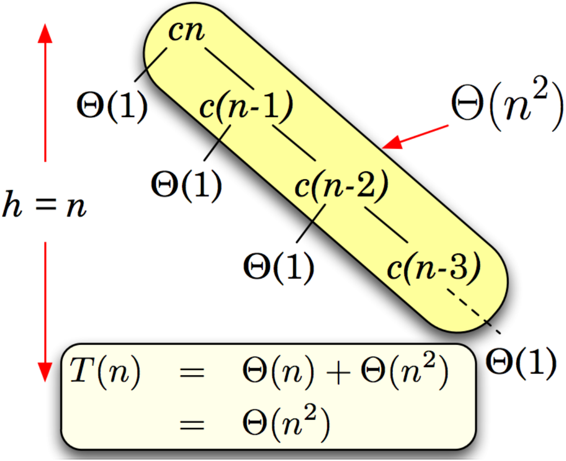

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Key observation

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How to ensure that we don't *usually* hit the worst-case?

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```
i ← RANDOM(p, r)              ▷ i = [p .. r]
exchange A[p] ↔ A[i]
return PARTITION(A, p, r)
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RANDOMIZED-QUICKSORT(A, p, r)

if $p < r$
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    RANDOMIZED-QUICKSORT$(A, p, q - 1)$
    RANDOMIZED-QUICKSORT$(A, q + 1, r)$

Conclusion

- One of the most widely used sorting algorithm.
- While it runs in $O(n^2)$ time in the worst-case, it runs in $O(n \lg n)$ time on the average.
- Runs almost twice as fast as merge-sort.
- Can be *tuned* substantially.
- Almost all program language runtime library provide some variant of Quicksort (java.util.Arrays.sort() in Java, qsort() in C, std::sort() in C++, etc).

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- Is it stable?