CSE 221: Algorithms

Greedy algorithms

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References

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Greedy design strategy

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Basic idea

- At each step of the solution, pick the best choice given the information currently available (i.e., *greedily*).

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- At each step of the solution, pick the best choice given the information currently available (i.e., *greedily*).
- Often leads to very efficient solutions to optimization problems.

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Basic idea

- At each step of the solution, pick the best choice given the information currently available (i.e., *greedily*).
- Often leads to very efficient solutions to optimization problems.
- However, not all problems have greedy solutions.

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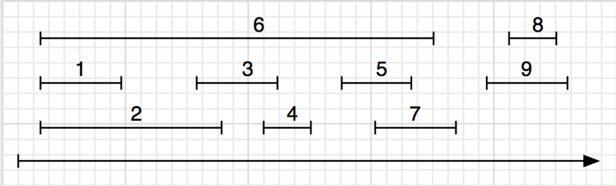
Designing a greedy algorithm

Definition (Interval scheduling problem)

Given a set of schedules $I = \{I_i\}$, find the largest set $A \subseteq I$ such that the members of A are non-conflicting.

Example



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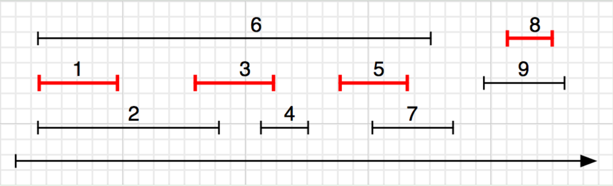

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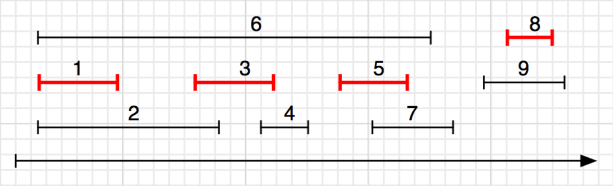

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Question

Is this the only "correct" answer?

Designing a greedy algorithm

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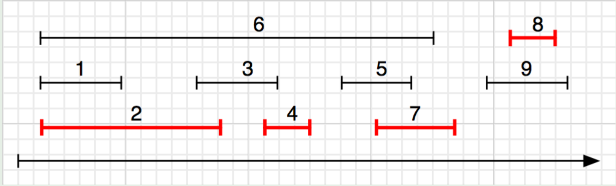

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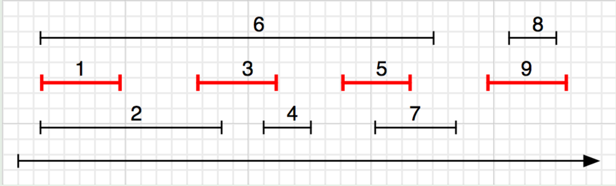

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Question

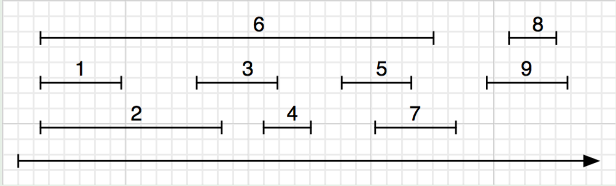
How about $\{2, 4, 7, 8\}$? $\{1, 3, 5, 9\}$?

Designing a greedy algorithm

Definition (Interval scheduling problem)

Given a set of schedules $I = \{I_i\}$, find $A \subseteq I$ such that the members of A are non-conflicting *and* $|A|$ is maximized.

Example



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Question

$\{1, 3, 5, 8\}$? $\{2, 4, 7, 8\}$? $\{1, 3, 5, 9\}$? . . . **How many?**

Brute force solution

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1. Enumerate all possible *configurations* (i.e., all possible subsets of the intervals).
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3. Pick (any one of) the largest subset from the ones that survive.

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Complexity

- There are $2^n - 1$ non-empty subsets, one or more of which may be a feasible solution.

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- There are $2^n - 1$ non-empty subsets, one or more of which may be a feasible solution.
- Each feasible solution must be scanned for conflict, which takes $O(n)$ time.
- The algorithm runs in $\Theta(n2^n)$ time $\Rightarrow$ an exponential time algorithm!

Designing a greedy algorithm (continued)

Basic steps

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1. Use a "simple" rule (or strategy) to select the first interval i_1 to be accepted.
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3. Select the second interval i_2 to be accepted, and remove all the intervals that conflict with i_2.

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4. And so on until there are no more requests remain.

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4. And so on until there are no more requests remain.

Key challenge

How to choose the "simple" rule to select the next interval that leads to an optimal solution?

Designing a greedy algorithm (continued)

Strategy 1. *Earliest First*

The idea is to start using the resource as early as possible.

1. Sort the intervals by starting time, breaking ties arbitrarily.
2. Pick the first one, removing it from the list along with all the intervals that conflict with it.
3. Repeat Step 2, until the list is empty.

Example

$|A| = ???.$

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Example (using *Earliest First* strategy)

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Designing a greedy algorithm (continued)

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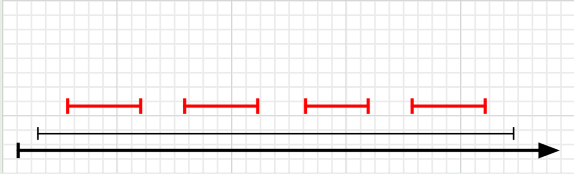
Example (using an optimal strategy)



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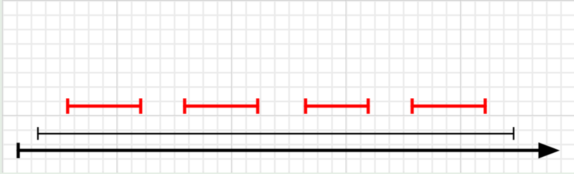
This strategy does not lead to an optimal solution.

Example (using an optimal strategy)



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Designing a greedy algorithm (continued)

Strategy 2. *Shortest First*

The *Earliest First* strategy failed perhaps because it missed the shorter intervals, which would accommodate more intervals.

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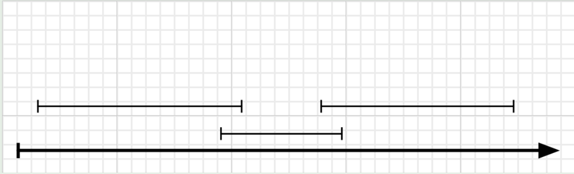

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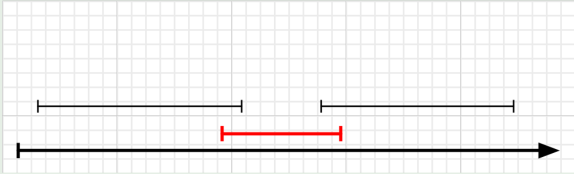

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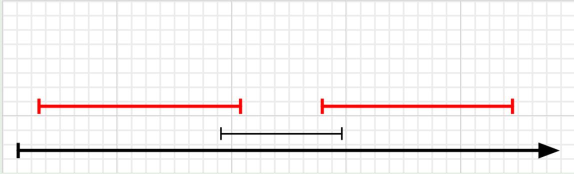

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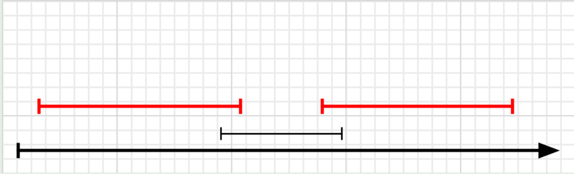

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Designing a greedy algorithm (continued)

Strategy 3. *Least-conflict First*

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1. Sort the intervals by the number of other intervals which conflict with it.
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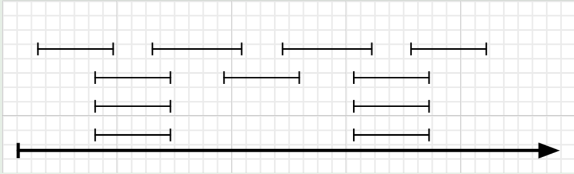

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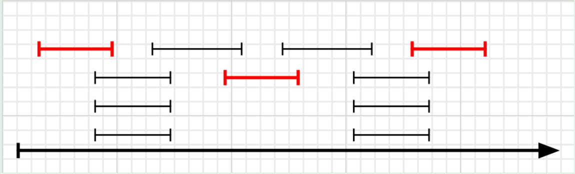

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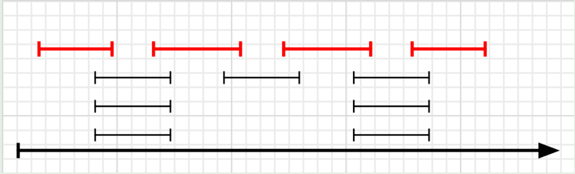

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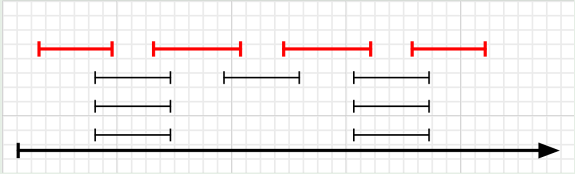

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Strategy 4. *Finish First*

The idea is to free up the resource as early as possible.

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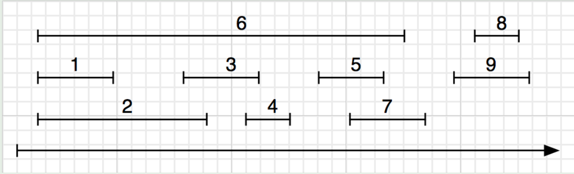

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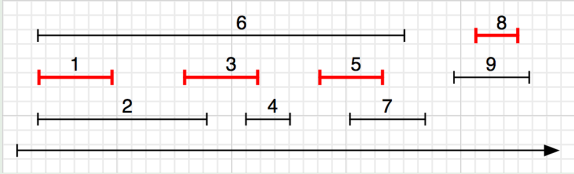

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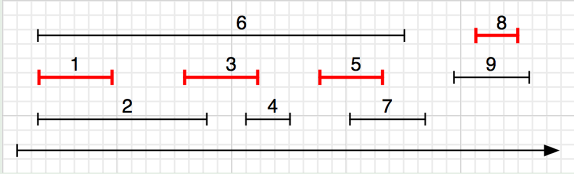

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This strategy is the one that works. But can you prove that it works?

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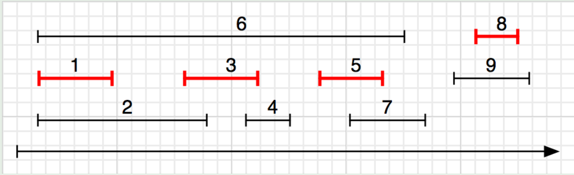

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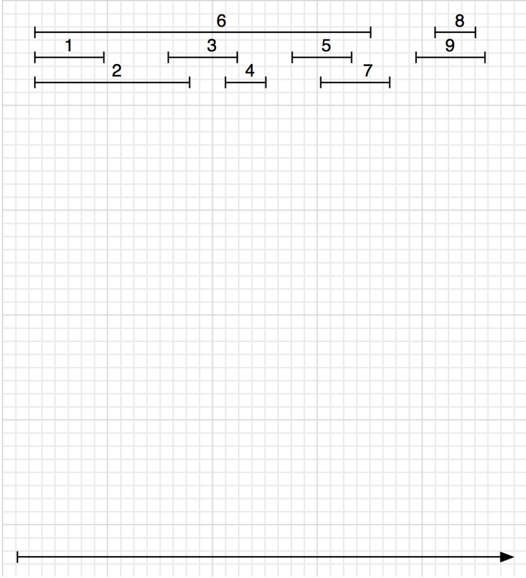
Interval scheduling in action



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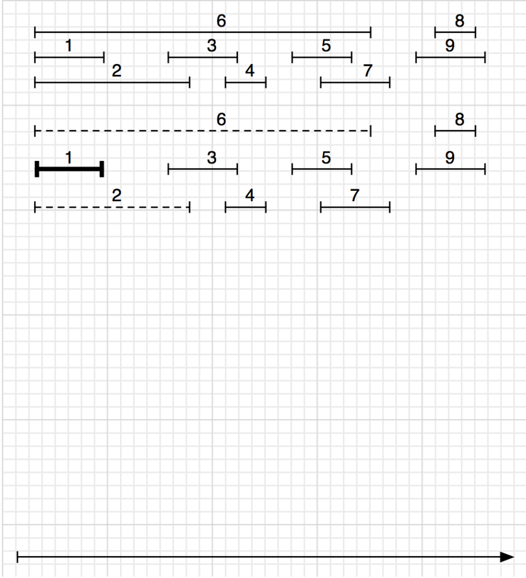

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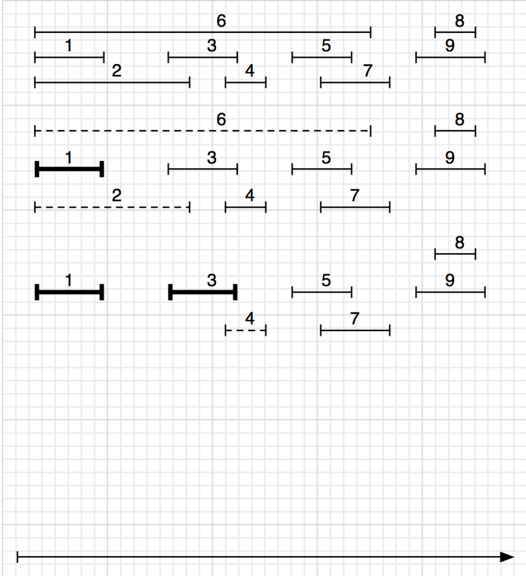

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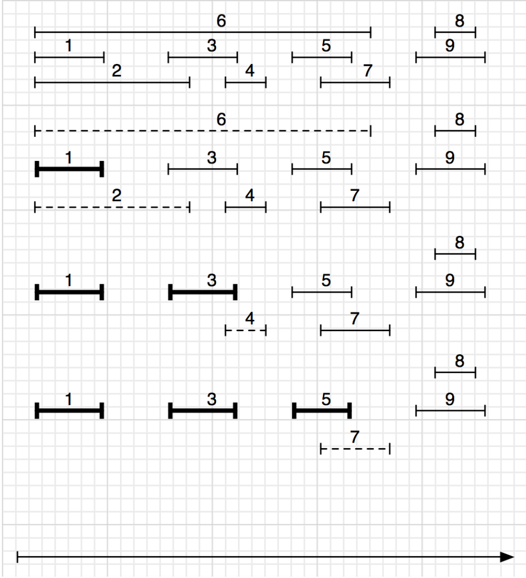

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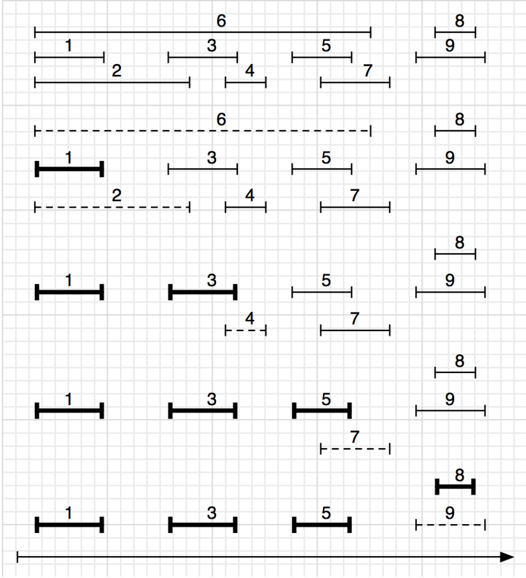

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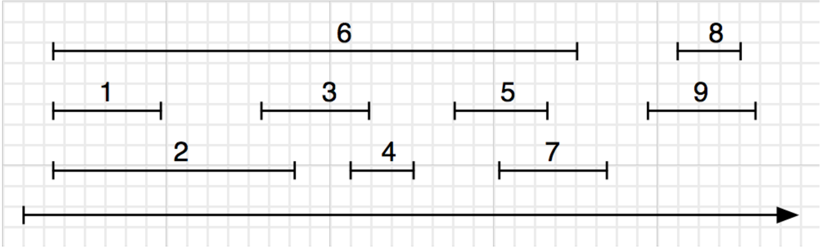

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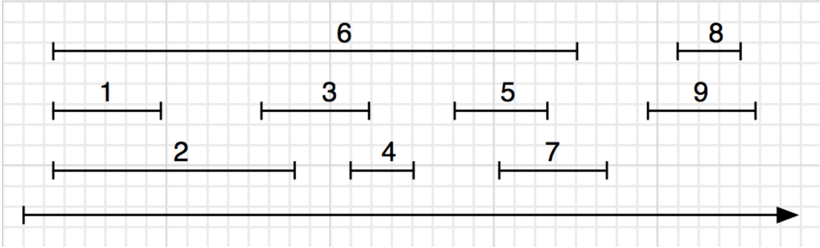
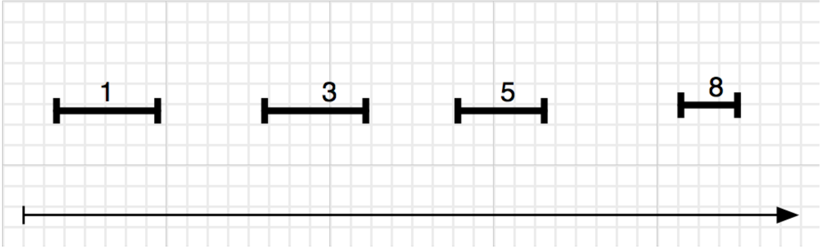


$\Downarrow$



An $O(n \lg n)$ greedy algorithm for interval scheduling

SCHEDULE-INTERVALS(I) $\quad \triangleright\ I = \{I_i\}, I_i = (s_i, f_i)$

$R =$ Sorted requests in order of finishing times such that $f_i \leq f_j$ when $i < j$.
Create an array $S[1 \,..\, n]$ with starting times such that $S[i]$ contains s_i.
$A = \{R_1\}$ $\quad \triangleright$ select first interval from sorted list
$f = f_1$
while there are more intervals in S to look at
    do $j =$ first interval for which $s_j \geq f$
        $A \leftarrow A \cup \{j\}$
        $f \leftarrow f_j$
return A

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- The sorting step in takes $O(n \lg n)$ time.

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- The sorting step in takes $O(n \lg n)$ time.
- Creating the starting time array $S[1..n]$ takes $O(n)$ time.
- The single pass through the array S takes $O(n)$ time

An $O(n \lg n)$ greedy algorithm for interval scheduling

SCHEDULE-INTERVALS(I) $\triangleright$ $I = \{I_i\}, I_i = (s_i, f_i)$

R = Sorted requests in order of finishing times such that $f_i \leq f_j$ when $i < j$.
Create an array $S[1\,.\,.\,n]$ with starting times such that $S[i]$ contains s_i.
$A = \{R_1\}$ $\triangleright$ select first interval from sorted list
$f = f_1$
while there are more intervals in S to look at
    do j = first interval for which $s_j \geq f$
        $A \leftarrow A \cup \{j\}$
        $f \leftarrow f_j$
return A

Analysis

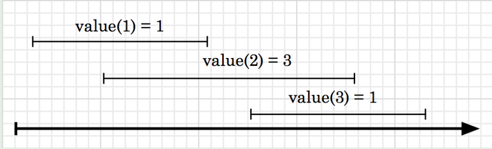
- The sorting step in takes $O(n \lg n)$ time.
- Creating the starting time array $S[1\,.\,.\,n]$ takes $O(n)$ time.
- The single pass through the array S takes $O(n)$ time
- An $\boxed{O(n \lg n)}$ time algorithm for a problem with a natural search space of $O(n2^n)$.

Extension: weighted interval scheduling problem

Definition (Weighted interval scheduling problem)

Given a set of schedules $I = \{I_i\}$, with associated weights $W = \{w_i\}$, find $A \subseteq I$ such that the members of A are non-conflicting *and* the total weight $\sum_{i \in A} w_i$ is maximized.

Example



$|A| =???$, $\sum_{i \in A} w_i =???$.

Extension: weighted interval scheduling problem

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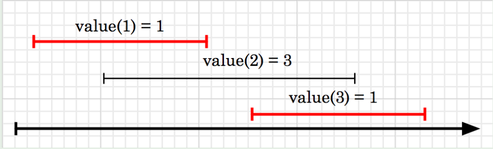
Example (using our greedy strategy)



$|A| = 2$, $\sum_{i \in A} w_i = 2$.

Extension: weighted interval scheduling problem

Definition (Weighted interval scheduling problem)

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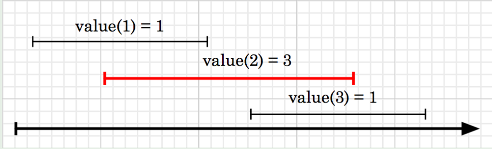
Example (using an optimal strategy)



$$|A| = 1, \sum_{i \in A} w_i = 3.$$

Extension: weighted interval scheduling problem

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Given a set of schedules $I = \{I_i\}$, with associated weights $W = \{w_i\}$, find $A \subseteq I$ such that the members of A are non-conflicting *and* the total weight $\sum_{i \in A} w_i$ is maximized.

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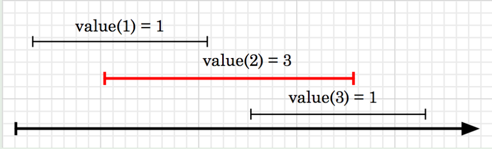

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Hmmm...

There is no greedy solution for the weighted interval scheduling problem!

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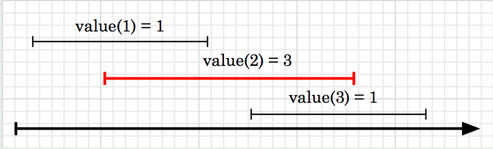

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Hmmm...

There is no greedy solution for the weighted interval scheduling problem! Why? (see Greedy Choice property later)

Contents

Scheduling all intervals greedy algorithm

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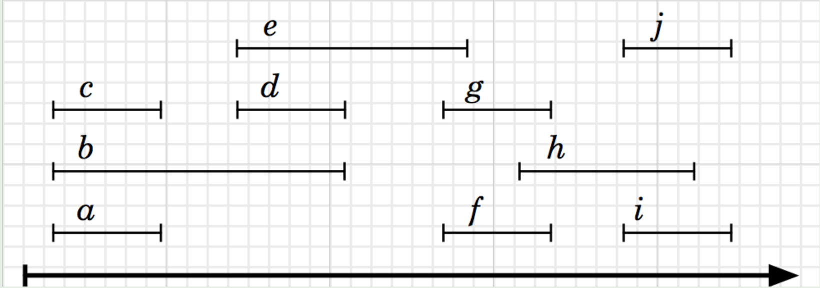
Given a set of schedules $I = \{I_i\}$, find the minimum number of resources needed to schedule I such that the intervals on each resource are non-conflicting.

Example



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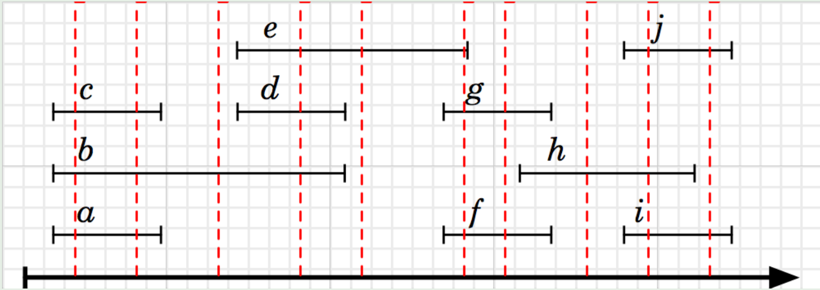

Depth = Maximum number of intervals at any point in time.

Scheduling all intervals greedy algorithm

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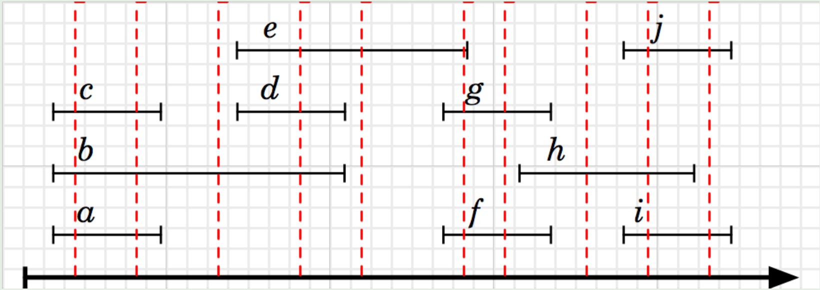
Given a set of schedules $I = \{I_i\}$, find the minimum number of resources needed to schedule I such that the intervals on each resource are non-conflicting.

Example



Depth $= 3$

Scheduling all intervals greedy algorithm

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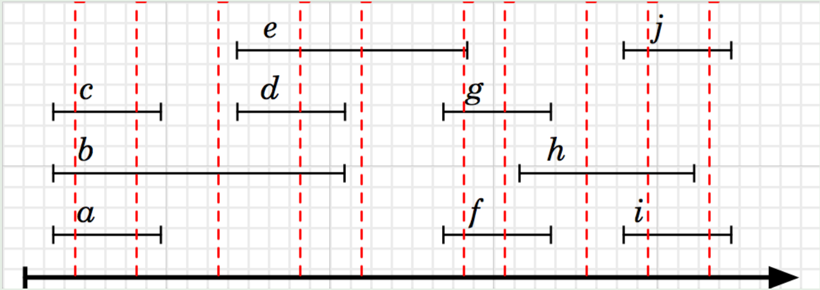
Given a set of schedules $I = \{I_i\}$, find the minimum number of resources needed to schedule I such that the intervals on each resource are non-conflicting.

Example



Depth $= 3 \implies$ Minimum # of resources needed $= 3$

A greedy algorithm for scheduling all intervals

SCHEDULE-INTERVALS(I) $\quad \triangleright\ I = \{I_i\}, I_i = (s_i, f_i)$

R = Sorted requests in order of starting times, breaking ties arbitrarily, such that $s_i \leq s_j$ when $i < j$.

$m \leftarrow 0$ $\triangleright$ the optimal number of resources needed to schedule R

while $R \neq \emptyset$

  do req = extract the next element in R

    if there is a resource j with no interval conflicting with req

      then schedule interval req on resource j

      else

        $m \leftarrow m + 1$ $\quad \triangleright$ allocate a new resource

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Complexity

$T(n) = O(n \lg n)$.

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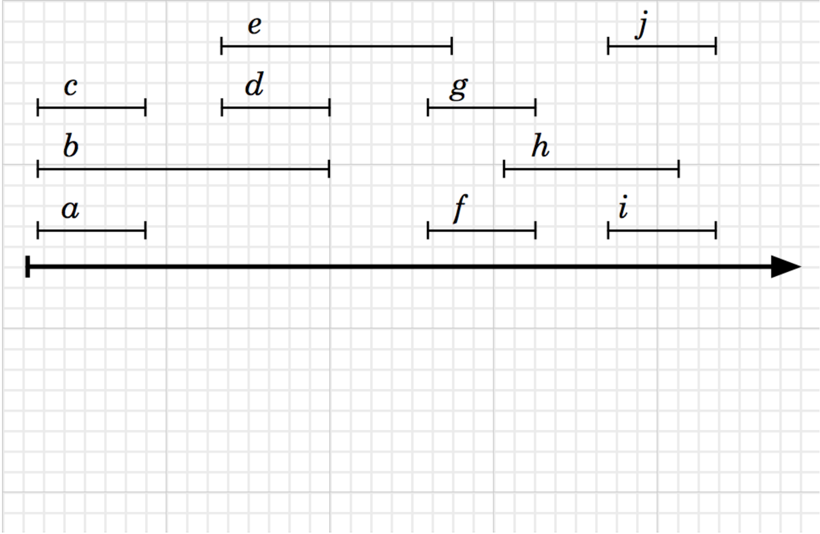
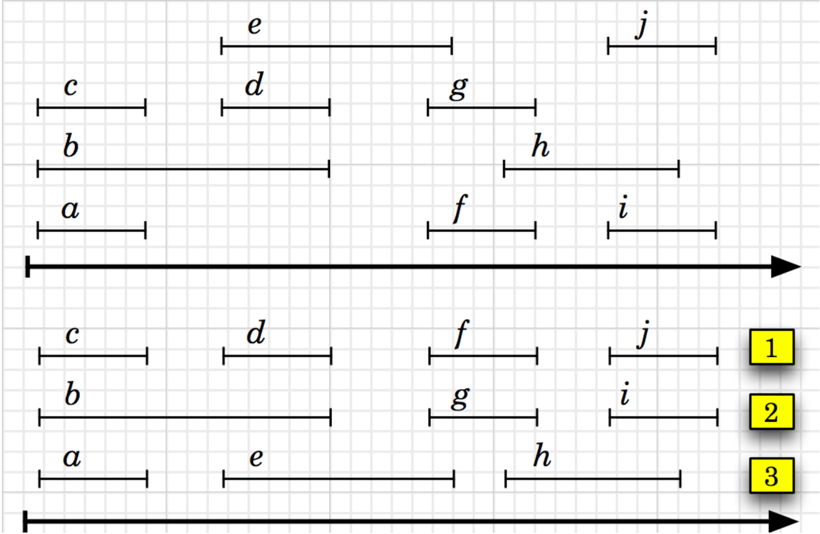
Complexity

$T(n) = O(n \lg n)$. Prove it.

Scheduling all intervals in action



Scheduling all intervals in action



Contents

Fractional knapsack problem

Definition (fractional knapsack problem)

Given a set S of n items, such that each item i has a positive benefit b_i and a positive weight w_i, the goal is to find the maximum-benefit subset that does not exceed a given weight W, allowing for fractional items.

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Key question

- What strategy to use to select the next item (and the amount of it)?

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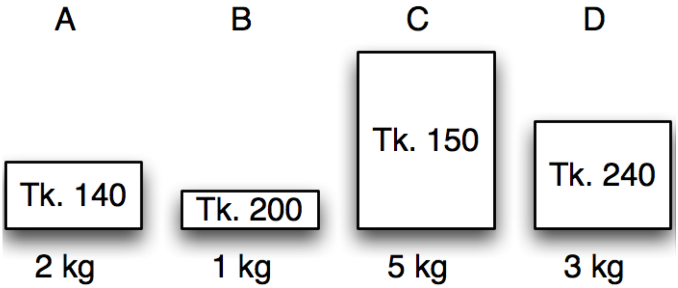
- Maximum-benefit subset is then maximizing $\sum_{i \in S} b_i(x_i/w_i)$.

Key question

- What strategy to use to select the next item (and the amount of it)?
- Since we're maximizing the benefit, select the next item with the highest benefit per weight – b_i/w_i.

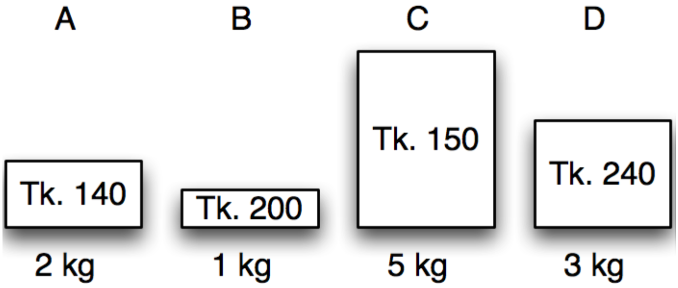
Fractional knapsack in action



Item	Price	Weight
A	140	2 kg
B	200	1 kg
C	150	5 kg
D	240	3 kg

Calculate price/kg – the value index.

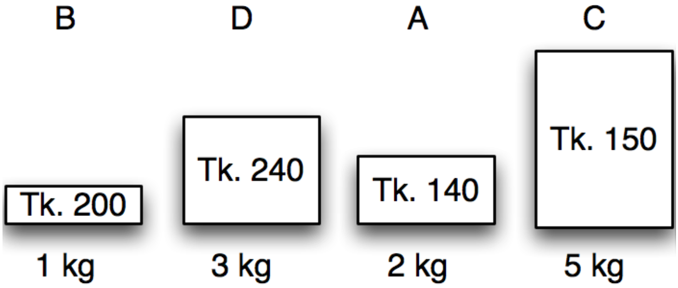
Fractional knapsack in action



Item	Price	Weight	Value index
A	140	2 kg	70
B	200	1 kg	200
C	150	5 kg	30
D	240	3 kg	80

Sort by non-increasing value index.

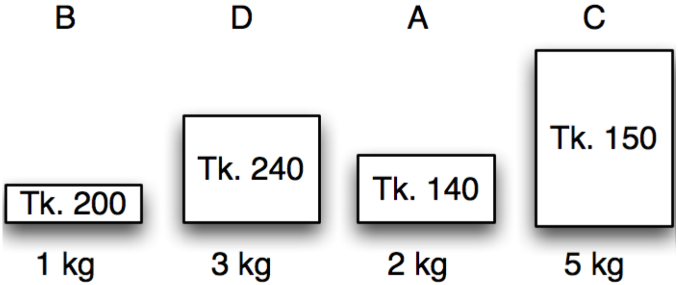
Fractional knapsack in action



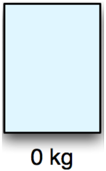
Item	Price	Weight	Value index
B	200	1 kg	200
D	240	3 kg	80
A	140	2 kg	70
C	150	5 kg	30

Maximum weight: 5 kg

Fractional knapsack in action

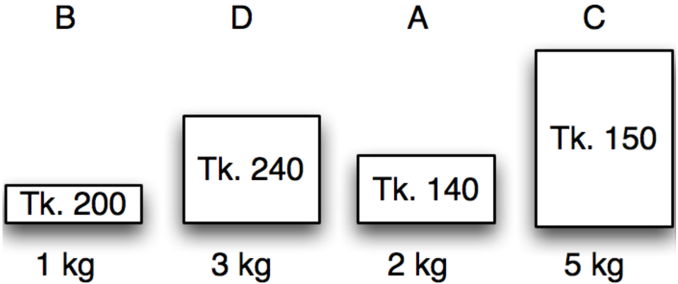


Item	Price	Weight	Value index	Chosen
B	200	1 kg	200	0 kg
D	240	3 kg	80	0 kg
A	140	2 kg	70	0 kg
C	150	5 kg	30	0 kg

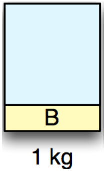


Maximum weight: 5 kg Remaining: 5 kg Benefit: 0 kg

Fractional knapsack in action

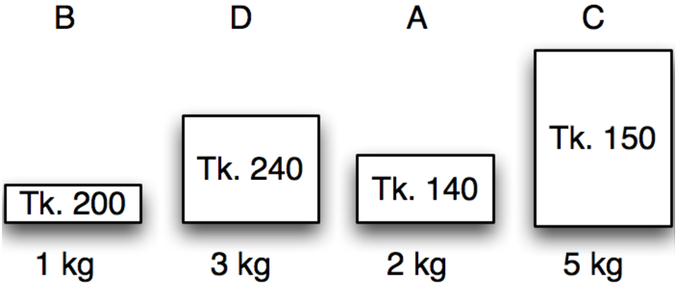


Item	Price	Weight	Value index	Chosen
B	200	1 kg	200	1 kg
D	240	3 kg	80	0 kg
A	140	2 kg	70	0 kg
C	150	5 kg	30	0 kg



Maximum weight: 5 kg Remaining: 4 kg Benefit: 200 kg

Fractional knapsack in action



Item	Price	Weight	Value index	Chosen
B	200	1 kg	200	1 kg
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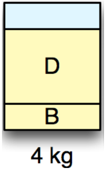
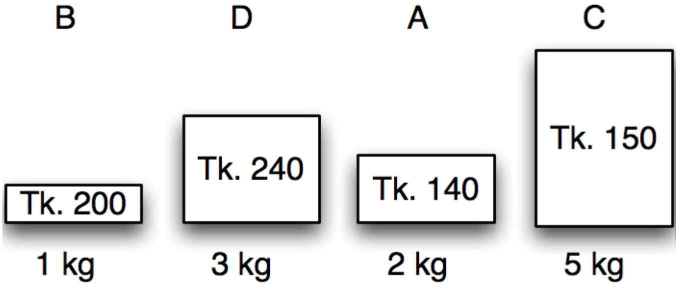
C	150	5 kg	30	0 kg



Maximum weight: 5 kg Remaining: 1 kg Benefit: 440 kg

Fractional knapsack in action



Item	Price	Weight	Value index	Chosen
B	200	1 kg	200	1 kg
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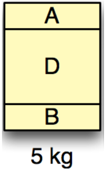
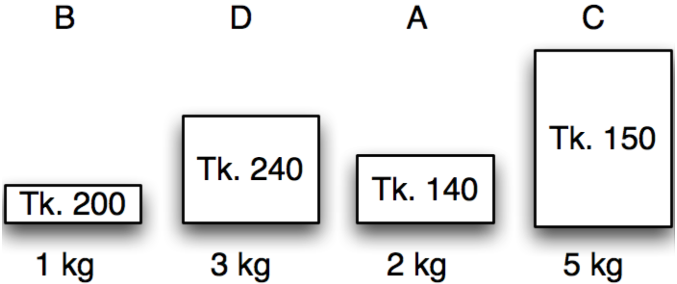


Maximum weight: 5 kg Remaining: 0 kg Benefit: 510 kg

Fractional knapsack in action



Item	Price	Weight	Value index	Chosen
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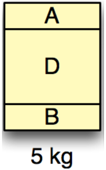

Maximum weight: 5 kg   Remaining: 0 kg   Benefit: 510 kg

Fractional knapsack greedy algorithm

FRACTIONAL-KNAPSACK(S, W) $\quad \triangleright$ $S = \{(w_i, b_i)\}$

for each item $i \in S$
    do $x_i \leftarrow 0$ $\quad \triangleright$ amount of item i chosen $(0 \leq x \leq w_i)$
        $v_i \leftarrow b_i / w_i$ $\quad \triangleright$ compute *value index*
$w \leftarrow 0$
while $w < W$
    do i = extract from S the item with highest value index
            $\triangleright$ greedy choice
        if $w + w_i \leq W$
            then $x_i = w_i$
            else $x_i = W - w$ $\triangleright$ fill up the remaining with i
        $w \leftarrow w + x_i$
return x $\quad \triangleright$ x_i contains amount of item i chosen

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Complexity

$T(n) = O(n \lg n)$.

Fractional knapsack greedy algorithm

```
FRACTIONAL-KNAPSACK(S, W)    ▷ S = {(w_i, b_i)}
for each item i ∈ S
        do x_i ← 0        ▷ amount of item i chosen (0 ≤ x ≤ w_i)
           v_i ← b_i/w_i                        ▷ compute value index
w ← 0
while w < W
        do i = extract from S the item with highest value index
                 ▷ greedy choice
           if w + w_i ≤ W
              then x_i = w_i
              else  x_i = W − w ▷ fill up the remaining with i
           w ← w + x_i
return x      ▷ x_i contains amount of item i chosen
```

Complexity

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Extension: 0/1 knapsack problem

Exactly the same as the Fractional Knapsack Problem, except that fractional quantities are not allowed.

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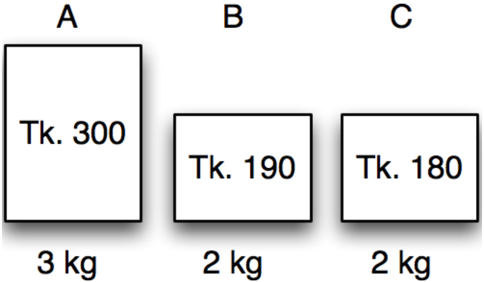
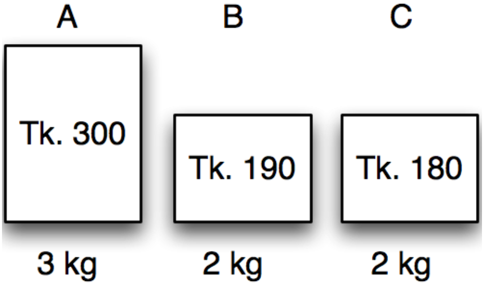


Maximum weight: 4 kg

Extension: 0/1 knapsack problem

Exactly the same as the Fractional Knapsack Problem, except that fractional quantities are not allowed.



Maximum weight: 4 kg

Greedy solution: item A Benefit: 300

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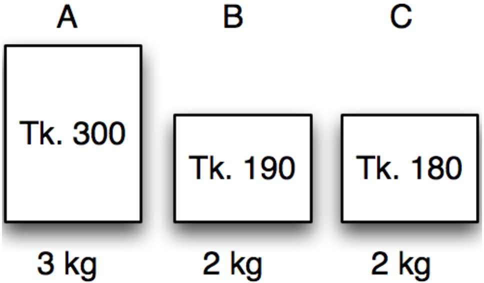
Exactly the same as the Fractional Knapsack Problem, except that fractional quantities are not allowed.



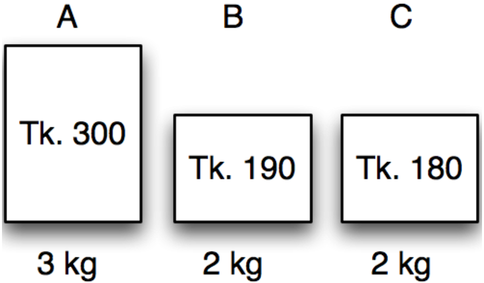
Maximum weight: $\boxed{4 \text{ kg}}$

Greedy solution: item A — Benefit: $\boxed{300}$

Optimal solution: items B and C — Benefit: $\boxed{370}$

Extension: 0/1 knapsack problem

Exactly the same as the Fractional Knapsack Problem, except that fractional quantities are not allowed.



Maximum weight: $\boxed{4 \text{ kg}}$

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The 0/1 Knapsack Problem does not have a greedy solution!

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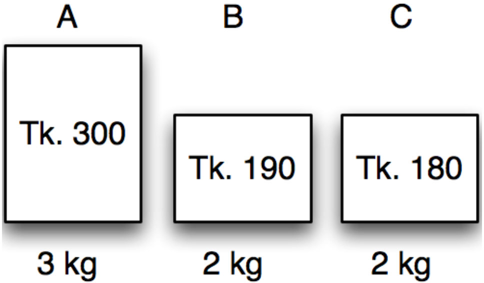
Exactly the same as the Fractional Knapsack Problem, except that fractional quantities are not allowed.



Maximum weight: $\boxed{4 \text{ kg}}$

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Given coin denominations in $\{C\}$, make change for a given amount A with the minimum number of coins.

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Example

Coin denominations, $C = \{25, 10, 5, 1\}$ Amount to change, $A = 73$

Coin changing problem

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Given coin denominations in $\{C\}$, make change for a given amount A with the minimum number of coins.

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Coin denominations, $C = \{25, 10, 5, 1\}$ Amount to change, $A = 73$

1. Choose 2 25 coins, so remaining is $73 - 2 * 25 = 23$

Coin changing problem

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Coin denominations, $C = \{25, 10, 5, 1\}$ Amount to change, $A = 73$

1. Choose 2 25 coins, so remaining is $73 - 2 * 25 = 23$
2. Choose 2 10 coins, so remaining is $23 - 2 * 10 = 3$

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Given coin denominations in $\{C\}$, make change for a given amount A with the minimum number of coins.

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Coin denominations, $C = \{25, 10, 5, 1\}$ Amount to change, $A = 73$

1. Choose 2 25 coins, so remaining is $73 - 2 * 25 = 23$
2. Choose 2 10 coins, so remaining is $23 - 2 * 10 = 3$
3. Choose 0 5 coins, so remaining is 3

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Given coin denominations in $\{C\}$, make change for a given amount A with the minimum number of coins.

Example

Coin denominations, $C = \{25, 10, 5, 1\}$ Amount to change, $A = 73$

1. Choose 2 25 coins, so remaining is $73 - 2 * 25 = 23$
2. Choose 2 10 coins, so remaining is $23 - 2 * 10 = 3$
3. Choose 0 5 coins, so remaining is 3
4. Choose 3 1 coins, so remaining is $3 - 1 * 3 = 0$

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Key question

Does a greedy approach always produce the optimal solution?

Coin changing problem (continued)

Coin denominations, $C = \{12, 5, 1\}$ Amount to change, $A = 15$

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1. Choose 1 12 coins, so remaining is $15 - 1 * 12 = 3$

Coin changing problem (continued)

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1. Choose 1 12 coins, so remaining is $15 - 1 * 12 = 3$
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Solution: 4 coins.

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Key observation

Correctness depends on the choice of coins, so greedy strategy does not provide a general solution to this problem!

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  Greedy choice property
  : If the global optimal solution can be reached by making locally optimal choices, then it has the greedy choice property.

  Subproblem optimality
  : If the optimal solution to the entire problem contain optimal solution to the subproblems, then it has the subproblem optimality property.

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- The algorithm must be rigorously proven to be correct!
- Except for a few select problems, it is far better to use Dynamic Programming to solve such optimization problems.
- So why study greedy algorithms? Because there are very efficient provably correct greedy algorithms for many common problems .