(11)

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 3 \end{bmatrix} \longleftarrow \begin{matrix} R_2' = -3R_1 + R_2 \\ R_3' = 2R_1 + R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & \cancel{-2}\ 1 & 1/-2 & 5/-2 \\ 0 & 7 & 2 & 3 \end{bmatrix} \xleftarrow{R_2' = \frac{1}{-2} R_2}$$

$$= \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & \frac{1}{-2} & 5/-2 \\ 0 & 0 & 11/2 & 41/2 \end{bmatrix} \xleftarrow{R_3' = -7R_2 + R_3}$$

$$= \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & -1/2 & -5/2 \\ 0 & 0 & 1 & 41/11 \end{bmatrix} \xleftarrow{R_3' = \frac{1}{11/2} R_3}$$

Basis for row space

$r_1 = [1\ 2\ 0\ -1]$

$r_2 = [0\ 1\ -1/2\ -5/2]$

$r_3 = [0\ 0\ 1\ 41/11]$

Basis for column space of A

$c_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ $c_2 = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ $c_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

rank = 3

(iii)

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1/2 & 3/2 & 4/2 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix} \xleftarrow{R_1' = \frac{1}{2} R_1}$$

$$= \begin{bmatrix} 1 & -1/2 & 3/2 & 4/2 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 6 & 8 & 2 \end{bmatrix} \xleftarrow[R_4' = -2R_1 + R_4]{R_3' = -2R_1 + R_3}$$

$$= \begin{bmatrix} 1 & -1/2 & 3/2 & 4/2 \\ 0 & 1 & 4/3 & 1/3 \\ 0 & 4 & 4 & 1 \\ 0 & 6 & 8 & 2 \end{bmatrix} \xleftarrow{R_2' = \frac{1}{3} R_2}$$

$$= \begin{bmatrix} 1 & -1/2 & 3/2 & 4/2 \\ 0 & 1 & 4/3 & 1/3 \\ 0 & 0 & -4/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xleftarrow[R_4' = -6R_2 + R_4]{R_3' = -4R_2 + R_3}$$

$$= \begin{bmatrix} 1 & -1/2 & 3/2 & 4/2 \\ 0 & 1 & 4/3 & 1/3 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for row space of A

$r_1 = [1 \;\; -1/2 \;\; 3/2 \;\; 4/2]$

$r_2 = [0 \;\; 1 \;\; 4/3 \;\; 1/3]$

$r_3 = [0 \;\; 0 \;\; 1 \;\; 1/4]$

Basis for column space of A

$$c_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \end{bmatrix} \quad c_2 = \begin{bmatrix} -1 \\ 3 \\ 3 \\ 5 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 3 \\ 4 \\ 7 \\ 11 \end{bmatrix}$$

rank = 3

(v)

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad R_1 \rightleftarrows R_4$$

$$= \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 6 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad \begin{array}{l} R_2' = -1R_1 + R_2 \\ R_3' = -3R_1 + R_3 \end{array}$$

$$= \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & -2 & 6 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad R_2' = -1R_2$$

$R_3' = 2R_1 + R_3$

$R_4' = -R_2 + R_3$

$$= \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for row space

$r_1 = [1 \ \ 1 \ \ -2 \ \ 0]$

$r_2 = [0 \ \ 1 \ \ -3 \ \ -1]$

Basis for column space

$$c_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} \quad c_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Rank = 2

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix} \longleftarrow \begin{array}{l} R_2' = -R_1 + R_2 \\ R_3' = -2R_1 + R_3 \\ R_4' = -3R_1 + R_4 \end{array}$$

$$= \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \begin{array}{l} R_3' = 3R_2 + R_3 \\ R_4' = R_2 + R_4 \end{array}$$

Basis for row space

$r_1 = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \end{bmatrix}$

$r_2 = \begin{bmatrix} 0 & 1 & 2 & 1 & -1 \end{bmatrix}$

Basis for column space

$c_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ $c_2 = \begin{bmatrix} 3 \\ 4 \\ 3 \\ 8 \end{bmatrix}$

Rank = 2

8

(1)

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix} \quad R_1' = (-1)R_1$$

$$= \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \end{bmatrix} \quad \begin{array}{l} R_2' = -3R_1 + R_2 \\ R_3' = -2R_1 + R_2 \\ R_4' = -4R_1 + R_4 \end{array}$$

$$= \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & 1 & -2 & -12 & -16 & +5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_2' = -1R_2 \\ R_3' = R_2 + R_3 \\ R_4' = R_3 + R_4 \end{array}$$

$x_1 \cancel{-2x_2 - 4x_4 +}$

$A \cdot x = 0$

$\cancel{x = A^{-1}}\ 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - 2x_2 - 4x_4 - 5x_5 + 3x_6 = 0$

$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$

$x_3 = p$

$x_4 = q$

$x_5 = r$

$x_6 = s$

$x_1 = 2x_2 + 4x_4 + 5x_5 - 3x_6$

$= 2(2p + 12q + 16r - 5s) + 4q + 5r - 3s$

$= 4p + 24q + 32r - 10s + 4q + 5r - 3s = 4p + 28q + 37r - 13s$

$x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6 = 0$

$= 2p + 12q + 16r - 5s$

$\therefore x_1 = 4p + 28q + 37r - 13s$

$x_2 = 2p + 12q + 16r - 5s$

$x_3 = p + 0 + 0 + 0$

$x_4 = 0 + q + 0 + 0$

$x_5 = 0 + 0 + r + 0$

$x_6 = 0 + 0 + 0 + s$

$\therefore$ basis for null space

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = p\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + q\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

nullity $= 4$

Rank $= 2$

(11)

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix} \xleftarrow[R_3' = -7R_1 + R_3]{R_2' = -5R_1 + R_2}$$

$$= \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_3' = -R_2 + R_3}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} A = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 + 3x_3 = 0$$

$$x_2 - 19x_3 = 0 \qquad x_2 = 19r$$

$$x_3 = r$$

$$\therefore x_1 = x_2 - 3r = 19r - 3r = 16r$$

Basis for null space

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = r \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} \qquad \text{Rank} = 2 \qquad \text{nullity} = 1$$

(iii)

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \longleftarrow \begin{array}{l} R_2' = -2R_1 + R_2 \\ R_3' = R_1 + R_3 \end{array}$$

$$= \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & 4/7 \\ 0 & 7 & 7 & 4 \end{bmatrix} \longleftarrow R_2' = \frac{1}{-7} R_2$$

$$= \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow R_3' = -7R_2 + R_3$$

$\therefore$ $A \cdot x = \underline{0}$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 4x_2 + 5x_3 + 2x_4 = 0$

$x_2 + x_3 + 4/7\, x_4 = 0$

$$4/7 - 2 \qquad \frac{4-14}{7} \qquad -10/7$$

$x_3 = r$

$x_4 = s$

$x_1 = -4x_2 - 5x_3 - 2x_4 \quad = -4(-r - 4/7 s) - 5r - 2s$

$= +4r + \cancel{16}/7\, s - 5r - 2s$

$x_2 = -x_3 - 4/7\, x_4 = -r - 4/7\, s$

$x_1 = -\cancel{6}r + \overset{2}{\cancel{10}}/7\, s$

$x_2 = -r - 4/7\, s$

$x_3 = r + 0$

$x_4 = 0 + s$

Basis for null space

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} -\cancel{1} \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} \cancel{0}\,\overset{2}{\cancel{10}}/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix}$$

nullity = 2

rank = 2

(iv)

$$A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & 6 & 0 & -3 \end{bmatrix} \longleftarrow \begin{matrix} R_3' = -2R_1 + R_3 \\ R_4' = -3R_1 + R_4 \\ R_5' = 2R_1 + R_5 \end{matrix}$$

$$= \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & 6 & 0 & -3 \end{bmatrix} \longleftarrow R_2' = \frac{1}{3} R_2$$

$$= \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \begin{matrix} R_3' = -3R_2 + R_3 \\ R_4' = -3R_2 + R_4 \\ R_5' = -3R_2 + R_5 \end{matrix}$$

$$= \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -5/12 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow R_3' = \frac{1}{-12} R_3$$

$$= \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -5/12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -5/12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - 3x_2 + 2x_3 + 2x_4 + x_5 = 0$

$x_2 + 2x_3 - x_5 = 0$

$x_3 + -5/12\, x_5 = 0 \quad x_3 = 5/12\, t$

~~$x_3 = r$~~

$x_4 = s$

$x_5 = t$

$$\boxed{\begin{aligned} x_2 &= -2r + t \\ x_1 &= 3(-2r + t) - 2r - 2s - t \\ &= -6r + 3t - 2r - 2s - t \\ &= -8r + 2t - 2s \end{aligned}}$$

$x_1 = -6r + 2s + 2t$

$x_2 = -2r + 0 + t$

$x_2 = -2\,\frac{5}{12}t + t$

$= -\frac{5}{6}t + t$

$= \frac{(-5+6)t}{6}$ $\qquad$ $6, 12$

$= \frac{1}{6}t$

$x_1 = 3\left(\frac{1}{6}t\right) - 2\left(\frac{5}{12}t\right) - 2s - t$

$= \frac{1}{2}t - \frac{5}{6}t - 2s - t$

$= \frac{6t - 10t - 12t}{12} - 2s$

$= \frac{-16t}{12} - 2s$

$= -\frac{4t}{3} - 2s$

$\therefore x_1 = -\frac{4}{3}t - 2s = -2s - \frac{4}{3}t$

$x_2 = \qquad = 0 + \frac{1}{6}t$

$x_3 = \qquad = 0 + \frac{5}{12}t$

$x_4 = \qquad = s + 0$

$x_5 = \qquad = 0 + t$

Basis for null space

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4/3 \\ 1/6 \\ 5/12 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ nullity $= 2$

Rank $= 3$

9)

$$S = \{(1,2,1),(3,1,2),(1,-3,4)\}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & -5 & 3 \end{bmatrix} \xleftarrow[R_3' = -R_1 + R_2]{R_2' = -3R_1 + R_2}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/5 \\ 0 & -5 & 3 \end{bmatrix} \xleftarrow{R_2' = -1/5 R_1}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/5 \\ 0 & 0 & \cancel{16/5}\,4 \end{bmatrix} \longleftarrow \quad R_3' = 5R_2 + R_3$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/5 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3' = \frac{1}{4} R_3$$

Basis for row space

$r_1 = [1 \quad 2 \quad 1]$

$r_2 = [0 \quad 1 \quad 1/5]$

$r_3 = [0 \quad 0 \quad 1]$

dimention = ~~3e~~ 3 $\longleftarrow$ check it (verify it)

$S = \{(1,2,1), (0,-1,0), (2,0,2)\}$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -4 & 0 \end{bmatrix} \xleftarrow[R_2' = -R_2]{R_3' = -2R_1 + R_2}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_3' = 4R_2 + R_3}$$

Basis for row space [sub space]

$r_1 = [1\ 2\ 1]$

$r_2 = [0\ 1\ 0]$ $\quad$ dim = 2

11

$S = \{ (1, -2, 5, -3)$