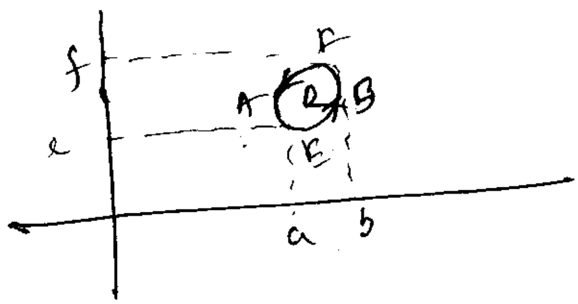
15.4

Suppose R is a closed region in the xy-plane bounded by a simple closed curve C and suppose M and N are continuous functions of x and y having continuous derivatives in R. then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

where C is traversed in the positive direction



Suppose R is a closed region in the xy plane bounded by a simple closed curve C. Suppose M and N are continuous function of x and y having a continuous derivative in R.

* Use Green's theorem to evaluate

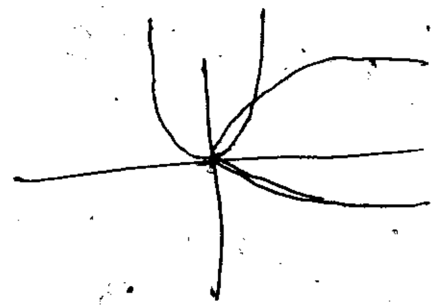
$$\oint_C x^2y\,dx + (y + xy^2)\,dy$$

where C is the boundary of the region enclosed by $y = x^2$ and $x = y^2$

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\frac{\partial N}{\partial x} = y^2$$

$$\frac{\partial M}{\partial y} = x^2$$



$$\int_0^1 \int_{y^2}^{\sqrt{y}} (y^2 - x^2)\,dx\,dy$$

$x^2 = \sqrt{x}$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$

$x = 1$

$$= \int_0^1 \left[xy^2 - \frac{x^3}{3}\right]_{y^2}^{\sqrt{y}}$$

$2 + \frac{1}{2} = \frac{5}{2} + 1 = \frac{5+}{2}$

$$= \int_0^1 \sqrt{y}\,y^2 + y^6/3 \; dy$$

$$= \left[\frac{2y^{7/2}}{7}\right]_0^1 - \left[\frac{y^7}{27}\right]_0^1$$

$$= \int_0^1 y^{5/2} - y^6/7 \; dy$$

$$= \frac{2}{7} - \frac{1}{27}$$

$$= 20/189$$

$$= \int_0^1 \left(y^2\sqrt{y} - \frac{(\sqrt{y})^3}{3} - y^4 + \frac{y^6}{3}\right) dy$$

$$= \int_0^1 \left(y^2\sqrt{y} - \frac{y\sqrt{y}}{3} - y^4 + \frac{y^6}{3}\right) dy$$

$2+\frac{1}{2}$

$$= \int_0^1 \left(\frac{3y^2\sqrt{y} - y\sqrt{y} - 3y^4 + y^6}{3}\right) dy$$

$$= \frac{1}{3}\int_0^1 3y^{5/2} - y^{3/2} - 3y^4 + y^6 \; dy$$

$$= \frac{1}{3}\int_0^1 \left[3\times\frac{2}{7}y^{7/2} - \frac{2}{5}y^{5/2} - \frac{3}{4}\cdot y^5 + \frac{1}{7}y^7\right]_0^1$$

$$= \frac{1}{3}\left(\frac{6}{7} - \frac{2}{5} - \frac{3}{4} + \frac{1}{7}\right)$$

$$\int_0^1 \int_{y^2}^{\sqrt{y}} (y^2 - x^2)\, dx\, dy$$

$$= \int_0^1 \left[xy^2 - \frac{x^3}{3} \right]_{y^2}^{\sqrt{y}} dy$$

$$= \int_0^1 \left(y^2\sqrt{y} - \frac{y\sqrt{y}}{3} - y^4 + \frac{y^6}{3} \right) dy$$

$2 + \frac{1}{2}$

$\frac{4+}{2}$

$$= \frac{1}{3}\int_0^1 3y^2 \cdot y^{1/2} - y y^{1/2} - 3y^4 + y^6 \; dy$$

$\frac{5}{2} + 1$

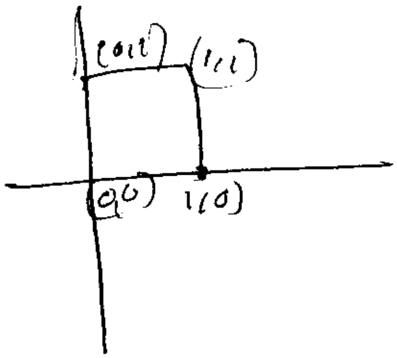
$\frac{5+2}{2}$

$$= \frac{1}{3}\int_0^1 3y^{5/2} - y^{3/2} - 3y^4 + y^6$$

$$= \frac{1}{3}\left[3 \times \frac{2}{7} y^{7/2} - \frac{2}{5} y^{5/2} - \frac{3}{5} y^5 + \frac{1}{7} y^6 \right]_0^1$$

$$= \frac{1}{3}\left[\frac{6}{7} - \frac{2}{5} - \frac{3}{5} + \frac{1}{7} \right]$$

$$= \frac{1}{3}[0] = 0 \quad Ans$$



11 10

$$\frac{x-1}{1-1} = \frac{y-1}{1-0}$$

$$\frac{x-1}{0} = y-1$$

$x = 1$

$M = y^2$

$N = x^2$

$$\frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} = 2y$$

$2x - 2y$

$$2\int_0^1\int_0^1 x - y \, dx \, dy$$

$$2\int_0^1 \left[\frac{x^2}{2} - xy\right]_0^1 dy$$

$$2\int_0^1 \left(\frac{1}{2} - y\right) dy$$

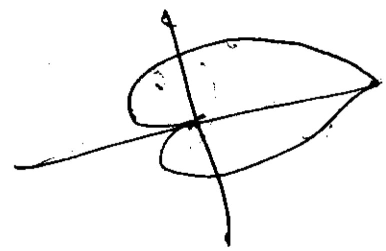
$$2\left[\frac{1}{2}y - \frac{y^2}{2}\right]_0^1$$

$$2\left[\frac{1}{2} - \frac{1}{2}\right]$$

$= 0$

$$\int_{\pi/2}^{\pi}\int_0^{x^2} \frac{1}{x}\cos\frac{y}{x}\,dy\,dx$$

$$\int_{\pi/2}^{\pi} \frac{1}{x}\,x\left[\sin y/x\right]_0^{x^2} dx$$



$$\int_{\pi/2}^{\pi} \sin x \; dx$$

$$-\left[\cos x\right]_{\pi/2}^{\pi} \qquad -(-1 - \;= \underline{1}$$

$$\int_0^1\int_0^1 \frac{x}{(xy+1)^2}\,dy\,dx \qquad p = xy+1 \quad dp = x\,dy$$

$$\int_0^1\int_0^1 \frac{1}{p^2}\,dx \qquad \int_0^1 \left[\frac{1}{xy+1}\right]_0^1 dx$$

$$\ln 2 - \ln 1 - 1 \qquad \int_0^1 \left[\frac{1}{x+1} - 1\right] dx$$

$$-(\ln 2 - 1 - \ln 1) \qquad \left[\ln x+1 - x\right]_0^1$$

$$-(\ln 2 - 1)$$

$$\int_1^{2}\int_0^{y^2} e^{x/y^2}\, dx\, dy$$

$$= \int_1^{2} y^2 \left[e^{x/y^2} \right]_0^{y^2} \cancel{dx}\, dy$$

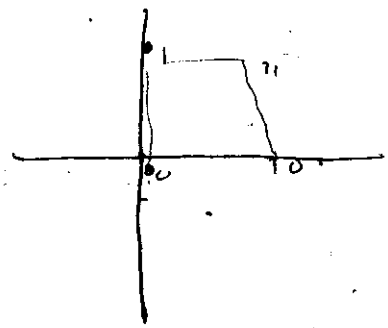
$$= \int_1^{2} y^2 (e - 1)\, dy$$

$$= \frac{7(e-1)}{3}$$

(11) $\oint \tan^{-1} y \, dx - \frac{y^2 x}{1+y^2} \, dy$

$\frac{\partial N}{\partial x} = -\frac{y^2}{1+y^2}$

$\frac{\partial M}{\partial y} = \frac{1}{1+y^2}$



$$\int_0^1 \int_0^1 \frac{1}{1+y^2} + \frac{y^2}{1+y^2} \, dx \, dy$$

$$- \frac{y^2}{1+y^2} - \frac{1}{1+y^2}$$

$$- \int_0^1 \int_0^1 dx \, dy$$

$$- \int_0^1 x \, dy$$

$- 1$ Ans.

(12) $\oint \cos x \sin y \, dx + \sin x \cos y \, dx$

$M = \cos x \sin y$ $\quad \frac{\partial M}{\partial y} = \cos^2 y$

$N = \sin x \cos y$ $\quad \frac{\partial N}{\partial x} = \cos^2 x$

$$\int_0^3 \int_0^x \cos^2 x - \cos^2 y \quad \cos x \, dx dy$$

cos y

$= 0$

$\boxed{\begin{matrix}0 \\ 3\end{matrix}} \begin{matrix}0 \\ 3\end{matrix}$

$\frac{x-0}{0-3} = \frac{y-0}{0-3}$

$x = y$

$\frac{x-0}{0-0} = \frac{y-0}{0-3}$

$x = \frac{y}{3}$

$x = y$

(19)

(a) $\oint 6xy - y^2 \, dx$

$\frac{\partial M}{\partial y} = 6x - 2y$

$2x = x^2$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$

$x = 2$

$$\int_0^2 \int_{x^2}^{2x} -6x + 2y \; dy \, dx$$

$A = \oint x \, dy$

$A = -\oint y \, dx$

$A = \frac{1}{2} \int -y \, dx + x \, dy$

(b) $\oint x^2 y \, dx - y^2 x \, dy$

$M = x^2 y \qquad \frac{\partial M}{\partial y} = x^2$

$N = -y^2 x \qquad \frac{\partial N}{\partial x} = -y^2$

$$-\int_0^{2\pi} \int_0^4 x^2 + y^2 r \, dr \, d\theta$$

$$-\int_0^{2\pi} \int_0^4 r^3 \, dr \, d\theta$$

$\frac{1}{4} - \int_0^{\pi/4} \left[r^4 \right]_0^4$

$\frac{256}{4} \quad \frac{2\pi}{4} = \cancel{128\pi}$

$= -\frac{256\pi}{4}$