Basic Laws of Differentiation

If $f(x)$ and $g(x)$ are both differentiable functions of x then,

- ✓ Law 1 Constant function Law

  If $f(x) = c$ is real number then

$$f'(x) = 0 \quad \text{or} \quad \frac{d}{dx}(c) = 0$$

  For example

$$\frac{d}{dx}(5) = 0$$

- ✓ Law 2 Sum Law

$$[f(x) + g(x)]' = f'(x) + g'(x) \quad \text{or} \quad \frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

  For example

$$\frac{d}{dx}(x^2 + \sin x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin x) = 2x + \cos x$$

- ✓ Law 4 Product Law

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \quad \text{or,} \quad \frac{d}{dx}[f(x)g(x)] = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$$

$$\frac{d}{dx}(x^2 \sin x) = \frac{d}{dx}(x^2)\sin x + x^2\frac{d}{dx}(\sin x) = 2x\sin x + x^2\cos x$$

  The Rule can be extended for three or more finite number of functions. For example

$$\frac{d}{dx}[f(x)g(x)h(x)] = \frac{df(x)}{dx}g(x)h(x) + f(x)\frac{dg(x)}{dx}h(x) + f(x)g(x)\frac{dh(x)}{dx}$$

- ✓ Law 5 Constant Multiple Law

$$[cf(x)]' = cf'(x) \quad \text{or,} \quad \frac{d}{dx}[cf(x)] = c\frac{df(x)}{dx}$$

$$\frac{d}{dx}(2x^2) = 2\frac{d}{dx}(x^2) = 2 \times 2x = 4x$$

- ✓ Law 6 Quotient Law

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - g'(x)f(x)}{[g(x)]^2} \quad \text{or,} \quad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{\big(g(x)\big)^2}$$

provided that $g(x) \neq 0$

$$\frac{d}{dx}\left(\frac{x^2}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin x)}{(\sin x)^2} = \frac{2x \sin x - x^2 \cos x}{(\sin x)^2}$$

CHAIN RULE FOR DIFFERENTIATION

Example

1. Find $F'(x)$

$$F(x) = (x^2+1)^5 \quad F(x) = \sin(x^2+1) \quad F(x) = \sin\sqrt{(x^2+1)}$$

2. Find $F'(t)$

$$F(t) = \left(\frac{2t-1}{t^2+1}\right)^3$$

3. Find dy/dx when

1. $y(x) = \tan^3(3x^2+1)$ **2.** $y(x) = \ln(\ln(x))$ **3.** $y(x) = \ln(\sin(x^2))$

4. $y(x) = \sin(\cos(\tan^5 x))$ **5.** $y(x) = \sqrt{\sec\sqrt{x}}$ **6.** $y(x) = \ln\sqrt{\frac{(1+\sin ax)}{(1-\sin ax)}}$

IMPLICIT DIFFERENTIATION

Sometimes we are not given y as a function of x explicitly, but instead have an equation connecting them which we may be unable to solve explicitly for either x or y. We may still want to find dy/dx, but we shall find that the resulting expression still involves both variables. The following example illustrates what is meant.

Example Find dy/dx

$$x^3 - 5xy^2 + y^3 + 11 = 0$$

Solution:

Differentiating both sides, by considering y as a function of x, we obtain,

$$\frac{d}{dx}(x^3 - 5xy^2 + y^3 + 11) = 0$$

$$\frac{d}{dx}(x^3) - \frac{d}{dx}(5xy^2) + \frac{d}{dx}(y^3) + \frac{d}{dx}(11) = 0$$

$$3x^2 - 5\left(y^2 + 2xy\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} + 0 = 0$$

$$3x^2 - 5y^2 - 10xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$3x^2 - 5y^2 + (3y^2 - 10xy)\frac{dy}{dx} = 0$$

Now solving for dy/dx, we get,

$$\boxed{\frac{dy}{dx} = \frac{(5y^2 - 3x^2)}{(3y^2 - 10xy)}}$$

Problems Find dy/dx

1. $x^2 - 4y^3 = 0$ **2.** $x^2y^2 = 4x$ **3.** $x^2y^2 = 4y$ **4.** $x^2y^3 = \sin y$

5. $x^2 - y^3 - 3y = 4$ **6.** $y^2 = x^3 - 6x + 4\cos y$ **7.** $\cos xy = e^{xy}$

Problem Find d^2y/dx^2 in terms of x and y for the following equation

$$xy + e^{3y} = 0$$

LOGARITHMIC DIFFERENTIATION

The laws of logarithms can help to simplify the work involved in differentiating logarithmic expressions, we now look at a procedure that takes advantage of these same laws to help us differentiate functions that at first blush do not necessarily involve logarithms. This method, called **logarithmic differentiation,** is especially useful for differentiating functions involving complicated products, quotients, and/or powers that can be simplified by using logarithms.

Basic laws of logarithm

If A and B are two positive real numbers,

1. $\ln(AB) = \ln A + \ln B$ **2.** $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$ **3.** $\ln(A^n) = n\ln A$

4. $\ln 1 = 0$ **5.** $\ln e = 1$

where e is a real number (truncated to 10 decimal places)

$$e = 2.8182818284$$

Example Use logarithm to differentiate the following

$$y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$$

Solution:

Taking the natural logarithm of both sides of the given equation, we get,

$$\ln y = \ln\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5} = \ln\left(x^{\frac{3}{4}}\sqrt{x^2+1}\right) - \ln(3x+2)^5 = \ln x^{\frac{3}{4}} + \ln\sqrt{x^2+1} - \ln(3x+2)^5$$

$$= \ln x^{\frac{3}{4}} + \ln(x^2+1)^{\frac{1}{2}} - \ln(3x+2)^5 = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Now differentiating implicitly we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{3}{4}\frac{1}{x}+\frac{1}{2}\frac{1}{(x^2+1)}2x-5\frac{1}{(3x+2)}3$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{3}{4x}+\frac{x}{(x^2+1)}-\frac{15}{(3x+2)}$$

$$\frac{dy}{dx}=y\left\{\frac{3}{4x}+\frac{x}{(x^2+1)}-\frac{15}{(3x+2)}\right\}$$

$$\boxed{\frac{dy}{dx}=\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left\{\frac{3}{4x}+\frac{x}{(x^2+1)}-\frac{15}{(3x+2)}\right\}}$$

Problems Differentiate the following functions

1. $y=\sqrt{x}e^{x^2}(x^2+1)^{10}$ **2**. $y=\dfrac{e^{\sqrt{x}}\sin(x^2)}{(x^3+1)^{\frac{3}{4}}}$

Example Differentiate $y=x^x$

Solution:

Taking the natural logarithm of both sides of the given equation, we get,

$$\ln y=\ln x^x=x\ln x$$

Now differentiating implicitly we get

$$\frac{1}{y}\frac{dy}{dx}=x\frac{1}{x}+(\ln x).1=1+\ln x\Rightarrow\frac{dy}{dx}=y(1+\ln x)$$

$$\boxed{\therefore\frac{dy}{dx}=x^x(1+\ln x)}$$

which is the required result.

Example Differentiate $y=x^{x^x}$

Solution:

Taking the natural logarithm of both sides of the given equation, we get,

$\ln y=\ln x^{x^x}=x^x\ln x$

Again taking the natural logarithm on both sides,

$\ln(\ln y) = \ln(x^x \ln x) = \ln x^x + \ln(\ln x) = x \ln x + \ln(\ln x)$

Now differentiating implicitly, we get,

$$\frac{1}{\ln y}\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + (\ln x).1 + \frac{1}{\ln x}\frac{1}{x} = 1 + \ln x + \frac{1}{x \ln x} \Rightarrow \frac{dy}{dx} = y \ln y \left(1 + \ln x + \frac{1}{x \ln x}\right)$$

$$\boxed{\therefore \frac{dy}{dx} = x^{x^x} x^x \ln x \left(1 + \ln x + \frac{1}{x \ln x}\right)}$$

which is the required result.

Problem Differentiate

$$y = x^{\sin x}, \qquad y = x^{\sqrt{\sin x}}, \qquad y = (\sin x)^{(\sin x)^{\sin x}}, \qquad y = \sin(x^x)$$

Example Differentiate

$$y = \sin(x^x)$$

Solution:

Given that

$$y = \sin(x^x)$$

Let us consider,

$$y = \sin u\,, \qquad u = x^x$$

Now using chain rule for derivative,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{1}$$

Now to find du/dx we take the natural logarithm of both sides of the equation (2i),

$$\ln u = \ln x^x = x \ln x$$

Now differentiating implicitly we get

$$\frac{1}{u}\frac{du}{dx} = x\frac{1}{x} + (\ln x).1 = 1 + \ln x \Rightarrow \frac{du}{dx} = u(1 + \ln x)$$

$$\therefore \frac{du}{dx} = x^x(1 + \ln x)$$

$$\frac{dy}{dx} = \cos u\, x^x(1 + \ln x) = \cos(x^x)\, x^x(1 + \ln x)$$

Example Differentiate $y = x^x + (\sin x)^{\sin x}$

Solution:

Given function is

$$y(x) = x^x + (\sin x)^{\sin x} \tag{1}$$

Let us consider,

$$u(x) = x^x \tag{2i}$$

$$v(x) = (\sin x)^{\sin x} \tag{2ii}$$

Now (1) becomes,

$$y(x) = u(x) + v(x) \tag{3}$$

Differentiating equation (3) w. r. t. x we get,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \tag{4}$$

Now taking the natural logarithm of both sides of the equation (2i), we get,

$$\ln u = \ln x^x = x \ln x$$

Now differentiating implicitly we get

$$\frac{1}{u}\frac{du}{dx} = x\frac{1}{x} + (\ln x).1 = 1 + \ln x \Rightarrow \frac{du}{dx} = u(1 + \ln x)$$

$$\therefore \frac{du}{dx} = x^x(1 + \ln x)$$

Again taking the natural logarithm of both sides of the equation (2ii), we get,

$$\ln v = \ln(\sin x)^{\sin x} = (\sin x)\ln(\sin x)$$

Now differentiating implicitly we get

$$\frac{1}{v}\frac{dv}{dx} = (\sin x)\frac{1}{(\sin x)}(\cos x) + \ln(\sin x)\cos x = \cos x\{1 + \ln(\sin x)\}$$

$$\frac{dv}{dx} = v\cos x\{1 + \ln(\sin x)\} = (\sin x)^{\sin x}\cos x\{1 + \ln(\sin x)\}$$

Now putting the values of du/dx and dv/dx into (4), we obtain,

$$\boxed{\therefore \frac{dy}{dx} = x^x(1 + \ln x) + (\sin x)^{\sin x}\cos x\{1 + \ln(\sin x)\}}$$

which is the required result.

Problem Differentiate the following

$$y = x^{\sin x} + (\sin x)^x, \qquad y = \cos\left(x^{\sin x} + (\sin x)^x\right), \qquad y = x^{\sqrt{\sin x}} - \left(\sqrt{\sin x}\right)^{\sec^{-1} x}$$

Problem If $x^y = e^{x-y}$, then show that

$$\frac{dy}{dx} = \frac{\ln x}{(1 + \ln x)^2}$$

Problem Find dy/dx

1. $x^y = \cos(e^{x-y})$, **2**. $x^y y^x = 1$