1.3 Limits at Infinity; End Behavior of a Function
Solutions to the Selected Problems

9–40. Find the limits.

9.

$$\lim_{x\to+\infty}(1+2x-3x^5)$$

Solution

$$\begin{aligned}\lim_{x\to+\infty}(1+2x-3x^5) &= \lim_{x\to+\infty}\left[x^5\left(\frac{1}{x^5}+\frac{2x}{x^5}-3\right)\right]\\ &= \lim_{x\to+\infty}(x^5)\times\lim_{x\to+\infty}\left(\frac{1}{x^5}+\frac{2}{x^4}-3\right)\\ &= +\infty\times\lim_{x\to+\infty}(0+0-3)\\ &= -\infty\end{aligned}$$

$$\boxed{\lim_{x\to+\infty}(1+2x-3x^5)=-\infty}$$

10.

$$\lim_{x\to+\infty}(2x^3-100x+5)$$

Solution

$$\begin{aligned}\lim_{x\to+\infty}(2x^3-100x+5) &= \lim_{x\to+\infty}\left[x^3\left(2-\frac{100x}{x^3}+\frac{5}{x^3}\right)\right]\\ &= \lim_{x\to+\infty}(x^3)\times\lim_{x\to+\infty}\left(2-\frac{100x}{x^3}+\frac{5}{x^3}\right)\\ &= +\infty\times 2\\ &= +\infty\end{aligned}$$

$$\boxed{\lim_{x\to+\infty}(2x^3-100x+5)=+\infty}$$

14.

$$\lim_{x\to+\infty}\frac{5x^2-4x}{2x^2+3}$$

Solution

$$\lim_{x\to+\infty}\frac{5x^2-4x}{2x^2+3}=\lim_{x\to+\infty}\frac{x^2\left(5-\frac{4x}{x^2}\right)}{x^2\left(2+\frac{3}{x^2}\right)}$$

$$= \lim_{x\to+\infty} \frac{\left(5-\frac{4}{x}\right)}{\left(2+\frac{3}{x^2}\right)}$$

$$= \frac{\lim_{x\to+\infty}\left(5-\frac{4}{x}\right)}{\lim_{x\to+\infty}\left(2+\frac{3}{x^2}\right)}$$

$$= \frac{(5-0)}{(2+0)}$$

$$\boxed{\lim_{x\to+\infty} \frac{5x^2-4x}{2x^2+3} = \frac{5}{2}}$$

17.

$$\lim_{x\to-\infty} \frac{x-2}{x^2+2x+1}$$

Solution

$$\lim_{x\to-\infty} \frac{x-2}{x^2+2x+1} = \lim_{x\to-\infty} \frac{x\left(1-\frac{2}{x}\right)}{x^2\left(1+\frac{2x}{x^2}+\frac{1}{x^2}\right)}$$

$$= \lim_{x\to-\infty} \frac{\left(1-\frac{2}{x}\right)}{x\left(1+\frac{2}{x}+\frac{1}{x^2}\right)}$$

$$= \lim_{x\to-\infty} \frac{1}{x} \times \lim_{x\to-\infty} \frac{\left(1-\frac{2}{x}\right)}{\left(1+\frac{2}{x}+\frac{1}{x^2}\right)}$$

$$= \lim_{x\to-\infty} \frac{1}{x} \times \frac{\lim_{x\to-\infty}\left(1-\frac{2}{x}\right)}{\lim_{x\to-\infty}\left(1+\frac{2}{x}+\frac{1}{x^2}\right)}$$

$$= 0 \times \frac{1-0}{1+0+0}$$

$$\boxed{\lim_{x\to-\infty} \frac{x-2}{x^2+2x+1} = 0}$$

1.3 Limits at Infinity; End Behavior of a Function
Solutions to the Selected Problems

29.

$$\lim_{x\to-\infty}\frac{\sqrt{3x^4+x}}{x^2-8}$$

Solution

$$\begin{aligned}\lim_{x\to-\infty}\frac{\sqrt{3x^4+x}}{x^2-8} &= \lim_{x\to-\infty}\frac{\sqrt{x^4\left(3+\frac{x}{x^4}\right)}}{x^2\left(1-\frac{8}{x^2}\right)}\\ &= \lim_{x\to-\infty}\frac{\sqrt{x^4}\sqrt{\left(3+\frac{1}{x^3}\right)}}{x^2\left(1-\frac{8}{x^2}\right)}\\ &= \lim_{x\to-\infty}\frac{|x^2|\sqrt{\left(3+\frac{1}{x^3}\right)}}{x^2\left(1-\frac{8}{x^2}\right)}\\ &= \lim_{x\to-\infty}\frac{\sqrt{\left(3+\frac{1}{x^3}\right)}}{\left(1-\frac{8}{x^2}\right)}\\ &= \frac{\lim_{x\to-\infty}\sqrt{\left(3+\frac{1}{x^3}\right)}}{\lim_{x\to-\infty}\left(1-\frac{8}{x^2}\right)}\\ &= \sqrt{3}\end{aligned}$$

$$\boxed{\lim_{x\to-\infty}\frac{\sqrt{3x^4+x}}{x^2-8}=\sqrt{3}}$$

31.

$$\lim_{x\to+\infty}\left(\sqrt{x^2+3}-x\right)$$

Solution

$$\lim_{x\to+\infty}\left(\sqrt{x^2+3}-x\right)=\lim_{x\to+\infty}\frac{\left(\sqrt{x^2+3}-x\right)\left(\sqrt{x^2+3}+x\right)}{\left(\sqrt{x^2+3}+x\right)}$$

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$$= \lim_{x\to+\infty} \frac{\left(\sqrt{x^2+3}\right)^2 - (x)^2}{\left(\sqrt{x^2+3}+x\right)}$$

$$= \lim_{x\to+\infty} \frac{x^2+3-x^2}{\sqrt{x^2+3}+x}$$

$$= \lim_{x\to+\infty} \frac{3}{\sqrt{x^2\left(1+\frac{3}{x^2}\right)}+x}$$

$$= \lim_{x\to+\infty} \frac{3}{x\sqrt{\left(1+\frac{3}{x^2}\right)}+x} \qquad [x>0]$$

$$= \lim_{x\to+\infty} \frac{3}{x\left[\sqrt{\left(1+\frac{3}{x^2}\right)}+1\right]}$$

$$= \lim_{x\to+\infty} \frac{1}{x} \times \lim_{x\to+\infty} \frac{3}{\left[\sqrt{\left(1+\frac{3}{x^2}\right)}+1\right]}$$

$$= 0 \times \frac{3}{2}$$

$$\boxed{\lim_{x\to+\infty} \left(\sqrt{x^2+3}-x\right) = 0}$$

Mathematica Check

$$\text{In}[1] = \text{Limit}[\text{Sqrt}[x\text{^}2+3]-x, x \to \text{Infinity}]$$

$$\text{Out}[1] = 0$$

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