# 1 Systems of Linear Equations



**CHAPTER OBJECTIVES**

- Recognize, graph, and solve a system of linear equations in $n$ variables.
- Use back-substitution to solve a system of linear equations.
- Determine whether a system of linear equations is consistent or inconsistent.
- Determine if a matrix is in row-echelon form or reduced row-echelon form.
- Use elementary row operations with back-substitution to solve a system in row-echelon form.
- Use elimination to rewrite a system in row-echelon form.
- Write an augmented or coefficient matrix from a system of linear equations, or translate a matrix into a system of linear equations.
- Solve a system of linear equations using Gaussian elimination and Gaussian elimination with back-substitution.
- Solve a homogeneous system of linear equations.
- Set up and solve a system of equations to fit a polynomial function to a set of data points, as well as to represent a network.

## 1.1 Introduction to Systems of Linear Equations

**HISTORICAL NOTE**

**Carl Friedrich Gauss (1777–1855)**

is often ranked—along with Archimedes and Newton—as one of the greatest mathematicians in history. To read about his contributions to linear algebra, visit *college.hmco.com/pic/larsonELA6e.*

Linear algebra is a branch of mathematics rich in theory and applications. This text strikes a balance between the theoretical and the practical. Because linear algebra arose from the study of systems of linear equations, you shall begin with linear equations. Although some material in this first chapter will be familiar to you, it is suggested that you carefully study the methods presented here. Doing so will cultivate and clarify your intuition for the more abstract material that follows.

The study of linear algebra demands familiarity with algebra, analytic geometry, and trigonometry. Occasionally you will find examples and exercises requiring a knowledge of calculus; these are clearly marked in the text.

Early in your study of linear algebra you will discover that many of the solution methods involve dozens of arithmetic steps, so it is essential to strive to avoid careless errors. A computer or calculator can be very useful in checking your work, as well as in performing many of the routine computations in linear algebra.

## Linear Equations in n Variables

Recall from analytic geometry that the equation of a line in two-dimensional space has the form

$$a_1x + a_2y = b, \quad a_1, a_2, \text{ and } b \text{ are constants.}$$

This is a **linear equation in two variables** $x$ and $y$. Similarly, the equation of a plane in three-dimensional space has the form

$$a_1x + a_2y + a_3z = b, \quad a_1, a_2, a_3, \text{ and } b \text{ are constants.}$$

Such an equation is called a **linear equation in three variables** $x$, $y$, and $z$. In general, a linear equation in $n$ variables is defined as follows.

### Definition of a Linear Equation in *n* Variables

A **linear equation in *n* variables** $x_1, x_2, x_3, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

The **coefficients** $a_1, a_2, a_3, \ldots, a_n$ are real numbers, and the **constant term** $b$ is a real number. The number $a_1$ is the **leading coefficient,** and $x_1$ is the **leading variable.**

REMARK: Letters that occur early in the alphabet are used to represent constants, and letters that occur late in the alphabet are used to represent variables.

Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions. Variables appear only to the first power. Example 1 lists some equations that are linear and some that are not linear.

### EXAMPLE 1 Examples of Linear Equations and Nonlinear Equations

Each equation is linear.

(a) $3x + 2y = 7$

(b) $\frac{1}{2}x + y - \pi z = \sqrt{2}$

(c) $x_1 - 2x_2 + 10x_3 + x_4 = 0$

(d) $\left(\sin\frac{\pi}{2}\right)x_1 - 4x_2 = e^2$

Each equation is not linear.

(a) $xy + z = 2$

(b) $e^x - 2y = 4$

(c) $\sin x_1 + 2x_2 - 3x_3 = 0$

(d) $\dfrac{1}{x} + \dfrac{1}{y} = 4$

A **solution** of a linear equation in $n$ variables is a sequence of $n$ real numbers $s_1, s_2, s_3, \ldots, s_n$ arranged so the equation is satisfied when the values

$$x_1 = s_1, \qquad x_2 = s_2, \qquad x_3 = s_3, \qquad \ldots, \qquad x_n = s_n$$

are substituted into the equation. For example, the equation

$$x_1 + 2x_2 = 4$$

is satisfied when $x_1 = 2$ and $x_2 = 1$. Some other solutions are $x_1 = -4$ and $x_2 = 4$, $x_1 = 0$ and $x_2 = 2$, and $x_1 = -2$ and $x_2 = 3$.

The set of *all* solutions of a linear equation is called its **solution set,** and when this set is found, the equation is said to have been **solved.** To describe the entire solution set of a linear equation, a **parametric representation** is often used, as illustrated in Examples 2 and 3.

### *EXAMPLE 2* Parametric Representation of a Solution Set

Solve the linear equation $x_1 + 2x_2 = 4$.

*SOLUTION* To find the solution set of an equation involving two variables, solve for one of the variables in terms of the other variable. If you solve for $x_1$ in terms of $x_2$, you obtain

$$x_1 = 4 - 2x_2.$$

In this form, the variable $x_2$ is **free,** which means that it can take on any real value. The variable $x_1$ is not free because its value depends on the value assigned to $x_2$. To represent the infinite number of solutions of this equation, it is convenient to introduce a third variable $t$ called a **parameter.** By letting $x_2 = t$, you can represent the solution set as

$$x_1 = 4 - 2t, \qquad x_2 = t, \quad t \text{ is any real number.}$$

Particular solutions can be obtained by assigning values to the parameter $t$. For instance, $t = 1$ yields the solution $x_1 = 2$ and $x_2 = 1$, and $t = 4$ yields the solution $x_1 = -4$ and $x_2 = 4$.

The solution set of a linear equation can be represented parametrically in more than one way. In Example 2 you could have chosen $x_1$ to be the free variable. The parametric representation of the solution set would then have taken the form

$$x_1 = s, \qquad x_2 = 2 - \tfrac{1}{2}s, \quad s \text{ is any real number.}$$

For convenience, choose the variables that occur last in a given equation to be free variables.

### *EXAMPLE 3* Parametric Representation of a Solution Set

Solve the linear equation $3x + 2y - z = 3$.

*SOLUTION* Choosing $y$ and $z$ to be the free variables, begin by solving for $x$ to obtain

$$\begin{aligned} 3x &= 3 - 2y + z \\ x &= 1 - \tfrac{2}{3}y + \tfrac{1}{3}z. \end{aligned}$$

Letting $y = s$ and $z = t$, you obtain the parametric representation

$$x = 1 - \tfrac{2}{3}s + \tfrac{1}{3}t, \qquad y = s, \qquad z = t$$

where $s$ and $t$ are any real numbers. Two particular solutions are

$$x = 1, y = 0, z = 0 \qquad \text{and} \qquad x = 1, y = 1, z = 2.$$

## *Systems of Linear Equations*

A **system of $m$ linear equations in $n$ variables** is a set of $m$ equations, each of which is linear in the same $n$ variables:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

REMARK: The double-subscript notation indicates $a_{ij}$ is the coefficient of $x_j$ in the $i$th equation.

A **solution** of a system of linear equations is a sequence of numbers $s_1, s_2, s_3, \ldots, s_n$ that is a solution of each of the linear equations in the system. For example, the system

$$\begin{aligned}
3x_1 + 2x_2 &= 3\\
-x_1 + x_2 &= 4
\end{aligned}$$

has $x_1 = -1$ and $x_2 = 3$ as a solution because *both* equations are satisfied when $x_1 = -1$ and $x_2 = 3$. On the other hand, $x_1 = 1$ and $x_2 = 0$ is not a solution of the system because these values satisfy only the first equation in the system.

**Discovery**

*Graph the two lines*

$$\begin{aligned}
3x - y &= 1\\
2x - y &= 0
\end{aligned}$$

*in the xy-plane. Where do they intersect? How many solutions does this system of linear equations have?*

*Repeat this analysis for the pairs of lines*

$$\begin{aligned}
3x - y &= 1 \qquad & 3x - y &= 1\\
3x - y &= 0 \qquad & 6x - 2y &= 2.
\end{aligned}$$

*In general, what basic types of solution sets are possible for a system of two equations in two unknowns?*

It is possible for a system of linear equations to have exactly one solution, an infinite number of solutions, or no solution. A system of linear equations is called **consistent** if it has at least one solution and **inconsistent** if it has no solution.

### EXAMPLE 4 Systems of Two Equations in Two Variables

Solve each system of linear equations, and graph each system as a pair of straight lines.

(a) $\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$ (b) $\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$ (c) $\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$

*SOLUTION*

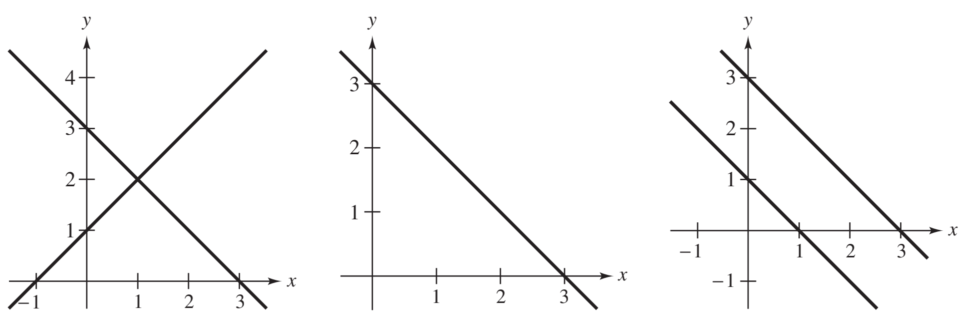
(a) This system has exactly one solution, $x = 1$ and $y = 2$. The solution can be obtained by adding the two equations to give $2x = 2$, which implies $x = 1$ and so $y = 2$. The graph of this system is represented by two *intersecting* lines, as shown in Figure 1.1(a).

(b) This system has an infinite number of solutions because the second equation is the result of multiplying both sides of the first equation by 2. A parametric representation of the solution set is shown as

$$x = 3 - t, \quad y = t, \quad t \text{ is any real number.}$$

The graph of this system is represented by two *coincident* lines, as shown in Figure 1.1(b).

(c) This system has no solution because it is impossible for the sum of two numbers to be 3 and 1 simultaneously. The graph of this system is represented by two *parallel* lines, as shown in Figure 1.1(c).




(a) Two intersecting lines: $\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$

(b) Two coincident lines: $\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$

(c) Two parallel lines: $\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$

Figure 1.1

Example 4 illustrates the three basic types of solution sets that are possible for a system of linear equations. This result is stated here without proof. (The proof is provided later in Theorem 2.5.)

**Number of Solutions of a System of Linear Equations**

For a system of linear equations in $n$ variables, precisely one of the following is true.

1. The system has exactly one solution (consistent system).
2. The system has an infinite number of solutions (consistent system).
3. The system has no solution (inconsistent system).

## Solving a System of Linear Equations

Which system is easier to solve algebraically?

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \qquad &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned} \qquad\qquad \begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

The system on the right is clearly easier to solve. This system is in **row-echelon form,** which means that it follows a stair-step pattern and has leading coefficients of 1. To solve such a system, use a procedure called **back-substitution.**

### *EXAMPLE 5* Using Back-Substitution to Solve a System in Row-Echelon Form

Use back-substitution to solve the system.

$$\begin{aligned} x - 2y &= 5 && \text{Equation 1} \\ y &= -2 && \text{Equation 2} \end{aligned}$$

*SOLUTION* From Equation 2 you know that $y = -2$. By substituting this value of $y$ into Equation 1, you obtain

$$\begin{aligned} x - 2(-2) &= 5 && \text{Substitute } y = -2. \\ x &= 1. && \text{Solve for } x. \end{aligned}$$

The system has exactly one solution: $x = 1$ and $y = -2$.

The term "back-substitution" implies that you work *backward.* For instance, in Example 5, the second equation gave you the value of $y$. Then you substituted that value into the first equation to solve for $x$. Example 6 further demonstrates this procedure.

### *EXAMPLE 6* Using Back-Substitution to Solve a System in Row-Echelon Form

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 && \text{Equation 1} \\ y + 3z &= 5 && \text{Equation 2} \\ z &= 2 && \text{Equation 3} \end{aligned}$$

*SOLUTION* From Equation 3 you already know the value of $z$. To solve for $y$, substitute $z = 2$ into Equation 2 to obtain

$$\begin{aligned} y + 3(2) &= 5 && \textbf{Substitute } z = 2. \\ y &= -1. && \textbf{Solve for } y. \end{aligned}$$

Finally, substitute $y = -1$ and $z = 2$ in Equation 1 to obtain

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 && \textbf{Substitute } y = -1,\ z = 2. \\ x &= 1. && \textbf{Solve for } x. \end{aligned}$$

The solution is $x = 1$, $y = -1$, and $z = 2$.

Two systems of linear equations are called **equivalent** if they have precisely the same solution set. To solve a system that is not in row-echelon form, first change it to an *equivalent* system that is in row-echelon form by using the operations listed below.

### Operations That Lead to Equivalent Systems of Equations

Each of the following operations on a system of linear equations produces an *equivalent* system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

Rewriting a system of linear equations in row-echelon form usually involves a *chain* of equivalent systems, each of which is obtained by using one of the three basic operations. This process is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss (1777–1855).

### EXAMPLE 7 Using Elimination to Rewrite a System in Row-Echelon Form

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{+ 3z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

*SOLUTION* Although there are several ways to begin, you want to use a systematic procedure that can be applied easily to large systems. Work from the upper left corner of the system, saving the $x$ in the upper left position and eliminating the other $x$'s from the first column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \quad \longleftarrow \\ 2x - 5y + 5z &= 17 \end{aligned}$$

**Adding the first equation to the second equation produces a new second equation.**

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \quad \longleftarrow \end{aligned}$$

**Adding $-2$ times the first equation to the third equation produces a new third equation.**

Now that everything but the first $x$ has been eliminated from the first column, work on the second column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Adding the second equation to the third equation produces a new third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

This is the same system you solved in Example 6, and, as in that example, the solution is

$$x = 1, \qquad y = -1, \qquad z = 2.$$

Each of the three equations in Example 7 is represented in a three-dimensional coordinate system by a plane. Because the unique solution of the system is the point

$$(x, y, z) = (1, -1, 2),$$

the three planes intersect at the point represented by these coordinates, as shown in Figure 1.2.

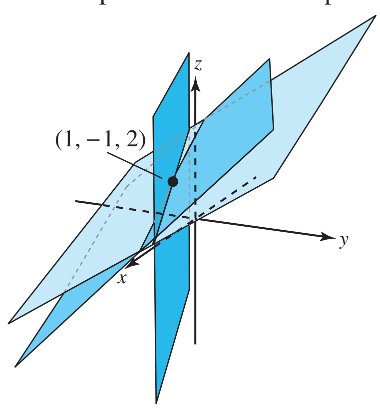



Figure 1.2

**Technology Note**

Many graphing utilities and computer software programs can solve a system of $m$ linear equations in $n$ variables. Try solving the system in Example 7 using the simultaneous equation solver feature of your graphing utility or computer software program. Keystrokes and programming syntax for these utilities/programs applicable to Example 7 are provided in the **Online Technology Guide,** available at *college.hmco.com/pic/larsonELA6e*.

Because many steps are required to solve a system of linear equations, it is very easy to make errors in arithmetic. It is suggested that you develop the habit of *checking your solution by substituting it into each equation in the original system.* For instance, in Example 7, you can check the solution $x = 1$, $y = -1$, and $z = 2$ as follows.

$$\begin{aligned}
&\text{Equation 1:} && (1) - 2(-1) + 3(2) = 9\\
&\text{Equation 2:} && -(1) + 3(-1) = -4\\
&\text{Equation 3:} && 2(1) - 5(-1) + 5(2) = 17
\end{aligned}$$

**Substitute solution in each equation of the original system.**

Each of the systems in Examples 5, 6, and 7 has exactly one solution. You will now look at an inconsistent system—one that has no solution. The key to recognizing an inconsistent system is reaching a false statement such as $0 = 7$ at some stage of the elimination process. This is demonstrated in Example 8.

## EXAMPLE 8 An Inconsistent System

Solve the system.

$$\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
2x_1 - x_2 - 2x_3 &= 2\\
x_1 + 2x_2 - 3x_3 &= -1
\end{aligned}$$

*SOLUTION*

$$\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
5x_2 - 4x_3 &= 0\\
x_1 + 2x_2 - 3x_3 &= -1
\end{aligned}$$

**Adding $-2$ times the first equation to the second equation produces a new second equation.**

$$\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
5x_2 - 4x_3 &= 0\\
5x_2 - 4x_3 &= -2
\end{aligned}$$

**Adding $-1$ times the first equation to the third equation produces a new third equation.**

(Another way of describing this operation is to say that you *subtracted* the first equation from the third equation to produce a new third equation.) Now, continuing the elimination process, add $-1$ times the second equation to the third equation to produce a new third equation.

$$\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
5x_2 - 4x_3 &= 0\\
0 &= -2
\end{aligned}$$

**Adding $-1$ times the second equation to the third equation produces a new third equation.**

Because the third "equation" is a false statement, this system has no solution. Moreover, because this system is equivalent to the original system, you can conclude that the original system also has no solution.

As in Example 7, the three equations in Example 8 represent planes in a three-dimensional coordinate system. In this example, however, the system is inconsistent. So, the planes do not have a point in common, as shown in Figure 1.3 on the next page.

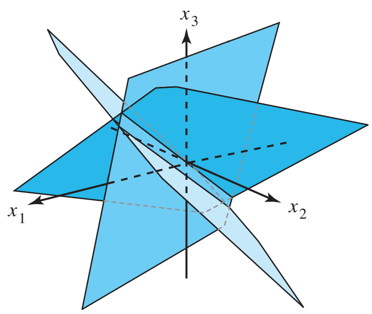



Figure 1.3

This section ends with an example of a system of linear equations that has an infinite number of solutions. You can represent the solution set for such a system in parametric form, as you did in Examples 2 and 3.

### EXAMPLE 9 A System with an Infinite Number of Solutions

Solve the system.

$$\begin{aligned} x_2 - x_3 &= 0 \\ x_1 \quad - 3x_3 &= -1 \\ -x_1 + 3x_2 \quad &= 1 \end{aligned}$$

*SOLUTION* Begin by rewriting the system in row-echelon form as follows.

$$\begin{aligned} x_1 \quad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ -x_1 + 3x_2 \quad &= 1 \end{aligned}$$

**The first two equations are interchanged.**

$$\begin{aligned} x_1 \quad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 3x_2 - 3x_3 &= 0 \end{aligned}$$

**Adding the first equation to the third equation produces a new third equation.**

$$\begin{aligned} x_1 \quad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

**Adding $-3$ times the second equation to the third equation eliminates the third equation.**

Because the third equation is unnecessary, omit it to obtain the system shown below.

$$\begin{aligned} x_1 \quad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \end{aligned}$$

To represent the solutions, choose $x_3$ to be the free variable and represent it by the parameter $t$. Because $x_2 = x_3$ and $x_1 = 3x_3 - 1$, you can describe the solution set as

$$x_1 = 3t - 1, \qquad x_2 = t, \qquad x_3 = t, \quad t \text{ is any real number.}$$

**Discovery**

*Graph the two lines represented by the system of equations.*

$$\begin{aligned} x - 2y &= 1 \\ -2x + 3y &= -3 \end{aligned}$$

*You can use Gaussian elimination to solve this system as follows.*

$$\begin{aligned} x - 2y &= 1 \\ -1y &= -1 \end{aligned} \qquad \begin{aligned} x - 2y &= 1 \\ y &= 1 \end{aligned} \qquad \begin{aligned} x &= 3 \\ y &= 1 \end{aligned}$$

*Graph the system of equations you obtain at each step of this process. What do you observe about the lines? You are asked to repeat this graphical analysis for other systems in Exercises 91 and 92.*

## SECTION 1.1 Exercises

In Exercises 1–6, determine whether the equation is linear in the variables $x$ and $y$.

**1.** $2x - 3y = 4$

**2.** $3x - 4xy = 0$

**3.** $\frac{3}{y} + \frac{2}{x} - 1 = 0$

**4.** $x^2 + y^2 = 4$

**5.** $2 \sin x - y = 14$

**6.** $(\sin 2)x - y = 14$

In Exercises 7–10, find a parametric representation of the solution set of the linear equation.

**7.** $2x - 4y = 0$

**8.** $3x - \frac{1}{2}y = 9$

**9.** $x + y + z = 1$

**10.** $13x_1 - 26x_2 + 39x_3 = 13$

In Exercises 11–16, use back-substitution to solve the system.

**11.** $\begin{aligned} x_1 - x_2 &= 2 \\ x_2 &= 3 \end{aligned}$

**12.** $\begin{aligned} 2x_1 - 4x_2 &= 6 \\ 3x_2 &= 9 \end{aligned}$

**13.** $\begin{aligned} -x + y - z &= 0 \\ 2y + z &= 3 \\ \tfrac{1}{2}z &= 0 \end{aligned}$

**14.** $\begin{aligned} x - y \phantom{+ z} &= 4 \\ 2y + z &= 6 \\ 3z &= 6 \end{aligned}$

**15.** $\begin{aligned} 5x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 \phantom{+ x_3} &= 0 \end{aligned}$

**16.** $\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 &= 0 \end{aligned}$

In Exercises 17–30, graph each system of equations as a pair of lines in the $xy$-plane. Solve each system and interpret your answer.

**17.** $\begin{aligned} 2x + y &= 4 \\ x - y &= 2 \end{aligned}$

**18.** $\begin{aligned} x + 3y &= 2 \\ -x + 2y &= 3 \end{aligned}$

**19.** $\begin{aligned} x - y &= 1 \\ -2x + 2y &= 5 \end{aligned}$

**20.** $\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{3}y &= 1 \\ -2x + \tfrac{4}{3}y &= -4 \end{aligned}$

**21.** $\begin{aligned} 3x - 5y &= 7 \\ 2x + y &= 9 \end{aligned}$

**22.** $\begin{aligned} -x + 3y &= 17 \\ 4x + 3y &= 7 \end{aligned}$

**23.** $\begin{aligned} 2x - y &= 5 \\ 5x - y &= 11 \end{aligned}$

**24.** $\begin{aligned} x - 5y &= 21 \\ 6x + 5y &= 21 \end{aligned}$

**25.** $\begin{aligned} \frac{x+3}{4} + \frac{y-1}{3} &= 1 \\ 2x - y &= 12 \end{aligned}$

**26.** $\begin{aligned} \frac{x-1}{2} + \frac{y+2}{3} &= 4 \\ x - 2y &= 5 \end{aligned}$

**27.** $\begin{aligned} 0.05x - 0.03y &= 0.07 \\ 0.07x + 0.02y &= 0.16 \end{aligned}$

**28.** $\begin{aligned} 0.2x - 0.5y &= -27.8 \\ 0.3x + 0.4y &= 68.7 \end{aligned}$

**29.** $\begin{aligned} \frac{x}{4} + \frac{y}{6} &= 1 \\ x - y &= 3 \end{aligned}$

**30.** $\begin{aligned} \frac{2}{3}x + \frac{1}{6}y &= \frac{2}{3} \\ 4x + y &= 4 \end{aligned}$

In Exercises 31–36, complete the following set of tasks for each system of equations.

(a) Use a graphing utility to graph the equations in the system.
(b) Use the graphs to determine whether the system is consistent or inconsistent.
(c) If the system is consistent, approximate the solution.
(d) Solve the system algebraically.
(e) Compare the solution in part (d) with the approximation in part (c). What can you conclude?

**31.** $\begin{aligned} -3x - y &= 3 \\ 6x + 2y &= 1 \end{aligned}$

**32.** $\begin{aligned} 4x - 5y &= 3 \\ -8x + 10y &= 14 \end{aligned}$

**33.** $\begin{aligned} 2x - 8y &= 3 \\ \tfrac{1}{2}x + y &= 0 \end{aligned}$

**34.** $\begin{aligned} 9x - 4y &= 5 \\ \tfrac{1}{2}x + \tfrac{1}{3}y &= 0 \end{aligned}$

**35.** $\begin{aligned} 4x - 8y &= 9 \\ 0.8x - 1.6y &= 1.8 \end{aligned}$

**36.** $\begin{aligned} -5.3x + 2.1y &= 1.25 \\ 15.9x - 6.3y &= -3.75 \end{aligned}$

The symbol indicates an exercise in which you are instructed to use a graphing utility or a symbolic computer software program.

In Exercises 37–56, solve the system of linear equations.

**37.** $$\begin{aligned} x_1 - x_2 &= 0 \\ 3x_1 - 2x_2 &= -1 \end{aligned}$$

**38.** $$\begin{aligned} 3x + 2y &= 2 \\ 6x + 4y &= 14 \end{aligned}$$

**39.** $$\begin{aligned} 2u + v &= 120 \\ u + 2v &= 120 \end{aligned}$$

**40.** $$\begin{aligned} x_1 - 2x_2 &= 0 \\ 6x_1 + 2x_2 &= 0 \end{aligned}$$

**41.** $$\begin{aligned} 9x - 3y &= -1 \\ \tfrac{1}{5}x + \tfrac{2}{5}y &= -\tfrac{1}{3} \end{aligned}$$

**42.** $$\begin{aligned} \tfrac{2}{3}x_1 + \tfrac{1}{6}x_2 &= 0 \\ 4x_1 + x_2 &= 0 \end{aligned}$$

**43.** $$\begin{aligned} \frac{x-1}{2} + \frac{y+2}{3} &= 4 \\ x - 2y &= 5 \end{aligned}$$

**44.** $$\begin{aligned} \frac{x_1+3}{4} + \frac{x_2-1}{3} &= 1 \\ 2x_1 - x_2 &= 12 \end{aligned}$$

**45.** $$\begin{aligned} 0.02x_1 - 0.05x_2 &= -0.19 \\ 0.03x_1 + 0.04x_2 &= 0.52 \end{aligned}$$

**46.** $$\begin{aligned} 0.05x_1 - 0.03x_2 &= 0.21 \\ 0.07x_1 + 0.02x_2 &= 0.17 \end{aligned}$$

**47.** $$\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ 3x \qquad - z &= 0 \end{aligned}$$

**48.** $$\begin{aligned} x + y + z &= 2 \\ -x + 3y + 2z &= 8 \\ 4x + y \qquad &= 4 \end{aligned}$$

**49.** $$\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= 1 \\ x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - 3x_2 + 6x_3 &= 8 \end{aligned}$$

**50.** $$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$$

**51.** $$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 4 \\ 4x_1 \qquad + 2x_3 &= 10 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$$

**52.** $$\begin{aligned} x_1 \qquad + 4x_3 &= 13 \\ 4x_1 - 2x_2 + x_3 &= 7 \\ 2x_1 - 2x_2 - 7x_3 &= -19 \end{aligned}$$

**53.** $$\begin{aligned} x - 3y + 2z &= 18 \\ 5x - 15y + 10z &= 18 \end{aligned}$$

**54.** $$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 2 \\ 3x_1 + 2x_2 - x_3 &= -2 \end{aligned}$$

**55.** $$\begin{aligned} x + y + z + w &= 6 \\ 2x + 3y \qquad - w &= 0 \\ -3x + 4y + z + 2w &= 4 \\ x + 2y - z + w &= 0 \end{aligned}$$

**56.** $$\begin{aligned} x_1 \qquad\qquad + 3x_4 &= 4 \\ 2x_2 - x_3 - x_4 &= 0 \\ 3x_2 \qquad - 2x_4 &= 1 \\ 2x_1 - x_2 + 4x_3 \qquad &= 5 \end{aligned}$$

In Exercises 57–64, use a computer software program or graphing utility to solve the system of linear equations.

**57.** $$\begin{aligned} x_1 + 0.5x_2 + 0.33x_3 + 0.25x_4 &= 1.1 \\ 0.5x_1 + 0.33x_2 + 0.25x_3 + 0.21x_4 &= 1.2 \\ 0.33x_1 + 0.25x_2 + 0.2x_3 + 0.17x_4 &= 1.3 \\ 0.25x_1 + 0.2x_2 + 0.17x_3 + 0.14x_4 &= 1.4 \end{aligned}$$

The symbol HM indicates that electronic data sets for these exercises are available at *college.hmco.com/pic/larsonELA6e*. These data sets are compatible with each of the following technologies: MATLAB, *Mathematica*, Maple, Derive, TI-83/TI-83 Plus, TI-84/TI-84 Plus, TI-86, TI-89, TI-92, and TI-92 Plus.

**58.** $$\begin{aligned} 0.1x - 2.5y + 1.2z - 0.75w &= 108 \\ 2.4x + 1.5y - 1.8z + 0.25w &= -81 \\ 0.4x - 3.2y + 1.6z - 1.4w &= 148.8 \\ 1.6x + 1.2y - 3.2z + 0.6w &= -143.2 \end{aligned}$$

**59.** $$\begin{aligned} 123.5x + 61.3y - 32.4z &= -262.74 \\ 54.7x - 45.6y + 98.2z &= 197.4 \\ 42.4x - 89.3y + 12.9z &= 33.66 \end{aligned}$$

**60.** $$\begin{aligned} 120.2x + 62.4y - 36.5z &= 258.64 \\ 56.8x - 42.8y + 27.3z &= -71.44 \\ 88.1x + 72.5y - 28.5z &= 225.88 \end{aligned}$$

**61.** $$\begin{aligned} \tfrac{1}{2}x_1 - \tfrac{3}{7}x_2 + \tfrac{2}{9}x_3 &= \tfrac{349}{630} \\ \tfrac{2}{3}x_1 + \tfrac{4}{9}x_2 - \tfrac{2}{5}x_3 &= -\tfrac{19}{45} \\ \tfrac{4}{5}x_1 - \tfrac{1}{8}x_2 + \tfrac{4}{3}x_3 &= \tfrac{139}{150} \end{aligned}$$

**62.** $$\begin{aligned} \tfrac{1}{4}x_1 - \tfrac{3}{5}x_2 + \tfrac{1}{3}x_3 &= \tfrac{43}{60} \\ \tfrac{2}{5}x_1 + \tfrac{1}{4}x_2 - \tfrac{5}{6}x_3 &= -\tfrac{331}{600} \\ \tfrac{3}{4}x_1 - \tfrac{2}{5}x_2 + \tfrac{1}{5}x_3 &= \tfrac{81}{100} \end{aligned}$$

**63.** $$\begin{aligned} \tfrac{1}{8}x - \tfrac{1}{7}y + \tfrac{1}{6}z - \tfrac{1}{5}w &= 1 \\ \tfrac{1}{7}x + \tfrac{1}{6}y - \tfrac{1}{5}z + \tfrac{1}{4}w &= 1 \\ \tfrac{1}{6}x - \tfrac{1}{5}y + \tfrac{1}{4}z - \tfrac{1}{3}w &= 1 \\ \tfrac{1}{5}x + \tfrac{1}{4}y - \tfrac{1}{3}z + \tfrac{1}{2}w &= 1 \end{aligned}$$

**64.** $$\begin{aligned} \tfrac{1}{8}x + \tfrac{1}{7}y - \tfrac{1}{6}z + \tfrac{1}{5}w &= 1 \\ \tfrac{1}{7}x - \tfrac{1}{6}y + \tfrac{1}{5}z - \tfrac{1}{4}w &= 1 \\ \tfrac{1}{6}x + \tfrac{1}{5}y - \tfrac{1}{4}z + \tfrac{1}{3}w &= 1 \\ \tfrac{1}{5}x - \tfrac{1}{4}y + \tfrac{1}{3}z - \tfrac{1}{2}w &= 1 \end{aligned}$$

In Exercises 65–68, state why each system of equations must have at least one solution. Then solve the system and determine if it has exactly one solution or an infinite number of solutions.

**65.** $$\begin{aligned} 4x + 3y + 17z &= 0 \\ 5x + 4y + 22z &= 0 \\ 4x + 2y + 19z &= 0 \end{aligned}$$

**66.** $$\begin{aligned} 2x + 3y \qquad &= 0 \\ 4x + 3y - z &= 0 \\ 8x + 3y + 3z &= 0 \end{aligned}$$

**67.** $$\begin{aligned} 5x + 5y - z &= 0 \\ 10x + 5y + 2z &= 0 \\ 5x + 15y - 9z &= 0 \end{aligned}$$

**68.** $$\begin{aligned} 12x + 5y + z &= 0 \\ 12x + 4y - z &= 0 \end{aligned}$$

**True or False?** In Exercises 69 and 70, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

**69.** (a) A system of one linear equation in two variables is always consistent.

(b) A system of two linear equations in three variables is always consistent.

(c) If a linear system is consistent, then it has an infinite number of solutions.

**70.** (a) A system of linear equations can have exactly two solutions.
(b) Two systems of linear equations are equivalent if they have the same solution set.
(c) A system of three linear equations in two variables is always inconsistent.

**71.** Find a system of two equations in two variables, $x_1$ and $x_2$, that has the solution set given by the parametric representation $x_1 = t$ and $x_2 = 3t - 4$, where $t$ is any real number. Then show that the solutions to your system can also be written as

$$x_1 = \frac{4}{3} + \frac{t}{3} \text{ and } x_2 = t.$$

**72.** Find a system of two equations in three variables, $x_1$, $x_2$, and $x_3$, that has the solution set given by the parametric representation

$$x_1 = t, \quad x_2 = s, \quad \text{and} \quad x_3 = 3 + s - t,$$

where $s$ and $t$ are any real numbers. Then show that the solutions to your system can also be written as

$$x_1 = 3 + s - t, \; x_2 = s, \; \text{and} \; x_3 = t.$$

In Exercises 73–76, solve the system of equations by letting $A = 1/x$, $B = 1/y$, and $C = 1/z$.

**73.** $$\begin{aligned} \frac{12}{x} - \frac{12}{y} &= 7 \\ \frac{3}{x} + \frac{4}{y} &= 0 \end{aligned}$$

**74.** $$\begin{aligned} \frac{2}{x} + \frac{3}{y} &= 0 \\ \frac{3}{x} - \frac{4}{y} &= -\frac{25}{6} \end{aligned}$$

**75.** $$\begin{aligned} \frac{2}{x} + \frac{1}{y} - \frac{3}{z} &= 4 \\ \frac{4}{x} \qquad + \frac{2}{z} &= 10 \\ -\frac{2}{x} + \frac{3}{y} - \frac{13}{z} &= -8 \end{aligned}$$

**76.** $$\begin{aligned} \frac{2}{x} + \frac{1}{y} - \frac{2}{z} &= 5 \\ \frac{3}{x} - \frac{4}{y} \qquad &= -1 \\ \frac{2}{x} + \frac{1}{y} + \frac{3}{z} &= 0 \end{aligned}$$

In Exercises 77 and 78, solve the system of linear equations for $x$ and $y$.

**77.** $$\begin{aligned} (\cos\theta)x + (\sin\theta)y &= 1 \\ (-\sin\theta)x + (\cos\theta)y &= 0 \end{aligned}$$

**78.** $$\begin{aligned} (\cos\theta)x + (\sin\theta)y &= 1 \\ (-\sin\theta)x + (\cos\theta)y &= 1 \end{aligned}$$

In Exercises 79–84, determine the value(s) of $k$ such that the system of linear equations has the indicated number of solutions.

**79.** An infinite number of solutions
$$\begin{aligned} 4x + ky &= 6 \\ kx + y &= -3 \end{aligned}$$

**80.** An infinite number of solutions
$$\begin{aligned} kx + y &= 4 \\ 2x - 3y &= -12 \end{aligned}$$

**81.** Exactly one solution
$$\begin{aligned} x + ky &= 0 \\ kx + y &= 0 \end{aligned}$$

**82.** No solution
$$\begin{aligned} x + ky &= 2 \\ kx + y &= 4 \end{aligned}$$

**83.** No solution
$$\begin{aligned} x + 2y + kz &= 6 \\ 3x + 6y + 8z &= 4 \end{aligned}$$

**84.** Exactly one solution
$$\begin{aligned} kx + 2ky + 3kz &= 4k \\ x + y + z &= 0 \\ 2x - y + z &= 1 \end{aligned}$$

**85.** Determine the values of $k$ such that the system of linear equations does not have a unique solution.
$$\begin{aligned} x + y + kz &= 3 \\ x + ky + z &= 2 \\ kx + y + z &= 1 \end{aligned}$$

**86.** Find values of $a$, $b$, and $c$ such that the system of linear equations has (a) exactly one solution, (b) an infinite number of solutions, and (c) no solution.
$$\begin{aligned} x + 5y + z &= 0 \\ x + 6y - z &= 0 \\ 2x + ay + bz &= c \end{aligned}$$

**87.** **Writing** Consider the system of linear equations in $x$ and $y$.
$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \\ a_3x + b_3y &= c_3 \end{aligned}$$
Describe the graphs of these three equations in the $xy$-plane when the system has (a) exactly one solution, (b) an infinite number of solutions, and (c) no solution.

**88.** **Writing** Explain why the system of linear equations in Exercise 87 must be consistent if the constant terms $c_1$, $c_2$, and $c_3$ are all zero.

**89.** Show that if $ax^2 + bx + c = 0$ for all $x$, then $a = b = c = 0$.

**90.** Consider the system of linear equations in $x$ and $y$.
$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$
Under what conditions will the system have exactly one solution?

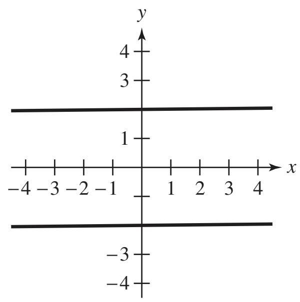
In Exercises 91 and 92, sketch the lines determined by the system of linear equations. Then use Gaussian elimination to solve the system. At each step of the elimination process, sketch the corresponding lines. What do you observe about these lines?

**91.** $$\begin{aligned} x - 4y &= -3 \\ 5x - 6y &= 13 \end{aligned}$$

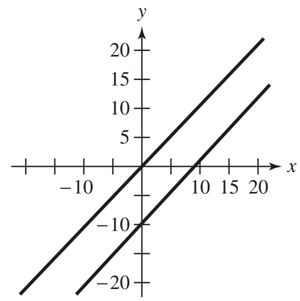
**92.** $$\begin{aligned} 2x - 3y &= 7 \\ -4x + 6y &= -14 \end{aligned}$$

**Writing** In Exercises 93 and 94, the graphs of two equations are shown and appear to be parallel. Solve the system of equations algebraically. Explain why the graphs are misleading.

**93.** $\begin{aligned} 100y - x &= \phantom{-}200 \\ 99y - x &= -198 \end{aligned}$




**94.** $\begin{aligned} 21x - 20y &= \phantom{12}0 \\ 13x - 12y &= 120 \end{aligned}$




## 1.2 Gaussian Elimination and Gauss-Jordan Elimination

In Section 1.1, Gaussian elimination was introduced as a procedure for solving a system of linear equations. In this section you will study this procedure more thoroughly, beginning with some definitions. The first is the definition of a **matrix.**

### Definition of a Matrix

If $m$ and $n$ are positive integers, then an $m \times n$ **matrix** is a rectangular array

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{bmatrix} a \\ a \\ a \\ \vdots \\ a \end{bmatrix}}\right\} m \text{ rows}$$

in which each **entry,** $a_{ij}$, of the matrix is a number. An $m \times n$ matrix (read "$m$ by $n$") has $m$ **rows** (horizontal lines) and $n$ **columns** (vertical lines).

REMARK: The plural of matrix is *matrices.* If each entry of a matrix is a *real* number, then the matrix is called a **real matrix.** Unless stated otherwise, all matrices in this text are assumed to be real matrices.

The entry $a_{ij}$ is located in the $i$th row and the $j$th column. The index $i$ is called the **row subscript** because it identifies the row in which the entry lies, and the index $j$ is called the **column subscript** because it identifies the column in which the entry lies.

A matrix with $m$ rows and $n$ columns (an $m \times n$ matrix) is said to be of **size** $m \times n$. If $m = n$, the matrix is called **square** of **order** $n$. For a square matrix, the entries $a_{11}$, $a_{22}$, $a_{33}$, . . . are called the **main diagonal** entries.

### EXAMPLE 1 Examples of Matrices

Each matrix has the indicated size.

(a) Size: $1 \times 1$

$$\begin{bmatrix} 2 \end{bmatrix}$$

(b) Size: $2 \times 2$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(c) Size: $1 \times 4$

$$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$$

(d) Size: $3 \times 2$

$$\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$$

One very common use of matrices is to represent systems of linear equations. The matrix derived from the coefficients and constant terms of a system of linear equations is called the **augmented matrix** of the system. The matrix containing only the coefficients of the system is called the **coefficient matrix** of the system. Here is an example.

*System*

$$\begin{aligned} x - 4y + 3z &= 5 \\ -x + 3y - z &= -3 \\ 2x \phantom{{}+3y} - 4z &= 6 \end{aligned}$$

*Augmented Matrix*

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix}$$

*Coefficient Matrix*

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

REMARK: Use 0 to indicate coefficients of zero. The coefficient of $y$ in the third equation is zero, so a 0 takes its place in the matrix. Also note the fourth column of constant terms in the augmented matrix.

When forming either the coefficient matrix or the augmented matrix of a system, you should begin by aligning the variables in the equations vertically.

*Given System*

$$\begin{aligned} x_1 + 3x_2 &= 9 \\ -x_2 + 4x_3 &= -2 \\ x_1 - 5x_3 &= 0 \end{aligned}$$

*Align Variables*

$$\begin{array}{rrrcr} x_1 & +\,3x_2 & & = & 9 \\ & -x_2 & +\,4x_3 & = & -2 \\ x_1 & & -\,5x_3 & = & 0 \end{array}$$

*Augmented Matrix*

$$\begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 1 & 0 & -5 & 0 \end{bmatrix}$$

## Elementary Row Operations

In the previous section you studied three operations that can be used on a system of linear equations to produce equivalent systems.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

In matrix terminology these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are said to be **row-equivalent** if one can be obtained from the other by a finite sequence of elementary row operations.

## Elementary Row Operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Although elementary row operations are simple to perform, they involve a lot of arithmetic. Because it is easy to make a mistake, you should get in the habit of noting the elementary row operation performed in each step so that it is easier to check your work.

Because solving some systems involves several steps, it is helpful to use a shorthand method of notation to keep track of each elementary row operation you perform. This notation is introduced in the next example.

### *EXAMPLE 2* Elementary Row Operations

(a) Interchange the first and second rows.

*Original Matrix* — *New Row-Equivalent Matrix* — *Notation*

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

(b) Multiply the first row by $\frac{1}{2}$ to produce a new first row.

*Original Matrix* — *New Row-Equivalent Matrix* — *Notation*

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \qquad \left(\tfrac{1}{2}\right)R_1 \rightarrow R_1$$

(c) Add $-2$ times the first row to the third row to produce a new third row.

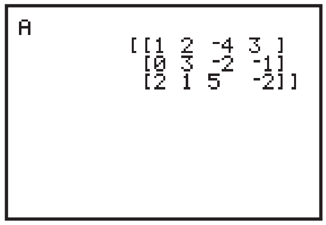
*Original Matrix* — *New Row-Equivalent Matrix* — *Notation*

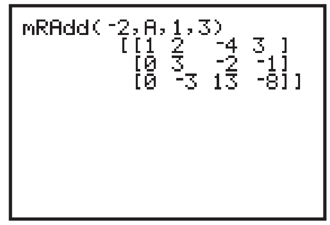
$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix} \qquad R_3 + (-2)R_1 \rightarrow R_3$$

REMARK: Notice in Example 2(c) that adding $-2$ times row 1 to row 3 does not change row 1.

**Technology Note**

Many graphing utilities and computer software programs can perform elementary row operations on matrices. If you are using a graphing utility, your screens for Example 2(c) may look like those shown below. Keystrokes and programming syntax for these utilities/programs applicable to Example 2(c) are provided in the **Online Technology Guide,** available at *college.hmco.com/pic/larsonELA6e*.




In Example 7 in Section 1.1, you used Gaussian elimination with back-substitution to solve a system of linear equations. You will now learn the matrix version of Gaussian elimination. The two methods used in the next example are essentially the same. The basic difference is that with the matrix method there is no need to rewrite the variables over and over again.

## EXAMPLE 3 Using Elementary Row Operations to Solve a System

*Linear System*

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{{}+5z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

*Associated Augmented Matrix*

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

Add the first equation to the second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Add the first row to the second row to produce a new second row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix} \quad R_2 + R_1 \rightarrow R_2$$

Add $-2$ times the first equation to the third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

Add $-2$ times the first row to the third row to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \quad R_3 + (-2)R_1 \rightarrow R_3$$

Add the second equation to the third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Add the second row to the third row to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad R_3 + R_2 \rightarrow R_3$$

Multiply the third equation by $\frac{1}{2}$.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiply the third row by $\frac{1}{2}$ to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \left(\tfrac{1}{2}\right)R_3 \rightarrow R_3$$

Now you can use back-substitution to find the solution, as in Example 6 in Section 1.1. The solution is $x = 1$, $y = -1$, and $z = 2$.

The last matrix in Example 3 is said to be in **row-echelon form.** The term *echelon* refers to the stair-step pattern formed by the nonzero elements of the matrix. To be in row-echelon form, a matrix must have the properties listed below.

### Definition of Row-Echelon Form of a Matrix

A matrix in **row-echelon form** has the following properties.

1. All rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

REMARK: A matrix in row-echelon form is in **reduced row-echelon form** if every column that has a leading 1 has zeros in every position above and below its leading 1.

### EXAMPLE 4 Row-Echelon Form

The matrices below are in row-echelon form.

(a) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The matrices shown in parts (b) and (d) are in *reduced* row-echelon form. The matrices listed below are not in row-echelon form.

(e) $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

**Technology Note**

Use a graphing utility or a computer software program to find the reduced row-echelon form of the matrix in part (f) of Example 4. Keystrokes and programming syntax for these utilities/programs applicable to Example 4(f) are provided in the **Online Technology Guide,** available at *college.hmco.com/pic/larsonELA6e.*

It can be shown that every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 4 you could change the matrix in part (e) to row-echelon form by multiplying the second row in the matrix by $\frac{1}{2}$.

The method of using Gaussian elimination with back-substitution to solve a system is as follows.

REMARK: For keystrokes and programming syntax regarding specific graphing utilities and computer software programs involving Example 4(f), please visit *college.hmco.com/pic/larsonELA6e*. Similar exercises and projects are also available on the website.

### Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Gaussian elimination with back-substitution works well as an algorithmic method for solving systems of linear equations. For this algorithm, the order in which the elementary row operations are performed is important. Move from *left to right by columns*, changing all entries directly below the leading 1's to zeros.

**EXAMPLE 5** Gaussian Elimination with Back-Substitution

Solve the system.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 \phantom{- 3x_4} &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

*SOLUTION* The augmented matrix for this system is

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}.$$

Obtain a leading 1 in the upper left corner and zeros elsewhere in the first column.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

The first two rows are interchanged. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

Adding $-2$ times the first row to the third row produces a new third row. $R_3 + (-2)R_1 \rightarrow R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{bmatrix} \longleftarrow$$

**Adding $-1$ times the first row to the fourth row produces a new fourth row.** $R_4 + (-1)R_1 \to R_4$

Now that the first column is in the desired form, you should change the second column as shown below.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix} \longleftarrow$$

**Adding 6 times the second row to the fourth row produces a new fourth row.** $R_4 + (6)R_2 \to R_4$

To write the third column in proper form, multiply the third row by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix} \longleftarrow$$

**Multiplying the third row by $\frac{1}{3}$ produces a new third row.** $\left(\frac{1}{3}\right)R_3 \to R_3$

Similarly, to write the fourth column in proper form, you should multiply the fourth row by $-\frac{1}{13}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \longleftarrow$$

**Multiplying the fourth row by $-\frac{1}{13}$ produces a new fourth row.** $\left(-\frac{1}{13}\right)R_4 \to R_4$

The matrix is now in row-echelon form, and the corresponding system of linear equations is as shown below.

$$\begin{aligned} x_1 + 2x_2 - x_3 \qquad\quad &= \phantom{-}2 \\ x_2 + x_3 - 2x_4 &= -3 \\ x_3 - \phantom{2}x_4 &= -2 \\ x_4 &= \phantom{-}3 \end{aligned}$$

Using back-substitution, you can determine that the solution is

$$x_1 = -1, \qquad x_2 = 2, \qquad x_3 = 1, \qquad x_4 = 3.$$

When solving a system of linear equations, remember that it is possible for the system to have no solution. If during the elimination process you obtain a row with all zeros except for the last entry, it is unnecessary to continue the elimination process. You can simply conclude that the system is inconsistent and has no solution.

### EXAMPLE 6 A System with No Solution

Solve the system.

$$\begin{aligned} x_1 - \phantom{3}x_2 + 2x_3 &= 4 \\ x_1 \phantom{{}-3x_2} + \phantom{2}x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - \phantom{2}x_3 &= 1 \end{aligned}$$

*SOLUTION* The augmented matrix for this system is

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix}.$$

Apply Gaussian elimination to the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \qquad R_2 + (-1)R_1 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \qquad R_3 + (-2)R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad R_4 + (-3)R_1 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad R_3 + R_2 \rightarrow R_3$$

Note that the third row of this matrix consists of all zeros except for the last entry. This means that the original system of linear equations is *inconsistent.* You can see why this is true by converting back to a system of linear equations.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_2 - x_3 &= 2 \\ 0 &= -2 \\ 5x_2 - 7x_3 &= -11 \end{aligned}$$

Because the third "equation" is a false statement, the system has no solution.

**Discovery**

*Consider the system of linear equations.*

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 0 \\ -5x_1 + 6x_2 - 17x_3 &= 0 \\ 7x_1 - 4x_2 + 3x_3 &= 0 \end{aligned}$$

*Without doing any row operations, explain why this system is consistent.*

*The system below has more variables than equations. Why does it have an infinite number of solutions?*

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 + 2x_4 &= 0 \\ -5x_1 + 6x_2 - 17x_3 - 3x_4 &= 0 \\ 7x_1 - 4x_2 + 3x_3 + 13x_4 &= 0 \end{aligned}$$

## Gauss-Jordan Elimination

With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called **Gauss-Jordan elimination** after Carl Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. This procedure is demonstrated in the next example.

### EXAMPLE 7 Gauss-Jordan Elimination

Use Gauss-Jordan elimination to solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{+ 3z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

*SOLUTION* In Example 3, Gaussian elimination was used to obtain the row-echelon form

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now, rather than using back-substitution, apply elementary row operations until you obtain a matrix in reduced row-echelon form. To do this, you must produce zeros above each of the leading 1's, as follows.

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1 + (2)R_2 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_2 + (-3)R_3 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1 + (-9)R_3 \rightarrow R_1$$

Now, converting back to a system of linear equations, you have

$$\begin{aligned} x \qquad\qquad &= \phantom{-}1 \\ y \qquad &= -1 \\ z &= \phantom{-}2. \end{aligned}$$

The Gaussian and Gauss-Jordan elimination procedures employ an algorithmic approach easily adapted to computer use. These elimination procedures, however, make no effort to avoid fractional coefficients. For instance, if the system in Example 7 had been listed as

$$\begin{aligned} 2x - 5y + 5z &= \phantom{-}17 \\ x - 2y + 3z &= \phantom{-}9 \\ -x + 3y \qquad &= -4 \end{aligned}$$

both procedures would have required multiplying the first row by $\frac{1}{2}$, which would have introduced fractions in the first row. For hand computations, fractions can sometimes be avoided by judiciously choosing the order in which elementary row operations are applied.

REMARK: No matter which order you use, the reduced row-echelon form will be the same.

The next example demonstrates how Gauss-Jordan elimination can be used to solve a system with an infinite number of solutions.

## EXAMPLE 8 A System with an Infinite Number of Solutions

Solve the system of linear equations.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 \qquad &= 1 \end{aligned}$$

SOLUTION The augmented matrix of the system of linear equations is

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix}.$$

Using a graphing utility, a computer software program, or Gauss-Jordan elimination, you can verify that the reduced row-echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}.$$

The corresponding system of equations is

$$\begin{aligned} x_1 \qquad + 5x_3 &= \phantom{-}2 \\ x_2 - 3x_3 &= -1. \end{aligned}$$

Now, using the parameter $t$ to represent the *nonleading* variable $x_3$, you have

$$x_1 = 2 - 5t, \qquad x_2 = -1 + 3t, \qquad x_3 = t, \qquad \text{where } t \text{ is any real number.}$$

REMARK: Note that in Example 8 an arbitrary parameter was assigned to the nonleading variable $x_3$. You subsequently solved for the leading variables $x_1$ and $x_2$ as functions of $t$.

You have looked at two elimination methods for solving a system of linear equations. Which is better? To some degree the answer depends on personal preference. In real-life applications of linear algebra, systems of linear equations are usually solved by computer. Most computer programs use a form of Gaussian elimination, with special emphasis on ways to reduce rounding errors and minimize storage of data. Because the examples and exercises in this text are generally much simpler and focus on the underlying concepts, you will need to know both elimination methods.

## Homogeneous Systems of Linear Equations

As the final topic of this section, you will look at systems of linear equations in which each of the constant terms is zero. We call such systems **homogeneous.** For example, a homogeneous system of $m$ equations in $n$ variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

It is easy to see that a homogeneous system must have at least one solution. Specifically, if all variables in a homogeneous system have the value zero, then each of the equations must be satisfied. Such a solution is called **trivial** (or **obvious**). For instance, a homogeneous system of three equations in the three variables $x_1$, $x_2$, and $x_3$ must have $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ as a trivial solution.

### EXAMPLE 9 Solving a Homogeneous System of Linear Equations

Solve the system of linear equations.

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

*SOLUTION* Applying Gauss-Jordan elimination to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

yields the matrix shown below.

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \qquad R_2 + (-2)R_1 \to R_2$$

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad \left(\tfrac{1}{3}\right)R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad R_1 + R_2 \to R_1$$

The system of equations corresponding to this matrix is

$$\begin{aligned} x_1 \qquad + 2x_3 &= 0 \\ x_2 - \phantom{2}x_3 &= 0. \end{aligned}$$

Using the parameter $t = x_3$, the solution set is

$$x_1 = -2t, \qquad x_2 = t, \qquad x_3 = t, \quad t \text{ is any real number.}$$

This system of equations has an infinite number of solutions, one of which is the trivial solution (given by $t = 0$).

Example 9 illustrates an important point about homogeneous systems of linear equations. You began with two equations in three variables and discovered that the system has an infinite number of solutions. In general, a homogeneous system with fewer equations than variables has an infinite number of solutions.

**THEOREM 1.1**
**The Number of Solutions of a Homogeneous System**

Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have an infinite number of solutions.

A proof of Theorem 1.1 can be done using the same procedures as those used in Example 9, but for a general matrix.

## SECTION 1.2 Exercises

In Exercises 1–8, determine the size of the matrix.

**1.** $\begin{bmatrix} 1 & 2 & -4 \\ 3 & -4 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

**2.** $\begin{bmatrix} 2 & -1 & 4 & 2 \\ 1 & 0 & 2 & -6 \end{bmatrix}$

**3.** $\begin{bmatrix} 2 & -1 & -1 & 1 \\ -6 & 2 & 0 & 1 \end{bmatrix}$

**4.** $\begin{bmatrix} 1 & 2 & 3 & 0 & 1 \end{bmatrix}$

**5.** $\begin{bmatrix} 1 & 2 & 3 & 4 & -10 \end{bmatrix}$

**6.** $\begin{bmatrix} -1 \end{bmatrix}$

**7.** $\begin{bmatrix} 8 & 6 & 4 & 1 & 3 \\ 2 & 1 & -7 & 4 & 1 \\ 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 2 & 0 & 0 \end{bmatrix}$

**8.** $\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$

In Exercises 9–14, determine whether the matrix is in row-echelon form. If it is, determine whether it is also in reduced row-echelon form.

**9.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**10.** $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$

**11.** $\begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**12.** $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

**13.** $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$

**14.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 15–22, find the solution set of the system of linear equations represented by the augmented matrix.

**15.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

**16.** $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

**17.** $\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

**18.** $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**19.** $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

**20.** $\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & -2 & 1 & -2 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

**21.** $\begin{bmatrix} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

**22.** $\begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 23–36, solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

**23.** $\begin{aligned} x + 2y &= 7 \\ 2x + y &= 8 \end{aligned}$

**24.** $\begin{aligned} 2x + 6y &= 16 \\ -2x - 6y &= -16 \end{aligned}$

**25.** $\begin{aligned} -x + 2y &= 1.5 \\ 2x - 4y &= 3 \end{aligned}$

**26.** $\begin{aligned} 2x - y &= -0.1 \\ 3x + 2y &= 1.6 \end{aligned}$

**27.** $\begin{aligned} -3x + 5y &= -22 \\ 3x + 4y &= 4 \\ 4x - 8y &= 32 \end{aligned}$

**28.** $\begin{aligned} x + 2y &= 0 \\ x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$

**29.** $\begin{aligned} x_1 \phantom{{}+x_2} - 3x_3 &= -2 \\ 3x_1 + x_2 - 2x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 4 \end{aligned}$

**30.** $\begin{aligned} 2x_1 - x_2 + 3x_3 &= 24 \\ 2x_2 - x_3 &= 14 \\ 7x_1 - 5x_2 \phantom{{}-x_3} &= 6 \end{aligned}$

**31.** $\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 \phantom{{}+x_2} - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$

**32.** $\begin{aligned} 2x_1 \phantom{{}-3x_2} + 3x_3 &= 3 \\ 4x_1 - 3x_2 + 7x_3 &= 5 \\ 8x_1 - 9x_2 + 15x_3 &= 10 \end{aligned}$

**33.** $\begin{aligned} 4x + 12y - 7z - 20w &= 22 \\ 3x + 9y - 5z - 28w &= 30 \end{aligned}$

**34.** $\begin{aligned} x + 2y + z &= 8 \\ -3x - 6y - 3z &= -21 \end{aligned}$

**35.** $\begin{aligned} 3x + 3y + 12z &= 6 \\ x + y + 4z &= 2 \\ 2x + 5y + 20z &= 10 \\ -x + 2y + 8z &= 4 \end{aligned}$

**36.** $\begin{aligned} 2x + y - z + 2w &= -6 \\ 3x + 4y \phantom{{}-z} + w &= 1 \\ x + 5y + 2z + 6w &= -3 \\ 5x + 2y - z - w &= 3 \end{aligned}$

In Exercises 37–42, use a computer software program or graphing utility to solve the system of linear equations.

**37.** $\begin{aligned} x_1 - 2x_2 + 5x_3 - 3x_4 &= 23.6 \\ x_1 + 4x_2 - 7x_3 - 2x_4 &= 45.7 \\ 3x_1 - 5x_2 + 7x_3 + 4x_4 &= 29.9 \end{aligned}$

**38.** $\begin{aligned} 23.4x - 45.8y + 43.7z &= 87.2 \\ 86.4x + 12.3y - 56.9z &= 14.5 \\ 93.6x - 50.7y + 12.6z &= 44.4 \end{aligned}$

**39.** 
$$\begin{aligned} x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 &= 6\\ 3x_1 - 2x_2 + 4x_3 + 4x_4 + 12x_5 &= 14\\ x_2 - x_3 - x_4 - 3x_5 &= -3\\ 2x_1 - 2x_2 + 4x_3 + 5x_4 + 15x_5 &= 10\\ 2x_1 - 2x_2 + 4x_3 + 4x_4 + 13x_5 &= 13 \end{aligned}$$

**40.** 
$$\begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 + 2x_5 &= 9\\ 3x_1 + 3x_2 - x_3 + x_4 + x_5 &= 5\\ 2x_1 + 2x_2 - x_3 + x_4 - 2x_5 &= 1\\ 4x_1 + 4x_2 + x_3 \phantom{+ x_4} - 3x_5 &= 4\\ 8x_1 + 5x_2 - 2x_3 - x_4 + 2x_5 &= 3 \end{aligned}$$

**41.** 
$$\begin{aligned} 4x_1 - 3x_2 + x_3 - x_4 + 2x_5 - x_6 &= 8\\ x_1 - 2x_2 + x_3 - 3x_4 + x_5 - 4x_6 &= 4\\ 2x_1 + x_2 - 3x_3 + x_4 - 2x_5 + 5x_6 &= 2\\ -2x_1 + 3x_2 - x_3 + x_4 - x_5 + 2x_6 &= -7\\ x_1 - 3x_2 + x_3 - 2x_4 + x_5 - 2x_6 &= 9\\ 5x_1 - 4x_2 - x_3 - x_4 + 4x_5 + 5x_6 &= 9 \end{aligned}$$

**42.** 
$$\begin{aligned} x_1 + 2x_2 - 2x_3 + 2x_4 - x_5 + 3x_6 &= 0\\ 2x_1 - x_2 + 3x_3 + x_4 - 3x_5 + 2x_6 &= 17\\ x_1 + 3x_2 - 2x_3 + x_4 - 2x_5 - 3x_6 &= -5\\ 3x_1 - 2x_2 + x_3 - x_4 + 3x_5 - 2x_6 &= -1\\ -x_1 - 2x_2 + x_3 + 2x_4 - 2x_5 + 3x_6 &= 10\\ x_1 - 3x_2 + x_3 + 3x_4 - 2x_5 + x_6 &= 11 \end{aligned}$$

In Exercises 43–46, solve the homogeneous linear system corresponding to the coefficient matrix provided.

**43.** $\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0 \end{bmatrix}$ **44.** $\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 \end{bmatrix}$

**45.** $\begin{bmatrix} 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$ **46.** $\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}$

**47.** Consider the matrix $A = \begin{bmatrix} 1 & k & 2\\ -3 & 4 & 1 \end{bmatrix}$.

(a) If $A$ is the *augmented* matrix of a system of linear equations, determine the number of equations and the number of variables.

(b) If $A$ is the *augmented* matrix of a system of linear equations, find the value(s) of $k$ such that the system is consistent.

(c) If $A$ is the *coefficient* matrix of a *homogeneous* system of linear equations, determine the number of equations and the number of variables.

(d) If $A$ is the *coefficient* matrix of a *homogeneous* system of linear equations, find the value(s) of $k$ such that the system is consistent.

**48.** Consider the matrix $A = \begin{bmatrix} 2 & -1 & 3\\ -4 & 2 & k\\ 4 & -2 & 6 \end{bmatrix}$.

(a) If $A$ is the *augmented* matrix of a system of linear equations, determine the number of equations and the number of variables.

(b) If $A$ is the *augmented* matrix of a system of linear equations, find the value(s) of $k$ such that the system is consistent.

(c) If $A$ is the *coefficient* matrix of a *homogeneous* system of linear equations, determine the number of equations and the number of variables.

(d) If $A$ is the *coefficient* matrix of a *homogeneous* system of linear equations, find the value(s) of $k$ such that the system is consistent.

In Exercises 49 and 50, find values of $a$, $b$, and $c$ (if possible) such that the system of linear equations has (a) a unique solution, (b) no solution, and (c) an infinite number of solutions.

**49.** 
$$\begin{aligned} x + y \phantom{+ z} &= 2\\ y + z &= 2\\ x \phantom{+ y} + z &= 2\\ ax + by + cz &= 0 \end{aligned}$$

**50.** 
$$\begin{aligned} x + y \phantom{+ z} &= 0\\ y + z &= 0\\ x \phantom{+ y} + z &= 0\\ ax + by + cz &= 0 \end{aligned}$$

**51.** The system below has one solution: $x = 1$, $y = -1$, and $z = 2$.

$4x - 2y + 5z = 16$ **Equation 1**

$x + y \phantom{+ 2z} = 0$ **Equation 2**

$-x - 3y + 2z = 6$ **Equation 3**

Solve the systems provided by (a) Equations 1 and 2, (b) Equations 1 and 3, and (c) Equations 2 and 3. (d) How many solutions does each of these systems have?

**52.** Assume the system below has a unique solution.

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ **Equation 1**

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$ **Equation 2**

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$ **Equation 3**

Does the system composed of Equations 1 and 2 have a unique solution, no solution, or an infinite number of solutions?

In Exercises 53 and 54, find the unique reduced row-echelon matrix that is row-equivalent to the matrix provided.

**53.** $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$

**54.** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

**55. Writing** Describe all possible $2 \times 2$ reduced row-echelon matrices. Support your answer with examples.

**56. Writing** Describe all possible $3 \times 3$ reduced row-echelon matrices. Support your answer with examples.

**True or False?** In Exercises 57 and 58, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

**57.** (a) A $6 \times 3$ matrix has six rows.

(b) Every matrix is row-equivalent to a matrix in row-echelon form.

(c) If the row-echelon form of the augmented matrix of a system of linear equations contains the row $[1 \ \ 0 \ \ 0 \ \ 0 \ \ 0]$, then the original system is inconsistent.

(d) A homogeneous system of four linear equations in six variables has an infinite number of solutions.

**58.** (a) A $4 \times 7$ matrix has four columns.

(b) Every matrix has a unique reduced row-echelon form.

(c) A homogeneous system of four linear equations in four variables is always consistent.

(d) Multiplying a row of a matrix by a constant is one of the elementary row operations.

In Exercises 59 and 60, determine conditions on $a$, $b$, $c$, and $d$ such that the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

will be row-equivalent to the given matrix.

**59.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**60.** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

In Exercises 61 and 62, find all values of $\lambda$ (the Greek letter lambda) such that the homogeneous system of linear equations will have nontrivial solutions.

**61.** $\begin{aligned} (\lambda - 2)x + \phantom{(\lambda - 2)}y &= 0 \\ x + (\lambda - 2)y &= 0 \end{aligned}$

**62.** $\begin{aligned} (\lambda - 1)x + 2y &= 0 \\ x + \lambda y &= 0 \end{aligned}$

**63. Writing** Is it possible for a system of linear equations with fewer equations than variables to have no solution? If so, give an example.

**64. Writing** Does a matrix have a unique row-echelon form? Illustrate your answer with examples. Is the reduced row-echelon form unique?

**65. Writing** Consider the $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Perform the sequence of row operations.

(a) Add $(-1)$ times the second row to the first row.

(b) Add 1 times the first row to the second row.

(c) Add $(-1)$ times the second row to the first row.

(d) Multiply the first row by $(-1)$.

What happened to the original matrix? Describe, in general, how to interchange two rows of a matrix using only the second and third elementary row operations.

**66.** The augmented matrix represents a system of linear equations that has been reduced using Gauss-Jordan elimination. Write a system of equations with nonzero coefficients that is represented by the reduced matrix.

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are many correct answers.

**67. Writing** Describe the row-echelon form of an augmented matrix that corresponds to a system of linear equations that is inconsistent.

**68. Writing** Describe the row-echelon form of an augmented matrix that corresponds to a system of linear equations that has infinitely many solutions.

**69. Writing** In your own words, describe the difference between a matrix in row-echelon form and a matrix in reduced row-echelon form.

# 2 Matrices



**CHAPTER OBJECTIVES**

- Write a system of linear equations represented by a matrix, as well as write the matrix form of a system of linear equations.
- Write and solve a system of linear equations in the form $A\mathbf{x} = \mathbf{b}$.
- Use properties of matrix operations to solve matrix equations.
- Find the transpose of a matrix, the inverse of a matrix, and the inverse of a matrix product (if they exist).
- Factor a matrix into a product of elementary matrices, and determine when they are invertible.
- Find and use the *LU*-factorization of a matrix to solve a system of linear equations.
- Use a stochastic matrix to measure consumer preference.
- Use matrix multiplication to encode and decode messages.
- Use matrix algebra to analyze economic systems (Leontief input-output models).
- Use the method of least squares to find the least squares regression line for a set of data.

## 2.1 Operations with Matrices

In Section 1.2 you used matrices to solve systems of linear equations. Matrices, however, can be used to do much more than that. There is a rich mathematical theory of matrices, and its applications are numerous. This section and the next introduce some fundamentals of matrix theory.

It is standard mathematical convention to represent matrices in any one of the following three ways.

1. A matrix can be denoted by an uppercase letter such as

$$A, B, C, \dots .$$

2. A matrix can be denoted by a representative element enclosed in brackets, such as

$$[a_{ij}], [b_{ij}], [c_{ij}], \dots .$$

3. A matrix can be denoted by a rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

As mentioned in Chapter 1, the matrices in this text are primarily *real matrices.* That is, their entries contain real numbers.

Two matrices are said to be **equal** if their corresponding entries are equal.

**Definition of Equality of Matrices**

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if they have the same size $(m \times n)$ and

$$a_{ij} = b_{ij}$$

for $1 \le i \le m$ and $1 \le j \le n$.

## EXAMPLE 1 Equality of Matrices

Consider the four matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

$$C = [1 \quad 3], \qquad \text{and} \qquad D = \begin{bmatrix} 1 & 2 \\ x & 4 \end{bmatrix}.$$

Matrices $A$ and $B$ are **not** equal because they are of different sizes. Similarly, $B$ and $C$ are not equal. Matrices $A$ and $D$ are equal if and only if $x = 3$.

REMARK: The phrase "if and only if" means the statement is true in both directions. For example, "$p$ if and only if $q$" means that $p$ implies $q$ and $q$ implies $p$.

A matrix that has only one column, such as matrix $B$ in Example 1, is called a **column matrix** or **column vector.** Similarly, a matrix that has only one row, such as matrix $C$ in Example 1, is called a **row matrix** or **row vector.** Boldface lowercase letters are often used to designate column matrices and row matrices. For instance, matrix $A$ in Example 1 can be partitioned into the two column matrices $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, as follows.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \left[\begin{array}{c:c} 1 & 2 \\ 3 & 4 \end{array}\right] = [\mathbf{a}_1 \ \vdots \ \mathbf{a}_2]$$

## Matrix Addition

You can **add** two matrices (of the same size) by adding their corresponding entries.

**Definition of Matrix Addition**

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of size $m \times n$, then their **sum** is the $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of different sizes is undefined.

**EXAMPLE 2** **Addition of Matrices**

(a) $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) The sum of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

is undefined.

## Scalar Multiplication

When working with matrices, real numbers are referred to as **scalars.** You can multiply a matrix $A$ by a scalar $c$ by multiplying each entry in $A$ by $c$.

**Definition of Scalar Multiplication**

If $A = [a_{ij}]$ is an $m \times n$ matrix and $c$ is a scalar, then the **scalar multiple** of $A$ by $c$ is the $m \times n$ matrix given by

$$cA = [ca_{ij}].$$

You can use $-A$ to represent the scalar product $(-1)A$. If $A$ and $B$ are of the same size, $A - B$ represents the sum of $A$ and $(-1)B$. That is,

$$A - B = A + (-1)B. \qquad \text{Subtraction of matrices}$$

**EXAMPLE 3** **Scalar Multiplication and Matrix Subtraction**

For the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

find (a) $3A$, (b) $-B$, and (c) $3A - B$.

SOLUTION

(a) $$3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

(b) $$-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

(c) $$3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

## Matrix Multiplication

REMARK: It is often convenient to rewrite a matrix $B$ as $cA$ by factoring $c$ out of every entry in matrix $B$. For instance, the scalar $\frac{1}{2}$ has been factored out of the matrix below.

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ 5 & 1 \end{bmatrix}$$
$$B = cA$$

The third basic matrix operation is **matrix multiplication.** To see the usefulness of this operation, consider the following application in which matrices are helpful for organizing information.

A football stadium has three concession areas, located in the south, north, and west stands. The top-selling items are peanuts, hot dogs, and soda. Sales for a certain day are recorded in the first matrix below, and the prices (in dollars) of the three items are given in the second matrix.

| | *Number of Items Sold* | | | *Selling Price* | |
|---|---|---|---|---|---|
| | *Peanuts* | *Hot Dogs* | *Soda* | | |
| *South stand* | 120 | 250 | 305 | 2.00 | *Peanuts* |
| *North stand* | 207 | 140 | 419 | 3.00 | *Hot Dogs* |
| *West stand* | 29 | 120 | 190 | 2.75 | *Soda* |

To calculate the total sales of the three top-selling items at the south stand, you can multiply each entry in the first row of the matrix on the left by the corresponding entry in the price column matrix on the right and add the results. The south stand sales are

$$(120)(2.00) + (250)(3.00) + (305)(2.75) = \$1828.75. \qquad \text{South stand sales}$$

Similarly, you can calculate the sales for the other two stands as follows.

$$(207)(2.00) + (140)(3.00) + (419)(2.75) = \$1986.25 \qquad \text{North stand sales}$$
$$(29)(2.00) + (120)(3.00) + (190)(2.75) = \$940.50 \qquad \text{West stand sales}$$

The preceding computations are examples of matrix multiplication. You can write the product of the $3 \times 3$ matrix indicating the number of items sold and the $3 \times 1$ matrix indicating the selling prices as follows.

$$\begin{bmatrix} 120 & 250 & 305 \\ 207 & 140 & 419 \\ 29 & 120 & 190 \end{bmatrix} \begin{bmatrix} 2.00 \\ 3.00 \\ 2.75 \end{bmatrix} = \begin{bmatrix} 1828.75 \\ 1986.25 \\ 940.50 \end{bmatrix}$$

The product of these matrices is the $3 \times 1$ matrix giving the total sales for each of the three stands.

The general definition of the product of two matrices shown below is based on the ideas just developed. Although at first glance this definition may seem unusual, you will see that it has many practical applications.

### Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the **product** $AB$ is an $m \times p$ matrix

$$AB = [c_{ij}]$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj}.$$

This definition means that the entry in the $i$th row and the $j$th column of the product $AB$ is obtained by multiplying the entries in the $i$th row of $A$ by the corresponding entries in the $j$th column of $B$ and then adding the results. The next example illustrates this process.

### EXAMPLE 4 Finding the Product of Two Matrices

Find the product $AB$, where

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}.$$

*SOLUTION* First note that the product $AB$ is defined because $A$ has size $3 \times 2$ and $B$ has size $2 \times 2$. Moreover, the product $AB$ has size $3 \times 2$ and will take the form

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

To find $c_{11}$ (the entry in the first row and first column of the product), multiply corresponding entries in the first row of $A$ and the first column of $B$. That is,

$$c_{11} = (-1)(-3) + (3)(-4) = -9$$

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

Similarly, to find $c_{12}$, multiply corresponding entries in the first row of $A$ and the second column of $B$ to obtain

$$c_{12} = (-1)(2) + (3)(1) = 1$$

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

Continuing this pattern produces the results shown below.

$$\begin{aligned} c_{21} &= (4)(-3) + (-2)(-4) = -4 \\ c_{22} &= (4)(2) + (-2)(1) = 6 \\ c_{31} &= (5)(-3) + (0)(-4) = -15 \\ c_{32} &= (5)(2) + (0)(1) = 10 \end{aligned}$$

The product is

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}.$$

**HISTORICAL NOTE**

**Arthur Cayley (1821–1895)**

showed signs of mathematical genius at an early age, but ironically wasn't able to find a position as a mathematician upon graduating from college. Ultimately, however, Cayley made major contributions to linear algebra. To read about his work, visit *college.hmco.com/pic/larsonELA6e*.

Be sure you understand that for the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix. That is,

$$\underset{m \times n}{A} \quad \underset{n \times p}{B} \quad = \quad \underset{m \times p}{AB}.$$

equal

size of $AB$

So, the product $BA$ is not defined for matrices such as $A$ and $B$ in Example 4.

The general pattern for matrix multiplication is as follows. To obtain the element in the $i$th row and the $j$th column of the product $AB$, use the $i$th row of $A$ and the $j$th column of $B$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = c_{ij}$$

**Discovery**

*Let*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

*Calculate $A + B$ and $B + A$.*

*In general, is the operation of matrix addition commutative? Now calculate $AB$ and $BA$. Is matrix multiplication commutative?*

### EXAMPLE 5 Matrix Multiplication

(a) $$\underset{2\times 3}{\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}}$$

(b) $$\underset{2\times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}$$

(c) $$\underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

(d) $$\underset{1\times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} = \underset{1\times 1}{\begin{bmatrix} 1 \end{bmatrix}}$$

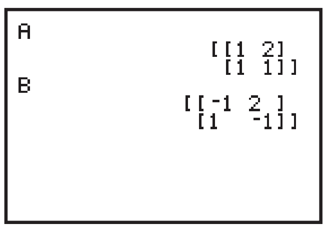
(e) $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}$

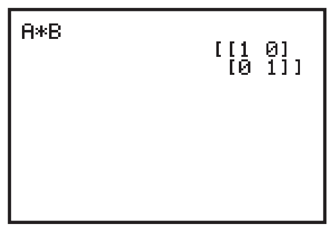
$\quad 3 \times 1 \qquad 1 \times 3 \qquad\qquad 3 \times 3$

REMARK: Note the difference between the two products in parts (d) and (e) of Example 5. In general, matrix multiplication is not commutative. It is usually not true that the product $AB$ is equal to the product $BA$. (See Section 2.2 for further discussion of the noncommutativity of matrix multiplication.)

**Technology Note**

Most graphing utilities and computer software programs can perform matrix addition, scalar multiplication, and matrix multiplication. If you are using a graphing utility, your screens for Example 5(c) may look like:




Keystrokes and programming syntax for these utilities/programs applicable to Example 5(c) are provided in the **Online Technology Guide,** available at *college.hmco.com/pic/larsonELA6e*.

## *Systems of Linear Equations*

One practical application of matrix multiplication is representing a system of linear equations. Note how the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

can be written as the matrix equation $A\mathbf{x} = \mathbf{b}$, where $A$ is the coefficient matrix of the system, and $\mathbf{x}$ and $\mathbf{b}$ are column matrices. You can write the system as

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} \underset{=}{=} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}.$$

**EXAMPLE 6** **Solving a System of Linear Equations**

Solve the matrix equation $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

*SOLUTION* As a system of linear equations, $A\mathbf{x} = \mathbf{0}$ looks like

$$\begin{aligned} x_1 - 2x_2 + \phantom{2}x_3 &= 0 \\ 2x_1 + 3x_2 - 2x_3 &= 0. \end{aligned}$$

Using Gauss-Jordan elimination on the augmented matrix of this system, you obtain

$$\begin{bmatrix} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & -\frac{4}{7} & 0 \end{bmatrix}.$$

So, the system has an infinite number of solutions. Here a convenient choice of a parameter is $x_3 = 7t$, and you can write the solution set as

$$x_1 = t, \qquad x_2 = 4t, \qquad x_3 = 7t, \qquad t \text{ is any real number.}$$

In matrix terminology, you have found that the matrix equation

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has an infinite number of solutions represented by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 4t \\ 7t \end{bmatrix} = t\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \qquad t \text{ is any scalar.}$$

That is, any scalar multiple of the column matrix on the right is a solution.

## Partitioned Matrices

The system $A\mathbf{x} = \mathbf{b}$ can be represented in a more convenient way by partitioning the matrices $A$ and $\mathbf{x}$ in the following manner. If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

are the coefficient matrix, the column matrix of unknowns, and the right-hand side, respectively, of the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$, then you can write

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{b}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \mathbf{b}$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \mathbf{b}.$$

In other words,

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b},$$

where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are the columns of the matrix $A$. The expression

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

is called a **linear combination** of the column matrices $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ with **coefficients** $x_1, x_2, \ldots, x_n$.

In general, the matrix product $A\mathbf{x}$ is a linear combination of the column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ that form the coefficient matrix $A$. Furthermore, the system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be expressed as such a linear combination, where the coefficients of the linear combination are a solution of the system.

## *EXAMPLE 7* Solving a System of Linear Equations

The linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 4x_1 + 5x_2 + 6x_3 &= 3 \\ 7x_1 + 8x_2 + 9x_3 &= 6 \end{aligned}$$

**Technology Note**

Many real-life applications of linear systems involve enormous numbers of equations and variables. For example, a flight crew scheduling problem for American Airlines required the manipulation of matrices with 837 rows and more than 12,750,000 columns. This application of *linear programming* required that the problem be partitioned into smaller pieces and then solved on a CRAY Y-MP supercomputer.

(Source: Very-Large Scale Linear Programming, A Case Study in Combining Interior Point and Simplex Methods, Bixby, Robert E., et al., *Operations Research,* 40, no. 5, 1992.)

can be rewritten as a matrix equation $A\mathbf{x} = \mathbf{b}$, as follows.

$$x_1\begin{bmatrix}1\\4\\7\end{bmatrix} + x_2\begin{bmatrix}2\\5\\8\end{bmatrix} + x_3\begin{bmatrix}3\\6\\9\end{bmatrix} = \begin{bmatrix}0\\3\\6\end{bmatrix}$$

Using Gaussian elimination, you can show that this system has an infinite number of solutions, one of which is $x_1 = 1$, $x_2 = 1$, $x_3 = -1$.

$$1\begin{bmatrix}1\\4\\7\end{bmatrix} + 1\begin{bmatrix}2\\5\\8\end{bmatrix} + (-1)\begin{bmatrix}3\\6\\9\end{bmatrix} = \begin{bmatrix}0\\3\\6\end{bmatrix}$$

That is, $\mathbf{b}$ can be expressed as a linear combination of the columns of $A$. This representation of one column vector in terms of others is a fundamental theme of linear algebra.

Just as you partitioned $A$ into columns and $\mathbf{x}$ into rows, it is often useful to consider an $m \times n$ matrix partitioned into smaller matrices. For example, the matrix on the left below can be partitioned as shown below at the right.

$$\begin{bmatrix}1 & 2 & 0 & 0\\3 & 4 & 0 & 0\\-1 & -2 & 2 & 1\end{bmatrix} \qquad \left[\begin{array}{cc|cc}1 & 2 & 0 & 0\\3 & 4 & 0 & 0\\ \hline -1 & -2 & 2 & 1\end{array}\right]$$

The matrix could also be partitioned into column matrices

$$\left[\begin{array}{c|c|c|c}1 & 2 & 0 & 0\\3 & 4 & 0 & 0\\-1 & -2 & 2 & 1\end{array}\right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$

or row matrices

$$\left[\begin{array}{cccc}1 & 2 & 0 & 0\\ \hline 3 & 4 & 0 & 0\\ \hline -1 & -2 & 2 & 1\end{array}\right] = \begin{bmatrix}\mathbf{r}_1\\\mathbf{r}_2\\\mathbf{r}_3\end{bmatrix}.$$

## SECTION 2.1 Exercises

In Exercises 1–6, find (a) $A + B$, (b) $A - B$, (c) $2A$, (d) $2A - B$, and (e) $B + \frac{1}{2}A$.

**1.** $A = \begin{bmatrix}1 & -1\\2 & -1\end{bmatrix}$, $B = \begin{bmatrix}2 & -1\\-1 & 8\end{bmatrix}$

**2.** $A = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$, $B = \begin{bmatrix}-3 & -2\\4 & 2\end{bmatrix}$

**3.** $A = \begin{bmatrix}6 & -1\\2 & 4\\-3 & 5\end{bmatrix}$, $B = \begin{bmatrix}1 & 4\\-1 & 5\\1 & 10\end{bmatrix}$

**4.** $A = \begin{bmatrix}2 & 1 & 1\\-1 & -1 & 4\end{bmatrix}$, $B = \begin{bmatrix}2 & -3 & 4\\-3 & 1 & -2\end{bmatrix}$

**5.** $A = \begin{bmatrix}3 & 2 & -1\\2 & 4 & 5\\0 & 1 & 2\end{bmatrix}$, $B = \begin{bmatrix}0 & 2 & 1\\5 & 4 & 2\\2 & 1 & 0\end{bmatrix}$

**6.** $A = \begin{bmatrix}2 & 3 & 4\\0 & 1 & -1\\2 & 0 & 1\end{bmatrix}$, $B = \begin{bmatrix}0 & 6 & 2\\4 & 1 & 0\\-1 & 2 & 4\end{bmatrix}$

**7.** Find (a) $c_{21}$ and (b) $c_{13}$, where $C = 2A - 3B$,

$$A = \begin{bmatrix} 5 & 4 & 4 \\ -3 & 1 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 2 & -7 \\ 0 & -5 & 1 \end{bmatrix}.$$

**8.** Find (a) $c_{23}$ and (b) $c_{32}$, where $C = 5A + 2B$,

$$A = \begin{bmatrix} 4 & 11 & -9 \\ 0 & 3 & 2 \\ -3 & 1 & 1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 0 & 5 \\ -4 & 6 & 11 \\ -6 & 4 & 9 \end{bmatrix}.$$

**9.** Solve for $x$, $y$, and $z$ in the matrix equation

$$4\begin{bmatrix} x & y \\ z & -1 \end{bmatrix} = 2\begin{bmatrix} y & z \\ -x & 1 \end{bmatrix} + 2\begin{bmatrix} 4 & x \\ 5 & -x \end{bmatrix}.$$

**10.** Solve for $x$, $y$, $z$, and $w$ in the matrix equation

$$\begin{bmatrix} w & x \\ y & x \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix} + 2\begin{bmatrix} y & w \\ z & x \end{bmatrix}.$$

In Exercises 11–18, find (a) $AB$ and (b) $BA$ (if they are defined).

**11.** $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

**12.** $A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

**13.** $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$

**14.** $A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

**15.** $A = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$

**16.** $A = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

**17.** $A = \begin{bmatrix} 6 \\ -2 \\ 1 \\ 6 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 12 \end{bmatrix}$

**18.** $A = \begin{bmatrix} 1 & 0 & 3 & -2 & 4 \\ 6 & 13 & 8 & -17 & 20 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}$

In Exercises 19 and 20, find (a) $2A + B$, (b) $3B - A$, (c) $AB$, and (d) $BA$ (if they are defined).

**19.** $A = \begin{bmatrix} 2 & -2 & 4 & 1 & 0 & 3 \\ -1 & 4 & 2 & -2 & -1 & 3 \\ 3 & -3 & 1 & 2 & 3 & -4 \\ 2 & -1 & 3 & 0 & 1 & 2 \\ 5 & 1 & -2 & -4 & 1 & 3 \\ 2 & 2 & 3 & -4 & -1 & -2 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 & -3 & 4 & 1 & 2 \\ 2 & -3 & 1 & 3 & -1 & 2 \\ 0 & -2 & -3 & 0 & 1 & -1 \\ 1 & 2 & 3 & 2 & 1 & -1 \\ 2 & -1 & -3 & 0 & 4 & 2 \\ 1 & -2 & 4 & -2 & -4 & -1 \end{bmatrix}$

**20.** $A = \begin{bmatrix} 2 & 1 & 3 & 2 & -1 \\ 3 & -1 & 0 & 1 & 2 \\ 2 & 1 & -3 & 3 & -2 \\ -4 & 0 & 2 & -3 & 1 \\ 1 & 0 & -1 & 2 & 4 \\ 2 & -3 & 2 & 1 & -4 \end{bmatrix}$

$B = \begin{bmatrix} 5 & 2 & 1 & 3 & 2 & 1 \\ -4 & -2 & 2 & -1 & 3 & -1 \\ 4 & 0 & 1 & 3 & -2 & 1 \\ -1 & 2 & -3 & -1 & 2 & 3 \\ -2 & 1 & 4 & 3 & -2 & 2 \\ 1 & -2 & 3 & 4 & -2 & -1 \end{bmatrix}$

In Exercises 21–28, let $A, B, C, D$, and $E$ be matrices with the provided orders.

$A$: $3 \times 4$ $B$: $3 \times 4$ $C$: $4 \times 2$ $D$: $4 \times 2$ $E$: $4 \times 3$

If defined, determine the size of the matrix. If not defined, provide an explanation.

**21.** $A + B$ **22.** $C + E$ **23.** $\frac{1}{2}D$ **24.** $-4A$

**25.** $AC$ **26.** $BE$ **27.** $E - 2A$ **28.** $2D + C$

In Exercises 29–36, write the system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ and solve this matrix equation for $\mathbf{x}$.

**29.** $\begin{aligned} -x_1 + x_2 &= 4 \\ -2x_1 + x_2 &= 0 \end{aligned}$

**30.** $\begin{aligned} 2x_1 + 3x_2 &= 5 \\ x_1 + 4x_2 &= 10 \end{aligned}$

**31.** $\begin{aligned} -2x_1 - 3x_2 &= -4 \\ 6x_1 + x_2 &= -36 \end{aligned}$

**32.** $\begin{aligned} -4x_1 + 9x_2 &= -13 \\ x_1 - 3x_2 &= 12 \end{aligned}$

**33.** $\begin{aligned} x_1 - 2x_2 + 3x_3 &= 9 \\ -x_1 + 3x_2 - x_3 &= -6 \\ 2x_1 - 5x_2 + 5x_3 &= 17 \end{aligned}$

**34.** $\begin{aligned} x_1 + x_2 - 3x_3 &= -1 \\ -x_1 + 2x_2 \phantom{- 3x_3} &= 1 \\ x_1 - x_2 + x_3 &= 2 \end{aligned}$

**35.** $\begin{aligned} x_1 - 5x_2 + 2x_3 &= -20 \\ -3x_1 + x_2 - x_3 &= 8 \\ -2x_2 + 5x_3 &= -16 \end{aligned}$

**36.** $\begin{aligned} x_1 - x_2 + 4x_3 &= 17 \\ x_1 + 3x_2 \phantom{+ 4x_3} &= -11 \\ -6x_2 + 5x_3 &= 40 \end{aligned}$

In Exercises 37 and 38, solve the matrix equation for $A$.

**37.** $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**38.** $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

In Exercises 39 and 40, solve the matrix equation for $a$, $b$, $c$, and $d$.

**39.** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 19 & 2 \end{bmatrix}$

**40.** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 17 \\ 4 & -1 \end{bmatrix}$

**41.** Find conditions on $w$, $x$, $y$, and $z$ such that $AB = BA$ for the matrices below.

$$A = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

**42.** Verify $AB = BA$ for the following matrices.

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

A square matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

is called a **diagonal matrix** if all entries that are not on the main diagonal are zero. In Exercises 43 and 44, find the product $AA$ for the diagonal matrix.

**43.** $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**44.** $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

In Exercises 45 and 46, find the products $AB$ and $BA$ for the diagonal matrices.

**45.** $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 0 \\ 0 & 4 \end{bmatrix}$

**46.** $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix}$

**47. Guided Proof** Prove that if $A$ and $B$ are diagonal matrices (of the same size), then $AB = BA$.

Getting Started: To prove that the matrices $AB$ and $BA$ are equal, you need to show that their corresponding entries are equal.

(i) Begin your proof by letting $A = [a_{ij}]$ and $B = [b_{ij}]$ be two diagonal $n \times n$ matrices.

(ii) The $ij$th entry of the product $AB$ is $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

(iii) Evaluate the entries $c_{ij}$ for the two cases $i \neq j$ and $i = j$.

(iv) Repeat this analysis for the product $BA$.

**48. Writing** Let $A$ and $B$ be $3 \times 3$ matrices, where $A$ is diagonal.

(a) Describe the product $AB$. Illustrate your answer with examples.

(b) Describe the product $BA$. Illustrate your answer with examples.

(c) How do the results in parts (a) and (b) change if the diagonal entries of $A$ are all equal?

In Exercises 49–52, find the trace of the matrix. The **trace** of an $n \times n$ matrix $A$ is the sum of the main diagonal entries. That is, $\mathrm{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

**49.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 3 & 1 & 3 \end{bmatrix}$

**50.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**51.** $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 4 & 2 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}$

**52.** $\begin{bmatrix} 1 & 4 & 3 & 2 \\ 4 & 0 & 6 & 1 \\ 3 & 6 & 2 & 1 \\ 2 & 1 & 1 & -3 \end{bmatrix}$

**53.** Prove that each statement is true if $A$ and $B$ are square matrices of order $n$ and $c$ is a scalar.

(a) $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$

(b) $\mathrm{Tr}(cA) = c\mathrm{Tr}(A)$

**54.** Prove that if $A$ and $B$ are square matrices of order $n$, then $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

**55.** Show that the matrix equation has no solution.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**56.** Show that no $2 \times 2$ matrices $A$ and $B$ exist that satisfy the matrix equation

$$AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**57.** Let $i = \sqrt{-1}$ and let $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$.

(a) Find $A^2$, $A^3$, and $A^4$. Identify any similarities among $i^2$, $i^3$, and $i^4$.

(b) Find and identify $B^2$.

**58.** **Guided Proof** Prove that if the product $AB$ is a square matrix, then the product $BA$ is defined.

Getting Started: To prove that the product $BA$ is defined, you need to show that the number of columns of $B$ equals the number of rows of $A$.

(i) Begin your proof by noting that the number of columns of $A$ equals the number of rows of $B$.

(ii) You can then assume that $A$ has size $m \times n$ and $B$ has size $n \times p$.

(iii) Use the hypothesis that the product $AB$ is a square matrix.

**59.** Prove that if both products $AB$ and $BA$ are defined, then $AB$ and $BA$ are square matrices.

**60.** Let $A$ and $B$ be two matrices such that the product $AB$ is defined. Show that if $A$ has two identical rows, then the corresponding two rows of $AB$ are also identical.

**61.** Let $A$ and $B$ be $n \times n$ matrices. Show that if the $i$th row of $A$ has all zero entries, then the $i$th row of $AB$ will have all zero entries. Give an example using $2 \times 2$ matrices to show that the converse is not true.

**62.** The columns of matrix $T$ show the coordinates of the vertices of a triangle. Matrix $A$ is a transformation matrix.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \end{bmatrix}$$

(a) Find $AT$ and $AAT$. Then sketch the original triangle and the two transformed triangles. What transformation does $A$ represent?

(b) A triangle is determined by $AAT$. Describe the transformation process that produces the triangle determined by $AT$ and then the triangle determined by $T$.

**63.** A corporation has three factories, each of which manufactures acoustic guitars and electric guitars. The number of guitars of type $i$ produced at factory $j$ in one day is represented by $a_{ij}$ in the matrix

$$A = \begin{bmatrix} 70 & 50 & 25 \\ 35 & 100 & 70 \end{bmatrix}.$$

Find the production levels if production is increased by 20%.

**64.** A corporation has four factories, each of which manufactures sport utility vehicles and pickup trucks. The number of vehicles of type $i$ produced at factory $j$ in one day is represented by $a_{ij}$ in the matrix

$$A = \begin{bmatrix} 100 & 90 & 70 & 30 \\ 40 & 20 & 60 & 60 \end{bmatrix}.$$

Find the production levels if production is increased by 10%.

**65.** A fruit grower raises two crops, apples and peaches. Each of these crops is shipped to three different outlets. The number of units of crop $i$ that are shipped to outlet $j$ is represented by $a_{ij}$ in the matrix

$$A = \begin{bmatrix} 125 & 100 & 75 \\ 100 & 175 & 125 \end{bmatrix}.$$

The profit per unit is represented by the matrix

$B = [\$3.50 \quad \$6.00].$

Find the product $BA$ and state what each entry of the product represents.

**66.** A company manufactures tables and chairs at two locations. Matrix $C$ gives the total cost of manufacturing each product at each location.

$$\begin{array}{cc} & \begin{array}{cc} \textit{Location 1} & \textit{Location 2} \end{array} \\ C = \begin{array}{c} \textit{Tables} \\ \textit{Chairs} \end{array} & \begin{bmatrix} 627 & 681 \\ 135 & 150 \end{bmatrix} \end{array}$$

(a) If labor accounts for about $\frac{2}{3}$ of the total cost, determine the matrix $L$ that gives the labor cost for each product at each location. What matrix operation did you use?

(b) Find the matrix $M$ that gives material costs for each product at each location. (Assume there are only labor and material costs.)

**True or False?** In Exercises 67 and 68, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

**67.** (a) For the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix.

(b) The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be expressed as a linear combination, where the coefficients of the linear combination are a solution of the system.

**68.** (a) If $A$ is an $m \times n$ matrix and $B$ is an $n \times r$ matrix, then the product $AB$ is an $m \times r$ matrix.

(b) The matrix equation $A\mathbf{x} = \mathbf{b}$, where $A$ is the coefficient matrix and $\mathbf{x}$ and $\mathbf{b}$ are column matrices, can be used to represent a system of linear equations.

**69. Writing** The matrix

$$P = \begin{array}{r} \\ \textit{From R} \\ \textit{From D} \\ \textit{From I} \end{array} \begin{array}{c} \begin{array}{ccc} \textit{To R} & \textit{To D} & \textit{To I} \end{array} \\ \begin{bmatrix} 0.75 & 0.15 & 0.10 \\ 0.20 & 0.60 & 0.20 \\ 0.30 & 0.40 & 0.30 \end{bmatrix} \end{array}$$

represents the proportions of a voting population that change from party $i$ to party $j$ in a given election. That is, $p_{ij}$ $(i \neq j)$ represents the proportion of the voting population that changes from party $i$ to party $j$, and $p_{ii}$ represents the proportion that remains loyal to party $i$ from one election to the next. Find the product of $P$ with itself. What does this product represent?

**70.** The matrices show the numbers of people (in thousands) who lived in various regions of the United States in 2005 and the numbers of people (in thousands) projected to live in those regions in 2015. The regional populations are separated into three age categories. (Source: U.S. Census Bureau)

| | 2005 | | |
|---|---|---|---|
| | *0–17* | *18–64* | *65+* |
| *Northeast* | 12,607 | 34,418 | 6286 |
| *Midwest* | 16,131 | 41,395 | 7177 |
| *South* | 26,728 | 63,911 | 11,689 |
| *Mountain* | 5306 | 12,679 | 2020 |
| *Pacific* | 12,524 | 30,741 | 4519 |

| | 2015 | | |
|---|---|---|---|
| | *0–17* | *18–64* | *65+* |
| *Northeast* | 12,441 | 35,289 | 8835 |
| *Midwest* | 16,363 | 42,250 | 9955 |
| *South* | 29,373 | 73,496 | 17,572 |
| *Mountain* | 5263 | 14,231 | 3337 |
| *Pacific* | 12,826 | 33,292 | 7086 |

(a) The total population in 2005 was 288,131,000 and the projected total population in 2015 is 321,609,000. Rewrite the matrices to give the information as percents of the total population.

(b) Write a matrix that gives the projected changes in the percents of the population in the various regions and age groups from 2005 to 2015.

(c) Based on the result of part (b), which age group(s) is (are) projected to show relative growth from 2005 to 2015?

In Exercises 71 and 72, perform the indicated block multiplication of matrices $A$ and $B$. If matrices $A$ and $B$ have each been partitioned into four submatrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

then you can **block multiply** $A$ and $B$, provided the sizes of the submatrices are such that the matrix multiplications and additions are defined.

$$\begin{aligned} AB &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \end{aligned}$$

**71.** $A = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \end{array}\right], \quad B = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 3 \end{array}\right]$

**72.** $A = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array}\right]$

In Exercises 73–76, express the column matrix **b** as a linear combination of the columns of $A$.

**73.** $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

**74.** $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

**75.** $A = \begin{bmatrix} 1 & 1 & -5 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

**76.** $A = \begin{bmatrix} -3 & 5 \\ 3 & 4 \\ 4 & -8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -22 \\ 4 \\ 32 \end{bmatrix}$

## 2.2 Properties of Matrix Operations

In Section 2.1 you concentrated on the mechanics of the three basic matrix operations: matrix addition, scalar multiplication, and matrix multiplication. This section begins to develop the **algebra of matrices.** You will see that this algebra shares many (but not all) of the properties of the algebra of real numbers. Several properties of matrix addition and scalar multiplication are listed below.

**THEOREM 2.1 Properties of Matrix Addition and Scalar Multiplication**

If $A$, $B$, and $C$ are $m \times n$ matrices and $c$ and $d$ are scalars, then the following properties are true.

1. $A + B = B + A$ — **Commutative property of addition**
2. $A + (B + C) = (A + B) + C$ — **Associative property of addition**
3. $(cd)A = c(dA)$ — **Associative property of multiplication**
4. $1A = A$ — **Multiplicative identity**
5. $c(A + B) = cA + cB$ — **Distributive property**
6. $(c + d)A = cA + dA$ — **Distributive property**

*PROOF*

The proofs of these six properties follow directly from the definitions of matrix addition, scalar multiplication, and the corresponding properties of real numbers. For example, to prove the commutative property of *matrix addition,* let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then, using the commutative property of *addition of real numbers,* write

$$A + B = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A.$$

Similarly, to prove Property 5, use the distributive property (for real numbers) of multiplication over addition to write

$$c(A + B) = [c(a_{ij} + b_{ij})] = [ca_{ij} + cb_{ij}] = cA + cB.$$

The proofs of the remaining four properties are left as exercises. (See Exercises 47–50.)

In the preceding section, matrix addition was defined as the sum of *two* matrices, making it a binary operation. The associative property of matrix addition now allows you to write expressions such as $A + B + C$ as $(A + B) + C$ or as $A + (B + C)$. This same reasoning applies to sums of four or more matrices.

### EXAMPLE 1 Addition of More than Two Matrices

By adding corresponding entries, you can obtain the sum of four matrices shown below.

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

One important property of the addition of real numbers is that the number 0 serves as the additive identity. That is, $c + 0 = c$ for any real number $c$. For matrices, a similar property holds. Specifically, if $A$ is an $m \times n$ matrix and $O_{mn}$ is the $m \times n$ matrix consisting entirely of zeros, then $A + O_{mn} = A$. The matrix $O_{mn}$ is called a **zero matrix,** and it serves as the **additive identity** for the set of all $m \times n$ matrices. For example, the following matrix serves as the additive identity for the set of all $2 \times 3$ matrices.

$$O_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

When the size of the matrix is understood, you may denote a zero matrix simply by 0.

The following properties of zero matrices are easy to prove, and their proofs are left as an exercise. (See Exercise 51.)

**THEOREM 2.2 Properties of Zero Matrices**

If $A$ is an $m \times n$ matrix and $c$ is a scalar, then the following properties are true.

1. $A + O_{mn} = A$
2. $A + (-A) = O_{mn}$
3. If $cA = O_{mn}$, then $c = 0$ or $A = O_{mn}$.

REMARK: Property 2 can be described by saying that matrix $-A$ is the **additive inverse** of $A$.

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the solutions below.

| Real Numbers (Solve for $x$.) | $m \times n$ Matrices (Solve for $X$.) |
|---|---|
| $x + a = b$ | $X + A = B$ |
| $x + a + (-a) = b + (-a)$ | $X + A + (-A) = B + (-A)$ |
| $x + 0 = b - a$ | $X + O = B - A$ |
| $x = b - a$ | $X = B - A$ |

The process of solving a matrix equation is demonstrated in Example 2.

**EXAMPLE 2** **Solving a Matrix Equation**

Solve for $X$ in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

*SOLUTION* Begin by solving the equation for $X$ to obtain

$$3X = B - A \longrightarrow X = \tfrac{1}{3}(B - A).$$

Now, using the given matrices $A$ and $B$, you have

$$X = \frac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$

## *Properties of Matrix Multiplication*

In the next theorem, the algebra of matrices is extended to include some useful properties of matrix multiplication. The proof of Property 2 is presented below. The proofs of the remaining properties are left as an exercise. (See Exercise 52.)

**THEOREM 2.3 Properties of Matrix Multiplication**

If $A$, $B$, and $C$ are matrices (with sizes such that the given matrix products are defined) and $c$ is a scalar, then the following properties are true.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $c(AB) = (cA)B = A(cB)$

*PROOF* To prove Property 2, show that the matrices $A(B + C)$ and $AB + AC$ are equal by showing that their corresponding entries are equal. Assume $A$ has size $m \times n$, $B$ has size $n \times p$, and $C$ has size $n \times p$. Using the definition of matrix multiplication, the entry in the $i$th row and $j$th column of $A(B + C)$ is $a_{i1}(b_{1j} + c_{1j}) + \cdot \cdot \cdot + a_{in}(b_{nj} + c_{nj})$. Moreover, the entry in the $i$th row and $j$th column of $AB + AC$ is

$$(a_{i1}b_{1j} + \cdot \cdot \cdot + a_{in}b_{nj}) + (a_{i1}c_{1j} + \cdot \cdot \cdot + a_{in}c_{nj}).$$

By distributing and regrouping, you can see that these two $ij$th entries are equal. So,

$$A(B + C) = AB + AC.$$

The associative property of matrix multiplication permits you to write such matrix products as $ABC$ without ambiguity, as demonstrated in Example 3.

### EXAMPLE 3 Matrix Multiplication Is Associative

Find the matrix product $ABC$ by grouping the factors first as $(AB)C$ and then as $A(BC)$. Show that the same result is obtained from both processes.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$$

*SOLUTION* Grouping the factors as $(AB)C$, you have

$$\begin{aligned} (AB)C &= \left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}\right)\begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix}. \end{aligned}$$

Grouping the factors as $A(BC)$, you obtain the same result.

$$\begin{aligned} A(BC) &= \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix} \end{aligned}$$

Note that no commutative property for matrix multiplication is listed in Theorem 2.3. Although the product $AB$ is defined, it can easily happen that $A$ and $B$ are not of the proper sizes to define the product $BA$. For instance, if $A$ is of size $2 \times 3$ and $B$ is of size $3 \times 3$, then the product $AB$ is defined but the product $BA$ is not. The next example shows that even if both products $AB$ and $BA$ are defined, they may not be equal.

### EXAMPLE 4 Noncommutativity of Matrix Multiplication

Show that $AB$ and $BA$ are not equal for the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

*SOLUTION*

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

$$AB \neq BA$$

Do not conclude from Example 4 that the matrix products $AB$ and $BA$ are *never* the same. Sometimes they are the same. For example, try multiplying the following matrices, first in the order $AB$ and then in the order $BA$.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 4 \\ 2 & -2 \end{bmatrix}$$

You will see that the two products are equal. The point is this: Although $AB$ and $BA$ are sometimes equal, $AB$ and $BA$ are usually not equal.

Another important quality of matrix algebra is that it does not have a general cancellation property for matrix multiplication. That is, if $AC = BC$, it is not necessarily true that $A = B$. This is demonstrated in Example 5. (In the next section you will see that, for some special types of matrices, cancellation is valid.)

### EXAMPLE 5 An Example in Which Cancellation Is Not Valid

Show that $AC = BC$.

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

SOLUTION

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$AC = BC$, even though $A \neq B$.

You will now look at a special type of *square* matrix that has 1's on the main diagonal and 0's elsewhere.

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

For instance, if $n = 1$, 2, or 3, we have

$$\underset{1 \times 1}{I_1 = [1]}, \qquad \underset{2 \times 2}{I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \qquad \underset{3 \times 3}{I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

When the order of the matrix is understood to be $n$, you may denote $I_n$ simply as $I$.

As stated in Theorem 2.4 on the next page, the matrix $I_n$ serves as the **identity** for matrix multiplication; it is called the **identity matrix of order *n*.** The proof of this theorem is left as an exercise. (See Exercise 53.)

**THEOREM 2.4 Properties of the Identity Matrix**

If $A$ is a matrix of size $m \times n$, then the following properties are true.

1. $AI_n = A$
2. $I_mA = A$

As a special case of this theorem, note that if $A$ is a *square* matrix of order $n$, then

$$AI_n = I_nA = A.$$

**EXAMPLE 6** **Multiplication by an Identity Matrix**

(a) $\begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

For repeated multiplication of *square* matrices, you can use the same exponential notation used with real numbers. That is, $A^1 = A$, $A^2 = AA$, and for a positive integer $k$, $A^k$ is

$$A^k = \underbrace{AA \cdot \cdot \cdot A}_{k \text{ factors}}.$$

It is convenient also to define $A^0 = I_n$ (where $A$ is a square matrix of order $n$). These definitions allow you to establish the properties

1. $A^jA^k = A^{j+k}$ and 2. $(A^j)^k = A^{jk}$

where $j$ and $k$ are nonnegative integers.

**EXAMPLE 7** **Repeated Multiplication of a Square Matrix**

Find $A^3$ for the matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$.

SOLUTION

$$A^3 = \left(\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -6 \end{bmatrix}$$

In Section 1.1 you saw that a system of linear equations must have exactly one solution, an infinite number of solutions, or no solution. Using the matrix algebra developed so far, you can now prove that this is true.

**THEOREM 2.5**
**Number of Solutions of a System of Linear Equations**

For a system of linear equations in $n$ variables, precisely one of the following is true.

1. The system has exactly one solution.
2. The system has an infinite number of solutions.
3. The system has no solution.

*PROOF*

Represent the system by the matrix equation $A\mathbf{x} = \mathbf{b}$. If the system has exactly one solution or no solution, then there is nothing to prove. So, you can assume that the system has at least two distinct solutions $\mathbf{x}_1$ and $\mathbf{x}_2$. The proof will be complete if you can show that this assumption implies that the system has an infinite number of solutions. Because $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, you have $A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{b}$ and $A(\mathbf{x}_1 - \mathbf{x}_2) = O$. This implies that the (nonzero) column matrix $\mathbf{x}_h = \mathbf{x}_1 - \mathbf{x}_2$ is a solution of the homogeneous system of linear equations $A\mathbf{x} = O$. It can now be said that for any scalar $c$,

$$A(\mathbf{x}_1 + c\mathbf{x}_h) = A\mathbf{x}_1 + A(c\mathbf{x}_h) = \mathbf{b} + c(A\mathbf{x}_h) = \mathbf{b} + cO = \mathbf{b}.$$

So $\mathbf{x}_1 + c\mathbf{x}_h$ is a solution of $A\mathbf{x} = \mathbf{b}$ for any scalar $c$. Because there are an infinite number of possible values of $c$ and each value produces a different solution, you can conclude that the system has an infinite number of solutions.

## The Transpose of a Matrix

The **transpose** of a matrix is formed by writing its columns as rows. For instance, if $A$ is the $m \times n$ matrix shown by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix},$$

Size: $m \times n$

then the transpose, denoted by $A^T$, is the $n \times m$ matrix below

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

Size: $n \times m$

**EXAMPLE 8** **The Transpose of a Matrix**

Find the transpose of each matrix.

(a) $A = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$

*SOLUTION* (a) $A^T = \begin{bmatrix} 2 & 8 \end{bmatrix}$ (b) $B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ (c) $C^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $D^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$

**Discovery**

*Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$.

*Calculate* $(AB)^T$, $A^TB^T$, *and* $B^TA^T$. *Make a conjecture about the transpose of a product of two square matrices. Select two other square matrices to check your conjecture.*

REMARK: Note that the square matrix in part (c) of Example 8 is equal to its transpose. Such a matrix is called **symmetric.** A matrix $A$ is symmetric if $A = A^T$. From this definition it is clear that a symmetric matrix must be square. Also, if $A = [a_{ij}]$ is a symmetric matrix, then $a_{ij} = a_{ji}$ for all $i \neq j$.

**THEOREM 2.6 Properties of Transposes**

If $A$ and $B$ are matrices (with sizes such that the given matrix operations are defined) and $c$ is a scalar, then the following properties are true.

1. $(A^T)^T = A$ **Transpose of a transpose**
2. $(A + B)^T = A^T + B^T$ **Transpose of a sum**
3. $(cA)^T = c(A^T)$ **Transpose of a scalar multiple**
4. $(AB)^T = B^TA^T$ **Transpose of a product**

*PROOF* Because the transpose operation interchanges rows and columns, Property 1 seems to make sense. To prove Property 1, let $A$ be an $m \times n$ matrix. Observe that $A^T$ has size $n \times m$ and $(A^T)^T$ has size $m \times n$, the same as $A$. To show that $(A^T)^T = A$ you must show that the $ij$th entries are the same. Let $a_{ij}$ be the $ij$th entry of $A$. Then $a_{ij}$ is the $ji$th entry of $A^T$, and the $ij$th entry of $(A^T)^T$. This proves Property 1. The proofs of the remaining properties are left as an exercise. (See Exercise 54.)

REMARK: Remember that you *reverse the order* of multiplication when forming the transpose of a product. That is, the transpose of $AB$ is $(AB)^T = B^TA^T$ and is *not* usually equal to $A^TB^T$.

Properties 2 and 4 can be generalized to cover sums or products of any finite number of matrices. For instance, the transpose of the sum of three matrices is

$$(A + B + C)^T = A^T + B^T + C^T,$$

and the transpose of the product of three matrices is

$$(ABC)^T = C^TB^TA^T.$$

## EXAMPLE 9 Finding the Transpose of a Product

Show that $(AB)^T$ and $B^TA^T$ are equal.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

*SOLUTION*

$$AB = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & -1 \\ -1 & 2 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$(AB)^T = B^TA^T$$

## EXAMPLE 10 The Product of a Matrix and Its Transpose

For the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix}$$

find the product $AA^T$ and show that it is symmetric.

*SOLUTION* Because

$$AA^T = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -6 & -5 \\ -6 & 4 & 2 \\ -5 & 2 & 5 \end{bmatrix}$$

it follows that $AA^T = (AA^T)^T$, so $AA^T$ is symmetric.

REMARK: The property demonstrated in Example 10 is true in general. That is, for any matrix $A$, the matrix given by $B = AA^T$ is symmetric. You are asked to prove this result in Exercise 55.

## SECTION 2.2 Exercises

In Exercises 1–6, perform the indicated operations when $a = 3$, $b = -4$, and

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

**1.** $aA + bB$ **2.** $A + B$ **3.** $ab(B)$

**4.** $(a + b)B$ **5.** $(a - b)(A - B)$ **6.** $(ab)O$

**7.** Solve for $X$ when

$$A = \begin{bmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{bmatrix}.$$

(a) $3X + 2A = B$ (b) $2A - 5B = 3X$

(c) $X - 3A + 2B = O$ (d) $6X - 4A - 3B = O$

**8.** Solve for $X$ when

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}.$$

(a) $X = 3A - 2B$ (b) $2X = 2A - B$

(c) $2X + 3A = B$ (d) $2A + 4B = -2X$

In Exercises 9–14, perform the indicated operations, provided that $c = -2$ and

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

**9.** $B(CA)$ **10.** $C(BC)$ **11.** $(B + C)A$

**12.** $B(C + O)$ **13.** $(cB)(C + C)$ **14.** $B(cA)$

In Exercises 15 and 16, demonstrate that if $AC = BC$, then $A$ is *not* necessarily equal to $B$ for the following matrices.

**15.** $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

**16.** $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -6 & 3 \\ 5 & 4 & 4 \\ -1 & 0 & 1 \end{bmatrix},$

$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix}$

In Exercises 17 and 18, demonstrate that if $AB = O$, then it is *not* necessarily true that $A = O$ or $B = O$ for the following matrices.

**17.** $A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

**18.** $A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$

In Exercises 19–22, perform the indicated operations when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**19.** $A^2$ **20.** $A^4$

**21.** $A(I + A)$ **22.** $A + IA$

In Exercises 23–28, find (a) $A^T$, (b) $A^TA$, and (c) $AA^T$.

**23.** $A = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ **24.** $A = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix}$

**25.** $A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 4 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ **26.** $A = \begin{bmatrix} -7 & 11 & 12 \\ 4 & -3 & 1 \\ 6 & -1 & 3 \end{bmatrix}$

**27.** $A = \begin{bmatrix} 0 & -4 & 3 & 2 \\ 8 & 4 & 0 & 1 \\ -2 & 3 & 5 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

**28.** $A = \begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ -1 & -2 & 0 & 3 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix}$

**Writing** In Exercises 29 and 30, explain why the formula is *not* valid for matrices. Illustrate your argument with examples.

**29.** $(A + B)(A - B) = A^2 - B^2$

**30.** $(A + B)(A + B) = A^2 + 2AB + B^2$

In Exercises 31–34, verify that $(AB)^T = B^TA^T$.

**31.** $A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$

**32.** $A = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$

**33.** $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & -1 \end{bmatrix}$

**34.** $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 4 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

**True or False?** In Exercises 35 and 36, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

**35.** (a) Matrix addition is commutative.

(b) Matrix multiplication is associative.

(c) The transpose of the product of two matrices equals the product of their transposes; that is, $(AB)^T = A^TB^T$.

(d) For any matrix $C$, the matrix $CC^T$ is symmetric.

**36.** (a) Matrix multiplication is commutative.

(b) Every matrix $A$ has an additive inverse.

(c) If the matrices $A$, $B$, and $C$ satisfy $AB = AC$, then $B = C$.

(d) The transpose of the sum of two matrices equals the sum of their transposes.

**37.** Consider the matrices shown below.

$$X = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad W = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(a) Find scalars $a$ and $b$ such that $Z = aX + bY$.

(b) Show that there do not exist scalars $a$ and $b$ such that $W = aX + bY$.

(c) Show that if $aX + bY + cW = O$, then $a = b = c = 0$.

(d) Find scalars $a$, $b$, and $c$, not all equal to zero, such that $aX + bY + cZ = O$.

**38.** Consider the matrices shown below.

$$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}, \quad W = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(a) Find scalars $a$ and $b$ such that $Z = aX + bY$.

(b) Show that there do not exist scalars $a$ and $b$ such that $W = aX + bY$.

(c) Show that if $aX + bY + cW = O$, then $a = b = c = 0$.

(d) Find scalars $a$, $b$, and $c$, not all equal to zero, such that $aX + bY + cZ = O$.

In Exercises 39 and 40, compute the power of $A$ for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**39.** $A^{19}$

**40.** $A^{20}$

An $n$th root of a matrix $B$ is a matrix $A$ such that $A^n = B$. In Exercises 41 and 42, find the $n$th root of the matrix $B$.

**41.** $B = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}, \quad n = 2$

**42.** $B = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 27 \end{bmatrix}, \quad n = 3$

In Exercises 43–46, use the given definition to find $f(A)$: If $f$ is the polynomial function,

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

then for an $n \times n$ matrix $A$, $f(A)$ is defined to be

$$f(A) = a_0I_n + a_1A + a_2A^2 + \cdots + a_nA^n.$$

**43.** $f(x) = x^2 - 5x + 2, \quad A = \begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}$

**44.** $f(x) = x^2 - 7x + 6, \quad A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

**45.** $f(x) = x^2 - 3x + 2, \quad A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

**46.** $f(x) = x^3 - 2x^2 + 5x - 10, \quad A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}$

**47. Guided Proof** Prove the associative property of matrix addition: $A + (B + C) = (A + B) + C$.

Getting Started: To prove that $A + (B + C)$ and $(A + B) + C$ are equal, show that their corresponding entries are the same.

(i) Begin your proof by letting $A$, $B$, and $C$ be $m \times n$ matrices.
(ii) Observe that the $ij$th entry of $B + C$ is $b_{ij} + c_{ij}$.
(iii) Furthermore, the $ij$th entry of $A + (B + C)$ is $a_{ij} + (b_{ij} + c_{ij})$.
(iv) Determine the $ij$th entry of $(A + B) + C$.

**48.** Prove the associative property of scalar multiplication: $(cd)A = c(dA)$.

**49.** Prove that the scalar 1 is the identity for scalar multiplication: $1A = A$.

**50.** Prove the following distributive property: $(c + d)A = cA + dA$.

**51.** Prove Theorem 2.2.

**52.** Complete the proof of Theorem 2.3.

(a) Prove the associative property of multiplication: $A(BC) = (AB)C$.
(b) Prove the distributive property: $(A + B)C = AC + BC$.
(c) Prove the property: $c(AB) = (cA)B = A(cB)$.

**53.** Prove Theorem 2.4.

**54.** Prove Properties 2, 3, and 4 of Theorem 2.6.

**55. Guided Proof** Prove that if $A$ is an $m \times n$ matrix, then $AA^T$ and $A^TA$ are symmetric matrices.

Getting Started: To prove that $AA^T$ is symmetric, you need to show that it is equal to its transpose, $(AA^T)^T = AA^T$.

(i) Begin your proof with the left-hand matrix expression $(AA^T)^T$.
(ii) Use the properties of the transpose operation to show that it can be simplified to equal the right-hand expression, $AA^T$.
(iii) Repeat this analysis for the product $A^TA$.

**56.** Give an example of two $2 \times 2$ matrices $A$ and $B$ such that $(AB)^T \neq A^TB^T$.

In Exercises 57–60, determine whether the matrix is symmetric, skew-symmetric, or neither. A square matrix is called **skew-symmetric** if $A^T = -A$.

**57.** $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

**58.** $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

**59.** $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}$

**60.** $A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}$

**61.** Prove that the main diagonal of a skew-symmetric matrix consists entirely of zeros.

**62.** Prove that if $A$ and $B$ are $n \times n$ skew-symmetric matrices, then $A + B$ is skew-symmetric.

**63.** Let $A$ be a square matrix of order $n$.

(a) Show that $\frac{1}{2}(A + A^T)$ is symmetric.
(b) Show that $\frac{1}{2}(A - A^T)$ is skew-symmetric.
(c) Prove that $A$ can be written as the sum of a symmetric matrix $B$ and a skew-symmetric matrix $C$, $A = B + C$.
(d) Write the matrix

$$A = \begin{bmatrix} 2 & 5 & 3 \\ -3 & 6 & 0 \\ 4 & 1 & 1 \end{bmatrix}$$

as the sum of a skew-symmetric matrix and a symmetric matrix.

**64.** Prove that if $A$ is an $n \times n$ matrix, then $A - A^T$ is skew-symmetric.

**65.** Let $A$ and $B$ be two $n \times n$ symmetric matrices.

(a) Give an example to show that the product $AB$ is not necessarily symmetric.
(b) Prove that $AB$ is symmetric if and only if $AB = BA$.

**66.** Consider matrices of the form

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & 0 & a_{34} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{(n-1)n} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

(a) Write a $2 \times 2$ matrix and a $3 \times 3$ matrix in the form of $A$.
(b) Use a graphing utility or computer software program to raise each of the matrices to higher powers. Describe the result.
(c) Use the result of part (b) to make a conjecture about powers of $A$ if $A$ is a $4 \times 4$ matrix. Use a graphing utility to test your conjecture.
(d) Use the results of parts (b) and (c) to make a conjecture about powers of $A$ if $A$ is an $n \times n$ matrix.

## 2.3 The Inverse of a Matrix

Section 2.2 discussed some of the similarities between the algebra of real numbers and the algebra of matrices. This section further develops the algebra of matrices to include the solutions of matrix equations involving matrix multiplication. To begin, consider the real number equation $ax = b$. To solve this equation for $x$, multiply both sides of the equation by $a^{-1}$ (provided $a \neq 0$).

$$\begin{aligned} ax &= b \\ (a^{-1}a)x &= a^{-1}b \\ (1)x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The number $a^{-1}$ is called the *multiplicative inverse* of $a$ because $a^{-1}a$ yields 1 (the identity element for multiplication). The definition of a multiplicative inverse of a matrix is similar.

**Definition of the Inverse of a Matrix**

An $n \times n$ matrix $A$ is **invertible** (or **nonsingular**) if there exists an $n \times n$ matrix $B$ such that

$$AB = BA = I_n$$

where $I_n$ is the identity matrix of order $n$. The matrix $B$ is called the (multiplicative) **inverse** of $A$. A matrix that does not have an inverse is called **noninvertible** (or **singular**).

*Nonsquare matrices do not have inverses.* To see this, note that if $A$ is of size $m \times n$ and $B$ is of size $n \times m$ (where $m \neq n$), then the products $AB$ and $BA$ are of different sizes and cannot be equal to each other. Indeed, not all square matrices possess inverses. (See Example 4.) The next theorem, however, tells you that if a matrix *does* possess an inverse, then that inverse is unique.

**THEOREM 2.7 Uniqueness of an Inverse Matrix**

If $A$ is an invertible matrix, then its inverse is unique. The inverse of $A$ is denoted by $A^{-1}$.

*PROOF* Because $A$ is invertible, you know it has at least one inverse $B$ such that

$$AB = I = BA.$$

Suppose $A$ has another inverse $C$ such that

$$AC = I = CA.$$

Then you can show that $B$ and $C$ are equal, as follows.

$$\begin{aligned} AB &= I \\ C(AB) &= CI \\ (CA)B &= C \\ IB &= C \\ B &= C \end{aligned}$$

Consequently $B = C$, and it follows that the inverse of a matrix is unique.

Because the inverse $A^{-1}$ of an invertible matrix $A$ is unique, you can call it *the* inverse of $A$ and write

$$AA^{-1} = A^{-1}A = I.$$

**EXAMPLE 1** **The Inverse of a Matrix**

Show that $B$ is the inverse of $A$, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

*SOLUTION* Using the definition of an inverse matrix, you can show that $B$ is the inverse of $A$ by showing that $AB = I = BA$, as follows.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

REMARK: Recall that it is not always true that $AB = BA$, even if both products are defined. If $A$ and $B$ are both square matrices and $AB = I_n$, however, then it can be shown that $BA = I_n$. Although the proof of this fact is omitted, it implies that in Example 1 you needed only to check that $AB = I_2$.

The next example shows how to use a system of equations to find the inverse of a matrix.

**EXAMPLE 2** **Finding the Inverse of a Matrix**

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$$

*SOLUTION* To find the inverse of $A$, try to solve the matrix equation $AX = I$ for $X$.

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, by equating corresponding entries, you obtain the two systems of linear equations shown below.

$$\begin{aligned} x_{11} + 4x_{21} &= 1 \\ -x_{11} - 3x_{21} &= 0 \end{aligned} \qquad \begin{aligned} x_{12} + 4x_{22} &= 0 \\ -x_{12} - 3x_{22} &= 1 \end{aligned}$$

Solving the first system, you find that the first column of $X$ is $x_{11} = -3$ and $x_{21} = 1$. Similarly, solving the second system, you find that the second column of $X$ is $x_{12} = -4$ and $x_{22} = 1$. The inverse of $A$ is

$$X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}.$$

Try using matrix multiplication to check this result.

Generalizing the method used to solve Example 2 provides a convenient method for finding an inverse. Notice first that the two systems of linear equations

$$\begin{aligned} x_{11} + 4x_{21} &= 1 \\ -x_{11} - 3x_{21} &= 0 \end{aligned} \qquad \begin{aligned} x_{12} + 4x_{22} &= 0 \\ -x_{12} - 3x_{22} &= 1 \end{aligned}$$

have the *same coefficient matrix*. Rather than solve the two systems represented by

$$\left[\begin{array}{rr:r} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{rr:r} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right]$$

separately, you can solve them simultaneously. You can do this by **adjoining** the identity matrix to the coefficient matrix to obtain

$$\left[\begin{array}{rr:rr} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right].$$

By applying Gauss-Jordan elimination to this matrix, you can solve *both* systems with a single elimination process, as follows.

$$\left[\begin{array}{rr:rr} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \qquad R_2 + R_1 \rightarrow R_2$$

$$\left[\begin{array}{rr:rr} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right] \qquad R_1 + (-4)R_2 \rightarrow R_1$$

Applying Gauss-Jordan elimination to the "doubly augmented" matrix $[A \,\vdots\, I]$, you obtain the matrix $[I \,\vdots\, A^{-1}]$.

$$\underset{A}{\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}} \ \vdots \ \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \longrightarrow \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \ \vdots \ \underset{A^{-1}}{\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}}$$

This procedure (or algorithm) works for an arbitrary $n \times n$ matrix. If $A$ cannot be row reduced to $I_n$, then $A$ is noninvertible (or singular). This procedure will be formally justified in the next section, after the concept of an elementary matrix is introduced. For now the algorithm is summarized as follows.

### Finding the Inverse of a Matrix by Gauss-Jordan Elimination

Let $A$ be a square matrix of order $n$.

1. Write the $n \times 2n$ matrix that consists of the given matrix $A$ on the left and the $n \times n$ identity matrix $I$ on the right to obtain $[A \ \vdots \ I]$. Note that you separate the matrices $A$ and $I$ by a dotted line. This process is called **adjoining** matrix $I$ to matrix $A$.
2. If possible, row reduce $A$ to $I$ using elementary row operations on the *entire* matrix $[A \ \vdots \ I]$. The result will be the matrix $[I \ \vdots \ A^{-1}]$. If this is not possible, then $A$ is noninvertible (or singular).
3. Check your work by multiplying $AA^{-1}$ and $A^{-1}A$ to see that $AA^{-1} = I = A^{-1}A$.

## EXAMPLE 3 Finding the Inverse of a Matrix

Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

*SOLUTION* Begin by adjoining the identity matrix to $A$ to form the matrix

$$[A \ \vdots \ I] = \left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right].$$

Now, using elementary row operations, rewrite this matrix in the form $[I \ \vdots \ A^{-1}]$, as follows.

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \qquad R_2 + (-1)R_1 \rightarrow R_2$$

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \qquad R_3 + (6)R_1 \rightarrow R_3$$

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \qquad R_3 + (4)R_2 \rightarrow R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad (-1)R_3 \rightarrow R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad R_2 + R_3 \rightarrow R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad R_1 + R_2 \rightarrow R_1$$

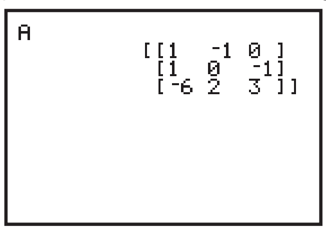
The matrix $A$ is invertible, and its inverse is

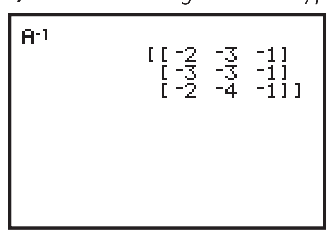
$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

Try confirming this by showing that $AA^{-1} = I = A^{-1}A$.

**Technology Note**

Most graphing utilities and computer software programs can calculate the inverse of a square matrix. If you are using a graphing utility, your screens for Example 3 may look like the images below. Keystrokes and programming syntax for these utilities/programs applicable to Example 3 are provided in the **Online Technology Guide,** available at *college.hmco.com/pic/larsonELA6e*.




The process shown in Example 3 applies to any $n \times n$ matrix and will find the inverse of matrix $A$, if possible. If matrix $A$ has no inverse, the process will also tell you that. The next example applies the process to a singular matrix (one that has no inverse).

## EXAMPLE 4 A Singular Matrix

Show that the matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

SOLUTION Adjoin the identity matrix to $A$ to form

$$[A \vdots I] = \left[\begin{array}{rrr:rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right]$$

and apply Gauss-Jordan elimination as follows.

$$\left[\begin{array}{rrr:rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \qquad R_2 + (-3)R_1 \rightarrow R_2$$

$$\left[\begin{array}{rrr:rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array}\right] \qquad R_3 + (2)R_1 \rightarrow R_3$$

Now, notice that adding the second row to the third row produces a row of zeros on the left side of the matrix.

$$\left[\begin{array}{rrr:rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \qquad R_3 + R_2 \rightarrow R_3$$

Because the "$A$ portion" of the matrix has a row of zeros, you can conclude that it is not possible to rewrite the matrix $[A \vdots I]$ in the form $[I \vdots A^{-1}]$. This means that $A$ has no inverse, or is noninvertible (or singular).

Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of size $3 \times 3$ or greater. For $2 \times 2$ matrices, however, you can use a formula to find the inverse instead of using Gauss-Jordan elimination. This simple formula is explained as follows.

If $A$ is a $2 \times 2$ matrix represented by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then $A$ is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is represented by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Try verifying this inverse by finding the product $AA^{-1}$.

REMARK: The denominator $ad - bc$ is called the **determinant** of $A$. You will study determinants in detail in Chapter 3.

**EXAMPLE 5** **Finding the Inverse of a 2 x 2 Matrix**

If possible, find the inverse of each matrix.

(a) $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

*SOLUTION* (a) For the matrix $A$, apply the formula for the inverse of a $2 \times 2$ matrix to obtain $ad - bc = (3)(2) - (-1)(-2) = 4$. Because this quantity is not zero, the inverse is formed by interchanging the entries on the main diagonal and changing the signs of the other two entries, as follows.

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

(b) For the matrix $B$, you have $ad - bc = (3)(2) - (-1)(-6) = 0$, which means that $B$ is noninvertible.

## *Properties of Inverses*

Some important properties of inverse matrices are listed below.

**THEOREM 2.8 Properties of Inverse Matrices**

If $A$ is an invertible matrix, $k$ is a positive integer, and $c$ is a scalar not equal to zero, then $A^{-1}$, $A^k$, $cA$, and $A^T$ are invertible and the following are true.

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = A^{-1}A^{-1} \cdot \cdot \cdot A^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \dfrac{1}{c}A^{-1}, \ c \neq 0$
4. $(A^T)^{-1} = (A^{-1})^T$

*PROOF* The key to the proofs of Properties 1, 3, and 4 is the fact that the inverse of a matrix is unique (Theorem 2.7). That is, if $BC = CB = I$, then $C$ is the inverse of $B$.

Property 1 states that the inverse of $A^{-1}$ is $A$ itself. To prove this, observe that $A^{-1}A = AA^{-1} = I$, which means that $A$ is the inverse of $A^{-1}$. Thus, $A = (A^{-1})^{-1}$.

Similarly, Property 3 states that $\dfrac{1}{c}A^{-1}$ is the inverse of $(cA)$, $c \neq 0$. To prove this, use the properties of scalar multiplication given in Theorems 2.1 and 2.3, as follows.

$$(cA)\left(\frac{1}{c}A^{-1}\right) = \left(c\frac{1}{c}\right)AA^{-1} = (1)I = I$$

and

$$\left(\frac{1}{c}A^{-1}\right)(cA) = \left(\frac{1}{c}c\right)A^{-1}A = (1)I = I$$

So $\frac{1}{c}A^{-1}$ is the inverse of $(cA)$, which implies that

$$\frac{1}{c}A^{-1} = (cA)^{-1}.$$

Properties 2 and 4 are left for you to prove. (See Exercises 47 and 48.)

---

For nonsingular matrices, the exponential notation used for repeated multiplication of *square* matrices can be extended to include exponents that are negative integers. This may be done by defining $A^{-k}$ to be

$$A^{-k} = \underbrace{A^{-1}A^{-1} \cdot \cdot \cdot A^{-1}}_{k \text{ factors}} = (A^{-1})^k.$$

Using this convention you can show that the properties $A^jA^k = A^{j+k}$ and $(A^j)^k = A^{jk}$ hold true for any integers $j$ and $k$.

## EXAMPLE 6 The Inverse of the Square of a Matrix

Compute $A^{-2}$ in two different ways and show that the results are equal.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

*SOLUTION* One way to find $A^{-2}$ is to find $(A^2)^{-1}$ by squaring the matrix $A$ to obtain

$$A^2 = \begin{bmatrix} 3 & 5 \\ 10 & 18 \end{bmatrix}$$

and using the formula for the inverse of a $2 \times 2$ matrix to obtain

$$(A^2)^{-1} = \tfrac{1}{4}\begin{bmatrix} 18 & -5 \\ -10 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{2} & -\frac{5}{4} \\ -\frac{5}{2} & \frac{3}{4} \end{bmatrix}.$$

Another way to find $A^{-2}$ is to find $(A^{-1})^2$ by finding $A^{-1}$

$$A^{-1} = \tfrac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

and then squaring this matrix to obtain

$$(A^{-1})^2 = \begin{bmatrix} \frac{9}{2} & -\frac{5}{4} \\ -\frac{5}{2} & \frac{3}{4} \end{bmatrix}.$$

Note that each method produces the same result.

**Discovery**

*Let* $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}.$

*Calculate* $(AB)^{-1}$, $A^{-1}B^{-1}$, *and* $B^{-1}A^{-1}$. *Make a conjecture about the inverse of a product of two nonsingular matrices. Select two other nonsingular matrices and see whether your conjecture holds.*

The next theorem gives a formula for computing the inverse of a product of two matrices.

**THEOREM 2.9 The Inverse of a Product**

If $A$ and $B$ are invertible matrices of size $n$, then $AB$ is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

*PROOF* To show that $B^{-1}A^{-1}$ is the inverse of $AB$, you need only show that it conforms to the definition of an inverse matrix. That is,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I.$$

In a similar way you can show that $(B^{-1}A^{-1})(AB) = I$ and conclude that $AB$ is invertible and has the indicated inverse.

Theorem 2.9 states that the inverse of a product of two invertible matrices is the product of their inverses taken in the *reverse* order. This can be generalized to include the product of several invertible matrices:

$$(A_1A_2A_3 \cdot \cdot \cdot A_n)^{-1} = A_n^{-1} \cdot \cdot \cdot A_3^{-1}A_2^{-1}A_1^{-1}.$$

(See Example 4 in Appendix A.)

### EXAMPLE 7 Finding the Inverse of a Matrix Product

Find $(AB)^{-1}$ for the matrices

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

using the fact that $A^{-1}$ and $B^{-1}$ are represented by

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}.$$

*SOLUTION* Using Theorem 2.9 produces

$$(AB)^{-1} = B^{-1}A^{-1} = \overset{B^{-1}}{\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}} \overset{A^{-1}}{\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 8 & -5 & -2 \\ -8 & 4 & 3 \\ 5 & -2 & -\frac{7}{3} \end{bmatrix}.$$

REMARK: Note that you *reverse the order* of multiplication to find the inverse of $AB$. That is, $(AB)^{-1} = B^{-1}A^{-1}$, and the inverse of $AB$ is usually *not* equal to $A^{-1}B^{-1}$.

One important property in the algebra of real numbers is the cancellation property. That is, if $ac = bc$ $(c \neq 0)$, then $a = b$. *Invertible* matrices have similar cancellation properties.

## THEOREM 2.10 Cancellation Properties

If $C$ is an invertible matrix, then the following properties hold.

1. If $AC = BC$, then $A = B$. **Right cancellation property**
2. If $CA = CB$, then $A = B$. **Left cancellation property**

*PROOF* To prove Property 1, use the fact that $C$ is invertible and write

$$\begin{aligned} AC &= BC \\ (AC)C^{-1} &= (BC)C^{-1} \\ A(CC^{-1}) &= B(CC^{-1}) \\ AI &= BI \\ A &= B. \end{aligned}$$

The second property can be proved in a similar way; this is left to you. (See Exercise 50.)

Be sure to remember that Theorem 2.10 can be applied only if $C$ is an *invertible* matrix. If $C$ is not invertible, then cancellation is not usually valid. For instance, Example 5 in Section 2.2 gives an example of a matrix equation $AC = BC$ in which $A \neq B$, because $C$ is not invertible in the example.

## *Systems of Equations*

In Theorem 2.5 you were able to prove that a system of linear equations can have exactly one solution, an infinite number of solutions, or no solution. For *square* systems (those having the same number of equations as variables), you can use the theorem below to determine whether the system has a unique solution.

**THEOREM 2.11 Systems of Equations with Unique Solutions**

If $A$ is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution given by

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

*PROOF* Because $A$ is nonsingular, the steps shown below are valid.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

This solution is unique because if $\mathbf{x}_1$ and $\mathbf{x}_2$ were two solutions, you could apply the cancellation property to the equation $A\mathbf{x}_1 = \mathbf{b} = A\mathbf{x}_2$ to conclude that $\mathbf{x}_1 = \mathbf{x}_2$.

Theorem 2.11 is theoretically important, but it is not very practical for solving a system of linear equations. It would require more work to find $A^{-1}$ and then multiply by $\mathbf{b}$ than simply to solve the system using Gaussian elimination with back-substitution. A situation in which you might consider using Theorem 2.11 as a computational technique would be one in which you have *several* systems of linear equations, all of which have the same coefficient matrix $A$. In such a case, you could find the inverse matrix once and then solve each system by computing the product $A^{-1}\mathbf{b}$. This is demonstrated in Example 8.

### *EXAMPLE 8* Solving a System of Equations Using an Inverse Matrix

Use an inverse matrix to solve each system.

(a) $\begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned}$ (b) $\begin{aligned} 2x + 3y + z &= 4 \\ 3x + 3y + z &= 8 \\ 2x + 4y + z &= 5 \end{aligned}$ (c) $\begin{aligned} 2x + 3y + z &= 0 \\ 3x + 3y + z &= 0 \\ 2x + 4y + z &= 0 \end{aligned}$

*SOLUTION* First note that the coefficient matrix for each system is

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}.$$

Using Gauss-Jordan elimination, you can find $A^{-1}$ to be

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}.$$

To solve each system, use matrix multiplication, as follows.

(a) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

The solution is $x = 2$, $y = -1$, and $z = -2$.

(b) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$

The solution is $x = 4$, $y = 1$, and $z = -7$.

(c) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution is trivial: $x = 0$, $y = 0$, and $z = 0$.

## SECTION 2.3 Exercises

In Exercises 1–4, show that $B$ is the inverse of $A$.

**1.** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

**2.** $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

**3.** $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

**4.** $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

In Exercises 5–24, find the inverse of the matrix (if it exists).

**5.** $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

**6.** $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

**7.** $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$

**8.** $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$

**9.** $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$

**10.** $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

**11.** $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ 7 & 16 & -21 \end{bmatrix}$

**12.** $\begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

# 3 Determinants



**CHAPTER OBJECTIVES**

- Find the determinants of a $2 \times 2$ matrix and a triangular matrix.
- Find the minors and cofactors of a matrix and use expansion by cofactors to find the determinant of a matrix.
- Use elementary row or column operations to evaluate the determinant of a matrix.
- Recognize conditions that yield zero determinants.
- Find the determinant of an elementary matrix.
- Use the determinant and properties of the determinant to decide whether a matrix is singular or nonsingular, and recognize equivalent conditions for a nonsingular matrix.
- Verify and find an eigenvalue and an eigenvector of a matrix.
- Find and use the adjoint of a matrix to find its inverse.
- Use Cramer's Rule to solve a system of linear equations.
- Use determinants to find the area of a triangle defined by three distinct points, to find an equation of a line passing through two distinct points, to find the volume of a tetrahedron defined by four distinct points, and to find an equation of a plane passing through three distinct points.

## 3.1 The Determinant of a Matrix

Every *square* matrix can be associated with a real number called its *determinant.* Determinants have many uses, several of which will be explored in this chapter. The first two sections of this chapter concentrate on procedures for evaluating the determinant of a matrix.

Historically, the use of determinants arose from the recognition of special patterns that occur in the solutions of systems of linear equations. For instance, the general solution of the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

can be shown to be

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad \text{and} \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}},$$

provided that $a_{11}a_{22} - a_{21}a_{12} \neq 0$. Note that both fractions have the same denominator, $a_{11}a_{22} - a_{21}a_{12}$. This quantity is called the determinant of the coefficient matrix $A$.

**Definition of the Determinant of a $2 \times 2$ Matrix**

The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}.$$

REMARK: In this text, $\det(A)$ and $|A|$ are used interchangeably to represent the determinant of a matrix. Vertical bars are also used to denote the absolute value of a real number; the context will show which use is intended. Furthermore, it is common practice to delete the matrix brackets and write

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad \text{instead of} \quad \left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right|.$$

A convenient method for remembering the formula for the determinant of a $2 \times 2$ matrix is shown in the diagram below.

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

The determinant is the difference of the products of the two diagonals of the matrix. Note that the order is important, as demonstrated above.

## EXAMPLE 1 The Determinant of a Matrix of Order 2

Find the determinant of each matrix.

(a) $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ (c) $C = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$

*SOLUTION*

(a) $|A| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$

(b) $|B| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$

(c) $|C| = \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$

REMARK: The determinant of a matrix can be positive, zero, or negative.

The determinant of a matrix of order 1 is defined simply as the entry of the matrix. For instance, if $A = [-2]$, then

$$\det(A) = -2.$$

To define the determinant of a matrix of order higher than 2, it is convenient to use the notions of *minors* and *cofactors.*

## Definitions of Minors and Cofactors of a Matrix

If $A$ is a square matrix, then the **minor** $M_{ij}$ of the element $a_{ij}$ is the determinant of the matrix obtained by deleting the $i$th row and $j$th column of $A$. The **cofactor** $C_{ij}$ is given by

$$C_{ij} = (-1)^{i+j}M_{ij}.$$

For example, if $A$ is a $3 \times 3$ matrix, then the minors and cofactors of $a_{21}$ and $a_{22}$ are as shown in the diagram below.

*Minor of $a_{21}$*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Delete row 2 and column 1.

*Cofactor of $a_{21}$*

$$C_{21} = (-1)^{2+1}M_{21} = -M_{21}$$

*Minor of $a_{22}$*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Delete row 2 and column 2.

*Cofactor of $a_{22}$*

$$C_{22} = (-1)^{2+2}M_{22} = M_{22}$$

As you can see, the minors and cofactors of a matrix can differ only in sign. To obtain the cofactors of a matrix, first find the minors and then apply the checkerboard pattern of $+$'s and $-$'s shown below.

*Sign Pattern for Cofactors*

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$3 \times 3$ *matrix*

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$4 \times 4$ *matrix*

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ *matrix*

Note that *odd* positions (where $i + j$ is odd) have negative signs, and even positions (where $i + j$ is even) have positive signs.

### EXAMPLE 2 Find the Minors and Cofactors of a Matrix

Find all the minors and cofactors of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

*SOLUTION* To find the minor $M_{11}$, delete the first row and first column of $A$ and evaluate the determinant of the resulting matrix.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

Similarly, to find $M_{12}$, delete the first row and second column.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

Continuing this pattern, you obtain

$$\begin{array}{lll} M_{11} = -1 & M_{12} = -5 & M_{13} = \phantom{-}4 \\ M_{21} = \phantom{-}2 & M_{22} = -4 & M_{23} = -8 \\ M_{31} = \phantom{-}5 & M_{32} = -3 & M_{33} = -6. \end{array}$$

Now, to find the cofactors, combine the checkerboard pattern of signs with these minors to obtain

$$\begin{array}{lll} C_{11} = -1 & C_{12} = \phantom{-}5 & C_{13} = \phantom{-}4 \\ C_{21} = -2 & C_{22} = -4 & C_{23} = \phantom{-}8 \\ C_{31} = \phantom{-}5 & C_{32} = \phantom{-}3 & C_{33} = -6. \end{array}$$

Now that the minors and cofactors of a matrix have been defined, you are ready for a general definition of the determinant of a matrix. The next definition is called **inductive** because it uses determinants of matrices of order $n - 1$ to define the determinant of a matrix of order $n$.

### Definition of the Determinant of a Matrix

If $A$ is a square matrix (of order 2 or greater), then the determinant of $A$ is the sum of the entries in the first row of $A$ multiplied by their cofactors. That is,

$$\det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdot \cdot \cdot + a_{1n}C_{1n}.$$

REMARK: Try checking that, for $2 \times 2$ matrices, this definition yields $|A| = a_{11}a_{22} - a_{21}a_{12}$, as previously defined.

When you use this definition to evaluate a determinant, you are **expanding by cofactors in the first row.** This procedure is demonstrated in Example 3.

### EXAMPLE 3 The Determinant of a Matrix of Order 3

Find the determinant of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

*SOLUTION* This matrix is the same as the one in Example 2. There you found the cofactors of the entries in the first row to be

$$C_{11} = -1, \qquad C_{12} = 5, \qquad C_{13} = 4.$$

By the definition of a determinant, you have

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{First row expansion} \\ &= 0(-1) + 2(5) + 1(4) = 14. \end{aligned}$$

Although the determinant is defined as an expansion by the cofactors in the first row, it can be shown that the determinant can be evaluated by expanding by *any* row or column. For instance, you could expand the $3 \times 3$ matrix in Example 3 by the second row to obtain

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{Second row expansion} \\ &= 3(-2) + (-1)(-4) + 2(8) = 14 \end{aligned}$$

or by the first column to obtain

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} && \text{First column expansion} \\ &= 0(-1) + 3(-2) + 4(5) = 14. \end{aligned}$$

Try other possibilities to confirm that the determinant of $A$ can be evaluated by expanding by *any* row or column. This is stated in the theorem below, Laplace's Expansion of a Determinant, named after the French mathematician Pierre Simon de Laplace (1749–1827).

### THEOREM 3.1 Expansion by Cofactors

Let $A$ be a square matrix of order $n$. Then the determinant of $A$ is given by

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

or

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

# 3.5 Applications of Determinants

So far in this chapter, you have examined procedures for evaluating determinants, studied properties of determinants, and learned how determinants are used to find eigenvalues. In this section, you will study an explicit formula for the inverse of a nonsingular matrix and then use this formula to derive a theorem known as Cramer's Rule. You will then solve several applications of determinants using Cramer's Rule.

## *The Adjoint of a Matrix*

Recall from Section 3.1 that the cofactor $C_{ij}$ of a matrix $A$ is defined as $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the $i$th row and the $j$th column of $A$. If $A$ is a square matrix, then the **matrix of cofactors** of $A$ has the form

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

The transpose of this matrix is called the **adjoint** of $A$ and is denoted by $\operatorname{adj}(A)$. That is,

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

**EXAMPLE 1** Finding the Adjoint of a Square Matrix

Find the adjoint of

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}.$$

*SOLUTION* The cofactor $C_{11}$ is given by

$$\begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix} \longrightarrow C_{11} = (-1)^2 \begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} = 4.$$

Continuing this process produces the following matrix of cofactors of $A$.

$$\begin{bmatrix} \begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix}$$

The transpose of this matrix is the adjoint of $A$. That is,

$$\operatorname{adj}(A) = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}.$$

The adjoint of a matrix $A$ can be used to find the inverse of $A$, as indicated in the next theorem.

**THEOREM 3.10 The Inverse of a Matrix Given by Its Adjoint**

If $A$ is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).$$

*PROOF* Begin by proving that the product of $A$ and its adjoint is equal to the product of the determinant of $A$ and $I_n$.

Consider the product

$$A[\operatorname{adj}(A)] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}.$$

The entry in the $i$th row and $j$th column of this product is

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}.$$

If $i = j$, then this sum is simply the cofactor expansion of $A$ along its $i$th row, which means that the sum is the determinant of $A$. On the other hand, if $i \neq j$, then the sum is zero.

$$A[\operatorname{adj}(A)] = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} = \det(A)I$$

Because $A$ is invertible, $\det(A) \neq 0$ and you can write

$$\frac{1}{\det(A)} A[\operatorname{adj}(A)] = I \qquad \text{or} \qquad A\left[\frac{1}{\det(A)} \operatorname{adj}(A)\right] = I.$$

Multiplying both sides of the equation by $A^{-1}$ results in the equation

$$A^{-1}A\left[\frac{1}{\det(A)} \operatorname{adj}(A)\right] = A^{-1}I, \text{ which yields } \frac{1}{\det(A)} \operatorname{adj}(A) = A^{-1}.$$

If $A$ is a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the adjoint of $A$ is simply

$$\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Moreover, if $A$ is invertible, then from Theorem 3.10 you have

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

which agrees with the result in Section 2.3.

### EXAMPLE 2 Using the Adjoint of a Matrix to Find Its Inverse

Use the adjoint of

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

to find $A^{-1}$.

*SOLUTION* The determinant of this matrix is 3. Using the adjoint of $A$ (found in Example 1), you can find the inverse of $A$ to be

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{3} \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}.$$

You can check to see that this matrix is the inverse of $A$ by multiplying to obtain

$$AA^{-1} = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

# 4 Vector Spaces



**CHAPTER OBJECTIVES**

- Perform, recognize, and utilize vector operations on vectors in $R^n$.
- Determine whether a set of vectors with two operations is a vector space and recognize standard examples of vector spaces, such as: $R^n$, $M_{m,n}$, $P_n$, $P$, $C(-\infty, \infty)$, $C[a, b]$.
- Determine whether a subset $W$ of a vector space $V$ is a subspace.
- Write a linear combination of a finite set of vectors in $V$.
- Determine whether a set $S$ of vectors in a vector space $V$ is a spanning set of $V$.
- Determine whether a finite set of vectors in a vector space $V$ is linearly independent.
- Recognize standard bases in the vector spaces $R^n$, $M_{m,n}$, and $P_n$.
- Determine if a vector space is finite dimensional or infinite dimensional.
- Find the dimension of a subspace of $R^n$, $M_{m,n}$, and $P_n$.
- Find a basis and dimension for the column or row space and a basis for the nullspace (nullity) of a matrix.
- Find a general solution of a consistent system $A\mathbf{x} = \mathbf{b}$ in the form $\mathbf{x}_p + \mathbf{x}_h$.
- Find $[\mathbf{x}]_B$ in $R^n$, $M_{m,n}$, and $P_n$.
- Find the transition matrix from the basis $B$ to the basis $B'$ in $R^n$.
- Find $[\mathbf{x}]_{B'}$ for a vector $\mathbf{x}$ in $R^n$.
- Determine whether a function is a solution of a differential equation and find the general solution of a given differential equation.
- Find the Wronskian for a set of functions and test a set of solutions for linear independence.
- Identify and sketch the graph of a conic or degenerate conic section and perform a rotation of axes.

## 4.1 Vectors in $R^n$

In physics and engineering, a vector is characterized by two quantities (length and direction) and is represented by a directed line segment. In this chapter you will see that these are only two special types of vectors. Their geometric representations can help you understand the more general definition of a vector.

This section begins with a short review of vectors in the plane, which is the way vectors were developed historically.

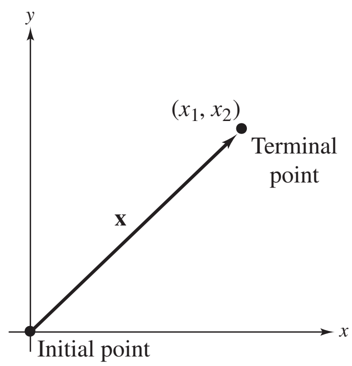



Figure 4.1

## Vectors in the Plane

A **vector in the plane** is represented geometrically by a **directed line segment** whose **initial point** is the origin and whose **terminal point** is the point $(x_1, x_2)$, as shown in Figure 4.1. This vector is represented by the same **ordered pair** used to represent its terminal point. That is,

$$\mathbf{x} = (x_1, x_2).$$

The coordinates $x_1$ and $x_2$ are called the **components** of the vector $\mathbf{x}$. Two vectors in the plane $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.

REMARK: The term *vector* derives from the Latin word *vectus,* meaning "to carry." The idea is that if you were to carry something from the origin to the point $(x_1, x_2)$, the trip could be represented by the directed line segment from $(0, 0)$ to $(x_1, x_2)$. Vectors are represented by lowercase letters set in boldface type (such as $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{x}$).

### EXAMPLE 1 Vectors in the Plane

Use a directed line segment to represent each vector in the plane.

(a) $\mathbf{u} = (2, 3)$ (b) $\mathbf{v} = (-1, 2)$

*SOLUTION* To represent each vector, draw a directed line segment from the origin to the indicated terminal point, as shown in Figure 4.2.

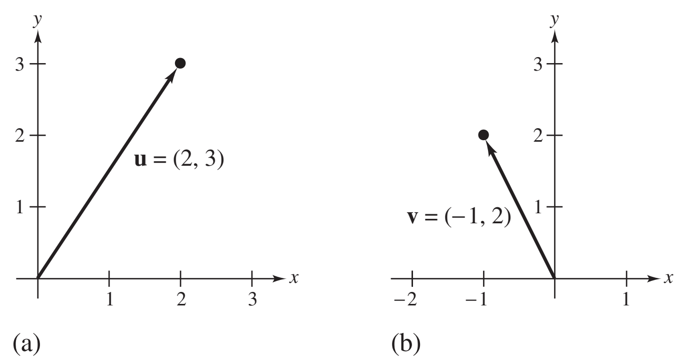



Figure 4.2

The first basic vector operation is **vector addition.** To add two vectors in the plane, add their corresponding components. That is, the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2).$$

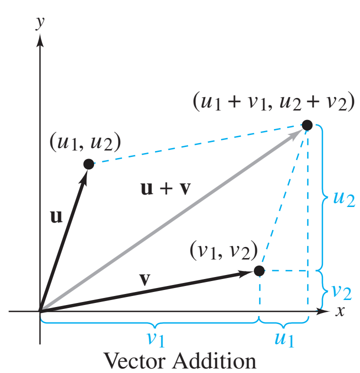



Figure 4.3

Geometrically, the sum of two vectors in the plane is represented as the diagonal of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as its adjacent sides, as shown in Figure 4.3.

In the next example, one of the vectors you will add is the vector $(0, 0)$, called the **zero vector.** The zero vector is denoted by $\mathbf{0}$.

## EXAMPLE 2 Adding Two Vectors in the Plane

Find the sum of the vectors.

(a) $\mathbf{u} = (1, 4)$, $\mathbf{v} = (2, -2)$ (b) $\mathbf{u} = (3, -2)$, $\mathbf{v} = (-3, 2)$ (c) $\mathbf{u} = (2, 1)$, $\mathbf{v} = (0, 0)$

***Simulation***

Explore this concept further with an electronic simulation available on the website *college.hmco.com/pic/larsonELA6e.* Please visit this website for keystrokes and programming syntax for specific graphing utilities and computer software programs applicable to Example 2. Similar exercises and projects are also available on this website.

*SOLUTION*

(a) $\mathbf{u} + \mathbf{v} = (1, 4) + (2, -2) = (3, 2)$

(b) $\mathbf{u} + \mathbf{v} = (3, -2) + (-3, 2) = (0, 0)$

(c) $\mathbf{u} + \mathbf{v} = (2, 1) + (0, 0) = (2, 1)$

Figure 4.4 gives the graphical representation of each sum.

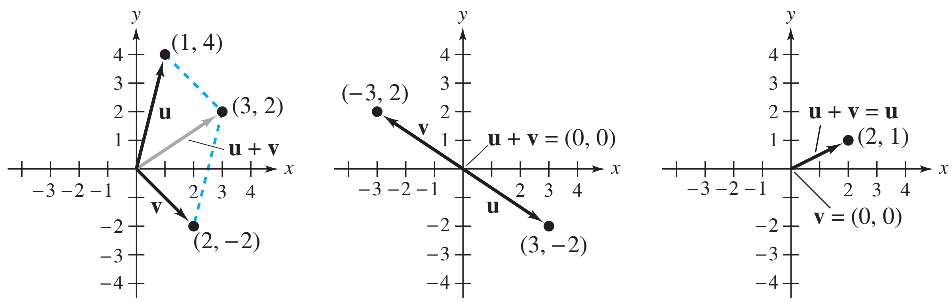



Figure 4.4

The second basic vector operation is called **scalar multiplication.** To multiply a vector $\mathbf{v}$ by a scalar $c$, multiply each of the components of $\mathbf{v}$ by $c$. That is,

$$c\mathbf{v} = c(v_1, v_2) = (cv_1, cv_2).$$

Recall from Chapter 2 that the word *scalar* is used to mean a real number. Historically, this usage arose from the fact that multiplying a vector by a real number changes the "scale" of the vector. For instance, if a vector $\mathbf{v}$ is multiplied by 2, the resulting vector $2\mathbf{v}$ is a vector having the same direction as $\mathbf{v}$ and twice the length. In general, for a scalar $c$, the vector $c\mathbf{v}$ will be $|c|$ times as long as $\mathbf{v}$. If $c$ is positive, then $c\mathbf{v}$ and $\mathbf{v}$ have the same direction, and if $c$ is negative, then $c\mathbf{v}$ and $\mathbf{v}$ have opposite directions. This is shown in Figure 4.5.

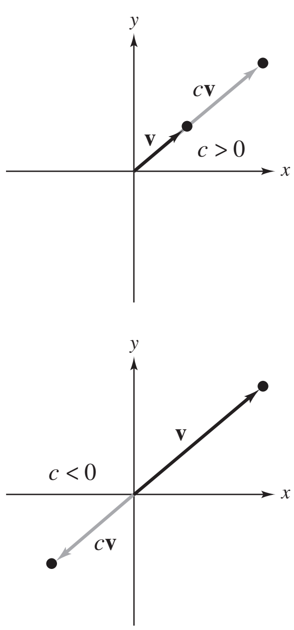



Figure 4.5

The product of a vector $\mathbf{v}$ and the scalar $-1$ is denoted by

$$-\mathbf{v} = (-1)\mathbf{v}.$$

The vector $-\mathbf{v}$ is called the **negative** of $\mathbf{v}$. The **difference** of $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}),$$

and you can say $\mathbf{v}$ is **subtracted** from $\mathbf{u}$.

## EXAMPLE 3 Operations with Vectors in the Plane

Provided with $\mathbf{v} = (-2, 5)$ and $\mathbf{u} = (3, 4)$, find each vector.

(a) $\frac{1}{2}\mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$ (c) $\frac{1}{2}\mathbf{v} + \mathbf{u}$

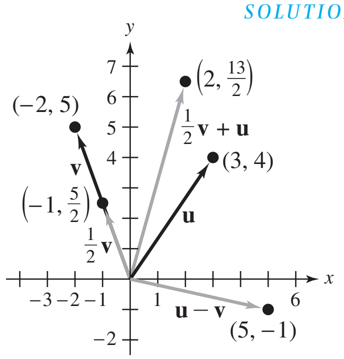



Figure 4.6

*SOLUTION* (a) Because $\mathbf{v} = (-2, 5)$, you have

$$\tfrac{1}{2}\mathbf{v} = \left(\tfrac{1}{2}(-2), \tfrac{1}{2}(5)\right) = \left(-1, \tfrac{5}{2}\right).$$

(b) By the definition of vector subtraction, you have

$$\mathbf{u} - \mathbf{v} = (3 - (-2), 4 - 5) = (5, -1).$$

(c) Using the result of part(a), you have

$$\tfrac{1}{2}\mathbf{v} + \mathbf{u} = \left(-1, \tfrac{5}{2}\right) + (3, 4) = \left(-1 + 3, \tfrac{5}{2} + 4\right) = \left(2, \tfrac{13}{2}\right).$$

Figure 4.6 gives a graphical representation of these vector operations.

Vector addition and scalar multiplication share many properties with matrix addition and scalar multiplication. The ten properties listed in the next theorem play a fundamental role in linear algebra. In fact, in the next section you will see that it is precisely these ten properties that have been abstracted from vectors in the plane to help define the general notion of a vector space.

### THEOREM 4.1 Properties of Vector Addition and Scalar Multiplication in the Plane

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let $c$ and $d$ be scalars.

| | |
|---|---|
| 1. $\mathbf{u} + \mathbf{v}$ is a vector in the plane. | **Closure under addition** |
| 2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | **Commutative property of addition** |
| 3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | **Associative property of addition** |
| 4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | **Additive identity property** |
| 5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ | **Additive inverse property** |
| 6. $c\mathbf{u}$ is a vector in the plane. | **Closure under scalar multiplication** |
| 7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | **Distributive property** |
| 8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ | **Distributive property** |
| 9. $c(d\mathbf{u}) = (cd)\mathbf{u}$ | **Associative property of multiplication** |
| 10. $1(\mathbf{u}) = \mathbf{u}$ | **Multiplicative identity property** |

*PROOF* The proof of each property is a straightforward application of the definition of vector addition and scalar multiplication combined with the corresponding properties of addition and multiplication of real numbers. For instance, to prove the associative property of vector addition, you can write

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= [(u_1, u_2) + (v_1, v_2)] + (w_1, w_2)\\ &= (u_1 + v_1, u_2 + v_2) + (w_1, w_2)\\ &= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2)\\ &= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2))\\ &= (u_1, u_2) + (v_1 + w_1, v_2 + w_2)\\ &= (u_1, u_2) + [(v_1, v_2) + (w_1, w_2)]\\ &= \mathbf{u} + (\mathbf{v} + \mathbf{w}).\end{aligned}$$

**REMARK:** Note that the associative property of vector addition allows you to write such expressions as $\mathbf{u} + \mathbf{v} + \mathbf{w}$ without ambiguity, because you obtain the same vector sum regardless of which addition is performed first.

Similarly, to prove the right distributive property of scalar multiplication over addition,

$$\begin{aligned}(c + d)\mathbf{u} &= (c + d)(u_1, u_2)\\ &= ((c + d)u_1, (c + d)u_2) = (cu_1 + du_1, cu_2 + du_2)\\ &= (cu_1, cu_2) + (du_1, du_2) = c(u_1, u_2) + d(u_1, u_2)\\ &= c\mathbf{u} + d\mathbf{u}.\end{aligned}$$

The proofs of the other eight properties are left as an exercise. (See Exercise 61.)

## Vectors in $R^n$

**HISTORICAL NOTE**

**William Rowan Hamilton (1805–1865)**

is considered to be Ireland's most famous mathematician. His work led to the development of modern vector notation. We still use his **i**, **j**, and **k** notation today. To read about his work, visit *college.hmco.com/pic/larsonELA6e.*

The discussion of vectors in the plane can now be extended to a discussion of vectors in $n$-space. A vector in $n$-space is represented by an **ordered *n*-tuple.** For instance, an ordered triple has the form $(x_1, x_2, x_3)$, an ordered quadruple has the form $(x_1, x_2, x_3, x_4)$, and a general ordered $n$-tuple has the form $(x_1, x_2, x_3, \ldots, x_n)$. The set of all $n$-tuples is called ***n*-space** and is denoted by $R^n$.

$$\begin{aligned}R^1 &= \text{1-space} = \text{set of all real numbers}\\ R^2 &= \text{2-space} = \text{set of all ordered pairs of real numbers}\\ R^3 &= \text{3-space} = \text{set of all ordered triples of real numbers}\\ R^4 &= \text{4-space} = \text{set of all ordered quadruples of real numbers}\\ &\ \vdots\\ R^n &= n\text{-space} = \text{set of all ordered } n\text{-tuples of real numbers}\end{aligned}$$

The practice of using an ordered pair to represent either a point or a vector in $R^2$ continues in $R^n$. That is, an $n$-tuple $(x_1, x_2, x_3, \ldots, x_n)$ can be viewed as a **point** in $R^n$ with the $x_i$'s as its coordinates or as a **vector**

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n) \qquad \text{Vector in } R^n$$

with the $x_i$'s as its components. As with vectors in the plane, two vectors in $R^n$ are **equal** if and only if corresponding components are equal. [In the case of $n = 2$ or $n = 3$, the familiar $(x, y)$ or $(x, y, z)$ notation is used occasionally.]

The sum of two vectors in $R^n$ and the scalar multiple of a vector in $R^n$ are called the **standard operations in $R^n$** and are defined as follows.

### Definitions of Vector Addition and Scalar Multiplication in $R^n$

Let $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n)$ be vectors in $R^n$ and let $c$ be a real number. Then the sum of $\mathbf{u}$ and $\mathbf{v}$ is defined as the vector

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots, u_n + v_n),$$

and the **scalar multiple** of $\mathbf{u}$ by $c$ is defined as the vector

$$c\mathbf{u} = (cu_1, cu_2, cu_3, \ldots, cu_n).$$

As with 2-space, the **negative** of a vector in $R^n$ is defined as

$$-\mathbf{u} = (-u_1, -u_2, -u_3, \ldots, -u_n)$$

and the **difference** of two vectors in $R^n$ is defined as

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, \ldots, u_n - v_n).$$

The **zero vector** in $R^n$ is denoted by $\mathbf{0} = (0, 0, \ldots, 0)$.

## EXAMPLE 4 Vector Operations in $R^3$

Provided that $\mathbf{u} = (-1, 0, 1)$ and $\mathbf{v} = (2, -1, 5)$ in $R^3$, find each vector.

(a) $\mathbf{u} + \mathbf{v}$ (b) $2\mathbf{u}$ (c) $\mathbf{v} - 2\mathbf{u}$

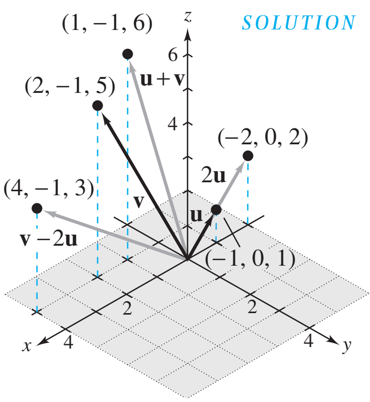



Figure 4.7

*SOLUTION* (a) To add two vectors, add their corresponding components, as follows.

$$\mathbf{u} + \mathbf{v} = (-1, 0, 1) + (2, -1, 5) = (1, -1, 6)$$

(b) To multiply a vector by a scalar, multiply each component by the scalar, as follows.

$$2\mathbf{u} = 2(-1, 0, 1) = (-2, 0, 2)$$

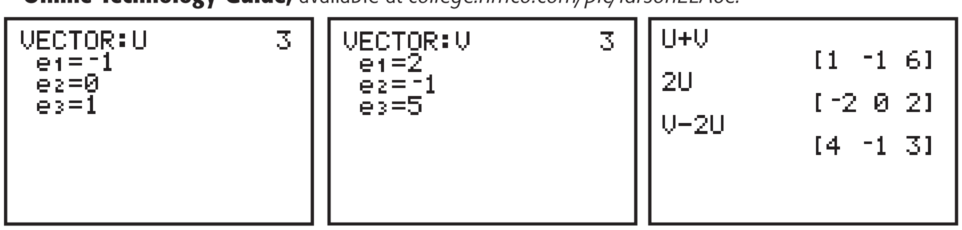
(c) Using the result of part (b), you have

$$\mathbf{v} - 2\mathbf{u} = (2, -1, 5) - (-2, 0, 2) = (4, -1, 3).$$

Figure 4.7 gives a graphical representation of these vector operations in $R^3$.

**Technology Note**

Some graphing utilities and computer software programs will perform vector addition and scalar multiplication. Using a graphing utility, you may verify Example 4 as follows. Keystrokes and programming syntax for these utilities/programs applicable to Example 4 are provided in the **Online Technology Guide,** available at *college.hmco.com/pic/larsonELA6e.*



The following properties of vector addition and scalar multiplication for vectors in $R^n$ are the same as those listed in Theorem 4.1 for vectors in the plane. Their proofs, based on the definitions of vector addition and scalar multiplication in $R^n$, are left as an exercise. (See Exercise 62.)

**THEOREM 4.2 Properties of Vector Addition and Scalar Multiplication in $R^n$**

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $R^n$, and let $c$ and $d$ be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in $R^n$. **Closure under addition**
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ **Commutative property of addition**
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ **Associative property addition**
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ **Additive identity property**
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ **Additive inverse property**
6. $c\mathbf{u}$ is a vector in $R^n$. **Closure under scalar multiplication**
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ **Distributive property**
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ **Distributive property**
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$ **Associative property of multiplication**
10. $1(\mathbf{u}) = \mathbf{u}$ **Multiplicative identity property**

Using the ten properties from Theorem 4.2, you can perform algebraic manipulations with vectors in $R^n$ in much the same way as you do with real numbers, as demonstrated in the next example.

### EXAMPLE 5 Vector Operations in $R^4$

Let $\mathbf{u} = (2, -1, 5, 0)$, $\mathbf{v} = (4, 3, 1, -1)$, and $\mathbf{w} = (-6, 2, 0, 3)$ be vectors in $R^4$. Solve for $\mathbf{x}$.

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$ (b) $3(\mathbf{x} + \mathbf{w}) = 2\mathbf{u} - \mathbf{v} + \mathbf{x}$

*SOLUTION* (a) Using the properties listed in Theorem 4.2, you have

$$\begin{aligned}
\mathbf{x} &= 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w}) \\
&= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
&= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9) \\
&= (4 - 4 + 18, -2 - 3 - 6, 10 - 1 - 0, 0 + 1 - 9) \\
&= (18, -11, 9, -8).
\end{aligned}$$

(b) Begin by solving for $\mathbf{x}$ as follows.

$$\begin{aligned}
3(\mathbf{x} + \mathbf{w}) &= 2\mathbf{u} - \mathbf{v} + \mathbf{x} \\
3\mathbf{x} + 3\mathbf{w} &= 2\mathbf{u} - \mathbf{v} + \mathbf{x} \\
3\mathbf{x} - \mathbf{x} &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
2\mathbf{x} &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
\mathbf{x} &= \tfrac{1}{2}(2\mathbf{u} - \mathbf{v} - 3\mathbf{w})
\end{aligned}$$

Using the result of part (a) produces

$$\begin{aligned}\mathbf{x} &= \tfrac{1}{2}(18, -11, 9, -8)\\ &= \left(9, -\tfrac{11}{2}, \tfrac{9}{2}, -4\right).\end{aligned}$$

The zero vector $\mathbf{0}$ in $R^n$ is called the **additive identity** in $R^n$. Similarly, the vector $-\mathbf{v}$ is called the **additive inverse** of $\mathbf{v}$. The theorem below summarizes several important properties of the additive identity and additive inverse in $R^n$.

**THEOREM 4.3 Properties of Additive Identity and Additive Inverse**

Let $\mathbf{v}$ be a vector in $R^n$, and let $c$ be a scalar. Then the following properties are true.

1. The additive identity is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{v}$, then $\mathbf{u} = \mathbf{0}$.
2. The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$.
3. $0\mathbf{v} = \mathbf{0}$
4. $c\mathbf{0} = \mathbf{0}$
5. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
6. $-(-\mathbf{v}) = \mathbf{v}$

*PROOF* To prove the first property, assume $\mathbf{v} + \mathbf{u} = \mathbf{v}$. Then the steps below are justified by Theorem 4.2.

| | |
|---|---|
| $\mathbf{v} + \mathbf{u} = \mathbf{v}$ | Given |
| $(\mathbf{v} + \mathbf{u}) + (-\mathbf{v}) = \mathbf{v} + (-\mathbf{v})$ | Add $-\mathbf{v}$ to both sides. |
| $(\mathbf{v} + \mathbf{u}) + (-\mathbf{v}) = \mathbf{0}$ | Additive inverse |
| $\mathbf{v} + [\mathbf{u} + (-\mathbf{v})] = \mathbf{0}$ | Associative property |
| $\mathbf{v} + [(-\mathbf{v}) + \mathbf{u}] = \mathbf{0}$ | Commutative property |
| $[\mathbf{v} + (-\mathbf{v})] + \mathbf{u} = \mathbf{0}$ | Associative property |
| $\mathbf{0} + \mathbf{u} = \mathbf{0}$ | Additive inverse |
| $\mathbf{u} + \mathbf{0} = \mathbf{0}$ | Commutative property |
| $\mathbf{u} = \mathbf{0}$ | Additive identity |

As you gain experience in reading and writing proofs involving vector algebra, you will not need to list this many steps. For now, however, it's a good idea. The proofs of the other five properties are left as exercises. (See Exercises 63–67.)

**REMARK:** In Properties 3 and 5 of Theorem 4.3, note that two different zeros are used, the scalar 0 and the vector $\mathbf{0}$.

The next example illustrates an important type of problem in linear algebra—writing one vector $\mathbf{x}$ as the sum of scalar multiples of other vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots,$ and $\mathbf{v}_n$. That is,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

The vector $\mathbf{x}$ is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots,$ and $\mathbf{v}_n$.

### EXAMPLE 6 Writing a Vector as a Linear Combination of Other Vectors

Provided that $\mathbf{x} = (-1, -2, -2)$, $\mathbf{u} = (0, 1, 4)$, $\mathbf{v} = (-1, 1, 2)$, and $\mathbf{w} = (3, 1, 2)$ in $R^3$, find scalars $a$, $b$, and $c$ such that

$$\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

*SOLUTION* By writing

$$\begin{aligned}\overbrace{(-1, -2, -2)}^{\mathbf{x}} &= a\overbrace{(0, 1, 4)}^{\mathbf{u}} + b\overbrace{(-1, 1, 2)}^{\mathbf{v}} + c\overbrace{(3, 1, 2)}^{\mathbf{w}}\\ &= (-b + 3c, a + b + c, 4a + 2b + 2c),\end{aligned}$$

you can equate corresponding components so that they form the system of three linear equations in $a$, $b$, and $c$ shown below.

$$\begin{aligned} -b + 3c &= -1 &\quad& \text{Equation from first component}\\ a + b + c &= -2 && \text{Equation from second component}\\ 4a + 2b + 2c &= -2 && \text{Equation from third component}\end{aligned}$$

Using the techniques of Chapter 1, solve for $a$, $b$, and $c$ to get

$$a = 1, \quad b = -2, \quad \text{and} \quad c = -1.$$

$\mathbf{x}$ can be written as a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

$$\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}$$

Try using vector addition and scalar multiplication to check this result.

**Discovery**

*Is the vector $(1, 1)$ a linear combination of the vectors $(1, 2)$ and $(-2, -4)$? Graph these vectors in the plane and explain your answer geometrically. Similarly, determine whether the vector $(1, 1)$ is a linear combination of the vectors $(1, 2)$ and $(2, 1)$. What is the geometric significance of these two questions? Is every vector in $R^2$ a linear combination of the vectors $(1, 2)$ and $(2, 1)$? Give a geometric explanation for your answer.*

You will often find it useful to represent a vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in $R^n$ as either a $1 \times n$ row matrix (row vector),

$$\mathbf{u} = [u_1 \quad u_2 \quad \cdots \quad u_n],$$

or an $n \times 1$ column matrix (column vector),

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

This approach is valid because the matrix operations of addition and scalar multiplication give the same results as the corresponding vector operations. That is, the matrix sums

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= [u_1 \quad u_2 \quad \cdots \quad u_n] + [v_1 \quad v_2 \quad \cdots \quad v_n] \\ &= [u_1 + v_1 \quad u_2 + v_2 \quad \cdots \quad u_n + v_n]\end{aligned}$$

and

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

yield the same results as the vector operation of addition,

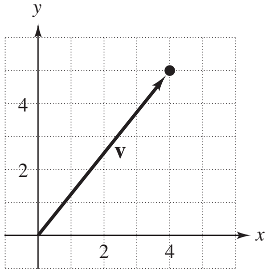
$$\mathbf{u} + \mathbf{v} = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

The same argument applies to scalar multiplication. The only difference in the three notations for vectors is how the components are displayed; the underlying operations are the same.

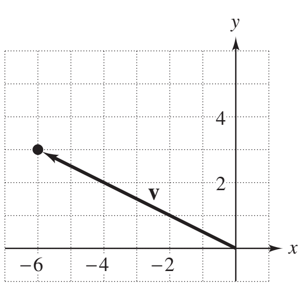
## SECTION 4.1 Exercises

In Exercises 1 and 2, find the component form of the vector shown.

**1.**




**2.**




In Exercises 3–6, use a directed line segment to represent the vector.

**3.** $\mathbf{u} = (2, -4)$ **4.** $\mathbf{v} = (-2, 3)$

**5.** $\mathbf{u} = (-3, -4)$ **6.** $\mathbf{v} = (-2, -5)$

In Exercises 7–10, find the sum of the vectors and illustrate the indicated vector operations geometrically.

**7.** $\mathbf{u} = (1, 3), \mathbf{v} = (2, -2)$ **8.** $\mathbf{u} = (-1, 4), \mathbf{v} = (4, -3)$

**9.** $\mathbf{u} = (2, -3), \mathbf{v} = (-3, -1)$

**10.** $\mathbf{u} = (4, -2), \mathbf{v} = (-2, -3)$

In Exercises 11–16, find the vector **v** and illustrate the indicated vector operations geometrically, where $\mathbf{u} = (-2, 3)$ and $\mathbf{w} = (-3, -2)$.

**11.** $\mathbf{v} = \frac{3}{2}\mathbf{u}$ **12.** $\mathbf{v} = \mathbf{u} + \mathbf{w}$ **13.** $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$

**14.** $\mathbf{v} = -\mathbf{u} + \mathbf{w}$ **15.** $\mathbf{v} = \frac{1}{2}(3\mathbf{u} + \mathbf{w})$ **16.** $\mathbf{v} = \mathbf{u} - 2\mathbf{w}$

**17.** Given the vector $\mathbf{v} = (2, 1)$, sketch (a) $2\mathbf{v}$, (b) $-3\mathbf{v}$, and (c) $\frac{1}{2}\mathbf{v}$.

**18.** Given the vector $\mathbf{v} = (3, -2)$, sketch (a) $4\mathbf{v}$, (b) $-\frac{1}{2}\mathbf{v}$, and (c) $0\mathbf{v}$.

In Exercises 19–24, let $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (2, 2, -1)$, and $\mathbf{w} = (4, 0, -4)$.

**19.** Find $\mathbf{u} - \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$. **20.** Find $\mathbf{u} - \mathbf{v} + 2\mathbf{w}$.

**21.** Find $2\mathbf{u} + 4\mathbf{v} - \mathbf{w}$. **22.** Find $5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w}$.

**23.** Find **z**, where $2\mathbf{z} - 3\mathbf{u} = \mathbf{w}$.

**24.** Find **z**, where $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = \mathbf{0}$.

**25.** Given the vector $\mathbf{v} = (1, 2, 2)$, sketch (a) $2\mathbf{v}$, (b) $-\mathbf{v}$, and (c) $\frac{1}{2}\mathbf{v}$.

**26.** Given the vector $\mathbf{v} = (2, 0, 1)$, sketch (a) $-\mathbf{v}$, (b) $2\mathbf{v}$, and (c) $\frac{1}{2}\mathbf{v}$.

**27.** Which of the vectors below are scalar multiples of $\mathbf{z} = (3, 2, -5)$?
(a) $\mathbf{u} = (-6, -4, 10)$
(b) $\mathbf{v} = \left(2, \frac{4}{3}, -\frac{10}{3}\right)$
(c) $\mathbf{w} = (6, 4, 10)$

**28.** Which of the vectors below are scalar multiples of $\mathbf{z} = \left(\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}\right)$?
(a) $\mathbf{u} = (6, -4, 9)$
(b) $\mathbf{v} = \left(-1, \frac{4}{3}, -\frac{3}{2}\right)$
(c) $\mathbf{w} = (12, 0, 9)$

In Exercises 29–32, find (a) $\mathbf{u} - \mathbf{v}$, (b) $2(\mathbf{u} + 3\mathbf{v})$, and (c) $2\mathbf{v} - \mathbf{u}$.

**29.** $\mathbf{u} = (4, 0, -3, 5)$, $\mathbf{v} = (0, 2, 5, 4)$

**30.** $\mathbf{u} = (0, 4, 3, 4, 4)$, $\mathbf{v} = (6, 8, -3, 3, -5)$

**31.** $\mathbf{u} = (-7, 0, 0, 0, 9)$, $\mathbf{v} = (2, -3, -2, 3, 3)$

**32.** $\mathbf{u} = (6, -5, 4, 3)$, $\mathbf{v} = \left(-2, \frac{5}{3}, -\frac{4}{3}, -1\right)$

In Exercises 33 and 34, use a graphing utility with matrix capabilities to find the following, where $\mathbf{u} = (1, 2, -3, 1)$, $\mathbf{v} = (0, 2, -1, -2)$, and $\mathbf{w} = (2, -2, 1, 3)$.

**33.** (a) $\mathbf{u} + 2\mathbf{v}$
(b) $\mathbf{w} - 3\mathbf{u}$
(c) $4\mathbf{v} + \frac{1}{2}\mathbf{u} - \mathbf{w}$
(d) $\frac{1}{4}(3\mathbf{u} + 2\mathbf{v} - \mathbf{w})$

**34.** (a) $\mathbf{v} + 3\mathbf{w}$
(b) $2\mathbf{w} - \frac{1}{2}\mathbf{u}$
(c) $2\mathbf{u} + \mathbf{w} - 3\mathbf{v}$
(d) $\frac{1}{2}(4\mathbf{v} - 3\mathbf{u} + \mathbf{w})$

In Exercises 35–38, solve for $\mathbf{w}$ provided that $\mathbf{u} = (1, -1, 0, 1)$ and $\mathbf{v} = (0, 2, 3, -1)$.

**35.** $2\mathbf{w} = \mathbf{u} - 3\mathbf{v}$

**36.** $\mathbf{w} + \mathbf{u} = -\mathbf{v}$

**37.** $\frac{1}{2}\mathbf{w} = 2\mathbf{u} + 3\mathbf{v}$

**38.** $\mathbf{w} + 3\mathbf{v} = -2\mathbf{u}$

In Exercises 39–44, write $\mathbf{v}$ as a linear combination of $\mathbf{u}$ and $\mathbf{w}$, if possible, where $\mathbf{u} = (1, 2)$ and $\mathbf{w} = (1, -1)$.

**39.** $\mathbf{v} = (2, 1)$

**40.** $\mathbf{v} = (0, 3)$

**41.** $\mathbf{v} = (3, 0)$

**42.** $\mathbf{v} = (1, -1)$

**43.** $\mathbf{v} = (-1, -2)$

**44.** $\mathbf{v} = (1, -4)$

In Exercises 45 and 46, find $\mathbf{w}$ such that $2\mathbf{u} + \mathbf{v} - 3\mathbf{w} = \mathbf{0}$.

**45.** $\mathbf{u} = (0, 2, 7, 5)$, $\mathbf{v} = (-3, 1, 4, -8)$

**46.** $\mathbf{u} = (0, 0, -8, 1)$, $\mathbf{v} = (1, -8, 0, 7)$

In Exercises 47–50, write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, if possible.

**47.** $\mathbf{u}_1 = (2, 3, 5)$, $\mathbf{u}_2 = (1, 2, 4)$, $\mathbf{u}_3 = (-2, 2, 3)$,
$\mathbf{v} = (10, 1, 4)$

**48.** $\mathbf{u}_1 = (1, 3, 5)$, $\mathbf{u}_2 = (2, -1, 3)$, $\mathbf{u}_3 = (-3, 2, -4)$,
$\mathbf{v} = (-1, 7, 2)$

**49.** $\mathbf{u}_1 = (1, 1, 2, 2)$, $\mathbf{u}_2 = (2, 3, 5, 6)$, $\mathbf{u}_3 = (-3, 1, -4, 2)$,
$\mathbf{v} = (0, 5, 3, 0)$

**50.** $\mathbf{u}_1 = (1, 3, 2, 1)$, $\mathbf{u}_2 = (2, -2, -5, 4)$, $\mathbf{u}_3 = (2, -1, 3, 6)$,
$\mathbf{v} = (2, 5, -4, 0)$

In Exercises 51–54, use a graphing utility or computer software program with matrix capabilities to write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, and $\mathbf{u}_5$, or of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$, and $\mathbf{u}_6$. Then verify your solution.

**51.** $\mathbf{u}_1 = (1, 2, -3, 4, -1)$
$\mathbf{u}_2 = (1, 2, 0, 2, 1)$
$\mathbf{u}_3 = (0, 1, 1, 1, -4)$
$\mathbf{u}_4 = (2, 1, -1, 2, 1)$
$\mathbf{u}_5 = (0, 2, 2, -1, -1)$
$\mathbf{v} = (5, 3, -11, 11, 9)$

**52.** $\mathbf{u}_1 = (1, 1, -1, 2, 1)$
$\mathbf{u}_2 = (2, 1, 2, -1, 1)$
$\mathbf{u}_3 = (1, 2, 0, 1, 2)$
$\mathbf{u}_4 = (0, 2, 0, 1, -4)$
$\mathbf{u}_5 = (1, 1, 2, -1, 2)$
$\mathbf{v} = (5, 8, 7, -2, 4)$

**53.** $\mathbf{u}_1 = (1, 2, -3, 4, -1, 2)$
$\mathbf{u}_2 = (1, -2, 1, -1, 2, 1)$
$\mathbf{u}_3 = (0, 2, -1, 2, -1, -1)$
$\mathbf{u}_4 = (1, 0, 3, -4, 1, 2)$
$\mathbf{u}_5 = (1, -2, 1, -1, 2, -3)$
$\mathbf{u}_6 = (3, 2, 1, -2, 3, 0)$
$\mathbf{v} = (10, 30, -13, 14, -7, 27)$

**54.** $\mathbf{u}_1 = (1, -3, 4, -5, 2, -1)$
$\mathbf{u}_2 = (3, -2, 4, -3, -2, 1)$
$\mathbf{u}_3 = (1, 1, 1, -1, 4, -1)$
$\mathbf{u}_4 = (3, -1, 3, -4, 2, 3)$
$\mathbf{u}_5 = (1, -2, 1, 5, -3, 4)$
$\mathbf{u}_6 = (4, 2, -1, 3, -1, 1)$
$\mathbf{v} = (8, 17, -16, 26, 0, -4)$

**True or False?** In Exercises 55 and 56, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

**55.** (a) Two vectors in $R^n$ are equal if and only if their corresponding components are equal.
(b) For a nonzero scalar $c$, the vector $c\mathbf{v}$ is $c$ times as long as $\mathbf{v}$ and has the same direction as $\mathbf{v}$ if $c > 0$ and the opposite direction if $c < 0$.

**56.** (a) To add two vectors in $R^n$, add their corresponding components.
(b) The zero vector $\mathbf{0}$ in $R^n$ is defined as the additive inverse of a vector.

In Exercises 57 and 58, the zero vector $\mathbf{0} = (0, 0, 0)$ can be written as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$. This is called the *trivial* solution. Can you find a *nontrivial* way of writing $\mathbf{0}$ as a linear combination of the three vectors?

**57.** $\mathbf{v}_1 = (1, 0, 1), \quad \mathbf{v}_2 = (-1, 1, 2), \quad \mathbf{v}_3 = (0, 1, 4)$

**58.** $\mathbf{v}_1 = (1, 0, 1), \quad \mathbf{v}_2 = (-1, 1, 2), \quad \mathbf{v}_3 = (0, 1, 3)$

**59.** Illustrate properties 1–10 of Theorem 4.2 for $\mathbf{u} = (2, -1, 3, 6)$, $\mathbf{v} = (1, 4, 0, 1)$, $\mathbf{w} = (3, 0, 2, 0)$, $c = 5$, and $d = -2$.

**60.** Illustrate properties 1–10 of Theorem 4.2 for $\mathbf{u} = (2, -1, 3)$, $\mathbf{v} = (3, 4, 0)$, $\mathbf{w} = (7, 8, -4)$, $c = 2$, and $d = -1$.

**61.** Complete the proof of Theorem 4.1.

**62.** Prove each property of vector addition and scalar multiplication from Theorem 4.2.
(a) Property 1: $\mathbf{u} + \mathbf{v}$ is a vector in $R^n$.
(b) Property 2: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
(c) Property 3: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
(d) Property 4: $\mathbf{u} + \mathbf{0} = \mathbf{u}$
(e) Property 5: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
(f) Property 6: $c\mathbf{u}$ is a vector in $R^n$.
(g) Property 7: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
(h) Property 8: $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
(i) Property 9: $c(d\mathbf{u}) = (cd)\mathbf{u}$
(j) Property 10: $1(\mathbf{u}) = \mathbf{u}$

In Exercises 63–67, complete the proofs of the remaining properties of Theorem 4.3 by supplying the justification for each step. Use the properties of vector addition and scalar multiplication from Theorem 4.2.

**63.** Property 2: The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$.

| | |
|---|---|
| $\mathbf{v} + \mathbf{u} = \mathbf{0}$ | Given |
| $(-\mathbf{v}) + (\mathbf{v} + \mathbf{u}) = (-\mathbf{v}) + \mathbf{0}$ | a. ________ |
| $((-\mathbf{v}) + \mathbf{v}) + \mathbf{u} = -\mathbf{v}$ | b. ________ |
| $\mathbf{0} + \mathbf{u} = -\mathbf{v}$ | c. ________ |
| $\mathbf{u} + \mathbf{0} = -\mathbf{v}$ | d. ________ |
| $\mathbf{u} = -\mathbf{v}$ | e. ________ |

**64.** Property 3: $0\mathbf{v} = \mathbf{0}$

| | |
|---|---|
| $0\mathbf{v} = (0 + 0)\mathbf{v}$ | a. ________ |
| $0\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$ | b. ________ |
| $0\mathbf{v} + (-0\mathbf{v}) = (0\mathbf{v} + 0\mathbf{v}) + (-0\mathbf{v})$ | c. ________ |
| $\mathbf{0} = 0\mathbf{v} + (0\mathbf{v} + (-0\mathbf{v}))$ | d. ________ |
| $\mathbf{0} = 0\mathbf{v} + \mathbf{0}$ | e. ________ |
| $\mathbf{0} = 0\mathbf{v}$ | f. ________ |

**65.** Property 4: $c\mathbf{0} = \mathbf{0}$

| | |
|---|---|
| $c\mathbf{0} = c(\mathbf{0} + \mathbf{0})$ | a. ________ |
| $c\mathbf{0} = c\mathbf{0} + c\mathbf{0}$ | b. ________ |
| $c\mathbf{0} + (-c\mathbf{0}) = (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0})$ | c. ________ |
| $\mathbf{0} = c\mathbf{0} + (c\mathbf{0} + (-c\mathbf{0}))$ | d. ________ |
| $\mathbf{0} = c\mathbf{0} + \mathbf{0}$ | e. ________ |
| $\mathbf{0} = c\mathbf{0}$ | f. ________ |

**66.** Property 5: If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$. If $c = 0$, you are done. If $c \neq 0$, then $c^{-1}$ exists, and you have

| | |
|---|---|
| $c^{-1}(c\mathbf{v}) = c^{-1}\mathbf{0}$ | a. ________ |
| $(c^{-1}c)\mathbf{v} = \mathbf{0}$ | b. ________ |
| $1\mathbf{v} = \mathbf{0}$ | c. ________ |
| $\mathbf{v} = \mathbf{0}$. | d. ________ |

**67.** Property 6: $-(-\mathbf{v}) = \mathbf{v}$

| | |
|---|---|
| $-(-\mathbf{v}) + (-\mathbf{v}) = \mathbf{0}$ and $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ | a. ________ |
| $-(-\mathbf{v}) + (-\mathbf{v}) = \mathbf{v} + (-\mathbf{v})$ | b. ________ |
| $-(-\mathbf{v}) + (-\mathbf{v}) + \mathbf{v} = \mathbf{v} + (-\mathbf{v}) + \mathbf{v}$ | c. ________ |
| $-(-\mathbf{v}) + ((-\mathbf{v}) + \mathbf{v}) = \mathbf{v} + ((-\mathbf{v}) + \mathbf{v})$ | d. ________ |
| $-(-\mathbf{v}) + \mathbf{0} = \mathbf{v} + \mathbf{0}$ | e. ________ |
| $-(-\mathbf{v}) = \mathbf{v}$ | f. ________ |

In Exercises 68 and 69, determine if the third column can be written as a linear combination of the first two columns.

**68.** $\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}$ **69.** $\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 7 \end{bmatrix}$

**70.** **Writing** Let $A\mathbf{x} = \mathbf{b}$ be a system of $m$ linear equations in $n$ variables. Designate the columns of $A$ as $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. If $\mathbf{b}$ is a linear combination of these $n$ column vectors, explain why this implies that the linear system is consistent. Illustrate your answer with appropriate examples. What can you conclude about the linear system if $\mathbf{b}$ is not a linear combination of the columns of $A$?

**71.** **Writing** How could you describe vector subtraction geometrically? What is the relationship between vector subtraction and the basic vector operations of addition and scalar multiplication?

## 4.2 Vector Spaces

In Theorem 4.2, ten special properties of vector addition and scalar multiplication in $R^n$ were listed. Suitable definitions of addition and scalar multiplication reveal that many other mathematical quantities (such as matrices, polynomials, and functions) also share these ten properties. *Any* set that satisfies these properties (or **axioms**) is called a **vector space,** and the objects in the set are called **vectors.**

It is important to realize that the next definition of vector space is precisely that—a *definition.* You do not need to prove anything because you are simply listing the axioms required of vector spaces. This type of definition is called an **abstraction** because you are abstracting a collection of properties from a particular setting $R^n$ to form the axioms for a more general setting.

### Definition of Vector Space

Let $V$ be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $V$ and every scalar (real number) $c$ and $d$, then $V$ is called a **vector space.**

Addition:

1. $\mathbf{u} + \mathbf{v}$ is in $V$. — **Closure under addition**
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — **Commutative property**
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ — **Associative property**
4. $V$ has a **zero vector** $\mathbf{0}$ such that for every $\mathbf{u}$ in $V$, $\mathbf{u} + \mathbf{0} = \mathbf{u}$. — **Additive identity**
5. For every $\mathbf{u}$ in $V$, there is a vector in $V$ denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. — **Additive inverse**

Scalar Multiplication:

6. $c\mathbf{u}$ is in $V$. — **Closure under scalar multiplication**
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — **Distributive property**
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — **Distributive property**
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$ — **Associative property**
10. $1(\mathbf{u}) = \mathbf{u}$ — **Scalar identity**

It is important to realize that a vector space consists of four entities: a set of vectors, a set of scalars, and two operations. When you refer to a vector space $V$, be sure all four entities are clearly stated or understood. Unless stated otherwise, assume that the set of scalars is the set of real numbers.

The first two examples of vector spaces on the next page are not surprising. They are, in fact, the models used to form the ten vector space axioms.

**47.** Let $V$ and $W$ be two subspaces of a vector space $U$. Prove that the set

$$V + W = \{\mathbf{u} : \mathbf{u} = \mathbf{v} + \mathbf{w}, \text{ where } \mathbf{v} \in V \text{ and } \mathbf{w} \in W\}$$

is a subspace of $U$. Describe $V + W$ if $V$ and $W$ are the subspaces of $U = R^2$:

$V = \{(x, 0) : x$ is a real number$\}$ and $W = \{(0, y) : y$ is a real number$\}$.

**48.** Complete the proof of Theorem 4.6 by showing that the intersection of two subspaces of a vector space is closed under scalar multiplication.

## 4.4 Spanning Sets and Linear Independence

This section begins to develop procedures for representing each vector in a vector space as a **linear combination** of a select number of vectors in the space.

**Definition of Linear Combination of Vectors**

A vector $\mathbf{v}$ in a vector space $V$ is called a **linear combination** of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in $V$ if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k,$$

where $c_1, c_2, \ldots, c_k$ are scalars.

Often, one or more of the vectors in a set can be written as linear combinations of other vectors in the set. Examples 1, 2, and 3 illustrate this possibility.

### EXAMPLE 1 Examples of Linear Combinations

(a) For the set of vectors in $R^3$,

$$S = \{\overset{\mathbf{v}_1}{(1, 3, 1)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(1, 0, -5)}\},$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ because

$$\begin{aligned}\mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3 &= 3(0, 1, 2) + (1, 0, -5)\\ &= (1, 3, 1).\end{aligned}$$

(b) For the set of vectors in $M_{2,2}$,

$$S = \left\{\overset{\mathbf{v}_1}{\begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}}, \overset{\mathbf{v}_4}{\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}}\right\},$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ because

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{v}_2 + 2\mathbf{v}_3 - \mathbf{v}_4\\ &= \begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix} + 2\begin{bmatrix}-1 & 3\\ 1 & 2\end{bmatrix} - \begin{bmatrix}-2 & 0\\ 1 & 3\end{bmatrix}\\ &= \begin{bmatrix}0 & 8\\ 2 & 1\end{bmatrix}.\end{aligned}$$

In Example 1 it was easy to verify that one of the vectors in the set $S$ was a linear combination of the other vectors because you were provided with the appropriate coefficients to form the linear combination. In the next example, a procedure for finding the coefficients is demonstrated.

## EXAMPLE 2 Finding a Linear Combination

Write the vector $\mathbf{w} = (1, 1, 1)$ as a linear combination of vectors in the set $S$.

$$S = \{\overset{\mathbf{v}_1}{(1, 2, 3)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(-1, 0, 1)}\}$$

*SOLUTION* You need to find scalars $c_1$, $c_2$, and $c_3$ such that

$$\begin{aligned}(1, 1, 1) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1)\\ &= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3).\end{aligned}$$

Equating corresponding components yields the system of linear equations below.

$$\begin{aligned}c_1 \qquad\quad - c_3 &= 1\\ 2c_1 + \phantom{2}c_2 \qquad &= 1\\ 3c_1 + 2c_2 + c_3 &= 1\end{aligned}$$

Using Gauss-Jordan elimination, you can show that this system has an infinite number of solutions, each of the form

$$c_1 = 1 + t, \qquad c_2 = -1 - 2t, \qquad c_3 = t.$$

To obtain one solution, you could let $t = 1$. Then $c_3 = 1$, $c_2 = -3$, and $c_1 = 2$, and you have

$$\mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

Other choices for $t$ would yield other ways to write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

**EXAMPLE 3** Finding a Linear Combination

If possible, write the vector $\mathbf{w} = (1, -2, 2)$ as a linear combination of vectors in the set $S$ from Example 2.

*SOLUTION* Following the procedure from Example 2 results in the system

$$\begin{aligned} c_1 \quad\quad\; - c_3 &= \phantom{-}1 \\ 2c_1 + \phantom{2}c_2 \quad\quad &= -2 \\ 3c_1 + 2c_2 + c_3 &= \phantom{-}2. \end{aligned}$$

The augmented matrix of this system row reduces to

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the third row you can conclude that the system of equations is inconsistent, and that means that there is no solution. Consequently, $\mathbf{w}$ *cannot* be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

## Spanning Sets

If every vector in a vector space can be written as a linear combination of vectors in a set $S$, then $S$ is called a **spanning set** of the vector space.

**Definition of Spanning Set of a Vector Space**

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of a vector space $V$. The set $S$ is called a **spanning set** of $V$ if *every* vector in $V$ can be written as a linear combination of vectors in $S$. In such cases it is said that $S$ **spans** $V$.

**EXAMPLE 4** Examples of Spanning Sets

(a) The set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans $R^3$ because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in $R^3$ can be written as

$$\mathbf{u} = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = (u_1, u_2, u_3).$$

(b) The set $S = \{1, x, x^2\}$ spans $P_2$ because any polynomial function $p(x) = a + bx + cx^2$ in $P_2$ can be written as

$$\begin{aligned} p(x) &= a(1) + b(x) + c(x^2) \\ &= a + bx + cx^2. \end{aligned}$$

The spanning sets in Example 4 are called the **standard spanning sets** of $R^3$ and $P_2$, respectively. (You will learn more about standard spanning sets in the next section.) In the next example you will look at a nonstandard spanning set of $R^3$.

### EXAMPLE 5 A Spanning Set of $R^3$

Show that the set $S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ spans $R^3$.

SOLUTION Let $\mathbf{u} = (u_1, u_2, u_3)$ be *any* vector in $R^3$. You need to find scalars $c_1$, $c_2$, and $c_3$ such that

$$\begin{aligned}(u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1)\\ &= (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3).\end{aligned}$$

This vector equation produces the system

$$\begin{aligned} c_1 \quad\quad\;\; - 2c_3 &= u_1\\ 2c_1 + \;\; c_2 \quad\quad\;\; &= u_2\\ 3c_1 + 2c_2 + \;\; c_3 &= u_3.\end{aligned}$$

The coefficient matrix for this system has a nonzero determinant, and it follows from the list of equivalent conditions given in Section 3.3 that the system has a unique solution. So, any vector in $R^3$ can be written as a linear combination of the vectors in $S$, and you can conclude that the set $S$ spans $R^3$.

### EXAMPLE 6 A Set That Does Not Span $R^3$

From Example 3 you know that the set

$$S = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$$

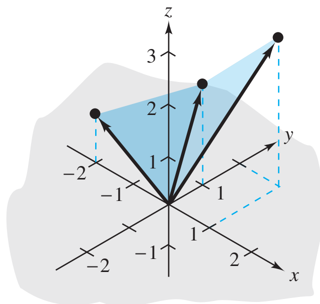
*does not* span $R^3$ because $\mathbf{w} = (1, -2, 2)$ is in $R^3$ and cannot be expressed as a linear combination of the vectors in $S$.

Comparing the sets of vectors in Examples 5 and 6, note that the sets are the same except for a seemingly insignificant difference in the third vector.

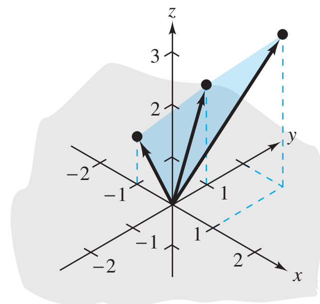
$$S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\} \qquad \textbf{Example 5}$$

$$S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\} \qquad \textbf{Example 6}$$

The difference, however, is significant, because the set $S_1$ spans $R^3$ whereas the set $S_2$ does not. The reason for this difference can be seen in Figure 4.16. The vectors in $S_2$ lie in a common plane; the vectors in $S_1$ do not.




$S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$
The vectors in $S_1$ do not lie in a common plane.




$S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$
The vectors in $S_2$ lie in a common plane.

Figure 4.16

Although the set $S_2$ does not span all of $R^3$, it does span a subspace of $R^3$—namely, the plane in which the three vectors of $S_2$ lie. This subspace is called the **span of $S_2$,** as indicated in the next definition.

**Definition of the Span of a Set**

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space $V$, then the **span of $S$** is the set of all linear combinations of the vectors in $S$,

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k : c_1, c_2, \ldots, c_k \text{ are real numbers}\}.$$

The span of $S$ is denoted by span($S$) or span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. If span($S$) $= V$, it is said that $V$ is **spanned** by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, or that $S$ **spans** $V$.

The theorem below tells you that the span of any finite nonempty subset of a vector space $V$ is a subspace of $V$.

**THEOREM 4.7**
**Span($S$) Is a Subspace of $V$**

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space $V$, then span($S$) is a subspace of $V$. Moreover, span($S$) is the smallest subspace of $V$ that contains $S$, in the sense that every other subspace of $V$ that contains $S$ must contain span($S$).

*PROOF* To show that span($S$), the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, is a subspace of $V$, show that it is closed under addition and scalar multiplication. Consider any two vectors $\mathbf{u}$ and $\mathbf{v}$ in span($S$),

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$
$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_k,$$

where $c_1, c_2, \ldots, c_k$ and $d_1, d_2, \ldots, d_k$ are scalars. Then

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \cdots + (c_k + d_k)\mathbf{v}_k$$

and

$$c\mathbf{u} = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2 + \cdots + (cc_k)\mathbf{v}_k,$$

which means that $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ are also in span($S$) because they can be written as linear combinations of vectors in $S$. So, span($S$) is a subspace of $V$. It is left to you to prove that span($S$) is the smallest subspace of $V$ that contains $S$. (See Exercise 50.)

## Linear Dependence and Linear Independence

For a given set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space $V$, the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

always has the **trivial solution**

$$c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

Often, however, there are also **nontrivial** solutions. For instance, in Example 1(a) you saw that in the set

$$S = \{\overset{\mathbf{v}_1}{(1, 3, 1)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(1, 0, -5)}\},$$

the vector $\mathbf{v}_1$ can be written as a linear combination of the other two as follows.

$$\mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3$$

The vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

has a nontrivial solution in which the coefficients are *not all zero:*

$$c_1 = 1, \qquad c_2 = -3, \qquad c_3 = -1.$$

This characteristic is described by saying that the set $S$ is **linearly dependent.** Had the *only* solution been the trivial one $(c_1 = c_2 = c_3 = 0)$, then the set $S$ would have been **linearly independent.** This notion is essential to the study of linear algebra, and is formally stated in the next definition.

**Definition of Linear Dependence and Linear Independence**

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_k\}$ in a vector space $V$ is called **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot \cdot \cdot + c_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution, $c_1 = 0, c_2 = 0, . . . , c_k = 0$. If there are also nontrivial solutions, then $S$ is called **linearly dependent.**

### EXAMPLE 7 Examples of Linearly Dependent Sets

(a) The set $S = \{(1, 2), (2, 4)\}$ in $R^2$ is linearly dependent because

$$-2(1, 2) + (2, 4) = (0, 0).$$

(b) The set $S = \{(1, 0), (0, 1), (-2, 5)\}$ in $R^2$ is linearly dependent because

$$2(1, 0) - 5(0, 1) + (-2, 5) = (0, 0).$$

(c) The set $S = \{(0, 0), (1, 2)\}$ in $R^2$ is linearly dependent because

$$1(0, 0) + 0(1, 2) = (0, 0).$$

The next example demonstrates a testing procedure for determining whether a set of vectors is linearly independent or linearly dependent.

### EXAMPLE 8 Testing for Linear Independence

Determine whether the set of vectors in $R^3$ is linearly independent or linearly dependent.

$$S = \{\overset{\mathbf{v}_1}{(1, 2, 3)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(-2, 0, 1)}\}$$

*SOLUTION* To test for linear independence or linear dependence, form the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

If the only solution of this equation is

$$c_1 = c_2 = c_3 = 0,$$

then the set $S$ is linearly independent. Otherwise, $S$ is linearly dependent. Expanding this equation, you have

$$\begin{aligned} c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) &= (0, 0, 0) \\ (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3) &= (0, 0, 0), \end{aligned}$$

which yields the homogeneous system of linear equations in $c_1$, $c_2$, and $c_3$ shown below.

$$\begin{aligned} c_1 \qquad\quad - 2c_3 &= 0 \\ 2c_1 + \phantom{2}c_2 \qquad\quad &= 0 \\ 3c_1 + 2c_2 + \phantom{2}c_3 &= 0 \end{aligned}$$

The augmented matrix of this system reduces by Gauss-Jordan elimination as follows.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This implies that the only solution is the trivial solution

$$c_1 = c_2 = c_3 = 0.$$

So, $S$ is linearly independent.

The steps shown in Example 8 are summarized as follows.

### Testing for Linear Independence and Linear Dependence

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space $V$. To determine whether $S$ is linearly independent or linearly dependent, perform the following steps.

1. From the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, write a homogeneous system of linear equations in the variables $c_1, c_2, \ldots,$ and $c_k$.
2. Use Gaussian elimination to determine whether the system has a unique solution.
3. If the system has only the trivial solution, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$, then the set $S$ is linearly independent. If the system also has nontrivial solutions, then $S$ is linearly dependent.

**EXAMPLE 9** **Testing for Linear Independence**

Determine whether the set of vectors in $P_2$ is linearly independent or linearly dependent.

$$S = \{\overset{\mathbf{v}_1}{1 + x - 2x^2}, \overset{\mathbf{v}_2}{2 + 5x - x^2}, \overset{\mathbf{v}_3}{x + x^2}\}$$

*SOLUTION* Expanding the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ produces

$$c_1(1 + x - 2x^2) + c_2(2 + 5x - x^2) + c_3(x + x^2) = 0 + 0x + 0x^2$$
$$(c_1 + 2c_2) + (c_1 + 5c_2 + c_3)x + (-2c_1 - c_2 + c_3)x^2 = 0 + 0x + 0x^2.$$

Equating corresponding coefficients of equal powers of $x$ produces the homogeneous system of linear equations in $c_1$, $c_2$, and $c_3$ shown below.

$$\begin{aligned} c_1 + 2c_2 \phantom{{}+c_3} &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ -2c_1 - c_2 + c_3 &= 0 \end{aligned}$$

The augmented matrix of this system reduces by Gaussian elimination as follows.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This implies that the system has an infinite number of solutions. So, the system must have nontrivial solutions, and you can conclude that the set $S$ is linearly dependent.

One nontrivial solution is

$$c_1 = 2, \quad c_2 = -1, \quad \text{and} \quad c_3 = 3,$$

which yields the nontrivial linear combination

$$(2)(1 + x - 2x^2) + (-1)(2 + 5x - x^2) + (3)(x + x^2) = 0.$$

## EXAMPLE 10 Testing for Linear Independence

Determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.

$$S = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}} \right\}$$

*SOLUTION* From the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0},$$

you have

$$c_1\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which produces the system of linear equations in $c_1$, $c_2$, and $c_3$ shown below.

$$\begin{aligned} 2c_1 + 3c_2 + \phantom{2}c_3 &= 0 \\ c_1 \phantom{+ 3c_2 + 2c_3} &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + \phantom{3}c_2 \phantom{+ 2c_3} &= 0 \end{aligned}$$

Using Gaussian elimination, the augmented matrix of this system reduces as follows.

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system has only the trivial solution and you can conclude that the set $S$ is linearly independent.

**68. Writing** Let $A$ be a nonsingular matrix of order 3. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set in $M_{3,1}$, then the set $\{A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3\}$ is also linearly independent. Explain, by means of an example, why this is not true if $A$ is singular.

**69.** Prove the corollary to Theorem 4.8: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if and only if one is a scalar multiple of the other.

# 4.5 Basis and Dimension

In this section you will continue your study of spanning sets. In particular, you will look at spanning sets (in a vector space) that both are linearly independent *and* span the entire space. Such a set forms a **basis** for the vector space. (The plural of *basis* is *bases.*)

**Definition of Basis**

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ in a vector space $V$ is called a **basis** for $V$ if the following conditions are true.

1. $S$ spans $V$.
2. $S$ is linearly independent.

REMARK: This definition tells you that a basis has two features. A basis $S$ must have *enough vectors* to span $V$, but *not so many vectors* that one of them could be written as a linear combination of the other vectors in $S$.

This definition does not imply that every vector space has a basis consisting of a finite number of vectors. In this text, however, the discussion of bases is restricted to those consisting of a finite number of vectors. Moreover, if a vector space $V$ has a basis consisting of a finite number of vectors, then $V$ is **finite dimensional.** Otherwise, $V$ is called **infinite dimensional.** [The vector space $P$ of *all* polynomials is infinite dimensional, as is the vector space $C(-\infty, \infty)$ of all continuous functions defined on the real line.] The vector space $V = \{\mathbf{0}\}$, consisting of the zero vector alone, is finite dimensional.

### EXAMPLE 1 The Standard Basis for $R^3$

Show that the following set is a basis for $R^3$.

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

SOLUTION Example 4(a) in Section 4.4 showed that $S$ spans $R^3$. Furthermore, $S$ is linearly independent because the vector equation

$$c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$. (Try verifying this.) So, $S$ is a basis for $R^3$. (See Figure 4.18.)

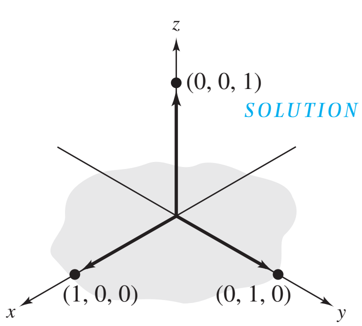


Figure 4.18

The basis $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is called the **standard basis** for $R^3$. This result can be generalized to $n$-space. That is, the vectors

$$\begin{aligned}
\mathbf{e}_1 &= (1, 0, \ldots, 0)\\
\mathbf{e}_2 &= (0, 1, \ldots, 0)\\
&\vdots\\
\mathbf{e}_n &= (0, 0, \ldots, 1)
\end{aligned}$$

form a basis for $R^n$ called the **standard basis** for $R^n$.

The next two examples describe nonstandard bases for $R^2$ and $R^3$.

### EXAMPLE 2 The Nonstandard Basis for $R^2$

Show that the set

$$S = \{\overset{\mathbf{v}_1}{(1, 1)}, \overset{\mathbf{v}_2}{(1, -1)}\}$$

is a basis for $R^2$.

*SOLUTION* According to the definition of a basis for a vector space, you must show that $S$ spans $R^2$ and $S$ is linearly independent.

To verify that $S$ spans $R^2$, let

$$\mathbf{x} = (x_1, x_2)$$

represent an arbitrary vector in $R^2$. To show that $\mathbf{x}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, consider the equation

$$\begin{aligned}
c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= \mathbf{x}\\
c_1(1, 1) + c_2(1, -1) &= (x_1, x_2)\\
(c_1 + c_2, c_1 - c_2) &= (x_1, x_2).
\end{aligned}$$

Equating corresponding components yields the system of linear equations shown below.

$$\begin{aligned}
c_1 + c_2 &= x_1\\
c_1 - c_2 &= x_2
\end{aligned}$$

Because the coefficient matrix of this system has a nonzero determinant, you know that the system has a unique solution. You can now conclude that $S$ spans $R^2$.

To show that $S$ is linearly independent, consider the linear combination

$$\begin{aligned}
c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= \mathbf{0}\\
c_1(1, 1) + c_2(1, -1) &= (0, 0)\\
(c_1 + c_2, c_1 - c_2) &= (0, 0).
\end{aligned}$$

Equating corresponding components yields the homogeneous system

$$\begin{aligned}
c_1 + c_2 &= 0\\
c_1 - c_2 &= 0.
\end{aligned}$$

Because the coefficient matrix of this system has a nonzero determinant, you know that the system has only the trivial solution

$$c_1 = c_2 = 0.$$

So, you can conclude that $S$ is linearly independent.

You can conclude that $S$ is a basis for $R^2$ because it is a linearly independent spanning set for $R^2$.

## EXAMPLE 3 A Nonstandard Basis for $R^3$

From Examples 5 and 8 in the preceding section, you know that

$$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

spans $R^3$ and is linearly independent. So, $S$ is a basis for $R^3$.

## EXAMPLE 4 A Basis for Polynomials

Show that the vector space $P_3$ has the basis

$$S = \{1, x, x^2, x^3\}.$$

*SOLUTION* It is clear that $S$ spans $P_3$ because the span of $S$ consists of all polynomials of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3, \quad a_0, a_1, a_2, \text{ and } a_3 \text{ are real,}$$

which is precisely the form of all polynomials in $P_3$.

To verify the linear independence of $S$, recall that the zero vector $\mathbf{0}$ in $P_3$ is the polynomial $\mathbf{0}(x) = 0$ for all $x$. The test for linear independence yields the equation

$$a_0 + a_1x + a_2x^2 + a_3x^3 = \mathbf{0}(x) = 0, \quad \text{for all } x.$$

This third-degree polynomial is said to be *identically equal to zero*. From algebra you know that for a polynomial to be identically equal to zero, all of its coefficients must be zero; that is,

$$a_0 = a_1 = a_2 = a_3 = 0.$$

So, $S$ is linearly independent and is a basis for $P_3$.

REMARK: The basis $S = \{1, x, x^2, x^3\}$ is called the **standard basis** for $P_3$. Similarly, the **standard basis** for $P_n$ is

$$S = \{1, x, x^2, \ldots, x^n\}.$$

**EXAMPLE 5** A Basis for $M_{2,2}$

The set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for $M_{2,2}$. This set is called the **standard basis** for $M_{2,2}$. In a similar manner, the standard basis for the vector space $M_{m,n}$ consists of the $mn$ distinct $m \times n$ matrices having a single 1 and all the other entries equal to zero.

**THEOREM 4.9 Uniqueness of Basis Representation**

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space $V$, then every vector in $V$ can be written in one and only one way as a linear combination of vectors in $S$.

*PROOF* The existence portion of the proof is straightforward. That is, because $S$ spans $V$, you know that an arbitrary vector $\mathbf{u}$ in $V$ can be expressed as $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$.

To prove uniqueness (that a vector can be represented in only one way), suppose $\mathbf{u}$ has another representation $\mathbf{u} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$. Subtracting the second representation from the first produces

$$\mathbf{u} - \mathbf{u} = (c_1 - b_1)\mathbf{v}_1 + (c_2 - b_2)\mathbf{v}_2 + \cdots + (c_n - b_n)\mathbf{v}_n = \mathbf{0}.$$

Because $S$ is linearly independent, however, the only solution to this equation is the trivial solution

$$c_1 - b_1 = 0, \qquad c_2 - b_2 = 0, \quad \ldots, \quad c_n - b_n = 0,$$

which means that $c_i = b_i$ for all $i = 1, 2, \ldots, n$. So, $\mathbf{u}$ has only one representation for the basis $S$.

**EXAMPLE 6** Uniqueness of Basis Representation

Let $\mathbf{u} = (u_1, u_2, u_3)$ be any vector in $R^3$. Show that the equation $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ has a unique solution for the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$.

*SOLUTION* From the equation

$$\begin{aligned}(u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) \\ &= (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3),\end{aligned}$$

the following system of linear equations is obtained.

$$\begin{aligned} c_1 \qquad - 2c_3 &= u_1 \\ 2c_1 + \phantom{2}c_2 \qquad &= u_2 \\ 3c_1 + 2c_2 + \phantom{2}c_3 &= u_3 \end{aligned} \qquad \underset{A}{\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}} \underset{c}{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}} = \underset{u}{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}$$

Because the matrix $A$ is invertible, you know this system has a unique solution $\mathbf{c} = A^{-1}\mathbf{u}$. Solving for $A^{-1}$ yields

$$A^{-1} = \begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 4 \\ -1 & 2 & -1 \end{bmatrix},$$

which implies

$$\begin{aligned} c_1 &= -u_1 + 4u_2 - 2u_3 \\ c_2 &= \phantom{-}2u_1 - 7u_2 + 4u_3 \\ c_3 &= -u_1 + 2u_2 - \phantom{2}u_3. \end{aligned}$$

For instance, the vector $\mathbf{u} = (1, 0, 0)$ can be represented uniquely as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as follows.

$$(1, 0, 0) = -\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$$

You will now study two important theorems concerning bases.

**THEOREM 4.10 Bases and Linear Dependence**

If $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_n\}$ is a basis for a vector space $V$, then every set containing more than $n$ vectors in $V$ is linearly dependent.

*PROOF* Let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, . . . , \mathbf{u}_m\}$ be any set of $m$ vectors in $V$, where $m > n$. To show that $S_1$ is linearly *dependent,* you need to find scalars $k_1, k_2, . . . , k_m$ (not all zero) such that

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdot \cdot \cdot + k_m\mathbf{u}_m = \mathbf{0}. \qquad \textbf{Equation 1}$$

Because $S$ is a basis for $V$, it follows that each $\mathbf{u}_i$ is a linear combination of vectors in $S$, and you can write

$$\begin{aligned} \mathbf{u}_1 &= c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \cdot \cdot \cdot + c_{n1}\mathbf{v}_n \\ \mathbf{u}_2 &= c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdot \cdot \cdot + c_{n2}\mathbf{v}_n \\ &\ \ \vdots \\ \mathbf{u}_m &= c_{1m}\mathbf{v}_1 + c_{2m}\mathbf{v}_2 + \cdot \cdot \cdot + c_{nm}\mathbf{v}_n. \end{aligned}$$

Substituting each of these representations of $\mathbf{u}_i$ into Equation 1 and regrouping terms produces

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdot \cdot \cdot + d_n\mathbf{v}_n = \mathbf{0},$$

where $d_i = c_{i1}k_1 + c_{i2}k_2 + \cdots + c_{im}k_m$. Because the $\mathbf{v}_i$'s form a linearly independent set, you can conclude that each $d_i = 0$. So, the system of equations shown below is obtained.

$$\begin{aligned}
c_{11}k_1 + c_{12}k_2 + \cdots + c_{1m}k_m &= 0\\
c_{21}k_1 + c_{22}k_2 + \cdots + c_{2m}k_m &= 0\\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\;\;\; \vdots\\
c_{n1}k_1 + c_{n2}k_2 + \cdots + c_{nm}k_m &= 0
\end{aligned}$$

But this homogeneous system has fewer equations than variables $k_1, k_2, \ldots, k_m$, and from Theorem 1.1 you know it must have *nontrivial* solutions. Consequently, $S_1$ is linearly dependent.

## EXAMPLE 7 Linearly Dependent Sets in $R^3$ and $P_3$

(a) Because $R^3$ has a basis consisting of three vectors, the set

$$S = \{(1, 2, -1), (1, 1, 0), (2, 3, 0), (5, 9, -1)\}$$

must be linearly dependent.

(b) Because $P_3$ has a basis consisting of four vectors, the set

$$S = \{1, 1 + x, 1 - x, 1 + x + x^2, 1 - x + x^2\}$$

must be linearly dependent.

Because $R^n$ has the standard basis consisting of $n$ vectors, it follows from Theorem 4.10 that every set of vectors in $R^n$ containing more than $n$ vectors must be linearly dependent. Another significant consequence of Theorem 4.10 is shown in the next theorem.

**THEOREM 4.11 Number of Vectors in a Basis**

If a vector space $V$ has one basis with $n$ vectors, then every basis for $V$ has $n$ vectors.

*PROOF* Let

$$S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

be the basis for $V$, and let

$$S_2 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$$

be any other basis for $V$. Because $S_1$ is a basis and $S_2$ is linearly independent, Theorem 4.10 implies that $m \leq n$. Similarly, $n \leq m$ because $S_1$ is linearly independent and $S_2$ is a basis. Consequently, $n = m$.

**EXAMPLE 8** **Spanning Sets and Bases**

Use Theorem 4.11 to explain why each of the statements below is true.

(a) The set $S_1 = \{(3, 2, 1), (7, -1, 4)\}$ is not a basis for $R^3$.

(b) The set

$$S_2 = \{x + 2, x^2, x^3 - 1, 3x + 1, x^2 - 2x + 3\}$$

is not a basis for $P_3$.

*SOLUTION* (a) The standard basis for $R^3$ has three vectors, and $S_1$ has only two. By Theorem 4.11, $S_1$ cannot be a basis for $R^3$.

(b) The standard basis for $P_3$, $S = \{1, x, x^2, x^3\}$, has four elements. By Theorem 4.11, the set $S_2$ has too many elements to be a basis for $P_3$.

## *The Dimension of a Vector Space*

The discussion of spanning sets, linear independence, and bases leads to an important notion in the study of vector spaces. By Theorem 4.11, you know that if a vector space $V$ has a basis consisting of $n$ vectors, then every other basis for the space also has $n$ vectors. The number $n$ is called the **dimension** of $V$.

**Definition of Dimension of a Vector Space**

If a vector space $V$ has a basis consisting of $n$ vectors, then the number $n$ is called the **dimension** of $V$, denoted by $\dim(V) = n$. If $V$ consists of the zero vector alone, the dimension of $V$ is defined as zero.

This definition allows you to observe the characteristics of the dimensions of the familiar vector spaces listed below. In each case, the dimension is determined by simply counting the number of vectors in the standard basis.

1. The dimension of $R^n$ with the standard operations is $n$.
2. The dimension of $P_n$ with the standard operations is $n + 1$.
3. The dimension of $M_{m,n}$ with the standard operations is $mn$.

If $W$ is a subspace of an $n$-dimensional vector space, then it can be shown that $W$ is finite dimensional and the dimension of $W$ is less than or equal to $n$. (See Exercise 81.) In the next three examples, you will look at a technique for determining the dimension of a subspace. Basically, you determine the dimension by finding a set of linearly independent vectors that spans the subspace. This set is a basis for the subspace, and the dimension of the subspace is the number of vectors in the basis.

## EXAMPLE 9 Finding the Dimension of a Subspace

Determine the dimension of each subspace of $R^3$.

(a) $W = \{(d, c - d, c): c \text{ and } d \text{ are real numbers}\}$

(b) $W = \{(2b, b, 0): b \text{ is a real number}\}$

*SOLUTION* The goal in each example is to find a set of linearly independent vectors that spans the subspace.

(a) By writing the representative vector $(d, c - d, c)$ as

$$\begin{aligned}(d, c - d, c) &= (0, c, c) + (d, -d, 0)\\ &= c(0, 1, 1) + d(1, -1, 0),\end{aligned}$$

you can see that $W$ is spanned by the set

$$S = \{(0, 1, 1), (1, -1, 0)\}.$$

Using the techniques described in the preceding section, you can show that this set is linearly independent. So, it is a basis for $W$, and you can conclude that $W$ is a two-dimensional subspace of $R^3$.

(b) By writing the representative vector $(2b, b, 0)$ as

$$(2b, b, 0) = b(2, 1, 0),$$

you can see that $W$ is spanned by the set $S = \{(2, 1, 0)\}$. So, $W$ is a one-dimensional subspace of $R^3$.

REMARK: In Example 9(a), the subspace $W$ is a two-dimensional plane in $R^3$ determined by the vectors $(0, 1, 1)$ and $(1, -1, 0)$. In Example 9(b), the subspace is a one-dimensional line.

## EXAMPLE 10 Finding the Dimension of a Subspace

Find the dimension of the subspace $W$ of $R^4$ spanned by

$$S = \{\overset{\mathbf{v}_1}{(-1, 2, 5, 0)}, \overset{\mathbf{v}_2}{(3, 0, 1, -2)}, \overset{\mathbf{v}_3}{(-5, 4, 9, 2)}\}.$$

*SOLUTION* Although $W$ is spanned by the set $S$, $S$ is not a basis for $W$ because $S$ is a linearly dependent set. Specifically, $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ as follows.

$$\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$$

This means that $W$ is spanned by the set $S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$. Moreover, $S_1$ is linearly independent because neither vector is a scalar multiple of the other, and you can conclude that the dimension of $W$ is 2.

**EXAMPLE 11** **Finding the Dimension of a Subspace**

Let $W$ be the subspace of all symmetric matrices in $M_{2,2}$. What is the dimension of $W$?

*SOLUTION* Every $2 \times 2$ symmetric matrix has the form listed below.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So, the set

$$S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

spans $W$. Moreover, $S$ can be shown to be linearly independent, and you can conclude that the dimension of $W$ is 3.

Usually, to conclude that a set

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

is a basis for a vector space $V$, you must show that $S$ satisfies two conditions: $S$ spans $V$ and is linearly independent. If $V$ is known to have a dimension of $n$, however, then the next theorem tells you that you do not need to check both conditions: either one will suffice. The proof is left as an exercise. (See Exercise 82.)

**THEOREM 4.12** **Basis Tests in an $n$-Dimensional Space**

Let $V$ be a vector space of dimension $n$.

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in $V$, then $S$ is a basis for $V$.
2. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans $V$, then $S$ is a basis for $V$.

**EXAMPLE 12** **Testing for a Basis in an $n$-Dimensional Space**

Show that the set of vectors is a basis for $M_{5,1}$.

$$S = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \\ 3 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \\ 5 \end{bmatrix}}, \overset{\mathbf{v}_4}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -3 \end{bmatrix}}, \overset{\mathbf{v}_5}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}} \right\}$$

*SOLUTION* Because $S$ has five vectors and the dimension of $M_{5,1}$ is five, you can apply Theorem 4.12 to verify that $S$ is a basis by showing either that $S$ is linearly independent or that $S$ spans $M_{5,1}$. To show the first of these, form the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{0},$$

which yields the homogeneous system of linear equations shown below.

$$\begin{aligned}
c_1 \qquad\qquad\qquad\qquad\qquad &= 0\\
2c_1 + c_2 \qquad\qquad\qquad\quad &= 0\\
-c_1 + 3c_2 + 2c_3 \qquad\qquad &= 0\\
3c_1 - 2c_2 - c_3 + 2c_4 \qquad &= 0\\
4c_1 + 3c_2 + 5c_3 - 3c_4 - 2c_5 &= 0
\end{aligned}$$

Because this system has only the trivial solution, $S$ must be linearly independent. So, by Theorem 4.12, $S$ is a basis for $M_{5,1}$.

## SECTION 4.5 Exercises

In Exercises 1–6, write the standard basis for the vector space.

**1.** $R^6$ **2.** $R^4$ **3.** $M_{2,4}$

**4.** $M_{4,1}$ **5.** $P_4$ **6.** $P_2$

**Writing** In Exercises 7–14, explain why $S$ is not a basis for $R^2$.

**7.** $S = \{(1, 2), (1, 0), (0, 1)\}$

**8.** $S = \{(-1, 2), (1, -2), (2, 4)\}$

**9.** $S = \{(-4, 5), (0, 0)\}$

**10.** $S = \{(2, 3), (6, 9)\}$

**11.** $S = \{(6, -5), (12, -10)\}$

**12.** $S = \{(4, -3), (8, -6)\}$

**13.** $S = \{(-3, 2)\}$

**14.** $S = \{(-1, 2)\}$

**Writing** In Exercises 15–20, explain why $S$ is not a basis for $R^3$.

**15.** $S = \{(1, 3, 0), (4, 1, 2), (-2, 5, -2)\}$

**16.** $S = \{(2, 1, -2), (-2, -1, 2), (4, 2, -4)\}$

**17.** $S = \{(7, 0, 3), (8, -4, 1)\}$

**18.** $S = \{(1, 1, 2), (0, 2, 1)\}$

**19.** $S = \{(0, 0, 0), (1, 0, 0), (0, 1, 0)\}$

**20.** $S = \{(6, 4, 1), (3, -5, 1), (8, 13, 6), (0, 6, 9)\}$

**Writing** In Exercises 21–24, explain why $S$ is not a basis for $P_2$.

**21.** $S = \{1, 2x, x^2 - 4, 5x\}$

**22.** $S = \{2, x, x + 3, 3x^2\}$

**23.** $S = \{1 - x, 1 - x^2, 3x^2 - 2x - 1\}$

**24.** $S = \{6x - 3, 3x^2, 1 - 2x - x^2\}$

**Writing** In Exercises 25–28, explain why $S$ is not a basis for $M_{2,2}$.

**25.** $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$

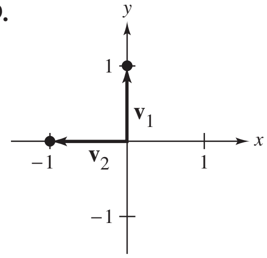
**26.** $S = \left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$

**27.** $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix}\right\}$

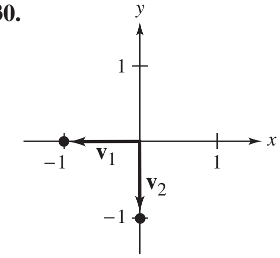
**28.** $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

In Exercises 29–34, determine whether the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $R^2$.

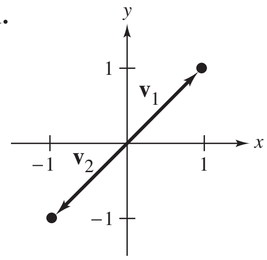
**29.**



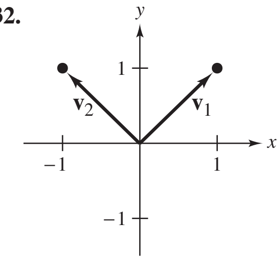

**30.**



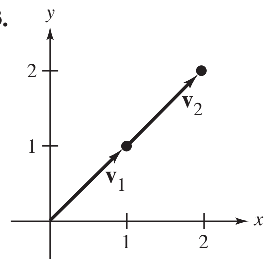

**31.**



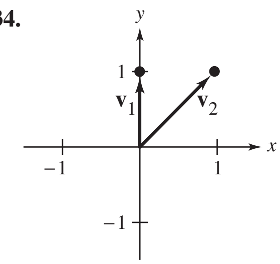

**32.**




**33.**




**34.**




In Exercises 35–42, determine whether $S$ is a basis for the indicated vector space.

**35.** $S = \{(3, -2), (4, 5)\}$ for $R^2$

**36.** $S = \{(1, 2), (1, -1)\}$ for $R^2$

**37.** $S = \{(1, 5, 3), (0, 1, 2), (0, 0, 6)\}$ for $R^3$

**38.** $S = \{(2, 1, 0), (0, -1, 1)\}$ for $R^3$

**39.** $S = \{(0, 3, -2), (4, 0, 3), (-8, 15, -16)\}$ for $R^3$

**40.** $S = \{(0, 0, 0), (1, 5, 6), (6, 2, 1)\}$ for $R^3$

**41.** $S = \{(-1, 2, 0, 0), (2, 0, -1, 0), (3, 0, 0, 4), (0, 0, 5, 0)\}$ for $R^4$

**42.** $S = \{(1, 0, 0, 1), (0, 2, 0, 2), (1, 0, 1, 0), (0, 2, 2, 0)\}$ for $R^4$

In Exercises 43 and 44, determine whether $S$ is a basis for $M_{2,2}$.

**43.** $S = \left\{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\right\}$

**44.** $S = \left\{\begin{bmatrix} 1 & 2 \\ -5 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -7 \\ 6 & 2 \end{bmatrix}, \begin{bmatrix} 4 & -9 \\ 11 & 12 \end{bmatrix}, \begin{bmatrix} 12 & -16 \\ 17 & 42 \end{bmatrix}\right\}$

In Exercises 45–48, determine whether $S$ is a basis for $P_3$.

**45.** $S = \{t^3 - 2t^2 + 1, t^2 - 4, t^3 + 2t, 5t\}$

**46.** $S = \{4t - t^2, 5 + t^3, 3t + 5, 2t^3 - 3t^2\}$

**47.** $S = \{4 - t, t^3, 6t^2, t^3 + 3t, 4t - 1\}$

**48.** $S = \{t^3 - 1, 2t^2, t + 3, 5 + 2t + 2t^2 + t^3\}$

In Exercises 49–54, determine whether $S$ is a basis for $R^3$. If it is, write $\mathbf{u} = (8, 3, 8)$ as a linear combination of the vectors in $S$.

**49.** $S = \{(4, 3, 2), (0, 3, 2), (0, 0, 2)\}$

**50.** $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$

**51.** $S = \{(0, 0, 0), (1, 3, 4), (6, 1, -2)\}$

**52.** $S = \{(1, 0, 1), (0, 0, 0), (0, 1, 0)\}$

**53.** $S = \left\{\left(\frac{2}{3}, \frac{5}{2}, 1\right), \left(1, \frac{3}{2}, 0\right), (2, 12, 6)\right\}$

**54.** $S = \{(1, 4, 7), (3, 0, 1), (2, 1, 2)\}$

In Exercises 55–62, determine the dimension of the vector space.

| | | | |
|---|---|---|---|
| **55.** $R^6$ | **56.** $R^4$ | **57.** $R$ | **58.** $R^3$ |
| **59.** $P_7$ | **60.** $P_4$ | **61.** $M_{2,3}$ | **62.** $M_{3,2}$ |

**63.** Find a basis for $D_{3,3}$ (the vector space of all $3 \times 3$ diagonal matrices). What is the dimension of this vector space?

**64.** Find a basis for the vector space of all $3 \times 3$ symmetric matrices. What is the dimension of this vector space?

**65.** Find all subsets of the set that forms a basis for $R^2$.
$S = \{(1, 0), (0, 1), (1, 1)\}$

**66.** Find all subsets of the set that forms a basis for $R^3$.
$S = \{(1, 3, -2), (-4, 1, 1), (-2, 7, -3), (2, 1, 1)\}$

**67.** Find a basis for $R^2$ that includes the vector $(1, 1)$.

**68.** Find a basis for $R^3$ that includes the set $S = \{(1, 0, 2), (0, 1, 1)\}$.

In Exercises 69 and 70, (a) give a geometric description of, (b) find a basis for, and (c) determine the dimension of the subspace $W$ of $R^2$.

**69.** $W = \{(2t, t)\colon t \text{ is a real number}\}$

**70.** $W = \{(0, t)\colon t \text{ is a real number}\}$

In Exercises 71 and 72, (a) give a geometric description of, (b) find a basis for, and (c) determine the dimension of the subspace $W$ of $R^3$.

**71.** $W = \{(2t, t, -t)\colon t \text{ is a real number}\}$

**72.** $W = \{(2t - t, s, t)\colon s \text{ and } t \text{ are real numbers}\}$

and the subspace spanned by the row vectors of $B$ is contained in the row space of $A$. But it is also true that the rows of $A$ can be obtained from the rows of $B$ by elementary row operations. So, you can conclude that the two row spaces are subspaces of each other, making them equal.

REMARK: Note that Theorem 4.13 says that the row space of a matrix is not changed by elementary row operations. Elementary row operations can, however, change the *column* space.

If a matrix $B$ is in row-echelon form, then its nonzero row vectors form a linearly independent set. (Try verifying this.) Consequently, they form a basis for the row space of $B$, and by Theorem 4.13 they also form a basis for the row space of $A$. This important result is stated in the next theorem.

**THEOREM 4.14 Basis for the Row Space of a Matrix**

If a matrix $A$ is row-equivalent to a matrix $B$ in row-echelon form, then the nonzero row vectors of $B$ form a basis for the row space of $A$.

## EXAMPLE 2 Finding a Basis for a Row Space

Find a basis for the row space of

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}.$$

*SOLUTION* Using elementary *row* operations, rewrite $A$ in row-echelon form as follows.

$$B = \begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \\ \\ \end{array}$$

By Theorem 4.14, you can conclude that the nonzero row vectors of $B$,

$$\mathbf{w}_1 = (1, 3, 1, 3), \quad \mathbf{w}_2 = (0, 1, 1, 0), \quad \text{and} \quad \mathbf{w}_3 = (0, 0, 0, 1),$$

form a basis for the row space of $A$.

The technique used in Example 2 to find the row space of a matrix can be used to solve the next type of problem. Suppose you are asked to find a basis for the subspace spanned by the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $R^n$. By using the vectors in $S$ to form the rows of a matrix $A$, you can use elementary row operations to rewrite $A$ in row-echelon form. The nonzero rows of this matrix will then form a basis for the subspace spanned by $S$. This is demonstrated in Example 3.

## EXAMPLE 3 Finding a Basis for a Subspace

Find a basis for the subspace of $R^3$ spanned by

$$S = \{\overset{\mathbf{v}_1}{(-1, 2, 5)}, \overset{\mathbf{v}_2}{(3, 0, 3)}, \overset{\mathbf{v}_3}{(5, 1, 8)}\}.$$

*SOLUTION* Use $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ to form the rows of a matrix $A$. Then write $A$ in row-echelon form.

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix}\begin{matrix}\mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3\end{matrix} \quad \longrightarrow \quad B = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix}\mathbf{w}_1 \\ \mathbf{w}_2 \\ \phantom{\mathbf{w}}\end{matrix}$$

So, the nonzero row vectors of $B$,

$$\mathbf{w}_1 = (1, -2, -5) \quad \text{and} \quad \mathbf{w}_2 = (0, 1, 3),$$

form a basis for the row space of $A$. That is, they form a basis for the subspace spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

To find a basis for the column space of a matrix $A$, you have two options. On the one hand, you could use the fact that the column space of $A$ is equal to the row space of $A^T$ and apply the technique of Example 2 to the matrix $A^T$. On the other hand, observe that although row operations can change the column space of a matrix, they do not change the dependency relationships between columns. You are asked to prove this fact in Exercise 71.

For example, consider the row-equivalent matrices $A$ and $B$ from Example 2.

$$A = \begin{array}{c}\begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \\ \begin{matrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{matrix}\end{array} \qquad B = \begin{array}{c}\begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{matrix}\end{array}$$

Notice that columns 1, 2, and 3 of matrix $B$ satisfy the equation $\mathbf{b}_3 = -2\mathbf{b}_1 + \mathbf{b}_2$, and so do the corresponding columns of matrix $A$; that is,

$$\mathbf{a}_3 = -2\mathbf{a}_1 + \mathbf{a}_2.$$

Similarly, the column vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_4$ of matrix $B$ are linearly independent, and so are the corresponding columns of matrix $A$.

The next example shows how to find a basis for the column space of a matrix using both of these methods.

### EXAMPLE 4 Finding a Basis for the Column Space of a Matrix

Find a basis for the column space of matrix $A$ from Example 2.

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

*SOLUTION 1* Take the transpose of $A$ and use elementary row operations to write $A^T$ in row-echelon form.

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix} \quad \longrightarrow \quad \begin{array}{cc} \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \phantom{0} \end{array} \end{array}$$

So, $\mathbf{w}_1 = (1, 0, -3, 3, 2)$, $\mathbf{w}_2 = (0, 1, 9, -5, -6)$, and $\mathbf{w}_3 = (0, 0, 1, -1, -1)$ form a basis for the row space of $A^T$. This is equivalent to saying that the column vectors

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

form a basis for the column space of $A$.

*SOLUTION 2* In Example 2, row operations were used on the original matrix $A$ to obtain its row-echelon form $B$. It is easy to see that in matrix $B$, the first, second, and fourth column vectors are linearly independent (these columns have the leading 1's). The corresponding columns of matrix $A$ are linearly independent, and a basis for the column space consists of the vectors

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \\ -2 \end{bmatrix}.$$

Notice that this is a different basis for the column space than that obtained in the first solution. Verify that these bases span the same subspace of $R^5$.

REMARK: Notice that in the second solution, the row-echelon form $B$ indicates which columns of $A$ form the basis of the column space. You do not use the column vectors of $B$ to form the basis.

Notice in Examples 2 and 4 that both the row space and the column space of $A$ have a dimension of 3 (because there are *three* vectors in both bases). This is generalized in the next theorem.

**THEOREM 4.15 Row and Column Spaces Have Equal Dimensions**

If $A$ is an $m \times n$ matrix, then the row space and column space of $A$ have the same dimension.

*PROOF* Let $\mathbf{v}_1, \mathbf{v}_2, \ldots,$ and $\mathbf{v}_m$ be the row vectors and $\mathbf{u}_1, \mathbf{u}_2, \ldots,$ and $\mathbf{u}_n$ be the column vectors of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Suppose the row space of $A$ has dimension $r$ and basis $S = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r\}$, where $\mathbf{b}_i = (b_{i1}, b_{i2}, \ldots, b_{in})$. Using this basis, you can write the row vectors of $A$ as

$$\begin{aligned} \mathbf{v}_1 &= c_{11}\mathbf{b}_1 + c_{12}\mathbf{b}_2 + \cdots + c_{1r}\mathbf{b}_r \\ \mathbf{v}_2 &= c_{21}\mathbf{b}_1 + c_{22}\mathbf{b}_2 + \cdots + c_{2r}\mathbf{b}_r \\ &\;\vdots \\ \mathbf{v}_m &= c_{m1}\mathbf{b}_1 + c_{m2}\mathbf{b}_2 + \cdots + c_{mr}\mathbf{b}_r. \end{aligned}$$

Rewrite this system of vector equations as follows.

$$\begin{aligned} [a_{11}a_{12} \cdots a_{1n}] &= c_{11}[b_{11}b_{12} \cdots b_{1n}] + c_{12}[b_{21}b_{22} \cdots b_{2n}] + \cdots + c_{1r}[b_{r1}b_{r2} \cdots b_{rn}] \\ [a_{21}a_{22} \cdots a_{2n}] &= c_{21}[b_{11}b_{12} \cdots b_{1n}] + c_{22}[b_{21}b_{22} \cdots b_{2n}] + \cdots + c_{2r}[b_{r1}b_{r2} \cdots b_{rn}] \\ &\;\vdots \\ [a_{m1}a_{m2} \cdots a_{mn}] &= c_{m1}[b_{11}b_{12} \cdots b_{1n}] + c_{m2}[b_{21}b_{22} \cdots b_{2n}] + \cdots + c_{mr}[b_{r1}b_{r2} \cdots b_{rn}] \end{aligned}$$

Now, take only entries corresponding to the first column of matrix $A$ to obtain the system of scalar equations shown below.

$$\begin{aligned} a_{11} &= c_{11}b_{11} + c_{12}b_{21} + c_{13}b_{31} + \cdots + c_{1r}b_{r1} \\ a_{21} &= c_{21}b_{11} + c_{22}b_{21} + c_{23}b_{31} + \cdots + c_{2r}b_{r1} \\ a_{31} &= c_{31}b_{11} + c_{32}b_{21} + c_{33}b_{31} + \cdots + c_{3r}b_{r1} \\ &\;\vdots \\ a_{m1} &= c_{m1}b_{11} + c_{m2}b_{21} + c_{m3}b_{31} + \cdots + c_{mr}b_{r1} \end{aligned}$$

Similarly, for the entries of the $j$th column you can obtain the system below.

$$\begin{aligned}
a_{1j} &= c_{11}b_{1j} + c_{12}b_{2j} + c_{13}b_{3j} + \cdots + c_{1r}b_{rj}\\
a_{2j} &= c_{21}b_{1j} + c_{22}b_{2j} + c_{23}b_{3j} + \cdots + c_{2r}b_{rj}\\
a_{3j} &= c_{31}b_{1j} + c_{32}b_{2j} + c_{33}b_{3j} + \cdots + c_{3r}b_{rj}\\
&\vdots\\
a_{mj} &= c_{m1}b_{1j} + c_{m2}b_{2j} + c_{m3}b_{3j} + \cdots + c_{mr}b_{rj}
\end{aligned}$$

Now, let the vectors $\mathbf{c}_i = [c_{1i}\, c_{2i} \cdots c_{mi}]^T$. Then the system for the $j$th column can be rewritten in a vector form as

$$\mathbf{u}_j = b_{1j}\mathbf{c}_1 + b_{2j}\mathbf{c}_2 + \cdots + b_{rj}\mathbf{c}_r.$$

Put all column vectors together to obtain

$$\begin{aligned}
\mathbf{u}_1 &= [a_{11}\quad a_{21} \cdots a_{m1}]^T = b_{11}\mathbf{c}_1 + b_{21}\mathbf{c}_2 + \cdots + b_{r1}\mathbf{c}_r\\
\mathbf{u}_2 &= [a_{12}\quad a_{22} \cdots a_{m2}]^T = b_{12}\mathbf{c}_1 + b_{22}\mathbf{c}_2 + \cdots + b_{r2}\mathbf{c}_r\\
&\vdots\\
\mathbf{u}_n &= [a_{1n}\quad a_{2n} \cdots a_{mn}]^T = b_{1n}\mathbf{c}_1 + b_{2n}\mathbf{c}_2 + \cdots + b_{rn}\mathbf{c}_r\,.
\end{aligned}$$

Because each column vector of $A$ is a linear combination of $r$ vectors, you know that the dimension of the column space of $A$ is less than or equal to $r$ (the dimension of the row space of $A$). That is,

$$\dim(\text{column space of } A) \leq \dim(\text{row space of } A).$$

Repeating this procedure for $A^T$, you can conclude that the dimension of the column space of $A^T$ is less than or equal to the dimension of the row space of $A^T$. But this implies that the dimension of the row space of $A$ is less than or equal to the dimension of the column space of $A$. That is,

$$\dim(\text{row space of } A) \leq \dim(\text{column space of } A).$$

So, the two dimensions must be equal.

The dimension of the row (or column) space of a matrix has the special name provided in the next definition.

### Definition of the Rank of a Matrix

The dimension of the row (or column) space of a matrix $A$ is called the **rank** of $A$ and is denoted by rank($A$).

REMARK: Some texts distinguish between the *row rank* and the *column rank* of a matrix. But because these ranks are equal (Theorem 4.15), this text will not distinguish between them.

**EXAMPLE 5** **Finding the Rank of a Matrix**

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix}.$$

*SOLUTION* Convert to row-echelon form as follows.

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix} \longrightarrow B = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Because $B$ has three nonzero rows, the rank of $A$ is 3.

## *The Nullspace of a Matrix*

The notions of row and column spaces and rank have some important applications to systems of linear equations. Consider first the homogeneous linear system

$$A\mathbf{x} = \mathbf{0}$$

where $A$ is an $m \times n$ matrix, $\mathbf{x} = [x_1 \ \ x_2 \ \ . \ . \ . \ \ x_n]^T$ is the column vector of unknowns, and $\mathbf{0} = [0 \ \ 0 \ \ . \ . \ . \ \ 0]^T$ is the zero vector in $R^m$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The next theorem tells you that the set of all solutions of this homogeneous system is a subspace of $R^n$.

**THEOREM 4.16 Solutions of a Homogeneous System**

If $A$ is an $m \times n$ matrix, then the set of all solutions of the homogeneous system of linear equations

$$A\mathbf{x} = \mathbf{0}$$

is a subspace of $R^n$ called the **nullspace** of $A$ and is denoted by $N(A)$. So,

$$N(A) = \{\mathbf{x} \in R^n : A\mathbf{x} = \mathbf{0}\}.$$

The dimension of the nullspace of $A$ is called the **nullity** of $A$.

*PROOF* Because $A$ is an $m \times n$ matrix, you know that $\mathbf{x}$ has size $n \times 1$. So, the set of all solutions of the system is a *subset* of $R^n$. This set is clearly nonempty, because $A\mathbf{0} = \mathbf{0}$. You can verify that it is a subspace by showing that it is closed under the operations of addition and scalar multiplication. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two solution vectors of the system $A\mathbf{x} = \mathbf{0}$, and let $c$ be a scalar. Because $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$, you know that

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0} \qquad \text{Addition}$$

and

$$A(c\mathbf{x}_1) = c(A\mathbf{x}_1) = c(\mathbf{0}) = \mathbf{0}. \qquad \text{Scalar multiplication}$$

So, both $(\mathbf{x}_1 + \mathbf{x}_2)$ and $c\mathbf{x}_1$ are solutions of $A\mathbf{x} = \mathbf{0}$, and you can conclude that the set of all solutions forms a subspace of $R^n$.

REMARK: The nullspace of $A$ is also called the **solution space** of the system $A\mathbf{x} = \mathbf{0}$.

## EXAMPLE 6 Finding the Solution Space of a Homogeneous System

Find the nullspace of the matrix.

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$$

*SOLUTION* The nullspace of $A$ is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

To solve this system, you need to write the augmented matrix $[A \ \vdots \ \mathbf{0}]$ in reduced row-echelon form. Because the system of equations is homogeneous, the right-hand column of the augmented matrix consists entirely of zeros and will not change as you do row operations. It is sufficient to find the reduced row-echelon form of $A$.

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system of equations corresponding to the reduced row-echelon form is

$$\begin{aligned} x_1 + 2x_2 \phantom{+ x_3} + 3x_4 &= 0 \\ x_3 + \phantom{3}x_4 &= 0. \end{aligned}$$

Choose $x_2$ and $x_4$ as free variables to represent the solutions in this parametric form.

$$x_1 = -2s - 3t, \quad x_2 = s, \quad x_3 = -t, \quad x_4 = t$$

This means that the solution space of $A\mathbf{x} = \mathbf{0}$ consists of all solution vectors $\mathbf{x}$ of the form shown below.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

A basis for the nullspace of $A$ consists of the vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

In other words, these two vectors are solutions of $A\mathbf{x} = \mathbf{0}$, and all solutions of this homogeneous system are linear combinations of these two vectors.

REMARK: Although the basis in Example 6 proved that the vectors spanned the solution set, it did not prove that they were linearly independent. When homogeneous systems are solved from the reduced row-echelon form, the spanning set is always independent.

In Example 6, matrix $A$ has four columns. Furthermore, the rank of the matrix is 2, and the dimension of the nullspace is 2. So, you can see that

$$\text{Number of columns} = \text{rank} + \text{nullity}.$$

One way to see this is to look at the reduced row-echelon form of $A$.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns with the leading 1's (columns 1 and 3) determine the rank of the matrix. The other columns (2 and 4) determine the nullity of the matrix because they correspond to the free variables. This relationship is generalized in the next theorem.

**THEOREM 4.17 Dimension of the Solution Space**

If $A$ is an $m \times n$ matrix of rank $r$, then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $n - r$. That is,

$$n = \text{rank}(A) + \text{nullity}(A).$$

*PROOF* Because $A$ has rank $r$, you know it is row-equivalent to a reduced row-echelon matrix $B$ with $r$ nonzero rows. No generality is lost by assuming that the upper left corner of $B$ has the form of the $r \times r$ identity matrix $I_r$. Moreover, because the zero rows of $B$ contribute nothing to the solution, you can discard them to form the $r \times n$ matrix $B'$, where $B' = [I_r \vdots C]$. The matrix $C$ has $n - r$ columns corresponding to the variables $x_{r+1}, x_{r+2}, \ldots, x_n$. So, the solution space of $A\mathbf{x} = \mathbf{0}$ can be represented by the system

$$\begin{array}{lllllll} x_1 + & & c_{11}x_{r+1} & + c_{12}x_{r+2} & + \cdots & + c_{1,n-r}x_n & = 0 \\ & x_2 + & c_{21}x_{r+1} & + c_{22}x_{r+2} & + \cdots & + c_{2,n-r}x_n & = 0 \\ & & \vdots & \vdots & & \vdots & \vdots \\ & & x_r + c_{r1}x_{r+1} & + c_{r2}x_{r+2} & + \cdots & + c_{r,n-r}x_n & = 0. \end{array}$$

Solving for the first $r$ variables in terms of the last $n - r$ variables produces $n - r$ vectors in the basis of the solution space. Consequently, the solution space has dimension $n - r$.

Example 7 illustrates this theorem and further explores the column space of a matrix.

## EXAMPLE 7 Rank and Nullity of a Matrix

Let the column vectors of the matrix $A$ be denoted by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, and $\mathbf{a}_5$.

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \\ \begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array} \end{array}$$

(a) Find the rank and nullity of $A$.
(b) Find a subset of the column vectors of $A$ that forms a basis for the column space of $A$.
(c) If possible, write the third column of $A$ as a linear combination of the first two columns.

*SOLUTION* Let $B$ be the reduced row-echelon form of $A$.

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \\ \begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array} \end{array} \longrightarrow B = \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \end{array} \end{array}$$

(a) Because $B$ has three nonzero rows, the rank of $A$ is 3. Also, the number of columns of $A$ is $n = 5$, which implies that the nullity of $A$ is $n - \text{rank} = 5 - 3 = 2$.

(b) Because the first, second, and fourth column vectors of $B$ are linearly independent, the corresponding column vectors of $A$,

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

form a basis for the column space of $A$.

(c) The third column of $B$ is a linear combination of the first two columns: $\mathbf{b}_3 = -2\mathbf{b}_1 + 3\mathbf{b}_2$. The same dependency relationship holds for the corresponding columns of matrix $A$.

$$\mathbf{a}_3 = \begin{bmatrix} -2 \\ -3 \\ 1 \\ 9 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix} = -2\mathbf{a}_1 + 3\mathbf{a}_2$$

## *Solutions of Systems of Linear Equations*

You now know that the set of all solution vectors of the *homogeneous* linear system $A\mathbf{x} = \mathbf{0}$ is a subspace. Is this true also of the set of all solution vectors of the *nonhomogeneous* system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} \neq \mathbf{0}$? The answer is "no," because the zero vector is never a solution of a nonhomogeneous system. There is a relationship, however, between the sets of solutions of the two systems $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$. Specifically, if $\mathbf{x}_p$ is a *particular* solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, then *every* solution of this system can be written in the form

$$\mathbf{x} = \underbrace{\mathbf{x}_p}_{\substack{\text{Solution} \\ \text{of } A\mathbf{x} = \mathbf{b}}} + \underbrace{\mathbf{x}_h}_{\substack{\text{Solution} \\ \text{of } A\mathbf{x} = \mathbf{0}}},$$

where $\mathbf{x}_h$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$. The next theorem states this important concept.

**THEOREM 4.18 Solutions of a Nonhomogeneous Linear System**

If $\mathbf{x}_p$ is a particular solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, then every solution of this system can be written in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

*PROOF* Let $\mathbf{x}$ be any solution of $A\mathbf{x} = \mathbf{b}$. Then $(\mathbf{x} - \mathbf{x}_p)$ is a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$, because

$$A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Letting $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p$, you have $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

## EXAMPLE 8 Finding the Solution Set of a Nonhomogeneous System

Find the set of all solution vectors of the system of linear equations.

$$\begin{aligned} x_1 \phantom{{}+2x_2} - 2x_3 + \phantom{5}x_4 &= 5 \\ 3x_1 + x_2 - 5x_3 \phantom{{}+5x_4} &= 8 \\ x_1 + 2x_2 \phantom{{}-2x_3} - 5x_4 &= -9 \end{aligned}$$

*SOLUTION* The augmented matrix for the system $A\mathbf{x} = \mathbf{b}$ reduces as follows.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system of linear equations corresponding to the reduced row-echelon matrix is

$$\begin{aligned} x_1 \phantom{{}+x_2} - 2x_3 + \phantom{3}x_4 &= 5 \\ x_2 + x_3 - 3x_4 &= -7. \end{aligned}$$

Letting $x_3 = s$ and $x_4 = t$, you can write a representative solution vector of $A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s - t + 5 \\ -s + 3t - 7 \\ s + 0t + 0 \\ 0s + t + 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

$$= s\mathbf{u}_1 + t\mathbf{u}_2 + \mathbf{x}_p$$

You can see that $\mathbf{x}_p$ is a *particular* solution vector of $A\mathbf{x} = \mathbf{b}$, and $\mathbf{x}_h = s\mathbf{u}_1 + t\mathbf{u}_2$ represents an arbitrary vector in the solution space of $A\mathbf{x} = \mathbf{0}$.

The final theorem in this section describes how the column space of a matrix can be used to determine whether a system of linear equations is consistent.

**THEOREM 4.19 Solutions of a System of Linear Equations**

The system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of $A$.

*PROOF* Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

be the coefficient matrix, the column matrix of unknowns, and the right-hand side, respectively, of the system $A\mathbf{x} = \mathbf{b}$. Then

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

So, $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{b}$ is a linear combination of the columns of $A$. That is, the system is consistent if and only if $\mathbf{b}$ is in the subspace of $R^m$ spanned by the columns of $A$.

## EXAMPLE 9 Consistency of a System of Linear Equations

Consider the system of linear equations

$$\begin{aligned} x_1 + x_2 - x_3 &= -1 \\ x_1 \phantom{+ x_2} + x_3 &= 3 \\ 3x_1 + 2x_2 - x_3 &= 1. \end{aligned}$$

The rank of the coefficient matrix is equal to the rank of the augmented matrix.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A \,\vdots\, \mathbf{b}] = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As shown above, $\mathbf{b}$ is in the column space of $A$, and the system of linear equations is consistent.