14.5

(1) $$\int_{-1}^{1}\int_{0}^{2}\int_{0}^{1}(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$=\int_{-1}^{1}\int_{0}^{2}\left[\frac{x^3}{3}+xy^2+xz^2\right]_0^1 dy\,dz$$

$$=\int_{-1}^{1}\int_{0}^{2}\left[\left(\frac{1}{3}+y^2+z^2\right)-0\right]dy\,dz$$

$$=\int_{-1}^{1}\int_{0}^{2}\left(\frac{1}{3}+y^2+z^2\right)dy\,dz$$

$$=\int_{-1}^{1}\left(\frac{1}{3}y+\frac{y^3}{3}+z^2y\right)_0^2 dz$$

$$=\int_{-1}^{1}\left[\left(\frac{2}{3}+\frac{8}{3}+2z^2\right)-0\right]dz$$

$$=\left[\frac{2}{3}z+\frac{8}{3}z+\frac{2z^3}{3}\right]_{-1}^{1}$$

$$=\left(\frac{2}{3}+\frac{8}{3}+\frac{2}{3}\right)-\left(-\frac{2}{3}-\frac{8}{3}-\frac{2}{3}\right)$$

$$=8$$

② $$\int_{1/3}^{1/2}\int_{0}^{\pi}\int_{0}^{1} zx\sin xy\,dz\,dy\,dx$$

$$= -\int_{1/3}^{1/2}\int_{0}^{\pi}\left[\left(x\,\overset{\cos}{\cancel{\sin}}\,xy\right)\frac{z^2}{2}\right]_0^1 dy\,dx$$

$$= -\int_{1/3}^{1/2}\int_{0}^{\pi}\frac{x\,\overset{\cos}{\cancel{\sin}}\,xy}{2}\,dy\,dx$$

$$\frac{\frac{1}{2}}{\frac{2\pi}{1}} \qquad \frac{1}{4\pi}$$

$$= -\int_{1/3}^{1/2}\frac{x}{x}\left[\frac{\overset{\sin}{\cancel{\cos}}\,xy}{2}\right]_0^{\pi} dx$$

$$= -\int_{1/3}^{1/2}\left(\frac{\overset{\sin}{\cancel{\cos}}\,\pi x}{2}-\frac{1}{2}\right)dx$$

$$= -\int_{1/3}^{1/2}\left(\frac{\overset{\sin}{\cancel{\cos}}\,\pi x}{2}-\frac{1}{2}\right)dx$$

$$= -\left[\frac{\overset{\sin}{\cancel{\cos}}\,\pi x}{2\pi}-\frac{1}{2}x\right]_{1/3}^{1/2}$$

$$= -\left(0-\frac{1}{4}-\frac{1}{4\pi}+\frac{1}{6}\right)$$

$$= \frac{1}{4}+\frac{1}{4\pi}-\frac{1}{6}$$

$$= \frac{1}{12}+\frac{1}{4\pi}$$

③ $$\int_0^2 \int_{-1}^{y^2} \int_{-1}^{z} yz \, dx \, dz \, dy$$

$$= \int_0^2 \int_{-1}^{y^2} \Big[xyz \Big]_{-1}^{z} dz \, dy$$

$$= \int_0^2 \int_{-1}^{y^2} \left(yz^2 + yz \right) dz \, dy$$

$$= \int_0^2 \left[\frac{yz^3}{3} + \frac{yz^2}{2} \right]_{-1}^{y^2} dy$$

$$= \int_0^2 \left[\frac{3yz^3 + 2yz^2}{6} \right]_{-1}^{y^2} dy$$

$$= \int_0^2 \left[\frac{3y^6 + 2y^4}{6} - \frac{-3y + 2y}{6} \right] dy$$

$$= \int_0^2 \frac{3y^6 + 2y^4 - 3y - 2y}{6} \, dy$$

$$= \int_0^2 \frac{3y^6 + 2y^4 - 5y}{6} \, dy$$

$$= \frac{1}{6} \left[\frac{3y^7}{7} + \frac{2y^5}{5} - \frac{5y^2}{2} \right]_0^2$$

(4) $$\int_0^{\pi/4}\int_0^1\int_0^{x^2} x\cos y \, dz\, dx\, dy$$

$$= \int_0^{\pi/4}\int_0^1 \cancel{\int} \Big[xz\cos y\Big]_0^{x^2} dx\, dy$$

$$= \int_0^{\pi/4}\int_0^1 x^3\cos y \, dx\, dy$$

$$= \frac{1}{4}\int_0^{\pi/4} \Big[x^4\cos y\Big]_0^1 dy$$

$$= \frac{1}{4}\int_0^{\pi/4} \cos y \, dy$$

$$= \frac{1}{4}\Big[\sin y\Big]_0^{\pi/4}$$

$$= \sqrt{2}/8$$

⑤ $$\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^x xy \, dy \, dx \, dz$$

$$= \frac{1}{2}\int_0^3 \int_0^{\sqrt{9-z^2}} \left[x y^2 \right]_0^x dx \, dz$$

$$= \frac{1}{2} \int_0^3 \int_0^{\sqrt{9-z^2}} x^3 \, dx \, dz$$

$$= \frac{1}{2} \times \frac{1}{4} \int_0^3 \left[x^4 \right]_0^{\sqrt{9-z^2}} dz$$

$$= \frac{1}{8} \int_0^3 (9-z^2)^2 \, dz$$

$$= \frac{1}{8} \int_0^3 81 - 18z^2 + z^4 \, dz$$

$$= \frac{1}{8} \left[81z - 6z^3 + z^5/5 \right]_0^3$$

$$= 81/15$$

⑥
$$\int_1^3 \int_x^{x^2} \int_0^{\ln z} x e^y \, dy \, dz \, dx$$

$$= \int_1^3 \int_x^{x^2} \Big[x e^y \Big]_0^{\ln z} dz \, dx$$

$$= \int_1^3 \int_x^{x^2} \Big[x e^{\ln z} - x e^0 \Big] dz \, dx$$

$$= \int_1^3 \int_x^{x^2} (xz - x) \, dz \, dx$$

$$= \int_1^3 \int_x^{x^2} xz - x \, dz \, dx$$

$$= \int_1^3 \Big[x z^2/2 - xz \Big]_x^{x^2} dx$$

$$= \int_1^3 \left(x^5/2 - x^3 - x^3/2 + x^2 \right) dx$$

$$= \int_1^3 \left(\frac{x^5}{2} - \frac{3x^3}{2} + x^2 \right) dx$$

$$= \Big[x^6/12 - 3x^4/8 + x^3/3 \Big]_1^3 \qquad = \frac{315}{8} - \frac{1}{24}$$

$$= 118/3$$

(6)

$$\int_1^3 \int_x^{x^2} \int_0^{\ln z} x e^y \, dy \, dz \, dx$$

$$= \int_1^3 \int_x^{x^2} \left[x e^y \right]_0^{\ln z} dz \, dx$$

$$= \int_1^3 \int_x^{x^2} \left[x e^{\ln z} - x e^0 \right] dz \, dx$$

$$= \int_1^3 \int_x^{x^2} (xz - x) \, dz \, dx$$

$$= \int_1^3 \int_x^{x^2} xz - x \, dz \, dx$$

$$= \int_1^3 \left[x z^2/2 - xz \right]_x^{x^2} dx$$

$$= \int_1^3 \left(x^5/2 - x^3 - x^3/2 + x^2 \right) dx$$

$$= \int_1^3 \left(\frac{x^5}{2} - \frac{3x^3}{2} + x^2 \right) dx$$

$$= \left[x^6/12 - 3x^4/8 + x^3/3 \right]_1^3 \qquad = \frac{315}{8} - \frac{1}{24}$$

$$= 118/3$$

(7) $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_{-5+x^2+y^2}^{3-x^2-y^2} x\, dz\, dy\, dx$

$$= \int_0^2\int_0^{\sqrt{4-x^2}} \Big[xz\Big]_{-5+x^2+y^2}^{3-x^2-y^2} dy\, dx$$

$$= \int_0^2\int_0^{\sqrt{4-x^2}} (3x - x^3 - xy^2 + 5x - x^3 - xy^2)\, dy\, dx$$

$$= \int_0^2\int_0^{\sqrt{4-x^2}} (8x - 2x^3 - 2xy^2)\, dy\, dx$$

$$= -2\int_0^2\int_0^{\sqrt{4-x^2}} -4x + x^3 + xy^2\, dy\, dx$$

$$= -2\int_0^2 \Big[-4xy + x^3 y + \frac{xy^3}{3}\Big]_0^{\sqrt{4-x^2}} dx$$

$$= -2\int_0^2 -4x\sqrt{4-x^2} + x^3\sqrt{4-x^2} + \frac{x\left(\sqrt{4-x^2}\right)^3}{3}\, dx$$

$$= \cancel{-2\Big[\frac{x^2}{2}\sqrt{4-x^2}} - 2\int_0^2 \sqrt{4-x^2}\left(-4x + x^3 + \frac{4x - x^3}{3}\right) dx$$

$$= -2\int_0^2 \sqrt{4-x^2}\left(\cancel{4x} - \frac{12x + 3x^3 + 4x - x^3}{3}\right) dx$$

$$= -\frac{2}{3}\int_0^2 \sqrt{4-x^2}\,(-8x + 2x^3)\,dx$$

$$= -\frac{4}{3}\int_0^2 \sqrt{4-x^2}\,(-4x + x^3)\,dx$$

$$= -\frac{4}{3}\int_0^2 x\sqrt{4-x^2}\,(-4 + x^2)\,dx$$

$$= \frac{4}{3}\int_0^2 x\cdot\sqrt{4-x^2}\,(4-x^2)\,dx$$

$$= \frac{4}{3}\int_0^2 x\,(4-x^2)^{3/2}\,dx$$

$4-x^2 = z$

$-2x = \frac{dz}{dx}$

$x = 0 \quad x = 2$

$z = 4 \quad z = 0$

$$= -\frac{4}{3\times 2}\int_0^2 z^{3/2}\,dz$$

$$= -\frac{2}{3}\times\frac{2}{5}\left[z^{5/2}\right]_4^0$$

$$= -\frac{4}{15}(-32)$$

$$= 128/15$$

(8)

$$\int_1^2 \int_z^2 \int_0^{\sqrt{3}y} \frac{y}{x^2+y^2}\, dx\, dy\, dz$$

$$= \int_1^2 \int_z^2 \int_0^{\sqrt{3}y} \frac{1}{y^2+x^2}\, dx\, dy\, dz$$

$$= \int_1^2 \int_z^2 y/y \left[\tan^{-1} x/y\right]_0^{\sqrt{3}y} dy\, dz$$

$$= \int_1^2 \int_z^2 \cancel{y} \left(\pi/3 - 0\right) dy\, dz$$

$$= \int_1^2 \int_z^2 \cancel{y}\,\pi/3\, dy\, dz$$

$$= \pi/3 \int_1^2 \left[y\cancel{^2/2}\right]_z^2 dz$$

$$= \pi/\cancel{6}3 \int_1^2 \left[\tfrac{2}{\cancel{4}} - z\right] dz$$

$$= \pi/\cancel{6}3 \left[\cancel{4z}\,2z - z^2/\cancel{3}\,2\right]_1^2$$

$$\cancel{= \pi/6 \left[\left(8 - 8/3 - 4 + 1/3\right)\right] = 5\pi/18}$$

$$= \pi/\cancel{6}3 \left(4 - 2 - \cancel{4}\,2 + 1/2\right) = \pi/6$$

(5) $\iiint_G xy\sin yz\,dV$ where G is the rectangular box defined by inequalities

$0 \le x \le \pi,\ 0 \le y \le 1,\ 0 \le z \le \pi/6$

$$\int_0^{\pi/6}\int_0^1\int_0^{\pi} xy\sin yz\,dx\,dy\,dz$$

$$= \tfrac{1}{2}\int_0^{\pi/6}\int_0^1 y\sin y\,[x^2]_0^{\pi}\,dy\,dz$$

$$= \pi^2/2 \int_0^{\pi/6}\int_0^1 y\sin y\,dy\,dz$$

$$\int y\sin y = y\int\sin y + \int\cos y\,dy$$

$$= -y\cos y + \sin y$$

$$= \pi^2/2 \int_0^{\pi/6}\left[-y\cos y + \sin y\right]_0^1 dz$$

$$= \pi^2/2 \int_0^{\pi/6}\left[-\cos 1 + \sin 1\right] dz.$$

$$\int_0^{\pi}\int_0^1\int_0^{\pi/6} xy \sin yz \; dz\, dy\, dx$$

$$= -\int_0^{\pi}\int_0^1 \left[xy \frac{1}{y} \cos yz \right]_0^{\pi/6} dy\, dx$$

$$= -\int_0^{\pi}\int_0^1 x \left(\cos \frac{y\pi}{6} - 1 \right) dy\, dx$$

$$= \int_0^{\pi}\int_0^1 x\left(1 - \cos \frac{y\pi}{6}\right) dy\, dx$$

$$= \int_0^{\pi} \left(xy - \frac{6x}{\pi} \sin \frac{y\pi}{6} \right)_0^1 dx$$

$$= \int_0^{\pi} x - \frac{6x}{\pi} \cdot \frac{1}{2} \, dx$$

$$= \int_0^{\pi} x - \frac{3x}{\pi} \, dx$$

$$= \left[\frac{x^2}{2} - \frac{3x^2}{2\pi} \right]_0^{\pi}$$

$$= \pi^2/2 - \frac{3\pi^2}{2\pi} = \frac{\pi^2 - 3\pi}{2}$$

Ans.

(10)

$\iiint_G y\, d$, where G is the solid enclosed

by the plane $z=y$, the ~~xz~~ plane and

parabolic cylinder $y = 1-x^2$

$z = 0$ (xy plane)

$z = y$

$y = 1-x^2$

$x^2 = 1-y$

$x^2 = 1-0$

$x = \pm 1$

$$\iiint y\, dV$$

$$\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{y} y \; dz\,dy\,dx \;\; \cancel{dy\,dx\,dz}$$

$$= \int_{-1}^{1}\int_{0}^{1-x} y(y)\, dy\, dx$$

$$= \int_{-1}^{1}\int_{0}^{1-x^2} y^2\, dy\, dx$$

$$= \tfrac{1}{3}\int_{-1}^{1} \left[y^3\right]_0^{1-x^2}$$

$$= \frac{1}{3}\int_{-1}^{1} (1-x^2)^3\, dx$$

$$= \frac{1}{3}\int_{-1}^{1} 1-3x^2+3x^4-x^6 \;dx = \frac{1}{3}\left[x - \frac{3}{3}x^3 + \frac{3}{5}x^5 - \frac{x^7}{7}\right]$$

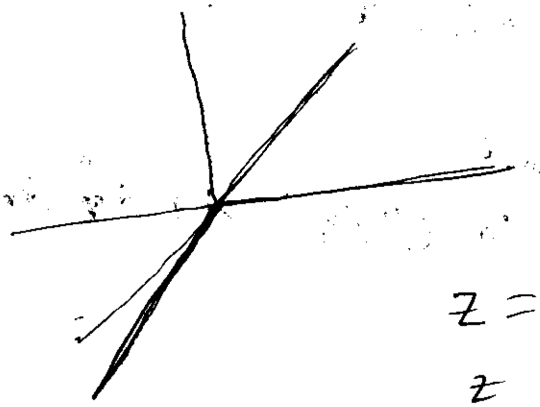
$$= \tfrac{1}{3}\left(16/35 + 16/35\right) = 32/105$$

$1-x^2 = p$

$-2x = p\,dz$

(17) $\iiint_G xyz\,dV$, where G is the solid in the first octant that is bounded by parabolic cylinders $z=2-x^2$ and the planes $z=0$, $y=x$ and $y=0$.

$$\int_0^{\sqrt{2}}\int_0^{x}\int_0^{2-x^2} xyz\,dz\,dy\,dx$$



$z=2-x^2$

$z=0$

$$=\frac{1}{2}\int_0^{\sqrt{2}}\int_0^{x} xy\left(4-4x^2+x^4\right)dy\,dx$$

$$=\frac{1}{2}\int_0^{\sqrt{2}}\int_0^{x} xy\left(2-x^2\right)dy\,dx$$

$$=\frac{1}{4}\int_0^{\sqrt{2}} x\left(2-x^2\right)x^2\,dx$$

$$=\frac{1}{4}\int_0^{\sqrt{2}} x^3\left(2-x^2\right)dx$$

$$=\frac{1}{4}\int_0^{\sqrt{2}} 2x^3-x^5\,dx=\frac{1}{4}\left[\frac{x^4}{2}-\frac{x^6}{6}\right]_0^{\sqrt{2}}$$

$=1/6$ Ans.

$z=2-x^2$

$=2-y^2$

$x^2=2-z$

$x=\pm\sqrt{2}$

$x^2=2-z$

$=2-2+y^2$

$=y^2$

$=x^2$

(12)

$$\iiint_G \cos(z/y)\, dV$$ where G is the solid defined by the inequalities

$\pi/6 \le y \le \pi/2,\ y \le x \le \pi/2,\ 0 \le z \le xy$

$$\int_{\pi/6}^{\pi/2}\int_{y}^{\pi/2}\int_{0}^{xy} \cos(z/y)\, dz\, dx\, dy$$

$$= \int_{\pi/6}^{\pi/2}\int_{y}^{\pi/2} \Big[y \sin z/y\Big]_0^{xy} dx\, dy$$

$$= \int_{\pi/6}^{\pi/2}\int_{y}^{\pi/2} y \sin x \, dx\, dy$$

$$= -\int_{\pi/6}^{\pi/2} y\left(\cos \pi/2 - \cos y\right) dy$$

$$= \int_{\pi/6}^{\pi/2} y \cos y \, dy$$

$$= \left[y\int \cos y - \int \frac{dy}{dy}\int \cos y\, dy\right]_{\pi/6}^{\pi/2} = (5\pi - 6\sqrt{3})$$

(15) Use the triple Integral to find the volume of the solid in the first octant bounded by the coordinate planes and the plane $3x+6y+4z=12$

$$\int_0^4\int_0^{4-\frac{x}{2}}\int_0^{\frac{12-3x-6y}{4}} dz\,dy\,dx$$

$$=\frac{1}{4}\int_0^4\int_0^{\frac{4-x}{2}} 12-3x-6y\,dy\,dx$$

$$=\frac{1}{4}\int_0^4\left[12y-3xy-3y^2\right]_0^{\frac{4-x}{2}}dx$$

$$=\frac{1}{4}\int_0^4\left[\frac{12}{2}(4-x)-\frac{3}{2}x(4-x)-\frac{3}{4}(4-x)^2\right]dx$$

$$=\frac{1}{8}\int_0^4 12(4-x)-3x(4-x)-\frac{3}{2}(4-x)^2\,dx$$

$$12-3x-6+\tfrac{3}{2}x$$

$$=\frac{1}{8}\int_0^4 \cancel{12p+}3(4-x)\left(\cancel{12-3x-12+6x}\right)dx$$

$$=\frac{1}{8}\int_0^4(4-x)\left(6+\frac{-6x+3x}{2}\right)dx=\frac{1}{8}\int_0^4(4-x)\left(6-\frac{3x}{2}\right)$$

$$=\frac{3}{16}\int_0^4(4-x)^2\,dx=4$$

$$z=\frac{12-3x-6y}{4}$$

$$6y=12-4z-3x$$

$$y=\frac{12-4z-3x}{6}$$

$$=\frac{12-3x}{6}$$

$$=\frac{4-x}{2}$$

$$x=12/3$$

$$=4$$

$$\frac{4-x}{\frac{12-3x}{2}}$$

let $4-x=p$

$-dx=dp$

(16) The solid bounded by the surface $z = \sqrt{y}$ and the planes $x+y=1$

$x=0$ and $z=0$

$x = 1-y$

$y=1$

$x+y=1$

$z=\sqrt{y}$

$z=0$

$x=0$

$y = z^2$

$y=0$

$y = 1-x$

$$\int_0^1 \int_0^{1-x} \int_0^{\sqrt{y}} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \sqrt{y}\, dy\, dx$$

$$= \frac{2}{3}\int_0^1 \left[y^{3/2}\right]_0^{1-x} dx$$

$$= \frac{2}{3}\int_0^1 (1-x)^{3/2}\, dx$$

let

$1-x = z$

$-dx = dz$

$x=0 \quad z=1$

$x=1 \quad z=0$

$$= -\frac{2}{3}\int_1^0 z^{3/2}\, \cancel{dz}$$

$$= -\frac{2}{3} \times \frac{2}{5}\left[z^{5/2}\right]_1^0$$

$$= \frac{4}{15} \quad \text{Ans.}$$

(17)

The solid bounded by the surface $y = x^2$ and the planes $y + z = 4$ and $z = 0$

$$\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{4-y} dz\, dy\, dx$$

$y = x^2$
$y + z = 4$
$z = 0$

$y = x^2$

$x = \sqrt{y}$
$= \sqrt{4}$
$= \pm\sqrt{2}$

$x = \pm\sqrt{y}$
$= 2$

$$= \int_{-2}^{2}\int_{x^2}^{4} 4 - y \; dy\, dx$$

$$= \int_{-2}^{2} \left[4y - \frac{y^2}{2}\right]_{x^2}^{4} dx$$

$$= \int_{-2}^{2} \left[16 - 8 - 4x^2 + x^4/2\right] dx$$

$$= \int_{-2}^{2} \left[8 - \frac{8x^2 - x^4}{2}\right] dx$$

$$= \int_{-2}^{2} \frac{16 - 8x^2 + x^4}{2} dx$$

$$= \frac{1}{2}\left[16x - \frac{8}{3}x^3 + x^5/5\right]_{-2}^{2} = \frac{1}{2}\left(\frac{256}{15} + \frac{256}{15}\right)$$

$$= 256/15$$

(18)

The wedge in the first octant that is cut from the solid cylinder $y^2+z^2\le 1$ by the planes $y=x$ and $x=0$

$y^2+z^2=1$

$y=x$

$x=0$

$y=\pm 1$

$x=\pm 1$

$$\int_0^1\int_0^y\int_0^{\sqrt{1-y^2}} dz\, dx\, dy$$

$$=\int_0^1\int_0^y \sqrt{1-y^2}\, dx\, dy$$

$$=\int_0^1 y\sqrt{1-y^2}$$

$1-y^2=t$

$-2y=dt$

$$=-\frac{1}{2}\int_1^0 \sqrt{t}\, dt$$

$$=\frac{1}{2}\times\frac{2}{3}$$

$$=\frac{1}{3}$$