CSE 221: Algorithms

Dynamic Programming

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References

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Contents

Dynamic Programming (DP)

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▷ *Greedy is evil, Dynamic Programming is good.* – Prof. Jeff Erickson, University of Illinois, Urbana-Champaign.

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Recursive solution to Fibonacci numbers

Definition (Fibonacci numbers)

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$$\langle 0, 1, 1, 2, 3, 5, 8, 21, 34, 55, 89, \ldots \rangle$$

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$$\textsc{Fib}(n) = \begin{cases} n & \text{if } n = 0 \text{ or } 1 \\ \textsc{Fib}(n-1) + \textsc{Fib}(n-2) & \text{if } n \geq 2 \end{cases}$$

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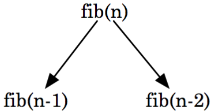
Straightforward recursive algorithm

```
FIBONACCI(n)                ▷ n ≥ 0
if n = 0 or n = 1
   then return n
   else return FIBONACCI(n − 1) + FIBONACCI(n − 2)
```

Recursion tree

fib(n)

Recursion tree



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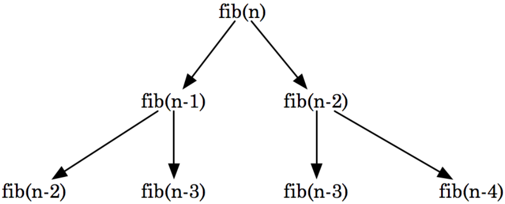

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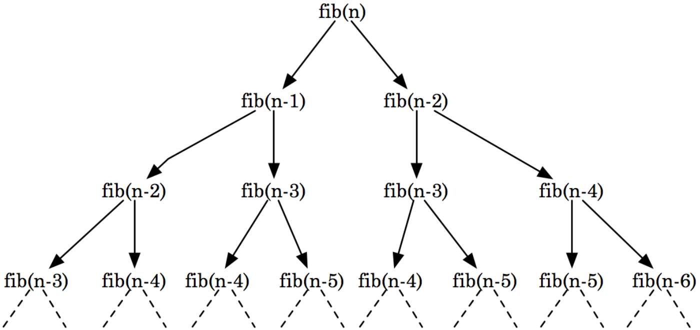

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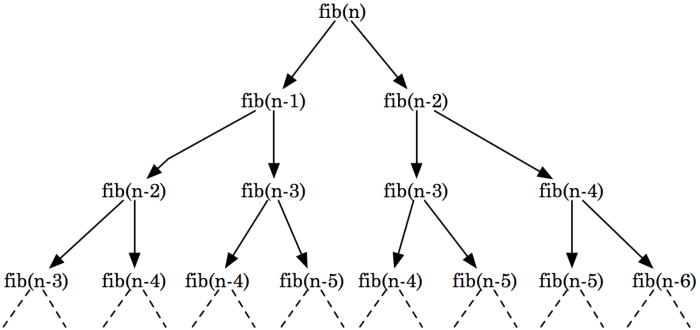

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This recursive algorithm for Fibonacci numbers has exponential running time!

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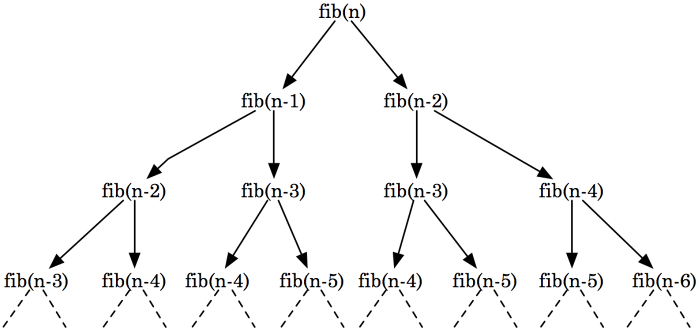

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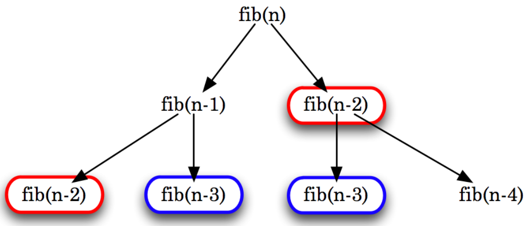
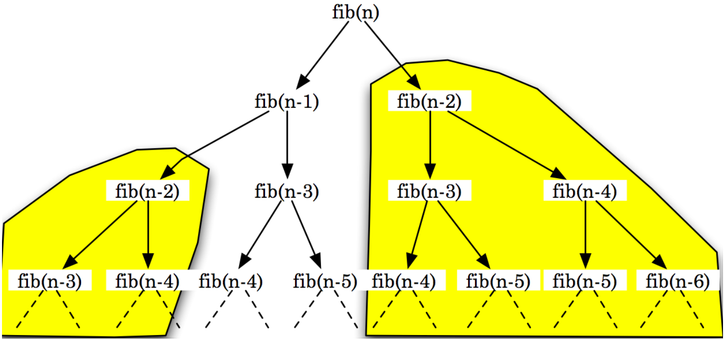
To be precise, $\boxed{T(n) = O(\varphi^n)}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Redundant computations



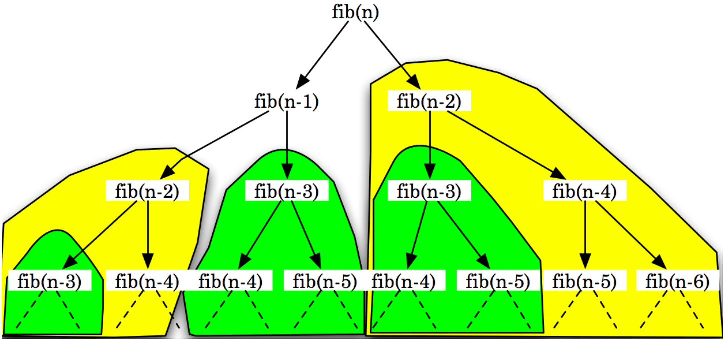
▷ Note how $\textsc{Fib}(n-2)$ and $\textsc{Fib}(n-3)$ are each being computed twice.

Redundant computations



▷ In fact, computing $\text{FIB}(n-2)$ involves computing a whole subtree.

Redundant computations



▷ Likewise for computing $\text{FIB}(n-3)$.

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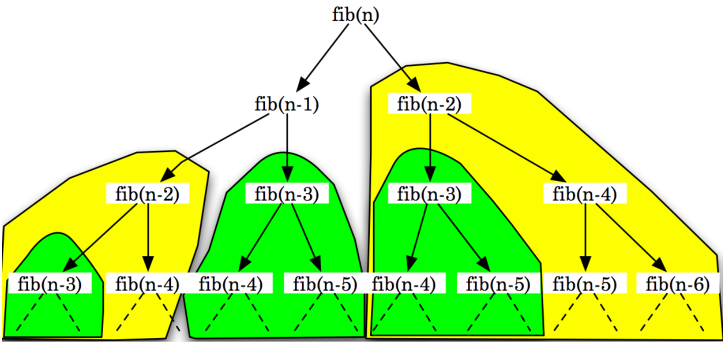

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- Spectacular redundancy in computation

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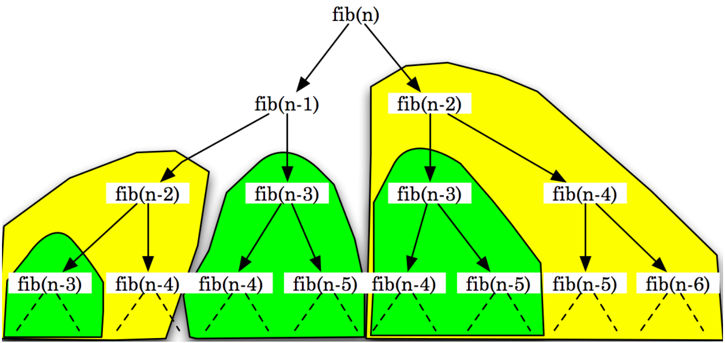

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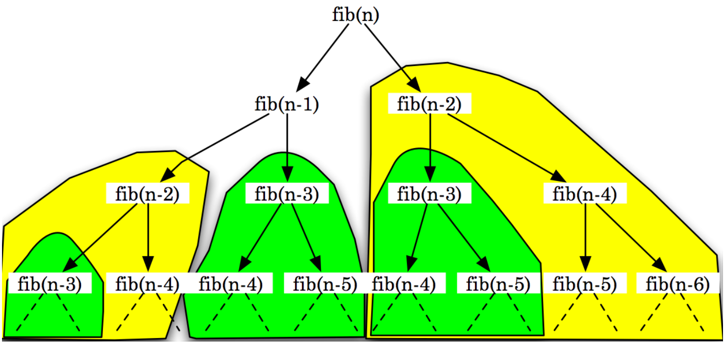

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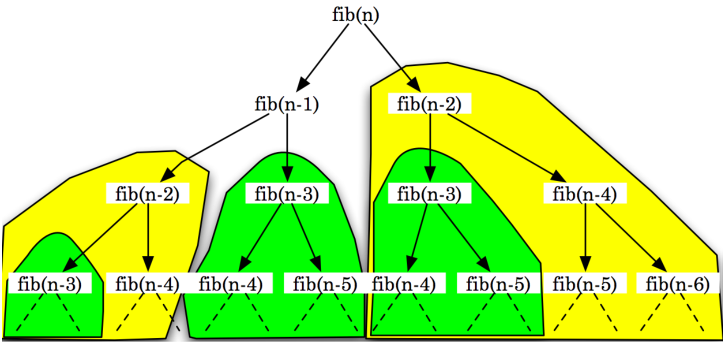

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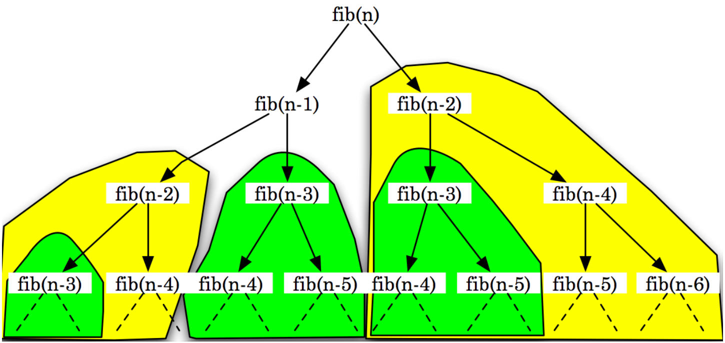

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- What if we compute and save the result of $\text{FIB}(i)$ for $i = \{2, 3, \ldots, n\}$ the first time, and then re-use it each time afterward?
- Ah, we've just (re)discovered Memo(r)ization!

Memoization

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1. At each step of computation, first see if the solution to the subproblem has already been found and saved.
2. If so, simply return the solution.
3. If not, compute the solution, and save it before returning the solution.

Memoized recursive algorithm for Fibonacci numbers

M-FIBONACCI(n) $\quad \triangleright$ $n \geq 0$, global $F = [0 \,..\, n]$

if $n = 0$ or $n = 1$
  then return n $\quad \triangleright$ Our base conditions.
if $F[n]$ is empty $\quad \triangleright$ No saved solution found for n.
  then $F[n] \leftarrow$ M-FIBONACCI($n - 1$) $+$ M-FIBONACCI($n - 2$)
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Questions

- What is this global array F?

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- What is this global array F? It's used store the values of the intermediate results, and must be initialized by the caller to all empty.

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M-FIBONACCI(n)        ▷ n ≥ 0, global F = [0 . . n]
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- What is an appropriate sentinel to indicate that $F[i], 0 \leq i \leq n$ has not been solved yet (i.e., empty)? Use -1, which is guaranteed to be an invalid value.

Memoized . . . Fibonacci numbers (continued)

```
FIBONACCI(n)            ▷ n ≥ 0
   ▷ Allocate an array F[0 . . n] to save results (LENGTH[F] = n + 1).
   for i ← 0 to n
        do F[i] ← −1            ▷ No solution computed for i yet (sentinel)
   return M-FIBONACCI(F, n)
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Memoized . . . Fibonacci numbers (continued)

FIBONACCI(n) $\quad \triangleright\ n \geq 0$

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Running time

Each element $F[2] \ldots F[n]$ is filled in just once in $\Theta(1)$ time, so $\boxed{T(n) = \Theta(n)}$.

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Questions to ask (and remember)

- What are the drawbacks, if any, of memoization?
- Would all recursive algorithms benefit from memoization? For example, would the recursive algorithm to compute the factorial of a number benefit from memoization?

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$$\boxed{T(n) = \Theta(n)}$$

Dynamic programming (continued)

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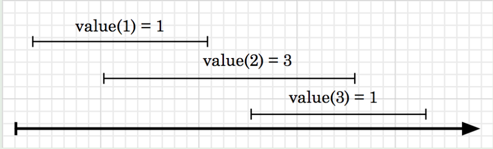
Definition (Weighted interval scheduling problem)

Given a set of schedules $I = \{I_i\}$, with associated weights $W = \{w_i\}$, find $A \subseteq I$ such that the members of A are non-conflicting *and* the total weight $\sum_{i \in A} w_i$ is maximized.

Example (an instance of weighted interval problem)



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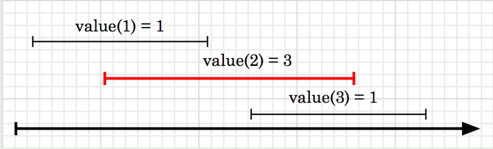

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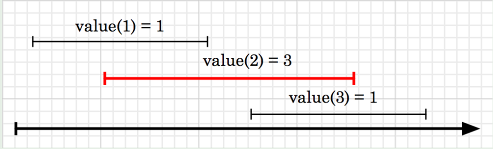

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What now?

First step is to formulate a recursive solution, but first we need to figure out what the subproblems are.

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  If $n \in \vartheta$ Then clearly all intervals that conflict with n are not members of ϑ. ϑ then contains n, plus an optimal solution to all intervals that do not conflict with n. We now need to have a quick way of computing list of conflicting intervals for n.

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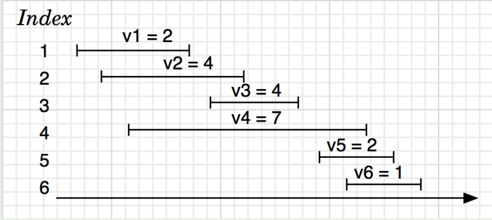
- Let ϑ be an optimal solution (even if we have no idea what it is yet).
- All we can say about ϑ is the following: interval n (the last interval) either belongs to ϑ, or it doesn't.

  If $n \in \vartheta$ Then clearly all intervals that conflict with n are not members of ϑ. ϑ then contains n, plus an optimal solution to all intervals that do not conflict with n. We now need to have a quick way of computing list of conflicting intervals for n.

  If $n \notin \vartheta$ Then ϑ contains an optimal solution for the intervals $\{i_1, i_2, \ldots, i_{n-1}\}$.

Developing a recursive solution (continued)

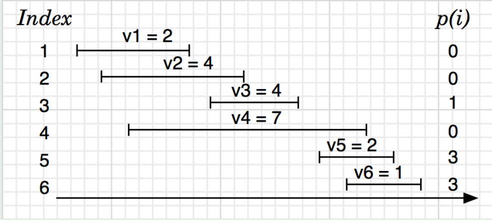
Example (an instance of a weighted interval problem)



▷ For each interval i, compute $p(i)$, the leftmost interval that does not conflict with i. Define $p(j) = 0$ if not request $i < j$ is disjoint from j.

Developing a recursive solution (continued)

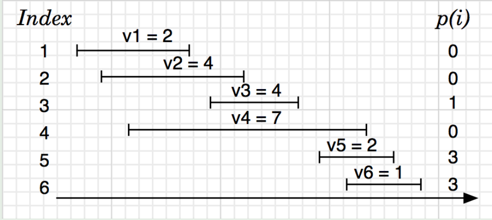
Example (an instance of a weighted interval problem)



▷ For a given interval i, $p(i)$ means that intervals $\{p(i)+1, p(i)+2, \ldots, i-1\}$ overlap with it. For example, $p(6) = 3$, which means that intervals $\{4, 5\}$ overlap interval 6.

Developing a recursive solution (continued)

Example (an instance of a weighted interval problem)



▷ Alternatively, intervals $\{1, 2, .., p(i)\}$ *do not* overlap interval i. For example, $p(6) = 3$ means that intervals $\{1, 2, 3\}$ do not overlap interval 6.

Developing a recursive solution (continued)

- If $n \in \vartheta$, then ϑ must include, in addition to interval n, an optimal solution to the subproblem consisting of intervals $\{1, 2, \ldots, p(n)\}$.

Developing a recursive solution (continued)

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Developing a recursive solution (continued)

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  ▷ $\boxed{\vartheta(n) = \text{MAX}(w_n + \vartheta(p(n)), \vartheta(n-1))}$

Developing a recursive solution (continued)

Recursive algorithm for an optimal value

If $OPT(j)$ is an optimal solution to the subproblem for intervals $\{1, 2, \ldots, j\}$, for any $j \in \{1, 2, \ldots, n\}$, then:

$$\boxed{OPT(j) = \textsc{max}(w_j + OPT(p(j)), OPT(j-1))}$$

Developing a recursive solution (continued)

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Extracting the intervals in an optimal solution

The interval j is in an optimal solution $OPT(j)$ **if and only if** the first of the two options is larger than the second.

Developing a recursive solution (continued)

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The interval j is in an optimal solution $OPT(j)$ **if and only if** the first of the two options is larger than the second.

Interval j belongs to an optimal solution on the set $\{1, 2, \ldots, j\}$ ***if and only if***

$$\boxed{w_j + OPT(p(j)) \geq OPT(j-1)}$$

A recursive algorithm

$\mathrm{WIS}(j)$

if $j = 0$

  then return 0

  else return $\textsc{max}(w_j + \mathrm{WIS}(p(j)),$
$\qquad\qquad\mathrm{WIS}(j-1))$

A recursive algorithm

$\text{WIS}(j)$

```
if j = 0
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- The initial call is $\text{WIS}(n)$ for intervals $\{1, 2, \ldots, n\}$ sorted in non-decreasing order of the finishing times.

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- The tree grows very rapidly, leading to exponential running time. The tree when $p(j) = j - 2$ for all j shows how quickly it grows.

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- The tree grows very rapidly, leading to exponential running time. The tree when $p(j) = j - 2$ for all j shows how quickly it grows.
- There are many overlapping subproblems, so the obvious choice is to memoize the recursion.

Memoizing the recursion

```
M-WIS(j)
if j = 0
    then return 0
elseif M[j] is empty
    then M[j] ← MAX(w_j + M-WIS(p(j)),
                      M-WIS(j − 1))
return M[j]
```

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- Each entry in $M[j]$ gets filled in only once at $\Theta(1)$ time, and there are $n+1$ entries, so $\text{M-WIS}(n)$ takes $\boxed{\Theta(n)}$ time.

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- Of course, sorting the intervals by the finish times takes $\Theta(n \lg n)$ time.
- This memoized algorithm *plus* sorting the intervals takes $\Theta(n \lg n) + \Theta(n) = \Theta(n \lg n)$ time.

Computing a solution in addition to its values

- The memoized algorithm only computes the optimal value, but does not extract the intervals that make up the solution.
- The key to extracting the solution is to note that item j is in ϑ if and only if $w_j + M[p(j)] \geq M[j-1]$. This provides two ways of extracting the intervals in the optimal solution:
  1. Trace back from $M[n]$ and extract the solution by checking which choice was made – $j-1$ or $p(j)$ – when $M[j]$ was included in the optimal set of intervals.
  2. Whenever a choice is made between two options, save in $pred[j]$, the predecessor pointer, the choice that was made between $j-1$ and $p(j)$.

Computing a solution in addition to its values (continued)

- The first way recursively extracts an optimal set of intervals for a problem size of $1 \leq j \leq n$.
- Calling $\text{WIS-FIND-SOLUTION}(n)$ extracts all the intervals in the optimal solution.

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$\text{WIS-FIND-SOLUTION}(j)$

if $j = 0$
  then Output nothing
  else
    if $w_j + M[p(j)] \ge M[j-1]$
      then Output j
        $\text{WIS-FIND-SOLUTION}(p(j))$
      else $\text{WIS-FIND-SOLUTION}(j-1)$

Computing a solution in addition to its values (continued)

- The second way requires that $\mathrm{M\text{-}WIS}$ use an auxiliary array $pred[0..n]$ to save the predecessor of each interval in the solution.
- Initialize $pred[j] = 0$ for all $0 \leq j \leq n$.

Computing a solution in addition to its values (continued)

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- Initialize $pred[j] = 0$ for all $0 \le j \le n$.

$\textsc{M-WIS}(j)$

if $j = 0$
    then return 0
elseif $M[j]$ is empty
    then if $w_j + \textsc{M-WIS}(p(j)) > \textsc{M-WIS}(j-1))$
        then $M[j] \leftarrow w_j + \textsc{M-WIS}(p(j)$
            $pred[j] \leftarrow p(j)$
        else $M[j] \leftarrow \textsc{M-WIS}(j-1)$
            $pred[j] \leftarrow j-1$
return $M[j]$

Computing a solution in addition to its values (continued)

Now that we have $pred[j]$ filled in, we start from $M[n]$ and work backwards.

1. If $pred[j] = p(j)$, then we did add the j^{th} interval in the final solution, and we continue with $pred[j] \leftarrow p(j)$.
2. if $pred[j] \neq p(j)$, then we did not add the j^{th} interval in the final solution, and we continue with $pred[j] \leftarrow j - 1$.

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Can you come up with an iterative version?

Developing a Dynamic Programming algorithm

- The value of an optimal solution $OPT(j)$ for any $j \in \{1, 2, 3, \ldots, n\}$ depends on the values of $OPT(p(j))$ and $OPT(j-1)$.

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Dynamic programming algorithm

$\textrm{WIS}(n)$

$M[0] \leftarrow 0$
for $j \leftarrow 1$ **to** n
 do $M[j] = \textsc{max}(w_j + M[p(j)], M[j-1])$
return $M[n]$

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```

$\boxed{T(n) = \Theta(n)}$

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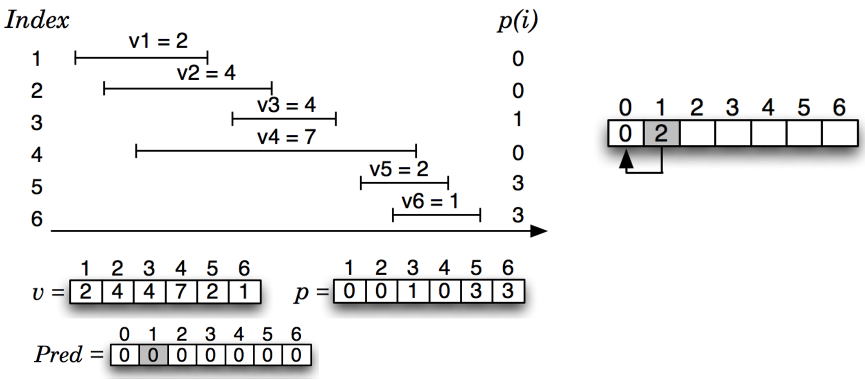
WIS-FIND-SOLUTION(j)

```
j ← n
while j > 0
      do if pred[j] = p(j)
            then Output j
         j ← pred[j]
```

Weighted Interval Scheduling DP algorithm in action



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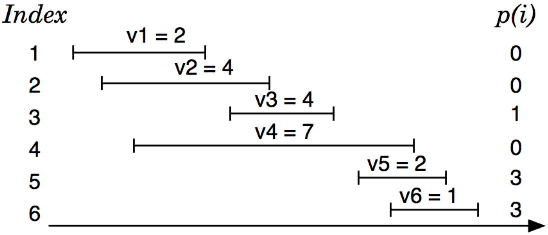

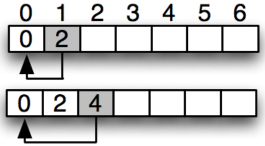

$v =$

1	2	3	4	5	6
2	4	4	7	2	1

$p =$

1	2	3	4	5	6
0	0	1	0	3	3

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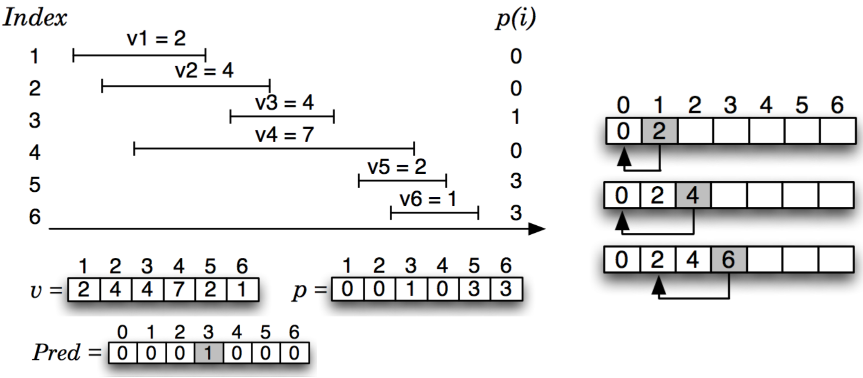

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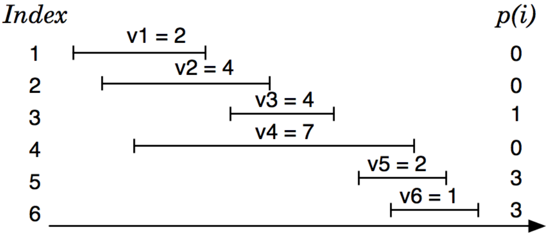

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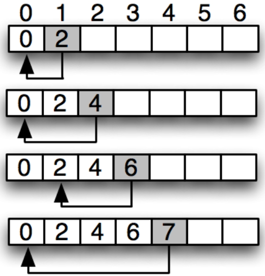

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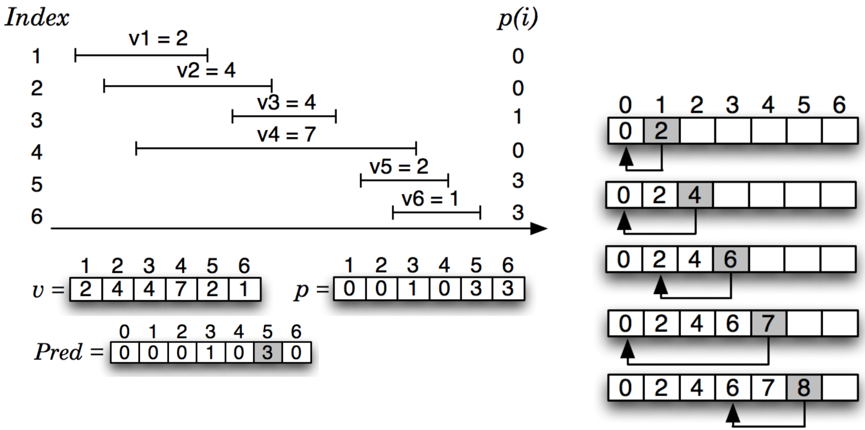

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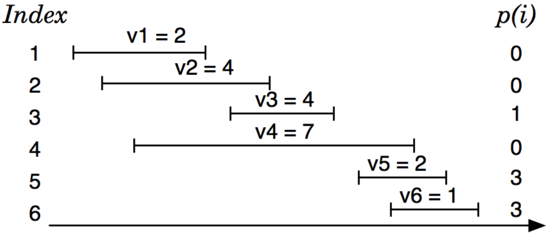

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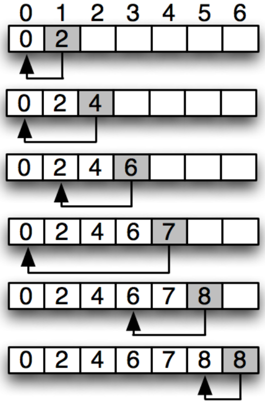

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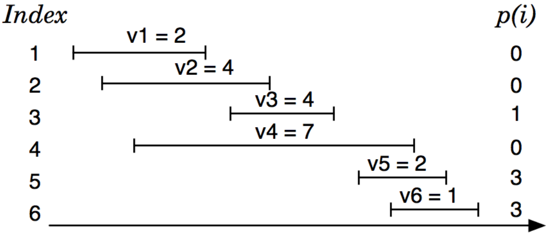

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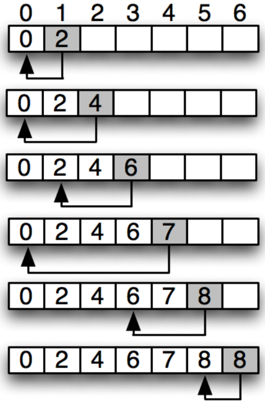

Optimal value: $\boxed{8}$

Optimal solution: $\boxed{\{5, 3, 1\}}$

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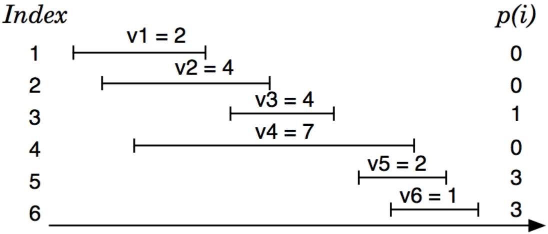

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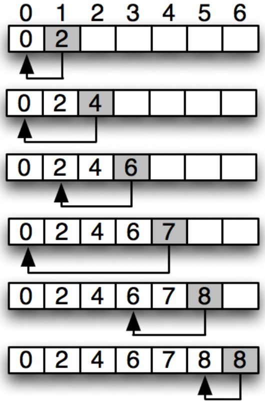

So, you think you understand Dynamic Programming now?

Answer the following questions

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Contents

0/1 knapsack problem

Definition (0/1 knapsack problem)

Given a set S of n items, such that each item i has a positive benefit v_i and a positive weight w_i, the goal is to find the maximum-benefit subset that does not exceed a given weight W.

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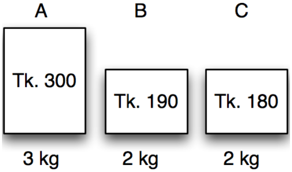

Maximum weight: $\boxed{W = 4 \text{ kg}}$

0/1 knapsack problem

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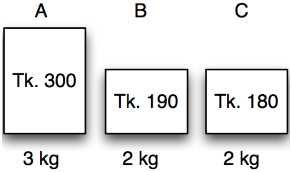
Given a set S of n items, such that each item i has a positive benefit v_i and a positive weight w_i, the goal is to find the maximum-benefit subset that does not exceed a given weight W. Formally, we wish to determine a subset of S that maximizes $\sum_{i \in S} v_i$, subject to $\sum_{i \in S} w_i \leq W$.



Maximum weight: $\boxed{W = 4 \text{ kg}}$

Optimal solution: items B and C Benefit: $\boxed{370}$

Developing a recursive solution

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  If $n \in \vartheta$ Then the optimal solution contains n, plus an optimal solution for the other $n - 1$ items, but with a reduced maximum weight of $W - w_n$.

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  If $n \notin \vartheta$ Then ϑ simply contains an optimal solution for the first $n-1$ items, with the maximum allowed weight W remaining unchanged.
- We have two parameters for each subproblem – the items S, and the maximum allowed weight W.

Developing a recursive solution (continued)

- $w_n > W \implies n \notin \vartheta$.
  - $\vartheta(n, W) = \vartheta(n-1, W)$

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  - If $n \in \vartheta$, then $\vartheta(n, W)$ is an optimal solution to the subproblem for items $\{1, 2, \ldots, n\}$:
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    $\triangleright$ $\boxed{\vartheta(n, W) = \textsc{max}(v_n + \vartheta(n-1, W - w_n), \vartheta(n-1, W))}$

Developing a recursive solution (continued)

Recursive algorithm for an optimal value

If $OPT(j, w)$ is an optimal solution to the subproblem for items $\{1, 2, \ldots, j\}$, for any $j \in \{1, 2, \ldots, n\}$, and with a maximum allowed weight of w, then:

$$OPT(j, w) = \begin{cases} OPT(j-1, w) & \text{if } w_j > w, \\ \textsc{max}(v_j + OPT(j-1, w - w_j), & \\ \qquad OPT(j-1, w)) & \text{otherwise.} \end{cases}$$

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Extracting the items in an optimal solution

The item j is in an optimal solution $OPT(j, w)$ **if and only if** the first of the two options is larger than the second.

$$\boxed{v_j + OPT(j-1, w-w_j) \geq OPT(j-1, w)}$$

A recursive algorithm

$\textsc{Knapsack}(j, w)$

```
if j = 0 or w = 0
    then return 0
elseif w_j > w
    then return KNAPSACK(j − 1, w))
else return MAX(v_j + KNAPSACK(j − 1, w − w_j),
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$\textsc{Knapsack}(j, w)$

1. **if** $j = 0$ or $w = 0$
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- The tree grows very rapidly, leading to exponential running time.
- There are many overlapping subproblems, so the obvious choice is to memoize the recursion.

Memoizing the recursion

M-KNAPSACK(j, w)

if $j = 0$ or $w = 0$
    then return 0
elseif $M[j, w]$ is empty
    then $M[j, w] \leftarrow$ MAX(v_j + M-KNAPSACK($j - 1, w - w_j$),
        M-KNAPSACK($j - 1, w$))
return $M[j, w]$

Memoizing the recursion

```
M-KNAPSACK(j, w)
if j = 0 or w = 0
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```

- Each entry in $M[j, w]$ gets filled in only once at $\Theta(1)$ time, and there are $n + 1 \times W + 1$ entries, so $\textsc{M-Knapsack}(n, W)$ takes $\boxed{\Theta(nW)}$ time.

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- Is this a linear-time algorithm?
- This is an example of a pseudo-polynomial problem, since it depends on another parameter W that is independent of the problem size.

Developing a Dynamic Programming algorithm

```
KNAPSACK(n, W)
for i ← 0 to n                ▷ no remaining capacity
      do M[i, 0] ← 0
for w ← 0 to W                ▷ no item to choose from
      do M[0, w] ← 0
for j ← 1 to n
      do for w ← 1 to W
            do if w_j > w
                  then M[j,w] = M[j − 1, w]
                  else  M[j, w] ← MAX(v_j + M[j − 1, w − w_j],
                                      M[j − 1, w])
return M[n, W]
```

0/1 Knapsack recursive algorithm in action

Given the following (from M. H. Alsuwaiyel, ex. 7.6):

$$W = 9$$
$$w_i = \{2, 3, 4, 5\}$$
$$v_i = \{3, 4, 5, 7\}$$

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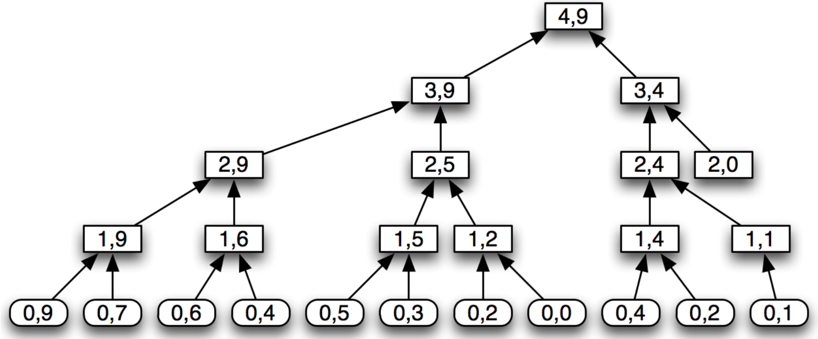

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	0	1	2	3	4	5	6	7	8	9
4	-	-	-	-	-	-	-	-	-	-
3	-	-	-	-	-	-	-	-	-	-
2	-	-	-	-	-	-	-	-	-	-
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4	0	0	3	4	5	7	8	10	11	12
3	0	0	3	4	5	7	8	9	9	12
2	0	0	3	4	4	7	7	7	7	7
1	0	0	3	3	3	3	3	3	3	3
0	0	0	0	0	0	0	0	0	0	0

Related problem: Subset Sums problem

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Given a set S of n items, such that each item i has a positive weight w_i, the goal is to find the maximum-weight subset that does not exceed a given weight W.

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- How is this similar to the 0/1 Knapsack problem?

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- How is this similar to the 0/1 Knapsack problem?
- Can you solve this using the same algorithm?

Contents

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Solution: 3 coins.

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Questions

What is the natural search space? Does this problem have a Dynamic Programming solution? If so, how do we develop it?

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  3. Best combination for $15 - 1 = 14$ paisa, plus a 1 paisa coin.

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- Since we're minimizing the number of coins, the best combination would be the minimum of these three choices.

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- By recursively solving for the best combination, this can be generalized to $|C|$ denominations to make change for any amount A.
- What are the subproblems?

Developing a recursive solution (continued)

If $OPT(p)$ is the minimum number of coins needed to make change for amount p with denominations $C = \{c_1, c_2, \ldots, c_k\}$, then:

Developing a recursive solution (continued)

If $OPT(p)$ is the minimum number of coins needed to make change for amount p with denominations $C = \{c_1, c_2, \ldots, c_k\}$, then:

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- Once we choose $c_i \leq p$, $OPT(p) = 1 + OPT(p - c_i)$, since we have to find the best combination for the remaining amount (picking a coin smaller than the amount at each step).

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Recurrence

$$OPT(p) = \begin{cases} 0 & \text{if } p = 0 \\ min_{i:c_i \leq p}\{1 + OPT(p - c_i)\} & \text{if } p > 0 \end{cases}$$

A recursive algorithm

$\textsc{Change}(n, C)$

if $n = 0$
  then return 0
  else $min \leftarrow \infty$
    for $i \leftarrow 1$ **to** $|C|$
      do if $c_i \leq n$ and $1 + \textsc{Change}(n - c_i, C) < min$
        then $min \leftarrow 1 + \textsc{Change}(n - c_i, C)$

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Memoizing the recursion

```
M-Change(n, C)

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   else  if M[n] is empty
            then min ← ∞
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                     do if c_i ≤ n and
                           1 + M-Change(n − c_i, C) < min
                          then min ← 1 + M-Change(n − c_i, C)
                 M[n] ← min
         return M[n]
```

Memoizing the recursion

```
M-CHANGE(n, C)
if n = 0
   then return 0
   else if M[n] is empty
           then min ← ∞
                for i ← 1 to |C|
                    do if c_i ≤ n and
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- Each entry in $M[n]$ gets filled in only once at $\Theta(|C|)$ time, and there are $n+1$ entries, so $\text{M-CHANGE}(n)$ takes $\boxed{\Theta(n|C|)}$ time.

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- Each entry in $M[n]$ gets filled in only once at $\Theta(|C|)$ time, and there are $n+1$ entries, so M-CHANGE(n) takes $\boxed{\Theta(n|C|)}$ time.
- Another pseudo-polynomial problem!

Developing a Dynamic Programming algorithm

```
CHANGE(n, C)
   ▷ M = [0 . . n], S = [0 . . n]
   M[0] ← 0          no amount to change
   for p ← 1 to n
       do min ← ∞
          for i ← 1 to |C|
              do if c_i ≤ p and 1 + M[p − c_i] < min
                    then min ← 1 + M[p − c_i]
                         coin ← i
          M[p] ← min
          S[p] ← coin
   return M and S
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```

- $M[p]$ for all $0 \leq p \leq n$ – minimum number of coins needed to change for p paisa.
- $S[p]$ for all $0 \leq p \leq n$ – the first coin chosen in computing an optimal solution for making change for p paise.

Computing a solution in addition to its values

- The S array in the algorithm "remembers" the first coin we use when computing an optimal value for a given amount.
- We go backwards using $S[n]$ until $n = 0$ and find the coin that was added at each step.

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COINS(S, C, n)

```
while n > 0
      do Output C[S[n]]
         n ← n − C_{S[n]}
```

Contents

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  Polynomially many subproblems
  : The total number of subproblems should be a polynomial, or else DP may not provide an efficient solution.

  Subproblem optimality
  : If the optimal solution to the entire problem contain optimal solution to the subproblems, then it has the subproblem optimality property. Also called the *principle of optimality*.

Dynamic Programming highlights

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- Unlike Memoization, which solves only the needed subproblems, DP solves all the subproblems, because it does it bottom-up.
- Dynamic Programming on the other hand may be much more efficient because its iterative, whereas Memoization must pay for the (often significant) overhead due to recursion.

Conclusion

- Memoization is the top-down technique, and dynamic programming is a bottom-up technique.
- The key to Dynamic programming is in "intelligent" recursion (the hard part), not in filling up the table (the easy part).
- Dynamic Programming has the potential to transform exponential-time brute-force solutions into polynomial-time algorithms.
- Greed does not pay, Dynamic Programming does!