BASIC LAWS OF LIMIT

The following limit laws allow us to find the limits of functions algebraically.

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then,

- ✓ **Law 1** Constant function Law

  If c is real number then

$$\lim_{x\to a} c = c$$

  For example

$$\lim_{x\to 3} 5 = 5$$

- ✓ **Law 2** Identity function Law

$$\lim_{x\to a} x = a$$

  For example

$$\lim_{x\to 3} x = 3$$

- ✓ **Law 3** Sum Law

$$\lim_{x\to a}[f(x) \pm g(x)] = \left[\lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)\right] = L \pm M$$

  For example

$$\lim_{x\to 3} x^2 = 9 \text{ and } \lim_{x\to 3} 2x = 6$$

$$\lim_{x\to 3}(x^2 + 2x) = \lim_{x\to 3}(x^2) + \lim_{x\to 3}(2x) = 9 + 6 = 15$$

- ✓ **Law 4** Product Law

$$\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x) \times \lim_{x\to a} g(x)\right] = LM$$

  If $f(x) = g(x)$ then

$$\lim_{x\to a}[f(x)]^2 = \lim_{x\to a}[f(x)f(x)] = \left[\lim_{x\to a} f(x) \times \lim_{x\to a} f(x)\right] = \left[\lim_{x\to a} f(x)\right]^2 = L^2$$

- ✓ **Law 5** Constant Multiple Law

$$\lim_{x\to a}[cf(x)] = c \times \lim_{x\to a}[f(x)] = cL$$

  For example

$$\lim_{x\to 3} 4x^2 = 4 \times \lim_{x\to 3} x^2 = 4 \times 9 = 36$$

- ✓ **Law 6** Quotient Law

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$$

  provided that $\lim_{x\to a} g(x) = M \neq 0$

✓ **Law 7** Root Law

$$\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{L}$$

provided that n is a positive integer, and $L > 0$ if n is even.

✓ **Law 8** If $\boldsymbol{n}$ is positive integer, then

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = [L]^n$$

For example

$$\lim_{x\to a}[f(x)]^2 = \left[\lim_{x\to a} f(x)\right]^2 = L^2$$

SOME USEFUL LIMITS

1. $\lim\limits_{x\to+\infty}\frac{1}{x} = 0$	2. $\lim\limits_{x\to-\infty}\frac{1}{x} = 0$	3. $\lim\limits_{x\to\infty}\frac{1}{x}$ **does not exist**
4. $\lim\limits_{x\to\infty}\frac{1}{x^2} = 0$	5. $\lim\limits_{x\to 0^+}\frac{1}{x} = +\infty$	6. $\lim\limits_{x\to 0^-}\frac{1}{x} = -\infty$
7. $\lim\limits_{x\to 0}\frac{1}{x}$ **does not exist**	8. $\lim\limits_{x\to 0^+}\ln x = -\infty$	9. $\lim\limits_{x\to+\infty}\ln x = +\infty$
10. $\lim\limits_{x\to 1}\ln x = 0$	11. $\lim\limits_{x\to e}\ln x = 1$	12. $\lim\limits_{x\to 0} e^x = 1$
13. $\lim\limits_{x\to+\infty} e^x = +\infty$	14. $\lim\limits_{x\to-\infty} e^x = 0$	15. $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$
16. $\lim\limits_{x\to 0}\sin x = 0$	17. $\lim\limits_{x\to\pi/2}\sin x = 1$	18. $\lim\limits_{x\to 0}\cos x = 1$
19. $\lim\limits_{x\to\pi/2}\cos x = 0$	14. $\lim\limits_{x\to 0}\frac{e^x - 1}{x} = 1$	15. $\lim\limits_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$
15. $\lim\limits_{x\to 0}(1 + x)^{\frac{1}{x}} = e$		

Example Calculate

$$\lim_{x\to 1}\frac{x}{\sqrt{x+1} - 1}$$

Solution

$$\lim_{x\to 1}\frac{x}{\sqrt{x+1} - 1}$$

Multiplying both the numerator and denominator by the conjugate of the denominator we get,

$$= \lim_{x\to 1}\frac{x\left(\sqrt{x+1} + 1\right)}{\left(\sqrt{x+1} - 1\right)\left(\sqrt{x+1} + 1\right)} = \lim_{x\to 1}\frac{x\left(\sqrt{x+1} + 1\right)}{\left(\sqrt{x+1}\right)^2 - 1^2} = \lim_{x\to 1}\frac{x\left(\sqrt{x+1} + 1\right)}{x + 1 - 1} = \lim_{x\to 1}\frac{x\left(\sqrt{x+1} + 1\right)}{x}$$

$$= \lim_{x\to 1}\left(\sqrt{x+1} + 1\right) = \lim_{x\to 1}\left(\sqrt{x+1}\right) + \lim_{x\to 1}(1) = \sqrt{2} + 1$$

Example Evaluate

$$\lim_{x\to 4}\left(2x^2+\sqrt[3]{x}-x^2+x-6\right)$$

Solution

$$\lim_{x\to 4}\left(2x^2+\sqrt[3]{x}-x-6\right)$$

$$=\lim_{x\to 4}(2x^2)+\lim_{x\to 4}\left(\sqrt[3]{x}\right)-\lim_{x\to 4}(x)-\lim_{x\to 4}(6)$$

$$=2\times\lim_{x\to 4}(x\times x)+\lim_{x\to 4}\left(\sqrt[3]{x}\right)-\lim_{x\to 4}(x)-\lim_{x\to 4}(6)$$

$$=2\times\lim_{x\to 4}(x)\times\lim_{x\to 4}(x)+\sqrt[3]{\lim_{x\to 4}(x)}-\lim_{x\to 4}(x)-\lim_{x\to 4}(6)$$

$$=2\times 4\times 4+\sqrt[3]{4}-4-6=22+\sqrt[3]{4}$$

Example Evaluate

$$\lim_{x\to 2}\frac{2x^2-5x+2}{5x^2-7x-6}$$

Solution

$$\lim_{x\to 2}\frac{2x^2-5x+2}{5x^2-7x-6}=\lim_{x\to 2}\frac{(x-2)(2x-1)}{(x-2)(5x+3)}=\lim_{x\to 2}\frac{(2x-1)}{(5x+3)}=\frac{\lim_{x\to 2}(2x-1)}{\lim_{x\to 2}(5x+3)}=\frac{4-1}{10+3}=\frac{3}{13}$$

Example Evaluate

$$\lim_{x\to 0}\frac{|x|}{x}$$

Solution

To evaluate the following limit

$$\lim_{x\to 0}\frac{|x|}{x}$$

We first calculate the left hand limit

$$\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}(-1)=-1$$

In the similar way the right hand limit yield

$$\lim_{x\to 0^+}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{x}{x}=\lim_{x\to 0^-}(1)=1$$

As

$$\lim_{x\to 0^-}\frac{|x|}{x}\neq\lim_{x\to 0^+}\frac{|x|}{x}$$

So we conclude that $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Example Evaluate

$$\lim_{x\to\infty}\frac{3x+5}{6x-7}$$

Solution

$$\lim_{x\to\infty}\frac{3x+5}{6x-7}=\lim_{x\to\infty}\frac{x\left(3+\frac{5}{x}\right)}{x\left(6-\frac{7}{x}\right)}=\lim_{x\to\infty}\frac{\left(3+\frac{5}{x}\right)}{\left(6-\frac{7}{x}\right)}=\frac{\lim_{x\to\infty}\left(3+\frac{5}{x}\right)}{\lim_{x\to\infty}\left(6-\frac{7}{x}\right)}=\frac{3}{6}=\frac{1}{2}$$

Example Evaluate

$$\lim_{x\to\infty}\frac{8x^3-3x^2+2x+5}{6x^3-7x^2+6x+9}$$

Solution

$$\lim_{x\to\infty}\frac{8x^3-3x^2+2x+5}{6x^3-7x^2+6x+9}=\lim_{x\to\infty}\frac{x^3\left(8-\frac{3}{x}+\frac{2}{x^2}+\frac{5}{x^3}\right)}{x^3\left(6-\frac{7}{x}+\frac{6}{x^2}+\frac{9}{x^3}\right)}=\lim_{x\to\infty}\frac{\left(8-\frac{3}{x}+\frac{2}{x^2}+\frac{5}{x^3}\right)}{\left(6-\frac{7}{x}+\frac{6}{x^2}+\frac{9}{x^3}\right)}$$

$$=\frac{\lim_{x\to\infty}\left(8-\frac{3}{x}+\frac{2}{x^2}+\frac{5}{x^3}\right)}{\lim_{x\to\infty}\left(6-\frac{7}{x}+\frac{6}{x^2}+\frac{9}{x^3}\right)}=\frac{8-0+0+0}{6-0+0+0}=\frac{4}{3}$$

Example Evaluate

$$\lim_{x\to\infty}\frac{3x^2-2x+5}{6x^3-7x^2+6x+9}$$

Solution

$$\lim_{x\to\infty}\frac{3x^2-2x+5}{6x^3-7x^2+6x+9}=\lim_{x\to\infty}\frac{x^2\left(3-\frac{2}{x}+\frac{5}{x^2}\right)}{x^3\left(6-\frac{7}{x}+\frac{6}{x^2}+\frac{9}{x^3}\right)}=\lim_{x\to\infty}\frac{\left(3-\frac{2}{x}+\frac{5}{x^2}\right)}{\left[x\left(6-\frac{7}{x}+\frac{6}{x^2}+\frac{9}{x^3}\right)\right]}$$

$$=\lim_{x\to\infty}\frac{\left(\frac{3}{x}-\frac{2}{x^2}+\frac{5}{x^3}\right)}{\left(6-\frac{7}{x}+\frac{6}{x^2}+\frac{9}{x^3}\right)}=\frac{\lim_{x\to\infty}\left(\frac{3}{x}-\frac{2}{x^2}+\frac{5}{x^3}\right)}{\lim_{x\to\infty}\left(6-\frac{7}{x}+\frac{6}{x^2}+\frac{9}{x^3}\right)}=\frac{0-0+0}{6-0+0+0}=0$$

Example Evaluate

$$\lim_{x\to 1}f(x)$$

Where

$$f(x)=\begin{cases}2-x & x<1\\ x^2+1 & x>1\end{cases}$$

Solution

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2-x) = \lim_{x\to 1^-}(2) - \lim_{x\to 1^-} x = 2-1 = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x^2+1) = \lim_{x\to 1^+}(x^2) + \lim_{x\to 1^+}(1) = 1+1 = 2$

Since

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

Therefore, we conclude that, $\lim_{x\to 1} f(x)$ does not exist.

Example Evaluate

$$\boldsymbol{\lim_{x\to 1} f(x)}$$

Where

$$\boldsymbol{f(x)} = \begin{cases} \boldsymbol{3x-1} & \boldsymbol{x} < 1 \\ \boldsymbol{3-x} & \boldsymbol{x} > 1 \end{cases}$$

Solution

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(3x-1) = \lim_{x\to 1^-}(3x) - \lim_{x\to 1^-} 1 = 3-1 = 2$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(3-x) = \lim_{x\to 1^+}(3) - \lim_{x\to 1^+}(x) = 3-1 = 2$

Since

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

Therefore, we conclude that, $\lim_{x\to 1} f(x)$ exists and

$$\lim_{x\to 1} f(x) = 2$$