Lecture (work sheet)

5

① Evaluate the line integral $\int_C (xy+z^3)\,ds$ from $(1,0,0)$ to $(-1,0,0)$ along the helix C that is represented by the parametric equation

$x = \cos t$, $y = \sin t$, $z = t$ $(0 \le t \le \pi)$

Solution:

$$ds = \sqrt{\left(\frac{\partial x}{dt}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2}\, dt$$

$$= \sqrt{\sin^2 t + \cos^2 t + 1}\, dt$$

$$= \sqrt{1+1}\, dt$$

$$= \sqrt{2}\, dt$$

$$\int_0^{\pi} (\cos t \sin t + t^3)\sqrt{2}\, dt$$

$$= \sqrt{2}\int_0^{\pi} \cos t \sin t + t^3 \, dt$$

$$= \sqrt{2}\int_0^{\pi} \cos t \sin t \, dt + \sqrt{2}\int_0^{\pi} t^2 \, dt$$

$$= \sqrt{2}\int_0^{\pi} \cancel{z} \, dz + \sqrt{2}\int_0^{\pi} t^2 \, dt$$

$z = \sin t$

$dz = \cos t \, dt$

$$= \frac{\sqrt{2}}{2}\left[z^2\right]_0^{\pi} + \frac{\sqrt{2}}{4}\left[t^4\right]_0^{\pi}$$

$$= \sqrt{2}/2 \left(\sin^2\pi - \sin^2 0\right) + \frac{\sqrt{2}}{4} \pi^4$$

$$= \frac{\sqrt{2}}{2} \times 0 + \sqrt{2}/4 \; \pi^4$$

$$= \sqrt{2}/4 \; \pi^4 \quad \text{An.}$$

(2) Evaluate $\int_C xy\,dx + x^2 dy$ if

(a) C consists of line segments from (2,1) to (4,1) and from (4,1) to (4,5)

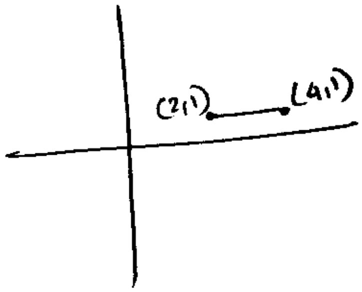
(b) C is the line segment from (2,1) and (4,5)

(c) parametric equation for C are $x = 3t - 1$

$y = 3t^2 - 2t$

$1 \le t \le 5/3$

(a)



$y = 1$

$dy = 0$

$\int_2^4 xy\,dx + x^2 dy$

$= \int_2^4 x \cdot 1\,dx + x^2 \times 0$

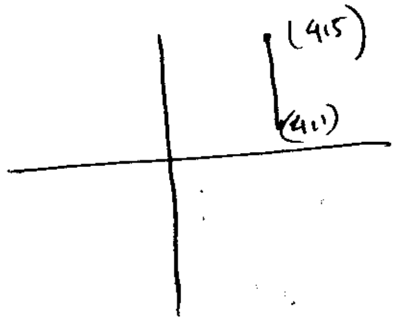
$= \int_2^4 x\,dx$

$= \frac{1}{2}\left[x^2\right]_2^4$

$= \frac{1}{2}(16 - 4)$

$= 6$

(b)



$x = 4$

$dx = 0$

$\int_1^5 xy\,dx + x^2 dy$

$= \int_1^5 4y \times 0 + 16\,dy$

$= 16\int_1^5 dy$

$= 16(5 - 1)$

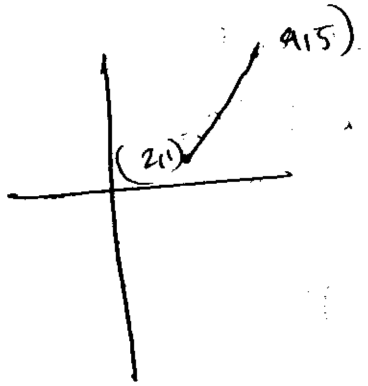
$= 16 \times 4 = 64$

Total $= 6 + 64$

$= 70$

Ans.

(6)



$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2}$$

$$\Rightarrow \frac{x-2}{2-4} = \frac{y-1}{1-5}$$

$\Rightarrow -(x-2)\times 4 = -(y-1)\cdot 2$

$\Rightarrow 4x-8 = 2y+2$

$\Rightarrow 4x-2y-8+2=0$

$\Rightarrow 4x-2y-6=0$

$\Rightarrow 2x-y-3=0$

$y=2x-3$

$dy=2dx$

$$\int_2^4 x(2x-3)\,dx + x^2\cdot 2dx$$

$$= \int_2^4 2x^2 - 3x\,dx + 2x^2\,dx$$

$$= \int_2^4 4x^2 - 3x\,dx$$

$$= \frac{4}{3}\left[x^3\right]_2^4 - \frac{3}{2}\left[x^2\right]_2^4$$

$$= \frac{4}{3}(64-8) - \frac{3}{2}(12)$$

$$= \cancel{206/3} \quad 170/3$$

(c) $x = 3t-1 \qquad y = 3t^2 - 2t \qquad 1 \le t \le 5/3$

$$\frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = 6t-2$$

$$dx = 3\,dt \qquad dy = (6t-2)\,dt$$

$$\int_1^{5/3} (3t-1)(3t^2-2t)\,3\,dt + (3t-1)^2(6t-2)\,dt$$

$$= \int_1^{5/3} 3(3t-1)(3t^2-2t) + (3t-1)^2(6t-2)\,dt$$

$$= \int_1^{5/3} (9t-3)(3t^2-2t) + (3t-1)^2\,(6t-2)\,dt$$

$$= \int_1^{5/3} 9t(3t^2-2t) - 9t^2 + 6t + (9t^2-6t+1)(6t-2)\,dt$$

$$= \int_1^{5/3} 27t^3 - 18t^2 - 9t^2 + 6 + 6t(9t^2 - 6t + 1) - 2(9t^2 - 6t \ldots \; dt$$

$$\cancel{= \int_1^{5/3} 27t^3 - 18t - 9t^2 + 6 + 54t^3 - 36t^2 + 6t - 18t^2 + 12t - 2 \; dt}$$

$\cancel{-18 + 6 + 12}$

$$\cancel{\int_1^{5/3} 84t^3 - 57t}$$

$$= \int_1^{5/3} 27t^3 - 18t^2 - 9t^2 + 6t + 54t^3 - 36t^2 + 6t - 18t^2 - 12t - 2 \; dt$$

$$= \int_1^{5/3} 27t^3 + 54t^3 - 18t^2 - 9t^2 - 36t^2 - 18t^2 + 6t + 6t - 12t - 2 \; dt$$

$$= \int_1^{5/3} \cancel{24} \; 81t^3 - 81t - 2 \; dt$$

$$= \int_1^{5/3} 27t^3 - 18t^2 - 9t^2 + 6 + 6t(9t^2 - 6t + 1) - 2(9t^2 - 6t \ldots$$

$$dt$$

$$\cancel{= \int_1^{5/3} 27t^3 - 18t^2 - 9t^2 + 6 + 54t^3 - 36t^2 + 6t - 18t^2}$$

$$\cancel{+ 12t - 2 \ dt}$$

$$\cancel{-18 + 6 + 12}$$

$$\cancel{= \int_1^{5/3} 84t^3 - 57t}$$

$$= \int_1^{5/3} 27t^3 - 18t^2 - 9t^2 + 6t + 54t^3 - 36t^2 + 6t$$

$$- 18t^2 - 12t - 2 \ dt$$

$$= \int_1^{5/3} 27t^3 + 54t^3 - 18t^2 - 9t^2 - 36t^2 - 18t^2$$

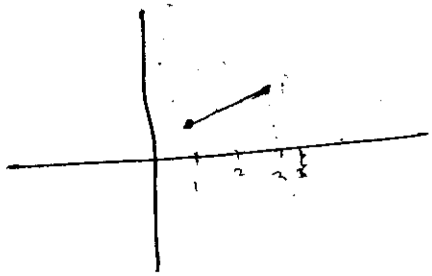
$$+ 6t + 6t - 12t - 2 \ dt$$

$$= \int_1^{5/3} \cancel{24} \ 81t^3 - 81t - 2 \ dt$$

(2) Show that

(a) $\int (6x^2y - 3xy^2)\,dy + (6xy^2 - y^3)\,dx$ is independent of the path joining the points $(1,2)$ and $(3,4)$

(b) Hence evaluate the integral



$\frac{\partial f}{\partial x} = \cancel{6x^2 - 3xy}\ 12xy - 3y^2$

$\frac{\partial g}{\partial y} = \cancel{6y^2}\ 12xy - 3y^2$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}$

$\frac{\partial \varphi}{\partial y} = 6x^2y - 3xy^2 \qquad \frac{\partial \varphi}{\partial x} = 6xy^2 - y^3$

$\varphi = \int 6x^2y - 3xy^2\,dx + K(y)$

$= \cancel{1\ 2x^3y - \frac{3}{2}x^2y^2 + K(y)}$

$\cancel{\frac{\partial \varphi}{\partial y} = 2x^3 - \frac{3}{2}x^2y + K'(y)}$

$\cancel{6xy}$

$\varphi = \int 6xy^2 - y^3\,dx + K(y)$

$= 3x^2y^2 - xy^3 + K(y)$

$\frac{\partial \varphi}{\partial y} = 6x^2y - 3xy^2 + K'(y)$

$K'(y) = 0$

$K(y) = C$

$\varphi = 3x^2y^2 - xy^3 + C$

$$\int_{(1,2)}^{(3,4)} (6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy$$

$$= \phi(3,4) - \phi(1,2) = 240 - 4 = 236$$

$$\left[\begin{array}{l} F = \nabla \phi_{(x,y)} \\ \int F \cdot d\bar{r} = \int_{(x,y)} \bar{\nabla}\phi \cdot d\bar{r} \\ = \phi(x_2, y_2) - \phi(x_1, y_1) \end{array}\right.$$

(4)

$$F(x,y) = (3x^2y + 2)\,\hat{i} + (x^3 + 4y^3)\,\hat{j}$$

$$\int F \cdot d\bar{r}$$

$$= \int \left[(3x^2y+2)\hat{i} + (x^3+4y^3)\hat{j}\right] \cdot \left[dx\,\hat{i} + dy\,\hat{j}\right]$$

$$= \int (3x^2y + 2)\,dx + (x^3 + 4y^3)\,dy$$

Let $P = 3x^2y + 2$ $\quad Q = x^3 + 4y^3$

$F(x,y)$ will be path independent if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial y} = 3x^2 \qquad \frac{\partial Q}{\partial x} = 3x^2$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$\therefore$ path independent

$$\frac{\partial \varphi}{\partial x} = 3x^2y + 2 \qquad \frac{\partial \varphi}{\partial y} = x^3 + 4y^3$$

$$\varphi = \int 3x^2y + 2 \cdot dx + K(y)$$

$$= \frac{3yx^3}{3} + 2x + K(y)$$

$$= x^3y + 2x + K(y)$$

$$\frac{\partial \varphi}{\partial y} = x^3 + K'(y)$$

$$x^3 + 4y^3 = K'(y)$$

$$K'(y) = 4y^3$$

$$K(y) = y^4 + C$$

$$\varphi = x^3y + 2x + y^4 + C \quad \text{Ans}$$

(2) Let $F(x,y) = 2xy^3\,\hat{i} + (1+3x^2y^2)\,\hat{j}$

(a) Show that F is a conservative vector field on the entire xy plane

(b) Find f by first integrating $\frac{\partial f}{\partial x}$

(c) find f by first integrating $\frac{\partial f}{\partial y}$

(a)

$$f(x,y) = 2xy^3$$
$$g(x,y) = 1+3x^2y^2$$

$$\frac{\partial f(x,y)}{\partial y} = 6xy^2$$

$$\frac{\partial g(x,y)}{\partial x} = 6xy^2$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

conservative vector field

(b) since the field is conservative there is a potential function ϕ. such that

$$\frac{\partial \phi}{\partial x} = 2xy^3, \quad \frac{\partial \phi}{\partial y} = 1+3x^2y^2$$

Integrating w.r.t x

$$\phi = \int 2xy^3 \, dx + K(y)$$

$$= 2\int xy^3 \, dx + K(y)$$

$$= \frac{2y^3}{2} x^2 + K(y)$$

$$= x^2 y^3 + K(y)$$

$$\frac{\partial \phi}{\partial y} = 3x^2y^2 + K'(y)$$

$$1 + 3x^2y^2 = 3x^2y^2 + K'(y)$$

$$K'(y) = 1$$

$$K(y) = y + C$$

$$\phi = x^2 y^3 + y + C$$

(c) Since the field is conservative there is a potential function ϕ

$$\therefore \frac{\partial \phi}{\partial x} = 2xy^3 \quad , \quad \frac{\partial \phi}{\partial y} = 1 + 3x^2y^2$$

$$\phi = \int 1 + 3x^2y^2\,dy + K(x)$$

$$= y + x^2y^3 + K(x)$$

$$\frac{d\phi}{dx} = \cancel{xy}^{\,0} + 2xy^3 + K'(x)$$

$$2xy^3 = \cancel{xy}^{\,0} + 2xy^3 + K'(x)$$

$$K'(x) = \cancel{xy}\ 0$$

$$0$$

$$K(x) = -\cancel{\frac{x^2y}{2}} + C$$

$$\phi = y + x^2y^3 + C$$

Ans.

$$\phi = \int 1 + 3x^2y^2\,dy + K(x)$$

$$= y + x^2y^3 + K(x)$$

$$\frac{d\phi}{dx} = \cancel{xy}^{\,0} + 2xy^3 + K'(x)$$

$$2xy^3 = \cancel{xy}^{\,0} + 2xy^3 + K'(x)$$

$$K'(x) = \cancel{-xy}\ 0$$

$$K(x) = \cancel{-\frac{x^2y}{2}} + C$$

$$\phi = y + x^2y^3 + C$$

Ans.

(6)
$$\int_{(1,2)}^{(3,1)} 2xy^3\,dx + (1+3x^2y^2)\,dy$$

$$f(x,y) = 2xy^3$$

$$g(x,y) = 1+3x^2y^2$$

$$\frac{\partial f}{\partial y} = 6xy^2$$

$$\frac{\partial g}{\partial x} = 6xy^2$$

so conservative

As this is conservative so there lies a function $\phi(x,y)$ such that

$$\frac{\partial \phi}{\partial x} = 2xy^3 \qquad \frac{\partial \phi}{\partial y} = 1+3x^2y^2$$

$$\phi = \int 2xy^3\,dx + K(y)$$

$$= \frac{2y^3x^2}{2} + K(y)$$

$$= x^2y^3 + K(y)$$

$$\frac{\partial \varphi}{\partial y} = 3x^2y^2 + k'(y)$$

$$1 + 3x^2y^2 = 3x^2y^2 + k'(y)$$

$$k'(y) = 1$$

$$k(y) = y + c$$

$$\therefore \varphi = x^2y^3 + y + c$$

$$\int_{(1,4)}^{(3,1)} 2xy^3\,dx + (1+3x^2y^2)\,dy$$

$$= \varphi(3,1) - \varphi(1,4)$$

$$= 9 + 1 + \cancel{c} - 64 - 4 - \cancel{c}$$

$$= 10 \cancel{+c} - 64 - 4$$

$$= -58$$