

Nondeterministic Finite Automata

Nondeterminism
Subset Construction

Nondeterminism

- ◆ A *nondeterministic finite automaton* has the ability to be in several states at once.
- ◆ Transitions from a state on an input symbol can be to any set of states.

Nondeterminism – (2)

- ◆ Start in one start state.
- ◆ Accept if any sequence of choices leads to a final state.
- ◆ **Intuitively**: the NFA always “guesses right.”

Formal NFA

- ◆ A finite set of states, typically Q .
- ◆ An input alphabet, typically Σ .
- ◆ A transition function, typically δ .
- ◆ A start state in Q , typically q_0 .
- ◆ A set of final states $F \subseteq Q$.

Transition Function of an NFA

- ◆ $\delta(q, a)$ is a set of states.
- ◆ Extend to strings as follows:
- ◆ **Basis**: $\delta(q, \epsilon) = \{q\}$
- ◆ **Induction**: $\delta(q, wa) =$ the union over all states p in $\delta(q, w)$ of $\delta(p, a)$

Language of an NFA

- ◆ A string w is accepted by an NFA if $\delta(q_0, w)$ contains at least one final state.
- ◆ The language of the NFA is the set of strings it accepts.

Example: Language of an NFA

1	2	3
4	5	6
7	8	9

- ◆ For our chessboard NFA we saw that rbb is accepted.
- ◆ If the input consists of only b's, the set of accessible states alternates between $\{5\}$ and $\{1,3,7,9\}$, so only even-length, nonempty strings of b's are accepted.
- ◆ What about strings with at least one r?

Equivalence of DFA's, NFA's

- ◆ A DFA can be turned into an NFA that accepts the same language.
- ◆ If $\delta_D(q, a) = p$, let the NFA have $\delta_N(q, a) = \{p\}$.
- ◆ Then the NFA is always in a set containing exactly one state – the state the DFA is in after reading the same input.

Equivalence – (2)

- ◆ Surprisingly, for any NFA there is a DFA that accepts the same language.
- ◆ Proof is the *subset construction*.
- ◆ The number of states of the DFA can be exponential in the number of states of the NFA.
- ◆ Thus, NFA's accept exactly the regular languages.

Subset Construction

- ◆ Given an NFA with states Q , inputs Σ , transition function δ_N , state state q_0 , and final states F , construct equivalent DFA with:
 - ◆ States 2^Q (Set of subsets of Q).
 - ◆ Inputs Σ .
 - ◆ Start state $\{q_0\}$.
 - ◆ Final states = all those with a member of F .

Critical Point

- ◆ The DFA states have *names* that are sets of NFA states.
- ◆ But as a DFA state, an expression like $\{p,q\}$ must be read as a single symbol, not as a set.
- ◆ **Analogy**: a class of objects whose values are sets of objects of another class.

Subset Construction – (2)

- ◆ The transition function δ_D is defined by:
 $\delta_D(\{q_1, \dots, q_k\}, a)$ is the union over all $i = 1, \dots, k$ of $\delta_N(q_i, a)$.
- ◆ **Example:** We'll construct the DFA equivalent of our “chessboard” NFA.

Example: Subset Construction

	r	b
→ 1	2,4	5
2	4,6	1,3,5
3	2,6	5
4	2,8	1,5,7
5	2,4,6,8	1,3,7,9
6	2,8	3,5,9
7	4,8	5
8	4,6	5,7,9
* 9	6,8	5

	r	b
→ {1}	{2,4}	{5}
{2,4}		
{5}		

Alert: What we're doing here is the *lazy* form of DFA construction, where we only construct a state if we are forced to.

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{5}		
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* {1,3,5,7,9}	{2,4,6,8}	{1,3,5,7,9}

Proof of Equivalence: Subset Construction

◆ The proof is almost a pun.

◆ Show by induction on $|w|$ that

$$\delta_N(q_0, w) = \delta_D(\{q_0\}, w)$$

◆ **Basis:** $w = \epsilon$: $\delta_N(q_0, \epsilon) = \delta_D(\{q_0\}, \epsilon) = \{q_0\}$.

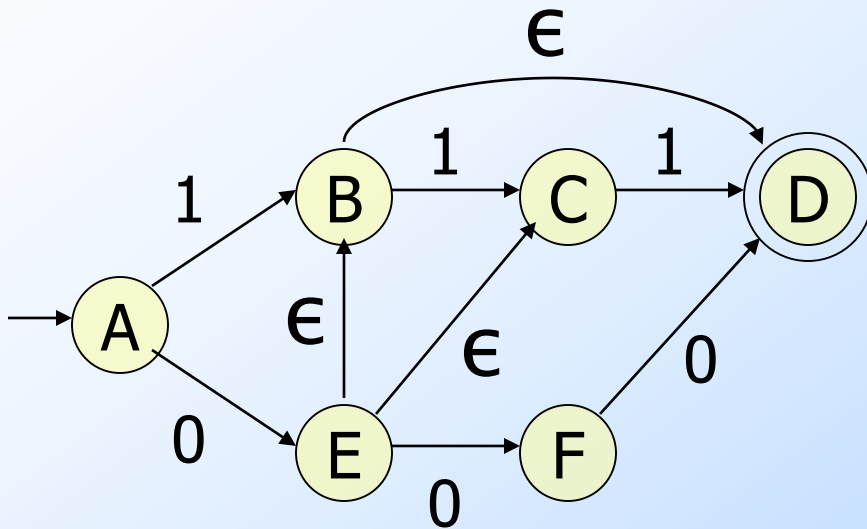
Induction

- ◆ Assume IH for strings shorter than w .
- ◆ Let $w = xa$; IH holds for x .
- ◆ Let $\delta_N(q_0, x) = \delta_D(\{q_0\}, x) = S$.
- ◆ Let $T =$ the union over all states p in S of $\delta_N(p, a)$.
- ◆ Then $\delta_N(q_0, w) = \delta_D(\{q_0\}, w) = T$.
 - ◆ For NFA: the extension of δ_N .
 - ◆ For DFA: definition of δ_D plus extension of δ_D .
 - That is, $\delta_D(S, a) = T$; then extend δ_D to $w = xa$.

NFA's With ϵ -Transitions

- ◆ We can allow state-to-state transitions on ϵ input.
- ◆ These transitions are done spontaneously, without looking at the input string.
- ◆ A convenience at times, but still only regular languages are accepted.

Example: ϵ -NFA

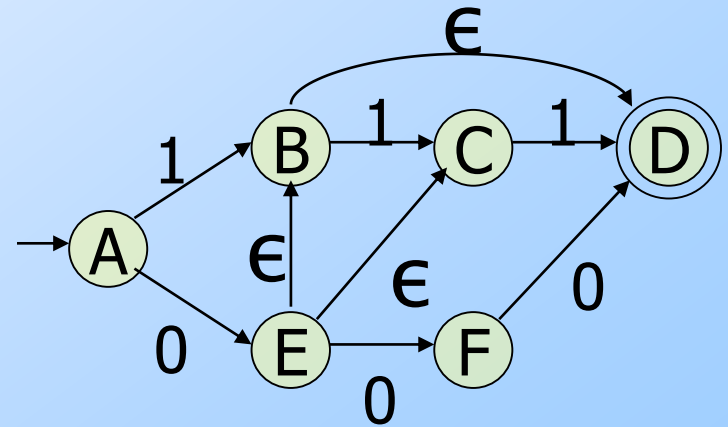


	0	1	ϵ
\rightarrow A	{E}	{B}	\emptyset
B	\emptyset	{C}	{D}
C	\emptyset	{D}	\emptyset
* D	\emptyset	\emptyset	\emptyset
E	{F}	\emptyset	{B, C}
F	{D}	\emptyset	\emptyset

Closure of States

- ◆ $CL(q)$ = set of states you can reach from state q following only arcs labeled ϵ .

- ◆ **Example:** $CL(A) = \{A\}$;
 $CL(E) = \{B, C, D, E\}$.

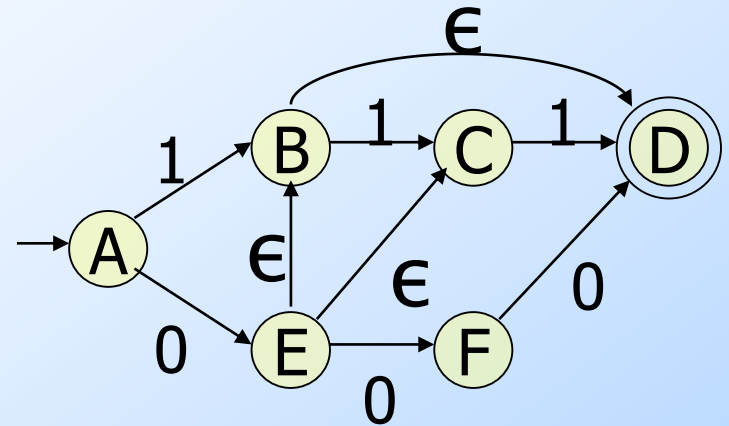


- ◆ Closure of a set of states = union of the closure of each state.

Extended Delta

- ◆ **Basis:** $\delta(q, \epsilon) = CL(q)$.
 - ◆ **Induction:** $\delta(q, xa)$ is computed as follows:
 1. Start with $\delta(q, x) = S$.
 2. Take the union of $CL(\delta(p, a))$ for all p in S .
 - ◆ **Intuition:** $\delta(q, w)$ is the set of states you can reach from q following a path labeled w .
- And notice that $\delta(q, a)$ is *not* that set of states, for symbol a .

Example: Extended Delta



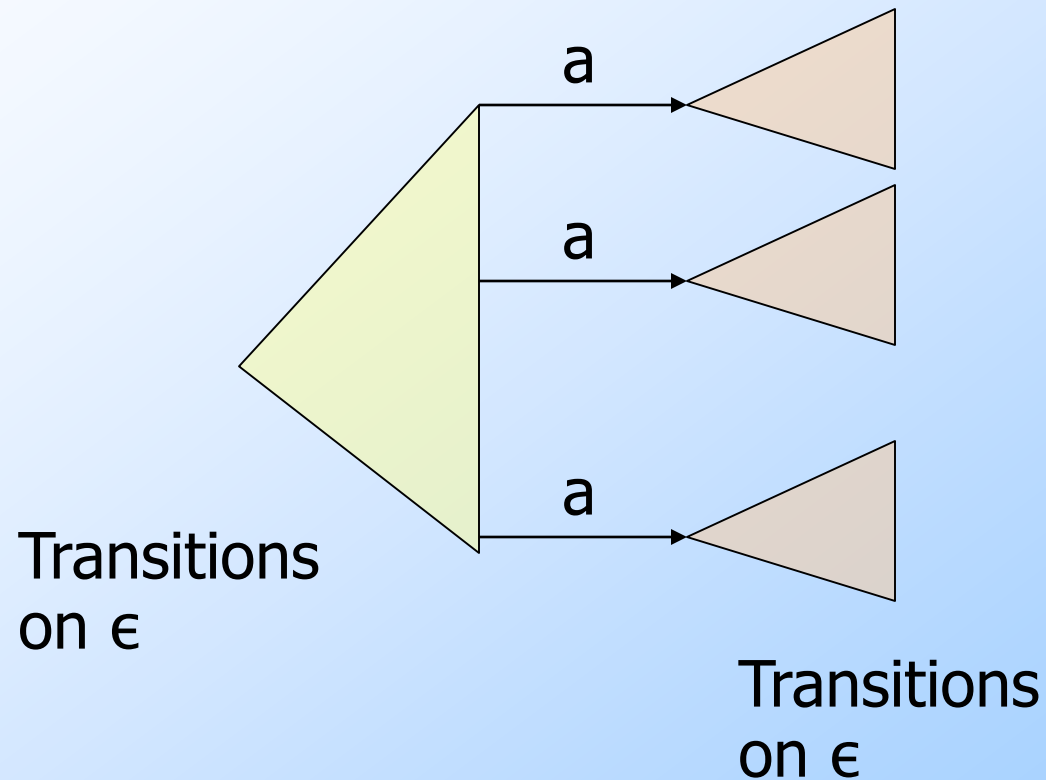
- ◆ $\delta(A, \epsilon) = CL(A) = \{A\}$.
- ◆ $\delta(A, 0) = CL(\{E\}) = \{B, C, D, E\}$.
- ◆ $\delta(A, 01) = CL(\{C, D\}) = \{C, D\}$.
- ◆ *Language* of an ϵ -NFA is the set of strings w such that $\delta(q_0, w)$ contains a final state.

Equivalence of NFA, ϵ -NFA

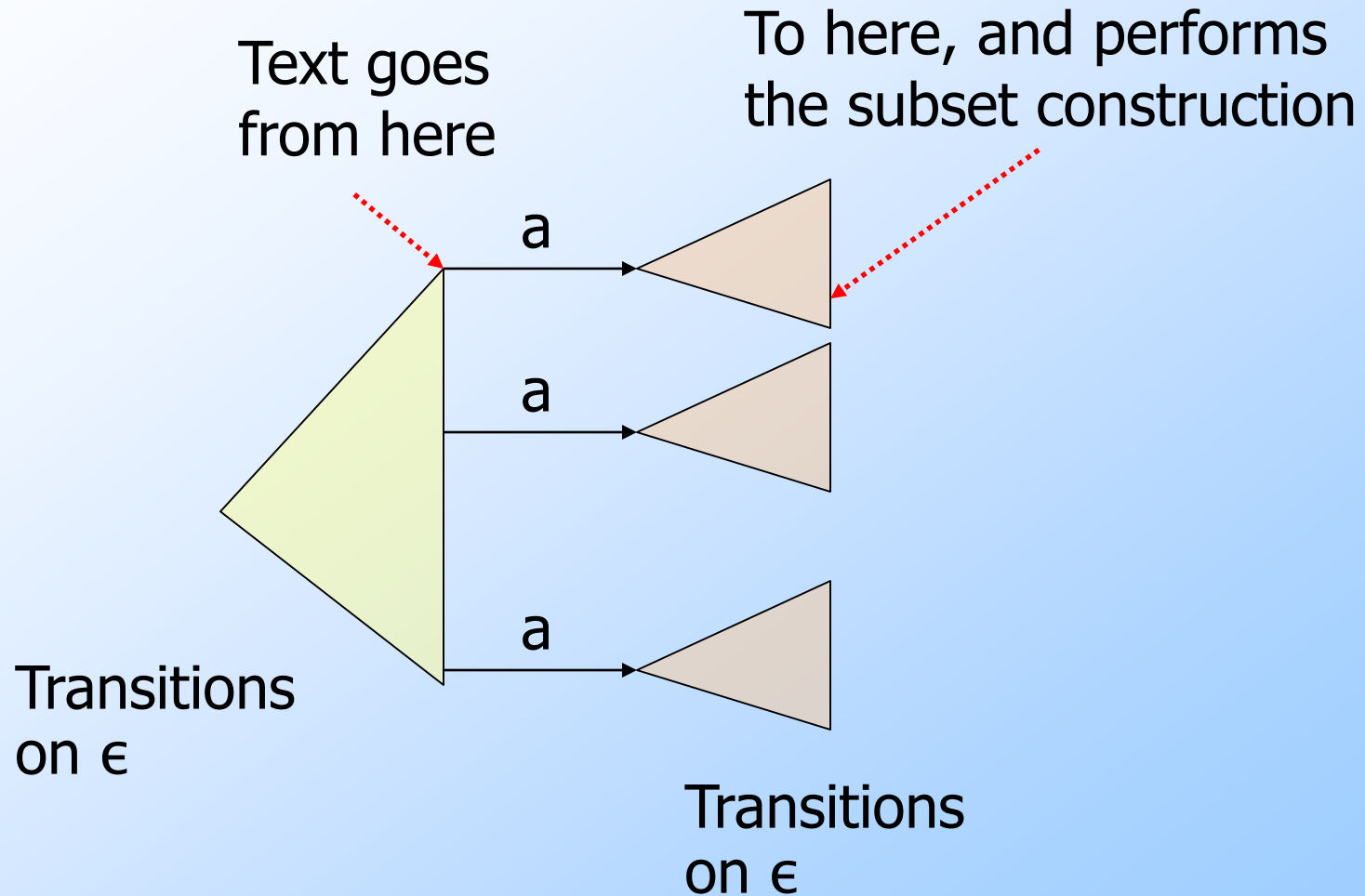
- ◆ Every NFA **is** an ϵ -NFA.
 - ◆ It just has no transitions on ϵ .
- ◆ Converse requires us to take an ϵ -NFA and construct an NFA that accepts the same language.
- ◆ We do so by combining ϵ -transitions with the next transition on a real input.

Warning: This treatment is a bit different from that in the text.

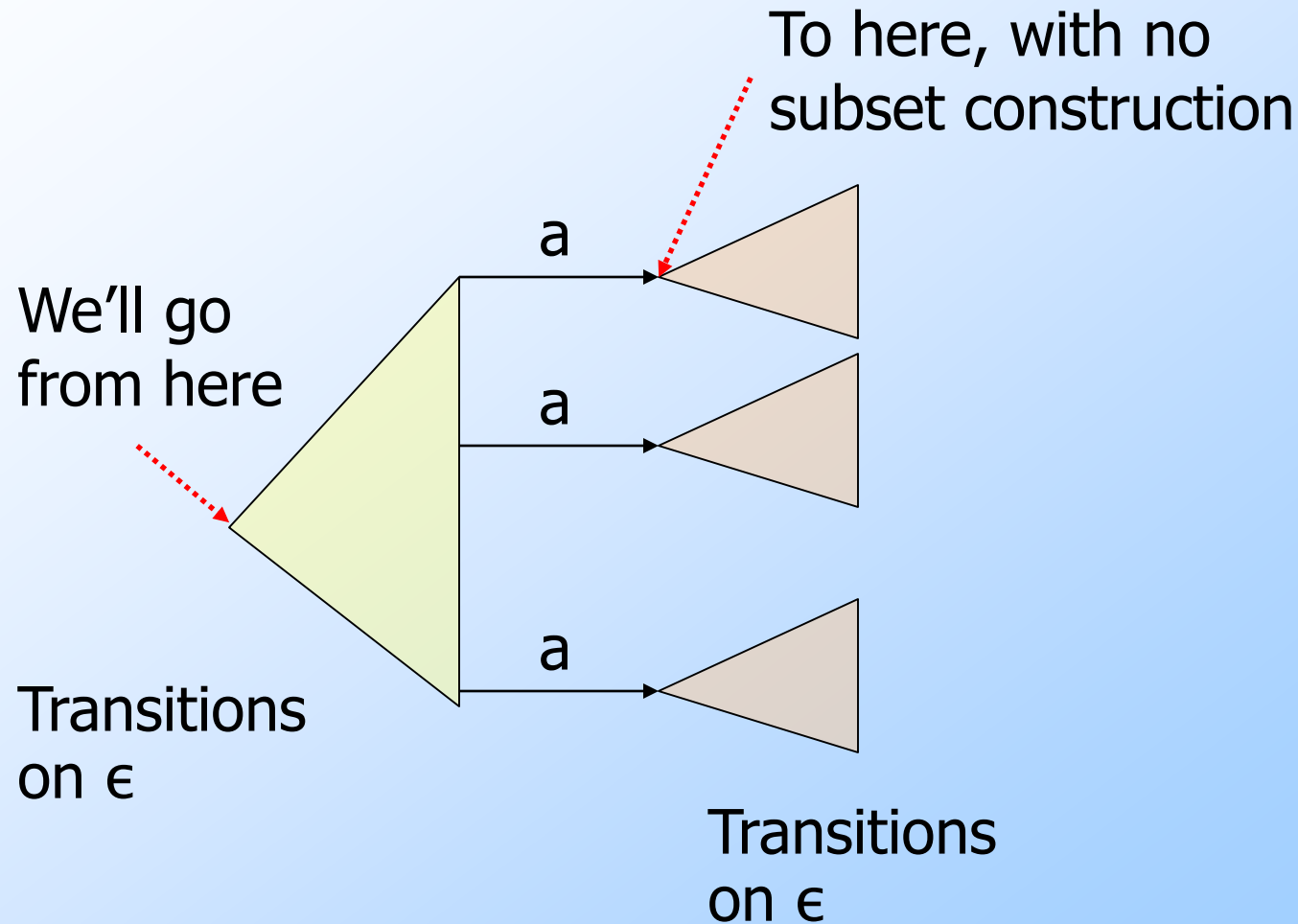
Picture of ϵ -Transition Removal



Picture of ϵ -Transition Removal



Picture of ϵ -Transition Removal



Equivalence – (2)

- ◆ Start with an ϵ -NFA with states Q , inputs Σ , start state q_0 , final states F , and transition function δ_E .
- ◆ Construct an “ordinary” NFA with states Q , inputs Σ , start state q_0 , final states F' , and transition function δ_N .

Equivalence – (3)

- ◆ Compute $\delta_N(q, a)$ as follows:
 1. Let $S = CL(q)$.
 2. $\delta_N(q, a)$ is the union over all p in S of $\delta_E(p, a)$.
- ◆ $F' =$ the set of states q such that $CL(q)$ contains a state of F .
- ◆ **Intuition:** δ_N incorporates ϵ -transitions before using a but not after.

Equivalence – (4)

◆ Prove by induction on $|w|$ that

$$\text{CL}(\delta_N(q_0, w)) = \delta_E(q_0, w).$$

◆ Thus, the ϵ -NFA accepts w if and only if the “ordinary” NFA does.

Interesting
closures: $CL(B)$
 $= \{B, D\}$; $CL(E)$
 $= \{B, C, D, E\}$

Example: ϵ -NFA- to-NFA

	0	1	ϵ
\rightarrow A	{E}	{B}	\emptyset
B	\emptyset	{C}	{D}
C	\emptyset	{D}	\emptyset
* D	\emptyset	\emptyset	\emptyset
E	{F}	\emptyset	{B, C}
F	{D}	\emptyset	\emptyset

ϵ -NFA

Since closures of
B and E include
final state D.

	0	1
\rightarrow A	{E}	{B}
B	\emptyset	{C}
C	\emptyset	{D}
* D	\emptyset	\emptyset
E	{F}	{C, D}
F	{D}	\emptyset

Since closure of
E includes B and
C; which have
transitions on 1
to C and D.

Summary

- ◆ DFA's, NFA's, and ϵ -NFA's all accept exactly the same set of languages: the regular languages.
- ◆ The NFA types are easier to design and may have exponentially fewer states than a DFA.
- ◆ But only a DFA can be implemented!