39. Show that the volume of the solid bounded by the coordinate planes and a plane tangent to the portion of the surface $xyz = k$, $k > 0$, in the first octant does not depend on the point of tangency.

40. Writing Discuss the role of the chain rule in defining a tangent plane to a level surface.

41. Writing Discuss the relationship between tangent planes and local linear approximations for functions of two variables.

✔ QUICK CHECK ANSWERS 13.7

1. $2(x-1) + y + (z+1) = 0;\ x = 1 + 2t;\ y = t;\ z = -1 + t$
2. $z = 4 + 2(x-3) - 3(y-1);\ x = 3 + 2t;\ y = 1 - 3t;\ z = 4 - t$
3. $z = 8 + 8(x-2) + (y-4);\ x = 2 + 8t;\ y = 4 + t;\ z = 8 - t$ **4.** $x = 2 + t;\ y = 1;\ z = 2 - t$

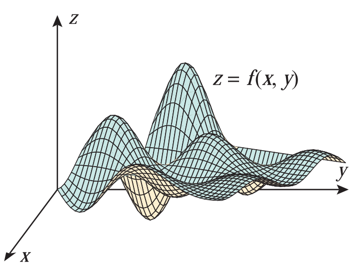
13.8 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Earlier in this text we learned how to find maximum and minimum values of a function of one variable. In this section we will develop similar techniques for functions of two variables.

EXTREMA

If we imagine the graph of a function f of two variables to be a mountain range (Figure 13.8.1), then the mountaintops, which are the high points in their immediate vicinity, are called *relative maxima* of f, and the valley bottoms, which are the low points in their immediate vicinity, are called *relative minima* of f.

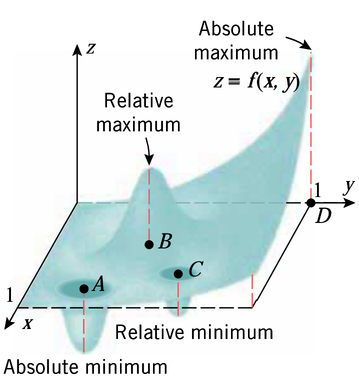
Just as a geologist might be interested in finding the highest mountain and deepest valley in an entire mountain range, so a mathematician might be interested in finding the largest and smallest values of $f(x, y)$ over the *entire* domain of f. These are called the *absolute maximum* and *absolute minimum values* of f. The following definitions make these informal ideas precise.



▲ **Figure 13.8.1**

13.8.1 DEFINITION A function f of two variables is said to have a ***relative maximum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute maximum*** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.

13.8.2 DEFINITION A function f of two variables is said to have a ***relative minimum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute minimum*** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.



▲ **Figure 13.8.2**

If f has a relative maximum or a relative minimum at (x_0, y_0), then we say that f has a ***relative extremum*** at (x_0, y_0), and if f has an absolute maximum or absolute minimum at (x_0, y_0), then we say that f has an ***absolute extremum*** at (x_0, y_0).

Figure 13.8.2 shows the graph of a function f whose domain is the square region in the xy-plane whose points satisfy the inequalities $0 \leq x \leq 1$, $0 \leq y \leq 1$. The function f has

FINDING RELATIVE EXTREMA

Recall that if a function g of one variable has a relative extremum at a point x_0 where g is differentiable, then $g'(x_0) = 0$. To obtain the analog of this result for functions of two variables, suppose that $f(x, y)$ has a relative maximum at a point (x_0, y_0) and that the partial derivatives of f exist at (x_0, y_0). It seems plausible geometrically that the traces of the surface $z = f(x, y)$ on the planes $x = x_0$ and $y = y_0$ have horizontal tangent lines at (x_0, y_0) (Figure 13.8.4), so

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$$

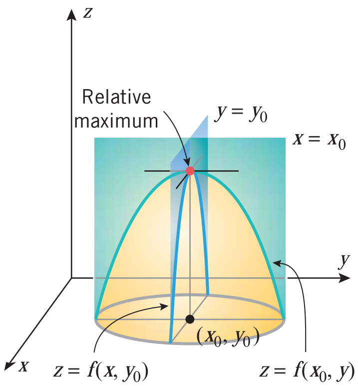


Figure 13.8.4

The same conclusion holds if f has a relative minimum at (x_0, y_0), all of which suggests the following result, which we state without formal proof.

13.8.4 THEOREM *If f has a relative extremum at a point (x_0, y_0), and if the first-order partial derivatives of f exist at this point, then*

$$f_x(x_0, y_0) = 0 \quad \textit{and} \quad f_y(x_0, y_0) = 0$$

Recall that the *critical points* of a function f of one variable are those values of x in the domain of f at which $f'(x) = 0$ or f is not differentiable. The following definition is the analog for functions of two variables.

Explain why

$$D_{\mathbf{u}} f(x_0, y_0) = 0$$

for all $\mathbf{u}$ if (x_0, y_0) is a critical point of f and f is differentiable at (x_0, y_0).

13.8.5 DEFINITION A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a ***critical point*** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

It follows from this definition and Theorem 13.8.4 that relative extrema occur at critical points, just as for a function of one variable. However, recall that for a function of one variable a relative extremum need not occur at *every* critical point. For example, the function might have an inflection point with a horizontal tangent line at the critical point (see Figure 4.2.6). Similarly, a function of two variables need not have a relative extremum at every critical point. For example, consider the function

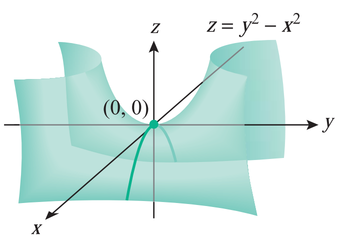
$$f(x, y) = y^2 - x^2$$

This function, whose graph is the hyperbolic paraboloid shown in Figure 13.8.5, has a critical point at $(0, 0)$, since

$$f_x(x, y) = -2x \quad \text{and} \quad f_y(x, y) = 2y$$

from which it follows that

$$f_x(0, 0) = 0 \quad \text{and} \quad f_y(0, 0) = 0$$

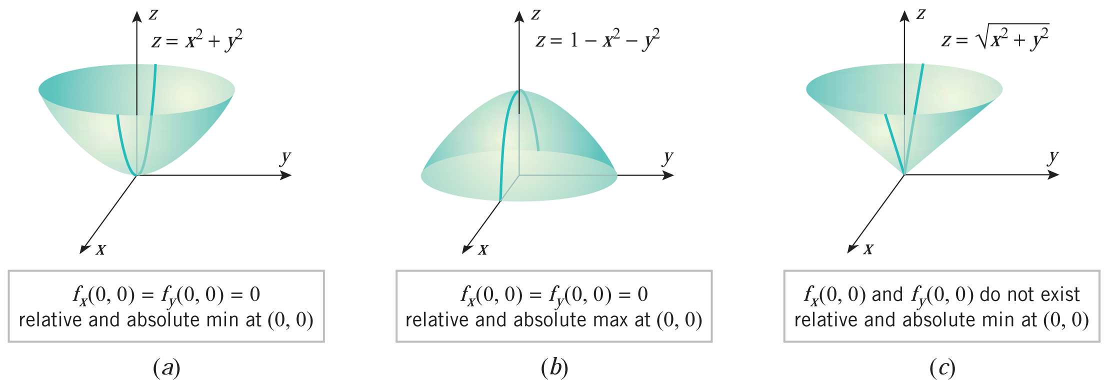


The function $f(x, y) = y^2 - x^2$ has neither a relative maximum nor a relative minimum at the critical point $(0, 0)$.

Figure 13.8.5

However, the function f has neither a relative maximum nor a relative minimum at $(0, 0)$. For obvious reasons, the point $(0, 0)$ is called a *saddle point* of f. In general, we will say that a surface $z = f(x, y)$ has a ***saddle point*** at (x_0, y_0) if there are two distinct vertical planes through this point such that the trace of the surface in one of the planes has a relative maximum at (x_0, y_0) and the trace in the other has a relative minimum at (x_0, y_0).

Example 2 The three functions graphed in Figure 13.8.6 all have critical points at $(0, 0)$. For the paraboloids, the partial derivatives at the origin are zero. You can check this algebraically by evaluating the partial derivatives at $(0, 0)$, but you can see it geometrically by observing that the traces in the xz-plane and yz-plane have horizontal tangent lines at $(0, 0)$. For the cone neither partial derivative exists at the origin because the traces in the xz-plane and the yz-plane have corners there. The paraboloid in part (a) and the cone in part (c) have a relative minimum and absolute minimum at the origin, and the paraboloid in part (b) has a relative maximum and an absolute maximum at the origin. ◄



▲ **Figure 13.8.6**

THE SECOND PARTIALS TEST

For functions of one variable the second derivative test (Theorem 4.2.4) was used to determine the behavior of a function at a critical point. The following theorem, which is usually proved in advanced calculus, is the analog of that theorem for functions of two variables.

13.8.6 THEOREM (*The Second Partials Test*) *Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point (x_0, y_0), and let*

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

(a) *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).*

(b) *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).*

(c) *If $D < 0$, then f has a saddle point at (x_0, y_0).*

(d) *If $D = 0$, then no conclusion can be drawn.*

With the notation of Theorem 13.8.6, show that if $D > 0$, then $f_{xx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$ have the same sign. Thus, we can replace $f_{xx}(x_0, y_0)$ by $f_{yy}(x_0, y_0)$ in parts (*a*) and (*b*) of the theorem.

► **Example 3** Locate all relative extrema and saddle points of

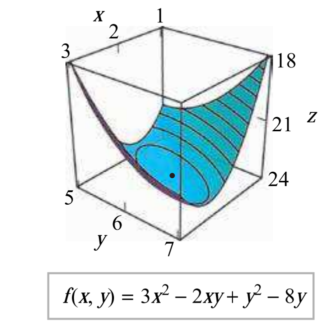
$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

Solution. Since $f_x(x, y) = 6x - 2y$ and $f_y(x, y) = -2x + 2y - 8$, the critical points of f satisfy the equations

$$6x - 2y = 0$$
$$-2x + 2y - 8 = 0$$

Solving these for x and y yields $x = 2$, $y = 6$ (verify), so $(2, 6)$ is the only critical point. To apply Theorem 13.8.6 we need the second-order partial derivatives

$$f_{xx}(x, y) = 6, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -2$$



▲ **Figure 13.8.7**

At the point (2, 6) we have

$$D = f_{xx}(2,6)\,f_{yy}(2,6) - f_{xy}^2(2,6) = (6)(2) - (-2)^2 = 8 > 0$$

and

$$f_{xx}(2,6) = 6 > 0$$

so f has a relative minimum at (2, 6) by part (a) of the second partials test. Figure 13.8.7 shows a graph of f in the vicinity of the relative minimum. ◀

▶ **Example 4** Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

Solution. Since

$$\begin{aligned} f_x(x, y) &= 4y - 4x^3 \\ f_y(x, y) &= 4x - 4y^3 \end{aligned} \tag{1}$$

the critical points of f have coordinates satisfying the equations

$$\begin{aligned} 4y - 4x^3 &= 0 \\ 4x - 4y^3 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} y &= x^3 \\ x &= y^3 \end{aligned} \tag{2}$$

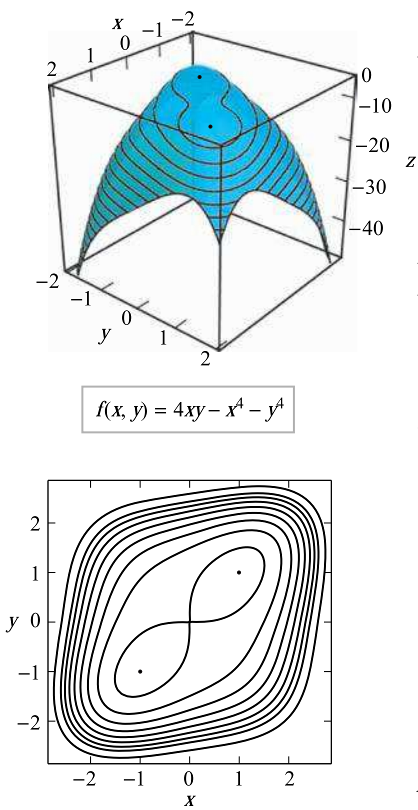
Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0$, $x = 1$, $x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0$, $y = 1$, $y = -1$. Thus, the critical points of f are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

From (1),

$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
(0, 0)	0	0	4	−16
(1, 1)	−12	−12	4	128
(−1, −1)	−12	−12	4	128



▲ **Figure 13.8.8**

At the points $(1, 1)$ and $(-1, -1)$, we have $D > 0$ and $f_{xx} < 0$, so relative maxima occur at these critical points. At $(0, 0)$ there is a saddle point since $D < 0$. The surface and a contour plot are shown in Figure 13.8.8. ◀

The "figure eight" pattern at $(0, 0)$ in the contour plot for the surface in Figure 13.8.8 is typical for level curves that pass through a saddle point. If a bug starts at the point $(0, 0, 0)$ on the surface, in how many directions can it walk and remain in the xy-plane?

The following theorem, which is the analog for functions of two variables of Theorem 4.4.3, will lead to an important method for finding absolute extrema.

13.8.7 THEOREM *If a function f of two variables has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.*

P (KILOPASCALS)	134	142	155	160	171	184
T (°CELSIUS)	0	20	40	60	80	100

(a) Use a calculating utility to find the regression line of P as a function of T.
(b) Use a graphing utility to make a graph that shows the data points and the regression line.
(c) Use the regression line to estimate the value of absolute zero in degrees Celsius.

58. Find
(a) a continuous function $f(x, y)$ that is defined on the entire xy-plane and has no absolute extrema on the xy-plane;
(b) a function $f(x, y)$ that is defined everywhere on the rectangle $0 \le x \le 1, 0 \le y \le 1$ and has no absolute extrema on the rectangle.

59. Show that if f has a relative maximum at (x_0, y_0), then $G(x) = f(x, y_0)$ has a relative maximum at $x = x_0$ and $H(y) = f(x_0, y)$ has a relative maximum at $y = y_0$.

60. **Writing** Explain how to determine the location of relative extrema or saddle points of $f(x, y)$ by examining the contours of f.

61. **Writing** Suppose that the second partials test gives no information about a certain critical point (x_0, y_0) because $D(x_0, y_0) = 0$. Discuss what other steps you might take to determine whether there is a relative extremum at that critical point.

✔ QUICK CHECK ANSWERS 13.8

1. $(0, 0)$ and $\left(\frac{1}{6}, -\frac{1}{12}\right)$ **2.** (a) no information (b) a saddle point at $(0, 0)$ (c) a relative minimum at $(0, 0)$ (d) a relative maximum at $(0, 0)$ **3.** (a) a saddle point at $(0, 0)$ (b) no information, since $(-1, -1)$ is not a critical point (c) a relative minimum at $(1, 1)$ **4.** $V = \dfrac{xy(1 - xy)}{x + y}$

13.9 LAGRANGE MULTIPLIERS

In this section we will study a powerful new method for maximizing or minimizing a function subject to constraints on the variables. This method will help us to solve certain optimization problems that are difficult or impossible to solve using the methods studied in the last section.

EXTREMUM PROBLEMS WITH CONSTRAINTS

In Example 6 of the last section, we solved the problem of minimizing

$$S = xy + 2xz + 2yz \tag{1}$$

subject to the constraint

$$xyz - 32 = 0 \tag{2}$$

This is a special case of the following general problem:

13.9.1 ***Three-Variable Extremum Problem with One Constraint***
Maximize or minimize the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

We will also be interested in the following two-variable version of this problem:

13.9.2 ***Two-Variable Extremum Problem with One Constraint***
Maximize or minimize the function $f(x, y)$ subject to the constraint $g(x, y) = 0$.

LAGRANGE MULTIPLIERS

One way to attack problems of these types is to solve the constraint equation for one of the variables in terms of the others and substitute the result into f. This produces a new function of one or two variables that incorporates the constraint and can be maximized or minimized by applying standard methods. For example, to solve the problem in Example 6 of the last section we substituted (2) into (1) to obtain

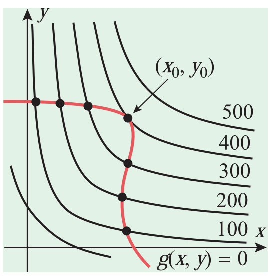
$$S = xy + \frac{64}{y} + \frac{64}{x}$$

which we then minimized by finding the critical points and applying the second partials test. However, this approach hinges on our ability to solve the constraint equation for one of the variables in terms of the others. If this cannot be done, then other methods must be used. One such method, called the *method of Lagrange multipliers*, will be discussed in this section.

To motivate the method of Lagrange multipliers, suppose that we are trying to maximize a function $f(x, y)$ subject to the constraint $g(x, y) = 0$. Geometrically, this means that we are looking for a point (x_0, y_0) on the graph of the constraint curve at which $f(x, y)$ is as large as possible. To help locate such a point, let us construct a contour plot of $f(x, y)$ in the same coordinate system as the graph of $g(x, y) = 0$. For example, Figure 13.9.1*a* shows some typical level curves of $f(x, y) = c$, which we have labeled $c = 100, 200, 300, 400$, and 500 for purposes of illustration. In this figure, each point of intersection of $g(x, y) = 0$ with a level curve is a candidate for a solution, since these points lie on the constraint curve. Among the seven such intersections shown in the figure, the maximum value of $f(x, y)$ occurs at the intersection (x_0, y_0) where $f(x, y)$ has a value of 400. Note that at (x_0, y_0) the constraint curve and the level curve just touch and thus have a *common* tangent line at this point. Since $\nabla f(x_0, y_0)$ is normal to the level curve $f(x, y) = 400$ at (x_0, y_0), and since $\nabla g(x_0, y_0)$ is normal to the constraint curve $g(x, y) = 0$ at (x_0, y_0), we conclude that the vectors $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ must be parallel. That is,

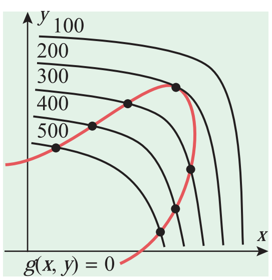
$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \tag{3}$$

for some scalar λ. The same condition holds at points on the constraint curve where $f(x, y)$ has a minimum. For example, if the level curves are as shown in Figure 13.9.1*b*, then the minimum value of $f(x, y)$ occurs where the constraint curve just touches a level curve.



Maximum of $f(x, y)$ is 400

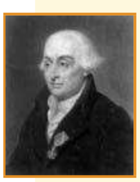
(*a*)



Minimum of $f(x, y)$ is 200

(*b*)

▲ **Figure 13.9.1**



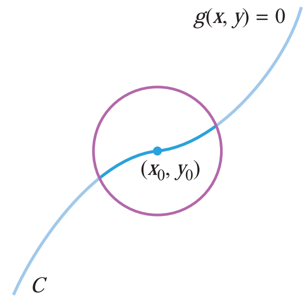
Joseph Louis Lagrange (1736–1813) French–Italian mathematician and astronomer. Lagrange, the son of a public official, was born in Turin, Italy. (Baptismal records list his name as Giuseppe Lodovico Lagrangia.) Although his father wanted him to be a lawyer, Lagrange was attracted to mathematics and astronomy after reading a memoir by the astronomer Halley. At age 16 he began to study mathematics on his own and by age 19 was appointed to a professorship at the Royal Artillery School in Turin. The following year Lagrange sent Euler solutions to some famous problems using new methods that eventually blossomed into a branch of mathematics called calculus of variations. These methods and Lagrange's applications of them to problems in celestial mechanics were so monumental that by age 25 he was regarded by many of his contemporaries as the greatest living mathematician.

In 1776, on the recommendations of Euler, he was chosen to succeed Euler as the director of the Berlin Academy. During his stay in Berlin, Lagrange distinguished himself not only in celestial mechanics, but also in algebraic equations and the theory of numbers. After twenty years in Berlin, he moved to Paris at the invitation of Louis XVI. He was given apartments in the Louvre and treated with great honor, even during the revolution.

Napoleon was a great admirer of Lagrange and showered him with honors—count, senator, and Legion of Honor. The years Lagrange spent in Paris were devoted primarily to didactic treatises summarizing his mathematical conceptions. One of Lagrange's most famous works is a memoir, *Mécanique Analytique*, in which he reduced the theory of mechanics to a few general formulas from which all other necessary equations could be derived.

It is an interesting historical fact that Lagrange's father speculated unsuccessfully in several financial ventures, so his family was forced to live quite modestly. Lagrange himself stated that if his family had money, he would not have made mathematics his vocation. In spite of his fame, Lagrange was always a shy and modest man. On his death, he was buried with honor in the Pantheon.

[*Image: http://commons.wikimedia.org/wiki/File:Joseph_Louis_Lagrange.jpg*]



A constrained relative maximum occurs at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ on some segment of C that extends on both sides of (x_0, y_0).

Figure 13.9.2

Thus, to find the maximum or minimum of $f(x, y)$ subject to the constraint $g(x, y) = 0$, we look for points at which (3) holds—this is the method of Lagrange multipliers.

Our next objective in this section is to make the preceding intuitive argument more precise. For this purpose it will help to begin with some terminology about the problem of maximizing or minimizing a function $f(x, y)$ subject to a constraint $g(x, y) = 0$. As with other kinds of maximization and minimization problems, we need to distinguish between relative and absolute extrema. We will say that f has a ***constrained absolute maximum (minimum)*** at (x_0, y_0) if $f(x_0, y_0)$ is the largest (smallest) value of f on the constraint curve, and we will say that f has a ***constrained relative maximum (minimum)*** at (x_0, y_0) if $f(x_0, y_0)$ is the largest (smallest) value of f on some segment of the constraint curve that extends on both sides of the point (x_0, y_0) (Figure 13.9.2).

Let us assume that a constrained relative maximum or minimum occurs at the point (x_0, y_0), and for simplicity let us further assume that the equation $g(x, y) = 0$ can be smoothly parametrized as

$$x = x(s), \quad y = y(s)$$

where s is an arc length parameter with reference point (x_0, y_0) at $s = 0$. Thus, the quantity

$$z = f(x(s), y(s))$$

has a relative maximum or minimum at $s = 0$, and this implies that $dz/ds = 0$ at that point. From the chain rule, this equation can be expressed as

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right) = 0$$

where the derivatives are all evaluated at $s = 0$. However, the first factor in the dot product is the gradient of f, and the second factor is the unit tangent vector to the constraint curve. Since the point (x_0, y_0) corresponds to $s = 0$, it follows from this equation that

$$\nabla f(x_0, y_0) \cdot \mathbf{T}(0) = 0$$

which implies that the gradient is either $\mathbf{0}$ or is normal to the constraint curve at a constrained relative extremum. However, the constraint curve $g(x, y) = 0$ is a level curve for the function $g(x, y)$, so that if $\nabla g(x_0, y_0) \neq \mathbf{0}$, then $\nabla g(x_0, y_0)$ is normal to this curve at (x_0, y_0). It then follows that there is some scalar λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \tag{4}$$

This scalar is called a ***Lagrange multiplier***. Thus, the ***method of Lagrange multipliers*** for finding constrained relative extrema is to look for points on the constraint curve $g(x, y) = 0$ at which Equation (4) is satisfied for some scalar λ.

13.9.3 THEOREM (***Constrained-Extremum Principle for Two Variables and One Constraint***) *Let f and g be functions of two variables with continuous first partial derivatives on some open set containing the constraint curve $g(x, y) = 0$, and assume that $\nabla g \neq \mathbf{0}$ at any point on this curve. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0) on the constraint curve at which the gradient vectors $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel; that is, there is some number λ such that*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

▶ **Example 1** At what point or points on the circle $x^2 + y^2 = 1$ does $f(x, y) = xy$ have an absolute maximum, and what is that maximum?

Solution. The circle $x^2 + y^2 = 1$ is a closed and bounded set and $f(x, y) = xy$ is a continuous function, so it follows from the Extreme-Value Theorem (Theorem 13.8.3) that f has an absolute maximum and an absolute minimum on the circle. To find these extrema, we will use Lagrange multipliers to find the constrained relative extrema, and then we will evaluate f at those relative extrema to find the absolute extrema.

We want to maximize $f(x, y) = xy$ subject to the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0 \tag{5}$$

First we will look for constrained *relative* extrema. For this purpose we will need the gradients

$$\nabla f = y\mathbf{i} + x\mathbf{j} \quad \text{and} \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$$

From the formula for ∇g we see that $\nabla g = \mathbf{0}$ if and only if $x = 0$ and $y = 0$, so $\nabla g \neq \mathbf{0}$ at any point on the circle $x^2 + y^2 = 1$. Thus, at a constrained relative extremum we must have

$$\nabla f = \lambda \nabla g \quad \text{or} \quad y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j})$$

which is equivalent to the pair of equations

$$y = 2x\lambda \quad \text{and} \quad x = 2y\lambda$$

It follows from these equations that if $x = 0$, then $y = 0$, and if $y = 0$, then $x = 0$. In either case we have $x^2 + y^2 = 0$, so the constraint equation $x^2 + y^2 = 1$ is not satisfied. Thus, we can assume that x and y are nonzero, and we can rewrite the equations as

$$\lambda = \frac{y}{2x} \quad \text{and} \quad \lambda = \frac{x}{2y}$$

from which we obtain

$$\frac{y}{2x} = \frac{x}{2y}$$

or

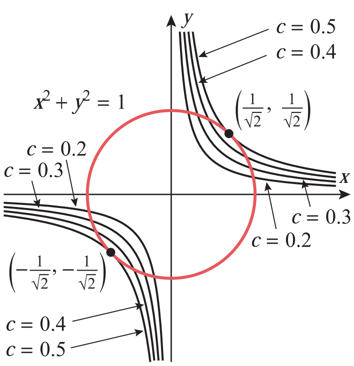
$$y^2 = x^2 \tag{6}$$

Substituting this in (5) yields

$$2x^2 - 1 = 0$$

from which we obtain $x = \pm 1/\sqrt{2}$. Each of these values, when substituted in Equation (6), produces y-values of $y = \pm 1/\sqrt{2}$. Thus, constrained relative extrema occur at the points $(1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$, and $(-1/\sqrt{2}, -1/\sqrt{2})$. The values of xy at these points are as follows:

(x, y)	$(1/\sqrt{2}, 1/\sqrt{2})$	$(1/\sqrt{2}, -1/\sqrt{2})$	$(-1/\sqrt{2}, 1/\sqrt{2})$	$(-1/\sqrt{2}, -1/\sqrt{2})$
xy	$1/2$	$-1/2$	$-1/2$	$1/2$



▲ **Figure 13.9.3**

Give another solution to Example 1 using the parametrization

$$x = \cos\theta, \quad y = \sin\theta$$

and the identity

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Thus, the function $f(x, y) = xy$ has an absolute maximum of $\frac{1}{2}$ occurring at the two points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. Although it was not asked for, we can also see that f has an absolute minimum of $-\frac{1}{2}$ occurring at the points $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$. Figure 13.9.3 shows some level curves $xy = c$ and the constraint curve in the vicinity of the maxima. A similar figure for the minima can be obtained using negative values of c for the level curves $xy = c$. ◀

REMARK If c is a constant, then the functions $g(x, y)$ and $g(x, y) - c$ have the same gradient since the constant c drops out when we differentiate. Consequently, it is *not* essential to rewrite a constraint of the form $g(x, y) = c$ as $g(x, y) - c = 0$ in order to apply the constrained-extremum principle. Thus, in the last example, we could have kept the constraint in the form $x^2 + y^2 = 1$ and then taken $g(x, y) = x^2 + y^2$ rather than $g(x, y) = x^2 + y^2 - 1$.

Example 2 Use the method of Lagrange multipliers to find the dimensions of a rectangle with perimeter p and maximum area.

Solution. Let

x = length of the rectangle, y = width of the rectangle, A = area of the rectangle

We want to maximize $A = xy$ on the line segment

$$2x + 2y = p, \quad 0 \leq x, y \tag{7}$$

that corresponds to the perimeter constraint. This segment is a closed and bounded set, and since $f(x, y) = xy$ is a continuous function, it follows from the Extreme-Value Theorem (Theorem 13.8.3) that f has an absolute maximum on this segment. This absolute maximum must also be a constrained relative maximum since f is 0 at the endpoints of the segment and positive elsewhere on the segment. If $g(x, y) = 2x + 2y$, then we have

$$\nabla f = y\mathbf{i} + x\mathbf{j} \quad \text{and} \quad \nabla g = 2\mathbf{i} + 2\mathbf{j}$$

Noting that $\nabla g \neq \mathbf{0}$, it follows from (4) that

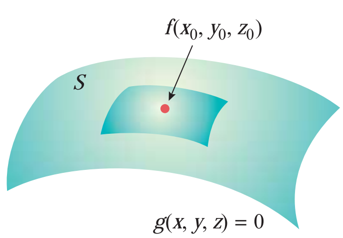
$$y\mathbf{i} + x\mathbf{j} = \lambda(2\mathbf{i} + 2\mathbf{j})$$

at a constrained relative maximum. This is equivalent to the two equations

$$y = 2\lambda \quad \text{and} \quad x = 2\lambda$$

Eliminating λ from these equations we obtain $x = y$, which shows that the rectangle is actually a square. Using this condition and constraint (7), we obtain $x = p/4$, $y = p/4$. ◄

THREE VARIABLES AND ONE CONSTRAINT



A constrained relative maximum occurs at (x_0, y_0, z_0) if $f(x_0, y_0, z_0) \geq f(x, y, z)$ at all points of S near (x_0, y_0, z_0).

▲ **Figure 13.9.4**

The method of Lagrange multipliers can also be used to maximize or minimize a function of three variables $f(x, y, z)$ subject to a constraint $g(x, y, z) = 0$. As a rule, the graph of $g(x, y, z) = 0$ will be some surface S in 3-space. Thus, from a geometric viewpoint, the problem is to maximize or minimize $f(x, y, z)$ as (x, y, z) varies over the surface S (Figure 13.9.4). As usual, we distinguish between relative and absolute extrema. We will say that f has a ***constrained absolute maximum (minimum)*** at (x_0, y_0, z_0) if $f(x_0, y_0, z_0)$ is the largest (smallest) value of $f(x, y, z)$ on S, and we will say that f has a ***constrained relative maximum (minimum)*** at (x_0, y_0, z_0) if $f(x_0, y_0, z_0)$ is the largest (smallest) value of $f(x, y, z)$ at all points of S "near" (x_0, y_0, z_0).

The following theorem, which we state without proof, is the three-variable analog of Theorem 13.9.3.

13.9.4 THEOREM (***Constrained-Extremum Principle for Three Variables and One Constraint***) *Let f and g be functions of three variables with continuous first partial derivatives on some open set containing the constraint surface $g(x, y, z) = 0$, and assume that $\nabla g \neq \mathbf{0}$ at any point on this surface. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0, z_0) on the constraint surface at which the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ are parallel; that is, there is some number λ such that*

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

Example 3 Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point $(1, 2, 2)$.

Solution. To avoid radicals, we will find points on the sphere that minimize and maximize the *square* of the distance to (1, 2, 2). Thus, we want to find the relative extrema of

$$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2$$

subject to the constraint

$$x^2 + y^2 + z^2 = 36 \tag{8}$$

If we let $g(x, y, z) = x^2 + y^2 + z^2$, then $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. Thus, $\nabla g = \mathbf{0}$ if and only if $x = y = z = 0$. It follows that $\nabla g \neq \mathbf{0}$ at any point of the sphere (8), and hence the constrained relative extrema must occur at points where

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

That is,

$$2(x-1)\mathbf{i} + 2(y-2)\mathbf{j} + 2(z-2)\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$$

which leads to the equations

$$2(x-1) = 2x\lambda, \quad 2(y-2) = 2y\lambda, \quad 2(z-2) = 2z\lambda \tag{9}$$

We may assume that x, y, and z are nonzero since $x = 0$ does not satisfy the first equation, $y = 0$ does not satisfy the second, and $z = 0$ does not satisfy the third. Thus, we can rewrite (9) as

$$\frac{x-1}{x} = \lambda, \quad \frac{y-2}{y} = \lambda, \quad \frac{z-2}{z} = \lambda$$

The first two equations imply that

$$\frac{x-1}{x} = \frac{y-2}{y}$$

from which it follows that

$$y = 2x \tag{10}$$

Similarly, the first and third equations imply that

$$z = 2x \tag{11}$$

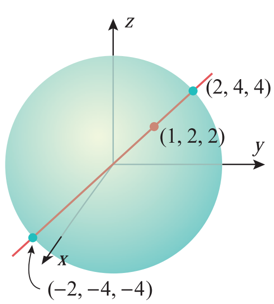
Substituting (10) and (11) in the constraint equation (8), we obtain

$$9x^2 = 36 \quad \text{or} \quad x = \pm 2$$

Substituting these values in (10) and (11) yields two points:

$$(2, 4, 4) \quad \text{and} \quad (-2, -4, -4)$$

Since $f(2, 4, 4) = 9$ and $f(-2, -4, -4) = 81$, it follows that (2, 4, 4) is the point on the sphere closest to (1, 2, 2), and (−2, −4, −4) is the point that is farthest (Figure 13.9.5). ◄



▲ **Figure 13.9.5**

REMARK Solving nonlinear systems such as (9) usually involves trial and error. A technique that sometimes works is demonstrated in Example 3. In that example the equations were solved for a common variable (λ), and we then derived relationships between the remaining variables (x, y, and z). Substituting those relationships in the constraint equation led to the value of one of the variables, and the values of the other variables were then computed.

Next we will use Lagrange multipliers to solve the problem of Example 6 in the last section.

► **Example 4** Use Lagrange multipliers to determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution. With the notation introduced in Example 6 of the last section, the problem is to minimize the surface area

$$S = xy + 2xz + 2yz$$

subject to the volume constraint

$$xyz = 32 \tag{12}$$

If we let $f(x, y, z) = xy + 2xz + 2yz$ and $g(x, y, z) = xyz$, then

$$\nabla f = (y + 2z)\mathbf{i} + (x + 2z)\mathbf{j} + (2x + 2y)\mathbf{k} \quad \text{and} \quad \nabla g = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

It follows that $\nabla g \neq \mathbf{0}$ at any point on the surface $xyz = 32$, since x, y, and z are all nonzero on this surface. Thus, at a constrained relative extremum we must have $\nabla f = \lambda \nabla g$, that is,

$$(y + 2z)\mathbf{i} + (x + 2z)\mathbf{j} + (2x + 2y)\mathbf{k} = \lambda(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k})$$

This condition yields the three equations

$$y + 2z = \lambda yz, \quad x + 2z = \lambda xz, \quad 2x + 2y = \lambda xy$$

Because x, y, and z are nonzero, these equations can be rewritten as

$$\frac{1}{z} + \frac{2}{y} = \lambda, \quad \frac{1}{z} + \frac{2}{x} = \lambda, \quad \frac{2}{y} + \frac{2}{x} = \lambda$$

From the first two equations,

$$y = x \tag{13}$$

and from the first and third equations,

$$z = \tfrac{1}{2}x \tag{14}$$

Substituting (13) and (14) in the volume constraint (12) yields

$$\tfrac{1}{2}x^3 = 32$$

This equation, together with (13) and (14), yields

$$x = 4, \quad y = 4, \quad z = 2$$

which agrees with the result that was obtained in Example 6 of the last section. ◀

There are variations in the method of Lagrange multipliers that can be used to solve problems with two or more constraints. However, we will not discuss that topic here.

✔ QUICK CHECK EXERCISES 13.9 *(See page 997 for answers.)*

1. (a) Suppose that $f(x, y)$ and $g(x, y)$ are differentiable at the origin and have nonzero gradients there, and that $g(0, 0) = 0$. If the maximum value of $f(x, y)$ subject to the constraint $g(x, y) = 0$ occurs at the origin, how is the tangent line to the graph of $g(x, y) = 0$ related to the tangent line at the origin to the level curve of f through $(0, 0)$?

(b) Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable at the origin and have nonzero gradients there, and that $g(0, 0, 0) = 0$. If the maximum value of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ occurs at the origin, how is the tangent plane to the graph of the constraint $g(x, y, z) = 0$ related to the tangent plane at the origin to the level surface of f through $(0, 0, 0)$?

2. The maximum value of $x + y$ subject to the constraint $x^2 + y^2 = 1$ is ________.

3. The maximum value of $x + y + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$ is ________.

4. The maximum and minimum values of $2x + 3y$ subject to the constraint $x + y = 1$, where $0 \leq x, 0 \leq y$, are ________ and ________, respectively.