

Permutations & Combinations

Extension 1 Mathematics

HSC Revision

Multiplication Rule

If one event can occur in **m** ways, a second event in **n** ways and a third event in **r**, then the three events can occur in **m × n × r** ways.

Example Erin has 5 tops, 6 skirts and 4 caps from which to choose an outfit.

In how many ways can she select one top, one skirt and one cap?

Solution: **Ways = 5 × 6 × 4**

Repetition of an Event

If one event with **n** outcomes occurs **r** times with repetition allowed, then the number of ordered arrangements is **n^r**

Example 1 What is the number of arrangements if a die is rolled

(a) 2 times ? $6 \times 6 = 6^2$

(b) 3 times ? $6 \times 6 \times 6 = 6^3$

(b) r times ? $6 \times 6 \times 6 \times \dots = 6^r$

Repetition of an Event

Example 2

(a) How many different car number plates are possible

with 3 letters followed by 3 digits?

Solution: $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 26^3 \times 10^3$

(b) How many of these number plates begin with ABC?
?

Solution: $1 \times 1 \times 1 \times 10 \times 10 \times 10 = 10^3$

(c) If a plate is chosen at random, what is the probability that it begins with ABC?

Solution: $\frac{10^3}{26^3 \times 10^3} = \frac{1}{26^3}$

Factorial Representation

$$n! = n(n - 1)(n - 2).....3 \times 2 \times 1$$

For example $5! = 5.4.3.2.1$

Note $0! = 1$

Example

a) In how many ways can 6 people be arranged in a row?

Solution : $6.5.4.3.2.1 = 6!$

b) How many arrangements are possible if only 3 of them are chosen?

Solution: $6.5.4 = 120$

Arrangements or Permutations

Distinctly ordered sets are called **arrangements** or **permutations**.

The number of permutations of **n** objects taken **r** at a time is given by:

$${}^n\mathbf{P}_r = \frac{\mathbf{n}!}{(\mathbf{n} - \mathbf{r})!}$$

where n = number of objects
 r = number of positions

Arrangements or Permutations

Eg 1. A maths debating team consists of 4 speakers.

a) In how many ways can all 4 speakers be arranged in a row for a photo?

Solution : $4.3.2.1 = 4!$ or 4P_4

b) How many ways can the captain and vice-captain be chosen?

Solution : $4.3 = 12$ or 4P_2



Arrangements or Permutations

Eg 2. A flutter on the horses
There are 7 horses in a race.

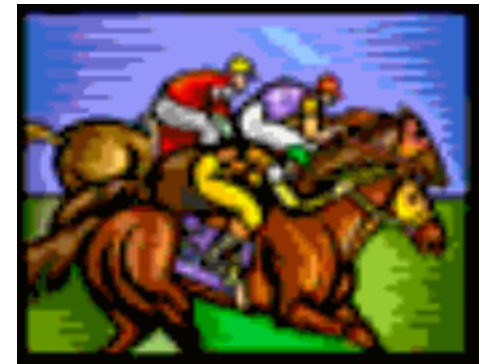


a) In how many different orders can the horses finish?

Solution : $7.6.5.4.3.2.1 = 7!$ or 7P_7

b) How many trifectas (1st , 2nd and 3rd) are possible?

Solution : $7.6.5 = 210$ or 7P_3



Permutations with Restrictions

Eg. In how many ways can 5 boys and 4 girls be arranged on a bench if

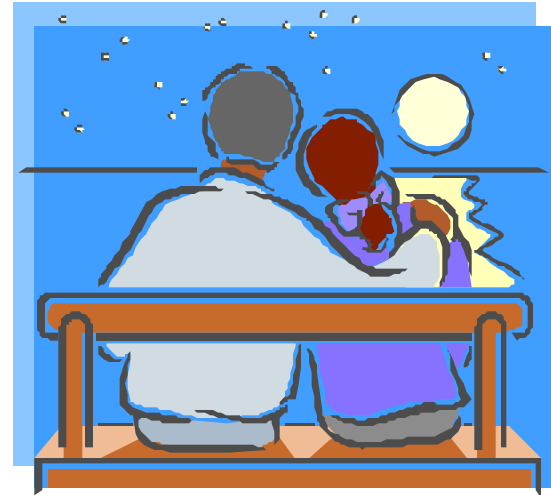
a) there are no restrictions?

Solution : 9! or 9P_9

c) boys and girls alternate?

Solution : A boy will be on each end

$$\begin{aligned} \text{BGBGBGBGB} &= 5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 \\ &= 5! \times 4! \text{ or } {}^5P_5 \times {}^4P_4 \end{aligned}$$



Permutations with Restrictions

Eg. In how many ways can 5 boys and 4 girls be arranged on a bench if

c) boys and girls are in separate groups?

Solution : Boys & Girls or Girls & Boys

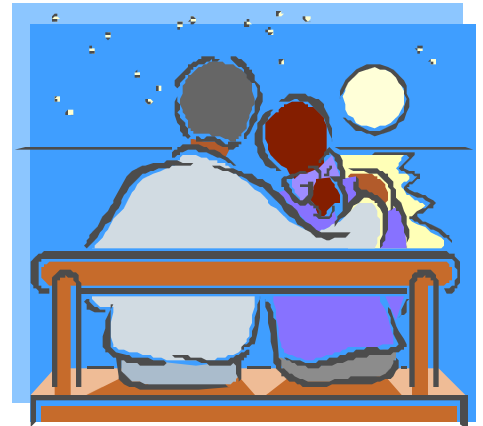
$$= 5! \times 4! + 4! \times 5! = 5! \times 4! \times 2$$

$$\text{or } {}^5P_5 \times {}^4P_4 \times 2$$

d) Anne and Jim wish to stay together?

Solution : (AJ) _ _ _ _ _

$$= 2 \times 8! \text{ or } 2 \times {}^8P_8$$



Arrangements with Repetitions

If we have n elements of which x are alike of one kind, y are alike of another kind, z are alike of another kind,

..... then the number of ordered selections or permutations is given by:

$$\frac{n!}{x! y! z!}$$

Arrangements with Repetitions

Eg.1 How many different arrangements of the word **PARRAMATTA** are possible?

Solution : 10 letters but note repetition
(4 A's, 2 R's, 2 T's)

P

A A A A

R R

M

T T

No. of
arrangements = $\frac{10!}{4! 2! 2!}$

= 37 800



Arrangements with Restrictions

Eg 1. How many arrangements of the letters of the word REMAND are possible if:

a) there are no restrictions?

Solution : ${}^6P_6 = 720$ or $6!$

b) they begin with RE?

Solution : $\underline{R} \underline{E} _ _ _ _ = {}^4P_4 = 24$ or $4!$

c) they do not begin with RE?

Solution : $\text{Total} - (b) = 6! - 4! = 696$

Arrangements with Restrictions

Eg 1. How many arrangements of the letters of the word REMAND are possible if:

d) they have RE together in order?

Solution : **(RE)** _ _ _ _ = ${}^5P_5 = 120$ or $5!$

e) they have REM together in any order?

Solution : **(REM)** _ _ _ = ${}^3P_3 \times {}^4P_4 = 144$

f) R, E and M are not to be together?

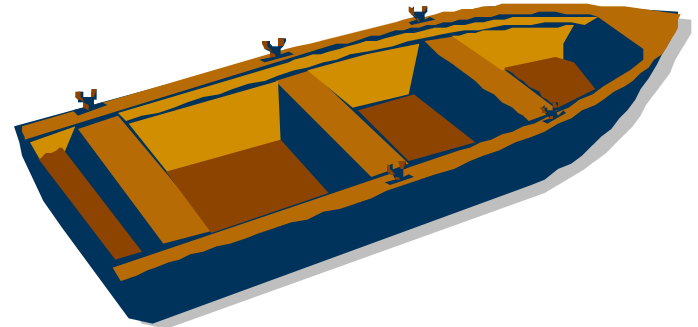
Solution : **Total – (e) = $6! - 144 = 576$**

Arrangements with Restrictions

Eg 2. There are 6 boys who enter a boat with 8 seats, 4 on each side. In how many ways can

a) they sit anywhere?

Solution : 8P_6



b) two boys A and B sit on the port side and another boy W sit on the starboard side?

Solution : $A \ \& \ B = {}^4P_2$

$W = {}^4P_1$

$\text{Others} = {}^5P_3$

Total $= {}^4P_2 \times {}^4P_1 \times {}^5P_3$



Arrangements with Restrictions

Eg 3. From the digits 2, 3, 4, 5, 6

a) how many numbers greater than 4 000 can be formed?

Solution : 5 digits (any) = 5P_5

4 digits (must start with digit ≥ 4) = ${}^3P_1 \times {}^4P_3$

Total = ${}^5P_5 + {}^3P_1 \times {}^4P_3$

b) how many 4 digit numbers would be even?

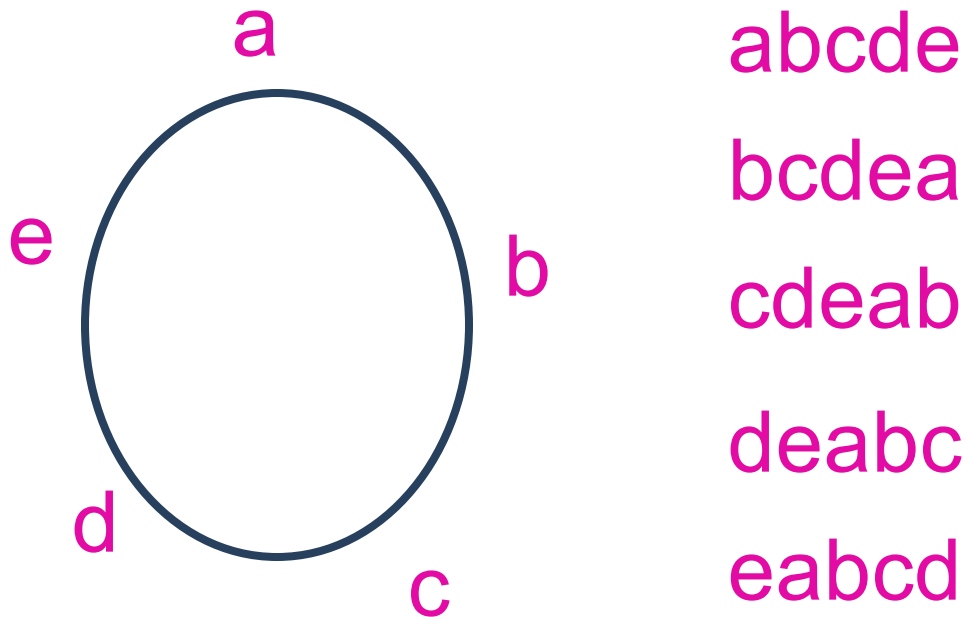
Even (ends with 2, 4 or 6) = $_ _ _ {}^3P_1$

$= {}^4P_3 \times {}^3P_1$

Circular Arrangements

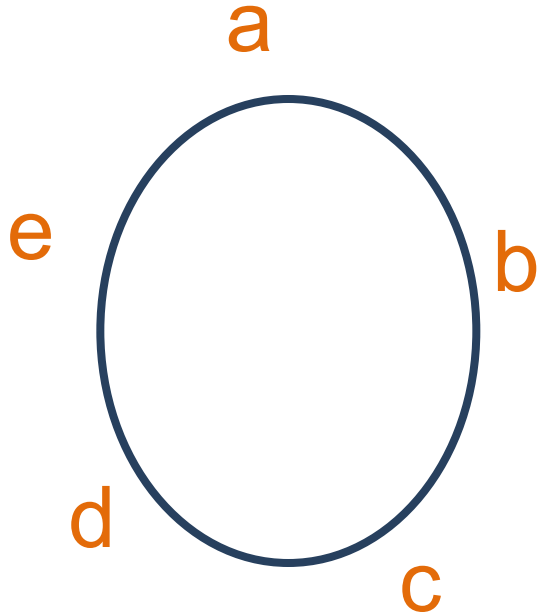
Circular arrangements are permutations in which objects are arranged in a circle.

Consider arranging 5 objects (a, b, c, d, e) around a circular table. The arrangements



are different in a line, but are **identical** around a circle.

Circular Arrangements



To calculate the number of ways in which n objects can be arranged in a circle, we arbitrarily fix the position of one object, so the remaining $(n-1)$ objects can be arranged as if they were on a straight line in $(n-1)!$ ways.

**i.e. the number of arrangements = $(n - 1) !$
in a circle**

Circular Arrangements

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

a) there are no restrictions

Solution :

$$(12 - 1)! = 11!$$

b) men and women alternate

Solution : $(6 - 1)! \times 6! = 5! \times 6!$



Circular Arrangements

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

c) Ted and Carol must sit together

Solution : (TC) & other 10 = $2! \times 10!$

d) Bob, Ted and Carol must sit together

Solution : (BTC) & other 9 = $3! \times 9!$

Circular Arrangements

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

d) Neither Bob nor Carol can sit next to Ted.

Solution : Seat 2 of the other 9 people next to Ted in (9×8) ways or 9P_2

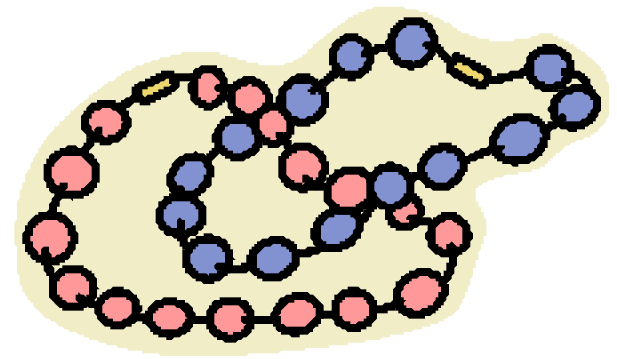
Then sit the remaining 9 people (including Bob and Carol) in $9!$ ways

Ways = $(9 \times 8) \times 9!$ or ${}^9P_2 \times 9!$

Circular Arrangements

Eg 2. In how many ways can 8 differently coloured beads be threaded on a string?

Solution :



As necklace can be turned over, clockwise and anti-clockwise arrangements are the same

$$= (8-1)! \div 2 = 7! \div 2$$

Unordered Selections

The number of different **combinations** (i.e. unordered sets) of **r** objects from **n** distinct objects is represented by :

No. of Combinations	=	$\frac{\text{number of permutations}}{\text{arrangements of } r \text{ objects}}$
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and is denoted by

${}^n\mathbf{C}_r = \frac{{}^n\mathbf{P}_r}{r!} = \frac{n!}{r! (n - r)!}$

Combinations

Eg 1. How many ways can a basketball team of 5 players be chosen from 8 players?

Solution :

$8C_5$

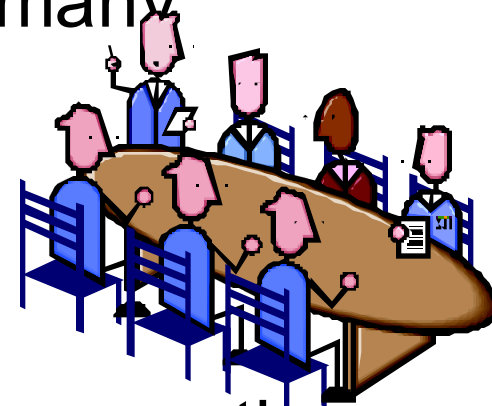


Combinations

Eg 2. A committee of 5 people is to be chosen from a group of 6 men and 4 women. How many committees are possible if

a) there are no restrictions?

Solution : $^{10}C_5$



b) one particular person must be chosen on the committee?

Solution : $\underline{1} \times {}^9C_4$

c) one particular woman must be excluded from the committee?

Solution : 9C_5

Combinations

Eg 2. A committee of 5 people is to be chosen from a group of 6 men and 4 women. How many committees are possible if:

d) there are to be 3 men and 2 women?

Solution : **Men & Women = ${}^6C_3 \times {}^4C_2$**

e) there are to be men only?

Solution : **6C_5**

f) there is to be a majority of women?

Solution :

3 Women & 2 men Or 4 Women & 1 man

$$= {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1$$

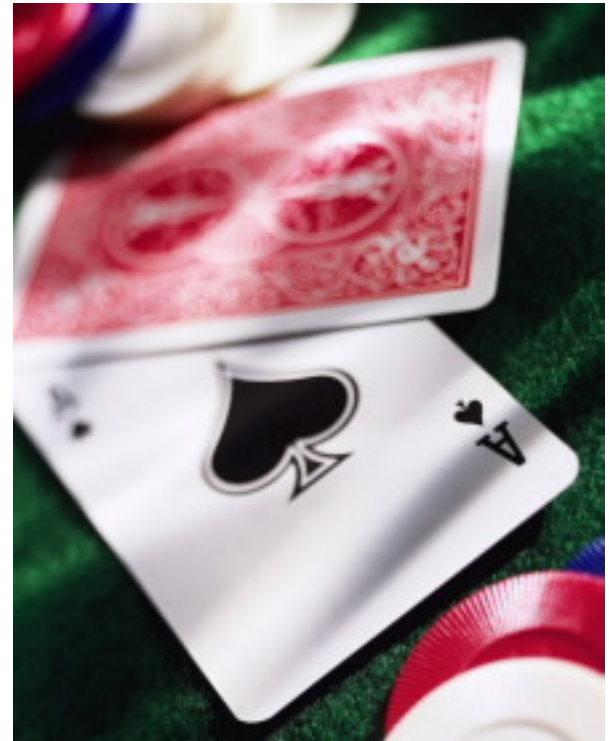
Combinations

Eg 3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

(i) What is the total possible number of hands if there are no restrictions?

Solution :

$${}^{52}C_5$$



Combinations

Eg 3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

ii) In how many of these hands are there:

a) 4 Kings?

Solution : ${}^4C_4 \times {}^{48}C_1$ or 1×48

b) 2 Clubs and 3 Hearts?

Solution : ${}^{13}C_2 \times {}^{13}C_3$

Combinations

Eg 3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

- ii) In how many of these hands are there:
- c) all Hearts?

Solution : $^{13}C_5$



- d) all the same colour?

Solution : **Red or Black** $^{26}C_5 + ^{26}C_5 = 2 \times ^{26}C_5$
=

Combinations

Eg 3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

ii) In how many of these hands are there

e) four of the same kind?

Solution :

$${}^4C_4 \times {}^{48}C_1 \times 13 = 1 \times 48 \times 13$$

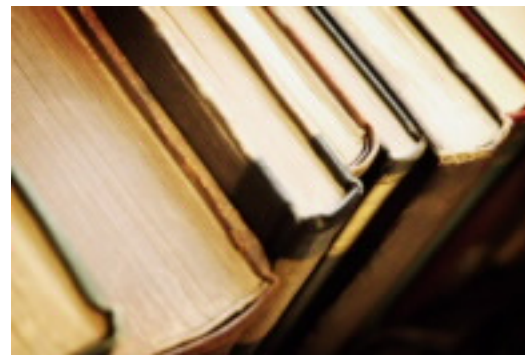
f) 3 Aces and two Kings?

Solution : ${}^4C_3 \times {}^4C_2$



Further Permutations and Combinations

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf:



a) If there are no restrictions?

Solution : ${}^6C_4 \times {}^5C_3 \times 7!$

c) If the 4 Maths books remain together?

Solution : $= (MMMM) _ _ _$

$$= {}^6P_4 \times {}^5C_3 \times 4! \text{ or } ({}^6C_4 \times 4!) \times {}^5C_3 \times 4!$$

Further Permutations and Combinations

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf if:

c) a Maths book is at the beginning of the shelf?



Solution : **= M _ _ _ _ _**

$$= 6 \times {}^5C_3 \times {}^5C_3 \times 6!$$

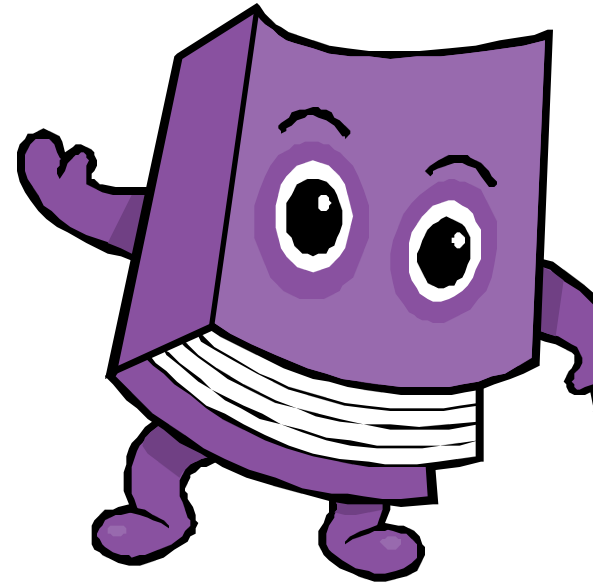
Further Permutations and Combinations

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf if:

d) Maths and English books alternate

Solution : **= M E M E M E M**

$$= {}^6P_4 \times {}^5P_3$$



Further Permutations and Combinations

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf if:

e) A Maths is at the beginning and an English book is in the middle of the shelf.

Solution :

$$\text{M} _ _ \text{E} _ _ _$$
$$= 6 \times 5 \times {}^5\text{C}_3 \times {}^4\text{C}_2 \times 5!$$

Further Permutations and Combinations

Eg 2. (i) How many different 8 letter words are possible using the letters of the word SYLLABUS ?

Solution : **2 S's & 2 L's**

$$\begin{aligned}\text{Words} &= \frac{8!}{2! \times 2!} \\ &= 10\,080\end{aligned}$$

Further Permutations and Combinations

SYLLABUS = 10 080 permutations

(ii) If a word is chosen at random, find the probability that the word:

a) contains the two S's together

Solution : **(SS) _ _ _ _ _** **(Two L's)**

$$\text{Words} = \frac{7!}{2!} = 2520 \qquad \text{Prob} = \frac{2520}{10080} = \frac{1}{4}$$

b) begins and ends with L

Solution : **L _ _ _ _ _ L** **(Two S's)**

$$\text{Words} = \frac{6!}{2!} = 360 \qquad \text{Prob} = \frac{360}{10080} = \frac{1}{28}$$