Exercise #4.6:

Q1
$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2+2R_1} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[R_3-R_2]{R_1-R_2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

let $x_3 = t$, $x_1 = t$, $x_2 = 0$

$(x_1, x_2, x_3) = (t, 0, t) = t(1,0,1)$

$\therefore$ Basis $\rightarrow$ $v_1 = (1,0,1)$

Dimension of Sol. Space $= 1$

Q3
$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow[R_2-R_1]{\frac{1}{2}R_1} \begin{bmatrix} 1 & 1/2 & 3/2 & 0 \\ 0 & -1/2 & 7/2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow[R_3-R_2]{-2R_2,\ R_1-\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix}$$

$\frac{1}{9}R_3$, R_2+7R_3, R_1-5R_3
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ $x_1 = x_2 = x_3 = 0$

Hence, no basis and dimension of sol space $= 0$

Q5
$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

let $x_3 = t$, $x_2 = s$; $x_1 = 3s - t$

$(x_1, x_2, x_3) = (3s-t, s, t) = t(-1,0,1) + s(3,1,0)$

Basis : $v_1 = (3,1,0)$
$v_2 = (-1,0,1)$

Dimension of sol space $= 2$

Q7 a) $3x - 2y + 5z = 0$

let $y = s$, $z = t$ $\therefore$ $x = \frac{2s - 5t}{3}$

$(x,y,z) = \left(\frac{2s-5t}{3}, s, t\right) = s\left(\frac{2}{3}, 1, 0\right) + t(-5/3, 0, 1)$

Basis $v_1 = \left(\frac{2}{3}, 1, 0\right)$
$v_2 = (-5/3, 0, 1)$

Dimension of Subspace $= 2$.

b) $x - y = 0$

$x - y - 0z = 0$ let $z = t$, $y = s$ $\therefore$ $x = s$

$(x,y,z) = (s,s,t) = t(0,0,1) + s(1,1,0)$

Basis : $v_1 : (1,1,0)$
$v_2 : (0,0,1)$

Dimension of subspace : 2

c) $x = 2t$, $y = -t$, $z = 4t$

$(x,y,z) = (2t, -t, 4t) = t(2,-1,4)$

Basis : $v_1 = (2,-1,4)$ Dimension of subspace $= 1$

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d) (a,b,c) where $b = a+c$

$(a, a+c, c) \rightarrow (a,b,c) = (a, a+c, c) = a(1,1,0) + c(0,1,1)$

Basis: $v_1 = (1,1,0)$ Dimension of subspace $= 2$

$v_2 = (0,1,1)$

Q9 a) Diagonal $n \times n$ matrices.

$$\begin{bmatrix} n_1 & 0 & \cdots & 0 \\ 0 & n_2 & & 0 \\ 0 & 0 & & n_n \end{bmatrix} \rightarrow \underbrace{n_1 \begin{bmatrix} 1 & 0 & & 0 \\ 0 & n_2 & & 0 \\ 0 & & \cdots & n_n \end{bmatrix} + n_2 \begin{bmatrix} n_1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \cdots & n_n \end{bmatrix}}_{A_1 \cdots A_n} + n_n \begin{bmatrix} n_1 & 0 & \cdots & 0 \\ 0 & n_2 & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Basis $A_1 - A_n$ Dimension: n.

b) Symmetric $n \times n$ matrices

Using part a) solution and adding all combinations of matrices where $i = j$

Basis $A_1 \quad A_n$ Dimension: $n + \frac{n(n-1)}{2}$

c) Upper triangle of $n \times n$ matrices

Using part a) solution and adding all combinations that satisfy upper triangle. Basis: $A_1 \cdots A_n$ Dimension $n + \frac{n(n-1)}{2}$.

Q11 a) Let $W \Rightarrow a_0 + a_1 x + a_2 x^2$ where $a_1 + a_2 + a_0 = 0$

$\therefore a_0 = -a_1 - a_2$

$u = (-a_1 - a_2 + a_1 x + a_2 x^2)$ $\quad v = (-b_1 - b_2 + b_1 x + b_2 x^2)$.

$u+v = (-a_1 - a_2 - b_1 - b_2 + a_1 x + b_1 x + a_2 x^2 + b_2 x^2)$.

$\therefore (-a_1 - a_2 - b_1 - b_2) + (a_1 + b_1) + (a_2 + b_2) = 0$

$ku = k(-a_1 - a_2 + a_1 x + a_2 x) = (-a_1 k - a_2 k + a_1 k x + a_2 k x^2)$

$\therefore (-a_1 k - a_2 k + a_1 k + a_2 k) = 0$

Since both condition is satisfied $\therefore W$ is subspace of P_2

b) Since W is subspace of P_2 $\therefore \dim(W) \leq \dim(P_2)$.

c) $-a_1 - a_2 + a_1(x) + a_2(x^2) = a_1(-1+x) + a_2(-1+x^2)$

Basis: $v_1 = (-1+x)$

$v_2 = (-1+x^2)$ Dimension $= 2$

$1 - \frac{2}{3}(1)$ $\quad -3 + 6/5 \quad \frac{-15+6}{5} = \frac{-9}{5} \quad -\frac{2}{3} \to -\frac{3}{2}$

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Q13 $k_1v_1 + k_2v_2 + k_3e_1 + k_4e_2 + k_5e_3 + k_6e_4 = 0$

$$\begin{bmatrix} 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ -4 & 8 & 0 & 1 & 0 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 & 0 \\ -3 & 6 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[\substack{R_3 - 2R_1 \\ R_4 + 3R_1}]{R_2 + 4R_1} \begin{bmatrix} 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -4 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\substack{-\frac{1}{4}R_2 \\ R_3 - 2R_2 \\ R_4 - 3R_2 \\ R_1 + 3R_2} \begin{bmatrix} 1 & 0 & -2 & -3/4 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3/4 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[\substack{R_4 - 3/4 R_3 \\ R_1 + 3/4 R_3 \\ R_2 + 1/4 R_3}]{2R_3} \begin{bmatrix} 1 & 0 & -2 & 0 & 3/2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3/2 & 1 & 0 \end{bmatrix}$$

$$\substack{-2/3 R_4 \\ R_1 - 3/2 R_4 \\ R_2 - 1/2 R_4 \\ R_3 - 2R_4} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4/3 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2/3 & 0 \end{bmatrix}$$

Pivots in : $\underbrace{k_1v_1, k_2v_2, k_4e_2, k_5e_3}_{\text{trivial solution}}$

$\therefore$ v_1, v_2, e_2 and e_3 are linearly independent and form basis of R^4 but answer is not unique solution.

Q15 $k_1v_1 + k_2v_2 + k_3e_1 + k_4e_2 + k_5e_3 = 0$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ -2 & 5 & 0 & 1 & 0 & 0 \\ 3 & -3 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 + 2R_1} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 5 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & 0 & 1 & 0 \end{bmatrix}$$

$$\substack{1/5 R_2 \\ R_3 + 3R_2} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2/5 & 1/5 & 0 & 0 \\ 0 & 0 & -9/5 & 3/5 & 1 & 0 \end{bmatrix} \xrightarrow[\substack{R_2 - 2/5 R_3 \\ R_1 - R_3}]{-5/9 R_3} \begin{bmatrix} 1 & 0 & 0 & 1/3 & 5/9 & 0 \\ 0 & 1 & 0 & 1/3 & 2/9 & 0 \\ 0 & 0 & 1 & -1/3 & -5/9 & 0 \end{bmatrix}$$

Pivots : $k_1v_1 + k_2v_2 + k_3e_1 = 0$

$\therefore$ v_1, v_2 and e_1 are linearly independent and form bases of R^3 but answer is not unique solution

Q17 $$\begin{bmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 - R_3} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

let $k_3 = s$, $k_4 = t$ $\quad$ $k_1 = -s - t$, $k_2 = -s + t$

V_1 and V_2 contain pivot value hence they are linearly independent and form the basis for span $\{v_1, v_2, v_3, v_4\}$ but answer is not unique

$V_3 = v_1 + v_2$ & $v_4 = v_1 - v_2$ $\rightarrow$ linear combination

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Q19 a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_3 - R_2,\ R_1 - R_2]{-R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

let $x_3 = t$; $x_2 = t$, $x_1 = -t \rightarrow (x_1, x_2, x_3) = (-t, t, t) = t(-1, 1, 1)$

Basis $\{(-1, 1, 1)\}$ Dimension = 1.

b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

let $x_2 = s$, $x_3 = t$; $x_1 = -2s \rightarrow (x_1, x_2, x_3) = (-2s, s, t) = t(0,0,1) + s(-2,1,0)$

Basis = $\{(-2,1,0), (0,0,1)\}$ Dimension = 2

c) $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$x_1 = x_2 = x_3 = 0$ ∴ dimension = 0.

Exercise # 4.8:

Q1 a) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = -2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

Q3 a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ $b = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \xrightarrow[R_1 - R_2,\ R_3 + R_2]{-R_2} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow[R_2 + R_3,\ 2R_3 + R_1]{\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$Ax = b$ is inconsistent ∴ b is not column space of A.

b) $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 9R_1} \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 12 & -8 & -44 \\ 0 & 2 & 0 & -6 \end{bmatrix} \xrightarrow[R_1 + R_2,\ R_3 - 2R_2]{\frac{1}{12}R_2} \begin{bmatrix} 1 & 0 & 1/3 & 4/3 \\ 0 & 1 & -2/3 & -11/3 \\ 0 & 0 & 4/3 & 4/3 \end{bmatrix}$

$\xrightarrow[R_2 + \frac{2}{3}R_3,\ R_1 - \frac{1}{3}R_3]{\frac{3}{4}R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $x_3 = 1$, $x_2 = -3$, $x_1 = 1$

∴ b is a column space of A

$1\begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

Q5 $x_1 = 3,\ x_2 = 0,\ x_3 = -1,\ x_4 = 5$

a) $Ax = 0 \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

b) $Ax = b \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -1 \\ 5 \end{bmatrix}$

Q7 a) $\begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ let $x_2 = t$

$x_1 = 1 + 3t$

S.set $\Rightarrow (x_1, x_2) = (1+3t, t) = (1,0) + t(3,1)$.

General sol. of $Ax = 0 \rightarrow t(3,1)$

b) $\begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 0 & 1 & -2 \\ 2 & 1 & 3 & 3 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 & 5 \\ 0 & -1 & -1 & -7 \\ 0 & -1 & -1 & -7 \end{bmatrix} \xrightarrow[R_1 - R_2,\ R_3 + R_2]{-R_2} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

let $x_3 = t$; $x_2 = 7 - t$, $x_1 = -2 - t$

S.Set $\Rightarrow (x_1, x_2, x_3) = (-2-t, 7-t, +t) = (-2, 7, 0) + t(-1,-1,1)$

General sol. of $Ax = 0 \rightarrow t(-1,-1,1)$

Q9 a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \xrightarrow[R_3 - 7R_1]{R_2 - 5R_1} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix} \xrightarrow[R_1 + R_2]{R_3 - R_2} \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$

let $x_3 = t$, $x_2 = 19t$, $x_1 = 16t$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} \rightsquigarrow$ Basis for null space

Basis of row space $\Rightarrow [1\ \ 0\ \ -16]$, $[0\ \ 1\ \ -19]$

b) $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 - 4R_1]{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

let $x_3 = t$, $x_2 = s$; $x_1 = \frac{1}{2}t$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}}_{\text{Basis for null space}}$

Basis of row space $\Rightarrow [1\ \ 0\ \ -1/2]$

Q11 a) Row space : $[1\ 0\ 2], [0\ 0\ 1]$ Column space : $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}$

b) Row space : $[1\ -3\ 0\ 0], [0\ 1\ 0\ 0]$ Column space $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\1\\0\\0\end{bmatrix}$

Q13 a) $\begin{bmatrix}1 & -2 & 5 & 0 & 3\\ -2 & 5 & -7 & 0 & -6\\ -1 & 3 & -2 & 1 & -3\\ -3 & 8 & -9 & 1 & -9\end{bmatrix} \xrightarrow[R_4+3R_1]{R_2+2R_1,\ R_3+R_1} \begin{bmatrix}1 & -2 & 5 & 0 & 3\\ 0 & 1 & 3 & 0 & 0\\ 0 & 1 & 3 & 1 & 0\\ 0 & 2 & 6 & 1 & 0\end{bmatrix} \xrightarrow[R_4-2R_2]{R_1+2R_2,\ R_3-R_2} \begin{bmatrix}1 & 0 & 11 & 0 & 3\\ 0 & 1 & 3 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0\end{bmatrix}$

$\xrightarrow{R_4-R_3} \begin{bmatrix}1 & 0 & 11 & 0 & 3\\ 0 & 1 & 3 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$ Row space : $[1\ 0\ 11\ 0\ 3], [0\ 1\ 3\ 0\ 0], [0\ 0\ 0\ 1\ 0]$

Col space $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$

Col space of A $\begin{bmatrix}1\\-2\\-1\\3\end{bmatrix}, \begin{bmatrix}-2\\5\\3\\8\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$

b) $A^T = \begin{bmatrix}1 & -2 & -1 & -3\\ -2 & 5 & 3 & 8\\ 5 & -7 & -2 & -9\\ 0 & 0 & 1 & 1\\ 3 & -6 & -3 & -9\end{bmatrix} \xrightarrow[R_5-3R_1]{R_2+2R_1,\ R_3-5R_1} \begin{bmatrix}1 & -2 & -1 & 3\\ 0 & 1 & 1 & 2\\ 0 & 3 & 3 & 6\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} \xrightarrow{R_1+2R_2,\ R_3-3R_2} \begin{bmatrix}1 & 0 & 1 & 7\\ 0 & 1 & 1 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$

$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix}1 & 0 & 1 & 7\\ 0 & 1 & 1 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ Row space $[1\ 0\ 0\ 0\ 0], [0\ 1\ 0\ 0\ 0], [1\ 1\ 1\ 0\ 0]$

Row space of A $[1\ -2\ 5\ 0\ 3], [-2\ 5\ -7\ 0\ -6], [-1\ 3\ -2\ 1\ -3]$

Q15 $\begin{bmatrix}1 & 0 & -2 & 0\\ 1 & 0 & 0 & -3\\ 0 & 1 & 2 & 0\\ 0 & 1 & 2 & 3\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 0 & 2 & -3\\ 0 & 1 & 2 & 0\\ 0 & 1 & 2 & 3\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3,\ R_4-R_2} \begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 2 & -3\\ 0 & 0 & 0 & 3\end{bmatrix} \xrightarrow[R_2-2R_3]{\frac{1}{2}R_3,\ R_1+2R_3} \begin{bmatrix}1 & 0 & 0 & -3\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & -3/2\\ 0 & 0 & 0 & 3\end{bmatrix}$

$\xrightarrow[R_1+3R_4,\ R_2-3R_4]{\frac{1}{3}R_4,\ R_3+3/2R_3} \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ Row space of A :

$[1\ 1\ 0\ 0], [0\ 0\ 1\ 1], [-2\ 0\ 2\ 2], [0\ -3\ 0\ 3]$

Q17 $\begin{bmatrix}1 & -2 & 4 & 0 & -7\\ -1 & 3 & -5 & 4 & 18\\ 5 & 1 & 9 & 2 & 2\\ 2 & 0 & 4 & -3 & -8\end{bmatrix} \xrightarrow[R_4-2R_1]{R_2+R_1,\ R_3-5R_1} \begin{bmatrix}1 & -2 & 4 & 0 & -7\\ 0 & 1 & -1 & 4 & 11\\ 0 & 11 & -11 & 2 & 37\\ 0 & 4 & -4 & -3 & 6\end{bmatrix} \xrightarrow[R_4-4R_2]{R_1+2R_2,\ R_3-11R_2} \begin{bmatrix}1 & 0 & 2 & 8 & 15\\ 0 & 1 & -1 & 4 & 11\\ 0 & 0 & 0 & -42 & -84\\ 0 & 0 & 0 & -19 & -38\end{bmatrix}$

$\xrightarrow[R_4+19R_3,\ R_1-8R_3]{-\frac{1}{42}R_3,\ R_2-4R_3} \begin{bmatrix}1 & 0 & 2 & 0 & -1\\ 0 & 1 & -1 & 0 & 3\\ 0 & 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$

v_1, v_2 and v_4 are basis for span $\{v_1, v_2, v_3, v_4, v_5\}$

$v_3 = 2v_1 - v_2$; $v_5 = -v_1 + 3v_2 + 2v_4$

Q19 $A^T = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 4 & -2 & 0 & 3 \\ 5 & 1 & -1 & 5 \\ 6 & 4 & -2 & 7 \\ 9 & -1 & -1 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/7 & 13/14 \\ 0 & 1 & -2/7 & 5/14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row space of A: $[1\ 4\ 5\ 6\ 9]$, $[3\ -2\ 1\ 4\ -1]$

Q21 a) $b = (0,0)$

$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & -1 & 4 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 4 & 0 \end{bmatrix} \xrightarrow[R_1 - 2R_2]{-1/3 R_2} \begin{bmatrix} 1 & 0 & 8/3 & 0 \\ 0 & 1 & -4/3 & 0 \end{bmatrix}$

let $x_3 = t$, $x_2 = 4/3 t$, $x_1 = -8/3 t$ $\therefore$ $x = t(-8/3, 4/3, 1)$.

b) $b = (1, 3)$

$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & 4 & 3 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -3 & 4 & 2 \end{bmatrix} \xrightarrow[R_1 - 2R_2]{-1/3 R_2} \begin{bmatrix} 1 & 0 & 8/3 & 7/3 \\ 0 & 1 & -4/3 & -2/3 \end{bmatrix}$

let $x_3 = t$, $x_2 = -2/3 + 4/3 t$, $x_1 = 7/3 - 8/3 t$ $\therefore$ $x = (7/3, -2/3, 0) + t(-8/3, 4/3, 1)$

c) $b = (-1, 1)$

$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & -1 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -3 & 4 & 2 \end{bmatrix} \xrightarrow[R_1 - 2R_2]{-1/3 R_2} \begin{bmatrix} 1 & 0 & 8/3 & 1/3 \\ 0 & 1 & -4/3 & -2/3 \end{bmatrix}$

let $x_3 = t$, $x_2 = -2/3 + 4/3 t$, $x_1 = 1/3 - 8/3 t$ $\therefore$ $x = (1/3, -2/3, 0) + t(-8/3, 4/3, 1)$

Q23 a) $x + y + z = 1$

let $y = s$, $z = t$ $\therefore$ $x = 1 - s - t$; $(x, y, z) = (1,0,0) + s(-1,1,0) + t(-1,0,1)$

b) It represents a plane passing through $(1,0,0)$ parallel to vectors $(-1,1,0)$ & $(-1,0,1)$

Q25 a) $\begin{bmatrix} 3 & 2 & -1 & 0 \\ 6 & 4 & -2 & 0 \\ -3 & -2 & 1 & 0 \end{bmatrix} \xrightarrow[\substack{R_2 - 6R_1 \\ R_3 + 3R_1}]{1/3 R_1} \begin{bmatrix} 1 & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

let $x_2 = s$, $x_3 = t$ $\therefore$ $x_1 = 1/3 t - 2/3 s$

S. Set : $(x_1, x_2, x_3) = (1/3 t - 2/3 s, s, t)$

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b) $\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-1 \\ 6-2 \\ -3+1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$ verified

c) $(x_1, x_2, x_3) = (1,0,1) + (1/3\,t - 2/3\,s,\ s,\ t)$

d) $\begin{bmatrix} 3 & 2 & -1 & 2 \\ 6 & 4 & -2 & 4 \\ -3 & -2 & 1 & -2 \end{bmatrix} \xrightarrow{\substack{1/3 R_1 \\ R_2 - 6R_1 \\ R_3 + 3R_1}} \begin{bmatrix} 1 & 2/3 & -1/3 & 2/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

let $x_2 = p$, $x_3 = q$ $\therefore$ $x_1 = 2/3 + 1/3\,p - 2/3\,q$

If $p = s$ and $q = t+1$, solution of (c) will be confirmed

Q27 $\begin{bmatrix} 3 & 4 & 1 & 2 & 3 \\ 6 & 8 & 2 & 5 & 7 \\ 9 & 12 & 3 & 10 & 13 \end{bmatrix} \xrightarrow{\substack{1/3 R_1 \\ R_2 - 6R_1 \\ R_3 - 9R_1}} \begin{bmatrix} 1 & 4/3 & 1/3 & 2/3 & 1/1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 & 4 \end{bmatrix}$

$\xrightarrow{\substack{R_3 - R_2 \\ R_1 - 2/3 R_2}} \begin{bmatrix} 1 & 4/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

let $x_2 = r$, $x_3 = s$ $\therefore$ $x_4 = 1$, $x_1 = 1/3 - 4/3\,r - 1/3\,s$

$(x_1, x_2, x_3, x_4) = (-4/3\,r - 1/3\,s,\ r,\ s,\ 0) + (1/3, 0, 0, 1)$

Q29 a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

let $z = t$; $y = 0$, $x = 0$ $\rightarrow$ $x = t(0,0,1)$

$\therefore$ all points are on z-axis

Column space of A = $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q31 a) $\begin{bmatrix} 3a & -5a \\ 3b & -5b \end{bmatrix}$

b) $A \rightarrow$ Zero vector $\Big\}$ ~~det $\neq 0$~~
$B \rightarrow$ Zero vector

$C \rightarrow 3x + y = 0$ {scalar multiple}

$D \rightarrow$ Entire plane

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Exercise # 4.9:

Q1 a) $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix} \xrightarrow[\substack{R_3-3R_1 \\ R_4-4R_1}]{R_2-2R_1} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank(A) = 1, nullity(A) = 3

Let $x_2 = a$, $x_3 = b$, $x_4 = c$

b) $\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ \cancel{-3} & \cancel{6} & \cancel{-1} & \cancel{1} & \cancel{-7} \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2+3R_1} \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix} \xrightarrow[\substack{R_3-R_2 \\ R_1-2R_2}]{\frac{1}{5}R_2} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ rank(A) = 2, nullity(A) = 3

Q3 a) rank(A) = 3, nullity(A) = 0

b) rank(A) + nullity(A) = n → 3 + 0 = 3 proved

c) 3 ~~le~~ pivots and no parameter in solution.

Q5 a) rank(A) = 1, nullity(A) = 2.

b) rank(A) + nullity(A) = n → 1 + 2 = 3 proved

c) 1 pivot and 2 parameter in solution

Q7 a) 4×4 → rank(A) = 4 ; nullity(A) = 0 {pivot in every rows}

b) 3×5 → rank(A) = 3 ; nullity(A) = 2. {max rows}

c) 5×3 → rank(A) = 3 ; nullity(A) = 0 {max cols}

Q9

	a	b	c	d	e	f	g
(i) dimension of row space (A)	3	2	1	2	2	2	2
dimension of col. space (A)	3	2	1	2	2	2	2
dimension of null space (A)	0	1	2	7	7	0	0
dimension of null space (A^T)	0	1	2	3	3	4	4
(ii) Ax = b consistency	Yes	No	Yes	Yes	No	Yes	Yes
(iii) No. of parameters	0	—	2	7	—	0	0

Q11 $A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ -9 & 0 \end{bmatrix} \xrightarrow{R_3+9R_1} \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ 0 & 36 \end{bmatrix} \xrightarrow[\substack{R_1-4R_2 \\ R_3-36R_2}]{\frac{1}{3}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 0 & -9 \\ 4 & 3 & 0 \end{bmatrix} \xrightarrow{R_2-4R_1} \begin{bmatrix} 1 & 0 & -9 \\ 0 & 3 & 36 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 12 \end{bmatrix}$

dim[row(A)] = dim[col(A)] = 2, dim[null(A)] = 0, dim[null(A^T)] = 1

Basis row(A) = {[1 0] [0 1]} Basis null(A) = $\varnothing$

Basis col(A) = $\left\{ \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \right\}$ Basis null(A^T) = $\begin{bmatrix} 9 \\ -12 \\ 1 \end{bmatrix}$

Q13 $A = \begin{bmatrix} 0 & -1 & -4 \\ -1 & 0 & -4 \\ -2 & 3 & 4 \end{bmatrix} \xrightarrow{\substack{R_3 \leftrightarrow R_1 \\ -1/2 R_1 \\ R_2 + R_1}} \begin{bmatrix} 1 & -3/2 & -2 \\ 0 & -3/2 & -6 \\ 0 & -1 & -4 \end{bmatrix} \xrightarrow{\substack{-2/3 R_2 \\ R_1 + 3/2 R_2 \\ R_3 + R_2}} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & -1 & -2 \\ -1 & 0 & 3 \\ -4 & -4 & 4 \end{bmatrix} \xrightarrow{\substack{R_3 \leftrightarrow R_1 \\ -1/4 R_1 \\ R_2 + R_1}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{\substack{R_3 + R_2 \\ R_1 - R_2}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\dim[\text{row}(A)] = \dim[\text{col}(A)] = 2$, $\dim[\text{null}(A)] = 1$, $\dim[\text{null}(A^T)] = 1$

Basis row(A) : $\{[1\ 0\ 4], [0\ 1\ 4]\}$ Basis null(A) = $\begin{bmatrix} -4 \\ -4 \\ 1 \end{bmatrix}$

Basis col(A) = $\left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$ Basis null(A^T) = $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

Q19 $\begin{bmatrix} 0 & 2 & 8 & -7 & 1 & 0 & 0 \\ 2 & -2 & 4 & 0 & 0 & 1 & 0 \\ -3 & 4 & -2 & 5 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 6 & 0 & 5/17 & 19/17 & 7/17 \\ 0 & 1 & 4 & 0 & 5/17 & 21/34 & 7/17 \\ 0 & 0 & 0 & 1 & -1/17 & 3/17 & 2/17 \end{bmatrix}$ Augmented matrix

Row(A) = $\{[1\ 0\ 6\ 0], [0\ 1\ 4\ 0], [0\ 0\ 0\ 1]\}$

Col(A) = $\left\{ \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 5 \end{bmatrix} \right\}$

Nullspace(A) = $\left\{ \begin{bmatrix} -6 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$ Basis for nullspace(A^T) = $\emptyset$

Q21 a) nullity(A) − nullity(A^T) = 1

b) nullity(A) − nullity(A^T) = $n - m$.

Q23 a) rank(T) = 3 b) nullity(T) = 2.

Q25 Rank 1 not possible

Rank 2 possible if $r = 2$ and $s = 1$

Q27 No, both row and column spaces of A must be plane because nullity(A) = 1

Q29 a) 3 → max rows b) 5 → zero vector

c) 3 → max rows d) 3 → max cols.

Q31 a) 3 b) No because there is one free vector outside column space

Q33 $\begin{bmatrix} x & y & z \\ 1 & x & y \end{bmatrix}$

$\begin{bmatrix} x & y \\ 1 & x \end{bmatrix} \to x^2 - y = 0$ $\begin{bmatrix} x & z \\ 1 & y \end{bmatrix} \Rightarrow xy - z = 0$ $\begin{bmatrix} y & z \\ x & y \end{bmatrix} \to y^2 - zx = 0$

$\therefore y = x^2$ and $z = x^3$

If $x = t$ then $y = t^2$ and $z = t^3$.

Q37 a) Over determined because $m > n$.

$\begin{bmatrix} 1 & -1 & b_1 \\ -3 & 1 & b_2 \\ 0 & 1 & b_3 \end{bmatrix} \xrightarrow[\text{equivalent}]{\text{row}} \begin{bmatrix} 1 & 0 & b_1 + b_3 \\ 0 & 1 & b_3 \\ 0 & 0 & 3b_1 + b_2 + 2b_3 \end{bmatrix}$ system is inconsistent if $3b_1 + b_2 + 2b_3 \neq 0$

b) Under determined because $m < n$

$\begin{bmatrix} 1 & -3 & 4 & b_1 \\ -2 & -6 & 8 & b_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}b_1 - \frac{1}{4}b_2 \\ 0 & 1 & -\frac{4}{3} & -\frac{1}{6}b_1 - \frac{1}{12}b_2 \end{bmatrix}$ system is consistent due to infinite solutions of b

c) Underdetermined because $m < n$.

$\begin{bmatrix} 1 & -3 & 0 & b_1 \\ -1 & 1 & 1 & b_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3/2 & -\frac{1}{2}b_1 - \frac{3}{2}b_2 \\ 0 & 1 & -1/2 & -\frac{1}{2}b_1 - \frac{1}{2}b_2 \end{bmatrix}$ system is consistent due to infinite solution of b.

Note: Skipped orthogonal questions.