Linear Algebra

Exercise 1.

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 8 \\ 4x_1 + 7x_2 + 5x_3 = 20 \\ -2x_2 + 2x_3 = 0 \end{cases}$$

Rewrite as: $Ax = b$

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \\ 0 \end{bmatrix}$$

Using Gauss method:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 4 & 7 & 5 & 20 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 8 & 8 \end{array}\right]$$

Back to the system:

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 8 \\ x_2 + 3x_3 = 4 \\ 8x_3 = 8 \end{cases}$$

$x_3 = 1$

$x_2 + 3 = 4 \Rightarrow x_2 = 1$

$2x_1 + 3 + 1 = 8$

$2x_1 = 4$

$x_1 = 1$

Answer:

$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the only solution of the system.

Exercise 2.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Find eigenvalues:

$\det(\lambda I - A) = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \cdot \begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \cdot \left((2-\lambda)(1-\lambda) - 1 \right) - (\lambda - 1) = 0$$

$$(1-\lambda)(2 - 3\lambda + \lambda^2 - 1) + (1-\lambda) = 0$$

$$(1-\lambda)(\lambda^2 - 3\lambda + 2) = 0$$

$\lambda_1 = 1$

$\lambda_2 = 2$

$\lambda_3 = 1$

So, 1 and 2 are eigenvalues of matrix A

Find eigenvectors:

$(\lambda_i I - A)\vartheta = 0$

for $\lambda_1 = 1$:

$(I - A)\vartheta = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{bmatrix} = 0$$

Using gaussian method we find that:

$\vartheta_3 = k_1$

$\vartheta_2 = 2k_1$

$\vartheta_1 = -k_1$

thus, the eigenvector is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} K_1$ (let $k = 1$)

Similarly, find eigenvector for $\lambda_2 = 2$:

$$\vartheta_2 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} K_2$$

Since there are only 2 eigenvectors for a 3×3 matrix, the matrix A is not diagonalizable