Fourier series as rotations

Sanjoy Mahajan ⟨**sanjoy@mit.edu**⟩

This document has two purposes: (1) to show how Fourier series are a change of representation, and how that change is like a rotation of axes, and (2) to illustrate several ideas and skills useful for understanding many mathematical and scientific concepts: discretization, continuity, and representation.

Contents

⟨`http://web.mit.edu/sanjoy/www/teaching/fourier/`⟩ has the source code for this document. It is version 0.82 from `2006-10-25 04:58:13`, changeset `83c2786f778b`, and was produced using free software: the `emacs` editor, the `mercurial` revision-control system, and the `ConTEXt` typesetting system.

1 Why study Fourier series

Much of our knowledge of the world is contained in functions. The voltage produced by a microphone, as a function of time, represents music that can then be encoded digitally and stored on a compact disc. The position of a planet, as a function of time, represents an orbit. Functions are everywhere. They are also extremely complex objects. Any sufficiently complex object is too difficult to comprehend in one glance, for our minds are limited. We therefore study functions from many vantage points and use multiple representations.

Explaining why anyone would care about functions and how to study them (the context). Providing context is a low-key way to integrate curricula.

Transforms, such as Fourier series and Laplace transforms, generate alternative representations of functions. Which representation you use depends on which questions you want to answer. A representation may be ideal for answering one question but lousy for answering another question. Therefore, build a library of representations, of function-analyzing tools, and know when and how to use each tool.

Setting transforms in the context of understanding functions.

The purpose of this unit is to show, using many examples, how Fourier transforms are a change of representation, and how that change is like a rotation of axes. Another purpose is to illustrate several ideas and skills useful for understanding many mathematical and scientific concepts:

- **Discretization**. Discretized (or lumped) systems are simpler to understand than fully general, continuous systems. For example, in analyzing a circuit, we may base it on the inductor, capacitor, or resistor values in the circuit. But the fundamental equations of electromagnetism know nothing of inductors, capacitors, or resistors. There is only Maxwell's equations. But we can approximate by dividing the system into lumps, and convenient lumps turn out to be what we call capacitors, resistors, and inductors. Typically, a discrete system can be analyzed with an ordinary differential equation; and a continuous system requires a partial differential equation (much harder!).

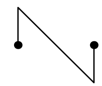
Naming the tools that we use, to give students a placeholder for each idea. These tools would be emphasized in other units, stitching another thread that integrates curricula.

- **Continuity**. Although continuous systems are hard to understand, continuous changes help you understand a system. Here, the idea is that if you make small changes to a system, it should produce only small changes to its behavior.
- **Representation**. If a picture is worth a thousand words, a good representations is worth a thousand pictures! Learn many representations and become fluid in converting among them. Fourier series provide a representation for functions, so this unit is an extended example of building a representation.

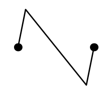
2 Making a representation

Before studying how to *change* representations, first create a function representation to be changed. Imagine all possible functions $g(x)$, where $0 \leq x \leq 1$. Sure, the set of all functions is even larger, but this limited class is already too big. So we study an even more limited class: functions where $g(0) = g(1) = 0$. They are the possible shapes of a unit-length guitar or violin string. The ideas developed in studying these functions work for the shapes and behavior or longer strings, of a membrane (a drum head), or of the output of analog filters (for example, in stereo systems).

The function that we use throughout the example is one tooth of a sawtooth wave, with the endpoints marked with dots. You might feel cheated because this function is not really 0 at its endpoints. Instead it has a discontinuity.



If that bothers you, and it should, imagine a similar function with the infinitely steep edge replaced by a very steep edge – and then make that edge steeper and steeper until it gets infinitely steep. We have just used the principle of continuity to make our life easier. From now on, we use the infinitely steep version of the sawtooth.

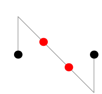


This restricted set of functions, of which the function above is merely one, forms a vast collection. To shrink the collection, sample each function and represent it by its values at several equally spaced values of x. Each function then becomes a point in a finite-dimensional space. So we are discretizing the continuous system (perhaps a unit-length string), thereby making our life easier.

To illustrate the ideas, sample at one point (besides the fixed endpoints). The functions differ only in $g(1/2)$, so each function is represented by one number, $g(1/2)$, or equivalently by a point on an axis. With only one axis, there is no way to rotate the axes, so one dimension is too simple to illustrate interesting transformations.

3 Two-sample representation

Therefore try $p = 2$ samples. To specify one of these functions requires two numbers: $g(1/3)$ and $g(2/3)$. A function becomes a point, or a vector, in a two-dimensional space. Here is the sawtooth function f sampled at two points (other than the easy endpoints). For comparison the original, unsampled sawtooth function is shown lightly in the background.

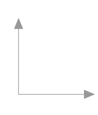


Building up to $p = \infty$ in slow motion. The case $p = 2$ is nice for exposition because one (teacher or student) can easily draw and mentally transform in two dimensions.

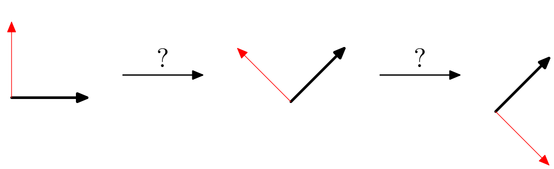
This drawing becomes a point in a two-dimensional space where $g(1/3)$ is the coordinate on the horizontal axis and $g(2/3)$ is the vertical coordinate.

What are $g(1/3)$ and $g(2/3)$?

Since $g(x) = 1 - 2x$, the coordinates are $g(1/3) = 1/3$ and $g(2/3) = -1/3$. Therefore in this two-dimensional space, the sampled function has coordinates $(1/3, -1/3)$. Changing representation means using new coordinate axes. The function is still the same (it is the same point). But it will have different coordinates if the axes change.



In the following change of axes, what transformations take you from the first to the second to the third (final) set of axes?



In this sequence, the first transform is a rotation by 45 degrees counterclockwise, and the second is a sign flip of the rotated second axis. The resulting coordinate system is the two-dimensional Fourier coordinate system. I chose it for several reasons that we (you!) investigate shortly. But first a few questions for you to check your understanding.

In the new coordinate system, what are the coordinates of the sampled sawtooth function?

These coordinates are different from its original coordinates $(1/3, -1/3)$. If the function remains the same, how can its coordinates change?

The new coordinates are $(0, \sqrt{2}/3)$. They are indeed different from the original coordinates $(1/3, -1/3)$. The function is still the same (sampled) sawtooth and therefore the same point in function space. However, its coordinates changed because **the axes (the representation) has changed underneath it.**

Almost any change of representation – change of axes – changes coordinates. The particular change is one of a special class of changes that have useful properties. Let's look at those properties, because we retain them when we sample at more points and, eventually, build the general Fourier representation based on an infinite number of samples.

Foreshadowing the $p = \infty$ properties, in a context that makes them simple.

To see the first property, try the following:

How long is each basis vector (colloquially speaking, each axis)?

There are several ways to answer this question. First, find the coordinates of the new basis vectors *in the old representation*. The old $(1, 0)$ basis vector becomes, after the 45-degree rotation, $(1/\sqrt{2}, 1/\sqrt{2})$ and remains the same after the flip because the flip affects only the rotated $(0, 1)$ axis. The length of $(1/\sqrt{2}, 1/\sqrt{2})$ is one. The old $(0, 1)$ basis vector becomes, after the 45-degree rotation, $(-1/\sqrt{2}, 1/\sqrt{2})$, and becomes, after the flip, $(1/\sqrt{2}, -1/\sqrt{2})$. It too has length one.

Alternatively and more elegantly, look at the sequence of transforms – a rotation and a flip – that produced the new axes. The first transform, a rotation, does not change lengths. The second transform, a flip, also does not change lengths. So the new axes have the same lengths (one) as the original axes.

Students become anxious in learning a derivation about whether they are straying from the one true path. So offering multiple methods reduces their anxiety. It is also a low-key form of curriculum integration, since it connects two approaches by using them to solve one problem.

The next property is illustrated in this question:

What is the angle between the basis vectors?

Neither transform (a rotation or a flip) changes angles, so each basis vector is perpendicular to the other. The new coordinate system, like the old, is therefore **orthogonal**. Since the basis vectors also have unit length, the new coordinate system, like the old, is also an **orthonormal** coordinate system.

Using the same method – here, looking at the consequence of each transform – promotes transfer.

An amazing property, which is a consequence of the previous ones, is the length of the vector representing the function. In the original representation, the vector is $(1/3, -1/3)$.

What is its length?

In the new representation, the vector is $(0, \sqrt{2}/3)$.

What is its length?

Here is the calculation for the original representation:

$$\text{length of } (1/3, -1/3) = \sqrt{(1/3)^2 + (-1/3)^2} = \sqrt{2/9} = \sqrt{2}/3.$$

In the new representation, the length is just the second coordinate, since the first coordinate is zero. So the lengths in the two representations are both $\sqrt{2}/3$.

Explain why **all** lengths are unchanged by our change of axes, not just the length of the particular function we are using (the sawtooth function).

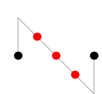
This length preservation may seem trivial, which is why we introduce it first in two dimensions (i.e. with two samples). When we generalize to an infinite number of samples, this seemingly trivial property turns into **Parseval's theorem**.

A further property of an orthonormal change of representation is that any function represented in the old system (here, by the two values $g(1/3)$ and $g(2/3)$) has a representation in the new system. The new system is therefore **complete**: every function can be represented.

This property, like length preservation, may seem trivial with only two samples. However, it becomes nonobvious and important in the general case with an infinite number of samples. There it says that any function, with a few disclaimers, has a Fourier representation: It has a representation as a sum of sines (and cosines). This property is counterintuitive. The great Euler, when faced with choosing whether to give up the principle of superposition for linear differential equations or to accept this completeness property, was willing to give up superposition. He said it was crazy to think that any function could be represented as a sum of sines (and cosines)!

Historical context to show students how hard this idea is and to validate worries that they may have in accepting it.

4 Three-sample representation



These same properties and patterns occur with $p = 3$ samples, although the transformations are harder to visualize because three dimensions are harder to draw than two dimensions (which is why we spent so much time investigating two dimensions, i.e. with $p = 2$.). In this three-sample world, a function can be represented as three numbers: $g(1/4)$, $g(2/4)$, and $g(3/4)$.

Scaffolding: Do three dimensions before the general case because it builds a bridge to a hard, abstract idea (rotations in infinite dimensions).

Why did I not include $g(0)$ and $g(1)$ in the representation?

What are the coordinates of the sampled sawtooth in this representation?

Here are the three new basis vectors, chosen for the moment by fiat, shown using their coordinates in the old representation:

$$v_1 = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}.$$

What is the length of each basis vector v_k?

What are the six angles between each pair of vectors?

As you've shown, the basis vectors are orthogonal (each vector is perpendicular to the others) and normalized (each vector has unit length), so they form an orthonormal coordinate system.

We need the representation (the coordinates) of the sawtooth function in this coordinate system, given its coordinates in the usual representation. These are its usual coordinates: $(1/2, 0, -1/2)$.

What are its coordinates in the new representation?

In an orthonormal coordinate system, it is easy to find the coordinates of a vector. Take the vector (i.e. the point or the function) and find its component along each new axis using the dot product:

$$k\text{th component} = g \cdot (\text{basis vector } k).$$

The dot product tells you the overlap between two functions. Here, it tells you the overlap between the candidate function g and the kth basis vector – or, what is the same object, the kth basis function. The sampled sawtooth sits on the second axis in the new representation, so its first and third coordinates are 0 in the new representation. Here is the second coordinate:

$$\text{2nd coordinate} = \underbrace{\begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}}_{g} \cdot \underbrace{\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}}_{v_k} = \frac{1}{\sqrt{2}}.$$

So the sampled sawtooth, in the new representation, is represented by these coordinates

$$g = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}.$$

The three components are the three dot products, each with respect to one basis vector. Each dot product measures how close the represented function g is to each basis vector, a.k.a. basis function. Remember that a vector, including a basis vector, is a function: Points, or vectors, are how we represent functions. So the three dot products give three pieces of information: how close g is to each basis function. Those three pieces of information locate g in function space by a process akin to triangulation. Earthquakes are located by measuring their distance from several (typically, three) seismographs, and then finding the one point, the source, that is has the correct distance from each seismograph. In function space, the three proximities (the three dot products) uniquely determine one function.

What are the lengths of g in the old and new representations?

As you've just shown, the length of the sawtooth function is the same $(1/\sqrt{2})$ in the old and new representations.

5 Many-sample representation

Now return to the original space of functions, before they got sampled at only a few points, by making p large. But before doing that, here's the recipe that I used to choose the basis vectors. Take the function $\sin(k\pi x)$, where $k = 1 \ldots p$, and sample it at p equally spaced points:

$$x = \frac{1}{p+1}, \frac{2}{p+1}, \frac{3}{p+1}, \ldots, \frac{p}{p+1}.$$

The values of g at these p samples produce the *unnormalized* basis vector. Then normalize it (make it have unit length) to get the vectors that I used.

Check that the v_k vectors (for $p = 3$) result using the preceding recipe.

To find the kth new coordinate, use the same recipe as when $p = 3$: Take the dot product of the normalized basis vector (or point or basis function) with the example function g.

As p goes to infinity, nothing essential changes. However, one issue needs to be dealt with carefully, which is the normalization.

Show that the new, *unnormalized* vectors produced by the above recipe have length $\sqrt{(p+1)/2}$. [Difficult unless you are fluent with series!]

As p goes to ∞, the unnormalized length goes to ∞. Normalizing the vector means dividing by the unnormalized length, which makes every component zero! So our results would be boring, often reducing to the grand conclusion that $0 = 0$.

This length problem is solved by slightly adjusting the definition of the dot product in the usual representation.

How could fixing the dot product solve a problem with length? In other words, what do length and dot product have to do with each other?

For finite p, the dot product means doing componentwise multiplication and then summing the results. That definition is fine for finite p. However, with an infinite number of terms in the sum, which happens when $p = \infty$, it is natural to replace the sum by an integral. We therefore define the dot product of two functions $g(x)$ and $h(x)$ by

$$g \cdot h = \int_0^1 g(x)h(x)\,dx.$$

Especially in quantum chemistry, such an integral is often called the *overlap integral*. The term overlap arises because functions that vary together have a positive dot product; functions that vary opposite to each other have a negative dot product; and functions that vary randomly with respect (no overlap) to each other have zero dot product.

Making a link to the chemistry curriculum (to quantum chemistry in particular).

The length of an unnormalized basis function u_k is $\sqrt{u_k \cdot u_k}$, and the dot product inside the square root is $\int_0^1 u_k(x)u_k(x)\,dx$, which does not blow up. In fact, with $u_k = \sin(k\pi x)$, the integral is $1/2$.

Show that

$$\int_0^1 \sin^2(k\pi x)\,dx = \frac{1}{2}.$$

Using this definition of dot product, the coordinates of the sawtooth g become

$$f_k \equiv k\text{th component} = \int_0^1 g(x)\,\sqrt{2}\sin(k\pi x)\,dx,$$

where the $\sqrt{2}$ comes from normalizing the sin functions to have unit length. These f_k are the **Fourier coefficients** of g.

Show that $f_k = 0$ for odd k and $2\sqrt{2}/k\pi$ for even k.

---

6 Parseval's theorem

Parseval's theorem relates the square of the original function and the squares of the Fourier coefficients. Go back to the two- or three-dimensional analyses to see its origin. In those cases, the old and new vector have the same length – because the change of representation is just a rotation perhaps with a flip. Let's look again at the squared lengths before and after, but with $p = \infty$. We use the squared length to minimize the square-root signs. The squared length of a vector is the sum of squares of the components. In the before (usual) representation, the sum of squares is $\int_0^1 g(x)^2\,dx$. In the after (Fourier) representation, the sum of squares is the squared sum of the Fourier coefficients: $\sum_1^\infty (a_k)^2$. The equality of the sum and integral is Parseval's theorem:

$$\sum f_k^2 = \int_0^1 g(x)^2\,dx.$$

Given the geometric meaning (length) attached to the integral and sum, why are their values equal?

The change from the old to the new representation is a rotation (perhaps with a flip). Parseval's theorem restates, in an infinite-dimensional space, the geometric fact that **rotations do not change lengths**. That requirement is 99% of the definition of a rotation.

Let's check that the theorem works. As they used to say in arms control negotiations (before that idea was replaced by 'terrorize the world into arms proliferation'): **trust but verify**. To verify the Fourier-representation half of the theorem, we need to find the Fourier coefficients, which you should already have done in a preceding question and which I calculate now. They are:

$$f_k = g \cdot v_k = \int_0^1 g(x) \cdot \underbrace{\sqrt{2}\ \sin(k\pi x)}_{v_k}\,dx,$$

where $g(x) = 1 - 2x$. We have to compute one integral for each k. But some integrals vanish. For example, look at the integral for f_1. The function $g(x)$ is antisymmetric about $x = 1/2$, whereas the Fourier function $\sqrt{2}\sin \pi x$ is symmetric about $x = 1/2$. The integrand, as the product of these two functions, is antisymmetric about $x = 1/2$, which is the midpoint of the integration range. So the integral vanishes. A similar argument works for any odd k. So those f_k's are zero. *Moral:* The Fourier functions that are used have some symmetries from the original function. If the original function has reflection symmetry, then only those Fourier functions with reflection symmetry will be used.

When k is even, we have integrals to do. The integral splits into two parts, one part from each term in $1 - 2x$:

$$\int_0^1 \sqrt{2}\,\sin(k\pi x)\,dx - 2\int_0^1 x\sqrt{2}\,\sin(k\pi x)\,dx.$$

When k is even, the first integral is zero, because $\sin k\pi x$ is antisymmetric about $x = 1/2$. So the only integral remaining is

$$-2\int_0^1 x\sqrt{2}\,\sin(k\pi x)\,dx,$$

and only for even k. Integrating by parts (or differentiating under the integral sign with respect to k) gives $2\sqrt{2}/k\pi$. That piece is the only contributor to the Fourier coefficient, so

$$f_k = \begin{cases} 2\sqrt{2}/k\pi, & k \text{ even;} \\ 0, & k \text{ odd.} \end{cases}.$$

Now we can do the Parseval sum. It is

$$\sum_{k \text{ even}} f_k^2 = \sum_{k \text{ even}} \frac{8}{k^2\pi^2}.$$

Do the sum!

To do the sum, a useful result is

$$\sum n^{-2} = \frac{\pi^2}{6},$$

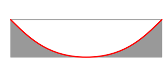
a famous result derived by Euler. The sum for f_k^2 uses only the even terms, and

$$\sum_{n \text{ even}} n^{-2} = \frac{\pi^2}{24}.$$

The factor of 8 in the Parseval sum, combined with this result, produces $\sum_{k \text{ even}} f_k^2 = 1/3$.

Let's hope that the other half of Parseval's theorem gives the same value. To do that half, square $g(x)$ and integrate:

$$|\text{length of } g \text{ in the usual rep}|^2 = \int_0^1 (1-2x)^2\,dx.$$



You can do this integral graphically. The function $(1 - 2x)^2$ is a parabola. A parabola occupies one-third of its bounding rectangle (which is the form in which Archimedes knew this result that we would usually do by integration), just as a triangle occupies one-half of its bounding rectangle. The bounding rectangle has unit area so the squared length of g is $1/3$. Which is the squared length of g as measured in the Fourier representation!

7 Why those functions?

Why did we use those particular functions? Nothing in this discussion seemed to depend on the functions being sines. So why not use other basis functions? And **you can**! Another set of basis functions would produce yet another function representation. For example, the Legendre polynomials are another set of basis vectors. So to chose which one to use, return to first principles: Use the representation that is most useful for answering the questions you have. Sines (and cosines) are a useful choice because they arise naturally in the equations that describe the motion of springs, waves, membranes, strings, LRC circuits, and much else. They arise because Newton's second law has a second time derivative in it; and because the net force on a piece of a stretched string depends on the local curvature, which incorporates a second space derivative. In either case the equation has the eigenvalue form

$$\text{second derivative of } f = \text{constant} \times f.$$

Show that the net force on a piece of a stretched string is proportional to a second derivative.

Sines and cosines satisfy this eigenvalue form. The benefit of representing functions in terms of sines and cosines, for these physical systems, is that the sine and cosine functions behave independently from one another. Each component does its own motion, at its own frequency, independent of what the others are doing. In terms of linear algebra, the Fourier basis functions diagonalize the second derivative operator. That is a large topic, and you will see more of it when you analyze many physical systems, such as the energy levels in hydrogen or the waves on a circular pond.

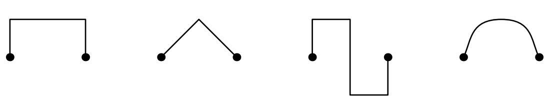
8 Problems

Add the explanation for $p = 4$ samples, using the same sawtooth g.

Near transfer of discretization

Redo the analysis, perhaps including $p = 4$, using another function for g. Here are several candidates (in order): the upper half of a square wave, a triangle wave, a symmetric square wave, and a parabolic hump.

Near transfer of all the ideas



Redo the analysis for functions over the domain $-\infty$ to ∞, instead of the limited domain 0 to 1 used here. This generalization results in **Fourier transforms**.

Medium transfer of continuity: The problem requires taking this analysis, extending it to, say, $[0, 2]$, then to $[0, n]$, and finally to $[-\infty, \infty]$.

Acknowledgements

Many thanks to Kate MacLeod, Hannah Dvorak, and Roian Egnor who, many years ago, asked me wonderful questions about Fourier series; to Julia Khodor for valuable suggestions for and improvements to this document; and to the talented and artistic authors of the TEX and ConTEXt typesetting systems.