**STAT 515**
**Homework #2 WITH SOLUTIONS**

1. Consider a Markov chain on $\Omega = \{1, 2, 3, 4, 5, 6\}$ specified by the transition probability matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   (a) What are the (communicating) classes of this Markov chain? Is the Markov chain irreducible ?

   **Solution:** The classes are $\{1, 2\}$, $\{3\}$, $\{4\}$, $\{5\}$, and $\{6\}$. In other words, states 3 through 6 do not communicate with any other states. Because there is more than one class, the chain is not irreducible.

   (b) Which states are transient and which are recurrent? Justify your answers.

   **Solution:** Only $\{1, 2\}$, $\{3\}$, and $\{6\}$ are recurrent. The other classes, $\{4\}$ and $\{5\}$, are transient. (Recall that recurrence is a class property, which means that we may describe entire classes, rather than merely states, as being recurrent or not.)

   (c) What is the period of each state of this Markov chain? Is the Markov chain aperiodic ?

   **Solution:** Whenever it is possible to stay in state $i$, given that you begin in state $i$, the period of the state is 1. A sufficient condition for this is that $P_{ii} > 0$, so a look at the diagonal of $P$ shows that all states other than state 5 have period 1. But state 5 is a bit odd in that $n = 0$ is the *only* value for which $P_{55}^n > 0$. Therefore, we never return to state 5 after starting there, and so the definition of the period of state 5 is somewhat arbitrary. One could make a case for calling the period zero or infinity; I prefer zero since at least zero is a number.

   Since not all states have period 1, this Markov chain is not aperiodic.

   (d) Let $X_0$ be the initial state with distribution $\pi_0 = (0, \frac{1}{4}, \frac{3}{4}, 0, 0, 0)^\top$ corresponding to the probability of being in states $1, 2, 3, 4, 5, 6$ respectively. Let $X_0, X_1, X_2, \ldots$ be the Markov chain constructed using $P$ above. What is $\mathrm{E}(X_1)$?

   **Solution:** We obtain $\pi_0^\top P = (1/8, 1/8, 3/4, 0, 0, 0)$. This is the vector of probabilities with which the random variable $X_1$ takes the values $(1, 2, 3, 4, 5, 6)$. Therefore,

$$E(X_1) = 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + 3 \times \frac{3}{4} = \frac{21}{8}.$$

   (e) What is $\mathrm{Var}(X_1)$?

   **Solution:** From the previous part, we obtain

$$E(X_1^2) = 1 \times \frac{1}{8} + 4 \times \frac{1}{8} + 9 \times \frac{3}{4} = \frac{59}{8}.$$

   Therefore, $\mathrm{Var}\, X_1 = 59/8 - (21/8)^2 = 31/64$.

   (f) What is $\mathrm{E}(X_3)$?

   **Solution:** The variable $X_3$ puts probabilities $\pi_0^\top P^3 = (13/128, 19/128, 3/4, 0, 0, 0)$ on $(1, 2, 3, 4, 5, 6)$. Therefore,

$$E(X_3) = 1 \times \frac{13}{128} + 2 \times \frac{19}{128} + 3 \times \frac{3}{4} = \frac{339}{128}.$$

2. Suppose that the probability of rain today depends on weather conditions from the previous three days. If it has rained for the past three days, then it will rain today with probability 0.7; if it did not rain for any of the past three days, then it will rain today with probability 0.1; if it rained each of the past two days but not three days ago, it will rain with probability 0.8; and, in any other case, the weather today will match yesterday's weather with probability 0.6.

(a) Describe this process using a Markov chain, i.e., define a state space and the corresponding transition probability matrix for the process.

**Solution:** We will need each "state" of the Markov chain to include the weather for each of the past three days. With this in mind, we define the following:

State 0: no rain, no rain, no rain
State 1: no rain, no rain, rain
State 2: no rain, rain, no rain
State 3: no rain, rain, rain
State 4: rain, no rain, no rain
State 5: rain, no rain, rain
State 6: rain, rain, no rain
State 7: rain, rain, rain

Incorporating all of the information given above, the transition matrix is

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.7 \end{bmatrix}$$

(b) Suppose you know that it rained on days one, two, and three. What is the probability that it will rain on day seven? (You are welcome to use a computer for this, but please explain what you did.)

**Solution:** The answer to this question will be found in the bottom row of $P^4$. This is because we start in state 7 (rain, rain, rain) at time $t = 3$ and take four steps of the Markov chain to get to time $t = 7$. Thus, the bottom row (corresponding to beginning in state 7) gives the probability of all eight states. The even states are "no rain" and the odd states are "rain". Here is the R code to calculate the answer:

```
> r <- rep(0,8)
> P <- matrix(c(9,1,r,4,6,r,6,4,r,2,8,6,4,r,4,6,r,6,4,r,3,7),8,8,byrow=T)/10
> P4 <- P %*% P %*% P %*% P
> sum(P4[8, c(2, 4, 6, 8)])
[1] 0.5305
```

3. Prove that if state $i$ is recurrent and state $i$ does not communicate with state $j$, then $P_{ij} = 0$.

*This implies that once a process enters a recurrent class of states, it can never leave that class. For this reason, a recurrent class is often referred to as a* closed *class.*

**Solution:** Assume $P_{ij} > 0$. Then it must be true that $P^n_{ji} = 0$ for all $n \geq 1$, since otherwise $i$ and $j$ would communicate. But this means that if the chain goes from state $i$ to state $j$, it will never return to state $i$. In other words,

$$P(\text{return to } i \mid \text{start in } i) < 1 - P_{ij} < 1.$$

therefore, $i$ is transient by definition, which is a contradiction. Thus, $P_{ij}$ cannot be larger than zero.

4. Computer problem: Use the Markov chain described in Problem 1 and the initial distribution in 1d.

(a) Simulate a realization of the random variable $X_3$. Repeat this 1000 times—i.e., generate 1000 instances of $X_3$—and calculate the average. This is your estimate of $E(X_3)$. (Ideally, you should report some sort of confidence interval, but this is not required.) Compare your estimate with your answer from Problem 1. Since this is a short program, include a printout of your code with your homework.

**Solution:**

```
> X3 <- rep(0, 1000) # This is where the realizations will be stored
> pi0 <- c(0, 1/4, 3/4, 0, 0, 0)
> P <- matrix(c(.25, .5, 0, 0, 1, 0, .75, 0.5, 0, 0, 0, 0, 0, 0, 1, 1/3, 0, 0,
+              0, 0, 0, 2/3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1), 6, 6)
> for(i in 1:1000) {
+   currentState <- sample(6, 1, prob=pi0)
+   for(j in 1:3) { # Sample according to the correct row in P
+     currentState <- sample(6, 1, prob=P[currentState,])
+   }
+   X3[i] <- currentState
+ }
> # Here is a confidence interval for the true expectation:
> mean(X3) + c(-1, 1)* sqrt(var(X3)/1000)
[1] 2.609898 2.652102
```

In this case, we happen to know that the true mean is $21/8 = 2.625$.

(b) Simulate the Markov chain according to Problem 1 and run it for 100,000 steps. Now calculate the proportion of times the Markov chain was in the states 1,2,3,4,5,6 respectively. Simulate two more realizations, each also of length 100,000, and again record the proportion of times the Markov chain was in the states 1,2,3,4,5,6 respectively. You only have to report the proportions for each of the three realizations (do not print out your Markov chains or your code for this!)

**Solution:** Here is one of the three realizations:

```
> X <- rep(0, 100000) # This is where the realizations will be stored
> pi0 <- c(0, 1/4, 3/4, 0, 0, 0)
> P <- matrix(c(.25, .5, 0, 0, 1, 0, .75, 0.5, 0, 0, 0, 0, 0, 0, 1, 1/3, 0, 0,
+              0, 0, 0, 2/3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1), 6, 6)
> currentState <- sample(6, 1, prob=pi0)
> for(j in 1:100000) { # Sample according to the correct row in P
+   X[j] <- currentState <- sample(6, 1, prob=P[currentState,])
+ }
> # Now summarize the states visited:
> table(X)
X
     3
100000
```

Sometimes, only states 1 and 2 are visited (in roughly a 4 to 6 ratio), and usually, only state 3 is visited. The latter happens three times more frequently than the former, on average.

(c) Use your answer to part (b) to find one or more vectors $\boldsymbol{v}$ such that $\boldsymbol{v}^\top P = \boldsymbol{v}^\top$. Explain your reasoning.

**Solution:** The equation $\boldsymbol{v}^\top P = \boldsymbol{v}^\top$ suggests that $\boldsymbol{v}$ is a "fixed point," i.e., a vector of probabilities that does not change after a step of the Markov chain. Assuming that the proportions of the six states converge to something after a long time, it stands to reason that this limiting vector of probabilities would provide a possible $\boldsymbol{v}$ vector. Thus, based on part (b), we could take $\boldsymbol{v}$ to be either $(0.4, 0.6, 0, 0, 0, 0)^\top$ or $(0, 0, 1, 0, 0, 0)^\top$; it is easy to check that each of these vectors satisfies $\boldsymbol{v}^\top P = \boldsymbol{v}^\top$.