Midterm STAT 515, Penn State Statistics
March 6, 2015.

Name:

(1) Show all your work for full credit.
(2) When possible, draw a box around your final answer to each question.
(3) You may leave your answer unsimplified, e.g. $0.7^4 \frac{18!}{4!8!}, 14e^{-23}, \sum_{i=1}^{n} \frac{i}{3^i}$.

1. Consider a Markov chain on a finite state space $\Omega = \{1, 2, \ldots, n\}$. Now assume that the chain has a symmetric transition probability matrix P and that $P_{ij} > 0$ for all $i, j \in \Omega$. Hint: read all parts of this problem before working on it.

   (a) Show that this Markov chain has a unique stationary distribution π that is (discrete) uniform on Ω. [3pts]

   (b) Assume the Markov chain has been running for infinitely long. Is it time reversible? Justify your answers, verifying all necessary properties. [3pts]

   (c) Is π also the limiting distribution of the chain? Why? [2pts]

2. A professor possesses 2 umbrellas, which she employs in going from her home to office and vice-versa (office to home). If she is at home at the beginning of a day and it is raining, then she will take an umbrella with her to the office, provided there is one to be taken. If it is not raining, then she never takes an umbrella. She does the same thing when she leaves the office. Assume that the probability of rain at the beginning of the day (also at the end of the day), p, is independent of rain on other days.

   (a) Define a Markov chain model for the number of umbrellas she has at her *current* location. E.g. if she has 0 umbrellas now, at the next location she will definitely have 2 umbrellas (since both are at the next location). [3pts]

   (b) In the long run, what fraction of the time will she get wet? (You may assume that the chain is irreducible, positive recurrent and apeiodic.) [2pts]

3. A car mechanic is only able to work on one car at a time. Suppose the amount of time he takes to work on each car is an exponential random variable with mean 30 minutes, and that these times are independent of each other.

   (a) Suppose there are always more cars to work on than the mechanic has time, and that all customers bring in their cars at 9am. What is the expected number of cars the mechanic will have completed work on from 9am to 10am ? [2pts]

   (b) Assume the same set up as in part (a), except now you know that the mechanic completed work on exactly 3 cars between 9am and 10am. What is the expected value of the time he finished work on the first car ? [2pts]

   (c) Now suppose he schedules two appointments, car A at 9am and car B at 9:30am. Assume both cars arrive on time for their appointments. Find the expected amount of time that car B spends at the mechanic's shop. [3pts]

4. Recall that a stochastic process $\{X(t), t \geq 0\}$ is said to be a compound Poisson process if it can be represented as $X(t) = \sum_{i=1}^{N(t)} Y_i$, $t \geq 0$ where $\{N(t), t \geq 0\}$ is a Poisson process with rate λ and $\{Y_i, i \geq 1\}$ is a family of independent and identically distributed random variables that is also independent of $\{N(t), t \geq 0\}$. Let $\mu_Y = E(Y_i)$ and $\sigma_Y^2 =$Var(Y_i) for $i = 1, 2, \ldots$

   (a) Find Var$(X(t))$. [2pts]

   (b) Find Cov$(X(t), X(t+s))$, where $t \geq 0$, $s > 0$. Hint: it may be useful to use the fact that this process has independent increments. [3pts]