Continuous-time Markov chains

We consider a continuous time, discrete state space Markov chain.

Chain stays in each state for a random time.

The random time is a continuous r.v. w/ a distr. that may depend on the state.

State of the chain at time t is $X(t)$ so M. chain is $\{X(t), t \in [0,\infty)\}$

<u>Defn.</u>: Let $\{X(t), t \geq 0\}$ be a collection of discrete r.v.'s, with $X(t) \in \Omega$ $\forall t$, and that evolves in time as follows:

(a) If the current state is i, the time until the state changes "holding time" has an exponential distr. ~~w/ parameter $\lambda(i)$~~.
If Expec. value of expon. r.v. is $\frac{1}{v_i}$, this is called the rate for i^{th} state.

(b) When the chain leaves state i, a new state $j \neq i$ is chosen according to transition probabilities of a discrete time M.C.

otherwise defn. does not make sense

Then $\{X(t), t \geq 0\}$ is a continuous-time Markov chain.

Intuition: M.chain is composed of exponential r.v.'s for the 'holding times' and a discrete time M.C., the 'jump chain', for the transitions.

Markov property: conditioned on current state and time, where and when the M.C. jumps next is indep. of the complete history of the chain.

Why are holding times exponential?

The exponential distr. of holding times follows from the Markovian property:

We want: $P\left(X(s+t)=j \mid X(s)=i,\ X(u)=x(u),\ 0 \le u < s\right)$

$$= P(X(s+t)=j \mid X(s)=i)$$

Furthermore, we impose time homogeneity so above only depends on $t-s$, so

$$P(X(s+t)=j \mid X(s)=i) = P(X(t)=j \mid X(0)=i) = p_{ij}(t).$$

Let T_i be time when chain leaves state i, given that it has been in state i at time 0.

$$P(T_i > t) = P\left(X(u)=i,\ 0 < u \le t \mid X(0)=i\right)$$

Why Exponential waiting / "holding" times?

Now, $P(T_i > s+t \mid T_i > s) = P(X(u)=i,\ s < u \le s+t \mid X(u)=i, u \in [0,s])$

Markov property $= P(X(u)=i,\ u \in (s, s+t] \mid X(s)=i)$

Time homogeneity $= P(X(u)=i,\ u \in (0,t] \mid X(0)=i)$

$= P(T_i > t)$

Memoryless property!

Hence T_i must be exponential r.v. (from before).

For a time-homogeneous continuous-time discrete state M.chain, for each $t \ge 0$, we have t.p.m. $P(t)$ w/ entries $p_{ij}(t)$, $i,j \in \Omega$.

w/ state space Ω

Transition prob. matrix, $P(t)$ for a cnts-time M.C., has the following properties:

(a) $P(0) = I$

(b) $\sum_{j\in\Omega} P_{ij}(t) = 1 \quad \forall i \in \Omega$ and $t \geq 0$

(c) $P(s+t) = P(s)\,P(t)$ (Chapman-Kolmogorov Eqns.)

(a) and (b) follow directly from defn.

(c) Consider $p_{ij}(s+t)$, (i,j)th element of $P(s+t)$.
(as before, condition on intermediate step)

$$p_{ij}(s+t) = \sum_{k\in\Omega} P\left(X(s+t)=j,\ X(t)=k \mid X(0)=i\right)$$

$$= \sum_{k\in\Omega} P\left(X(s+t)=j \mid X(t)=k, X(0)=i\right) P\left(X(t)=k \mid X(0)=i\right)$$

$$= \sum_{k\in\Omega} P\left(X(s+t)=j \mid X(t)=k\right) P\left(X(t)=k \mid X(0)=i\right)$$

$$= \sum_{k\in\Omega} P_{ik}(t)\,P_{kj}(s)$$

$$= (i,j)\text{th entry in matrix } P(s)\,P(t)$$

In general, $P(t)$ is difficult to compute.

Ref. ? Olifton

Examples of Cntns-time M.C.s

Poisson process: State space $\Omega = \{0, 1, 2, \dots\}$,
holding times: rate $v_i = \lambda$ (constant) $\forall i$.
T.p.m. for jump chain: $P_{i,i+1} = 1 \quad \forall i \in \Omega$.

On-off system: Stays OFF for a time that is $Exp(\lambda_0)$ and ON for a time that is $Exp(\lambda_1)$.
Only possible jumps are from 0 to 1 and 1 to 0.
T.p.m. for jump chain: $P_{10} = P_{01} = 1 \quad P_{00} = P_{11} = 0$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Holding times: rates $v_0 = 1/\lambda_0$, $v_1 = 1/\lambda_1$

Cntns-time r-walk: Discr. time r. walk: $X_n = X_{n-1} + Z_n$
where $Z_n = \begin{cases} 1 & \text{w/ prob } p \\ -1 & \text{" " } 1-p \end{cases}$

Now let steps be taken at random times, each w/ mean $1/\lambda$.
So, this process is simply $X_{N(t)}(t)$ for $t \geq 0$
where $N(t)$ is a Poisson process w/ expectation λ.
Useful model for diffusion of small particles in a fluid.

The Generator for a cntns-time MC.

Let jump chain have t.prob. p_{ij} for $i \neq j$ and consider the chain in a state i.

Holding time is $Exp(\lambda(i))$, when it leaves, chain jumps to state j w/ prob. p_{ij}.

Consider chain only when in state i (disregard everything else), we can view jumps from i as a Poisson process w/ rate $\lambda(i)$

For any $j \neq i$, jumps from i to j is a <u>thinned</u> Poisson process w/ rate $\lambda(i) p_{ij}$.

For any i, j, can define transition rate between i and j as $\gamma_{ij} = \lambda(i) p_{ij}$.

In addition, let $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$

Now, generator G is a matrix w/ i,j th element $= \gamma_{ij}$.

γ_{ii}'s were chosen so G has row sums equal to 0.

G has the following properties:

$\gamma_{ii} \le 0 \quad \forall i \in \Omega$ non-positive diagonal

$\gamma_{ij} \geq 0 \quad \forall i,j \in \Omega,\ i \neq j$ non-negative off-diagonals

$\sum_j \gamma_{ij} = 0 \quad \forall i$ row sums $= 0$.

Generator completely specifies Markov chain since

holding time params, $\lambda(i) = -\gamma_{ii}$, $i \in \Omega$ $\left(\because \lambda(i) = \sum_{j \neq i} \gamma_{ij}\right)$

jump prob. $p_{ij} = \dfrac{-\gamma_{ij}}{\gamma_{ii}}$, $j \neq i$ $\left(\because \gamma_{ij} = \lambda(i) p_{ij}\right)$

$p_{ii} = 0 \quad \forall i \in \Omega$

E.g. On-off system. Recall :

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda_0"] & 1 \arrow[l, bend left, "\lambda_1"] \end{tikzcd}$$

[But, λ_0, λ_1 are rates, not transition probs.]

Jump chain t.p.m. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

So, $\gamma_{01} = \lambda_0 \cdot 1 \qquad \gamma_{10} = \lambda_1 \cdot 1$

$\gamma_{00} = - \sum_{j \neq 0} \gamma_{0j} = -\lambda_0$

$\gamma_{11} = - \sum_{j \neq 0} \gamma_{1j} = -\lambda_1$

So, $G = \begin{bmatrix} -\lambda_0 & +\lambda_0 \\ +\lambda_1 & -\lambda_1 \end{bmatrix}$

E.g. Cntns-time M.C. on $\Omega = \{1,2,3\}$ w/

$$G = \begin{bmatrix} -6 & 2 & 4 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{bmatrix}$$

Suppose chain is in state $\underline{1}$.

E (time until it leaves the state) : $\lambda(1) = -\gamma_{11} = 6$, so

E (holding time) $= \frac{1}{6}$.

Prob (jump to state 2) $= -\frac{\gamma_{12}}{\gamma_{11}}$, $\frac{-2}{-6} = \frac{1}{3}$.

G plays the role of t.p.m. in discrete case: contains all info.

How does G relate to $P(t)$?

Transition matrix $P(t)$ and generator G satisfy

<u>backward eqns</u>: $P'(t) = G P(t), \quad t \geq 0$

<u>forward eqns</u>: $P'(t) = P(t) G \quad t \geq 0$

where $P'(t)$ is matrix of derivatives, $p_{ij}'(t)$.

Elementwise:

$$p_{ij}'(t) = \sum_{k \in \Omega} \gamma_{ik} \, p_{kj}(t) \qquad i,j \in \Omega, \; t \geq 0$$

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Intuitive argument:

Consider $p_{ij}(t+h) = P(X(t+h) = j \mid X_0 = i)$

Chapman-Kolmogorov gives: $p_{ij}(t+h) = \sum_{k \in \Omega} p_{ik}(t) \, p_{kj}(h)$

As $h \to 0$ and letting T_j denote holding time in state j.

$$p_{jj}(h) = P(X(h) = j \mid X(0) = j) \approx P(T_j > h) = e^{-\lambda(j) h}$$

$$\underset{\text{(Taylor series)}}{\approx} 1 - \lambda(j) h = 1 + \gamma_{jj} h$$

If chain is in state j at times 0 and h, for small h no event occurred in $(0, h)$.

Similarly, if there is a jump in $(0, h)$, Pr(more than 1 jump) ≈ 0.

Hence, $p_{kj}(h) \simeq \gamma_{kj} h$, $k \neq j$ (from 1st principles defn. of P.P.)

This gives $$p_{ij}(t+h) \approx p_{ij}(t)(1+\gamma_{jj}h) + \sum_{k \neq j} p_{ik}(t)\gamma_{kj}h$$
$$= p_{ij}(t) + h \sum_{k \in \Omega} p_{ik}(t)\gamma_{kj}$$

Hence, $$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \sum_{k \in \Omega} p_{ik}(t)\gamma_{kj}$$

$\lim_{h \to 0}$ results in forward eqns.

Similar argument for backward eqns.

Note: forward eqns usually easier to solve but may <u>not always exist.</u>

↑ ignored in above proof

Also, since $P(0) = I$, from these eqns, we obtain
$P'(0) = G\,P(0)$
and hence $G = P'(0)$, a way to obtain the generator from $P(t)$

Stationary distr. and limit distr. for cntns time M.C.s

Discrete time: stationary distr. $\underline{\pi}$ is soln. to

$$\pi = \pi P$$

Continuous-time: A prob. distr. π s.t.

$$\pi P(t) = \pi \quad \text{for all } t \geq 0.$$

is called a stationary distr. of the chain.

π_j is prop. of time spent in state j in the long run.

Problem: $P(t)$ is usually difficult to find.

Instead, differentiating on both sides w.r.t. t,

$$\frac{d}{dt} \underline{\pi} P(t) = \frac{d}{dt} \underline{\pi}$$

$$\text{So, } \quad \pi P'(t) = \underline{0} \quad \forall t \geq 0$$

But $P'(0) = G$, so stationary distr. satisfies

$$\therefore \quad \pi G = \underline{0}$$

(necessary & sufficient, see K. Lange "Applied Prob." p.155)

Elementwise: $\sum_{i \in \Omega} \gamma_{ij} \pi_i = 0 , \quad j \in \Omega$

and $\sum_{i \in \Omega} \pi_i = 1$

E.g. stationary distr. of ON-OFF system.

$$\underset{\sim}{\pi}\, G = \underset{\sim}{0} \quad \text{is}$$

$$(\pi_0 \quad \pi_1) \begin{bmatrix} -\lambda_0 & +\lambda_0 \\ +\lambda_1 & -\lambda_1 \end{bmatrix} = [0 \; 0]$$

1st eqn: $\pi_0 \lambda_0 - \pi_1 \lambda_1 = 0$

$$\pi_1 = \frac{\lambda_0}{\lambda_1} \pi_0$$

Also, $\pi_0 + \pi_1 = 1$

So, $\pi_0 \left(1 + \frac{\lambda_0}{\lambda_1}\right) = 1$

Hence $\pi_0 = \dfrac{\lambda_1}{\lambda_0 + \lambda_1} = \dfrac{1/\lambda_0}{1/\lambda_1 + 1/\lambda_0}$

$\pi_1 = \dfrac{\lambda_1}{\lambda_0 + \lambda_1} = \dfrac{1/\lambda_1}{1/\lambda_0 + 1/\lambda_1}$

Intuition: $E(\text{time in } 0) = 1/\lambda_0$

$E(\text{time in } 1) = 1/\lambda_1$

Note: jump chain $\underset{\sim}{\pi}^{(J)} = (1/2, 1/2)$ does not take holding times into account.

Existence of stationary & limiting distr. for continuous-time Markov chains

Thm: If a cntns-time M.C. $\{X_t, t \ge 0\}$ is irreducible and has a stationary distr. π, then

$$\lim_{t\to\infty} p_{ij}(t) = \pi_j \quad \forall i,j \in \Omega$$ where Ω is state space of

Furthermore, if g is a real-valued function, $g: \Omega \to \mathbb{R}$ s.t. $E_\pi |g(x)| < \infty$, then as $t \to \infty$

$$\frac{1}{t}\int_0^t g(X_s)\,ds \to E_\pi\{g(x)\}$$ (cf. R. Durrett, Essentials of Stoch. Proc.)

Thm: π is a stationary distr. iff $\pi G = 0$ where G is the generator for the cntns-time M.C.

Irreducibility: If $\{X_t, t\ge 0\}$ is irreducible, it means for any $i,j \in \Omega$ it is possible to get (w/ positive prob.) from i to j in a finite # of jumps.

Do not have to worry about aperiodicity.

Establishing recurrence/positive recurrence is usually complicated for cntns-time M.C.s

Refs for contins-time M.C.:

Durrett: Essentials of Stoch. Proc.
Guttorp: Stoch. Modeling of Sc. Data *
K. Lange: Applied Prob.

* has defn for persistence/recurrence

Computing transition probabilities for a cntus-time M.C.

We have: $P'(t) = P(t)\,G$ (forward eqn)

$P'(t) = G\,P(t)$ (backward ..)

These are eqns. of form $f'(t) = c\,f(t)$

w/ soln: $f(t) = f(0)\,e^{ct}$.

In matrix form, soln: $P(t) = P(0)\,e^{Gt}$

$\because P(0) = I$ $\qquad P'(0) = GP(0)$

$P(t) = e^{Gt}$

where matrix exponential, $e^{Gt} = \sum_{i=0}^{\infty} \frac{(Gt)^i}{i!}$

Approximations are available (e.g. see Ross book).

In practice, use software (e.g. expm in R using algorithm in Moler & Van Loan 20

Useful since: if we have a cntns time M.C., we can create a discrete time version, say w/ increments (steps) of length 1.

$X(0), X(1), X(2), \ldots$

w/ $P(X(t)=j \mid X(t-1)=i) = P_{ij}$ where $P = e^{G}$

Useful for simulation.

Birth-death processes

Integer valued cntns time M.C.'s that can only 'step up' (birth) or 'step down' (death) in discrete jumps.

Let state space, $\Omega = \{0, 1, 2, 3, \dots\}$.

E.g. Consider a popn. of cells. Each cell lives for a time Expon(α) and either splits into 2 new cells w/ prob. p or dies w/ prob. $1-p$, indep. of all other cells.

Let $\{X(t), t \geq 0\}$ be # of cells at time t.

If there are i cells, next change happens at time that is minimum of i lifetimes i.i.d. Expon(α).

Can show that min time $\sim$ Expon$(\underbrace{\alpha + \alpha + \dots + \alpha}_{i \text{ cells}})$

$= $ Expon$(i\alpha)$.

Hence, holding time param $\lambda(i) = i\alpha$, $i = 0, 1, 2, \dots$

w/ $\lambda(0) = 0$ meaning that 0 is an absorbing state.

This gives transition rates: $\gamma_{i,i+1} = (i\alpha)p$ $i = 1, 2, \dots$

$\gamma_{i,i-1} = (i\alpha)(1-p)$

$\left(\because \gamma_{ij} = \lambda(i) p_{ij} \right)$
$i \neq j$

$\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$

'Birth rate', $\beta_i = \gamma_{i,i+1}$

'Death rate', $\mu_i = \gamma_{i,i-1}$

$$G = \begin{bmatrix} -\beta_0 & \beta_0 & 0 & \cdots & & & \\ \mu_1 & -(\beta_1+\mu_1) & \lambda_1 & \cdots & & & \\ 0 & \mu_2 & -(\beta_2+\mu_2) & \beta_2 & 0 & \cdots & \\ 0 & 0 & \mu_3 & -(\beta_3+\mu_3) & \beta_3 & \cdots & \\ & & & & & & \end{bmatrix}$$

Can solve $\pi G = 0$ to find stationary distr.

Very general. Useful for queueing theory.

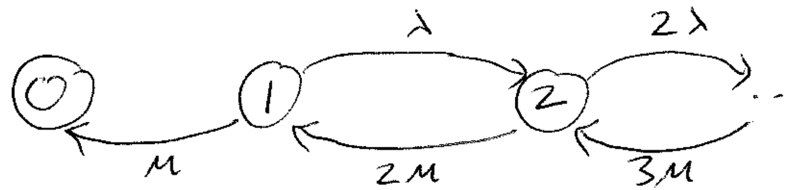
E.g. Individual death rate is μ and no births.
Constant immigration according to P.P. w/ rate λ.

$\lambda_i = \lambda \quad \forall i$
$\mu_i = i\mu \quad \forall i$

Linear birth-death process since linear in i.

Notation: Individual birth rate: $\lambda = \alpha p$
" death " : $\mu = \alpha(1-p)$ $\Big\}$ skip?

Transition graph:



and the generator,

$$G = \begin{array}{c|ccccc} & 0 & 1 & 2 & & \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ 2 & 0 & 2\mu & -2(\lambda+\mu) & 2\lambda & \cdots \\ & 0 & 0 & 3\mu & -3(\lambda+\mu) & 3\lambda \cdots \\ & \vdots & \vdots & \vdots & & \end{array}$$

(recall γ_{ij} = transition rate from i to j, $i \neq j$, $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$)

Jump chain is simple r.walk (transient if $p > 1/2$; only case when absorption in 0 can be avoided).

More generally, birth death process is characterized by new arrivals ('births') at an exponential rate λ_i, and departures ('deaths') " " " μ_i.
So, $\gamma_{i,i+1} = \lambda_i$ and $\gamma_{i,i-1} = \mu_i$.

Skip?

Detailed Balance

Solving $\pi G = 0$ may be challenging, especially for infinite state space M.C.s.

As in discrete-time setting, reversibility / detailed balance may be helpful in finding π when M.C. is time reversible.

Prop.: For an M.C. with generator matrix $G = \{\gamma_{ij}\}$ w/ state space Ω, If $\exists \underset{\sim}{\pi}$ s.t. $\sum_{i \in \Omega} \pi_i = 1$ and

$$\pi_i \gamma_{ij} = \pi_j \gamma_{ji} \quad \forall i,j \in \Omega$$

then, M.C. is time-reversible and $\underset{\sim}{\pi}$ also satisfies $\pi G = 0$, i.e., π is also the stationary distr. of the chain.

E.g. Birth-death processes

Let us try to find $\underset{\sim}{\pi}$ s.t. it satisfies detailed balance, so $\pi_i \gamma_{i\,i+1} = \pi_{i+1} \gamma_{i+1,i}$, $i = 0,1,2,\dots$

Equivalently, $\pi_i \lambda_i = \pi_{i+1} \underset{\gamma_{i+1,i} = \mu_{i+1}}{\mu_{i+1}}$ $i = 0,1,2,\dots$

Hence, $\pi_0 \lambda_0 = \pi_1 \mu_1$ and $\pi_1 \lambda_1 = \pi_2 \mu_2$

So, $\pi_2 = \dfrac{\lambda_1}{\mu_2} \pi_1 = \dfrac{\lambda_0 \lambda_1}{\mu_1 \mu_2} \pi_0$

In similar fashion, can obtain

$$\pi_i = \frac{\lambda_{i-1} \cdots \lambda_0}{\mu_i \cdots \mu_1} \pi_0$$

If we solve for π_0, we are done.

But, $1 = \pi_0 + \pi_0 \sum_{i=1}^{\infty} \frac{\lambda_{i-1} \cdots \lambda_0}{\mu_i \cdots \mu_1}$

Hence, $\pi_0 = \dfrac{1}{1 + \sum_{i=1}^{\infty} \frac{\lambda_{i-1} \cdots \lambda_0}{\mu_i \cdots \mu_1}}$

If $\sum_{i=1}^{\infty} \frac{\lambda_{i-1} \cdots \lambda_0}{\mu_i \cdots \mu_1}$ converges, we have a soln.

(~~Much~~ easier approach than trying to solve ergodic eqns.)

Now, if $\underline{\pi}$ is a soln. to ergodic eqns.
if M.C. is irreducible and jump chain is recurrent (need to check this),
it is positive recurrent by Proposition.
Then π is also limiting distr. of process. by previous theorem.