STAT 515
Homework #7 WITH SOLUTIONS

1. Customers arrive at a post office at a Poisson rate of 8 per hour. There is a single person serving customers, and service times are exponentially distributed (and independent) with mean 5 minutes. Suppose that an arriving customer will decide to wait in line if and only if there are two or fewer people already in line.

   (a) In the long run, what fraction of the time will there be at least 1 customer in the post office? Find the answer in two different ways:

   i. Write out the rate matrix (or generator) for the continuous-time Markov chain and find the stationary distribution using the generator.

   Solution: Although there was a bit of a misunderstanding about how many people should be allowed in the post office because of my ambiguous wording, the intent was to have 5 possible states, namely, 0 through 4 total people at the post office. Here is the rate matrix with states 0 through 4 in order:

$$R = \begin{bmatrix} -8 & 8 & 0 & 0 & 0 \\ 12 & -20 & 8 & 0 & 0 \\ 0 & 12 & -20 & 8 & 0 \\ 0 & 0 & 12 & -20 & 8 \\ 0 & 0 & 0 & 12 & -12 \end{bmatrix}.$$

   The stationary distribution π satisfies $\pi^\top R = 0$, which means that π is an eigenvector of $R^\top$ whose eigenvalue equals zero:

```
> R <- matrix(c(-8, 12, 0, 0, 0, 8, -20, 12, 0, 0, 0, 8, -20,
+                        12, 0, 0, 0, 8, -20, 12, 0, 0, 0, 8, -12),
+          5, 5)
> e <- eigen(t(R))
> pi <- e$vec[, abs(e$val)< 1e-15]
> pi <- pi/sum(pi)
```

   We want $1 - \pi_0$:

```
> 1 - pi[1]
[1] 0.6161137
```

   ii. Use the fact that this is a birth-death process to find the stationary distribution that satisfies the detailed balance equations.

   Solution: From detailed balance, we get the equations $\pi_1 = 8\pi_0/12$, $\pi_2 = 8\pi_1/12$, $\pi_3 = 8\pi_2/12$, and $\pi_4 = 8\pi_3/12$. Since the π vector must sum to one, we obtain

$$\pi_0 = \left(1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81}\right)^{-1} = \frac{81}{211}.$$

   Therefore, the desired answer is $1 - \pi_0 = 130/211 = 0.6161$.

   (b) In the long run, what is the expected number of customers in the post office (in line or being served) at any given time?

   Solution: From part (a), we know that

$$\pi^\top = \left(\frac{81}{211}, \frac{54}{211}, \frac{36}{211}, \frac{24}{211}, \frac{16}{211}\right).$$

   Therefore, the expectation equals

$$\frac{54}{211} + 2 \times \frac{36}{211} + 3 \times \frac{24}{211} + 4 \times \frac{16}{211} = \frac{262}{211} = 1.242.$$

   (c) What is the probability that an arriving potential customer will decide to leave because there are already 3 people in line?

Solution: Since the indicator of "next event is a customer arrival" is independent of the rest of the Markov chain, the answer to this question is simply π_4, the limiting probability of 3 people in line. This equals $16/211 = 0.0758$.

(d) If a new cash register is installed that decreases the mean service time to 4 minutes, how many more customers per hour, on average, can be served by this post office?

Solution: The wording of this question appears to be ambiguous. In one interpretation, we recalculate π_4 using the new value $\mu = 15$: From detailed balance, we get the equations $\pi_1 = 8\pi_0/15$, $\pi_2 = 8\pi_1/15$, $\pi_3 = 8\pi_2/15$, and $\pi_4 = 8\pi_3/15$. Since the π vector must sum to one, we obtain

$$\pi_0 = \left(1 + \frac{8}{15} + \frac{64}{225} + \frac{512}{3375} + \frac{4096}{50625}\right)^{-1} = \frac{50625}{103801}.$$

This gives $\pi_4 = (50625/103801)(8/15)^4 = 4096/103801$. Since customers arrive at the rate of 8 per hour, the average number of additional customers served is $8[(16/211) - (4096/103801)] = 0.291$ per hour.

According to a different interpretation, we might consider that customers are only being served when they are actually at the service window. Under this interpretation, the number served per hour is exactly 12 whenever there is at least one person in the post office under the first scenario, so we get a mean of $12(130/211)$. Under the second scenario, we obtain $15(53176/103801)$, which also yields a difference of 0.291 customers per hour.

At first, I was surprised that these two methods of solution give exactly the same answer! But after some more thought, this makes sense: The long-run rate equals limit of the total number of customers divided by the total number of hours as the latter goes to infinity. For this purpose, it does not matter whether the 1 to 3 people still in line at the end of the very-long time are counted among the total number of customers or not, since the limit will be the same whether or not the numerator is increased by 1 to 3 customers. So evidently, my wording was not quite as ambiguous as I had thought!

(e) The manager of the post office wants to be able to serve at least 95% of the potential customers who arrive at the post office. What mean service time will attain this goal?

Solution: Here, we want π_4 to equal 0.05. If x is the number of customers per hour, then

$$\pi_4 = \left(\frac{8}{x}\right)^4 \left(1 + \frac{8}{x} + \frac{64}{x^2} + \frac{512}{x^3} + \frac{4096}{x^4}\right)^{-1}.$$

With a bit of simplification, we find that the equation $\pi_4 = 1/20$ is the same as

$$x^4 + 8x^3 + 64x^2 + 512x = 19(4096).$$

Since the left hand side is a strictly increasing function when x is positive, there is a unique positive solution, which may be found numerically. This is easy to do using simple trial-and-error, but an alternative uses the `uniroot` function in R (and we know from parts (c) and (d) that the answer lies between 12 and 15):

```
> print(r <- uniroot(function(x) x^4+8*x^3+64*x^2+512*x-19*4096, lower=12, upper=15)$root)
[1] 13.87314
```

We now divide this number into 60 to obtain a mean service time, in minutes, of

```
> 60/r
[1] 4.324906
```

2. Suppose that a continuous-time Markov chain $X(t)$ has rate matrix

$$R = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}.$$

Given positive times s and t, calculate $\mathrm{Corr}\,[X(s), X(t)]$. Does your answer depend on the starting state of the chain (i.e., value of $X(0)$)?

(NB: The formula for $\text{Corr}\,(X,Y)$ is $\text{Corr}\,(X,Y) = \text{Cov}\,(X,Y)/\sqrt{\text{Var}\,(X)\,\text{Var}\,(Y)}$. Since the correlation is invariant to linear transformations, the two possible values that $X(t)$ may take do not influence your answer.)

Solution: If we let 0 and 1 be the two values that $X(t)$ can take, then the calculations will be simplest. Also, let us assume that $s < t$, since we may do so without loss of generality.

In the case $X(0) = 0$, using the notation in Section 6.4 of the textbook (the example about the two-state Markov chain), we obtain

$$EX(s) = P_{01}(s), \quad \text{Var } X(s) = P_{01}(s)P_{00}(s), \quad EX(t) = P_{01}(t), \quad \text{Var } X(t) = P_{01}(t)P_{00}(t),$$

and

$$E[X(s)X(t)] = P[X(t) = 1 \mid X(s) = 1]P[X(s) = 1] = P_{11}(t-s)P_{01}(s).$$

Putting everything together, we obtain

$$\text{Corr}\,[X(s), X(t)|X(0) = 0] = \frac{P_{01}(s)[P_{11}(t-s) - P_{01}(t)]}{\sqrt{P_{01}(s)P_{00}(s)P_{01}(t)P_{00}(t)}}.$$

Similarly, in the case $X(0) = 1$, we obtain

$$\text{Corr}\,[X(s), X(t)|X(0) = 1] = \frac{P_{11}(s)[P_{11}(t-s) - P_{11}(t)]}{\sqrt{P_{11}(s)P_{10}(s)P_{11}(t)P_{10}(t)}}.$$

To simplify things slightly, let us define $x = (\alpha+\beta)s$ and $y = (\alpha+\beta)t$. According to the example in Section 6.4, the two correlations are

$$\begin{aligned}
\text{Corr}\,[X(s), X(t)|X(0) = 0] &= \frac{(\alpha - \alpha e^{-x})(\beta e^{x-y} + \alpha e^{-y})}{\sqrt{(\alpha - \alpha e^{-x})(\beta + \alpha e^{-x})(\alpha - \alpha e^{-y})(\beta + \alpha e^{-y})}} \\
&= \frac{\beta e^{x-y} + (\alpha - \beta)e^{-y} - \alpha e^{-x-y}}{\sqrt{(1 - e^{-x})(\beta + \alpha e^{-x})(1 - e^{-y})(\beta + \alpha e^{-y})}}
\end{aligned}$$

and

$$\begin{aligned}
\text{Corr}\,[X(s), X(t)|X(0) = 1] &= \frac{(\alpha + \beta e^{-x})(\beta e^{x-y} - \beta e^{-y})}{\sqrt{(\beta - \beta e^{-x})(\alpha + \beta e^{-x})(\beta - \beta e^{-y})(\alpha + \beta e^{-y})}} \\
&= \frac{\alpha e^{x-y} + (\beta - \alpha)e^{-y} - \beta e^{-x-y}}{\sqrt{(1 - e^{-x})(\alpha + \beta e^{-x})(1 - e^{-y})(\alpha + \beta e^{-y})}}.
\end{aligned}$$

We see that these two expressions are identical except that the role of α and β is switched. This makes sense once we realize that an on-off process with parameters α and β is the same as an off-on process with parameters β and α. It also means that the two correlations are the same when $\alpha = \beta$. However, they are not generally the same. One way to see this is to consider the limit of each correlation as $\alpha \to 0$:

$$\lim_{\alpha\to 0} \text{Corr}\,[X(s), X(t)|X(0) = 0] = \frac{e^{x-y} - e^{-y}}{\sqrt{(1 - e^{-x})(1 - e^{-y})}}$$

and

$$\lim_{\alpha\to 0} \text{Corr}\,[X(s), X(t)|X(0) = 1] = \sqrt{\frac{e^{x-y} - e^{-y}}{(1 - e^{-x})(1 - e^{-y})}}.$$

3. At an amusement park, there are two video game machines. Suppose that for video game i, each period when it is being used is exponentially distributed with rate α_i and each period when it is not being used is exponentially distributed with rate β_i, independent of the other machine.

(a) Suppose that the vector $M(t)$ is given by

$$M(t) = [M_1(t), M_2(t)]^\top,$$

where $M_i(t) = I\{\text{machine } i \text{ is being used at time } t\}$ for $i = 1, 2$. The Markov chain $M(t)$ has four states; give its rate matrix R.

Solution: Let 0 denote "not being used" and 1 denote "being used". Then if we order the four states $(0,0), (0,1), (1,0), (1,1)$, we get

$$R = \begin{bmatrix} -(\beta_2+\beta_1) & \beta_2 & \beta_1 & 0 \\ \alpha_2 & -(\alpha_2+\beta_1) & 0 & \beta_1 \\ \alpha_1 & 0 & -(\alpha_1+\beta_2) & \beta_2 \\ 0 & \alpha_1 & \alpha_2 & -(\alpha_1+\alpha_2) \end{bmatrix}$$

(b) In the long run, what proportion of time are both machines being used?

Solution: One way to find the long-run probability vector π is to solve the system of equations $\pi^\top R = 0$. However, a simpler method is to consider the two on-off processes separately: In the long run, machine i spends a fraction $\beta_i/(\alpha_i + \beta_i)$ of the time being used. Since the two machines are independent, we conclude that

$$\pi_4 = \frac{\beta_1\beta_2}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}.$$

Incidentally, this process may be shown to be time-reversible (can you see why?), which means that the detailed balance equations give yet another alternative for finding the π vector.

(c) When the amusement park first opens, each machine is in its unused state. We can express this fact by $M_1(0) = M_2(0) = 0$. Simulate 10,000 independent realizations of this chain, until time $t = 3$, using $\alpha_1 = 2$, $\alpha_2 = 3$, $\beta_1 = 5$, and $\beta_2 = 6$. From your simulations, give an empirical estimate of the proportion μ of time in $(0, 3]$ that the machines are used. Report a 95% confidence interval for μ. How does this compare with the long-run value calculated in part (b)?

To find an approximate 95% confidence interval for a mean μ based on a i.i.d. sample of size n, take

$$\hat{\mu} \pm 1.96\frac{s}{\sqrt{n}},$$

where $\hat{\mu}$ is the sample mean and s is the sample standard deviation.

Solution: First, we'll set up the R matrix:

```
> R <- matrix(c(-11, 3, 2, 0, 6, -8, 0, 2, 5, 0, -8, 3, 0, 5, 6, -5), 4.4)
```

Next is the sample code I wrote that simulates 10,000 copies of the Markov chain:

```
> X <- list()  # Each item in X will consist of TWO vectors:  times and states
>              # Actually, technically, each item will be a list with two elements:
>              # A vector named times and a vector named states.
> maxTime <- 3 # This is the cutoff time.
> for (count in 1:10000) {
+   times <- 0
+   states <- 1 # Assume that we always start in state 1 at time 0
+   i <- 1 # which time/state are we currently in
+   finished <- FALSE # We'll set this to TRUE when it's time to stop.
+   while (!finished) {
+     currentState <- states[i]
+     currentTime <- times[i]
+     deltaTime <- rexp(1, rate = -R[currentState, currentState])
+     if (currentTime + deltaTime > maxTime) {
+       # Now we need to finish this chain
```

```
+       deltaTime <- maxTime - currentTime
+       finished <- TRUE
+     }
+     times <- c(times, currentTime + deltaTime)
+     possibleMoves <- (1:4)[-currentState]
+     states <- c(states, sample(possibleMoves, 1, prob=R[currentState,possibleMoves]))
+     i <- i+1
+   }
+   X[[count]] <- list(times=times, states=states)
+ }
```

Next, we need to figure out how much time was spent in the 4th state:

```
> f <- function(a) {
+   deltaTimes <- diff(a$times)
+   states <- a$states[-length(a$states)] # delete the final state, which is irrelevant
+   sum(deltaTimes[states==4])
+ }
> timesInState4 <- sapply(X, f)
```

Here is a 95% confidence interval for the true mean of the proportion of time in $(0, 3]$ that is spent in the fourth state:

```
> mean(timesInState4/3) + c(-1.96, 1.96)*sd(timesInState4/3)/100
[1] 0.4427792 0.4479613
```

This is a fairly narrow confidence interval, and it is not too close to the long-term mean $30/63 = 0.4762$. This suggests that three time units is not long enough for the chain to get close to its equilibrium state.