Homework 6 – Part I, Stat 515, Spring 2015

Due Wednesday, March 25, 2015 **beginning of class**

1. Consider a continuous-time Markov chain on $\{1, 2, 3\}$ with generator

$$G = \begin{bmatrix} -6 & 2 & 4 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{bmatrix}.$$

   (a) What is the distribution of the holding times? What is the transition probability matrix of the embedded "jump chain"?

   (b) Simulate two realizations of this Markov chain on the interval (0,50). Assume it starts at 1, that is, $X(0) = 1$. Provide pseudocode (sketch of the algorithm), `R` code and a plot for each of the realizations.

Part II

2. A small barbershop is operated by a single barber. It has room for at most two customers. Potential customers arrive as a Poisson process with a rate of three per hour, and the successive service times are independent exponential random variables with mean 0.25 hour. Find:

   (a) The average number of customers in the shop.

   (b) The proportion of potential customers that enter the shop (in the long run).

   (c) If the barber could work twice as fast, how much more business could he do?

3. Write `R code` to simulate a single realization over a period of 10 days for the above barbershop. What is the average number of customers in the shop for this realization?

4. Customers arrive at a small bank according to a Poisson process with rate 20 per hour. However, they will only enter the bank if there are no more than two customers (including the one being attended to) at the bank. Assume that the amount of time required to serve a customer is exponentially distributed with mean of 5 minutes. In the long run, what fraction of time will there be at least 1 customer in the bank? Solve this problem in two ways:

   (a) First write out the generator for this continous-time Markov chain and then find the stationary distribution using the generator.

   (b) Use the fact that it is a birth-death process and find the stationary distribution π that satisfies detailed balance conditions.

5. Consider a population where the individual death rate is $\mu > 0$ and there is constant immigration into the population (immigration always increases the size of the population) according to a Poisson process with rate $\lambda > 0$. Derive the stationary distribution of this process.