Gaussian Variational Approximate Inference and Monte Carlo EM Algorithm for Generalized Linear Mixed Models

STAT 540 Project Presentation

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Generalized Linear Mixed Models (GLMM)

- A generalized linear mixed model is an extension of the generalized linear model in which the linear predictor contains random effects in addition to the usual fixed effects.
- GLMMs are widely applied to the analysis of grouped data, since the differences among groups (from different distributions) can be modelled as random effects.
- The general form of the model is:

$$y = X\beta + Z\mu + \epsilon,\ \mu \sim N(0, G)$$

- Fitting GLMMs via maximum likelihood involves integrating over the random effects. In general, those integrals cannot be expressed in analytical forms.

GLMM

- Consider the exponential family models of the form:

$$\mathbf{y}|\mathbf{u} \sim \exp\{\mathbf{y}^T(\mathbf{X}\boldsymbol{\beta}+\mathbf{Z}\mathbf{u}) - \mathbf{1}^T b(\mathbf{X}\boldsymbol{\beta}+\mathbf{Z}\mathbf{u}) + \mathbf{1}^T c(\mathbf{y})\}, \qquad \mathbf{u} \sim N(\mathbf{0}, \mathbf{G}),$$

- The parameters in the exponential family models are the fixed effects vector \beta and the random effects covariance matrix G. Their loglikelihood is:

$$\ell(\boldsymbol{\beta}, \boldsymbol{\Sigma}) = \sum_{i=1}^{m} \{\mathbf{y}_i^T \mathbf{X}_i \boldsymbol{\beta} + \mathbf{1}_i^T c(\mathbf{y}_i)\} - \frac{m}{2}\log|\boldsymbol{\Sigma}| - \frac{mK}{2}\log(2\pi)$$

$$+ \sum_{i=1}^{m} \log \int_{\mathbb{R}^K} \exp\left\{\mathbf{y}_i^T \mathbf{Z}_i \mathbf{u} - \mathbf{1}_i^T b(\mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{u}) - \frac{1}{2}\mathbf{u}^T \boldsymbol{\Sigma}^{-1}\mathbf{u}\right\} d\mathbf{u}$$

- The K-dimensional integral in the loglikelihood cannot be solved analytically

Gaussian Variational Approximate (GVA)

- GVA introduces a pair of variational parameters μ_i, Λ_i. By Jensen's inequality and concavity of the logarithm function, we can get the lower bound:

$$
\begin{aligned}
\ell(\boldsymbol{\beta}, \boldsymbol{\Sigma}) &= \sum_{i=1}^{m} \{\mathbf{y}_i^T \mathbf{X}_i \boldsymbol{\beta} + \mathbf{1}_i^T c(\mathbf{y}_i)\} - \frac{m}{2} \log |\boldsymbol{\Sigma}| - \frac{mK}{2} \log(2\pi) \\
&\quad + \sum_{i=1}^{m} \log \int_{\mathbb{R}^K} \exp\left\{ \mathbf{y}_i^T \mathbf{Z}_i \mathbf{u} - \mathbf{1}_i^T b(\mathbf{X}_i \boldsymbol{\beta} + \mathbf{Z}_i \mathbf{u}) - \frac{1}{2} \mathbf{u}^T \boldsymbol{\Sigma}^{-1} \mathbf{u} \right\} \frac{\phi_{\boldsymbol{\Lambda}_i}(\mathbf{u} - \boldsymbol{\mu}_i)}{\phi_{\boldsymbol{\Lambda}_i}(\mathbf{u} - \boldsymbol{\mu}_i)} d\mathbf{u} \\
&\geq \sum_{i=1}^{m} \{\mathbf{y}_i^T \mathbf{X}_i \boldsymbol{\beta} + \mathbf{1}_i^T c(\mathbf{y}_i)\} - \frac{m}{2} \log |\boldsymbol{\Sigma}| - \frac{mK}{2} \log(2\pi) \\
&\quad + \sum_{i=1}^{m} E_{\mathbf{u} \sim N(\boldsymbol{\mu}_i, \boldsymbol{\Lambda}_i)} \Big(\mathbf{y}_i^T \mathbf{Z}_i \mathbf{u} - \mathbf{1}_i^T b(\mathbf{X}_i \boldsymbol{\beta} + \mathbf{Z}_i \mathbf{u}) \\
&\quad - \frac{1}{2} \mathbf{u}^T \boldsymbol{\Sigma}^{-1} \mathbf{u} - \log(\phi_{\boldsymbol{\Lambda}_i}(\mathbf{u} - \boldsymbol{\mu}_i)) \Big) \\
&\equiv \underline{\ell}(\boldsymbol{\beta}, \boldsymbol{\Sigma}, \boldsymbol{\mu}, \boldsymbol{\Lambda}),
\end{aligned}
$$

GVA

- The advantage of the lower-bound is that it no longer involves the integrals of size K. Hence, the computational speed is improved.
- We can use Newton-Raphson scheme to get the Gaussian variational approximate maximum likelihood estimators.

Monte Carlo EM algorithm

- Consider the random effects **u** to be the missing data. The complete data here is **W=(Y, u)**
- The monte carlo EM algorithm is as follows:
    1. Choosing starting values $\boldsymbol{\beta}^{(0)}$ and $\boldsymbol{\sigma}^{(0)}$, set **n=0**
    2. Generate *m* values, $\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}, \boldsymbol{u}^{(3)}, \ldots, \boldsymbol{u}^{(m)}$ from the conditional distribution of **u|Y** using a Metropolis algorithm (use **f(y|u)** as the proposal distribution) and using the current parameter values
    3. Choose:
        (1) $\boldsymbol{\beta}^{(n+1)}$ to maximize a Monte Carlo estimate of **E[ln(f(y|u))]**
        (2) $\boldsymbol{\sigma}^{(n+1)}$ to maximize **E[ln(f(u|$\boldsymbol{\sigma}$))]**

Simulation Study

- Dataset: *Epilepsy* dataset
- Description: 59 epilepsy patients; each of the patients was assigned to a control group (placebo) or a treatment group. The experiment recorded the number of seizures experienced by each patient over 4 two-week periods.
- Structure: 59 x 4 observations on the following 7 variables: count(y), log(base/4), trt, trt*log(base/4), log(age), subject (u), v4
- Each patient can be seen as a group, and we use subject, which is the id of the patient as the random effect
- Consider the Poisson random intercept model:

$$y_{ij}|u_i \sim Poisson(\exp(\beta^T x_{ij} + u_i))$$

Results

- Use adaptive Gauss-Hermite quadrature (AGHQ) here as "gold standard" when the true values of the parameters are not known.

	β_0	β_{base}	β_{trt}	$\beta_{base*trt}$	β_{age}	β_{v4}	σ_0	Time (seconds)
AGHQ	-1.325	0.883	-0.933	0.481	-0.160	0.339	0.251	0.901
GVA (6 ITER)	-1.325	0.883	-0.933	0.481	-0.160	0.339	0.251	0.066
Monte Carlo EM	-1.325	0.883	-0.933	0.481	-0.160	0.339	0.251	108

Modified EM algorithms for High-Dimensional Clustering

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Nov. 29. 2018

Motivation

- For **unsupervised learning** techniques, model-based approach is one of the most popular methods.
- Model-based approach exploits latent variable Gaussian mixture model (**GMM**) and estimates the model through expectation and maximization (**EM**) algorithm.

$$\text{E-step}: \quad Q(\theta|\theta^{(t)}) = \hat{E}_{Z|X,\theta^{(t)}} l_c(\theta; X, Z)$$
$$= \sum_i^n \sum_k^K \underbrace{\hat{E}_{\theta^{(t)}}[z_{ki}|x_i]}_{\tau_{ki}^{(t)}} \{\log \pi_k + \log f_k(x_i; \theta_k)\}$$
$$\text{M-step}: \quad \theta^{(t+1)} = \arg\max_\theta Q(\theta|\theta^{(t)})$$

- However, when data is **high dimensional**, EM algorithm is confronted with identifiability, stability and computational efficiency problems.

Motivation

- Literature approached this issue by imposing sparsity through modification of either E-step or M-step in EM algorithm.

  ⇒ **Modifying E-step**

  $$Q(\theta|\theta^{(t)}) = \hat{E}_{Z|X,\theta^{(t)}}\{l_c(\theta; X, Z) + p(\theta)\}$$

  ⇒ **Modifying M-step**

  $$\theta^{(t+0.5)} = \arg\max_{\theta} Q(\theta|\theta^{(t)})$$
  $$\theta^{(t+1)} = s(\theta^{(t+0.5)})$$

- Different modifications have different performances in stability and computational efficiency.

Comparison between different modifications

- Where the modification is applied is not critical.
- Different modifications result in different update rule.

E.g. [Modification of M-step]: truncation step (Wang et al. 2014)
The update rule changes as follows.

$$\tau_{ki}^{(t)} = \frac{\pi_k^{(t)} f_k(x_i;\theta_k^{(t)})}{\sum_k^K \pi_k^{(t)} f_k(x_i;\theta_k^{(t)})}, \quad \pi_k^{(t+1)} = \frac{1}{n}\sum_i^n \tau_{ki}^{(t)} \quad \mu_k^{(t+0.5)} = \frac{\sum_i^n \tau_{ki}^{(t)} x_i}{\sum_i^n \tau_{ki}^{(t)}},$$

And additional step impose sparcity.

$$\begin{aligned} \hat{\mathcal{S}}^{(t+0.5)} &= \text{set of index } j\text{'s of the top } s \text{ largest } |\mu_{kj}^{(t+0.5)}| \\ \hat{\mu}_k^{(t+1)} &= \begin{cases} \mu_k^{(t+0.5)} & j \in \hat{\mathcal{S}}^{(t+0.5)} \\ 0 & j \notin \hat{\mathcal{S}}^{(t+0.5)} \end{cases} \end{aligned}$$

Comparison between different modifications

E.g. [Modification of E-step]: L_1 penalty (Pan and Shen 2007)

$$Q_p(\theta;\theta^{(t)}) = \sum_i^n \sum_k^K \tau_{ki}^{(t)} \{\log \pi_k + \log f_k(x_i;\theta_k)\} - \lambda \sum_k^K \sum_j^p |\mu_{kj}|$$

The update rule is same as previous example except that the additional step changes as follows.

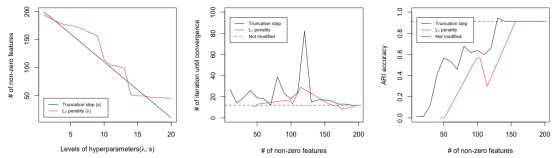
$$\hat{\mu}_k^{(t+1)} = sign(\mu_k^{(t+0.5)}) \left(|\mu_k^{(t+0.5)}| - \frac{\lambda}{\sum_i \tau_{ki}^{(t+1)}} V 1_p \right)_+ ,$$

For truncation step

$$\hat{\mu}_k^{(t+1)} = sign(\mu_k^{(t+0.5)}) \left(|\mu_k^{(t+0.5)}| - \mu_{k(n-s)}^{(t+0.5)} \right)_+ .$$

Simulation study

- $N = 100$, $p = 200$, $K = 3$(# of clusters), K-means initialization.

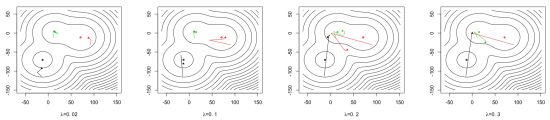


- Number of iteration till convergence is not proportional to dimension size.
- L1 penalized EM converges more quickly than EM with truncation step.
- Accuracy could not be improved by dim reduction.
- Any type of EM did not work well on random initialization.

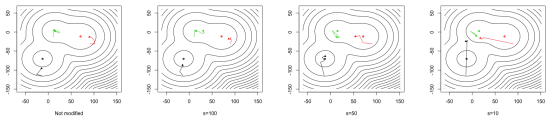
Simulation study

- Geometric interpretation on 2D principal components space.

L_1 penalty with different λ

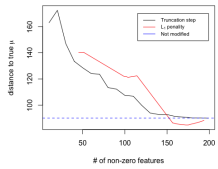


Truncation step with different thresholding s



Simulation study

- Geometric interpretation.



- Bias of estimated μ decreases at some level of hyperparameter and then increases as more dimensions are reduced.
- $\Rightarrow$ Bias corrected penalty such as SCAD, MCP may work better.

APPROXIMATE BAYESIAN COMPUTATION FOR ARCHIMEDEAN COPULA MODELS

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APPROXIMATE BAYESIAN COMPUTING (ABC)

$$\pi(\theta|y) \propto p(y|\theta)\pi(\theta)$$

Problem: How to perform Bayesian inference when the likelihood function $p(y|\theta)$ is computationally intractable?

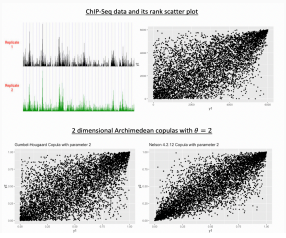
Solution: If we can easily simulate from the likelihood, ABC methods provide a possible way.

$$\pi_{ABC}(\theta|y) \propto \int \mathcal{I}(||y^* - y|| < \epsilon)\pi(\theta|y^*)dy^* \approx \pi(\theta|y)$$

MOTIVATION: ARCHIMEDEAN COPULA MODELS

J-dimensional copula: $C(u_1, \ldots, u_J) = P(U_1 \leq u_1, \ldots, U_J \leq u_J)$ where $U_1, \ldots, U_J$ are uniform random variables and $C : [0,1]^J \to [0,1]$.

J-dimensional Archimedean copula:
$\psi(C(u_1, \ldots, u_J); \theta) = \psi(u_1; \theta) + \ldots + \psi(u_J; \theta)$ where θ **is an association parameter** describing the strength of the dependence between U_j's, and ψ is a generator function specific to each Archimedean copula such that $\psi : [0,1] \to [0, \infty)$.



GUMBEL HOUGAARD COPULA

Gumbel Hougaard copula generator function:

$$\psi(u) = (-\log(u))^{\theta}\ ,\ \theta \in [1, \infty)$$

So its distribution function is,

$$C(u_1, ..., u_n) = \psi^{-1}(\psi(u_1)+...+\psi(u_n)) = \exp\{-((-\log(u_1))^{\theta}+...+(-\log(u_n))^{\theta})^{1/\theta}\}$$

Using ABC to estimate θ is possible because,

- likelihood is unavailable
- it easy to simulate from the model

ABC ALGORITHMS

Requirements:

- a proposal, q(.)
- the observed data, y
- a distance function, ||.||
- a tolerance level, ϵ

Rejection Sampling (Sisson et al. 2018)

1. Simulate $\theta^* \sim$ q(.)
2. Simulate y $\sim$ p(.|θ^*)
3. if $||y - y_{obs}|| < \epsilon$ then accept θ^* with probability $\frac{\pi(\theta^*)}{Kq(\theta^*)}$.
4. Repeat above steps N times.

MCMC (Marjoram et al. 2003)

1. Simulate $\theta^* \sim$ q(.)
2. Simulate y $\sim$ p(.|θ^*)
3. if $||y - y_{obs}|| < \epsilon$ then accept θ^* with probability $\min\{1, \frac{\pi(\theta^*)q(\theta|\theta^*)}{\pi(\theta)q(\theta^*|\theta)}\}$ else stay at θ.
4. Repeat above steps N times.

CHALLENGES OF ABC

- Should we use the entire dataset y or an appropriate summary statistic, s?
- Choice of distance function and summary statistic.
- Computationally expensive?
- Choice of tolerance level and other tuning parameters.

ABC MCMC: SIMULATION STUDY

- Prior on θ is **Gamma(shape=1, scale=2) truncated at 1**.
- Proposal is a random walk i.e. $N(\theta^{(i-1)}, \sigma^2)$ **truncated at 1**.
- Tolerance, ϵ and σ^2 are chosen such that the **acceptance rate was 20-35%**.
- **Absolute difference** is used as the distance function for two dimensions and for more than 2, **Frobenius norm** is used.
- For each simulation, y_{obs} consisted of 100 data points.
- Number of simulations = 100 and N=10000.

Dimensions:	2		3		5	
Summary statistic	Estimate	MSE	Estimate	MSE	Estimate	MSE
Spearman's rank ρ	3.06	1.32	2.94	1.12	2.19	0.07
Kendall's τ	2.60	0.47	2.55	0.39	2.16	0.05

Table: Results from ABC MCMC algorithm for Gumbel-Hougaard Copula with $\theta = 2$.

ABC MCMC WITH KERNEL FUNCTION

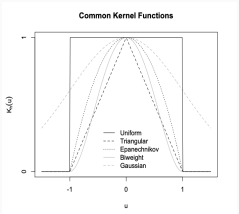
Kernel function: $K_\epsilon(u) = \frac{1}{\epsilon}K(\frac{u}{\epsilon})$ where $K(u) = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}$

Previously: Accept θ^* with probability $\min\{1, \frac{\pi(\theta^*)q(\theta|\theta^*)}{\pi(\theta)q(\theta^*|\theta)}\}$.

Now: Accept θ^* with probability $\min\{1, \frac{K_\epsilon(||s^*-s_{obs}||)\pi(\theta^*)q(\theta|\theta^*)}{K_\epsilon(||s^{(i-1)}-s_{obs}||)\pi(\theta)q(\theta^*|\theta)}\}$.

Dimension	Estimate	MSE
2	2.24	0.12
3	2.14	0.09
5	2.18	0.10

Table: Results from ABC MCMC algorithm (using Kendall's τ) with kernel function for Gumbel Hougaard Copula with $\theta = 2$.



Note: Smoothing functions are discussed in Sisson et al. 2018 for ABC Rejection algorithms and something similar in Fernhead and Prangle 2011.

CONCLUSION

- For large data sets, using y_{obs} instead of summary statistic is not advisable due to computational costs.
- ABC MCMC requires a lot of tuning to run well. If possible, a different ABC algorithm can be preferred.
- Kendall's τ estimates better than Spearman's rank correlation.
- Estimation for higher dimensional copula works better because there is more information for a single parameter.
- ABC MCMC with kernel function seems like a better approach in this case, at least for lower dimensional copula.

Sequential Monte Carlo Estimation of High Dimensional Latent Variable Models

Application to Stochastic Volatility Models

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Nov. 2018

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Motivation

- **Estimate** high-dimensional latent variable models

  Stochastic volatility models: Widely used in practice and option pricing. Application Given asset returns, study the underlying volatility.

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- **Estimate** high-dimensional latent variable models

  Stochastic volatility models: Widely used in practice and option pricing. Application Given asset returns, study the underlying volatility.

$$y_t = \exp\left(\frac{h_t}{2}\right)\varepsilon_t \quad (1)$$

$$h_{t+1} = \mu + \phi(h_t - \mu) + \sigma\eta_t \quad (2)$$

  where ε and η are iid standard normal distribution.

  - y_t : asset return in reality $\rightarrow$ Observable
  - h_t: risk underlying the asset $\rightarrow$ Latent variable
  - Parameter: $\Theta = \{\mu, \phi, \sigma^2\}$

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- **Goal**: learn parameters and latent variables recursively
  - use real time data and estimate efficiently

Computational challenge

The difficult part occurs in the latent variable (Equation (2)), i.e., likelihood function is not easily available:

$$f(h_t|y_t, \Theta) \propto \underbrace{f(y_t|h_t, \Theta)}_{\text{observed return}} \quad \underbrace{f(h_t|y_{t-1}, \Theta)}_{\text{latent variable: volatility}} \tag{3}$$

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$$f(h_t|y_{t-1},\Theta) = \int \underbrace{f(h_t|h_{t-1},\Theta)}_{\text{proposed AR(1)}} \underbrace{f(h_{t-1}|y_{t-1},\Theta)}_{\text{hard closed form}} dh_{t-1} \tag{4}$$

$\Rightarrow$ *Difficult to evaluate the objective function*

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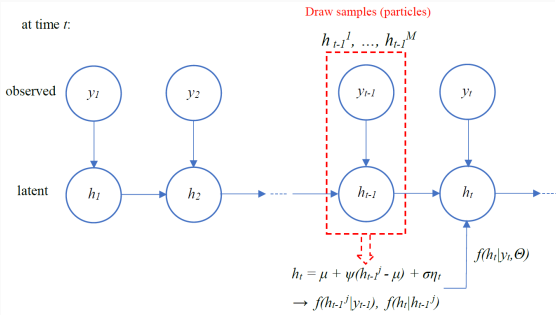
Challenge 1: Cannot directly draw **sample** from $f(h_t|y_{t-1},\Theta)$

Challenge 2: Computational **complexity** of simulation increases with the number of time points in data

Intuition for sequential Monte Carlo Method



Sequential Monte Carlo: particle filter algorithm

- Main Idea: draw particles $\{h_{t-1}^{j}\}$ from filtered distribution $f(h_{t-1}|y_{t-1})$, and derive $f(h_t|y_t)$ using Equation (2). Then sequentially draw particles $\{h_t^{j}\}$

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- Sequentially draw M particles at each time point t $\{h_{t-1}\} = \{h_{t-1}^1, \cdots, h_{t-1}^M\}$ from the filtered distributions: $f(h_{t-1}|y_{t-1})$, where M is the number of particles, then get

$$h_t = \mu + \phi(h_{t-1}^j - \mu)$$

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This gives Monte Carlo approximations

$$f(h_t|y_{t-1}, \Theta) \approx \frac{1}{M}\sum_{j=1}^{M} f(h_t|h_{t-1}^j, \Theta) \quad (5)$$

$$f(h_t|y_t, \Theta) \propto f(y_t|h_t, \Theta)\frac{1}{M}\sum_{j=1}^{M} f(h_t|h_{t-1}^j, \Theta) \quad (6)$$

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- I explore different sampling methods: importance-sampling, bootstrap, auxiliary variable Reference

Methodology: compare with MCMC

MCMC Algorithm

1. Initialize h_1 and $\Theta^{(1)}$
2. Sample h_t from $h_t^{(k+1)}|h_{t-1}^{(k)}, y, \Theta^{(k)}$ for $t = 1, \cdots, n$
3. Sample $\sigma^{2(k+1)}|y, h^{(k+1)}, \phi^{(k)}, \mu^{(k)}$
4. Sample $\phi^{(k+1)}|h^{(k+1)}, \mu^{(k)}, \sigma^{2(k+1)}$
5. Sample $\mu^{(k+1)}|h^{(k+1)}, \phi^{(k+1)}, \sigma^{2(k+1)}$
6. Goto 2

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4. Sample $\phi^{(k+1)}|h^{(k+1)}, \mu^{(k)}, \sigma^{2(k+1)}$
5. Sample $\mu^{(k+1)}|h^{(k+1)}, \phi^{(k+1)}, \sigma^{2(k+1)}$
6. Goto 2

Computing is more expensive

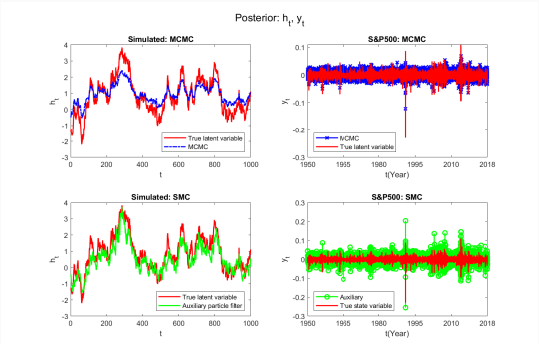
- Need to sample from $h_t^{(k+1)}|h_{t-1}^{(k)}, y, \Theta^{(k)}$ at each time point
- $\{h_t\}$,Θ are highly correlated, resulting in slow convergence ACF

Comparison: Efficiency and Accuracy

- Simulated data: $\mu_0 = 0.5$, $\phi_0 = 0.985$, $\sigma_0^2 = 0.04$
- Real data: Daily S&P 500 returns, 1950 – 2018

	Simulated Data				Real Data			
	MCMC	SISR	Boot	APF	MCMC	SISR	Boot	APF
Computational Efficiency								
ESS	5302	**724**	902	964	1107	**576**	756	826
time	113.106	**0.117**	0.228	0.311	193.733	**2.048**	3.766	4.274
MSE for last latent variable h_N								
mean	**0.013**	0.098	0.100	0.097	1.174	0.134	0.134	**0.132**
std	0.315	**0.041**	0.045	0.043	**0.047**	2.096	2.082	2.003
MSE for parameters								
μ	0.200	0.010	**0.000**	0.090	–	–	–	–
ϕ	0.006	**0.000**	0.000	0.000	–	–	–	–
σ^2	0.004	**0.001**	0.005	0.003	–	–	–	–

Comparison: Latent Variable Estimates



Conclusion

- SMC Strength
  - Efficient: large sample and iteration, more parameters
  - The algorithm can be easily parallelized
  - The computational complexity does not increase with increase in time
  - As the number of particles M increases, the accuracy increases.

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- Auxiliary particle filter may be better at handling outliers and heavy tails
  - I found some tentative evidence from simulation Density

Thank you

Appendix

- Application
- Auxiliary particle filter algorithm
- MCMC simulation autocorrelation
- Sequence density

Application: Stochastic Volatility Models

1. Option price: Black-Scholes, Heston and Hull-White model
2. Long run risk model
3. Industry

Back

Methodology: Sequential Monte Carlo

SMC Algorithm: auxiliary particle filter Back

1. Given $\{h_{t-1}^1, \cdots, h_{t-1}^M\}$ from $f(h_{t-1}|y_{t-1}, \Theta)$ calculate

$$\begin{aligned} \widehat{h}_t^{*j} &= \mu + \phi(h_{t-1}^j - \mu) \\ w_j &= f_N(y_t|exp(\widehat{h}_t^{*j})), \quad j = 1, \cdots, M \end{aligned}$$

   and sample R times with probability $\{w_j\}$. Let the sampled index be $k_1, \cdots, k_R$, and associate these with $\widehat{h}_t^{*k_1}, \cdots, \widehat{h}_t^{*k_R}$

2. For each value of k_j from Step 1 simulate

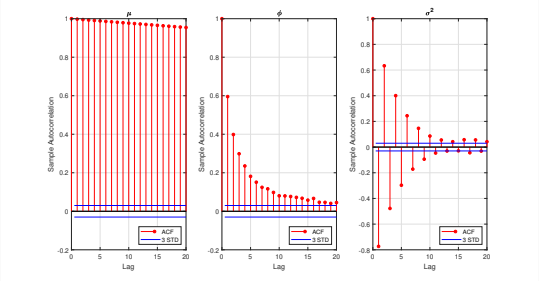
$$h_t^{*j} \sim \mathcal{N}(\mu + \phi(h_{t-1}^{*k_j} - \mu), \sigma^2), \quad j = 1, \cdots, R$$

3. Resample $\{h_t^{*1}, \cdots, h_t^{*R}\}$ with probability

$$\frac{\mathcal{N}(\mu + \phi(h_{t-1}^{*j} - \mu), \sigma^2)}{\mathcal{N}(\mu + \phi(h_{t-1}^{*k_j} - \mu), \sigma^2)}$$

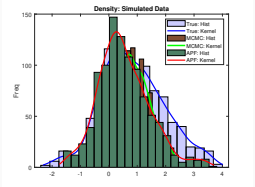
   to produce the filtered sample $\{h_t^1, \cdots, h_t^M\}$ from $f(h_t|y_t, \Theta)$

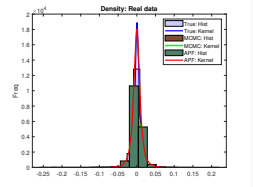
Slow convergence in MCMC: Autocorrelation



Back

Density analysis: simulated and real data





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Reference

- Gordon, N. J., Salmond, D. J., and Smith, A. F. M. (1993), Novel approach to nonlinear/non-Gaussian Bayesian state estimation, IEEE Proceedings Part F: Communications, Radar and Signal Processing, 140, 107–113.
- Kim, S., Shephard, N., and Chib, S. (1998). Stochastic volatility: likelihood inference and comparison with ARCH models. The review of economic studies, 65(3), 361-393.
- Pitt, M. K. and Shephard, N. (1999), Filtering via simulation: auxiliary particle filters, Journal of the American Statistical Association, 94, 590–599.
- Creal, D. (2012). A survey of sequential Monte Carlo methods for economics and finance. Econometric reviews, 31(3), 245-296.

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