1 PROBLEM 1

1.1 (A)

First need to derive the posterior distribution:

$$p(\beta_1|Y,X) = \frac{P(\beta_1) * p(Y,X|\beta_1)}{p(Y,X)} \propto P(\beta_1) * p(Y,X|\beta_1) \tag{1.1}$$

We have

$$p(\beta_1) = Norm(0,10) \tag{1.2}$$

and

$$p(Y,X|\beta_1) = \prod_{i=1}^{N} P(EMG(\beta_0 + \beta_1 * X_i, \sigma_i, \lambda) = Y_i) \tag{1.3}$$

M-H algorithm:
So we could use Metropolis-Hasting algorithm to approximate the posterior distribution first we define the proposal function, and we use Gaussian Distribution as our proposal function

$$Proposal(x) = norm(x, variance) \tag{1.4}$$

Steps:

- Produce initial value of β_1 based on Gaussian Distribution
- Set Number of Iteration as 30000
- Initialize a vector callsed chain to record the result samples
- Loop: i start from 1 to Number of Iteration
- Sampling according to the proposal distribution we get proposal value
- Calculate the propability of $ratio = \frac{posterior(proposal_value)}{posterior(chain[i])}$
- random sample from uniform distribution(0,1), get value rand
- If(rand < ratio) chain[i+1] = rand otherwise chain[i+1]=chain[i]

1.2 (B)

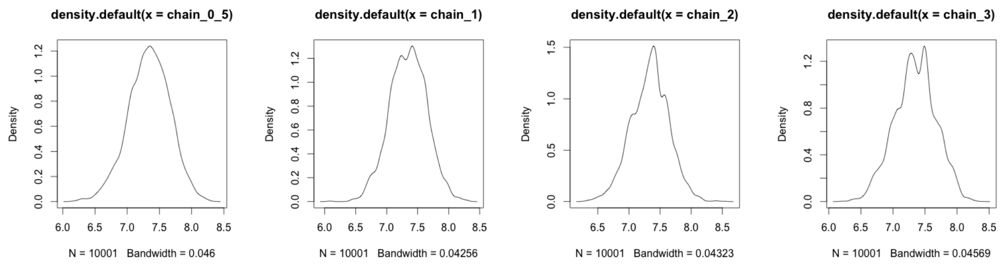
We test 4 variance: 0.5, 1, 2, 3. The result is shown in table 1.1:

1.3 (C)

the 0.95 credible interval with 4 different variance is shown in table 1.1

variance	mean	standard error	0.025	0.975
0.5	7.339742	0.007072086	6.671711	7.948032
1	7.346133	0.005802833	6.743288	7.934223
2	7.336704	0.008038947	6.731700	7.928981
3	7.345433	0.0110477	6.697364	7.956602

Table 1.1: Problem1(b and c). The estimated mean and stand error and 0.95 credible interval with different variance



(a) 0.5 (b) 1 (c) 2 (d) 3

Figure 1.1: The plot density with respect to different variance

1.4 (D)

The smoothed density function is shown in figure1.1

1.5 (E)

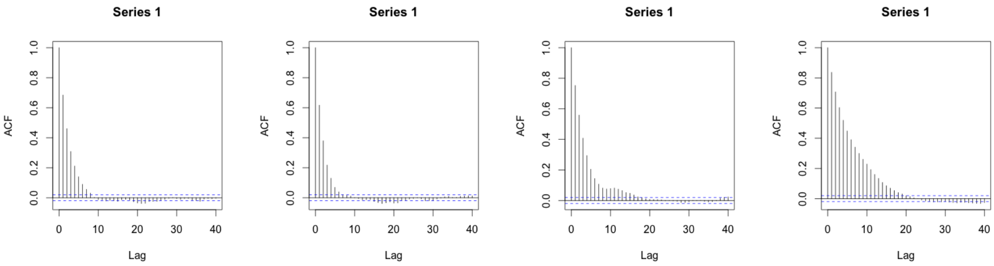
We chose **variance=1** and the acf result is as in figure 1.2. We could see that the **autocorrelation result is good**, and we could also find the **MCMC standard error is small** too. So the samples from variance equals to 1 is accurate. The result from **ess(samples) = 6679.076** which is larger than 5000. In terms of initialization, since I initialize after observation of the right hand side of equation 1, which means that I initialize β_1 with Gaussian(7.5, 1), and it has mean close to the final expectation of β_1.

2 Problem 2

According to Bayesian Theorem, we have the following equation:

$$p(\beta_0, \beta_1, \lambda / X, Y) = \frac{p(\beta_0, \beta_1, \lambda, X, Y)}{P(X, Y)} \propto p(X, Y / \beta_0, \beta_1, \lambda) * p(\beta_0) * p(\beta_1) * p(\lambda) \tag{2.1}$$

$$p(\beta_0 / \beta_1, \lambda, X, Y) \propto p(\beta_0) * EMG(\beta_0 + \beta_1 * X, 1, \lambda) \tag{2.2}$$



(a) 0.5 (b) 1 (c) 2 (d) 3

Figure 1.2: The plot of AutoCorrealtion with respect to different variance

$$p(\beta_1/\beta_0,\lambda,X,Y) \propto p(\beta_1) * EMG(\beta_0+\beta_1 * X,1,\lambda) \tag{2.3}$$

$$p(\lambda/\beta_1,\beta_1,X,Y) \propto p(\lambda) * EMG(\beta_0+\beta_1 * X,1,\lambda) \tag{2.4}$$

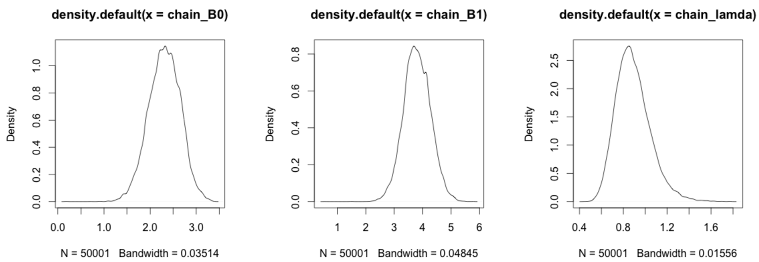
2.1 (A)

As mentioned before, we have all the paramters prior and their likelihood function given other paramteres. Here we only need to use "Variable-at-a-time" M-H algorithms to do sampling:
steps:

- Produce initial value of $\beta_1, \beta_0, \lambda$ based on uniform random sample form (0,1)
- Set Number of Iteration as 10000
- Initialize three vector callsed chainB0, chainB1, chainlambda to record the result samples
- Loop: i start from 1 to Number of Iteration
- textbfsampling β_0 according to current β_1 and λ
- Sampling according to the proposal distribution we get proposal value
- Calculate the propability of $ratio = \frac{posterior_B0(proposal_value)}{posterior_B0(chain_B0[i])}$
- random sample from uniform distribution(0,1), get value rand
- If(rand < ratio) chainB0[i+1] = rand otherwise chainB0[i+1]=chainB0[i]
- textbfsampling β_1 according to updated β_0 and λ
- Sampling according to the proposal distribution we get proposal value

variable	mean	standard error	0.025	0.975
β_0	2.34234	0.005767956	2.059455	2.616185
β_1	3.465962	0.007954217	3.058685	3.884110
λ	0.8035162	0.001290651	0.6946944	0.9322535

Table 2.1: Problem2(b). mean and standard error and 0.95 credible interval for 4 different variance



(a) distribution:B0 (b) Distribution:B1 (c) Distribution:lambda

Figure 2.1: Problem2(d): Posterior distribution of 3 parameter

- Calculate the propability of $ratio = \frac{posterior_B 1(proposal_value)}{posterior_B 1(chain_B 1[i])}$
- random sample from uniform distribution(0,1), get value rand
- If(rand < ratio) chainB1[i+1] = rand otherwise chainB1[i+1]=chainB1[i]
- textbfsampling λ according to updated β_0 and β_1
- Sampling according to the proposal distribution we get proposal value
- Calculate the propability of $ratio = \frac{posterior_lambda(proposal_value)}{posterior_lambda(chain_lambda[i])}$
- random sample from uniform distribution(0,1), get value rand
- If(rand < ratio) chainlambda[i+1] = rand otherwise chainlambda[i+1]=chainlambda[i]
- Back to the begining of the loop

2.2 (B)

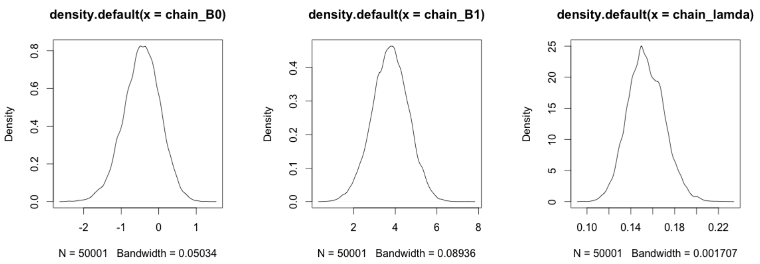
Table as shown in table 2.1

2.3 (C)

cor(chainB0,chainB1) = -0.8284228

variable	mean	standard error	0.025	0.975
β_0	-0.4528313	0.01067583	-1.4534577	0.4635137
β_1	3.752007	0.01899184	2.019387	5.416402
λ	0.1538976	0.0002073191	0.1232535	0.1872349

Table 3.1: Problem3(a). mean and standard error and 0.95 credible interval for 4 different variance



(a) distribution:B0 (b) Distribution:B1 (c) Distribution:lambda

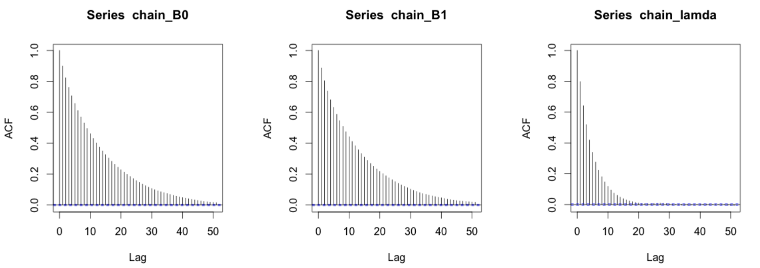
Figure 3.1: Problem3(b): Posterior distribution of 3 parameter

2.4 (E)

3 PROBLEM 3

3.1 (C)

The autocorrelation of B0, B1, lamda are shown in figure 3.2 which shows good autocorrelation result. The standard error of the MCMC estimate is also small. So basically the result is trustworthy.



(a) distribution:B0 (b) Distribution:B1 (c) Distribution:lambda

Figure 3.2: Problem3(c): autocorrelation result for 3 parameters