STAT 515
Homework #7, due Friday, Mar. 23 at 2:30pm
This homework may be submitted electronically to ANGEL, though this is not required. I strongly encourage the use of LaTeX in any case.

Please make every assignment easier to grade by neatly organizing your writeup and clearly labeling your final answers when appropriate. Try using LaTeX!

1. Customers arrive at a post office at a Poisson rate of 8 per hour. There is a single person serving customers, and service times are exponentially distributed (and independent) with mean 5 minutes. Suppose that an arriving customer will decide to wait in line if and only if there are two or fewer people already in line.

   (a) In the long run, what fraction of the time will there be at least 1 customer in the post office? Find the answer in two different ways:

      i. Write out the rate matrix (or generator) for the continuous-time Markov chain and find the stationary distribution using the generator.

      ii. Use the fact that this is a birth-death process to find the stationary distribution that satisfies the detailed balance equations.

   (b) In the long run, what is the expected number of customers in the post office (in line or being served) at any given time?

   (c) What is the probability that an arriving potential customer will decide to leave because there are already 3 people in line?

   (d) If a new cash register is installed that decreases the mean service time to 4 minutes, how many more customers per hour, on average, can be served by this post office?

   (e) The manager of the post office wants to be able to serve at least 95% of the potential customers who arrive at the post office. What mean service time will attain this goal?

2. Suppose that a continuous-time Markov chain $X(t)$ has rate matrix

$$R = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}.$$

   Given positive times s and t, calculate $\mathrm{Corr}\,[X(s), X(t)]$. Does your answer depend on the starting state of the chain (i.e., value of $X(0)$)?

   (NB: The formula for $\mathrm{Corr}\,(X,Y)$ is $\mathrm{Corr}\,(X,Y) = \mathrm{Cov}\,(X,Y)/\sqrt{\mathrm{Var}\,(X)\,\mathrm{Var}\,(Y)}$. Since the correlation is invariant to linear transformations, the two possible values that $X(t)$ may take do not influence your answer.)

3. At an amusement park, there are two video game machines. Suppose that for video game i, each period when it is being used is exponentially distributed with rate α_i and each period when it is not being used is exponentially distributed with rate β_i, independent of the other machines.

   (a) Suppose that the vector $M(t)$ is given by

$$M(t) = [M_1(t), M_2(t)]^\top,$$

   where $M_i(t) = I\{\text{machine } i \text{ is being used at time } t\}$ for $i = 1, 2$. The Markov chain $M(t)$ has four states; give its rate matrix R.

   (b) In the long run, what proportion of time are both machines being used?

   (c) When the amusement park first opens, each machine is in its unused state. We can express this fact by $M_1(0) = M_2(0) = 0$. Simulate 10,000 independent realizations of this chain, until time $t = 3$, using $\alpha_1 = 2$, $\alpha_2 = 3$, $\beta_1 = 5$, and $\beta_2 = 6$. From your simulations, give an empirical estimate of the proportion μ of time in $(0, 3]$ that the machines are used. Report a 95% confidence interval for μ. How does this compare with the long-run value calculated in part (b)?

   To find an approximate 95% confidence interval for a mean μ based on a i.i.d. sample of size n, take

$$\hat{\mu} \pm 1.96 \frac{s}{\sqrt{n}},$$

   where $\hat{\mu}$ is the sample mean and s is the sample standard deviation.