STAT 515
Homework #4, due Friday, Feb. 17 at 2:30pm

Please make every assignment easier to grade by neatly organizing your writeup and clearly labeling your final answers when appropriate. Try using LaTeX!

1. Suppose that in a branching process with $X_0 = 1$, each individual produces some number of offspring that is Poisson with mean 1, independently of all other individuals.

   (a) What is the expected number of generations until the process either dies out or attains size $X_n \geq 5$?

   (b) What is the probability that the process will ever attain size $X_n \geq 5$?

2. Define a Markov chain on the nonnegative integers as follows: $P_{0j} = I\{j = 1\}$, and for $i > 0$,

$$P_{ij} \quad = \quad \begin{cases} i/(i+1) & \text{if } j = i+1 \\ 1/(i+1) & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Argue that this chain is irreducible and aperiodic.

   (b) Prove that all states are recurrent.

   (c) Prove that all states are null recurrent. (You may assume without proof that null recurrence is a class property.)

3. Consider the symmetric one-dimensional random walk of Example 4.15 with $p = 1/2$.

   (a) Let T_i be the time at which the random walk first revisits state i given that it begins in state i. That is, $T_i = \inf\{n > 0 : X_n = i \mid X_0 = i\}$. For any $n > 0$, find $P(T_i = 2n)$.
   Hint: Read the ballot problem example of Section 3.5 and the discussion following it.

   (b) Prove that all states are null recurrent by showing that $E(T_i) = \infty$.
   Hint: Read the random walk example of Section 4.3 for an idea about how to show this.

4. Read the random walk example of Section 4.8 and the discussion of the Ehrenfest model following it.

   (a) Simulate the simple Ehrenfest diffusion process with total number of particles $M = 30$. Start the process at $X_0 = 10$. Run the process for 100,000 steps and draw a histogram of the resulting values of X_t.

   (b) On the same histogram, indicate the true values that would be expected from a sample of size 100,000 from a binomial($n = 30, p = 1/2$) distribution. See the example R code for ideas on how to do this.

   (c) Explain what you observe from the comparison in part (b). Is the Markov chain you are simulating ergodic?

5. Let Q be a transition "matrix" for an irreducible Markov chain on the set $\mathbb{Z}$ of all integers; i.e., $Q_{ij} = P(X_n = j \mid X_{n-1} = i)$ for all $i, j \in \mathbb{Z}$. Assume that $Q_{ij} > 0$ if and only if $Q_{ji} > 0$ and that $Q_{ii} > 0$ for all i. Also suppose that

$$\sum_{i \in \mathbb{Z}} \pi_i = 1 \quad \text{and } \pi_i > 0 \text{ for all } i \in \mathbb{Z}$$

   and define

$$\alpha(i, j) = \min\left(\frac{\pi_j Q_{ji}}{\pi_i Q_{ij}}, 1\right) \quad \text{for all } i, j \in \mathbb{Z}.$$

   Consider a second Markov chain with transition probabilities given by

$$P_{ij} = \begin{cases} \alpha(i,j) Q_{ij} & \text{if } j \neq i \\ Q_{ii} + \sum_{k \neq i} Q_{ik}(1 - \alpha(i,k)) & \text{if } i = j, \end{cases}$$

   Using time reversibility arguments, show that the second Markov chain has stationary probabilities $\{\pi_i\}$ and that these stationary probabilities are also the limiting probabilities of the Markov chain.

6. The lifetimes of two machines are independent with exponential distributions with rates λ_1 and λ_2, respectively. Suppose machine 1 starts working now and machine 2 starts working t units of time later. What is the probability that machine 1 will fail before machine 2?