Some Statistical Challenges in Studying the West Antarctic Ice Sheet

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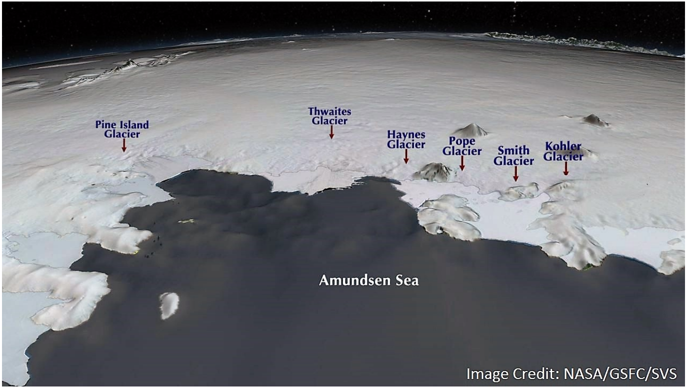
Ice Sheets

- Enormous mass of glacial land ice
- Antarctic ice sheet is over 14 million km^2.
  - $\approx$ United States + Mexico combined.
- Greenland ice sheet over 1.7 million km^2
- Combine for over 99% of the freshwater ice in the world
- Can have major impacts on sea level rise:
  - Melting entire Antarctic ice sheet: sea level rise $\approx$ 57 m.
  - Melting entire Greenland ice sheet: sea level rise $\approx$ 7 m
- Even a modest contribution to sea level rise can have a major impact on humans, e.g. on low-lying areas and areas prone to storm surges.

Ice Streams

- Corridors of fast flow within an ice sheet
- Discharge most of the ice and sediment from the ice sheets
- Flow orders of magnitude faster than their surrounding ice.
- Their behaviour and stability is important to overall ice sheet dynamics and mass balance. (cf. Bennett, 2003)

The West Antarctic Ice Sheet with Ice Streams



Consider Two Problems

1. Ice sheets
   - Sophisticated model for ice sheet, PSUICE3D (DeConto and Pollard, 2009, 2011). "Computer model": cannot work with the mathematical form of the model, treat it as a simulator.
   - Computer model emulation-calibration: infer parameters of the computer model using:
     - Model simulations
     - Observations of the ice sheet: modern (spatial) and paleo-reconstructed (temporal)
2. Ice streams
   - Simple dynamic model for ice stream. Can build hierarchical statistical model on top of it.
   - Full statistical inference: infer model parameters + ice sheet thickness.
     - Analytically tractable physical model
     - Surface observational data sets of the ice sheet

Ice Sheets Versus Ice Streams

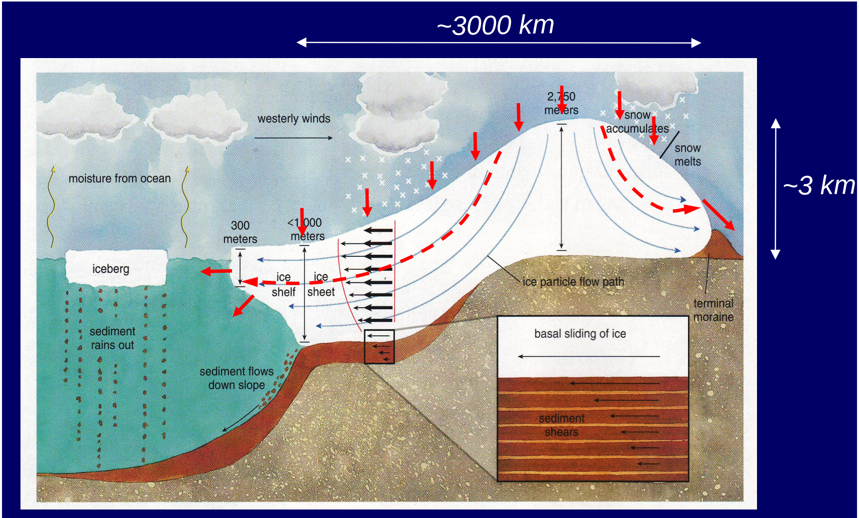
Major difference in scale/model complexity

- Physical models for the West Antarctic ice sheet based on Pollard and De Conto (2009, 2011).
  - Treat the physics like a **black box** and estimate parameters. Computer model emulation-calibration for high-dimensional binary spatial fields.
- With ice streams: we work **directly** with much simpler model. Are interested in parameter estimation as well as interpolation of ice thickness/bedrock topography.

Part 1 Overview: Statistics for Ice Sheets

- How can we project the future behavior of the West Antarctic Ice Sheet?
  - Ice sheet model: PSU3D-ICE (Pollard and DeConto, 2009).
- Key model input parameters are uncertain
- Observations:
  1. Satellite data on the modern ice sheet.
  2. Paleo reconstructions of ice sheet from 25,000 years ago to present time.
- Our research: methods to use observations of the ice sheet to infer important parameters of the ice sheet model.
- Challenges
  1. Two sets of data: spatial and temporal binary/non-Gaussian data. "Data" = Observations and computer model output.
  2. High-dimensional spatial data
- I will describe statistical methods to address these issues.

Ice Sheet Physics

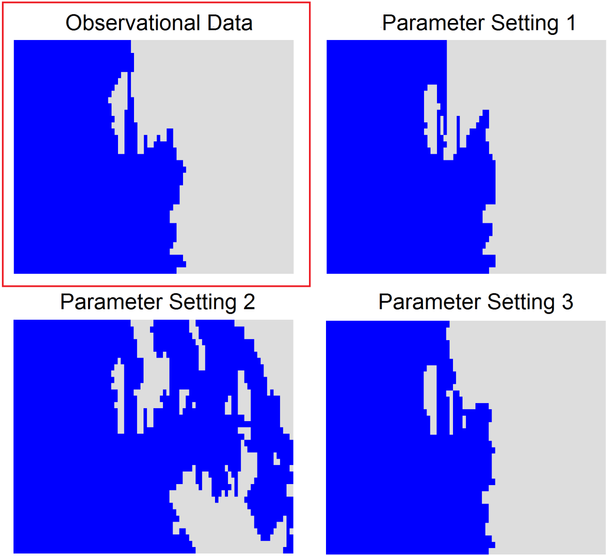


Siegert, 2002, Amer. Sci.

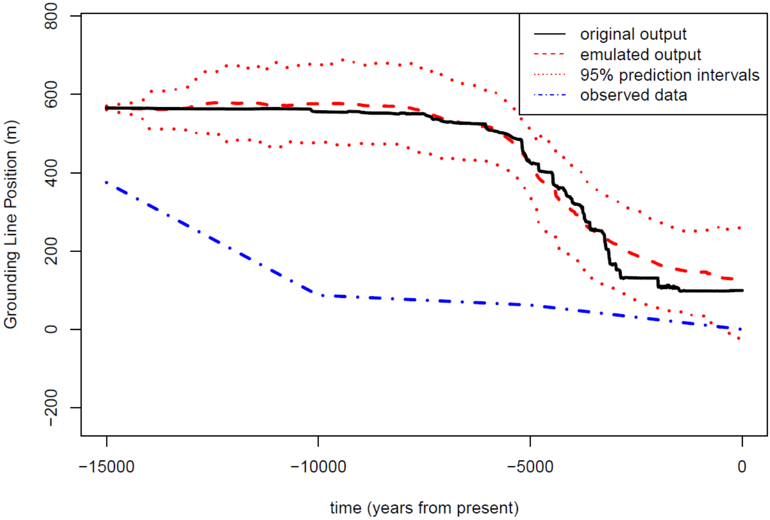
Ice Sheet Model

- The ice sheet's behavior is complex. Model simulations are slow.
- Model equations predict ice flow, thickness, temperatures, and bedrock elevation, through thousands to millions of years.
- Examples of key model parameters:
  - Ocean melt coefficient: sensitivity of ice sheet to temperature change in the surrounding ocean
  - Strength of the "calving" process. Calving = where ice breaks off and transitions from attached to floating
  - "Slipperiness" of the ocean floor

Satellite Observations versus Model Output



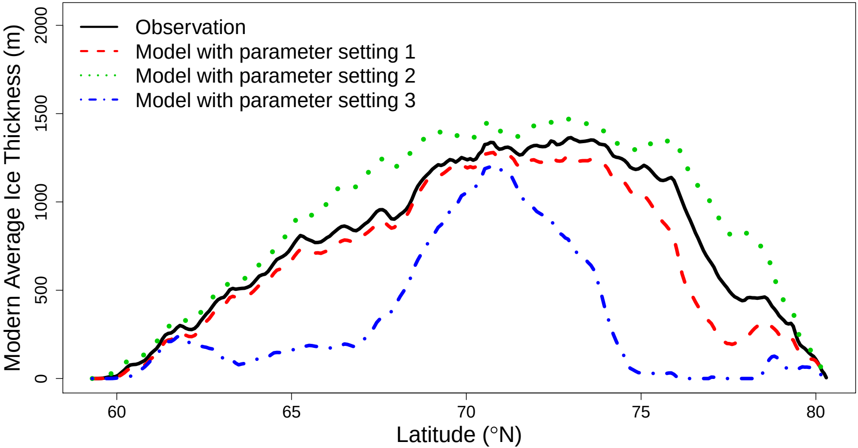
Paleo Data



original output
emulated output
95% prediction intervals
observed data
Grounding Line Position (m)
800
600
400
200
0
−200
−15000
−10000
−5000
0
time (years from present)

Aggregated Ice Sheet Data: Example

- To avoid binary spatial data: aggregate across longitude.

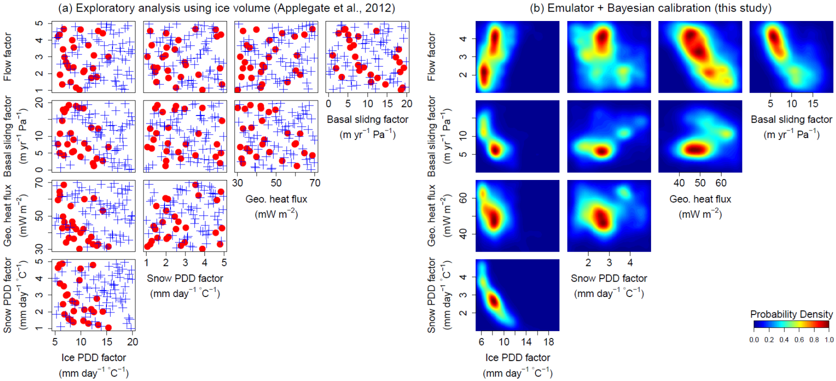


What is the value of using disaggregated data?

Aside: How Does Statistical Rigor Help Scientists?

1. We account for (epistemic) uncertainties in emulation
2. We provide *real* probability distributions, very important for impacts/risk quantification.
3. We use all available information (no aggregation): often reduces uncertainties.
4. We provide sharper/more useful results.
5. **Distributions are interpretable, resulting projections are interpretable.**

Example of Sharper Results



Left: previous ad-hoc methods. Right: statistical calibration
Chang, Haran, Olson, Keller (2014), *Annals of Applied Stats*

Two-stage Approach to Emulation-Calibration

1. Emulation step: Find fast approximation for computer model using a Gaussian process (GP).

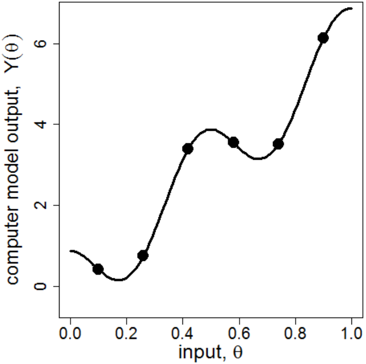
2. Calibration step: Infer climate parameter using emulator and observations, while accounting for data-model discrepancy

   Modularization

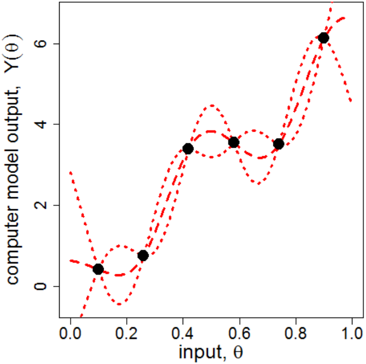
   - Liu, Bayarri and Berger (2009)
   - Bhat, Haran, Olson, Keller (2012)
   - Chang, Haran, Applegate, Pollard (2016a; 2016b)

Emulation Step

Toy example: model output is a scalar, and continuous.



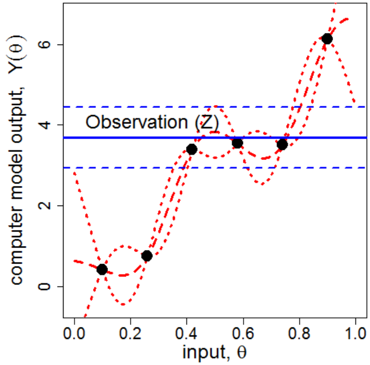
Computer model output (y-axis) vs. input (x-axis)



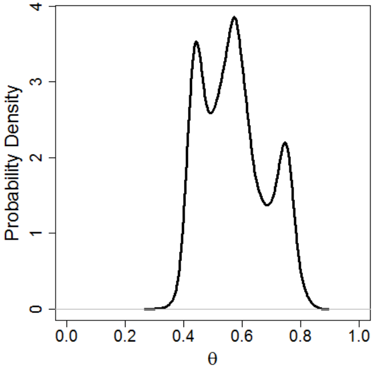
Emulation (approximation) of computer model using GP

Calibration Step

Toy example: model output, observations are scalars



Combining observation
and emulator



Posterior PDF of θ
given model output and observations

Summary of Statistical Problem

- **Goal**: Learn about $\boldsymbol{\theta}$ based on two sources of information:
  - **Observations**:
    1. Observed time series of past grounding line positions reconstructed from paleo records: $\mathbf{Z}_1 = (Z_1(t_1), \ldots, Z_1(t_n))^T$, $t_1, \ldots, t_n$ are time points locations.
    2. Observed modern ice-no ice from satellite data: $\mathbf{Z}_2 = [Z_2(\mathbf{s}_1), \ldots, Z_2(\mathbf{s}_m)]$, locations $\mathbf{s}_1, \ldots, \mathbf{s}_m$.
  - **Model output**
    1. $\mathbf{Y}_1(\boldsymbol{\theta}_1), \ldots, \mathbf{Y}_1(\boldsymbol{\theta}_p)$, where each $\mathbf{Y}_1(\boldsymbol{\theta}_i) = (Y_1(\boldsymbol{\theta}_i, t_1), \ldots, Y_1(\boldsymbol{\theta}_i, t_n))^T$ is a time series of grounding line positions at parameter setting $\boldsymbol{\theta}_i$.
    2. $\mathbf{Y}_2(\boldsymbol{\theta}_1), \ldots, \mathbf{Y}_2(\boldsymbol{\theta}_p)$, where each $\mathbf{Y}_2(\boldsymbol{\theta}_i) = (Y_2(\boldsymbol{\theta}_i, \mathbf{s}_1), \ldots, Y_2(\boldsymbol{\theta}_i, \mathbf{s}_m))^T$ is a vector of spatial data at parameter setting $\boldsymbol{\theta}_i$.

Step 1: Computer Model Emulation Basics

- Fit Gaussian process model for computer model output $\mathbf{Y}$ to interpolate the values at the parameter settings $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_p$ *and* the spatial locations $\mathbf{s}_1, \ldots, \mathbf{s}_n$

$$\text{vec}\,(\mathbf{Y}) \sim N(\mathbf{X}\beta, \boldsymbol{\Sigma}(\boldsymbol{\xi}_y)),$$

  vec$(\cdot)$ concatenates columns into one vector
- $\boldsymbol{\beta}$ and $\boldsymbol{\xi}_y$ estimated by maximum likelihood, $\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\xi}}_y$.
- Covariance interpolates across spatial surface *and* input space.

Result: Obtain a probability model (from predictive distribution) for model output at any input parameter θ, $\boldsymbol{\eta}(\boldsymbol{\theta}, \mathbf{Y})$.

Step 2: Calibration Basics

- Discrepancy $\approx$ mismatch between computer model output and data when parameters are perfectly calibrated and there is no observational error.
- Probability model for observations $\mathbf{Z}$ is then

$$\mathbf{Z} = \eta(\boldsymbol{\theta}, \mathbf{Y}) + \boldsymbol{\delta},$$

where n-dimensional spatial field $\boldsymbol{\delta}$ is model-observation discrepancy with covariance parameter $\boldsymbol{\xi}_\delta$.

- Inference for $\boldsymbol{\theta}$ based on posterior distribution

$$\pi(\boldsymbol{\theta}, \boldsymbol{\xi}_\delta | \mathbf{Z}, \mathbf{Y}, \hat{\boldsymbol{\xi}}_y) \propto \underbrace{\hat{\mathcal{L}}(\mathbf{Z}|\mathbf{Y}, \boldsymbol{\theta}, \boldsymbol{\xi}_\delta, \hat{\boldsymbol{\xi}}_y)}_{\text{likelihood given by above}} \times \underbrace{p(\boldsymbol{\theta}) \times p(\boldsymbol{\xi}_\delta)}_{\text{priors for } \boldsymbol{\theta} \text{ and } \boldsymbol{\xi}_\delta}$$

with emulator parameter $\hat{\boldsymbol{\xi}}_y$ fixed at MLE.

Some Statistical Methods/Challenges

- Emulation-calibration (Kennedy and O'Hagan, 2001)
- With climate models, output are vectors/spatial data
  - Bayarri et al. (2007, 2008, ...)
  - Climate model (4 scalar diagnostics): Sanso, Forest, Zantedeschi (2008)
  - Spatial/time series data: Bhat, Haran, Olson, Keller (2010a); Bhat, Haran, Goes (2010b); Olson et al. (2012)
  - High-dimensional spatial: Chang, Haran, Olson, Keller (2014)
  - Spatial binary (non-Gaussian): Chang, Haran, Applegate, Pollard (2016a, b)
- Major ongoing challenges, often includes computational issues
  - Very high-dimensional output, dependent, e.g. multivariate spatial, temporal; space-time output (dynamic)
  - Lots of parameters
  - Handling data-model discrepancies; complex errors/dependencies in data
  - Multiple models, multiple scales (EMICs of various kinds)

Principal Components for Emulation: Basic Idea

Popular approach for handling high-dimensional output (Higdon et al., 2008; Chang, Haran, Olson, Keller, 2012)

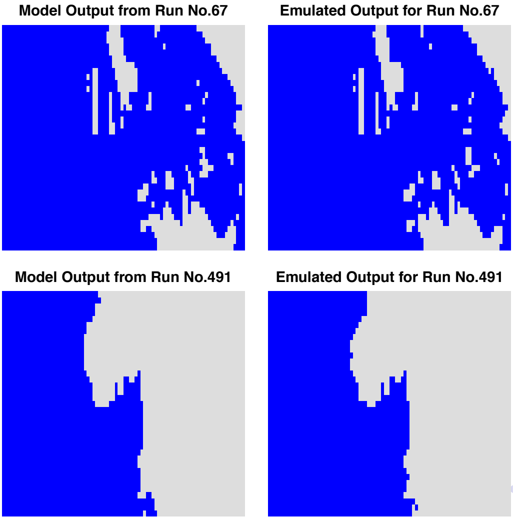
- Consider model outputs at $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_p$ as if they were replicates of a multivariate process, thereby obtaining their PCs

$$\begin{pmatrix} Y(\mathbf{s}_1, \boldsymbol{\theta}_1) & \dots & Y(\mathbf{s}_n, \boldsymbol{\theta}_1) \\ \vdots & \ddots & \vdots \\ Y(\mathbf{s}_1, \boldsymbol{\theta}_p) & \dots & Y(\mathbf{s}_n, \boldsymbol{\theta}_p) \end{pmatrix}_{p \times n} \Rightarrow \begin{pmatrix} Y_1^R(\boldsymbol{\theta}_1) & \dots & Y_{J_y}^R(\boldsymbol{\theta}_1) \\ \vdots & \ddots & \vdots \\ Y_1^R(\boldsymbol{\theta}_p) & \dots & Y_{J_y}^R(\boldsymbol{\theta}_p) \end{pmatrix}_{p \times J_y}$$

- PCs pick up characteristics of model output that vary most across input parameters $\theta_1, \dots, \theta_p$.
  Surprisingly flexible approach, allowing for non-separable covariance.

Emulation Examples

- Train on some, test on "hold-out" parameter settings
- Fast way to study how model behaves across parameter space

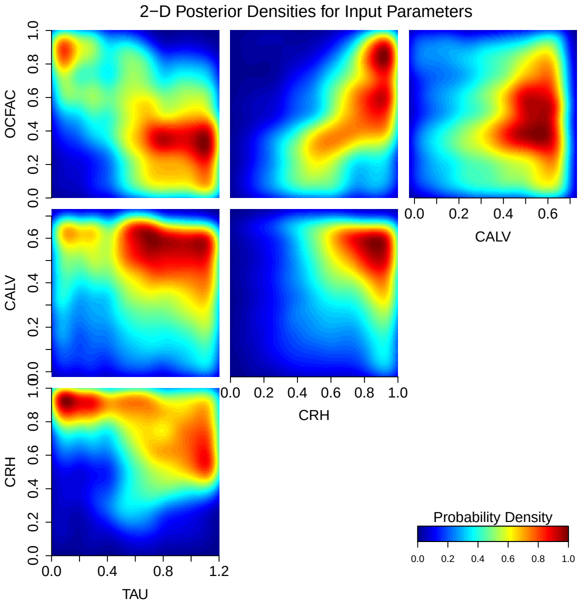


Efficient Calibration

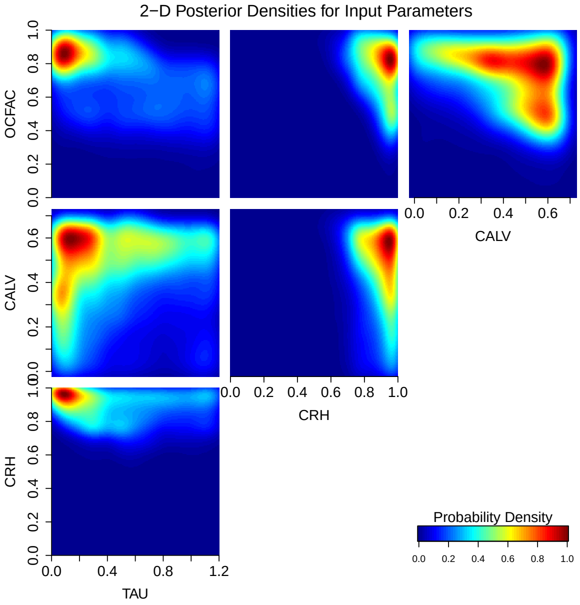
Main ideas:

- Utilize the data to derive information about the model-data discrepancy.
- Use a basis representation for model-data discrepancy term.
- Markov chain Monte Carlo

Calibration Results with Modern Data



Calibration Results with Modern and Paleo Data

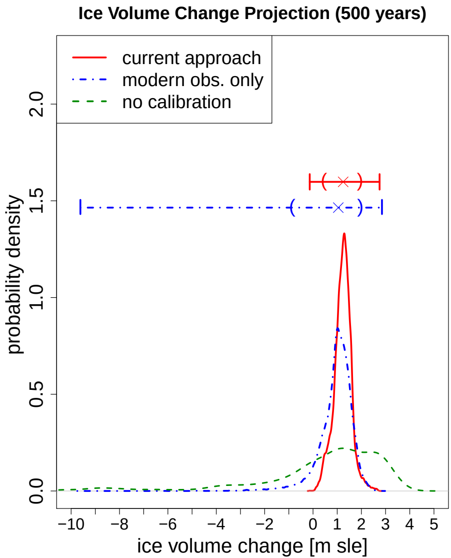


Paleo data eliminates unrealistic trajectories of the ice sheet.

Why are the Results Different?

- Unrealistic simulations with overshoots in past ice retreat and projected future regrowth are eliminated
- This “constrains” the probability distribution of the parameters

Ice Volume Projections: Paleo Cuts Off Left Tail



Possibility of "no sea level rise" is virtually eliminated.

Summary Remarks

- Statistical methods provide interpretable inference about parameters + probabilistic projections
- Have to think hard about computing
- Can study value of using more data (disaggregated) and multiple sources of data
- Central theme: combining physics with statistical modeling
- Lots of open challenges

Chang, W., Haran, M, Applegate, P., Pollard, D. (2016a, b)
Chang, W., Applegate, P., Haran, M. and Keller, K. (2016)
Chang, W., M. Haran, R. Olson, and K. Keller (2014)

Part 2 Overview: Ice Streams

- The West Antarctic Ice Sheet is drained by fast-flowing ice streams
- These are major contributors to ice loss
- Key components for understanding ice stream's stability and dynamics: ice thickness and bedrock topography.

Of interest:

- Interpolate ice thickness while obeying the underlying physics
- Estimate unknown quantities in a mathematical model that we use to understand ice stream dynamics.

Proposal:

- Bayesian approach that incorporates information from multiple data sets combined with a mathematical model that describes the physics of the ice stream.
- Focus on Thwaites glacier, covers an area of 182,000 km^2 ($\approx$ Italy/2). Estimated ice loss has doubled since the 1990s.
- This is preliminary work toward developing more widely applicable methodology.

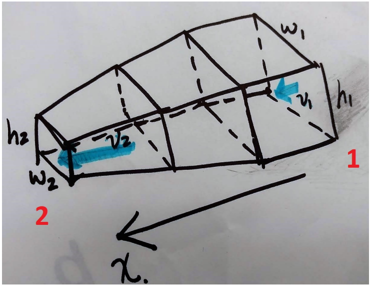
Basic Idea

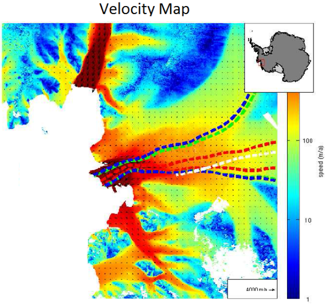
- Principal physics commonly applied is conservation of ice mass
- Given observations of surface elevation, ice velocity and surface mass balance, we can deduce the ice thickness assuming the ice sheet is in a steady state. Examples:
  - Using coarse grid in Antarctica (Warner and Budd, 2000)
  - Higher resolution in Greenland (Morlighem et al., 2011, 2013, 2014)
- Here we add to the physics by
  1. Adding a new component to dynamics model: shallow ice approximation (SIA).
  2. Including the varying glacier width to account for tributaries, which contribute to mass flux
- Model is still simple enough that we can solve it quickly.
- Our statistical model accounts for errors/uncertainties.

Mathematical Flowline Model

- Model ice as an incompressible material (constant density)
- Flux: the action or process of flowing or flowing out $=$Velocity Field ($\bar{V}$) $\times$ Surface Area ($H\omega$)
- Mass conservation along the flowline $\implies$ Flux at 1 - Flux at 2 $= 0$
- For an open rectangular channel: snow accumulation

$$\frac{\partial(\bar{V}_x H_x \omega_x)}{\partial x} = a_x \omega_x$$



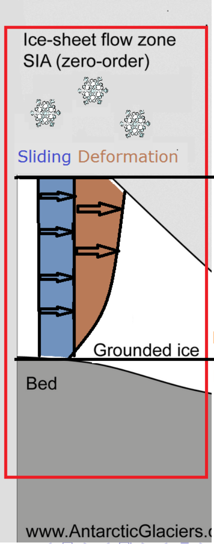


Flowline Model Adjustment

- Depth-averaged velocity cannot be observed
- We use surface velocity with adjustment as an approximation
- Adjustment accounts for ice deformation based on the Shallow Ice Approximation (e.g. Van der Veen, 2013)

$$\frac{\partial}{\partial x}\left(\overbrace{\left(V_x - \boxed{\frac{\alpha}{20}\left(\rho g |S_x|\right)^3 H_x^4}\right)}^{U_x} H_x \omega_x\right)$$

$$= a_x \omega_x$$



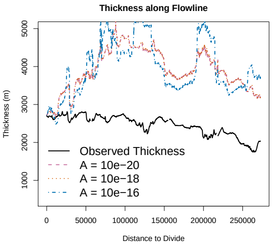
a_x : accumulation
E_x : surface elevation
S_x : |slope|
V_x : surface velocity

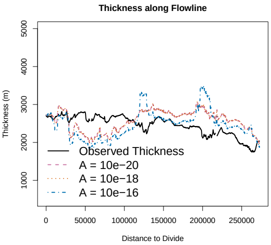
α : deformation coefficient
H_x : thickness

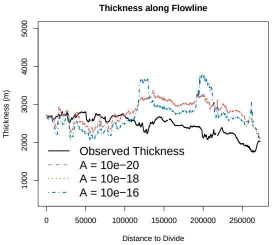
B_x : sliding coefficient
D_x : bed elevation

Non-Statistical Approach

- For each flow width, we can predict thickness according to the flowline model.
- Results for three different flow widths, wide, medium, narrow:







All are poor reconstructions
No uncertainty estimates
Solutions are not unique; do not exist for some locations

Data Sets

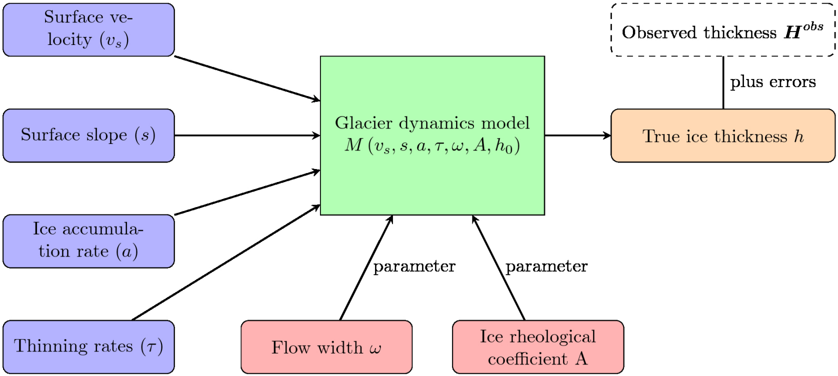
Data Set	Spatial resolution
Surface velocity, m/year (V_s)	450 m
Surface elevation, m (E)	$\sim$ 14 m
Net ice accumulation rate*/year (a)	55 km
Thinning rate/year (τ)	$\sim$ 1.5 - 5 km
Ice thickness, m (H)**	$\sim$ 14 m

1. Ice equivalent (ice eq.)
2. We use only 5, 10 and 25 observed thickness to fit our model to keep our method realistic for applying to other glaciers on WAIS

Uncertainties in Modeling an Ice Stream

- Spatially sparse thickness observations, $H_{x_i}^{obs}, i = 1, \ldots, m$, subject to observational errors
  - Assume model errors are normally distributed
- Unknown quantities in the mathematical model (**parameters**)
  - Deformation coefficient α
  - Flow width $\mathbf{s}\omega_x$, modeled with a latent Gaussian Process
- Discrepancy between mathematical model and true process
- Observational errors in the input processes (surface velocity, surface slope and snow accumulation rate)

Hierarchical Bayesian Model



Model Details

Goal: predict thickness and inference for deformation coefficient α

Flow width ("nuisance parameter") $\mathbf{s}\omega_x$ needed for thickness prediction

- Ice Thickness Model:
  - observations model: $\mathbf{s}H^{\text{obs}} \mid \mathbf{s}h, \mathbf{s}\theta \sim N(\mathbf{s}h, \sigma_H^2 \mathbf{s}I)$
  - physics (deterministic) model: $h \mid v, s, a, \omega, \mathbf{s}\theta = M\left(v, s, a, \omega, \alpha\right)$
- Flowline Width Model: $\mathbf{s}\omega \mid \mathbf{s}\theta \sim \text{GP}\left(0, C(\mathbf{s}\theta_\omega)\right)$
- Input Process Model: $v, s, a \mid \mathbf{s}\theta, \mathbf{s}V^{\text{obs}}, \mathbf{s}S^{\text{obs}}, \mathbf{s}a^{\text{obs}}$ $\sim f(v \mid \mathbf{s}V^{\text{obs}}, \mathbf{s}\theta_v) f(s \mid \mathbf{s}S^{\text{obs}}, \mathbf{s}\theta_s) f(a \mid \mathbf{s}a^{\text{obs}}, \mathbf{s}\theta_a)$
  - velocity model: $f(v \mid \mathbf{s}V^{\text{obs}}, \mathbf{s}\theta_v)$
  - slope model: $f(s \mid \mathbf{s}S^{\text{obs}}, \mathbf{s}\theta_s)$
  - accumulation rate model: $f(a \mid \mathbf{s}a^{\text{obs}}, \mathbf{s}\theta_a)$
- Prior: $p(\mathbf{s}\theta) = p(\sigma_H^2) p(\alpha) p(\mathbf{s}\theta_\omega) p(\mathbf{s}\theta_v) p(\mathbf{s}\theta_s) p(\mathbf{s}\theta_a)$

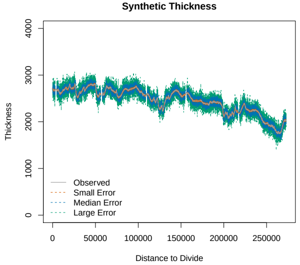
	Thickness	Velocity	Slope	Accum.rate
Observation:	$\mathbf{s}H^{obs}$	$\mathbf{s}V^{obs}$	$\mathbf{s}S^{obs}$	$\mathbf{s}a^{obs}$
True processes:	h	v	s	a

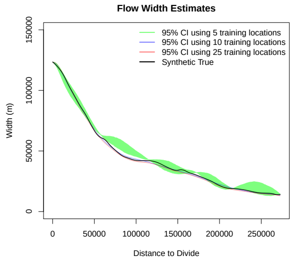
Simulated Example

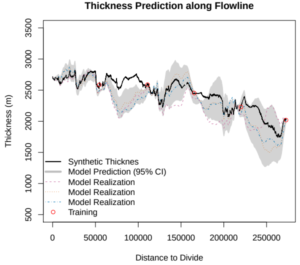
data sets

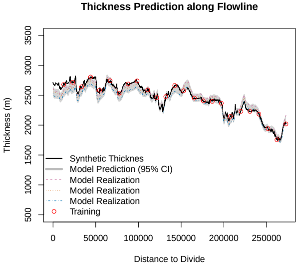
- Simulate synthetic data using the followings:
  - $A = 10^{-17}$
  - Smooth surface elevation, accumulation rate
  - Observed flow width
  - Observed ice thickness
- Simulate velocity from flowline model
- Add noise to observed ice thickness to create synthetic data

Simulation Results

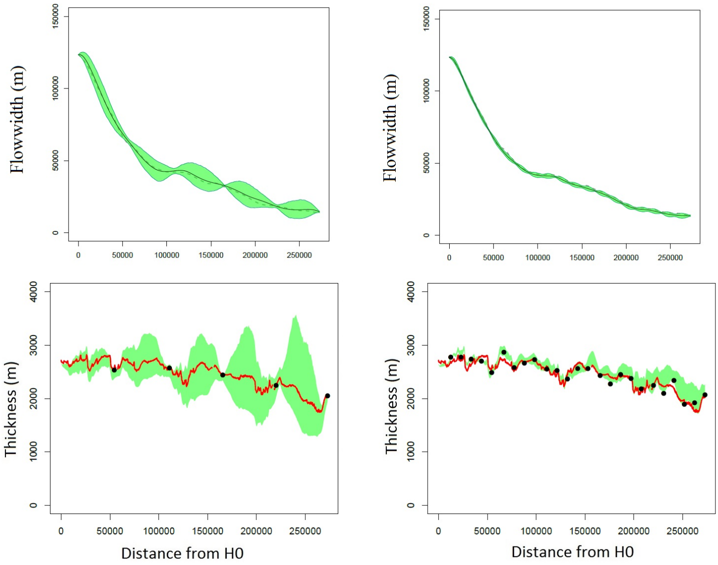








Thwaites Glacier



Conclusions

- Our approach can potentially be used for other glaciers in Antarctica with:
  - reliable surface data
  - sparse thickness observation
- Bayesian methods and computational tools allow us to combine:
  - multiple data sets
  - glacier dynamics model
  - estimate thickness while accounting for uncertainties
- Caveats:
  - Unable to account for all errors in the input processes
  - Model discrepancy is absent in hierarchical model

Guan, Y., Haran, M., Pollard, D. (2017), *Environmetrics*, in press.
`https://arxiv.org/abs/1612.01454`

Concluding Thoughts

- Theme: combining physical model + statistical model
  - (Note to statisticians:) physics is central to this research
  - (Note to modelers/domain scientists:) statisticians have nice methods for working with physics/mathematical models + complex error structures
- When mathematically tractable physical model: can build rich hierarchical model that directly handles error structures
- Even when model can only be studied through simulation (mathematically intractable), there are statistical methods for careful uncertainty quantification
- Lots of challenges:
  - size of data sets/model output
  - complexity of output: spatial, temporal, spatio-temporal, multiple spatio-temporal...
  - complexity of model
  - dimensionality of unknowns
  - dependencies, errors in (quality of) data

Appendix

BEGIN APPENDIX

Emulation-Calibration with Binary Spatial Output

- Now $Y(\boldsymbol{\theta}, \mathbf{s})$ is binary (0-1) model ouput, $Z(\mathbf{s})$ is data.
- Let $\boldsymbol{\Gamma}_{p \times n}$ be matrix of natural parameters for model output: $\gamma_{ij}^{Y} = \log\left(\frac{p_{ij}}{1-p_{ij}}\right)$ is logit for ith parameter setting at jth spatial location and $p_{ij} = P(Y(\boldsymbol{\theta}_i, \mathbf{s}_j) = 1)$.
- Given $\boldsymbol{\Gamma}$, $Y(\boldsymbol{\theta}_i, \mathbf{s}_j)$'s are conditionally independent Bernoulli.
- Approach (sketch):
  1. Assume it is possible to estimate $\boldsymbol{\Gamma}$ from the $n \times p$ matrix of computer model output.
  2. Emulate computer model by *interpolating natural parameters* using a Gaussian process across input parameter space and spatial locations.
  3. Calibration by using fitted Gaussian process $\boldsymbol{\eta}(\boldsymbol{\theta}, \mathbf{Y})$ + discrepancy $\boldsymbol{\delta}$ to obtain a likelihood function for the *natural parameter vector for observations*.

Challenges

- Step 1 (obtaining $\mathbf{\Gamma}$) is ill-posed: np parameters for np data points.
- Step 2 (emulation) is computationally infeasible: Cholesky factorization has computational cost of $\frac{1}{3} \times p^3 \times n^3 = \frac{1}{3} \times 499^3 \times 3,182^3 = 1.33 \times 10^{18}$ flops
- Step 3 (calibration): involves having to perform a high-dimensional integration + expensive matrix operations.

We propose dimension-reduction to address both ill-posedness and computational issues.

Efficient Emulation: Outline

- Rewrite $\mathbf{\Gamma}$ in terms of logistic principal components (Lee et al., 2010).
- Use maximum likelihood to perform logistic principal components. Non-trivial, requires majorization-minimization (MM) algorithm (Lange et al. 2000; Hunter and Lange, 2004).
- Emulate each column of (reduced-dimensional) principal component matrix with an independent Gaussian process. Very fast and easy to do.
- We can obtain an emulator for $\mathbf{\Gamma}$ by emulating these principal components.

Dimension-reduction

- Consider $\mathbf{\Gamma}$ the $p \times n$ matrix of natural parameters for model output. Using logistic principal components (Lee et al., 2010), rewrite as:
$$\mathbf{\Gamma} = \mathbf{1}_p \otimes \boldsymbol{\mu}^T + \mathbf{W}\mathbf{K}_y^T, \tag{1}$$
where $\mathbf{K}_y$ is an $n \times J_y$ orthogonal basis matrix, $\mathbf{W}$ is the $p \times J_y$ principal component matrix with (i,j)th element $w_j(\boldsymbol{\theta}_i)$, and $\boldsymbol{\mu}$ is the $n \times 1$ mean vector.
- Non-trivial and computationally challenging optimization to obtain matrices $\mathbf{W}$, $\mathbf{K}_y$ by maximizing log-likelihood. Use majorization-minimization (MM) algorithm (Lange et al. 2000; Hunter and Lange, 2004).
- Emulate each column of $\mathbf{W}$ using a separate Gaussian process.
- (Analogous to Gaussian emulation) By emulating these principal components we can emulate the original process.

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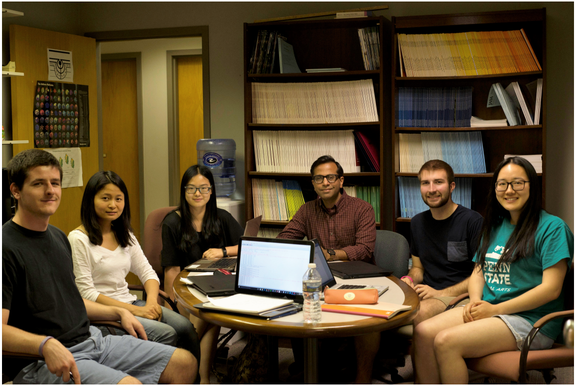
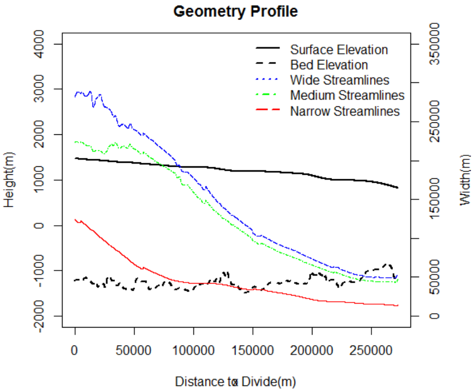
Acknowledgments

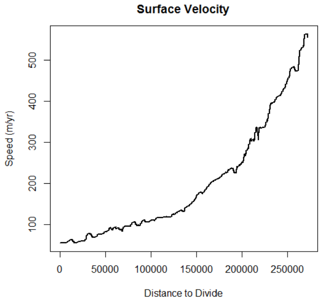


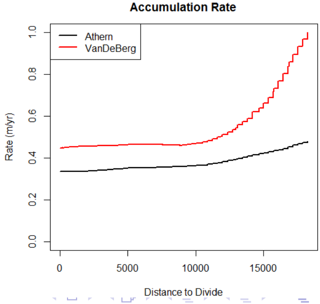
Research group consisting of grads and undergraduates
Partially supported by **NSF CDSE/DMS-1418090** Statistical Methods for Ice Sheet Projections

Ice Sheet Data

go back







Non-Statistical Approach Step 1: Solve for Flux

Goal: using surface observations and the flowline model to solve for the thickness along the flowline.

- Define a grid along the flowline $x_i, i = 1, \ldots, n$
- Assume we know initial flux $U_0 H_0 \omega_0$
- Plug in different values of α and ω
- Solve $UH\omega$ on x_i using finite-difference method:

$$\frac{\partial}{\partial x}\left(U_x H_x \omega_x\right) = a_x \omega_x$$

$$U_{x_{i+1}} H_{x_{i+1}} \omega_{x_{i+1}} - U_{x_i} H_{x_i} \omega_{x_i} = \int_{x_i}^{x_{i+1}} a_s \omega_s ds,$$

$$U_0 H_0 \omega_0 = C_0 \quad \text{initial value}$$

Non-Statistical Approach Step 2: Deduce Thickness

- We have obtained the $U_{x_i} H_{x_i} \omega_{x_i}$ (flux) on a grid
- This U is derived from our adjustment to surface velocity V
  - Recall $U_{x_i} H_{x_i} \omega_{x_i} = \left(V_{x_i} - \frac{\alpha}{20} \left(\rho g |S_{x_i}| \right)^3 H_{x_i}^4 \right) H_{x_i} \omega_{x_i}$
- Flux is a function of ice thickness H
  - This gives us $H_{x_i} = f^{-1}(\alpha, \omega; x_i)$
  - Involves inverting a fifth order polynomial

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  - Involves inverting a fifth order polynomial
- Compare $f^{-1}(\alpha, \omega; x_i)$ with sparsely observed thickness data
- Determine which α and ω minimizes the difference between model output and observed data.