Homework 5, Stat 515, Spring 2015

Due Wednesday, February 25, 2015 **beginning of class**

1. Consider a Poisson process $N(t)$ with rate λ. Prove the following result: Given that $N(t) = n$, the n arrival times $S_1, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$, i.e.,

$$P(S_1 = t_1, \ldots, S_n = t_n \mid N(t) = n) = \frac{n!}{t^n} I(0 < t_1 < \cdots < t_n).$$

   Use a "first principles" argument, i.e., I would like you to do a proof that utilizes assumption # 3 and # 4 in the "first principles" definition of the Poisson process. Do not use an argument based on assuming that the counting process over an interval is Poisson distributed (the second definition discussed in class).

2. The lifetime of a light bulb is known to be exponential with mean 3 years. If the lightbulb has been working for 4 years, what is the probability it will still be working 3 years later?

3. The lifetime of two machines are independent exponential(λ_1) and exponential(λ_2) respectively. Suppose machine 1 starts working now and the second machine is put to use t units of time later. What is the probability machine 1 will fail before machine 2?

4. Cars pass a certain street location according to a Poisson process with rate λ. A woman who wants to cross the street at that location waits until she can see that no cars will come by in the next T time units.
   (a) Find the probability that her waiting time is 0.
   (b) Find her expected waiting time.

5. Suppose $\{N(t)\}$ is a non-homogeneous Poisson process with intensity function $\lambda(x)$ (the intensity function is non-constant as it varies with x).
   (a) Derive the conditional distribution of the arrival times $S_1, \ldots, S_n \mid N(t) = n$.
   (b) For a *homogeneous* Poisson process, the conditional distribution of the arrival times $S_1, \ldots, S_n \mid N(t) = n$, are the order statistics of n i.i.d. Uniform$(0, t)$ random variables. Clearly describe the steps of a general algorithm this suggests for simulating a Poisson process on an interval $[0, t]$. **Hint**: you will simulate the process in two stages.
   (c) Consider a *homogeneous* Poisson process with $\lambda = 10$. Find the expected values of the number of events in the interval (0,1) and in the interval (4,5).
   (d) Using the algorithm from part (b), simulate 10,000 realizations of the above Poisson process on the interval $[0, 5]$.
      i. Report the sample mean for the number of events in the interval (0,1) and the number of events in the interval (4,5).
      ii. Plot a histogram each for the distribution of the number of events in the interval (0,1) and the interval (4,5) respectively, based on the 10,000 realizations.
   (e) Now, describe the steps of a general algorithm part (a) suggests for simulating a *non-homogeneous* Poisson process on an interval $[0, t]$.

6. Simulate a *single* realization of a Poisson process on the interval $(0, 100)$ with $\lambda = 0.2$. Do this in two ways as described below. For each part you need to submit your well-commented `R` code along with a plot displaying the process. See an `R` example for plotting here `http://www.stat.psu.edu/~mharan/515/hwdir/hw05ex.R`
   (a) Simulate the process using the characterization of a Poisson process in terms of uniform random variates.
   (b) Simulate the process using the characterization of a Poisson process in terms of exponential random variates. (Recall: the waiting times between successive events are exponential random variates.)