Problem 1a

First, we need to compute $h(\beta_1 \mid Y, X)$

$$h(\beta_1) \propto \pi(\beta_1 \mid Y, X) \propto f(Y \mid X, \beta_1) f(\beta_1)$$
$$\propto \prod_1^n dexpgauss(Y_i, \beta_0 + \beta_1 X_i, \sigma_i, \lambda) exp(-\beta_1^2/(2\sigma_1^2))$$
$$\Rightarrow log(h(\beta_1)) = \sum_1^n \log\left[dexpgauss(Y_i, \beta_0 + \beta_1 X_i, \sigma_i, \lambda)\right] - \beta_1^2/(2\sigma_1^2)$$

where n is the length of the data vector Y, σ_1^2 is the variance of the prior of β_1.

Now that we know $h(\beta_1)$, we can use the Metropolis-Hastings algorithm to approximate $\pi(\beta_1 \mid Y, X)$. The algorithm is:

1. Start with an initial guess value of β_1
2. Propose a trial value $\beta_1^* \sim N(\beta_1, 1)$. Let this pdf of β_1^* given β_1 be $q(\beta_1, \beta_1^*)$
3. Calculate
$$\alpha(\beta_1, \beta_1^*) = \min\left[1, \frac{q(\beta_1^*, \beta_1)h(\beta_1^*)}{q(\beta_1, \beta_1^*)h(\beta_1)}\right]$$
$$= \exp\{\min[0, \log(q(\beta_1^*, \beta_1)) + \log(h(\beta_1^*)) - \log(q(\beta_1, \beta_1^*)) - \log(h(\beta_1))]\}$$
4. Accept β_1^* with probability $(\alpha(\beta_1, \beta_1^*))$i.e., Set $\beta_1^{i+1} = \beta_1^*$ with probability $\alpha(\beta_1, \beta_1^*)$

The algorithm is simulated for $N = 30000$ iterations. The starting guess value for $\beta_1 = 0$. A random walk with $\sim N(\beta_1^i, 1)$ is used for the proposal distribution of β^*.

Problem 1b

E$[\beta_1] = 7.3464$

MCSE $= 0.0037$

Problem 1c

95% credible interval for β_1 is [6.695707 , 7.937655]

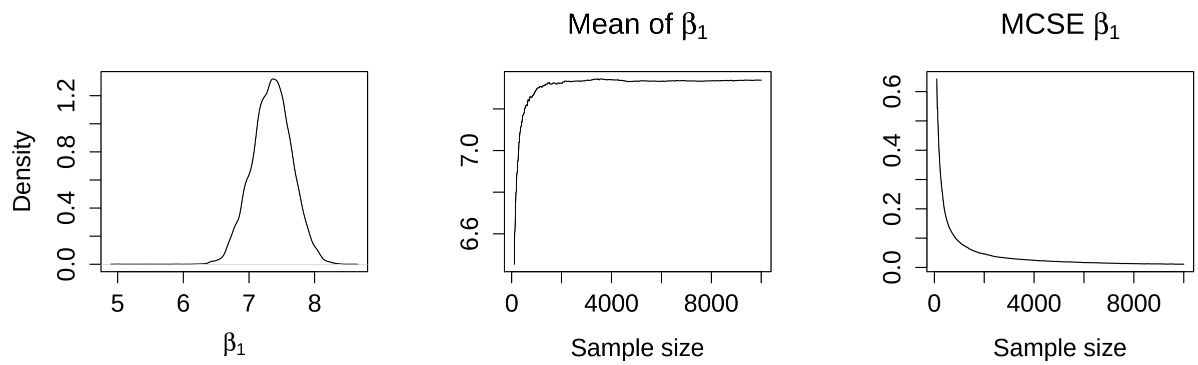
Problem 1d

The pdf for β_1 is shown in Fig. 1a

Problem 1e

The accuracy of the estimates was determined by the following two tests.

- The plots of mean and MCSE of the estimates with sample size are plotted in Fig. 1b and Fig. 1c. The estimates seem to stabilize after $N = 5000$.



(a) (b) (c)

Figure 1

- The effective sample size ESS for N = 30000 iterations is 6683.255. An estimate with ESS > 4000 is considered to be a good estimate.

Problem 2a

First, we need to compute the joint distribution $\pi(\beta_0, \beta_1, \lambda \mid Y, X)$ up to a proportionality constant.

$$h(\beta_0, \beta_1, \lambda) \propto \pi(\beta_0, \beta_1, \lambda \mid Y, X) \propto f(Y \mid X, \beta_0, \beta_1, \lambda) f(\beta_0) f(\beta_1) f(\lambda)$$

or

$$\log(h(\beta_0, \beta_1, \lambda)) = \sum_1^n \log\left[dexpgauss(Y_i, \beta_0 + \beta_1 X_i, \sigma_i, \lambda)\right] - \frac{\beta_0^2}{2\sigma_0^2} - \frac{\beta_1^2}{2\sigma_1^2} + (k-1)\log(\lambda) - \frac{\lambda}{\theta}$$

where n is the length of the data vector Y, $\sigma_0^2 = 100, \sigma_1^2 = 100$ are the variances of the priors of β_0 and β_1, respectively. $k = 0.01$ and $\theta = 100$ are the parameters of the prior of λ.

Now that we know $h(\beta_0, \beta_1, \lambda)$, we can use the variable Metropolis-Hastings algorithm to approximate $\pi(\beta_0, \beta_1, \lambda \mid Y, X)$. The algorithm is:

- Start with an initial guess values for $(\beta_0, \beta_1, \lambda)$
- At the i^{th} iteration,i.e., given $\beta_0^i, \beta_1^i, \lambda^i$, update $\beta_0^{i+1}, \beta_1^{i+1}, \lambda^{i+1}$ one at a time.
- Update β_0. Set $\beta_0 = \beta_0^i$
  1. Propose a trial value $\beta_0^* \sim N(\beta_0, 1)$. Let this pdf of β_0^* given β_0 be $q_0(\beta_0, \beta_0^*)$
  2. Calculate
  $$\alpha(\beta_0, \beta_0^*) = \exp\left\{ \min\left[0, \log\left(\frac{q_0(\beta_0^*, \beta_0) h(\beta_0^*, \beta_1^i, \lambda^i)}{q_0(\beta_0, \beta_0^*) h(\beta_0, \beta_1^i, \lambda^i)}\right)\right]\right\}$$
  3. Simulate $U \sim$ Uniform(0,1)
  4. Set $\beta_0^{i+1} = \beta_0^*$ if $U < \alpha(\beta_0, \beta_0^*)$
  5. Set $\beta_0^{i+1} = \beta_0$ if $U > \alpha(\beta_0, \beta_0^*)$
- Update β_1 using the updated β_0. Set $\beta_1 = \beta_1^i$

1. Propose a trial value $\beta_1^* \sim N(\beta_1, 1)$. Let this pdf of β_1^* given β_1 be $q_1(\beta_1, \beta_1^*)$
2. Calculate
$$\alpha(\beta_1, \beta_1^*) = \exp\left\{ \min\left[0, \log\left(\frac{q_1(\beta_1^*, \beta_1)h(\beta_0^{i+1}, \beta_1^*, \lambda^i)}{q_1(\beta_1, \beta_1^*)h(\beta_0^{i+1}, \beta_1, \lambda^i)}\right)\right]\right\}$$
3. Simulate $U \sim$ Uniform(0,1)
4. Set $\beta_1^{i+1} = \beta_1^*$ if $U < \alpha(\beta_1, \beta_1^*)$
5. Set $\beta_1^{i+1} = \beta_1$ if $U > \alpha(\beta_1, \beta_1^*)$

- Update λ using the updated β_0 and β_1. Set $\lambda = \lambda^i$

1. Propose a trial value $\lambda^* \sim Gamma(mean = \lambda, var = 1)$. Let this pdf of λ^* given λ be $q_2(\lambda, \lambda^*)$
2. Calculate
$$\alpha(\lambda, \lambda^*) = \exp\left\{ \min\left[0, \log\left(\frac{q_2(\lambda^*, \lambda)h(\beta_0^{i+1}, \beta_1^{i+1}, \lambda^*)}{q_2(\lambda, \lambda^*)h(\beta_0^{i+1}, \beta_1^{i+1}, \lambda)}\right)\right]\right\}$$
3. Simulate $U \sim$ Uniform(0,1)
4. Set $\lambda^{i+1} = \lambda^*$ if $U < \alpha(\lambda, \lambda^*)$
5. Set $\lambda^{i+1} = \lambda$ if $U > \alpha(\lambda, \lambda^*)$

The initial guesses for $(\beta_0, \beta_1, \lambda) = (1, 1, 1)$. The random walk proposals for β_0 and β_1 are $N(\beta_i, .5)$ and for λ is Gamma(mean = λ_i, variance = 1). The algorithm was run for N = 200000 iterations.

Problem 2b

The values are tabulated in Tab. 1

	Mean	MCSE	95% interval
β_0	2.334	0.0057	[1.653 , 2.968]
β_1	3.7611	0.0073	[2.851 , 4.685]
λ	0.8867	0.002	[0.626, 1.242]

Table 1: Problem 2b

Problem 2c

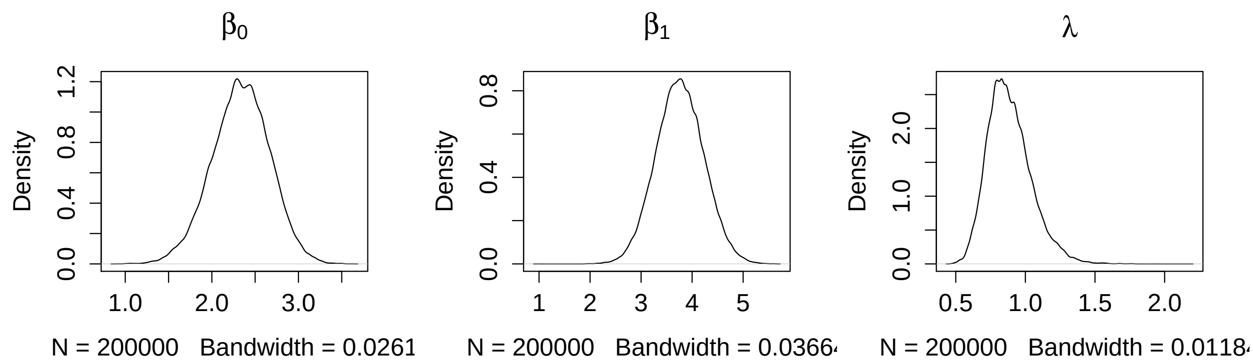
Cor(β_0, β_1) = -0.7824951. This shows that β_0 and β_1 are highly correlated.

Problem 2d

The probability densities are plotted in Fig. 2

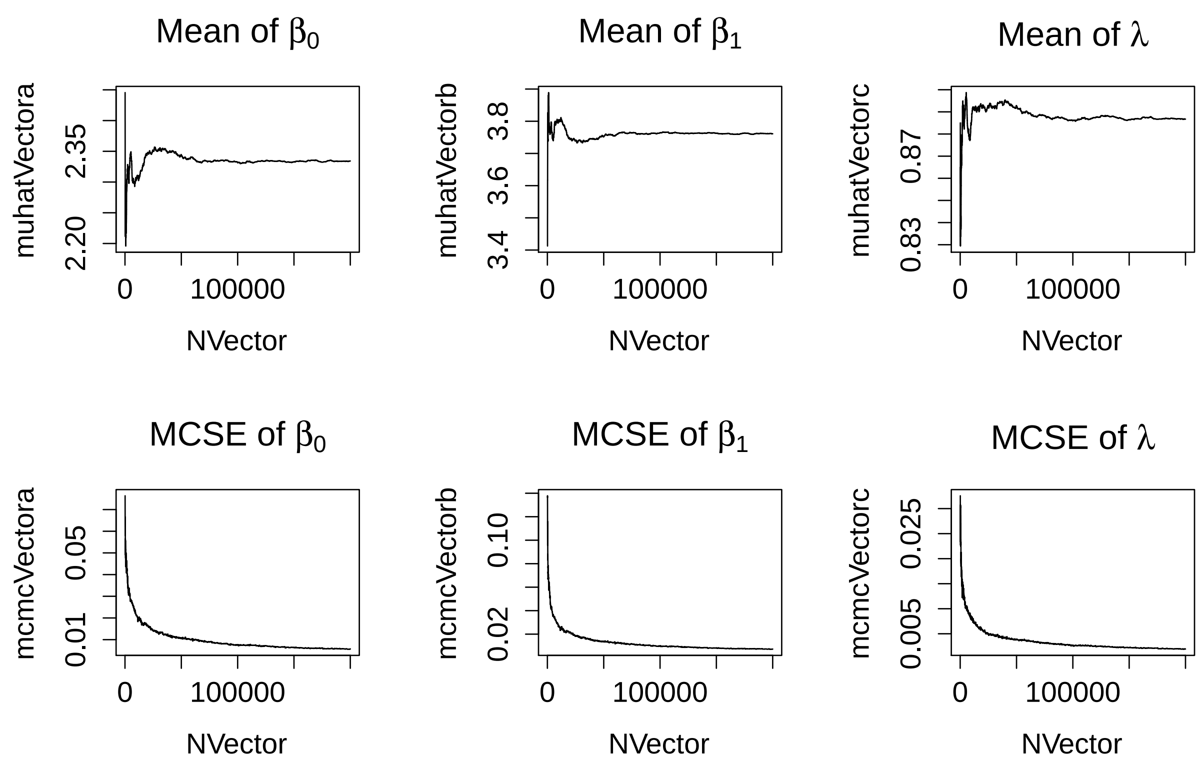
Problem 2e

The plots of mean and MCSE of the estimates with sample size are plotted in Fig. 3. The estimates seem to stabilize after $N = 80000$.



(a) (b) (c)

Figure 2: Problem 2d



(a) (b) (c)

Figure 3: Problem 2e

The effective sample size ESS for N = 200000 iterations for $(\beta_0, \beta_1, \lambda)$ are (4224, 4964, 7257). An estimate with ESS > 4000 is considered to be a good estimate.

Problem 3a

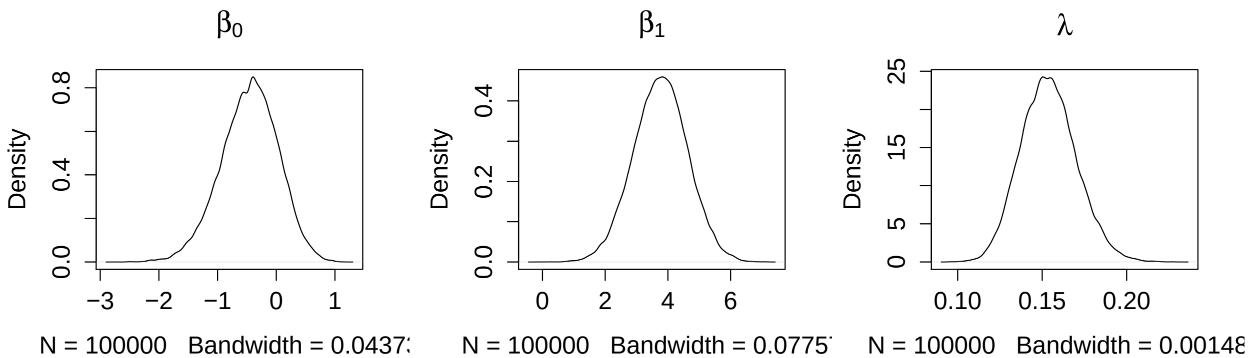
The values are tabulated in Tab. 2

Problem 3b

	Mean	MCSE	95% interval
β_0	-0.425	0.023	[-1.38 , 0.45]
β_1	3.715	0.039	[2.04 , 5.26]
λ	0.152	0.001	[0.122, 0.189]

Table 2: Problem 3a

The probability densities are plotted in Fig. 4



(a) (b) (c)

Figure 4: Problem 3b

Problem 3c

It was observed that the MCMC algorithm was accepting λ only for values ranging from 0.1 to 0.4 for dataset#3.

Therefore,the initial guess was changed from 1 to 0.1, and the random walk proposal for updating λ is changed to Beta(mean = λ, var = $\frac{\lambda^2-\lambda^3}{\lambda+10}$), so as to improve the acceptance rate of λ. It was made sure that the ESS was over 4000 for N = 200000 iterations.