Variance est. of ratio imp   Apr. 20, 2011

$$\sigma^2 = \frac{1}{n}\Big\{ Var_g(g(x)w(x)) - 2\mu Cov(g(x)w(x), w(x)) + \mu^2 Var(w(x)) \Big\}$$

M.O.M. estimation :

~~Est. for $Var_g\{g(x)w(x)\}$: $\frac{\sum_{i=1}^n \left[g(x_i)w(x_i) - \frac{\sum g(x_i)w(x_i)}{n}\right]}{n}$~~

~~" " μ : $\frac{\sum_{i=1}^n g(x_i)w(x_i)}{n}$~~

~~$Cov(g(x)w(x), w(x))$: $\frac{\sum_{i=1}^n g(x)w(x_i)w(x_i)}{n} - \frac{\sum g(x_i)w(x_i)}{n}\frac{\sum w(x_i)}{n}$~~

~~$Var\{w(x)\}$: $\frac{\sum_{i=1}^n \left[w(x_i) - \frac{\sum w(x_i)}{n}\right]^2}{n}$~~

$$\sigma^2 = \frac{1}{n}\Big\{ E g^2(x) w^2(x) - (E g(x)w(x))^2 - 2\mu\Big[E g(x) w^2(x) - E g(x)w(x) E w(x)\Big] + \mu^2\left[E w^2(x) - E^2(w)\right]\Big\}$$

$$= \frac{1}{n}\Big\{ E[g^2(x)w^2(x)] - \big[\underbrace{E\, g(x)w(x)}_{\mu}\big]^2 - 2\mu E[g(x)w^2(x)] + 2\mu \underbrace{E[g(x)w(x)]}_{\mu}\underbrace{E(w(x))}_{1} + \mu^2 E[w^2(x)] - \mu^2 \underbrace{E^2[w(x)]}_{1^2}\Big\}$$

$$= \frac{1}{n}\Big\{ E[g^2(x)w^2(x)] - \cancel{\mu^2} - 2\mu E[g(x)w^2(x)] + \cancel{2\mu^2} + \mu^2 E[w^2(x)] - \cancel{\mu^2}\Big\}$$

$$= \frac{1}{n}\Big\{ E[g^2(x)w^2(x)] - 2\mu E[g(x)w^2(x)] + \mu^2 E[w^2(x)]\Big\}$$

Want to replace : $w(x) = \frac{f(x)}{g(x)}$ w/ $\tilde{w}(x) = \frac{h(x)}{g(x)}$

$\tilde{w}(x) = c\, w(x)$

(1)

$$= \frac{1}{n}\mu^2 \left\{ \overset{(1)}{\frac{E[g^2(x)w^2(x)]}{\mu^2}} - 2\,\overset{(2)}{\frac{E[g(x)w^2(x)]}{\mu}} + \overset{(3)}{E[w^2(x)]} \right\}$$

(3) Estimate: $\dfrac{E[w^2(x)]}{[E(w(x))]^2}$: $\dfrac{\dfrac{\sum w^2(x_i)}{n}}{\left[\dfrac{\sum w(x_i)}{n}\right]^2} = \dfrac{\dfrac{\sum_{i=1}^{n} \frac{f^2(x_i)}{g^2(x_i)}}{n}}{\left[\dfrac{\sum_{i=1}^{n} f(x_i)/g(x_i)}{n}\right]^2}$

$$= \frac{\dfrac{\sum \tilde{w}^2(x_i)}{n}}{\left[\dfrac{\sum_{i=1}^{n} \tilde{w}(x_i)}{n}\right]^2}$$

$$= n\,\frac{\sum_{i=1}^{n} \tilde{w}^2(x_i)}{\left[\sum_{i=1}^{n} \tilde{w}(x_i)\right]^2} \quad ✓$$

(1) Estimate: $\dfrac{E[g^2(x)w^2(x)]}{\mu^2} \doteq \dfrac{\dfrac{\sum g^2(x_i)\,w^2(x_i)}{n}}{\left[\dfrac{\sum_{i=1}^{n} g(x_i)w(x_i)}{n}\right]^2}$

$$= n\,\frac{\sum_{i=1}^{n} g^2(x_i)\,\tilde{w}^2(x_i)}{\left[\sum g(x_i)\,\tilde{w}(x_i)\right]^2} \quad ✓$$

(2) Estimate $\dfrac{\left(\dfrac{E[g(X)\,w^2(X)]}{\mu}\right)}{E[w(X)]}$: $\dfrac{\dfrac{\sum_{i=1}^{n} g(X_i)\,w^2(X_i)}{n}}{\dfrac{\sum g(X_i)\,w(X_i)}{n}\;\dfrac{\sum w(X_i)}{n}}$

$$= n\,\frac{\sum_{i=1}^{n} g(X_i)\,\tilde{w}^2(X_i)}{\left[\sum_{i=1}^{n} g(X_i)\,\tilde{w}(X_i)\right]\left(\sum_{i=1}^{n} \tilde{w}(X_i)\right)} \quad \checkmark$$

Hence, estimate of σ^2 via M.O.M.,

$$\hat{\sigma}^2_{MOM} = \cancel{\tfrac{1}{n}}\cancel{n}\left[\frac{\sum_{i=1}^{n} g(X_i)\,\tilde{w}(X_i)}{\sum_{i=1}^{n} \tilde{w}(X_i)}\right]^2 \left\{ \frac{\sum_{i=1}^{n} g^2(X_i)\,\tilde{w}^2(X_i)}{\left[\sum_{i=1}^{n} g(X_i)\,\tilde{w}(X_i)\right]^2} - 2\,\frac{\sum g(X_i)\,\tilde{w}^2(X_i)}{\left(\sum_{i=1}^{n} g(X_i)\,\tilde{w}(X_i)\right)\left[\sum_{i=1}^{n} \tilde{w}(X_i)\right]} + \frac{\sum_{i=1}^{n} \tilde{w}^2(X_i)}{\left[\sum_{i=1}^{n} \tilde{w}(X_i)\right]^2} \right\}$$