STAT 515

Homework #5, due Friday, Feb. 24 at 2:30pm

This homework must be submitted electronically to ANGEL. I strongly encourage the use of LaTeX.

Please make every assignment easier to grade by neatly organizing your writeup and clearly labeling your final answers when appropriate. Try using LaTeX!

1. Let X_1 and X_2 be independent exponential random variables with rates λ_1 and λ_2, respectively. Let

$$X_{(1)} = \min\{X_1, X_2\} \quad \text{and} \quad X_{(2)} = \max\{X_1, X_2\}.$$

We have shown in class that $X_{(1)}$ is exponential with rate $\lambda_1 + \lambda_2$.

   (a) Find $EX_{(2)}$. (**Hint:** What is $E[X_{(1)} + X_{(2)}]$?)

   (b) Find a probability density function for $X_{(2)}$ and use it to calculate $\text{Var}\, X_{(2)}$.

   (c) Find $\text{Cov}\,(X_{(1)}, X_{(2)})$. (**Hint:** What is $\text{Var}\,[X_{(1)} + X_{(2)}]$?)

2. Theorem 5.2 in Section 5.3.5 states that in a Poisson process $N(t)$ with rate λ, given that $N(t) = n$, the n arrival times $S_1, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$, i.e.,

$$P(S_1 = t_1, \ldots, S_n = t_n \mid N(t) = n) = \frac{n!}{t^n} I(0 < t_1 < \cdots < t_n).$$

   (a) Clearly describe the general algorithm this suggests for simulating a Poisson process on an interval $[0, t]$. (**Hint**: you will simulate the process in two stages.)

   (b) Consider a *homogeneous* Poisson process with $\lambda = 10$. Using the algorithm from part (a), simulate 10,000 realizations of the above Poisson process on the interval $[0, 5]$.

   (c) Report the sample mean for the number of events in the interval (0,1) and the number of events in the interval (4,5). How do these means compare with the corresponding theoretical expectations?

   (d) Plot a histogram each for the distribution of the number of events in the interval (0,1) and the interval (4,5) respectively, based on the 10,000 realizations.

3. Cars pass a certain street location according to a Poisson process with rate λ. A woman who wants to cross the street at that location waits until she can see that no cars will come by in the next T time units.

   (a) Find the probability that her waiting time is 0.

   (b) Find her expected waiting time.