Homework 3, Stat 515, Spring 2015

Due Wednesday, February 11, 2015 **beginning of class**

1. Define μ_{ii} as in class, so μ_{ii} is the expected number of transitions until the Markov chain, starting in state i, makes a transition back to state i. Define π_i as the long run proportion of time the Markov chain, starting in state i, spends in state i. Assume $\pi_i = 1/\mu_{ii}$. Therefore, state i is positive recurrent iff $\pi_i > 0$.

   (a) Prove that positive recurrence is a class property: Suppose state i is positive recurrent and states i and j are in the same class. Prove that state j is also positive recurrent. Hint: show that, for state j, there exists $n \in \mathbb{Z}^+$ such that $\pi_j \geq \pi_i P_{ij}^n > 0$.

   (b) Prove that null recurrence is also a class property.

2. Consider the same problem from the previous homework: three urns, one colored red, one white, and one blue. The red urn contains 1 red and 4 blue balls; the white urn contains 3 white balls, 2 red balls, and 2 blue balls; the blue urn contains 4 white balls, 3 red balls, and 2 blue balls. At the initial stage, a ball is randomly selected from the red urn and then returned to that urn. At every subsequent stage, a ball is randomly selected from the urn whose color is the same as that of the ball previously selected and is then returned to that urn.

   (a) You have already derived the transition probability matrix and the initial distribution of the chain. Let X_0 be the initial state of the Markov chain. What is $E(X_2)$ if red, white and blue are coded as 1,2,3 respectively?

   (b) Does this process have a stationary distribution? Justify your answer. You will need to show how each of the requisite conditions is satisfied by the Markov chain.

   (c) Does this process have a limiting distribution? Justify your answer.

   (d) In the long run, what proportion of the selected balls are red? What proportion are white? What proportion are blue? Again, justify your answer (theoretical results, conditions satisfied).

   (e) Simulate a Markov chain of length 100,000 using the information provided above and count the proportion of times the chain was in each of the states. Compare this to your answer.

   (f) Suppose you have taken 3 steps, i.e., you start with the initial distribution to obtain X_0 and use the t.p.m. above to obtain state X_3 of the Markov chain. What proportion of times would you expect X_3 to be red, white and blue respectively? Similarly, what proportion of times would you expect X_5 to be red, white and blue respectively? Note that you should use `R` to solve this problem. (Follow the example from the previous homework, as well as the sample code discussed in lecture and available on Angel.)

   (g) Now simulate 10,000 realizations of the random variables X_2 and X_{20} using the initial distribution and transition probability matrix for this process. Calculate the proportion of times in your simulations that X_2 and X_{20} are red, white, and blue respectively. Compare these two sets of 3 proportions to their respective probabilities from above. *Note that here you are simulating multiple realizations of X_2, X_{20} in order to approximate their expected values. This is different from the simulation above where you were simulating a single Markov chain in order to approximate its stationary/limiting distribution, assuming one exists.*

3. Consider a Markov chain on states $0, 1, \ldots$ with transition probabilities given by:

$$P_{ij} = \frac{1}{i+2}$$

   for $j = 0, 1, \ldots, i, i+1$.

   (a) Is this Markov chain irreducible and ergodic? Provide a detailed answer.

   (b) What is its stationary distribution?