Dimension Reduction and Alleviation of Spatial Confounding for Spatial Generalized Linear Mixed Models

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What This Talk is About

- Modeling spatial data on a lattice is challenging.
- Spatial generalized linear mixed models (SGLMMs) provide a general framework. Widely used.
- Shortcomings of SGLMMs: (1) Inference presents difficult computational issues. (2) Parameter interpretation is generally misleading.
- I will describe an approach that simultaneously resolves both these issues.

Non-Gaussian Spatial Data

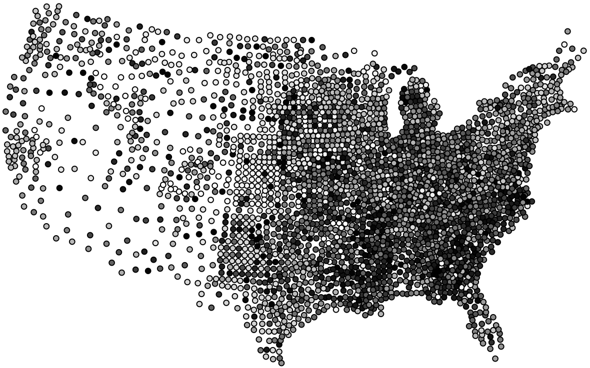


Figure: U.S. infant mortality data by county. $n = 3071$

Ratio of deaths to births, each averaged over 2002-2004.

Darker indicates higher rate.

Spatial Data on a Lattice

- Gaussian and non-Gaussian spatial data are very common and appear in a large number of disciplines.
- Common lattice data: binary, count, zero-inflated
- Purpose of the model
  1. regression while adjusting for residual spatial dependence
  2. smoothing the spatial field and “borrowing strength”
- These models are used widely and have become particularly important in disease epidemiology and ecology.

Spatial Linear Models

- Spatial process at location $\mathbf{s}$ is $Z(\mathbf{s}) = X(\mathbf{s})\boldsymbol{\beta} + W(\mathbf{s})$.
  - $X(\mathbf{s})$ are covariates at $\mathbf{s}$ and $\boldsymbol{\beta}$ is a vector of coefficients.
  - Model dependence among spatial random variables by imposing it on the errors (the $W(\mathbf{s})$'s).
- Gaussian Markov Random field (GMRF): Let Θ be the parameters for precision matrix $Q(\Theta)$. Then:

$$\mathbf{Z}_{n\times 1}|\Theta, \boldsymbol{\beta} \sim N(\mathbf{X}_{n\times p}\boldsymbol{\beta}_{p\times 1}, Q^{-1}(\Theta))$$

Spatial Linear Models: Dependence

- $Q = \text{diag}(A\mathbf{1}) - A$ where adjacency matrix A is such that $A_{ij} = 1$ if locations i and j are neighbors, 0 else
- Implications:
  - $W(\mathbf{s})$ is conditionally independent of all other Ws given its neighbors
  - uncertainty about $W(\mathbf{s})$ is inversely proportional to its number of neighbors.

Spatial Generalized Linear Mixed Models

Model for Z at location $\mathbf{s}_i$

1. $Z(\mathbf{s}_i)|\boldsymbol{\beta}, \Theta, W(\mathbf{s}_i), i = 1, \ldots, n$, conditionally independent
   E.g. $Z(\mathbf{s}_i) \mid \boldsymbol{\beta}, W(\mathbf{s}_i) \sim$Poisson$(\mu(\mathbf{s}_i))$
2. Link function $g(\mu(\mathbf{s}_i)) = X(\mathbf{s}_i)\boldsymbol{\beta} + W(\mathbf{s}_i)$
   E.g. $\log(\mu_i) = X(\mathbf{s}_i)\boldsymbol{\beta} + W(\mathbf{s}_i)$
3. Impose dependence: $\mathbf{W} = (W(\mathbf{s}_1), \ldots, W(\mathbf{s}_n))^T$

$$p(\mathbf{W}|\tau) \propto \tau^{(n-1)/2} \exp\left(-\frac{\tau}{2}\mathbf{W}' Q \mathbf{W}\right)$$

4. Priors for $\Theta, \boldsymbol{\beta}$

Inference based on $\pi(\Theta, \boldsymbol{\beta}, \mathbf{W} \mid \mathbf{Z})$

(Besag et al. (1991), Diggle et al. (1998))

SGLMMs: Challenges

SGLMMs have become very popular even outside mainstream statistics. Flexible models but some drawbacks:

(1) Confounding between spatial random effects and fixed effects (covariates)

(2) Computational challenges

Spatial Confounding in SGLMMs

- $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, orthogonal projection onto $C(\mathbf{X})$
- $\mathbf{P}^{\perp} = \mathbf{I} - \mathbf{P}$, orthogonal projection onto $C(\mathbf{X})$'s orthogonal complement
- Spectral decomposition to acquire orthogonal bases, $\mathbf{K}_{n \times p}$ and $\mathbf{L}_{n \times (n-p)}$, for $C(\mathbf{X})$ and $C(\mathbf{X})^{\perp}$. Rewrite:

$$g(\mathbb{E}(Z_i \mid \boldsymbol{\beta}, W_i)) = \mathbf{X}_i\boldsymbol{\beta} + W_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{K}_i\boldsymbol{\gamma} + \mathbf{L}_i\boldsymbol{\delta}.$$

  K is collinear with **X**.

This is the source of confounding. Appears to cause variance inflation.

Computing for SGLMMs

MCMC algorithms for SGLMMs are challenging to construct:

- Spatial random effects: one random effect for each data point. $n + p + 1$ dimensions where n=size of data, p =number of predictors. MCMC is slow per iteration due to high dimensionality
- Markov chain is slow mixing due to strong cross-correlations among the spatial random effects.

Several attempts to address these issues: Rue and Held (2005), Haran et al. (2003), Haran and Tierney (2010)

Observations

- Spatial random effects **W** are the cause of confounding issues as well as computational challenges.
- **W** are just a device to induce dependence. Not intrinsically important.
- Idea: reparameterize and reduce dimensions of **W**.

Spatial Confounding: Reparameterization Solution

- Reich, Hodges and Zadnik (2006) propose solution: since $\mathbf{K}$ have no scientific meaning, delete them from the model.
- $g(\mathbb{E}(Z_i \mid \beta, \delta)) = \mathbf{X}_i\beta + \mathbf{L}_i\delta$. Prior for random effects δ now

  $$p(\delta \mid \tau) \propto \tau^{(n-p)/2} \exp\left(-\frac{\tau}{2}\delta'\mathbf{Q}^*\delta\right),$$

  where $\mathbf{Q}^* = \mathbf{L}'\mathbf{Q}\mathbf{L}$.
- Corrects issues due to confounding
- # of parameters reduced (only slightly) from $n + p + 1$ to $n + 1$. Computational challenge remains.
- RHZ approach does not fully account for underlying graph

Our Sparse Reparameterization

- Represent graph $G = (V, E)$ using $\mathbf{A}$, $n \times n$ adjacency matrix with entries $\mathrm{diag}(\mathbf{A}) = \mathbf{0}$ and $\mathbf{A}_{ij} = 1\{(i, j) \in E, i \neq j\}$, with $1\{\cdot\}$ an indicator function
- Basic idea inspired by Griffith (2003): augment a generalized linear model with selected eigenvectors of $(\mathbf{I} - \mathbf{1}\mathbf{1}'/n)\mathbf{A}(\mathbf{I} - \mathbf{1}\mathbf{1}'/n)$. This appears in Moran's I statistic (nonparametric measure of spatial dependence),

$$I(\mathbf{A}) \propto \frac{\mathbf{Z}'(\mathbf{I} - \mathbf{1}\mathbf{1}'/n)\mathbf{A}(\mathbf{I} - \mathbf{1}\mathbf{1}'/n)\mathbf{Z}}{\mathbf{Z}'(\mathbf{I} - \mathbf{1}\mathbf{1}'/n)\mathbf{Z}},$$

Background for Sparse Reparameterization

- Griffith's goal: reveal the structure of missing spatial covariates. Our goal: smoothing orthogonal to $\mathbf{X}$
- Hence, we replace $\mathbf{I} - \mathbf{1}\mathbf{1}'/n$ with $\mathbf{P}^{\perp}$
- $\mathbf{M}_{\mathbf{X}}(\mathbf{A}) = \mathbf{P}^{\perp}\mathbf{A}\mathbf{P}^{\perp}$, Moran operator for $\mathbf{X}$ with respect to the graph G, appears in numerator of generalized Moran's I:

$$I_{\mathbf{X}}(\mathbf{A}) \propto \frac{\mathbf{Z}'\mathbf{P}^{\perp}\mathbf{A}\mathbf{P}^{\perp}\mathbf{Z}}{\mathbf{Z}'\mathbf{P}^{\perp}\mathbf{Z}}.$$

Applying the Sparse Reparameterization

- Replacing **L** with **M** in the RHZ model gives

  $$g(\mathbb{E}(Z_i \mid \boldsymbol{\beta}, \boldsymbol{\delta})) = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{M}_i\boldsymbol{\delta}.$$

  And the prior for the random effects is now

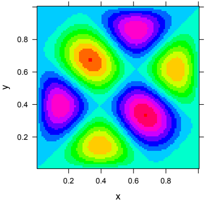
  $$p(\boldsymbol{\delta} \mid \tau) \propto \tau^{q/2} \exp\left(-\frac{\tau}{2}\boldsymbol{\delta}'\mathbf{Q}^{**}\boldsymbol{\delta}\right),$$

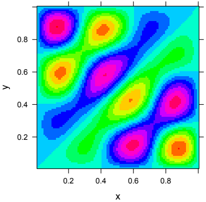
  where $\mathbf{Q}^{**} = \mathbf{M}'\mathbf{Q}\mathbf{M}$.
- Corrects issues due to confounding
- Potential for dimension reduction: if we reduce dimensions of $\mathbf{M}_i$ to q, the # parameters is reduced from $n + p + 1$ to $q + p + 1$ (q can be small)

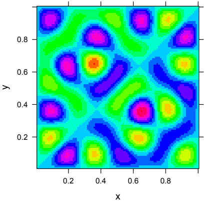
Interpreting the Resulting Reparameterization

- "Tailored" to $\mathbf{X}$ and G: eigenvectors comprise all possible patterns of clustering residual to $\mathbf{X}$ and accounting for G

Some selected basis vectors for the 30×30 lattice.



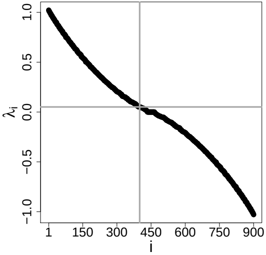




Interpreting the Resulting Reparameterization

- Positive (negative) eigenvalues correspond to varying degrees of positive (negative) spatial dependence (Boots and Tiefelsdorf, 2000)

The standardized eigenvalues for the 30×30 lattice.



Exploiting the New Parameterization

- If we assume positive spatial dependence, eigenvectors corresponding to negative spatial dependence (negative eigenvalues) should be removed.
- Small eigenvalues may not be meaningful. Remove corresponding eigenvectors.
- Result: much reduced dimensions

Study: Inference for Spatial Binary

30×30 lattice simulated from RHZ model with $\beta_1 = \beta_2 = 1$.

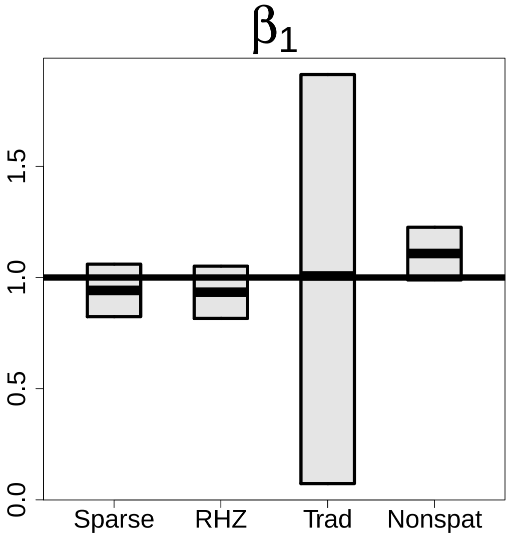
Predictors are the coordinates of unit square.

Model	$\hat{\beta}_1$ CI(β_1)	$\hat{\beta}_2$ CI(β_2)
Sparse	1.080 (0.613, 1.556)	1.130 (0.644, 1.635)
RHZ	1.120 (0.637, 1.606)	1.192 (0.679, 1.713)
Traditional	0.500 (-2.655, 3.616)	-0.605 (-3.698, 2.577)

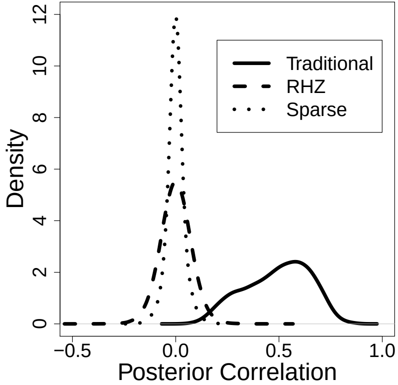
- Point and interval estimates for Traditional are very poor: 95% interval includes 0
- Sparse and RHZ produce similar (good) results

Similar results for Gaussian (linear) and Poisson

Spatial Count Data: Simulation Results



De-correlated Random Effects



Greatly improves efficiency of simple MCMC. No need for elaborate proposals (cf. Held and Rue (2005), Haran et al. (2003), Haran and Tierney (2010)).

Spatial Binary: Computational Efficiency

Model	Dimension	Running Time
Sparse	228	2.5 hours
RHZ	901	18.5 hours
Traditional	903	38.5 hours

- MCMC algorithm is
  - faster per iteration (far fewer random effects)
  - mixes faster (random effects are “decorrelated”)
- Far greater speed-ups with much smaller q, e.g. 25-50 is adequate for our examples (we are also being *extremely* careful by running very long chains!)

  Real data example: 14 days (traditional) versus 2-8 hours

Summary

- SGLMMs provide a very general approach for modeling non-Gaussian spatial data
- Our sparse approach results in more interpretable regression coefficients
- We allow for only meaningful spatial dependence and a natural approach to dimension reduction
- Automated MCMC is computationally efficient, allowing for routine analysis of large data sets

References

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- Griffith (2003) Spatial Autocorrelation and Spatial Filtering. *Springer*.
- Reich, Hodges and Zadnik (2006) Effects of residual smoothing on the posterior of the fixed effects in disease-mapping models. *Biometrics*

Hughes, J. and Haran, M. (2013) "Dimension Reduction and Alleviation of Confounding for Spatial Generalized Linear Mixed Models," *Journal of the Royal Statistical Society (B)*

Software: http://www.biostat.umn.edu/∼johnh/software.html