STAT 515
Homework #10 WITH SOLUTIONS

1. We wish to approximate $\mu = P(X > 4.5)$ where $X \sim N(0, 1)$. Suppose that $q(x)$ is a normal density with mean k, and suppose that $X_1, \ldots, X_n$ is a simple random sample from $q(\cdot)$.

   (a) Show that

   $$\tilde{\mu} = \frac{\frac{1}{n}\sum_{i=1}^n I\{X_i > 4.5\}\exp\{(X_i - k)^2/2 - X_i^2/2\}}{\frac{1}{n}\sum_{i=1}^n \exp\{(X_i - k)^2/2 - X_i^2/2\}}$$

   is a consistent estimator of μ. (To do this, it's enough to show that the true mean of the numerator divided by the true mean of the denominator equals μ.)

   Solution: For the numerator, we obtain for $X \sim N(k, 1)$,

   $$\begin{aligned} E\left[I\{X_i > 4.5\}\exp\{(X - k)^2/2 - X^2/2\}\right] &= \int_{4.5}^{\infty} \exp\{(x - k)^2/2 - x^2/2\}\frac{1}{\sqrt{2\pi}}\exp\{-(x - k)^2/2\}\,dx \\ &= \int_{4.5}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\{-x^2/2\}\,dx = \mu. \end{aligned}$$

   A similar derivation shows that for the denominator,

   $$E\left[\exp\{(X - k)^2/2 - X^2/2\}\right] = 1.$$

   This means that $\hat{\mu}$ consists of a fraction in which the numerator is a sample mean whose true mean is μ, and the denominator is a sample mean whose true mean is 1. This implies that $\hat{\mu}$ converges almost surely to μ.

   (b) Based on samples of size 100,000 from $q(\cdot)$, try using $\tilde{\mu}$ several times for $k = 0$, $k = 4.5$, and some intermediate values of k. What value of k seems to give the most precise estimates?

   Solution: Here is a function that returns 10 different estimates of μ based on a particular choice of k:

```
> f <- function(k, n=10) {
+   muhat <- 1:n
+   for (i in 1:n) {
+     x <- rnorm(1e5) + k
+     b <- exp((x-k)^2/2 - x^2/2)
+     a <- (x>4.5)*b
+     muhat[i] <- mean(a) / mean(b)
+   }
+   muhat
+ }
```

   Here are sample standard deviations for the 10 estimates for $k = 0, 0.5, 1.0, \ldots, 4.5$:

```
> mu <- 1-pnorm(4.5)
> print(out <- sapply((0:9)/2, function(a) mean((f(a)-mu)^2)))
 [1] 5.077211e-11 5.615992e-12 3.700880e-13 4.939637e-14 2.460070e-14
 [6] 5.623972e-14 3.708306e-13 2.690284e-13 2.135039e-12 2.495489e-11
> ((0:9)/2)[which.min(out)] # Which value gave the smallest mean squared error?
[1] 2
```

   So it appears that $k = 2$ gives the best results of the ten values tested.

   (c) Use the delta-method derivation

   $$\mathrm{Var}\left(\frac{\frac{1}{n}\sum_i A_i}{\frac{1}{n}\sum_i B_i}\right) \approx \frac{1}{n\mu_B^2}\begin{bmatrix}1 & \frac{-\mu_A}{\mu_B}\end{bmatrix}\begin{bmatrix}\sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2\end{bmatrix}\begin{bmatrix}1 \\ \frac{-\mu_A}{\mu_B}\end{bmatrix}$$

   to estimate the variances of your $\tilde{\mu}$ estimators from part (b). (Use sample estimates of μ_A, μ_B, and the covariance matrix.) Do the variance estimates correspond with your experience in part (b)?

Solution: The following code modifies that of part (a) so that an estimator of the variance is returned along with $\hat{\mu}$:

```
> f2 <- function(k, n=1e5) {
+   x <- rnorm(n) + k
+   b <- exp((x-k)^2/2 - x^2/2)
+   a <- (x>4.5)*b
+   muhat <- mean(a) / mean(b)
+   v <- (z <- c(1, -muhat)) %*% cov(cbind(a,b)) %*% z / (n*mean(b)^2)
+   cbind(muhat=muhat, var=as.numeric(v))
+ }
```

Now we may approximate the variance of the $\hat{\mu}$ estimator for various values of k:

```
> k <- (0:9)/2
> sapply(k, f2)[2,]  # Just print the variance estimates
 [1] 0.000000e+00 9.597689e-13 6.786709e-13 9.026692e-14 2.803545e-14
 [6] 3.270997e-14 1.033955e-13 8.247448e-13 1.017039e-12 3.865547e-12
```

The estimated variances here are of the same order of magnitude as the mean squared errors in part (b), but we know that some of the approximations are not very good because the true variance must always be smaller than the true MSE.

(d) Consider a modified estimator

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} I\{X_i > 4.5\}\exp\{(X_i - k)^2/2 - X_i^2/2\},$$

where once again $X_1, \ldots, X_n$ is a simple random sample from $q(\cdot)$. Verify that this estimator is a consistent estimator of μ. (Again, merely show that the true mean of each summand equals μ.) Using the best k you found earlier, compare the estimated variance of $\tilde{\mu}$ with the estimated variance of $\hat{\mu}$ (the latter should not be hard to find). Which estimator, $\tilde{\mu}$ or $\hat{\mu}$, appears to be more precise?

Solution: Using $k = 2$, we find:

```
> set.seed(12345) # Set seed so we can exactly replicate sample
> f2(2)[,2] # This is the ratio imp. samp. estimated variance
         var
2.911137e-14
> set.seed(12345)
> x <- rnorm(1e5)+2
> var((x>4.5) * dnorm(x) / dnorm(x-2)) / 1e5 # Plain imp. samp.
[1] 2.361843e-14
```

So it appears that the variance of the plain importance sampling estimator is smaller. NB: This does not consider which estimator is more *accurate*, which is related to the bias. However, since only the plain importance sampling estimate is theoretically unbiased, considering bias together with variance probably will not lead to a different conclusion.

Since part (c) suggests that the delta-method approximation of the variance might not be very accurate, let us try to find a better variance estimator by measuring the sample variance of 1000 independent replicates of the ratio importance sampling estimator:

```
> var(f(2,n=1000))
[1] 2.829011e-14
```

We find fairly good agreement between this empirical estimate and the asymptotic approximation based on the delta method.

2. Suppose that X is a binomial random variable with parameters n and p, where $p = e^\theta/(1 + e^\theta)$ for some real-valued parameter θ. The goal of this question will be to use ratio importance sampling to estimate the log-likelihood function $\ell(\theta) = \log P_\theta(X)$.

(a) Show that the log-likelihood function may be written as

$$\ell(\theta) = \theta X - \log c(\theta) + (\text{something not depending on } \theta),$$

and find the normalizing function $c(\theta)$.

Solution: Since $p = e^\theta/(1+e^\theta)$, the binomial mass function equals

$$f_\theta(x) = \binom{n}{x}\left(\frac{e^\theta}{1+e^\theta}\right)^x \left(\frac{1}{1+e^\theta}\right)^{n-x} = \binom{n}{x} e^{\theta x}\left(\frac{1}{1+e^\theta}\right)^n,$$

we conclude that

$$\ell(\theta) = \log f_\theta(X) = \theta X - \log[(1+e^\theta)^n] + \log\binom{n}{x},$$

which means that $c(\theta) = (1+e^\theta)^n$.

(b) Fix some θ_0. Show that

$$\ell(\theta) = \ell(\theta_0) + (\theta-\theta_0)X - \log E_{\theta_0}[\exp\{(\theta-\theta_0)Y\}],$$

where the notation above means that Y has a binomial distribution according to θ_0 (and X is the data, as usual).

Solution: From part (a), we conclude that

$$\ell(\theta) - \ell(\theta_0) = (\theta-\theta_0)X - \log\left(\frac{1+e^\theta}{1+e^{\theta_0}}\right)^n = (\theta-\theta_0)X - \log\frac{c(\theta)}{c(\theta_0)},$$

so the result follows from the fact that

$$E_{\theta_0}[\exp\{(\theta-\theta_0)Y\}] = \sum_{y=0}^n \exp\{(\theta-\theta_0)y\} f_{\theta_0}(y) = \frac{1}{c(\theta_0)}\sum_{y=0}^n \binom{n}{y}\exp\{\theta y\} = \frac{c(\theta)}{c(\theta_0)}\sum_{y=0}^n f_\theta(y) = \frac{c(\theta)}{c(\theta_0)}.$$

(c) The equation of part (b) suggests a method for approximating $\ell(\theta) - \ell(\theta_0)$, which is a function that can be maximized to find the MLE of θ. Suppose that $n = 100$ and $X = 80$, then take a random sample $Y_1, \ldots, Y_m$ using $m = 10^6$ and $\theta_0 = 1$ to approximate the function $\ell(\theta) - \ell(\theta_0)$. On the same set of axes, plot both the true $\ell(\theta) - \ell(\theta_0)$ and your approximation. How does your approximate MLE compare with the true MLE?

Solution: First, let us simulate $Y_1, \ldots, Y_m$:

```
> theta0 <- 1
> y <- rbinom(1e6, 100, exp(theta0)/(1+exp(theta0)))
```

Next, we create a function that approximates $\ell(\theta) - \ell(\theta_0)$:

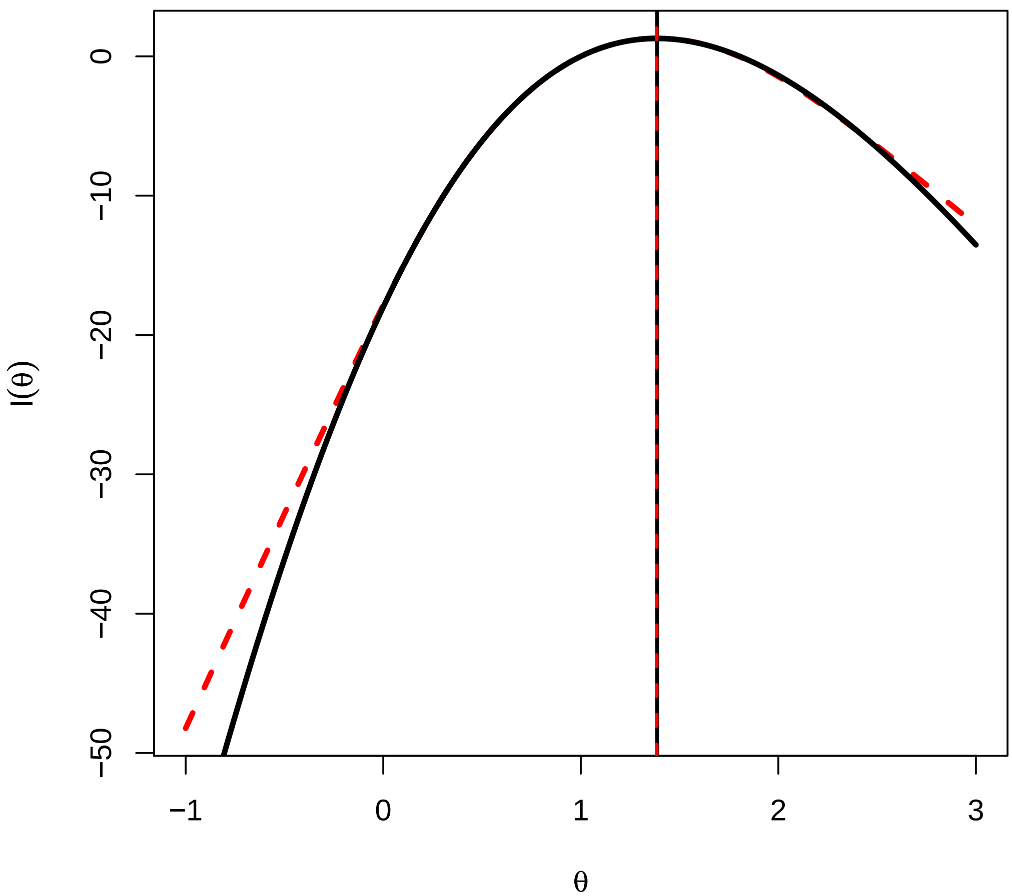
```
> xobs <- 80
> ell <- function(theta) (theta-theta0)*xobs - log(mean(exp((theta-theta0)*y)))
```

Finally, we must apply this function to a range of 501 θ values (with a timer to see how long it takes):

```
> th <- seq(-1, 3, len=501)
> system.time(ellth <- sapply(th, ell))
   user  system elapsed
 15.621   1.260  16.885
```

Finally, plot both curves on the same set of axes. The red dashed line is the approximate curve and the solid black line is the true $\ell(\theta) - \ell(\theta_0)$. The vertical lines on the plot are the true and approximate MLE:

```
> plot(th, ellth, type="l", lwd=3, lty=2, col=2, xlab=expression(theta),
+     ylab=expression(l(theta)))
> lines(th, (th-theta0)*xobs - 100*log((1+exp(th))/(1+exp(theta0))), lwd=3)
> abline(v=log(4), lwd=2)
> abline(v=th[which.max(ellth)], lwd=2, lty=2, col=2)
```

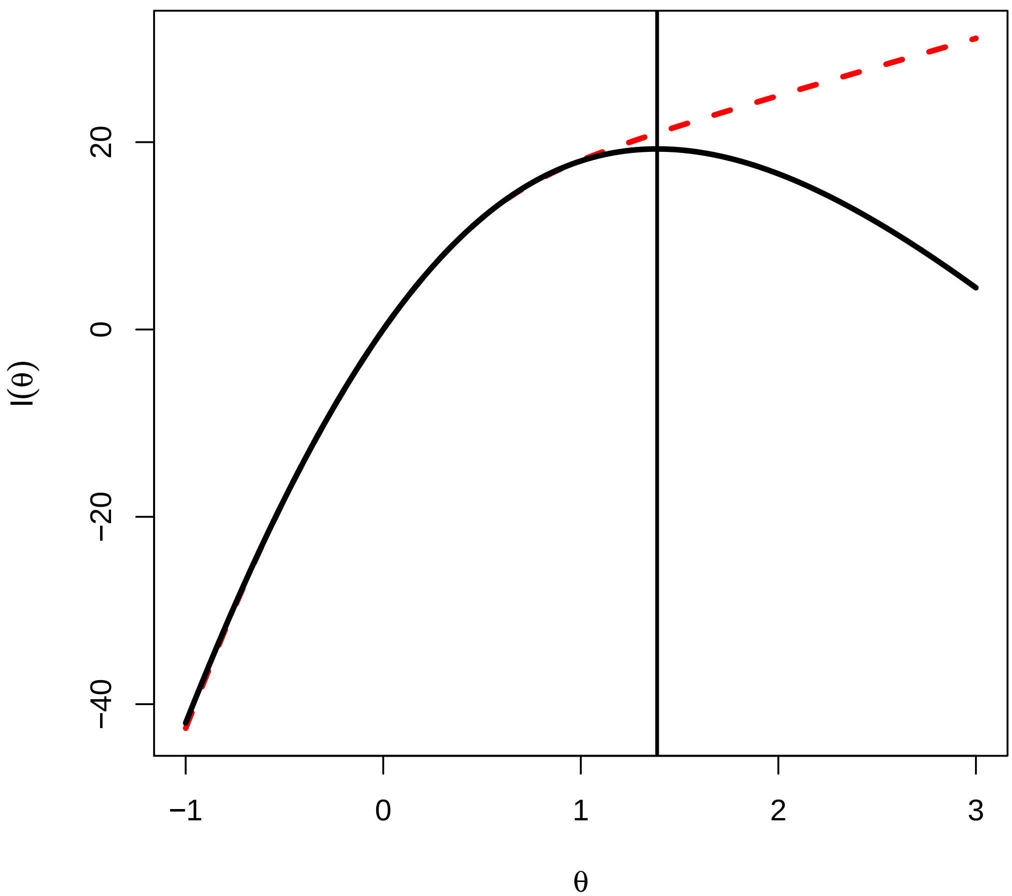


We observe that the two curves are very close together in the region of the true MLE, and in fact the approximate MLE is very close to the true MLE in this case.

(d) Try the same technique as in part (d) but use $\theta_0 = 0$. What do you observe?

Solution: Let us repeat most of the code from part (c):

```
> theta0 <- 0
> y <- rbinom(1e6, 100, exp(theta0)/(1+exp(theta0)))
> th <- seq(-1, 3, len=501)
> system.time(ellth <- sapply(th, ell))
   user  system elapsed
 15.536   1.245  16.787
> plot(th, ellth, type="l", lwd=3, lty=2, col=2, xlab=expression(theta),
+     ylab=expression(l(theta)))
> lines(th, (th-theta0)*xobs - 100*log((1+exp(th))/(1+exp(theta0))), lwd=3)
> abline(v=log(4), lwd=2)
```



In this case, the approximate (dashed red) curve is very close to the true curve in the neighborhood of θ_0, but near the true MLE, this approximation breaks down and in fact, the approximate curve has no maximizer at all in this case!