Stat 515 Take Home Final

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1 Problem 1

1.1 (a)

By Bayesian rule:

$$\pi(\beta_1|\mathbf{Y},X) = \frac{\pi(\mathbf{Y},X|\beta_1)\pi(\beta_1)}{\pi(\mathbf{Y},X)} \propto \pi(\mathbf{Y},X|\beta_1)\pi(\beta_1)$$

in which:

$$\pi(\mathbf{Y},X|\beta_1) = \prod_{i=1}^{n} \frac{\lambda}{2} exp(2(\beta_0 + \beta_1 X_i) + \lambda\sigma^2 - 2Y_i) erfc(\frac{\beta_0 + \beta_1 X_i + \lambda\sigma^2 - Y_i}{\sqrt{2}\sigma})$$

$$\pi(\beta_1) = \frac{1}{\sqrt{2\pi}10} exp(-\frac{\beta_1^2}{2 \cdot 10^2})$$

And I have the following Metropolis-Hastings algorithm to draw β_1 since it's analytically intractable:

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procedure MHSAMPLER(SAMPLESIZE)
Pick a starting value for the Markov Chain of β_1. After some trials, I choose 8.5 here.
i=0
$mchain = matrix(NA, 1, SampleSize)$
while i <SampleSize **do**
i=i+1
Propose a new value for β_1, β_1^* according to a proposal distribution, say $q(\beta_1|\mathbf{Y},X) = Normal(0,1)$
Compute the Metropolis-Hastings accept-reject ratio, $\alpha(\beta_1,\beta_1^*) = min(\frac{\pi(\beta_1^*|\mathbf{Y},X)q(\beta_1|\mathbf{Y},X)}{\pi(\beta_1|\mathbf{Y},X)q(\beta_1^*|\mathbf{Y},X)}, 1) = min(\frac{\pi(\beta_1^*|\mathbf{Y},X)}{\pi(\beta_1|\mathbf{Y},X)}, 1)$
Draw *temp* from Uniform(0,1)
if $temp < \alpha(\beta_1,\beta_1^*)$ **then**
mchain[i]=β_1^*
else
mchain[i]=β_1
return *mchain*

1.2 (b)

$\mathbb{E}(\beta_1) = 7.353536$ and the MCMC standard error associated with it is 0.002160992

1.3 (c)

95% credible interval is $(6.735108, 7.948522)$

1.4 (d)

See Figure 1.1.

1.5 (e)

The effective sample size for the generated Markov Chain is 20692.53, which is reasonable(> 5000) . And I can see from Figure 1.2 that auto-correlation function for β_1 decreases quickly as the lag increases. Also, When I run with different initail values for β_1, the estimation converges to nearly the same value and MCse decreases quickly as the sample size goes up. These all proves that I'm producing reliable estimation for β_1.

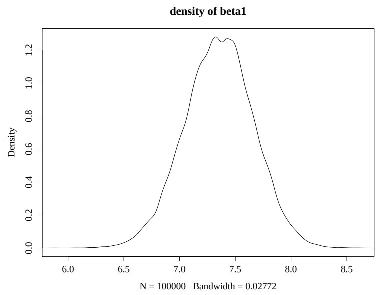

Figure 1.1 smoothed density plot for β_1

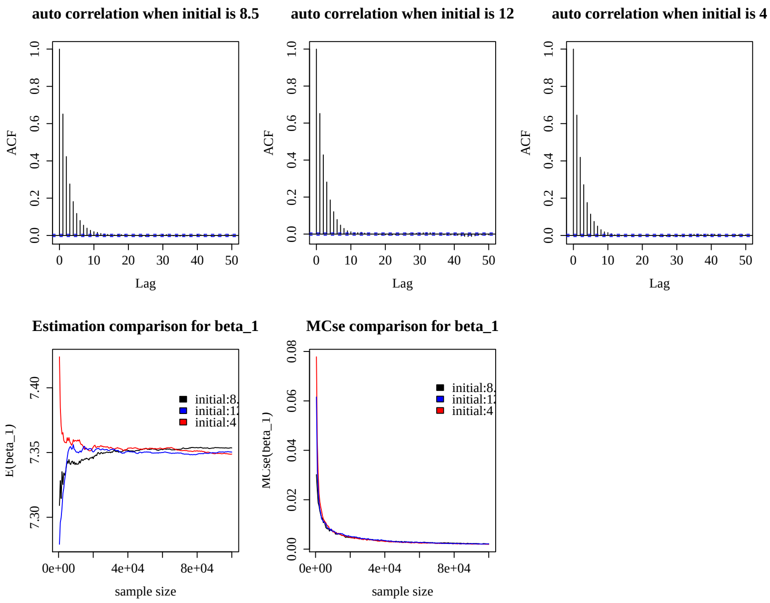

Figure 1.2 acf plot of the Markov Chain with initail value 8.5,9 and 12 respectively and comparison of $\mathbb{E}(\beta_1)$ w.r.t sample size and MCse w.r.t sample size. The initial values for β_1 corresponding to the black , blue and red lines are 8.5 , 12 , 4 respectively.

2 Problem 2

2.1 (a)

By Bayesian rule:

$$\pi(\beta_0, \beta_1, \lambda | \mathbf{Y}, X) \propto \pi(\mathbf{Y}, X | \beta_0, \beta_1, \lambda)\pi(\beta_0)\pi(\beta_1)\pi(\lambda)$$

in which:

$$\pi(\mathbf{Y}, X | \beta_0, \beta_1, \lambda) = \prod_{i=1}^{n} (\frac{\lambda}{2} exp(2(\beta_0 + \beta_1 X_i) + \lambda\sigma^2 - 2Y_i) erfc(\frac{\beta_0 + \beta_1 X_i + \lambda\sigma^2 - Y_i}{\sqrt{2}\sigma}))$$

So, the full conditional distribution of $\beta_0, \beta_1, \lambda$ are as follows:

$$\pi(\beta_0 | \beta_1, \lambda, \mathbf{Y}, X) \propto \prod_{i=1}^{n} (\frac{\lambda}{2} exp(2(\beta_0 + \beta_1 X_i) + \lambda\sigma^2 - 2Y_i) erfc(\frac{\beta_0 + \beta_1 X_i + \lambda\sigma^2 - Y_i}{\sqrt{2}\sigma})) N(\beta_0, 0, 10)$$

$$\pi(\beta_1 | \beta_0, \lambda, \mathbf{Y}, X) \propto \prod_{i=1}^{n} (\frac{\lambda}{2} exp(2(\beta_0 + \beta_1 X_i) + \lambda\sigma^2 - 2Y_i) erfc(\frac{\beta_0 + \beta_1 X_i + \lambda\sigma^2 - Y_i}{\sqrt{2}\sigma})) N(\beta_1, 0, 10)$$

$$\pi(\lambda | \beta_0, \beta_1, \mathbf{Y}, X) \propto \prod_{i=1}^{n} (\frac{\lambda}{2} exp(2(\beta_0 + \beta_1 X_i) + \lambda\sigma^2 - 2Y_i) erfc(\frac{\beta_0 + \beta_1 X_i + \lambda\sigma^2 - Y_i}{\sqrt{2}\sigma})) Gamma(\lambda, 0.01, 100)$$

And I have the following "Variate-at-a-time" Metropolis-Hastings algorithm to draw $\beta_0, \beta_1, \lambda$ since they are all analytically intractable:

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procedure MHSAMPLER(SAMPLESIZE)

Pick a starting value for the Markov Chain of β_0,β_1 and λ. After some trials, I choose 2,3.2,1 here.

i=0

$mchain = matrix(NA, 3, SampleSize)$;

while i <SampleSize **do**

i=i+1

Propose a new value for β_0, β_0^* according to a proposal distribution, say $q(\beta_0|\mathbf{Y}, X) = Normal(0, 1)$

Compute the Metropolis-Hastings accept-reject ratio, $\alpha(\beta_0, \beta_0^*) = min(\frac{\pi(\beta_0^*|\beta_1,\lambda,\mathbf{Y},X)q(\beta_0|\mathbf{Y},X)}{\pi(\beta_0|\beta_1,\lambda,\mathbf{Y},X)q(\beta_0^*|\mathbf{Y},X)}, 1) = min(\frac{\pi(\beta_0^*|\beta_1,\lambda,\mathbf{Y},X)}{\pi(\beta_0|\beta_1,\lambda,\mathbf{Y},X)}, 1)$

Draw $temp$ from Uniform(0,1)

if $temp < \alpha(\beta_0, \beta_0^*)$ **then**

mchain[i]=β_0^*

else

mchain[i]=β_0

Propose a new value for β_1, β_1^* according to a proposal distribution, say $q(\beta_1|\mathbf{Y}, X) = Normal(0, 1)$

Compute the Metropolis-Hastings accept-reject ratio, $\alpha(\beta_1, \beta_1^*) = min(\frac{\pi(\beta_1^*|\beta_0,\lambda,\mathbf{Y},X)q(\beta_1|\mathbf{Y},X)}{\pi(\beta_1|\beta_0,\lambda,\mathbf{Y},X)q(\beta_1^*|\mathbf{Y},X)}, 1) = min(\frac{\pi(\beta_1^*|\beta_0,\lambda,\mathbf{Y},X)}{\pi(\beta_1|\beta_0,\lambda,\mathbf{Y},X)}, 1)$

Draw $temp$ from Uniform(0,1)

if $temp < \alpha(\beta_1, \beta_1^*)$ **then**

mchain[i]=β_1^*

else

mchain[i]=β_1

Propose a new value for λ, λ^* according to a proposal distribution, say $q(\lambda|\mathbf{Y}, X) = Normal(0, 1)$

Compute the Metropolis-Hastings accept-reject ratio, $\alpha(\lambda, \lambda^*) = min(\frac{\pi(\lambda^*|\beta_0,\beta_1,\mathbf{Y},X)q(\lambda|\mathbf{Y},X)}{\pi(\lambda|\beta_0,\beta_1,\mathbf{Y},X)q(\lambda^*|\mathbf{Y},X)}, 1) = min(\frac{\pi(\lambda^*|\beta_0,\beta_1,\mathbf{Y},X)}{\pi(\lambda|\beta_0,\beta_1,\mathbf{Y},X)}, 1)$. (Set $\frac{\pi(\lambda^*|\beta_0,\beta_1,\mathbf{Y},X)}{\pi(\lambda|\beta_0,\beta_1,\mathbf{Y},X)}$ to 0 if $\lambda^* \leq 0$)

Draw $temp$ from Uniform(0,1)

if $temp < \alpha(\lambda, \lambda^*)$ **then**

mchain[i]=λ^*

else

mchain[i]=λ

return $mchain$

2.2 (b)

See summary in Table 1

variable	posterior mean	MCMC standard error	posterior 0.95 credible intervals
β_0	2.34943	0.002025202	$(2.081625, 2.610275)$
β_1	3.462513	0.002935254	$(0.056112, 3.869680)$
λ	0.8079838	0.0005653991	$(0.6978401, 0.9332141)$

Table 1 Table summary of β_0,β_1, λ's posterior mean , MCMC standard error and posterior 0.95 credible intervals

2.3 (c)

$Cor(\beta_0, \beta_1) = -0.7725857$

2.4 (d)

See Figure 2.3

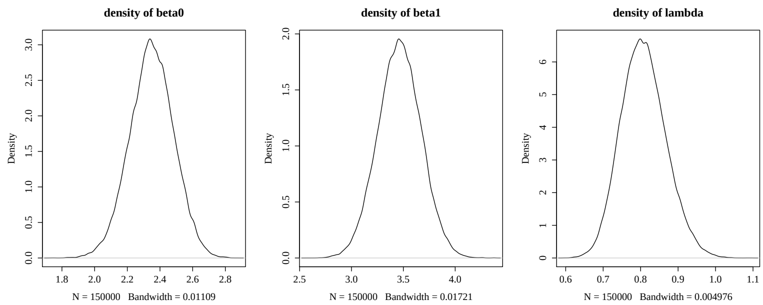

Figure 2.3 The marginal distribution of β_0,β_1 and λ for (d)

2.5 (e)

The effective sample sizes for the β_0,β_1 and λ are 4636.332, 5353.097 ,11490.77 respectively. I think they are reasonable (> 4500). The plot of auto-correlation function are shown in Figure 2.4 (a). I can see that the auto-correlation function for all three variables decrease quickly. The comaprison plot of Expectation and MCse w.r.t sample size of β_0,β_1 and λ with different initial values are shown in Figure 2.4 (b). The expectations converge to nearly the same value and the MCse decrease quite quickly. The marginal distribution of β_0,β_1 and λ of first 50000 samples and all samples are nearly the same. These all corroborates that I have chose the right initial value and reasonable proposal distribution and I are producing reliable approximations.

3 Problem 3

3.1 (a)

See summary in Table 2

variable	posterior mean	MCMC standard error	posterior 0.95 credible intervals
β_0	0.1481102	0.002033596	$(-0.1751833, 0.4581409)$
β_1	2.478279	0.003348214	$(1.937912, 3.023590)$
λ	0.1613289	$5.560394e-05$	$(0.1507897, 0.1723152)$

Table 2 Table summary of β_0,β_1, λ's posterior mean , MCMC standard error and posterior 0.95 credible intervals

3.2 (b)

See Figure 3.5

3.3 (c)

I set the initial value of β_0,β_1, λ to the expectation of β_0,β_1, λ in the first trial run. I increase the MH sample size to 200000 to increase the effective sample size (ESS) (I got 6763.383, 6781.89, 8107.92 for β_0,β_1, λ respectively). I didn't change anything else. I also include some plots similiar to Figure 2.4 in Figure 3.6 to prove that I'm producing reliable estimation.

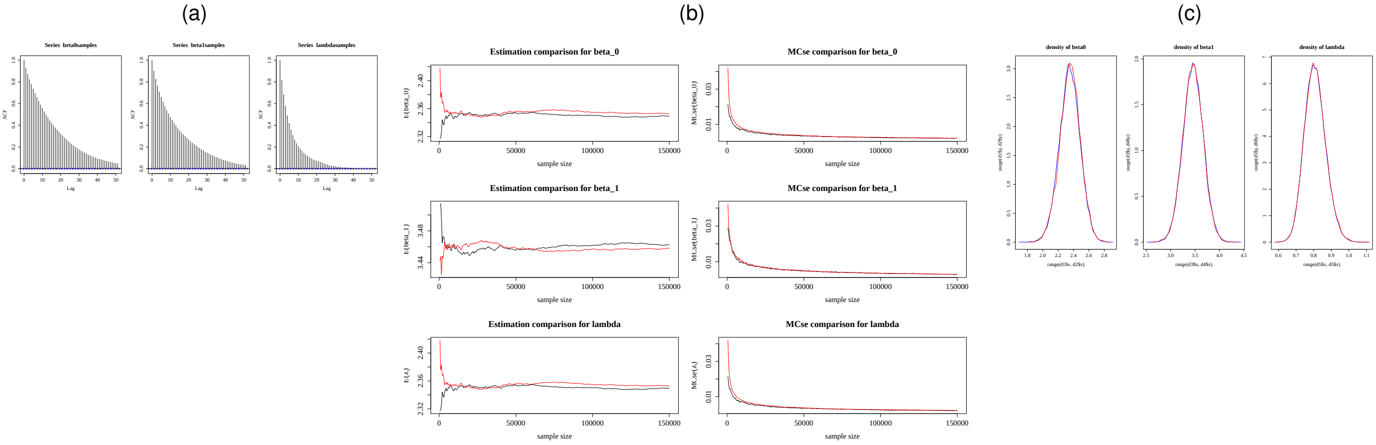

Figure 2.4 (a) The acf plot of β_0,β_1 and λ; (b) The Expectation and MCse w.r.t sample size plot of β_0,β_1 and λ with different initial values. The initial values corresponding to the black lines are $(\beta_0, \beta_1, \lambda) = (2, 3.2, 1)$. And the initial values corresponding to the red lines are $(\beta_0, \beta_1, \lambda) = (2.5, 4, 2)$. Plot (c) shows the marginal distribution comparison for first 50000 samples(red) and all samples(blue)

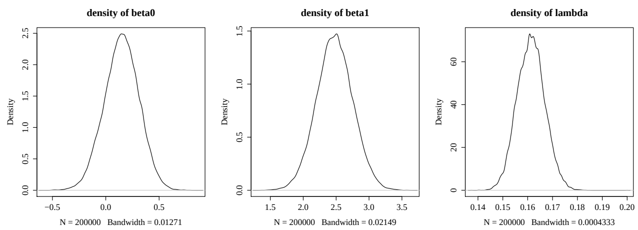

Figure 3.5 The marginal distribution of β_0,β_1 and λ for (b)

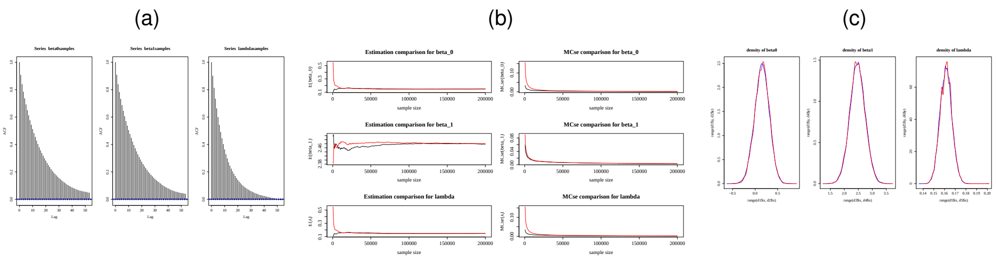

Figure 3.6 (a) The acf plot of β_0,β_1 and λ; (b) The Expectation and MCse w.r.t sample size plot of β_0,β_1 and λ with different initial values. The initial values corresponding to the black lines are $(\beta_0, \beta_1, \lambda) = (0.2, 2.5, 0.2)$. And the initial values corresponding to the red lines are $(\beta_0, \beta_1, \lambda) = (2.5, 4, 2)$. Plot (c) shows the marginal distribution comparison for first 50000 samples(red) and all samples(blue)