STAT 515

Homework #3, due Wednesday, Feb. 8 at 2:30pm

Please make every assignment easier to grade by neatly organizing your writeup and clearly labeling your final answers when appropriate.

1. Suppose that a population consists of a fixed number, $2m$, of genes in any generation. Each gene is one of two possible genetic types. If any generation has exactly i (of its $2m$) genes of type 1, then for any $0 \le j \le 2m$, the next generation will have exactly j genes of type 1 with binomial probability
$$\binom{2m}{j}\left(\frac{i}{2m}\right)^j\left(\frac{2m-i}{2m}\right)^{2m-j}.$$
Let X_n denote the number of type 1 genes in the nth generation, and assume $X_0 = m$.
   (a) Find $E(X_n)$.
   (b) Suppose that $m = 6$. What is the probability that $X_n = m$ for some $n > 0$?
   (c) If $m = 6$, what is the expected number of generations in which all genes except one are of the same type?

2. A transition matrix P is called *doubly stochastic* if each of its column sums equals one.
   (a) If an irreducible, aperiodic Markov chain has finitely many states and its transition matrix is doubly stochastic, prove that its limiting probability distribution is discrete uniform.
   (b) Find a doubly stochastic transition matrix P for a Markov chain with three states such that every entry of P is a different integer multiple of $1/12$ and such that $P_{11} = 0$ and $P_{22} = 1/2$. Calculate the matrix P^{10} (i.e., the tenth power of P). Explain why all nine entries of P^{10} should be nearly the same.

3. Consider three urns, one colored red, one white, and one blue. The red urn contains 1 red and 3 blue balls; the white urn contains 3 white balls, 2 red balls, and 1 blue ball; the blue urn contains 4 white balls, 3 red balls, and 2 blue balls. At the initial stage, a ball is randomly selected from the red urn and then returned to that (red) urn. At every subsequent stage, a ball is randomly selected from the urn whose color is the same as that of the ball previously selected and is then returned to the urn from which it was drawn.
   (a) Explain why this process is a Markov chain, then define an appropriate transition probability matrix to describe it.
   (b) Does this process have a stationary distribution? Justify your answer.
   (c) Explain why this process has a limiting distribution.
   (d) In the long run, what proportion of the selected balls are red? What proportion are white? What proportion are blue?
   (e) Simulate a Markov chain of length 100,000 using the information provided above and count the proportion of times the chain was in each of the states. Compare this to your answer.
   (f) Suppose you have taken 4 steps, i.e., you start with the initial distribution to obtain X_0 and use the transition probability matrix above to obtain state X_4 of the Markov chain. What proportion of times would you expect X_4 to be red, white, and blue, respectively?
   (g) Now simulate 10,000 realizations of the random variable X_4 using the initial distribution and transition probability matrix for this process. Calculate the proportion of times in your simulations that X_4 is red, white, and blue. Compare these proportions to your theoretically obtained answers above.

4. Suppose that in a branching process, the expected number of offspring of a given individual equals $4/5$. Find the expected number of individuals that ever exist in this population, assuming that $X_0 = n$.