Markov chains

We will consider discrete time discrete space M.C.s.
Stochastic process $\{X_n, n \in T\}$ where T is a countable set, usually $\{\ldots -2, -1, 0, 1, 2, \ldots\}$ or $\{0, 1, 2, \ldots\}$ and $X_n \in \Omega$ where Ω is a countable (~~or finite~~ no.) set.

A discrete time discrete space ~~M.C.~~ stochastic process $\{X_n, n \in T\}$ is a Markov chain if $P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, \ldots) = P_{ij}$ for all $i, i_{n-1}, i_{n-2}, \ldots, j \in \Omega$ and all $n \geq 0$, where $\{P_{ij}\}$ is a fixed set of transition probabilities.

Note that since P_{ij} is assumed to not vary (say w/ n) this is a homogeneous M.C.

Obviously, $P_{ij} \geq 0 \quad \forall i, j \in \Omega$, $\sum_{j \in \Omega} P_{ij} = 1 \quad \forall i \in \Omega$

Let $P = \{P_{ij}\}$ denote the set of onestep transition probabilities. If $T = \{0, 1, 2, \ldots\}$

$$P = \begin{array}{cc} & \text{next state} \\ \text{current state} & \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & & & \\ \vdots & & & \end{bmatrix} \end{array}$$

possibly infinite dimensional transition probability matrix (t.p.m.)

Markov chain idea: Future (X_{n+1} ...) and past ($X_{n-1}, X_{n-2}, \ldots$) are conditionally independent given the present (X_n)

Can generate a realization of chain if we have an initial distr. (Q) on state space and t.p.m. P.

$X_0 \sim Q$ $\quad X_{n+1} \mid X_n = x$ based on t.p.m P for $n=0,1,\ldots$

E.g. 1. Random walk. $\Omega = \{\ldots,-2,-1,0,1,2,\ldots\} = \mathbb{Z}$

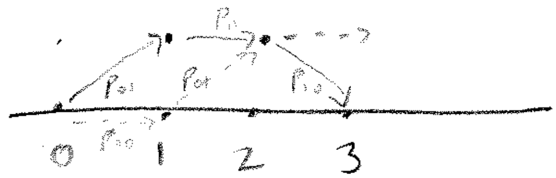
$$P_{i,i+1} = p$$
$$P_{i,i-1} = 1-p$$

Walking on straight line 1 step to right w/ prob. p
1 step " left " " $1-p$.

— end 1/25/08
lec #5

E.g. 2. 2 state M.C. w/ states = $\{0,1\}$

Markovian coin toss : use different coins if you are in state 0 or state $\underline{1}$.



$$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}$$

E.g. 3. Let $Z_1, Z_2, \ldots$ be iid r.v. on space of non-neg. integers,

so $P(Z_i = j) = p_j$ for $j \in \{0, 1, \ldots\}$

w/ $p_j \geq 0$, $\sum_{j=0}^{\infty} p_j = 1$

Let $S_n = \sum_{i=1}^{n} Z_i \quad , n = 1, 2, \ldots$

S_n is an M.C. w/ t.p.m.

$$P(S_{n+1} = j \mid S_n = j) = P\left(\sum_{i=1}^{n} Z_i + Z_{n+1} = j \;\middle|\; \sum_{i=1}^{n} Z_i = i\right)$$

$$= P(Z_{n+1} = j - i) = \begin{cases} p_{j-i} & j \geq i \\ 0 & \text{else} \end{cases}$$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & & \cdots & & \end{array} \\ \begin{bmatrix} p_0 & p_1 & p_2 & p_3 & \cdots & \\ 0 & p_0 & p_1 & p_2 & p_3 & \cdots \\ 0 & 0 & \cdots & & & \\ \vdots & \vdots & & & & \end{bmatrix} \end{array}$$

Gambler's ruin

Suppose gambler plays game where he wins \$1 w/ prob p
loses \$1 " " $1-p$

Let X_t = fortune at time t.

Gambler quits when he wins \$N (maximum) or when he has \$0 left. (he is 'ruined').

This is a modified random walk.

M.C. w/ t.p.m. $P_{i,i+1} = p$, $P_{i,i-1} = 1-p$ for $i = 1, \ldots, N-1$

and $P_{0,0} = 1 = P_{N,N}$ $0, N$ are 'absorbing states'.

Of interest: Pr gambler gets to goal (\$N) before ('going broke') 'ruin' (\$0).

Let $f_i = \Pr(\text{reach } \$N \text{ when already have } \$i)$, $i = 0, 1, \ldots, N$.

$f_0 = 0$, $f_N = 1$.

Condition on outcome of 1st bet.

$$f_i = f_{i+1} \underbrace{p}_{\text{win } \$1} + f_{i-1} \underbrace{(1-p)}_{\text{lose } \$1} \qquad i = 1, 2, \ldots, N-1$$

Let $q = 1-p$ for convenience

Now, since $p+q=1$.

$$p f_i + q f_i = p f_{i+1} + q f_{i-1}$$

$$\Rightarrow q\left(f_i - f_{i-1}\right) = p\left(f_{i+1} - f_i\right)$$

$$\Rightarrow f_{i+1} - f_i = \frac{q}{p}\left(f_i - f_{i-1}\right) \qquad i = 1, \dots, N-1.$$

Hence,

$$f_2 - f_1 = \frac{q}{p}\left(f_1 - f_0\right) = \frac{q}{p} f_1$$

$$f_3 - f_2 = \frac{q}{p}\left(f_2 - f_1\right) = \left(\frac{q}{p}\right)^2 f_1$$

$$\vdots \qquad\qquad \vdots$$

$$f_N - f_{N-1} = \left(\frac{q}{p}\right)^{N-1} f_1$$

Add first $i-1$ eqns: $\displaystyle f_i - f_1 = \sum_{k=1}^{i-1}\left(\frac{q}{p}\right)^k f_1$

$$\Rightarrow f_i = \underbrace{\sum_{k=0}^{i-1}\left(\frac{q}{p}\right)^k}\, f_1$$

sum of geometric series $\dfrac{1-r^{n+1}}{1-r}$

$$\Rightarrow f_i = \begin{cases} \left[\dfrac{1-(q/p)^i}{1-(q/p)}\right] f_1 & \dfrac{q}{p} \neq 1 \\ i\, f_1 & \dfrac{q}{p} = 1 \end{cases}$$

Need to find f_1.

Since $f_N = 1$

$$1 = f_N = \begin{cases} \dfrac{1-(q/p)^N}{1-q/p}\, f_1 & q/p \neq 1 \\ N f_1 & q/p = 1 \end{cases}$$

$$\Rightarrow f_1 = \begin{cases} \dfrac{1-q/p}{1-(q/p)^N} & \text{if } q/p \neq 1 \\ \dfrac{1}{N} & \text{if } q/p = 1 \end{cases}$$

Hence, $$f_i = \begin{cases} \dfrac{(1-(q/p)^i)}{\cancel{1-q/p}} \cdot \dfrac{\cancel{1-q/p}}{1-(q/p)^N} & \text{if } q/p \neq 1 \\ i/N & \text{if } q/p = 1 \end{cases}$$

Note: As $N \to \infty$ $$f_i \to \begin{cases} 1-(q/p)^i & \text{if } p > 1/2 \\ 0 & \text{if } p = 1/2 \\ 0 & \text{if } p < 1/2 \end{cases}$$

Implication: $p > 1/2$ means +ve prob. gambler can increase fortune indefinitely

$p \leq 1/2$ means gambler will be ruined w/ prob 1 if playing against infinitely wealthy opponent (casinos!)

— end of lec. 6

Chapman-Kolmogorov Equations

t.p.m. P describes $\underline{1}$ step transitions.

$$P_{ij} = P(X_{t+1}=j \mid X_t = i) \qquad i,j \in \Omega$$

More generally, n-step transitions:

$$P_{ij}^{(n)} = P(X_{t+n}=j \mid X_t = i) \qquad n \in \{1,2,3,\ldots\}$$

Note: $P^{(n)} = P^n$ so all information is in 1step t.p.m.

How do we calculate n-step transition probabilities?

Condition on intermediate steps using

C-K eqns.

$$P_{ij}^{n+m} = \sum_{\text{all } k} P_{ik}^n P_{kj}^m \quad , \forall\, n,m \geq 0 \, , \quad i,j \in \Omega$$

Pf:

$$\begin{aligned} P_{ij}^{n+m} &= P(X_{n+m}=j \mid X_0 = i) \\ &= \sum_k P(X_{n+m}=j, X_n = k \mid X_0 = i) \\ &= \sum_k P(X_{n+m}=j \mid X_n = k, X_0 = i)\, P(X_n = k \mid X_0 = i) \\ &= \sum_k P(X_{n+m}=j \mid X_n = k)\, P(X_n = k \mid X_0 = i) \\ &= \sum_k P_{kj}^m P_{ik}^n \end{aligned}$$

 Note next page # is 30

Classification of states of an M.C.

Accessible: state j is accessible from state i if for some integer $n \geq 0$, $P_{ij}^n > 0$

Can reach j from i in finite # of transitions w/ +ve probability.

Communicate: Two states i and j, accessible to each other, are said to communicate. $(i \leftrightarrow j)$

If i and j do not communicate:

either $P_{ij}^n = 0 \quad \forall n \geq 0$

or $P_{ji}^n = 0 \quad \forall n \geq 0$

or both are true.

Communication is an equivalence relation:

(1) $i \leftrightarrow i$: $P_{ij}^0 = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ by defn.

(reflexivity)

(2) $i \leftrightarrow j \Rightarrow j \leftrightarrow i$ (symmetry)

(3) $i \leftrightarrow j$ and $j \leftrightarrow k \Rightarrow i \leftrightarrow k$ (transitivity)

Pf: $i \leftrightarrow j$ and $j \leftrightarrow k \Rightarrow \exists\, n, m$ s.t. $P_{ij}^n > 0$ and $P_{jk}^m > 0$

$$\Rightarrow P_{ik}^{n+m} = \sum_r P_{ir}^n P_{rk}^m \geq P_{ij}^n P_{jk}^m > 0$$

$\Rightarrow i \rightarrow k$ Similarly, $k \rightarrow i$.

Hence Can find equivalence classes (of states that communicate w/ each other) that partition the state-space.

Two states that communicate are said to be in the same class.

~~So a two classes of states are either identical or disjoint~~

$\Rightarrow$ Communication ~~sta~~ partitions state space into a number of ~~different~~ separate classes.

Irreducible: M.C. is irreducible if there is only one class, i.e, all states communicate with each other.

E.g. 1.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{cc|ccc} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

Two classes: $\{1, 2\}$ and $\{3, 4, 5\}$ (not irreducible)

End of [illegible]

E.g. 2: Random walk w/ absorbing barriers (like gambler's ruin problem)

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ (a-1) \\ a \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \cdots & & \cdots & (a-1) & a & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ (1-p) & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & (1-p) & 0 & p & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & & & & & (1-p) & 0 & p \\ 0 & \cdots & & & & 0 & 0 & 1 \end{bmatrix} \end{array}$$

— end of lec 7

3 classes: $\underbrace{\{0\}}_{A}, \underbrace{\{1, 2, \ldots a-1\}}_{B}, \underbrace{\{a\}}_{C}$

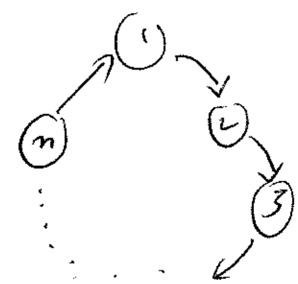
Can go from B to A or C but not A to B or C ~~to~~ B

Periodicity

Period of state i $(d(i))$ is the greatest common divisor (g.c.d.) of all integers $n \geq 1$ for which $P_{ii}^{n} > 0$. If $P_{ii}^{n} = 0$ $\forall n \geq 1$, define $d(i) = 0$.

E.g.

$$\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & \cdots & n \\
\hline
1 & 0 & 1 & 0 & 0 & \cdots & 0 \\
2 & 0 & 0 & 1 & 0 & \cdots & 0 \\
3 & \vdots & & & & & \\
4 & & & & & \cdots & 1 \\
\vdots & 0 & 0 & \cdots & & \cdots & 0 \\
\vdots & 1 & 0 & \cdots & & & \\
n & & & & & &
\end{array}$$



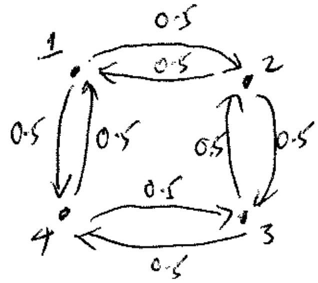
Each state has period n

$\because$ starting at i it is possibly to only re-enter i at $\{n, 2n, 3n, \ldots\}$ g.c.d. is n.

Periodicity is a class property, i.e. if $i \leftrightarrow j$ then $d(i) = d(j)$.

Aperiodic: M.C. in which each state has period 1 is called aperiodic. For irreducible M. chain only need to show period $= 1$ for any one state.

E.g. Random walk on a square



t.p.m. $P = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix}$

Start in 1 : in 1-step can get to 2 or 4, not return to 1

✓2-steps : " " " 3 or return to 1

~~3-steps : " " " 2 or 4~~ from 3, need two steps to get back to 1

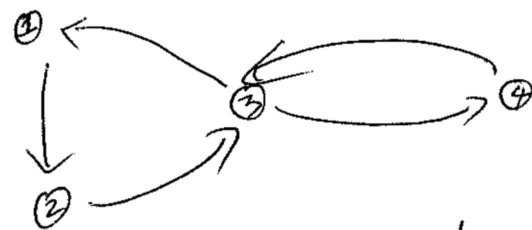
3-steps: cannot get back to 1

✓4-steps : can get back to 1

1 has period of 2.

Since 1, 2, 3, 4 are in same class, all have period of 2.

E.g. 2.



Is this M. chain aperiodic?

It is irreducible so only need to consider 1 state

say (3). $d(3) = \gcd\{2, 3, \dots\}$

$= 1$.

Yes, ~~irre~~ aperiodic. even though at 1st glance, it seems periodic (~~3~~

(1), (3) seems to have periods > 1)

Recurrence

For any states i & j, define

prob. that, starting in i, the 1st transition into j occurs at time n.

Define $f_{ij}^0 = 0$

$$f_{ij}^n = P(X_n = j,\ X_k \neq j,\ k = 1, \ldots, n-1 \mid X_0 = i)$$

Now define $f_{ij}^* = \sum_{n=1}^{\infty} f_{ij}^n$

f_{ij}^* = Prob of <u>ever</u> making a transition into j, given that process starts in state i.

$f_{ij}^* > 0 \iff i \to j \qquad (i \neq j)$

<u>Defn:</u> State i is <u>recurrent</u> if $f_{ii}^* = 1$

State " is <u>transient</u> if $f_{ii}^* < 1$

Note the difference between f_{ij}^n and P_{ij}^n

f_{ij}^n = prob. of <u>1st</u> transition to j at time n.

P_{ij}^n = prob. of a " " " " " ".

Thm: All states of a finite, irreducible M.C. are recurrent.

Pf: Since M.C. is irreducible ~~all~~ states are either all recurrent or all transient.

Suppose all states are transient.

For pf. by contradiction we need the following Lemma.

Define: Q_{ii} = Prob (M.C. returns ∞ often $| X_0 = i$)

Lemma: State is recurrent or transient according to whether $Q_{ii} = 1$ or 0 respectively.

Let Q_{ii}^N = Prob(M.C. returns to i at least N times $| X_0 = i$)

$$Q_{ii}^N = \sum_{k=1}^{\infty} \text{Prob (M.C. returns 1st time in } k^{th} \text{ step)} \times \text{Prob (M.C. returns to } i \text{ at least } N-1 \text{ times} \,|\, X_k = i)$$

$$= \sum_{k=1}^{\infty} f_{ii}^k \, Q_{ii}^{N-1} = Q_{ii}^{N-1} f_{ii}^*$$

Proceeding recursively, $Q_{ii}^N = (f_{ii}^*)^2 Q_{ii}^{N-2} = \ldots = (f_{ii}^*)^{N-1} Q_{ii}^1$

$= (f_{ii}^*)^N$

$$Q_{ii} = \lim_{N\to\infty} Q_{ii}^N = \lim_{N\to\infty} (f_{ii}^*)^N = \begin{cases} 1 & \text{if recurrent} \\ 0 & \text{if transient} \end{cases}$$

Hence, (from Lemma) if state is transient, M.C. does not return to state i after some finite time T_i. (w/prob 1) True for all states i

~~Similarly~~

So, M.C. does not return to any state after time $T = \cancel{\sup}\max\{T_i\}$

w/prob. 1 Contradiction!

Hence all states must be recurrent.

Thm: State i is recurrent iff
$$\sum_{n=1}^{\infty} P_{ii}^n = \infty$$

Sketch of Pf: $(\Rightarrow)$ If state i is recurrent, prob. (M.C. eventually returns to i) = 1.
Process probabilistically restarts itself when it enters state i so prob(it returns twice to state i) = 1.

So Prob (returns ∞ # of times to state i) = 1

$E(\#\text{ returns}) = \infty$

$$\Rightarrow E\left(\sum_{n=1}^{\infty} I(X_n = i \mid X_0 = i)\right) = \infty$$

(need Borell-Cantelli etc. for rigorous proof)

(careful about this!)
$$\Rightarrow \sum_{n=1}^{\infty} E(I(X_n = i \mid X_0 = i)) = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} P(X_n = i \mid X_0 = i) = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} P_{ii}^n = \infty$$

$(\Leftarrow)$ If state i is transient, prob. (M.C. never returns to i) = $1 - f_{ii}^* > 0$ $\quad (\because f_{ii}^* < 1)$

Prob (MC stays in state i for a total of k times $| X_0 = i$)
$$= \underbrace{(f_{ii}^*)^{k-1}}_{\text{Prob of getting to state } i \text{ } k-1 \text{ times}} \underbrace{(1 - f_{ii}^*)}_{\text{Prob of never returning to } i} > 0$$

($\Leftarrow$) Suppose state i is transient.

Each time process returns to i, there is positive probability, $1-f_{ii}^* > 0$ that it will never return.

Hence, # visits $\sim$ Geometric $(p = 1-f_{ii}^*)$

wait until 1^{st} success where "success" = leave and never return

$$E(\#\text{ visits}) = \frac{1}{1-f_{ii}^*} < \infty$$

$$\sum_{n=1}^{\infty} P_{ii}^n < \infty$$

Hence i is recurrent $\Leftrightarrow$ $\sum_{n=1}^{\infty} P_{ii}^n = \infty$

i is transient $\Leftrightarrow$ $\sum_{n=1}^{\infty} P_{ii}^n < \infty$

Thm: Recurrence/transience is a class property

Pf: Suppose i is reccurrent.

Assume $j \leftrightarrow i$ so j and i are in the same class

Remains to show that j is recurrent

Now, $P_{ij}^n > 0$ and $P_{ji}^m > 0$ for some n, m (by $i \leftrightarrow j$ assumption)

For any $s \geq 0$:

$$P_{jj}^{m+s+n} \geq P_{ji}^m P_{ii}^s P_{ij}^n \quad \text{(from Chapman-Kolmogorov)}$$

$$\Rightarrow \sum_{s=0}^{\infty} P_{jj}^{m+s+n} \geq \left(\sum_{s=0}^{\infty} P_{ii}^s\right) P_{ji}^m P_{ij}^n$$

But $\sum_{s=0}^{\infty} P_{ii}^s = \infty$ (by previous thm, since i is recurrent)

Hence $\sum_{s=0}^{\infty} P_{jj}^{m+s+n} = \infty$

and j is also recurrent.

Follows that transience is also a class property