Stat 515: Stochastic Processes I — Spring 2012
Take-Home Final Exam — April 21–May 2, 2012

This exam is worth 10 points. You have 11 days. Electronic submission is required. *Show all of your work for full credit; answers submitted without supporting work will receive little or no credit.* The rules of the exam are as follows:

A. You may not communicate about this exam with anyone other than the instructor, not even the grader. You may not receive help of any kind on this exam from anyone else except the instructor. You may not give help of any kind on this exam to anyone else.

B. You must submit your writeup and your code to ANGEL, in the appropriate dropboxes, before 5:00pm on Wednesday, May 2. Do not include your code in your writeup; the code must be easy to read (with comments as appropriate) and it should be clear how I can run it. You may use R, Matlab, or Python. I strongly encourage the use of a sensible text editor in preparing your code.

Problem 1. [2 points] Suppose that a player has five dollars and wishes to play craps until she either runs out of money or doubles her money (i.e., until the first time she has either zero dollars or some amount greater than or equal to ten dollars), at which point she stops playing. Betting in craps works as follows: If the player has x dollars and bets y dollars (where $y \le x$), then with probability $244/495$, she wins and her new total is $x + y$; with probability $251/495$, she loses and her new total is $x - y$.

(a) Suppose that the player always bets one dollar. What is the probability that she will eventually double her money?

Solution: This is an example of Gambler's ruin. With $p = 244/495$, the probability of getting to 10 before 0 if you start from 5 is

$$\frac{1 - \left(\frac{251}{244}\right)^5}{1 - \left(\frac{251}{244}\right)^{10}} = 0.4647$$

(b) Suppose that for each new game, the player bets 3, 2, or 1 dollar with probabilities 1/6, 2/6, and 3/6, respectively. (If she only has two dollars left, she bets 2 or 1 dollar with probabilities 1/3 and 2/3, respectively. If she only has one dollar left, she bets one dollar.) What is the expected number of games she will be able to play before stopping (at either zero dollars or ten or more dollars)?

Solution: If we make 0 and 10 absorbing states, and $p = 1-q = 244/495$, then the transition probability matrix for the Markov chain in which her current wealth is the state equals

$$P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q & 0 & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q/3 & 2q/3 & 0 & 2p/3 & p/3 & 0 & 0 & 0 & 0 & 0 & 0 \\
q/6 & q/3 & q/2 & 0 & p/2 & p/3 & p/6 & 0 & 0 & 0 & 0 \\
0 & q/6 & q/3 & q/2 & 0 & p/2 & p/3 & p/6 & 0 & 0 & 0 \\
0 & 0 & q/6 & q/3 & q/2 & 0 & p/2 & p/3 & p/6 & 0 & 0 \\
0 & 0 & 0 & q/6 & q/3 & q/2 & 0 & p/2 & p/3 & p/6 & 0 \\
0 & 0 & 0 & 0 & q/6 & q/3 & q/2 & 0 & p/2 & p/3 & p/6 \\
0 & 0 & 0 & 0 & 0 & q/6 & q/3 & q/2 & 0 & p/2 & p/2 \\
0 & 0 & 0 & 0 & 0 & 0 & q/6 & q/3 & q/2 & 0 & p \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

To find the expected time spent in states 1 through 9, we look at $S = (I - P_T)^{-1}$, where P_T is the 9×9 submatrix of P corresponding to states 1 through 9. The answer we are after is the sum of the whole 5th row of S:

```
> Pt=rbind( c(0, 6, 0, 0, 0, 0, 0, 0, 0),
+            c(4, 0, 4, 2, 0, 0, 0, 0, 0),
+            c(2, 3, 0, 3, 2, 1, 0, 0, 0),
+            c(1, 2, 3, 0, 3, 2, 1, 0, 0),
+            c(0, 1, 2, 3, 0, 3, 2, 1, 0),
+            c(0, 0, 1, 2, 3, 0, 3, 2, 1),
+            c(0, 0, 0, 1, 2, 3, 0, 3, 2),
+            c(0, 0, 0, 0, 1, 2, 3, 0, 3),
+            c(0, 0, 0, 0, 0, 1, 2, 3, 0)) / 6
> Pt[row(Pt)<col(Pt)] <- Pt[row(Pt)<col(Pt)] * 244/495
> Pt[row(Pt)>col(Pt)] <- Pt[row(Pt)>col(Pt)] * 251/495
> S <- solve(diag(rep(1,9)) - Pt)
> sum(S[5,])

[1] 8.948682
```

We conclude that the expected number of games is 8.95.

Problem 2. [8 points] Suppose we have parameters distributed as follows:

$$\begin{aligned} \theta_0, \theta_1 &\overset{\text{iid}}{\sim} N(0,1), \\ \lambda &\sim \text{beta}(2,2), \quad \text{independently of } \theta_0 \text{ and } \theta_1. \end{aligned}$$

Furthermore, suppose that, conditional on the parameters,

$$Z_1, \ldots, Z_{10} \overset{\text{iid}}{\sim} \text{Bernoulli}(\lambda).$$

(In other words, $P(Z_i = 1) = 1 - P(Z_i = 0) = \lambda$.) Finally, assume that $X_1, \ldots, X_{10}$ are conditionally independent—conditional on the parameters and the Z_i—with mass function

$$p(x_i \mid Z_i, \theta_0, \theta_1, \lambda) = \binom{20}{x_i} \left(\frac{e^{x_i \theta_0}}{(1 + e^{\theta_0})^{20}} \right)^{1 - Z_i} \left(\frac{e^{x_i \theta_1}}{(1 + e^{\theta_1})^{20}} \right)^{Z_i} \quad \text{for } i = 1, \ldots, 10.$$

Intuitively, this means that X_i is conditionally distributed as binomial$(20, p_i)$, where

$$p_i = \frac{\exp\{\theta_{Z_i}\}}{1 + \exp\{\theta_{Z_i}\}}.$$

(a) [4 points] Here are the data:

i	1	2	3	4	5	6	7	8	9	10
X_i	18	9	12	9	14	5	18	12	8	9
Z_i	1	0	0	0	1	0	1	0	0	0

Using these data:

(i) Demonstrate that λ, θ_0, and θ_1 are independent of one another in the posterior distribution.

Solution: To find the likelihood, we need to multiply $p(X_i \mid Z_i, \theta_0, \theta_1, \lambda)$ times $p(Z_i \mid \theta_0, \theta_1, \lambda)$ to obtain $p(X_i, Z_i \mid \theta_0, \theta_1, \lambda)$ for each i. Then the likelihood times the prior joint density, which is proportional to the posterior joint density, may be written

$$K\left[\lambda^{\sum_i Z_i}(1-\lambda)^{\sum_i(1-Z_i)}\lambda(1-\lambda)\right]\left[\left(\frac{\exp\{\theta_0\sum_i(1-Z_i)X_i\}}{(1+e^{\theta_0})^{20\sum_i(1-Z_i)}}\right)\exp\{-\theta_0^2/2\}\right]\left[\left(\frac{\exp\{\theta_1\sum_i Z_iX_i\}}{(1+e^{\theta_1})^{20\sum_i Z_i}}\right)\exp\{-\theta_1^2/2\}\right]$$

for a constant K, which may be rewritten as

$$K\left[\lambda^4(1-\lambda)^8\right]\left[\left(\frac{\exp\{64\theta_0\}}{(1+e^{\theta_0})^{140}}\right)\exp\{-\theta_0^2/2\}\right]\left[\left(\frac{\exp\{50\theta_1\}}{(1+e^{\theta_1})^{60}}\right)\exp\{-\theta_1^2/2\}\right].$$

It is written in this way to show clearly that the joint posterior density factors into a function of λ only times a function of θ_0 only times a function of θ_1 only. This proves that the three parameters are independent in the posterior.

(ii) Implement three separate importance samplers to estimate the posterior means of θ_0, θ_1, and λ, respectively. You may implement your samplers using any q distributions that you think are appropriate, but please explain what your choice is in each case.

Solution: Since the sample proportions for $Z = 0$ and $Z = 1$ are $64/140$ and $50/60$, respectively, let us suppose that the posterior densities for θ_0 and θ_1 will be peaked at roughly $\log(64/76) = -0.17$ and $\log(50/10) = 1.61$, respectively.

For λ, we already see that its posterior density is beta$(5, 9)$, which means that the exact posterior mean equals $5/14$ and we don't even have to use importance sampling. However, I'll go ahead and act as though we don't know this. Let's assume that the posterior density for λ has a peak around $3/10$, since that is the proportion of $Z_i = 1$. For the q densities, I'll select $N(-0.17, 1)$ and $N(1.61, 1)$ for θ_0 and θ_1 and beta$(1.5, 3.5)$ for λ. (The latter has mean $1.5/5 = 0.3$.)

The exact normalizing constant associated with each posterior density is not known, so we will have to use ratio importance sampling to estimate the posterior means. To be very cautious in calculating the ratios required, we could do all of the calculations on the logarithmic scale and then exponentiate; however, by using R's built-in density functions, we can avoid the need for this step:

```
> ## First, consider theta0 (with sample size one million):
> theta0 <- rnorm(n <- 1e6, mean=-.17, sd=1)
> p0<- exp(theta0)/(1+exp(theta0))
> a0 <- theta0 * dbinom(64, 140, p0) * dnorm(theta0) / dnorm(theta0, mean=-.17, sd=1)
> b0 <- dbinom(64,140,p0) * dnorm(theta0) / dnorm(theta0, mean=-.17, sd=1)
> meanTheta0 <- (muA0 <- mean(a0)) / (muB0 <- mean(b0))
> meanTheta0
[1] -0.1680111
> ## Next, same thing for theta1:
> theta1 <- rnorm(n, mean=1.61, sd=1)
> p1<- exp(theta1)/(1+exp(theta1))
> a1 <- theta1 * dbinom(50, 60, p1) * dnorm(theta1) / dnorm(theta1, mean=1.61, sd=1)
> b1 <- dbinom(50, 60,p1) * dnorm(theta1) / dnorm(theta1, mean=1.61, sd=1)
> meanTheta1 <- (muA1 <- mean(a1)) / (muB1 <- mean(b1))
> meanTheta1
[1] 1.472498
> ## Finally, for lambda:
> lambda <- rbeta(n, 1.5, 3.5)
> a2 <- lambda * dbeta(lambda, 4, 8) * dbeta(lambda, 2, 2) / dbeta(lambda, 1.5, 3.5)
> b2 <- dbeta(lambda, 4, 8) * dbeta(lambda, 2, 2) / dbeta(lambda, 1.5, 3.5)
> meanLambda <- (muA2 <- mean(a2)) / (muB2 <- mean(b2))
> c(meanLambda, 5/14) # These values should be close!
```

```
[1] 0.3573152 0.3571429
```

(iii) Based on your samplers, give 95% confidence intervals for each of the three true posterior means. Make sure that you have sampled enough to ensure that your confidence intervals are no wider than 0.01.

Solution: In each case, we'll use the delta-method approximation for ratio importance sampling, which is given by

$$\mathrm{Var}\left(\frac{\frac{1}{n}\sum_i A_i}{\frac{1}{n}\sum_i B_i}\right) \approx \frac{1}{n\mu_B^2}\begin{bmatrix}1 & \frac{-\mu_A}{\mu_B}\end{bmatrix}\begin{bmatrix}\sigma_A^2 & \sigma_{AB}\\ \sigma_{AB} & \sigma_B^2\end{bmatrix}\begin{bmatrix}1\\ \frac{-\mu_A}{\mu_B}\end{bmatrix}.$$

The 95% intervals will be equal to the estimators (from the previous part) plus or minus 1.96 times the square root of the approximation above in which each parameter (μ_A, μ_B, and the covariance matrix) is replaced by its sample estimate:

```
> # For theta0:
> tmp <- c(1, -muA0 / muB0)
> var0 <- tmp %*% var(cbind(a0, b0)) %*% tmp / n / muB0^2
> meanTheta0 + c(-1.96, 1.96) * sqrt(var0)
[1] -0.1684933 -0.1675290
> # For theta1:
> tmp <- c(1, -muA1 / muB1)
> var1 <- tmp %*% var(cbind(a1, b1)) %*% tmp / n / muB1^2
> meanTheta1 + c(-1.96, 1.96) * sqrt(var1)
[1] 1.471814 1.473182
> # For lambda:
> tmp <- c(1, -muA2 / muB2)
> var2 <- tmp %*% var(cbind(a2, b2)) %*% tmp / n / muB2^2
> meanLambda + c(-1.96, 1.96) * sqrt(var2)
[1] 0.3570799 0.3575506
```

In each case, the intervals are much narrower than 0.01.

(b) [4 points] Now, suppose that not all of the data have been observed. We only know the following:

i	1	2	3	4	5	6	7	8	9	10
X_i	18	9	12	9	14	5	18	12	8	9
Z_i	1	0	0	0	1	??	??	??	??	??

Using these data, in which $Z_6, \ldots, Z_{10}$ may now be considered to be parameters:

(i) Derive the full conditional densities (up to multiplicative constants) for λ, θ_0, and θ_1. Also derive the full conditional mass function for Z_i, where i can be any value from 6 to 10.

Solution: Starting from the solution to part (a)(i) and plugging in the values that are known, we obtain as the posterior joint density function

$$K\left[\lambda^{3+\sum_{i=6}^{10} Z_i}(1-\lambda)^{4+\sum_{i=6}^{10}(n-Z_i)}\right]\left[\left(\frac{\exp\{\theta_0[30+\sum_{i=6}^{10}(1-Z_i)X_i]\}}{(1+e^{\theta_0})^{60+20\sum_{i=6}^{10}(1-Z_i)}}\right)\exp\{-\theta_0^2/2\}\right]$$
$$\times\left[\left(\frac{\exp\{\theta_1[32+\sum_{i=6}^{10} Z_iX_i]\}}{(1+e^{\theta_1})^{40+20\sum_{i=6}^{10} Z_i}}\right)\exp\{-\theta_1^2/2\}\right]$$

We obtain the full conditionals for each of the parameters from this expression. In particular, λ has a beta density, θ_0 and θ_1 have difficult densities with no obvious family as in part (a), and Z_j has full conditional mass function

$$\left[\frac{\lambda \exp\{\theta_1 X_j\}}{(1+\exp\{\theta_1\})^{20}}\right]^{Z_j} \left[\frac{(1-\lambda)\exp\{\theta_0 X_j\}}{(1+\exp\{\theta_0\})^{20}}\right]^{1-Z_j},$$

which is of the form $\alpha^{Z_j}\beta^{1-Z_j}$, from which we conclude that Z_j is Bernoulli with mean $\alpha/(\alpha+\beta)$ (as explained in part ii).

(ii) Implement a variable-at-a-time Metropolis-Hastings algorithm to sample from the posterior distribution of $(\theta_0, \theta_1, \lambda)$. Describe the proposal distributions you use for this purpose and how you decided how long to run the chain. For the updates of $Z_6, \ldots, Z_{10}$, use Gibbs sampling together with the fact that for any Bernoulli variable Y with mass function proportional to $\alpha^y\beta^{1-y}$,

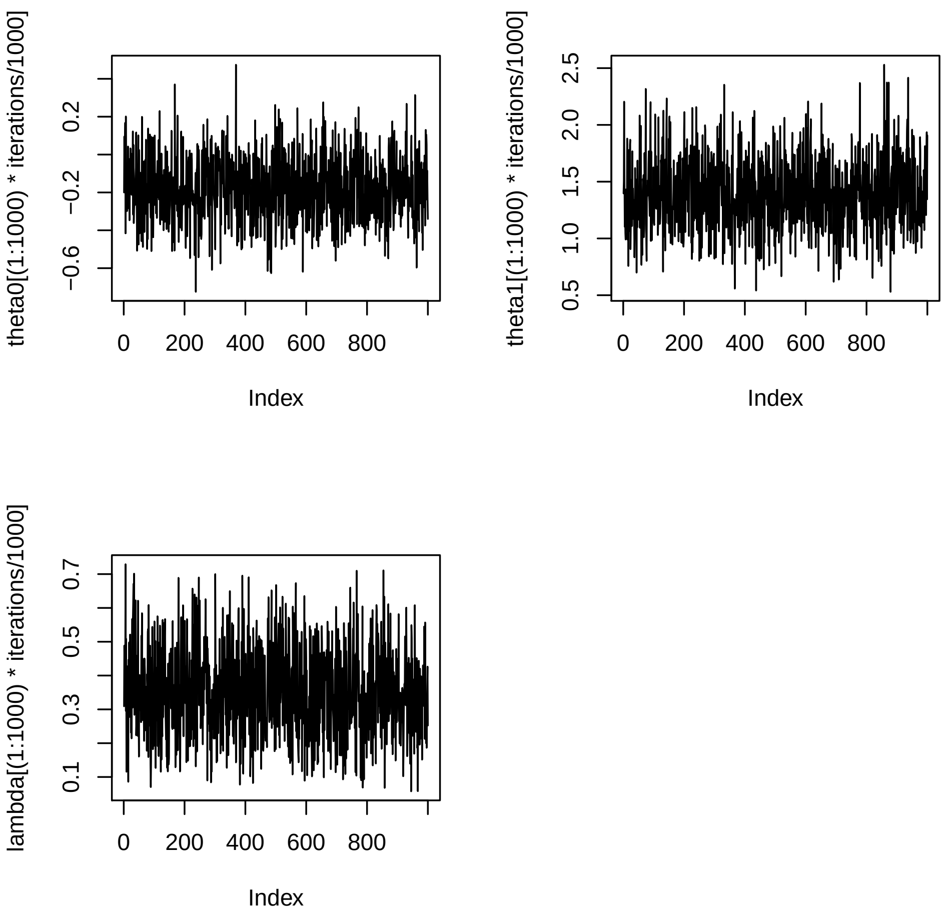
$$P(Y=1) = 1 - P(Y=0) = \frac{\alpha}{\alpha+\beta}.$$

Solution: We may use Gibbs sampling for updating λ, then Metropolis-Hastings for updating θ_0 and θ_1, then Gibbs for updating the Z_j. For the M-H iterations, we can propose new θ_0 and θ_1 values from a normal distribution with variance 0 centered at the current values. This proposal is symmetric, which means that the M-H ratio simplifies to a Metropolis ratio.

```
> # Initialization:
> x <- c(18, 9, 12, 9, 14, 5, 18, 12, 8, 9)
> z <- c(1, 0, 0, 0, 1, 0, 0, 0, 0, 0) # The last five of these are arbitrary starting values
> iterations <- 1e6
> z6toz10 <- matrix(0, 1+iterations, 5)
> theta0 <- theta1 <- rep(0, 1+iterations)
> lambda <- rep(1/2, iterations)
> for (i in 1:iterations) {
+   # Step 1:  Update lambda
+   lambda[i+1] <- rbeta(1, 1+sum(z), 11-sum(z))
+   # Step 2:  Update theta0
+   Proposal <- rnorm(1, mean=theta0[i])
+   logMHRatio <- (theta0[i]^2 - Proposal^2)/2 +
+                             (Proposal - theta0[i]) * sum((1-z) * x) -
+                             20 * sum(1-z) * log(1 + exp(Proposal)) +
+                             20 * sum(1-z) * log(1 + exp(theta0[i]))
+   theta0[i+1] <- ifelse (log(runif(1)) < logMHRatio, Proposal, theta0[i])
+   # Step 3:  Update theta1
+   Proposal <- rnorm(1, mean=theta1[i])
+   logMHRatio <- (theta1[i]^2 - Proposal^2)/2 +
+                             (Proposal - theta1[i]) * sum(z * x) -
+                             20 * sum(z) * log(1 + exp(Proposal)) +
+                             20 * sum(z) * log(1 + exp(theta1[i]))
+   theta1[i+1] <- ifelse (log(runif(1)) < logMHRatio, Proposal, theta1[i])
+   # Step 4:  Update Z_6 through Z_10
+   for (j in 6:10) {
+     alpha <- exp( log(lambda[i+1]) + x[j] * theta1[i+1] - 20*log(1+exp(theta1[i+1])))
+     beta <- exp( log(1-lambda[i+1]) + x[j] * theta0[i+1] - 20*log(1+exp(theta0[i+1])))
+     z6toz10[i+1, j-5] <- z[j] <- rbinom(1, 1, alpha/(alpha+beta))
+   }
+ }
```

Let's take a look at a trace plot to see how the chain appears to be mixing. (The plot thins the chain so that the pdf file is not too large.)

```
> par(mfrow=c(2,2))
> plot(theta0[(1:1000)*iterations/1000], type="l")
> plot(theta1[(1:1000)*iterations/1000], type="l")
> plot(lambda[(1:1000)*iterations/1000], type="l")
```



These plots look great, so the number of iterations seems okay. The only question is how narrow our confidence intervals are and whether they give estimates that are precise enough for our purposes. If not, we could always use more iterations.

(iii) Give 95% credible intervals for the two binomial proportions $\exp\{\theta_0\}/(1+\exp\{\theta_0\})$ and $\exp\{\theta_1\}/(1+\exp\{\theta_1\})$. Base these intervals on the 0.025 and 0.975 quantiles of the θ_0 and θ_1 parameters, respectively.

Solution:

```
> inverseLogit <- function(x) exp(x) / (1+exp(x))
> inverseLogit(quantile(theta0, c(.025, .975))) # p0 credible interval
     2.5%     97.5%
0.3695248 0.5382474
> inverseLogit(quantile(theta1, c(.025, .975))) # p1 credible interval
     2.5%     97.5%
0.6872755 0.8896077
```

(iv) Based on your MCMC run, give estimates of the posterior means of $Z_6, \ldots, Z_{10}$ along with corresponding confidence intervals.

Solution: For this, we'll use the batch means idea with $b = n/1000$ and $a = 1000$:

```
> phat <- colMeans(z6toz10)
> phat # These are the estimated posterior means
[1] 0.0002369998 0.9973890026 0.1915768084 0.0031239969 0.0087559912
> b <- iterations/1000
> a <- 1000
> varHat <- rep(0, 5)
> for (j in 1:5) {
+   y <- rowMeans(matrix(z6toz10[-1, j], nrow=b, byrow=TRUE))
+   varHat[j] <- b*var(y)
+ }
> lowerBounds <- phat - 1.96 * sqrt(varHat / iterations)
> upperBounds <- phat + 1.96 * sqrt(varHat / iterations)
> rbind(lowerBounds, upperBounds) # There are the conf intervals
                    [,1]      [,2]      [,3]        [,4]        [,5]
lowerBounds 0.0001984312 0.9972802 0.1902892 0.002999160 0.008537393
upperBounds 0.0002755684 0.9974978 0.1928644 0.003248833 0.008974590
```

Interestingly, the posterior means leave little doubt about the most likely classification of the four X_i values 5, 18, 8, and 9. Only 12 is somewhat in doubt, with a posterior mean (i.e., probability of coming from the θ_1 distribution) of around 19%.