- No new reading for Friday; make sure you've read through Section 4.8 (both editions)
- HW #3 is due Wednesday, Feb. 8
- No class one week from Friday.

Suppose you have \$2 and you bet on fair games of chance until you either go broke or have \$5.

- What is the expected number of time steps that you have \$2?
- How long will this experiment last, on average?
- What is the probability that you will at some point have \$1?

Notes: We saw questions one and three we saw on Monday; I added the middle question as an extra.

- For transient states i and j, let s_{ij} equal the expected number of time steps spent in j, given $X_0 = i$.
- Let $S = (s_i j)$ be the matrix of s_{ij} values.
- Last time, we showed that $S = I + P_T S$.
- Therefore, $S = (I = P_T)^{-1}$.

Notes: This is review from Monday's class.

In our example,

$$P_T = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \quad \text{and so} \quad S = (I - P_T)^{-1} = \frac{1}{5}\begin{bmatrix} 8 & 6 & 4 & 2 \\ 6 & 12 & 8 & 4 \\ 4 & 8 & 12 & 6 \\ 2 & 4 & 6 & 8 \end{bmatrix}.$$

Notes: Here, the theory developed so far is applied to the specific problem at hand (i.e., gambler's ruin with $N = 5$, $i = 2$, and $p = 1/2$). Important note: I did not *reorder the states, as is done in Section 4.6, so that all of the transient ones come first. Instead, this P_T matrix is the 4×4 submatrix in the middle of the 6×6 full P matrix, since the 4 transient states are all except for the first state (zero) and the last state (five).*

In our example,

$$S = \frac{1}{5}\begin{bmatrix} 8 & 6 & 4 & 2 \\ 6 & 12 & 8 & 4 \\ 4 & 8 & 12 & 6 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Starting with \$2...
- Mean # of steps with \$2: $\frac{12}{5}$
- Mean # of steps before end: 6

Notes: These answers may be read directly from the S matrix. For the second one, it's necessary to sum the entire second row because

$$E(\textit{time spent in all transient states}) = \sum_{i \textit{ transient}} E(\textit{time spent in state } i).$$

Let $f_{ij} \stackrel{\text{def}}{=} P(X_t = j \text{ for some } t > 0 \mid X_0 = t)$
- Demonstrate that $s_{ij} = I\{i = j\} + f_{ij} s_{jj}$.
- Solve to find f_{ij}.

Notes: The key is to argue that

$$E(\textit{\# steps in } j \mid X_0 = i, X_t = j \textit{ for some } t > 0) = \begin{cases} s_{jj} + 1 & \textit{if } i = j \\ s_{jj} & \textit{if } i \neq j \end{cases}$$

and

$$E(\textit{\# steps in } j \mid X_0 = i, X_t \textit{ is never } j \textit{ for any } t > 0) = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{if } i \neq j. \end{cases}$$

This leads to $s_{ij} = [s_{jj} + I\{i = j\}] f_{ij} + I\{i = j\}(1 - f_{ij})$ *using conditioning, as desired. Solving gives* $f_{ij} = [s_{ij} - I\{i = j\}]/s_{jj}$.

In our example,

$$S = \frac{1}{5}\begin{bmatrix} 8 & 6 & 4 & 2 \\ 6 & 12 & 8 & 4 \\ 4 & 8 & 12 & 6 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Starting with \$2...
- Probability of ever having \$1: $\frac{6}{8}$
- Probability of ever returning to \$2: $\frac{7}{12}$

- Particular type of Markov chain
- X_t =size of tth generation, $t = 0, 1, 2, \ldots$.
- Assume each individual produces offspring independently according to some distribution $\mathcal{P} = \{P_0, P_1, P_2, \ldots\}$ on the nonnegative integers.

For a branching process $X_0, X_1, X_2, \ldots$,

- Possible to demonstrate $EX_n = \mu^n$
- Possible to demonstrate $\mathrm{Var}\ X_n = \begin{cases} \sigma^2 \mu^{n-1} \left(\frac{1-\mu^n}{1-\mu}\right) & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$

...where μ and σ^2 are the mean and variance of $\mathcal{P}$.

Notes: The argument used for this derivation is simple conditioning on X_{n-1}, then recursively on $X_{n-2}, X_{n-3}, \ldots$.

Feb. 1 4.7 Branching Processes

Generally, the most interesting cases involve $P_0 > 0$. In this case...

- what are the classes of states?
- which are recurrent / transient?
- what does this mean for the Markov chain in the long term?

Notes: We argued that when $P_0 > 0$, all states other than zero form a single transient class. Zero is its own (recurrent) class, since it is an absorbing state. We conclude that either the process grows without bound, or it dies out by eventually hitting zero.

Feb. 1 4.7 Branching Processes

A commonly asked question is: What is

$$P(\text{population will die out} \mid X_0 = 1)?$$

Call this probability π_0 (as in book). Then

$$\pi_0 = \lim_{n\to\infty} P(X_n = 0 \mid X_0 = 1).$$

Feb. 1 4.7 Branching Processes

- Well-known fact: If $\mu < 1$, then $\pi_0 = 1$.
- Can you derive the equation $\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j$?

Notes: In an epidemiological context, μ is often called R_0. In a simple branching-process model, this means that $R_0 = 1$ is the threshold between an epidemic possibly occurring (X_n gets large with positive probability) and no epidemic possible ($X_n \to 0$ with probability one).