

A Bayesian change point model

Consider the following hierarchical changepoint model for the number of occurrences Y_i of some event during time interval i with change point k .

$$\begin{aligned} Y_i|k, \theta, \lambda &\sim \text{Poisson}(\theta) \text{ for } i = 1, \dots, k \\ Y_i|k, \theta, \lambda &\sim \text{Poisson}(\lambda) \text{ for } i = k + 1, \dots, n \end{aligned}$$

Assume the following prior distributions:

$$\begin{aligned} \theta|b_1 &\sim \text{Gamma}(0.5, b_1) & (\text{pdf}=g_1(\theta|b_1)) \\ \lambda|b_2 &\sim \text{Gamma}(0.5, b_2) & (\text{pdf}=g_2(\lambda|b_2)) \\ b_1 &\sim \text{IG}(0, 1) & (\text{pdf}=h_1(b_1)) \\ b_2 &\sim \text{IG}(0, 1) & (\text{pdf}=h_2(b_2)) \\ k &\sim \text{Uniform}(1, \dots, n) & (\text{pmf}=u(k)) \end{aligned}$$

k, θ, λ are conditionally independent and b_1, b_2 are independent.
Assume the Gamma density parameterization $\text{Gamma}(\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$
and IG (Inverse Gamma) density parameterization $\text{IG}(\alpha, \beta) = \frac{e^{-1/\beta x}}{\Gamma(\alpha)\beta^\alpha x^{\alpha+1}}$

Inference for this model is therefore based on the 5-dimensional **posterior** distribution $f(k, \theta, \lambda, b_1, b_2|\mathbf{Y})$ where $\mathbf{Y}=(Y_1, \dots, Y_n)$. The posterior distribution is obtained *upto a constant* by taking the product of all the conditional distributions. Thus we have

$$\begin{aligned} f(k, \theta, \lambda, b_1, b_2|\mathbf{Y}) &\propto \prod_{i=1}^k f_1(Y_i|\theta, \lambda, k) \prod_{i=k+1}^n f_2(Y_i|\theta, \lambda, k) \\ &\times g_1(\theta|b_1)g_2(\lambda|b_2)h_1(b_1)h_2(b_2)u(k) \\ &= \prod_{i=1}^k \frac{\theta^{Y_i} e^{-\theta}}{Y_i!} \prod_{i=k+1}^n \frac{\lambda^{Y_i} e^{-\lambda}}{Y_i!} \\ &\times \frac{1}{\Gamma(0.5)b_1^{0.5}} \theta^{-0.5} e^{-\theta/b_1} \times \frac{1}{\Gamma(0.5)b_2^{0.5}} \lambda^{-0.5} e^{-\lambda/b_2} \\ &\times \frac{e^{-1/b_1}}{b_1} \frac{e^{-1/b_2}}{b_2} \frac{1}{n} \end{aligned}$$

If we are able to draw samples from this distribution, we can answer questions of interest.