Scalable Bayes via Parallelization and Posterior Aggregation

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December 3, 2019

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Recall MCMC

- Goal: Sample from a posterior distribution / approximate the posterior
- Idea: Construct a Markov Chain whose stationary distribution is the target distribution

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(One) Solution: Parallelization

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Idea: Exploit multiple processors to run several independent chains and combine them.

Split data → MCMC Sampler → Aggregate all posteriors

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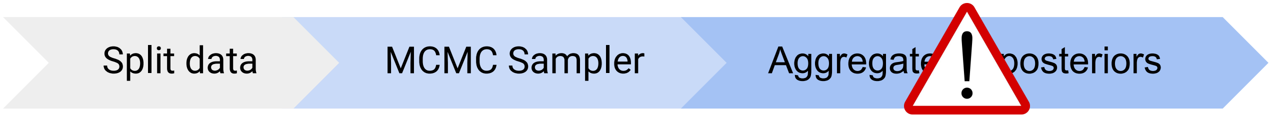

Easy!

You’re good at this

Aggregating Posteriors

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WASserstein **P**osterior (Srivastava et al., 2015)

Why it dominates:

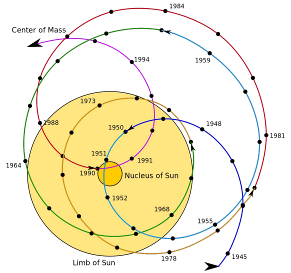
- ✅ No need for a kernel or tuning parameters (unlike other methods)
- ✅ WASP can be estimated “efficiently” via a linear program
- ✅ Resulting posterior is a “good” approximation

WASP calculates the **Wasserstein barycenter (WB)** of the subset posterior measures

First, some background

Wasserstein Barycenter?

Barycenter (Wikipedia)
the center of mass of two or more bodies that orbit one another and is the point about which the bodies orbit



Wasserstein metric
a distance function defined between probability measures on a given metric space

Wasserstein barycenter
The mean of a set of probability measures (the measure that minimizes the sum of its Wasserstein distances to each element in that set)

p^{th} Wasserstein distance

(Ω, d)	Metric space, metric
$P(\Omega)$	The set of Borel probability measures on Ω
$\Pi(\mu, \nu)$	The set of all probability measures on Ω^2 that have marginals μ and ν (The set of all "couplings" of μ and ν)

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Wasserstein Barycenter of N measures $\{\nu_1, \cdots, \nu_N\} \in P(\Omega)$: $\arg\min_{\tau} \frac{1}{N} \sum_{i=1}^{N} W_p^p(\tau, \nu_i)$

… in the the context of parallel MCMC

- Approximate subset posterior probability measures ν_i with empirical measures:

$$\hat{\nu}_i = \hat{\Pi}_i(\cdot) = \sum_{j=1}^{S} \frac{1}{S} \delta_{\theta ij}(\cdot), \qquad i = 1, \ldots, N$$

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A linear program! Yay!

Several existing algorithms to solve this linear program. See Cuturi (2014), Carlier et al. (2015), Srivastava et al. (2015)

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- 100,000 samples from mixture of two 2-dim Gaussians: $f_{\text{mix}}(\mathbf{y}|\theta) = \sum_{i=1}^{2} \pi_i N_2(\mathbf{y}|\boldsymbol{\mu}_i, \Sigma_i)$

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- Goal: approximate $p(\boldsymbol{\mu}_1|\mathbf{y}, \boldsymbol{\pi}, \Sigma_1, \Sigma_2)$ and $p(\boldsymbol{\mu}_2|\mathbf{y}, \boldsymbol{\pi}, \Sigma_1, \Sigma_2)$

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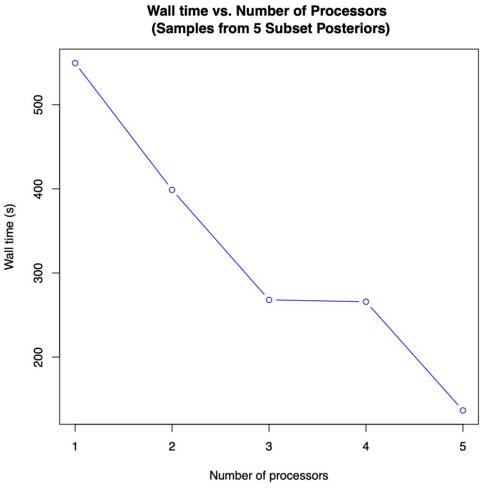
- Sampler: Gibbs
- Data split: 5 subsets, 5000 samples per chain

Preliminary results

5 subset posteriors



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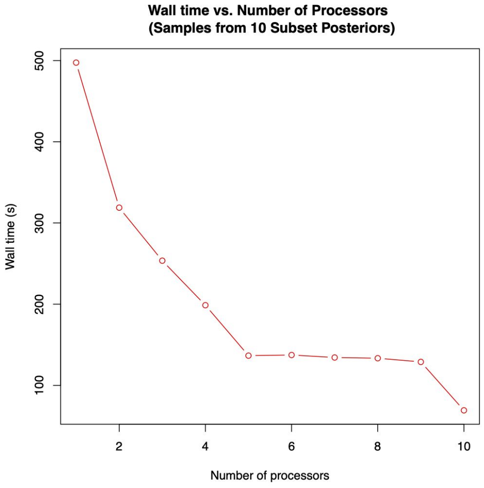

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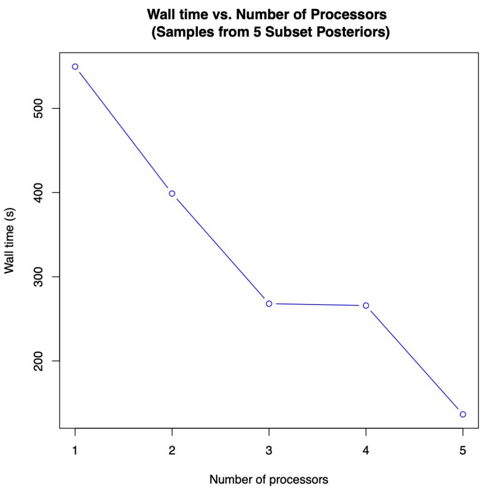

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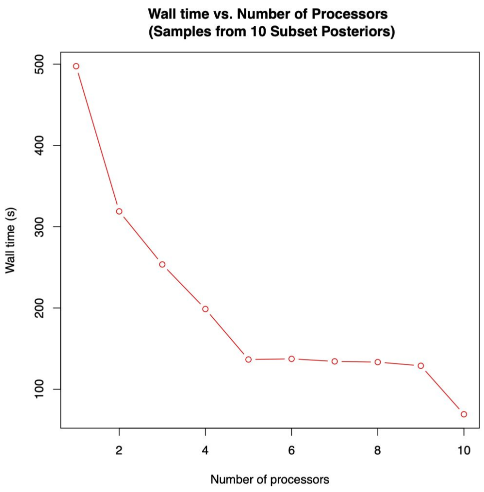
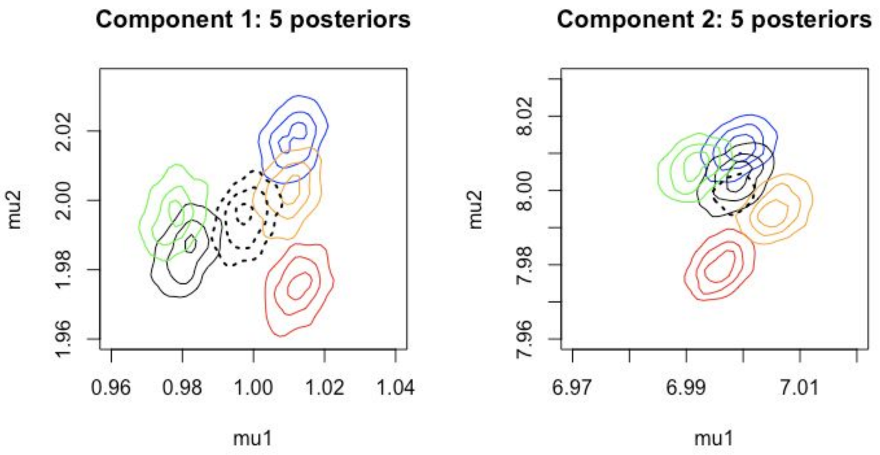


Subset Posterior Approximations



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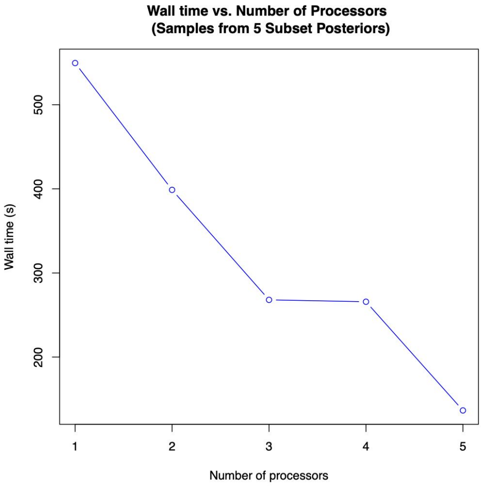

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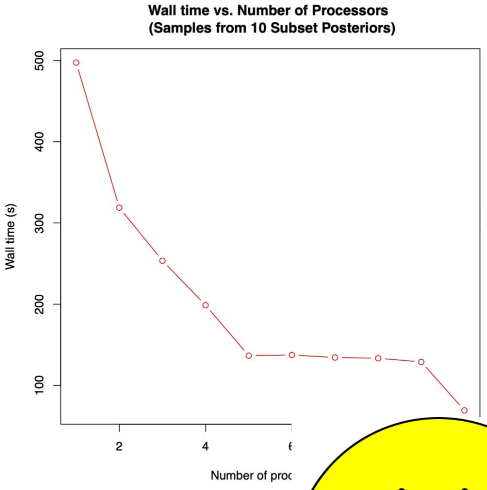

Not bad!

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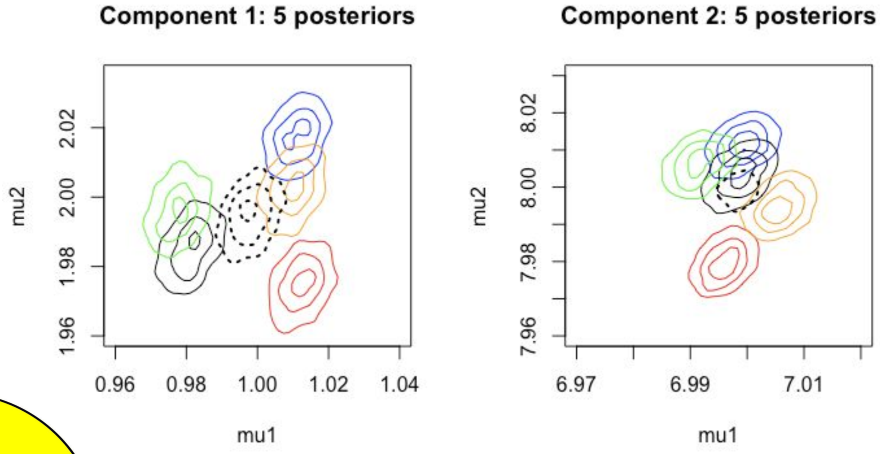

Makes sense!

Alternative algorithms to explore

See paper for theoretical justification of WASP and comparison to other methods:

- **Consensus Monte Carlo** (CMC, Scott et al., 2016)

  weighted average of samples

- **Semiparametric density product** (SDP, Neiswanger et al., 2014)

  kernel smooth each subset posterior density, multiply together to approximate the posterior density

- **Parallel MCMC with M-Posterior** (Minsker et al., 2017)

  posterior aggregation method, robust to outliers, but less efficient

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Note: Parallelizing can only take us so far. No substitute for good samplers.

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