Homework 2, Stat 515, Spring 2015

Due Wednesday, February 4th, 2015 **beginning of class**

1. Suppose that the probability of rain today depends on weather conditions from the previous three days. If it has rained for the past three days, then it will rain today with probability 0.8; if it did not rain for any of the past three days, then it will rain today with probability 0.2; and in any other case, the weather today will, with probability 0.6, be the same as the weather yesterday.

   (a) Describe this process using a Markov chain, i.e., define a state space and the corresponding transition probability matrix for the process.

   (b) Suppose you know that it rained the very first day but that it did not on the second and third days. What is the probability it will rain on the fifth day?

2. Consider a Markov chain on $\Omega = \{1, 2, 3, 4, 5, 6\}$ specified by the following transition probability matrix.

$$P = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

   (a) What are the (communicating) classes of this Markov chain? Is the Markov chain irreducible ?

   (b) Which states are transient and which are recurrent? Justify your answers.

   (c) What is the period of each state of this Markov chain? Is the Markov chain aperiodic ?

   (d) Let X_0 be the initial state with distribution $\boldsymbol{\pi}_0 = (0, \frac{1}{4}, \frac{3}{4}, 0, 0, 0)$ corresponding to the probability of being in states $1, 2, 3, 4, 5, 6$ respectively. Let $X_0, X_1, X_2, \ldots$ be the Markov chain constructed using P above. What is $\mathrm{E}(X_1)$?

   (e) What is $\mathrm{Var}(X_1)$?

   (f) What is $\mathrm{E}(X_3)$?

3. Let P be the transition probability matrix of a Markov chain. Show that if, for some positive integer r, P^r has all positive entries, then so does P^n, for all integers $n \geq r$.

4. Computer problem: Simulate a realization of the random variable X_3 according to the description in Problem 2, using the initial distribution in 1(d). Repeat this 1000 times (generate 1000 instances of X_3), and calculate the average. This is your estimate of $E(X_3)$. Compare it with your answer from Problem 2. Since this is a short program, include a printout of your code with your homework. *Please make your assignment easier to grade by neatly organizing your writeup and by drawing a box around and labeling your answers.*

5. Simulate the Markov chain according to Problem 2 and run it for 100,000 steps. Now calculate the proportion of times the Markov chain was in the states 1,2,3,4,5,6 respectively. Simulate two more realizations, each also of length 100,000, and again record the proportion of times the Markov chain was in the states 1,2,3,4,5,6 respectively. You only have to report the proportions for each of the three realizations (do not print out your Markov chains or your code for this!)

6. Prove that periodicity is a class property.

7. Consider two Markov chains on $\Omega = \{1, 2, 3, 4, 5\}$ specified by the following transition probability matrices.

$$P_1 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, P_2 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

   (a) Specify the classes of the two Markov chains with these transition probability matrices and determine whether the states are transient or recurrent.

   (b) List the periods of each state of the Markov chain. Are these Markov chains aperiodic?

8. Consider three urns, one colored red, one white, and one blue. The red urn contains 1 red and 4 blue balls; the white urn contains 3 white balls, 2 red balls, and 2 blue balls; the blue urn contains 4 white balls, 3 red balls, and 2 blue balls. At the initial stage, a ball is randomly selected from the red urn and then returned to that urn. At every subsequent stage, a ball is randomly selected from the urn whose color is the same as that of the ball previously selected and is then returned to that urn. Write the state space of this process and the associate initial distribution and transition probability matrix. Explain why this process is a Markov chain.