Positive and null recurrence

Define $\mu_{jj} = E($# transitions needed to return to state j given started in state $j)$

μ_{jj} is the "expected return time".

Recall: $f_{ij}^n = P(X_n = j, X_k \neq j, k = 1, \ldots, n-1 \mid X_0 = i)$

$= \text{Prob}($# transitions to reach j the first time, starting from state i $= n)$

$$\mu_{jj} = \begin{cases} \sum_{n=1}^{\infty} n f_{jj}^n & \text{if } j \text{ is recurrent} \\ \infty & \text{if } j \text{ is transient} \end{cases}$$

$(\because P(\text{never returning}) > 0 \Rightarrow P(\#\text{ transitions before returning} = \infty) > 0 \Rightarrow \mu_{jj} = \infty)$

If state j is recurrent:

Positive recurrent if $\mu_{jj} < \infty$ ← [Finite expected return time (desirable!)]

Null recurrent " $\mu_{jj} = \infty$

Thm: Both positive and null recurrence are class properties.

Pf: H.W. (assigned)

Thm: Finite state M.C.: all recurrent states are positive recurrent.

Pf: ~~H.W.~~ ??

E.g. of null recurrent chain

$$\begin{tikzcd} 0 \arrow[r, "1"'] & 1 \arrow[r, "1/2"'] \arrow[l, bend right, "1/2"'] & 2 \arrow[r, "2/3"'] \arrow[ll, bend right, "1/3"'] & 3 \arrow[r, "3/4"'] \arrow[lll, bend right, "1/4"'] & {} \end{tikzcd}$$

Let $f_{ij}^{n} = Pr$(# transitions to reach state j the 1st time from state i is n)

Note: chain is irreducible.

Is state 0 recurrent ?

$$f_{00}^{1} = 0 \qquad f_{00}^{2} = 1/2$$

$$f_{00}^{3} = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}, \quad f_{00}^{4} = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4} = \frac{1}{12}$$

$$\ldots \quad \text{So} \quad f_{00}^{n} = \frac{1}{(n-1)n}, \quad n > 1$$

$$\Rightarrow Pr(\text{eventually returning to } 0 / X_0 = 0) = \sum_{n=1}^{\infty} f_{00}^{n}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n-1}\cdot\frac{1}{n}$$

Can show $\displaystyle\sum_{n=2}^{k} \frac{1}{n-1}\cdot\frac{1}{n} = \frac{k-1}{k}$

So $\displaystyle\lim_{k\to\infty} \sum_{n=2}^{\infty} \frac{1}{(n-1)n} = 1 \Rightarrow$ recurrent chain

Now, $\displaystyle E(\text{time to return}) = \sum_{n=2}^{\infty} n\left(\frac{1}{n-1}\cdot\frac{1}{n}\right) = \sum_{n=1}^{\infty}\frac{1}{n} = \infty$

$\Rightarrow$ null recurrent chain

Recurrence of random walks

1-D random walk on $\mathbb{Z}$

$P_{i,i+1} = p \quad \forall i \in \mathbb{Z}, \quad 0 < p < 1.$
$P_{i,i-1} = 1-p$

All states communicate so all are ~~either~~ recurrent or all are transient.

2/3/08, end of lecture

Sufficient to study state 0.

Idea: gambler starts at 0. +1 if win, -1 if loss.

Return to 0 iff as many losses(n) as wins(n) in 2n trials.

$$\Rightarrow P_{00}^{2n} = \binom{2n}{n} p^n (1-p)^n \quad n = 1, 2, \ldots \quad \text{(even \#)}$$

$$P_{00}^{2n-1} = 0 \quad n = 1, 2, \ldots \quad \text{(odd \#)}$$

$$\Rightarrow P_{00}^{2n} = \frac{(2n)!}{n!\,n!} p^n (1-p)^n$$

Use Stirling's approximation: $n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi}$

where $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

$$\text{So,} \quad P_{00}^{2n} \sim \frac{(2n)^{(2n+1/2)} \cancel{e^{-2n}} \cancel{\sqrt{2\pi}}}{n^{n+1/2} \cdot n^{n+1/2} \cdot \cancel{e^{-2n}} \cancel{\sqrt{2\pi}} \sqrt{2\pi}} p^n (1-p)^n$$

$$= \frac{(2n)^{2n} \cdot (2n)^{1/2}}{n^{2n+1} \sqrt{2\pi}} p^n (1-p)^n$$

$$= \frac{4^n \cancel{n^{2n}} \cdot \cancel{\sqrt{2}} \sqrt{n}}{\cancel{n^{2n}} \cdot n \, \cancel{\sqrt{2}} \sqrt{\pi}} p^n (1-p)^n$$

$$= \frac{[4\,p(1-p)]^n}{\sqrt{\pi n}}$$

$\therefore\ \sum a_n < \infty \iff \sum b_n < \infty$ limit form of Comparison test. (pg. 40 DePree & Schwartz)

$$\sum_n P_{00}^n < \infty \iff \sum_{n=1}^{\infty} \frac{[4p(1-p)]^n}{\sqrt{\pi n}} < \infty$$

Now, $4p(1-p) = 1 \iff p = 1/2$

So, $\sum P_{00}^n = \infty \iff p = 1/2$ ($\because$ p-series w/ $p \leq 1$)

If $p \neq 1/2$, $4p(1-p) < 1$ so $\sum P_{00}^n < \infty$

Hence, r.walk is recurrent if $p = 1/2$, else transient.

For 2-D r.walk: If r.walk is symmetric, i.e., left or right, up or down w/ prob $1/4$ each, it is recurrent.

~~In similar fashion~~

However - all symmetric r.walks in higher dimensions, are transient.

Summary / intuition for classification of states:

① Irreducibility: all states communicate, i.e., M.C. can get from anywhere in state space to anywhere else.

Periodicity: Prob (returning to state) is 0 except at regular intervals.

② Aperiodicity prevents M.C. from oscillating between different states in a regular periodic movement (helps establish limiting behavior of M.C.).

Recurrence: Prob. of starting at state i and returning to state i in finite # steps is 1.

$P(\text{return time} < \infty) = 1$

Transient: $P(\text{return time} = \infty) > 0$

③ Positive recurrence: $E(\text{return time}) < \infty$

Null recurrence: $P(\text{return time} < \infty) = 1$ and $E(\text{return time}) = \infty$.

①, ②, ③ are needed for establishing M.C. has a limiting distr.

①, ③ are needed for establishing M.C. " " stationary " .

Ergodic state: A state that is pos. recurrent and aperiodic.

Ergodic M.C.: all states are ergodic.

Consider the set $\{\pi_i : i \in \Omega\}$ s.t. $\pi_i \geq 0$ and $\sum_{i \in \Omega} \pi_i = 1$.

For convenience, let $\underset{\sim}{\pi}$ be a vector of k dimensions, where k = # states (cardinality of Ω), so $\underset{\sim}{\pi}$ describes a pmf.

Limiting distr.: An M.C. w/ t.p.m. $P_{k \times k}$ and state space Ω has a limiting distr. $\underset{\sim}{\pi}$ if

$$\lim_{n \to \infty} P^n = \begin{bmatrix} \underset{\sim}{\pi} \\ \vdots \\ \underset{\sim}{\pi} \end{bmatrix}_{k \times k}$$ (where $\underset{\sim}{\pi}$ is $1 \times k$ dimensional).

Stationary distr.: $\underset{\sim}{\pi}$ is a stationary (or invariant) distribution of an M.C. w/ t.p.m. P if $\underset{\sim}{\pi}_{1 \times k} = \underset{\sim}{\pi}_{1 \times k} P_{k \times k}$

Note: If $\underset{\sim}{\pi}$ is a stationary distr. of M.C., $\underset{\sim}{\pi}$ is also a stationary distr. of sub-chain w/ t.p.m. P^k, $k = 2, 3, \ldots$

Easy to see: $\pi = \pi P = \pi P P = \pi P^2 \ldots = \pi P^k$.

Ergodic Thm: Consider an M.C. $\underset{\sim}{X} = (X_1, X_2, X_3, \dots)$ w/ t.p.m. P and state space Ω. If $\underset{\sim}{X}$ is irreducible, positive recurrent and aperiodic, it has a unique stationary distr. $\underset{\sim}{\pi}$. Furthermore, the limiting distr. $\left(\lim_{n\to\infty} P\right)$ exists and is equal to $\underset{\sim}{\pi}$. Also, for any function $g: \Omega \to \mathbb{R}$ s.t. $E_\pi |g| < \infty$,

$$\frac{1}{N} \sum_{i=1}^{N} g(X_i) \longrightarrow E_\pi(g) \quad \text{almost surely}$$

S.L.L.N. for Markov chains.

Interpretation: (1) An irreducible, ergodic chain converges to its stationary distribution.

(2) Sample averages (of the states) converge to their theoretical expectations under the stationary distr.

[This is the basis for Markov chain Monte Carlo.]

Note: Conditions for stationary distr. to exist is weaker: only need irreducibility and positive recurrence (M.C. may be periodic).

Discrete-time, discrete-space M.C.

Thm: An irreducible M.C. has a stationary distr. iff it is positive recurrent. The stationary distr. is unique. (Guttorp, pg. 37)

Cor: For an irreducible M.C., the following are equivalent: (cf. R. Durrett E. of S.P.)

(i) some state i is positive recurrent

(ii) all states are positive recurrent

(iii) there is a stationary distr.

For existence of limiting distr., need aperiodicity also in addition to irreducibility and pos. recurrence.

Ergodic equations

If $\pi_j = \lim_{n\to\infty} P_{ij}^{n} = \lim_{n\to\infty} P(X_n = j \mid X_0 = i)$ then

$\{\pi_j\}$ is the unique soln. to

$$\left.\begin{aligned} \pi_j &= \sum_i \pi_i P_{ij} \quad \forall j \\ \text{and} \quad \sum_j \pi_j &= 1 \end{aligned}\right\} \text{Ergodic eqns.}$$

Note: ① Last equation is redundant

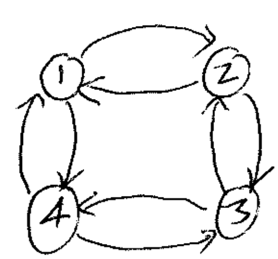
② $\pi_j = \lim_{N\to\infty} \sum_{k=1}^{N} I(X_k = j)/N$

Periodic chains: If M.C. is irreducible, positive recurrent but periodic, $\{\pi_j\}$ is still set of unique non-neg. soln. to ergodic eqns. and,

$$\pi_j = \lim_{N\to\infty} \sum_{k=1}^{N} I(X_k = j)/N \quad \leftarrow \text{[illegible]}$$

but $\lim_{n\to\infty} P(X_n = j \mid X_0 = i)$ no longer exists.

E.g. $P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$



Periodic w/ $d(i) = 2 \ \forall i$. Soln. to Ergodic eqn.: $\pi_i = 1/4 \ \forall i$.

$$P(X_n = 2 \mid X_0 = 1) = \begin{cases} 0 & \text{if } n \text{ even} \\ 1/2 & \text{if } n \text{ odd} \end{cases}$$

$\Rightarrow$ no limit exists.

Reducible chains: If M.C. is reducible stationary distr. may not be unique. E.g. $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ start at 0, $\pi = (1, 0)$

" " 1, $\pi = (0, 1)$.

Convention here: stat. distr. exists $\Leftrightarrow$ it is unique

E.g. Mobility table

$$\begin{array}{lc}
 & \text{Child's occupation} \\
 & \begin{array}{ccc} \text{upper} & \text{middle} & \text{lower} \end{array} \\
\text{Parent's occupation} \begin{array}{r} \text{upper} \\ \text{middle} \\ \text{lower class} \end{array} & \begin{bmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{bmatrix}
\end{array}$$

T.p.m. drives the steady state $\underset{\sim}{\pi} = (\pi_0, \pi_1, \pi_2)$

π_j = fraction of popn. in state j.

Ergodic eqns. : $\pi P = \pi \Rightarrow (\pi_0, \pi_1, \pi_2) P = (\pi_0, \pi_1, \pi_2)$

① $\pi_0 = \pi_0 \times 0.45 + \pi_1 \times 0.05 + \pi_2 \times 0.01$

② $\pi_1 = \pi_0 \times 0.48 + \pi_1 \times 0.7 + \pi_2 \times 0.5$

④ $1 = \pi_0 + \pi_1 + \pi_2$

(3rd eqn. redundant)

Soln.: $\underset{\sim}{\pi} = (0.07, 0.62, 0.31)$ $\underset{\sim}{\pi} = (0.0624, 0.6234, 0.3142)$

Ex. 4.19

```
## stationary distribution example 4.19 Mobility table
tpm <- rbind(c(0.45, 0.48, 0.07),c(0.05, 0.70, 0.25), c(0.01, 0.50, 0.49))
tpm2 <- tpm%*%tpm
       [,1]   [,2]   [,3]
[1,] 0.2272 0.5870 0.1858
[2,] 0.0600 0.6390 0.3010
[3,] 0.0344 0.5998 0.3658
tpm5 <- tpm%*%tpm%*%tpm%*%tpm%*%tpm
           [,1]      [,2]      [,3]
[1,] 0.07581789 0.6221158 0.3020663
[2,] 0.06238685 0.6235696 0.3140435
[3,] 0.05972524 0.6234466 0.3168281
tpm20 <-tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm
           [,1]      [,2]      [,3]
[1,] 0.06238865 0.6234403 0.3141711
[2,] 0.06238859 0.6234403 0.3141711
[3,] 0.06238858 0.6234403 0.3141711

tpm40 <- tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm%*%tpm
           [,1]      [,2]      [,3]
[1,] 0.06238859 0.6234403 0.3141711
[2,] 0.06238859 0.6234403 0.3141711
[3,] 0.06238859 0.6234403 0.3141711
```

P^2 $\boxed{P^5}$ P^{20} $\boxed{P^{40}}$

4.7 (b)

```
## calculating stationary distr.
pi0 <- 0.7/(20*0.56+0.02)
pi1 <- (20*0.56*pi0 - 0.2)/0.8
pi1
[1] 0.6234403
pi2 <- 1-pi0-pi1
pi2
[1] 0.3141711

pimat <- matrix(c(pi0,pi1,pi2),3,1)
pimat
           [,1]
[1,] 0.06238859
[2,] 0.62344029
[3,] 0.31417112

### similar to quantity from transition matrix above
pi0 <- 0.06238859
pi1 <- 0.6234403
pi2 <- 0.3141711

## example 1
tpm <- rbind(c(2/5, 1/2, 1/10),c(1/5, 7/10, 1/10), c(2/5, 2/5, 1/5))
tpmsq <- tpm%*%tpm
> tpmsq
     [,1] [,2] [,3]
[1,] 0.30 0.59 0.11
[2,] 0.26 0.63 0.11
[3,] 0.32 0.56 0.12
tpm4 <- tpmsq%*%tpmsq
       [,1]   [,2]   [,3]
[1,] 0.2786 0.6103 0.1111
[2,] 0.2770 0.6119 0.1111
[3,] 0.2800 0.6088 0.1112

tpm10 <- tpmsq%*%tpmsq%*%tpmsq%*%tpmsq%*%tpmsq%*%tpmsq%*%tpmsq%*%tpmsq%*%tpmsq%*%tpmsq
          [,1]      [,2]      [,3]
[1,] 0.2777778 0.6111111 0.1111111
[2,] 0.2777778 0.6111111 0.1111111
[3,] 0.2777778 0.6111111 0.1111111
> 18*tpm10
     [,1] [,2] [,3]
[1,]    5   11    2
[2,]    5   11    2
[3,]    5   11    2
```

Ex. #2

P^2

P^4

P^{10}

$\times \frac{1}{18}$ is soln to E.E.

$X_n | X_0 = P^{10}$

But, all rows are similar

$\Rightarrow X_n \sim \begin{cases} 1 & 5/18 \\ 2 & 11/18 \\ 3 & 2/18 \end{cases}$

ie, $P(X_n = 1) \approx 5/18$ and so on..

$$P = \begin{bmatrix} 2/5 & 1/2 & 1/10 \\ 1/5 & 7/10 & 1/10 \\ 2/5 & 2/5 & 1/5 \end{bmatrix}$$

Connecting existence of limiting distribution of an M.C. to its stationary distr.

(Heuristic argument)

$$P(X_{n+1}=j) = \sum_{i=0}^{\infty} P(X_{n+1}=j \mid X_n=i)\, P(X_n=i)$$

$$= \sum_i P_{ij}\, P(X_n=i)$$

Existence of limiting distr. $\Rightarrow$ $\lim_{n\to\infty} P(X_n=i) = \pi_i$

$\lim_{n\to\infty} P(X_n=j) = \pi_j$

Assume we can switch limits and summation (need Fatou's Lemma)

$$\lim_{n\to\infty} P(X_{n+1}=j) = \lim_{n\to\infty} \sum_i P_{ij}\, P(X_n=i)$$

$$= \sum_i P_{ij}\, \pi_i$$

$$\Rightarrow \quad \pi_j = \sum_i P_{ij}\, \pi_i$$

Time Reversible M.C.s.

Consider an M.C. $\{X_n\}$ that is

has initial distr. π stationary and ergodic, w/ t.p.m. $\{P_{ij}\}$

M.C. is in steady state.

Imagine process began at time $t = -\infty$

or equivalently that $X_0 \sim \pi$ stationary distr.

Now look at reverse chain: starting at time n trace sequence of states going backwards in time: $X_n, X_{n-1}, X_{n-2}, \ldots$

This sequence is also Markov w/ t.pm P^*_{ij}

$$P^*_{ij} = P(X_m = j \mid X_{m+1} = i)$$

$$= \frac{P(X_{m+1} = i \mid X_m = j)\, P(X_m = j)}{P(X_{m+1} = i)}$$

$$= \frac{\pi_j P_{ji}}{\pi_i}$$

due to stationarity

A <u>stationary ergodic</u> M.C. $\{X_n\}$ is time reversible if $P^*_{ij} = P_{ij} \ \forall i,j$

where $\{P_{ij}\}$ and $\{P^*_{ij}\}$ are t.p.m. for the forward and reverse chains respectively.

But $P^*_{ij} = P_{ij}$

$$\Leftrightarrow \boxed{\pi_j P_{ji} = \pi_i P_{ij}} \ \forall i,j$$

This is called the <u>detailed balance</u> condition.

Necessary and sufficient conditions for reversibility.

Interpretation: transition rate from $i \to j$

= transition rate from $j \to i$

M.C. 'looks' same going forwards or backwards in time. Note: assumption here is that chain is stationary & ergodic.

Simple e.g. of non-reversible chain

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

chain is irreducible but:

sequence $1 \to 3 \to 2 \to 1$ is possible

while " $1 \to 2 \to 3 \to 1$ is not possible

So for this sequence of states can tell which direction simulation occurred $\Rightarrow$ non-reversible (trivially).

Formal argument 'violates Kolmogorov's condition'. <1

Why is time reversibility useful?

<u>Thm:</u> Consider an irreducible ergodic chain $\{Y_n\}$ w/ t.p.m. P.

If we can find $x_i \geq 0$ s.t. $\sum_i x_i = 1$

and $x_i P_{ij} = x_j P_{ji} \quad \forall i,j$

then $\{Y_n\}$ is time reversible w/ stationary distr. $\underset{\sim}{\pi}$ s.t. $\pi_i = x_i \quad \forall i$. Also, $\underset{\sim}{\pi}$ is the limiting distr. of the M.C.

<u>Pf:</u> Since $x_i P_{ij} = x_j P_{ji} \quad \forall i,j$

summing over i gives:

$$\sum_i x_i P_{ij} = x_j \sum_i P_{ji} = x_j \quad \forall i,j \quad \text{ergodic eqns.!}$$

Also, since $\sum_i x_i = 1$ and stationary prob $\underset{\sim}{\pi}$ are unique soln. to ergodic eqns. follows that $x_i = \pi_i \quad \forall i$.

By ergodic thm. it follows that π_i's are also the limiting probabilities of the chain.

Usefulness:

(1) Suppose an M.C. seems to be reversible
(some physical processes are known to be reversible)
Can use detailed balance and solve for stationary distr. $\underset{\sim}{\pi}$. May be much simpler than solving ergodic eqns.

(2) Suppose we want to construct an M.C. w/ a particular stationary distr. $\underset{\sim}{\pi}$.
~~[illegible]~~
Hard problem in general.
May be much easier to construct reversible M.C. w/ stationary/limiting distr. $\underset{\sim}{\pi}$.

(3) Useful for simplifying theory e.g. studying eigen-structure of t.p.m. easier, connected to convergence rates.

Toy e.g. consider state space = $\{1, 2, 3, 4\}$

$$\underset{\sim}{\pi} = (1/4 \ \ 1/4 \ \ 1/4 \ \ 1/4)$$

Want P s.t. $\pi P = \pi$ (and M.C. ergodic, irred.)

Instead of finding P from above, limit M.C. to be reversible:

$$\pi_i P_{ij} = \pi_j P_{ji}$$

~~Hence~~, $\because \pi_i = \pi_j \quad \forall i,j$

$$P_{ij} = P_{ji} \quad \forall i,j$$

$\Rightarrow$ symmetric t.p.m.

If chain is irreducible and aperiodic we have satisfied conditions for ergodic thm. for time reversible chains.

(chain is automatically positive recurrent) $\because$ finite state space)

E.g. 1
$$\begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1/4 & 3/4 \end{bmatrix}$$

or E.g. 2. $P_{ij} = 1/4 \quad \forall i,j$ (trivial)

end Lec. 12

In general: irreducible M.C. w/ symmetric t.p.m. on a finite state space is a reversible M.C. w/ unique stationary distr. $\pi_i = \frac{1}{N}$ where $N = \#$ states.

(When is matrix = irreducible M.C.'s t.pm ?
Finite & irreduc $\Rightarrow$ multiplicity of eigenvalue $\lambda = 1$ is 1
How about $\Leftarrow$? See Karlin & Taylor)

Notes: although ergodic thm. lists positive recurrence as an assumption, practical approach to using thm. is to (i) check irreducibility, (ii) aperiodicity, (iii) solve $\pi P = \pi$ to get π.

If this can be done, π is also limiting distr. and we know positive recurrence also holds. pg. 427 Olofsson

[Thm: If an irreducible, aperiodic chain has a stationary distr $\underline{\pi}$ (satisfying $\pi_i \geq 0$, $\sum \pi_i = 1$ and ergodic equations). Then $\underline{\pi}$ is the limiting and unique stationary distr. and positive recurrence also holds. Billingsley, Thm. 8.6]

Reducible M.C.'s can be of great interest (e.g. gambler's ruin!) but we are then typically interested in questions like absorption probabilities.

Avoiding trivial cases:

<u>Limiting distr.</u>: Even when $\lim_{N \to \infty} P(X_N = j \mid X_0 = i)$ exists, if it depends on initial value / initial distr., we will not consider it a limiting distribution.

For M.C.'s on countable state spaces, if irred., aperiodic, has limiting distr., then $\pi_i > 0 \ \forall i$.
($\because$ it is positive recurrent + use Prob. #38 result).

<u>Stationary distr.</u>: We are interested in π only if it is <u>unique</u> stationary distr. of M.C.

E.g. (Physics) Ehrenfest Model of Diffusion

Describing movement of molecules

Total # particles $= m$

$X_t = $ # particles in A at time t

$m - X_t = $ " " " B " " "

Select one of m particles at random and transfer it to other box (to get X_{t+1})

$\{X_t\}$ is an M.C. w/ transition matrix P.

$$P_{ij} = 0 \quad \text{if} \quad j \notin \{i-1, i+1\}$$

54 (a) (ii)

We are interested in stationary distr. of $\{X_t\}$. Is it reversible?

Only transitions possible are

$X_t = i$ to $X_{t+1} = i+1$ $\quad i \to i+1$

or $X_t = i+1$ to $X_{t+1} = i$; $\quad i+1 \to i$

Notice: For any finite # of transitions, i.e., for $X_1, \ldots, X_N$

~~# trans~~ difference in # transitions : $i \to i+1$
and # " : $i+1 \to i$

is either 0 or 1 (at most 1).

Hence, in the long run:
rate of transitions : $i \to i+1$
= " " " : $i+1 \to i$

$$\Rightarrow \quad \pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i} \qquad \forall i$$

Now $P_{i,i+1} = P(X_{t+1} = i+1 \mid X_t = i)$
= P (selecting from B when A has i particles and B has m-i particles)

$$= \frac{m-i}{m}$$

Similarly, $P_{i+1,i} = P(X_{t+1} = i \mid X_t = i+1)$

$= P$ (select from A when A has $i+1$ particles)

$= \dfrac{i+1}{m}$

$\because \quad \pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$

we have $\pi_i \left(\dfrac{m-i}{m}\right) = \pi_{i+1} \left(\dfrac{i+1}{m}\right)$

$$\pi_{i+1} = \left(\frac{m-i}{i+1}\right) \pi_i$$

So, $\pi_1 = m \pi_0$

$$\pi_2 = \left(\frac{m-1}{2}\right) \pi_1 = \left(\frac{m-1}{2}\right) m \pi_0$$

$$\pi_3 = \left(\frac{m-2}{3}\right)\left(\frac{m-1}{2}\right) m \pi_0$$

$$\vdots$$

$$\pi_k = \frac{m(m-1)\dots(m-k+1)}{k!} \pi_0$$

$$= \binom{m}{k} \pi_0$$

$\because \quad \displaystyle\sum_{k=0}^{m} \pi_k = 1$

$$\sum_{k=0}^{m} \binom{m}{i} \pi_0 = 2^m \pi_0 = 1$$

Hence, $\pi_0 = 2^{-m}$

So, $\pi_i = \binom{m}{i} 2^{-m}$

$=$ pmf of $Bin(m, p = 1/2)$

Intuition: particles move independently of each other, equally likely to be in A or B in the long run.

Note: $\underset{\sim}{\pi}$ is not the limiting distr. ($\because$ chain is periodic, $d(i) = 2$)

Careful defn: Only ergodic chains can be time reversible.

(M.C. may satisfy detailed balance but is only ~~ergodic~~ ~~#~~ time reversible if it is also ergodic.)

E.g. 1. Random walk on $\mathbb{Z}$: $P_{i,i+1} = p \qquad P_{i,i-1} = 1-p$

Rate of transitions from $i \to i+1$

= " " " " $i+1 \to i$

Why? Similar argument: difference between # of $i \to i+1$ transitions and $i+1 \to i$ transitions is at most 1 for a finite time, i.e. $\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$

So detailed balance holds. regardless of value of $p \in (0,1)$

But M.C. is null-recurrent $(p = \frac{1}{2})$ or transient $(p \neq \frac{1}{2})$ and hence not ergodic. see 4.38 (problem)

$\Rightarrow$ not time reversible

~~Circular random walk~~

$P_{i,i+1} = p \qquad i = 1, \dots, N-1$

$P_{i,i-1} = 1-p \qquad i = 2, \dots, N$

$P_{N,1} = p$

$N_{1,N} = 1-p$

Irreducible on finite state space $\Rightarrow$ ergodic and Detailed balance holds $\Rightarrow$ time reversible.

Thm: An (irreducible) ergodic M.C. for which $P_{ij}=0$ iff $P_{ji}=0$ is time reversible iff roundtrips $i \to i$ have same probability as reverse round trip, i.e.,

$$P_{i,i_1} \times P_{i_1,i_2} \times \dots \times P_{i_k,i} = P_{i,i_k} \times \dots \times P_{i_1,i} \quad \forall i, i_1, \dots, i_k$$

Pf: ($\Rightarrow$) Assume M.C. is T-R: result follows immediately

E.g. trip of length 3:
$$\frac{P_{ij}\,P_{jk}\,P_{ki}}{P_{ik}\,P_{kj}\,P_{ji}} = \frac{\frac{\cancel{\pi_j} P_{ji}}{\cancel{\pi_i}}\;\frac{\cancel{\pi_k} P_{kj}}{\cancel{\pi_j}}\;\frac{\cancel{\pi_i} P_{ik}}{\cancel{\pi_k}}}{P_{ik}\;P_{kj}\;P_{ji}}$$

$$= 1$$

(Hence true for length 3).

($\Leftarrow$) Assume round trips have same prob as reverse round trip.

i.e, $P_{i i_1} P_{i_1 i_2} \dots P_{i_k,j} P_{ji} = P_{i,j} P_{j i_k} \dots P_{i_1 i}$

Summing over all $i_1, \dots, i_k$

$\Rightarrow P_{ij}^{k+1} P_{ji} = P_{ij} P_{ji}^{k+1}$

limit as $k \to \infty$:

$\pi_j P_{ji} = P_{ij} \pi_i$

$\therefore$ M.C. is ergodic, it is also time-rev.

Prop. Suppose $\underset{\sim}{X}$ is an irreducible M.C. w/ t.p.m. P,
If $\exists\, \pi_i \geq 0$, $\sum \pi_i = 1$ and t.p.m. Q
satisfying $\pi_i P_{ij} = \pi_j Q_{ji}$ $\forall i,j \in \Omega$ (Ω = state space of $\underset{\sim}{X}$)
then, ① Q ~~is~~ t.p.m. for reverse chain
② $\underset{\sim}{\pi}$ stationary for both forward and reverse chain.

Note: M.C. is not necessarily time reversible!

Useful trick if reverse chain is easier to work with than forward chain. E.g. bulbs e.g. in book