Department of Statistics
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515:Stochastic Processes and Monte Carlo Methods

Take Home Exam

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Exercise 1

(a)

First of all, I calculated the posterior distribution of β_1:

$$\pi(\beta_1|Y,X) \propto \prod f(Y_i;\beta_0+\beta_1 X_i,\sigma_i,\lambda)\pi(\beta_1)$$
$$\propto \prod f(Y_i;5+\beta_1 X_i,1,0.4)e^{\frac{-1}{200}\beta_1^2}$$

In order to approximate the posterior distribution, I used the following "All-at-once" Metropolis-Hastings Algorithm:

1. I select (arbitrarily) the initial value for β_1.
2. In each iteration I generate a new value for β_1 using a normal proposal with mean equal to the current value and variance 0.5.
3.I calculate the acceptance probability (in log-scale) which is equal to $a(x,y)=min(1,\frac{h(y)}{h(x)})$ where h is the unnormalised posterior distribution, x is the current value and y is the proposed value.
4.Then, I generate a standard uniform random variable U and I accept the y if $(U<a(x,y))$.

As regards the starting value I selected the value 1. Then, I tried different values for the variance of the proposal distribution in order to have an efficient MCMC sampler and good approximation to the posterior mean (The methods/plots that I used for that will be described in part (e)).

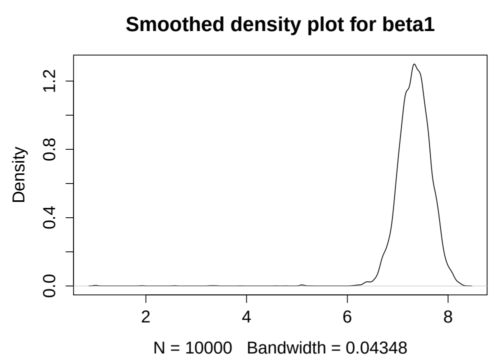
(b)

The estimate of the posterior expectation for β_1 is **7.342708** and the associated Monte Carlo standard error is **0.002159565** for run length equal to 100000.

(c)

The 95% credible interval for β_1 is (6.717798, 7.935114).

(d) The plot for the approximate posterior pdf for β_1 is the following:

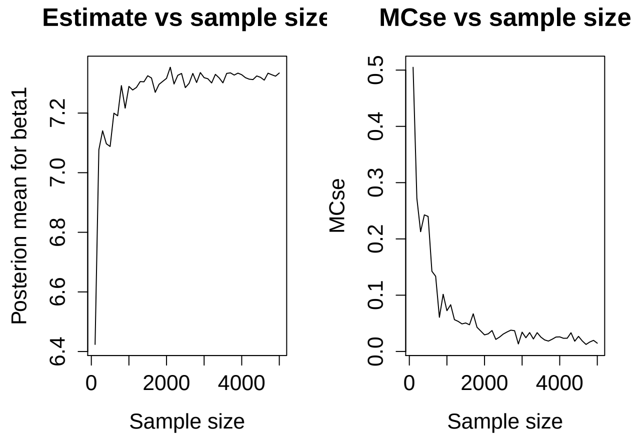


(e)

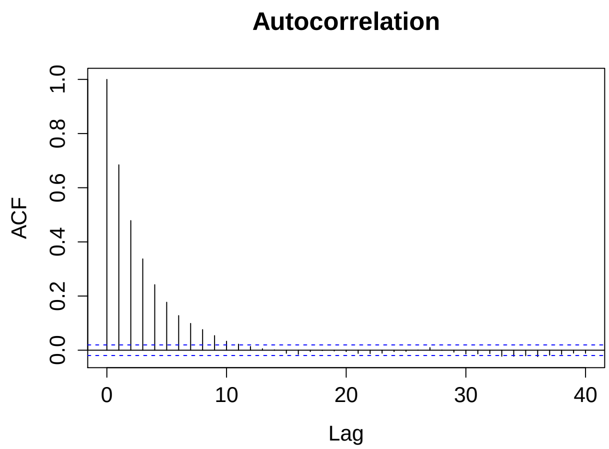
In order to determine if my approximation is accurate I used several plots.

1.First, I plotted the "Estimate vs Sample Size" and the "MCse vs Sample Size" in order to see if my MCMC estimator converges to some value and if the MCse converges (fast) to 0.

These seem to be true by the following plots:



2.Secondly,I plotted the autocorrelation of my sample and I calculated the effective sample size (for run length 10000). This seems to be approximately 3000 so I decided to run my algorithm for run length100000 (as seen in part (a)).



Exercise 2

(a) First of all, I calculated the posterior distribution of $(\beta_0, \beta_1, \lambda)$:

$$\pi(\beta_0, \beta_1, \lambda|Y, X) \propto \prod f(Y_i; \beta_0 + \beta_1 X_i, \sigma_i, \lambda)\pi(\beta_0, \beta_1, \lambda)$$
$$\propto \prod f(Y_i; \beta_0 + \beta_1 X_i, 1, \lambda)e^{\frac{-1}{200}(\beta_1^2+\beta_0^2)}\lambda^{-0.99}e^{\frac{-\lambda}{100}}$$

In order to approximate the posterior distribution, I used the following "Variables-at-a-time" Metropolis-Hastings Algorithm:
1. I select the initial value for $(\beta_0, \beta_1, \lambda)$.
2. In each iteration I calculate the full conditional distributions(in log-scale)
$\pi(\beta_0, \beta_1|\lambda, Y, X) \propto \prod f(Y_i; \beta_0 + \beta_1 X_i, 1, \lambda)e^{\frac{-1}{200}(\beta_1^2+\beta_0^2)}$ and
$\pi(\lambda|\beta_0, \beta_1, Y, X) \propto \prod f(Y_i; \beta_0 + \beta_1 X_i, 1, \lambda)\lambda^{-0.99}e^{\frac{-\lambda}{100}}$
where the "conditional part" contains the most recently updated values of the chain.
Also, I generate a new value for the (β_0, β_1) using a bivariate normal distribution with mean the current values,correlation 0 and marginal variances 0.05 and a new value for λ using a gamma distribution with parameters α=30*current value of λ and β=1/30.

3.I calculate the acceptance probability (in log-scale) which is equal to $a(x,y) = min(1, \frac{h(y)q(y,x)}{h(x)q(x,y)})$ where h is the unnormalised posterior distribution,x is the current value, y is the proposed value and q(x;) is the proposal distribution in each case.
4.Then, I generate a standard uniform random variable U and I accept the y if $(U < a(x,y))$ (in each case).

As regards the starting value I selected arbitrary ones(I used (1,1,1)). Then, I tried different values for the marginal variances and correlation of the proposal bivariate normal distribution as well as the parameters in the proposal gamma distribution in order to have an efficient MCMC sampler and good approximation to the posterior expectations.(The methods/plots that I used for that will be described in part (e)).

(b) The posterior expectations for β_0, β_1 and λ and associated Monte Carlo standard errors (for run length 100000) are provided in the following table:

Parameter	Estimate	MCse
β_0	2.319011	0.002055753
β_1	3.462065	0.003237788
λ	0.7776227	0.000660576

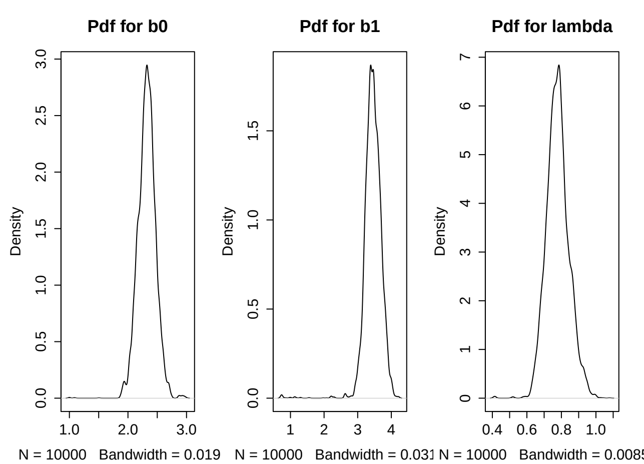
Table 1: *Estimates and MCse*

The 95% credible intervals for β_0, β_1 and λ are provided in the following table:

Parameter	2.5%	97.5%
β_0	2.047825	2.582667
β_1	3.051161	3.873282
λ	0.6696384	0.9026487

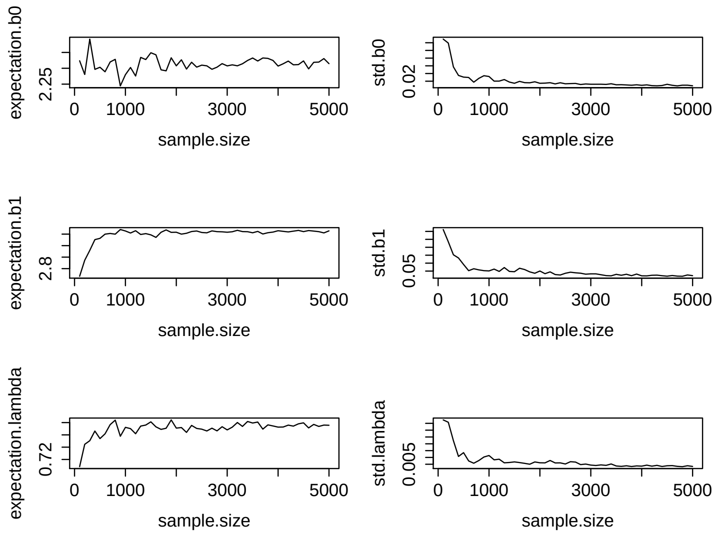
Table 2: *Credible Intervals*

(c) The approximated correlation of β_0 and β_1 (for n=100000) is: **-0.7542108**.
(d) The approximate density plots for the marginal distributions of β_0, β_1 and λ are the following:

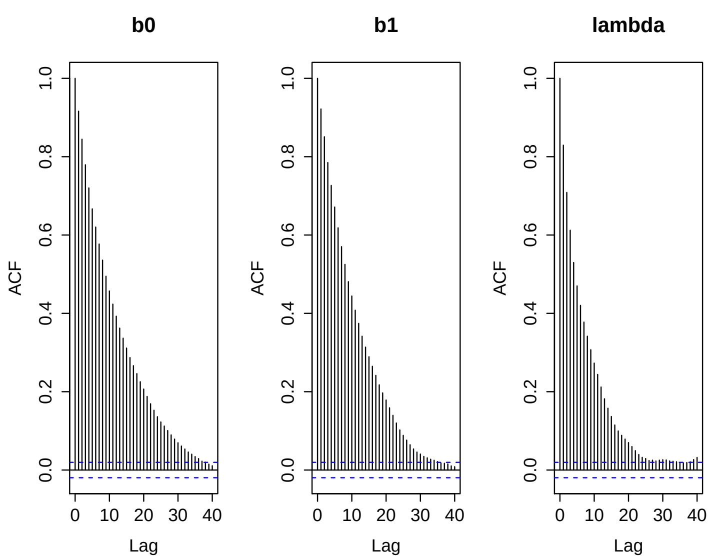


(e) In order to determine if my approximations are accurate I used several plots.

1.First, I plotted the "Estimate vs Sample Size" and the "MCse vs Sample Size" in order to see if my MCMC estimators converge to some value and if the MCses converge (fast) to 0.
These seem to be true by the following plots:



2.Secondly,I plotted the autocorrelation of my samples (separately for each parameter) and I calculated the effective sample size (for run length 10000). This seems to be approximately 1000 so I decided to run my algorithm for run length 100000 (as seen in part (a)).



Exercise 3

(a) For this data,the posterior expectations for β_0, β_1 and λ and associated Monte Carlo standard errors (for run length 100000) are provided in the following table:

Parameter	Estimate	MCse
β_0	0.1362627	0.002419721
β_1	2.479439	0.004267801
λ	0.1594534	0.0001002174

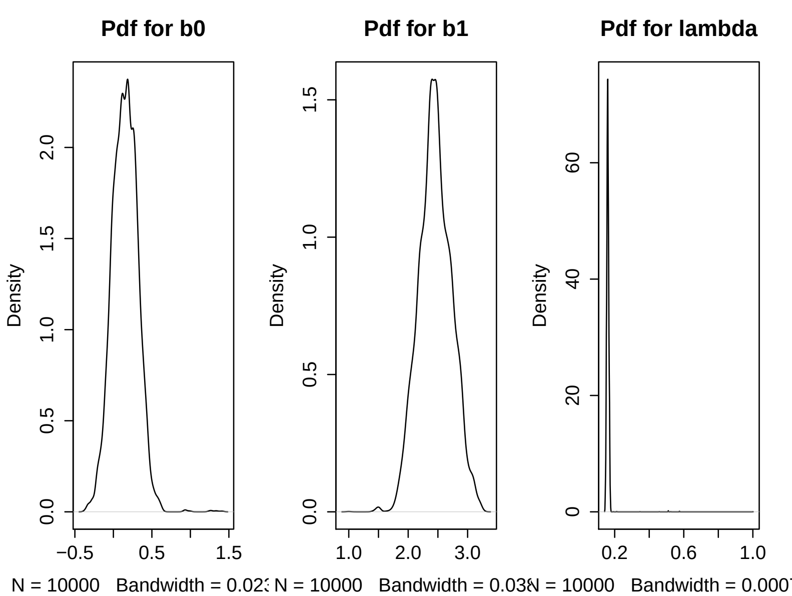
Table 3: *Estimates and MCse*

The 95% credible intervals for β_0, β_1 and λ are provided in the following table:

Parameter	2.5%	97.5%
β_0	-0.1880046	0.4556148
β_1	1.926740	3.018509
λ	0.1488295	0.1705996

Table 4: *Credible Intervals*

(b) The approximate density plots for the marginal distributions of β_0, β_1 and λ are the following:



(c) For this data, I decided that I had to increase a little bit the marginal variances in the proposal bivariate normal in order to have smaller autocorrelation. Hence, I changed the variances to 0.1.

Note: In all exercises, I tried also different starting values but I didn't notice any difference in my results. It seems that the algorithms are not sensitive to initial values (the estimators converge always to the same value).