Full conditional distributions for a Bayesian change point model

Our goal is to draw samples from the 5-dimensional **posterior** distribution $f(k, \theta, \lambda, b_1, b_2|\mathbf{Y})$ The posterior distribution is

$$
\begin{aligned}
f(k, \theta, \lambda, b_1, b_2|\mathbf{Y}) \propto & \prod_{i=1}^{k} \frac{\theta^{Y_i} e^{-\theta}}{Y_i!} \prod_{i=k+1}^{n} \frac{\lambda^{Y_i} e^{-\lambda}}{Y_i!} \\
& \times \frac{1}{\Gamma(0.5) b_1^{0.5}} \theta^{-0.5} e^{-\theta/b_1} \times \frac{1}{\Gamma(0.5) b_2^{0.5}} \lambda^{-0.5} e^{-\lambda/b_2} \\
& \times \frac{e^{-1/b_1}}{b_1} \frac{e^{-1/b_2}}{b_2} \times \frac{1}{n}
\end{aligned} \tag{1}
$$

From 1 we can obtain full conditional distributions for each parameter by ignoring all terms that are constant with respect to the parameter.
For θ:

$$
\begin{aligned}
f(\theta|k, \lambda, b_1, b_2, \mathbf{Y}) & \propto \prod_{i=1}^{k} \frac{\theta^{Y_i} e^{-\theta}}{Y_i!} \times \frac{1}{\Gamma(0.5) b_1^{0.5}} \theta^{-0.5} e^{-\theta/b_1} \\
& \propto \theta^{\sum_{i=1}^{k} Y_i - 0.5} e^{-\theta(k+1/b_1)} \\
& \propto \text{Gamma}\left(\sum_{i=1}^{k} Y_i + 0.5, \frac{b_1}{k b_1 + 1}\right)
\end{aligned} \tag{2}
$$

For λ:

$$
\begin{aligned}
f(\lambda|k, \theta, b_1, b_2, \mathbf{Y}) & \propto \prod_{i=k+1}^{n} \frac{\lambda^{Y_i} e^{-\lambda}}{Y_i!} \times \frac{1}{\Gamma(0.5) b_2^{0.5}} \lambda^{-0.5} e^{-\lambda/b_2} \\
& \propto \text{Gamma}\left(\sum_{i=k+1}^{n} Y_i + 0.5, \frac{b_2}{(n-k) b_2 + 1}\right)
\end{aligned} \tag{3}
$$

For k:

$$
f(k|\theta, \lambda, b_1, b_2, \mathbf{Y}) \propto \prod_{i=1}^{k} \frac{\theta^{Y_i} e^{-\theta}}{Y_i!} \prod_{i=k+1}^{n} \frac{\lambda^{Y_i} e^{-\lambda}}{Y_i!} \tag{4}
$$

For b_1:

$$
f(b_1|k, \theta, \lambda, b_2, \mathbf{Y}) \propto \frac{1}{b_1^{0.5}} e^{-\theta/b_1} \times \frac{e^{-1/b_1}}{b_1} \propto b_1^{-1.5} e^{-(1+\theta)/b_1} \propto IG(0.5, 1/(\theta+1)) \tag{5}
$$

For b_2:

$$f(b_2|k,\theta,\lambda,b_1|\mathbf{Y}) \propto \times \frac{1}{b_2^{0.5}} e^{-\lambda/b_2} \times \frac{e^{-1/b_2}}{b_2} \propto b_2^{-1.5} e^{-(1+\lambda)/b_2} \propto IG(0.5, 1/(\lambda+1)) \tag{6}$$

Note: The Inverse Gamma density is said to be a **conjugate** prior in this case since it results in a posterior that is also Inverse Gamma and therefore trivial to sample. As such, this density is mathematically convenient (due to its conjugacy property) but has poorly behaved moments; it may be better to adopt another prior density (such as a Gamma) instead.