Hint for Homework 5, Stat 515, Spring 2015

Due Wednesday, February 25, 2015 **beginning of class**

1. Consider a Poisson process $N(t)$ with rate λ. Prove the following result: Given that $N(t) = n$, the n arrival times $S_1, \dots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$, i.e.,

$$P(S_1 = t_1, \dots, S_n = t_n \mid N(t) = n) = \frac{n!}{t^n} I(0 < t_1 < \cdots < t_n).$$

Use a "first principles" argument, i.e., I would like you to do a proof that utilizes assumption # 3 and # 4 in the "first principles" definition of the Poisson process. Do not use an argument based on assuming that the counting process over an interval is Poisson distributed (the second definition discussed in class).

Hint: For any numbers s_i satisfying $0 \leq s_1 \leq s_2 \cdots \leq s_n \leq t$, show

$$P(S_i \leq s_i, i = 1, \dots, n \mid X(t) = n) = \frac{n!}{t^n} \int_0^{s_1} \cdots \int_{x_{n-2}}^{s_{n-1}} \int_{x_{n-1}}^{s_n} dx_n \dots dx_1,$$

which is the same as the distribution of the order statistics from a sample of n random variates taken from the Uniform$(0, t)$ distribution. First write each S_i in terms of inter-arrival times $(T_i, i = 0, \dots n)$. That is, $S_1 = T_0, S_2 = T_0 + T_1, \dots, S_n = \sum_{i=0}^{n-1} T_i$. In order to simplify things, while you may not use the Poisson distribution (definition 2), you are welcome to use the fact that inter-arrival times T_is are independent iid exponential random variables. You may also use known facts about exponential r.v.s from class, e.g. the distribution of the minimum or maximum of independent exponential r.v.s.