Homework 7, Stat 515, Spring 2015

Due Wednesday, April 8, 2015 **beginning of class**

1. Consider $\{W(t), t \geq 0\}$, a standard Brownian motion process.
   (a) Find the conditional distribution of $W(s) \mid W(t_1) = A, W(t_2) = B$, where $0 < t_1 < s < t_2$.
   (b) Find $E(W(t_1)W(t_2)W(t_3))$ where $0 < t_1 < t_2 < t_3$.

2. Let $\{X(t), t \geq 0\}$ be a Brownian motion with drift coefficient μ and variance parameter σ^2. Find the joint distribution of $X(s), X(t)$ where $0 < s < t$.

3. Distinguishing between Markov processes and martingales:
   (a) Provide one example of a martingale that is not a Markov process and show why this is the case.
   (b) Provide one example of a Markov process that is not a martingale and show why this is the case.

4. Prove the following result: If $X_i, i \geq 1$, are independent and identically distributed (iid) and if N is a bounded stopping time for $X_1, X_2, \ldots$ with $E(N) < \infty$, then

$$E\left(\sum_{i=1}^{N} X_i\right) = E(N)E(X)$$

   . Hint: consider the process $Z_n = \sum_{i=1}^{n}(X_i - \mu)$.

5. Let $\{X(t), t > 0\}$ be standard Brownian motion. Prove that the process $\{M(t), t > 0\}$ where $M(t) = \exp\left(\lambda X(t) - \frac{1}{2}\lambda^2 t\right)$, is a martingale.

6. Simulate 3 realizations for each of the following processes on the interval $[0, 10]$ on 20 equally spaced points on the interval.
   (a) Simulate 3 realizations of standard Brownian motion.
   (b) Simulate 3 realizations of Brownian motion with variance parameter $\mu = 0, \sigma^2 = 2$.
   (c) Simulate 3 realizations of Brownian motion with parameters $\sigma^2 = 2, \mu = 3$.

   Overlay the 3 realizations for each process on the same plot. Hence you should submit 3 clearly labeled plots. You do not have to submit your code for this problem but you have to provide pseudocode for each simulation algorithm above.

7. Consider a simple symmetric random walk, $S_n = \sum_{i=0}^{n} X_i$ where $X_1, X_2, \ldots$ are iid with $P(X_i = 1) = 1/2 = P(X_i = -1)$ and define a random time, $T \in [0, 3]$ at which S_n takes on its maximum value $\max\{S_n : 0 \leq n \leq 3\}$. If S_n takes its maximum value more than once, assume T is the last such time.
   (a) Show analytically that $E(X_T) > 0$ and hence $E(X_T) \neq E(X_0)$. This therefore results in an "unfair game", as discussed in class.
   (b) Find the expected value of T using Monte Carlo. Write pseudocode for the algorithm and report your estimate along with the Monte Carlo sample size and Monte Carlo standard error.