A Projection-based Approach for Modeling Non-Gaussian Spatial Data

Based on joint work with

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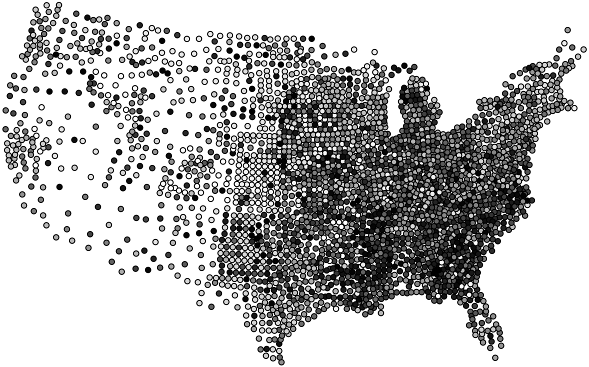
US Infant Mortality Data by County

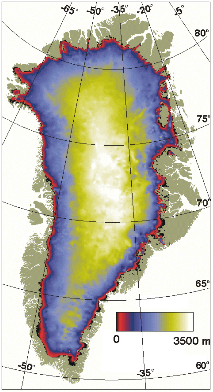


Ratio of deaths to births, each averaged over 2002-2004.

Darker indicates higher rate. $n = 3071$

Question (regression): which factors impact infant mortality?

Greenland Ice Sheet Thickness

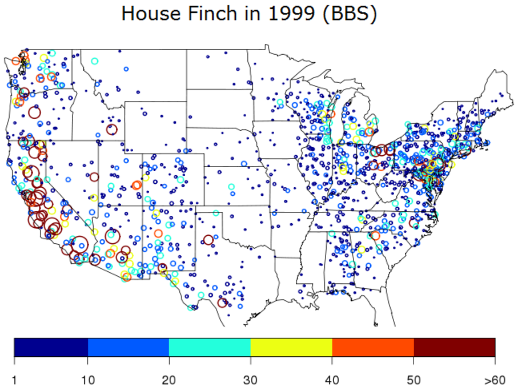


Bamber et al. (2001)

Question: How to interpolate this surface?

How to calibrate (infer parameters for) ice sheet model based on these data? (Chang, Haran, Applegate, Pollard, 2016a,b,c)

House Finch Abundances



Pardieck *et al.* 2015. *North American Breeding Bird Survey Dataset 1966 - 2014*

Question (interpolation): Abundance at unsampled location?

Talk Summary

- Spatial data are common in environmental science: disease modeling, ecology, climate...
- Spatial generalized linear mixed models (SGLMMs)
  - Popular for lattice or areal data
    Besag, York, Mollie (1991) $\approx$ 3,000 citations
  - and continuous-domain data
    Diggle et al. (1998) $\approx$ 2,000 citations
- Shortcomings of SGLMMs:
  1. Inference presents difficult computational issues, especially with large data sets
  2. Regression parameter interpretation is unreliable
- I will describe projection-based methods that simultaneously resolve both these issues

Spatial Generalized Linear Mixed Models

- Spatial linear mixed models (SLMMs): for Gaussian data
- Spatial generalized linear mixed models (SGLMMs): for non-Gaussian data
- What are these models used for?
  1. interpolation (continuous-domain) or smoothing the spatial field (lattice-domain)
  2. regression while adjusting for residual spatial dependence
  3. as a component in a complex hierarchical model

Spatial Linear Mixed Models (SLMMs)

- Spatial process at location $\mathbf{s} \in D \subset \mathbb{R}^d$ is

$$Z(\mathbf{s}) = X(\mathbf{s})\boldsymbol{\beta} + W(\mathbf{s})$$

  - $X(\mathbf{s})$ is covariate at $\mathbf{s}$, and $\boldsymbol{\beta}$ is a vector of coefficients
  - Model dependence among spatial random variables by imposing it on $W(\mathbf{s})$, the random effects
- Same framework works for both lattice data and continuous-domain data. Model for $W(\mathbf{s})$
  - Continuous domain: Gaussian process (GP)
  - Lattice data: Gaussian Markov Random field (GMRF)

Gaussian Processes

Infinite dimensional process $\{W(\mathbf{s}) : \mathbf{s} \in D\}$ such that

$$(W(\mathbf{s}_1), \dots W(\mathbf{s}_n))^T \mid \Theta \sim N(\mathbf{0}, \Sigma(\Theta))$$

- Covariance often specified via a positive definite covariance function with parameters Θ
- E.g. (stationary) exponential covariance function
- $\Theta = (\sigma^2, \phi)$

$$\Sigma_{ij}(\Theta) = \text{Cov}(W(\mathbf{s}_i), W(\mathbf{s}_j)) = \sigma^2 \exp(-|\mathbf{s}_i - \mathbf{s}_j|/\phi)$$

Gaussian Markov Random Fields

$$(W(\mathbf{s}_1), \ldots W(\mathbf{s}_n))^T \mid \Theta \sim N(\mathbf{0}, Q(\Theta)^{-1})$$

$Q(\Theta)$ is a precision matrix based on a graph that describes a neighborhood structure: adjacencies specify dependence (skip details....)

Inference for Spatial Linear Mixed Models

- MLE involves low-dimensional optimization

$$\arg\max_{\Theta,\beta} \mathcal{L}(\Theta, \boldsymbol{\beta}; \mathbf{Z})$$

- Bayesian inference:
  - Priors for $\Theta, \boldsymbol{\beta}$
  - Inference based on $\pi(\Theta, \boldsymbol{\beta} \mid \mathbf{Z}) \propto \mathcal{L}(\Theta, \boldsymbol{\beta}; \mathbf{Z}) p(\Theta) p(\boldsymbol{\beta})$
- Markov chain Monte Carlo with low-dimensional posterior

Literature on Computing for Spatial Linear Models

- Likelihood: high-dimensional matrices, $\mathcal{O}(n^3)$ operations
- Lots of excellent approaches that scale very well
  - Multiresolution methods, with parallelizations (Katzfuss, 2017; Katzfuss and Hammerling, 2014)
  - Nearest neighbor process (Datta et al., 2016)
  - Random projections (Banerjee, A., Tokdar, Dunson, 2013)
  - Stochastic PDEs (Lindgren et al., 2011)
  - Lattice kriging (Nychka et al., 2010)
  - Predictive process (Banerjee, Gelfand, Finley, Sang 2008)

Largely a “solved” problem

Spatial Generalized Linear Mixed Models (SGLMMs)

Model for Z at location $\mathbf{s}_i$

1. $Z(\mathbf{s}_i)|\boldsymbol{\beta}, \Theta, W(\mathbf{s}_i), i = 1, \ldots, n$, conditionally independent

   E.g. $Z(\mathbf{s}_i) \mid \boldsymbol{\beta}, W(\mathbf{s}_i) \sim$Poisson$(\mu(\mathbf{s}_i))$
2. Link function $g(\mu(\mathbf{s}_i)) = X(\mathbf{s}_i)\boldsymbol{\beta} + W(\mathbf{s}_i)$

   E.g. $\log(\mu_i) = X(\mathbf{s}_i)\boldsymbol{\beta} + W(\mathbf{s}_i)$
3. $\mathbf{W} = (W(\mathbf{s}_1), \ldots, W(\mathbf{s}_n))^T$ modeled as
   - Gaussian Markov random field model (Besag et al., 1991)
   - Gaussian processes (Diggle et al., 1998)
4. Priors for $\Theta, \boldsymbol{\beta}$

Commonly embedded within hierarchical models (cf. Banerjee, Carlin, Gelfand, 2014)

Problem 1. Computational Challenge

- MLE: low-dimensional optimization of *integrated* likelihood

$$\arg\max_{\Theta,\beta} \int \mathcal{L}(\Theta, \boldsymbol{\beta}, \mathbf{W}; \mathbf{Z}) d\mathbf{W}$$

  High-dimensional integration due to **W**

  MCMC-EM or MCMC-MLE: slow, challenging to implement

  (Zhang, 2002, 2003; Christensen, 2004)

- Bayesian inference based on

$$\pi(\Theta, \boldsymbol{\beta}, \mathbf{W} \mid \mathbf{Z})$$

Computing for SGLMMs

Bayes approach:

- Handle missing data easily
- Combine multiple data sets and uncertainties elegantly
- Rich inference about parameters, functions of parameters
- MCMC-based inference is easier than for MLE

But... MCMC algorithms are not easy/scalable

- MCMC is slow per iteration due to high-dimensional

$$\pi(\Theta, \boldsymbol{\beta}, \mathbf{W} \mid \mathbf{Z})$$

- Markov chain is slow mixing (need longer chain) due to strong cross-correlations among **W**
- Can become impractical for large N

MCMC for SGLMMs

- Markov chain is slow mixing (need longer Markov chain) due to strong cross-correlations among $\mathbf{W}$
- Block updating schemes may help. E.g. blocks:

$$\boxed{\pi(\mathbf{W} \mid \Theta, \boldsymbol{\beta}, \mathbf{Z})} \quad \boxed{\pi(\Theta \mid \boldsymbol{\beta}, \mathbf{W}, \mathbf{Z})} \quad \boxed{\pi(\boldsymbol{\beta} \mid \Theta, \mathbf{W}, \mathbf{Z})}$$

  - Challenging to obtain good proposals for $\mathbf{W}$, especially for high-dimensions
  - Computationally expensive per update

Attempts to address these issues: Rue and Held (2005), Christensen et al. (2006), Haran and Tierney (2012)
They do not scale well (problem for $N > 1000$)

Problem 2. Spatial Confounding

- Let $P = X(X^T X)^{-1} X^T$, and $P^{\perp} = I - P$

$$g\{E(\mathbf{Z} \mid \beta, \mathbf{W}, \Theta\} = X\beta + \mathbf{W} = X\beta + \boxed{P\mathbf{W}} + P^{\perp}\mathbf{W}$$

- $P\mathbf{W}$ is in span of X
- Basic regression issue: multicollinearity

Leads to variance inflation, unstable estimates of β

(Hodges and Reich 2010; Paciorek, 2010)

Hints of the symptom, without diagnosis, by others (e.g. Diggle, 1994)

Sketch of Our General Solution

- Culprit: **W** is cause of confounding as well as computational challenges
- **W**: just a device to induce dependence
- $\boxed{\text{Idea: project } \mathbf{W} \text{ on random effects } \delta \text{ such that}}$
  - Preserve spatial dependence implied by original **W**
  - δ is low-dimensional
  - δ is less dependent (“cross-correlated”)
  - Project orthogonal to space spanned by **X**
- Applies to both Gaussian process and GMRF models
  - GMRF models: projection based on Moran operator which uses neighborhood structure (Hughes and Haran, 2013)
  - GPs and GMRFs: general approach using eigendecomposition (Guan and Haran, 2017)

Outline of Projection-based Approach

1. Fast approximation to the principal components of Σ_ϕ
   - Approximate first m eigenvectors $U = (\mathbf{u}_1, \ldots, \mathbf{u}_m)$ and eigenvalues $D_m = \text{diag}(\lambda_1, ..., \lambda_m)$
2. Replace n-dimensional **W** with $UD_m^{1/2}\boldsymbol{\delta}$

   $\boldsymbol{\delta}$: lower dimensional and $\approx$ independent

   faster and better mixing MCMC algorithm
3. Project $UD_m^{1/2}\boldsymbol{\delta}$ to $C^\perp(X)$

   Makes random effects orthogonal to fixed effects

   handles confounding issues
4. Fit the reduced model under Bayesian framework

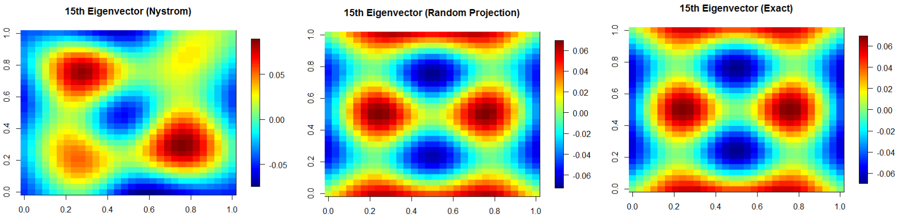
Step 1: Eigendecomposition

For speed we use a fast *approximate* eigendecomposition

Left: deterministic approximation

Center: **random approximation**

Right: exact eigendecomposition



- **Random projections** used in Banerjee, Tokdar, Dunson (2013); also Sarlos (2006), Halko et al. (2009)

Step 2: Reducing Dimensions via Projection

- Approximates the leading m eigencomponents of the covariance matrix Σ_ϕ
- **Replace W with** $UD_m^{1/2}\boldsymbol{\delta}$

Step 3: Projection to Handle Confounding

- Let $P = X(X^T X)^{-1} X^T$, and $P^{\perp} = I - P$
- Recall: $P\mathbf{W}$ is in span of X, causes confounding
- Solution: Remove it

$$g\{E(\mathbf{Z} \mid \boldsymbol{\beta}, \mathbf{W}, \sigma^2, \phi)\} = X\boldsymbol{\beta} + \mathbf{W} = X\boldsymbol{\beta} + \cancel{P\mathbf{W}} + P^{\perp}\mathbf{W}$$

[cf. Reich et al., 2006; Hughes and Haran, 2013]

Step 4: Inference Based on Reparameterization

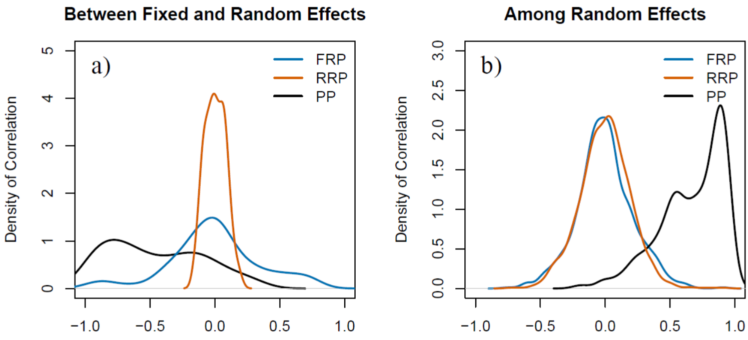
- Spatial generalized linear mixed models
  Usual: inference based on $\pi(\boldsymbol{\beta}, \sigma^2, \phi, \mathbf{W} \mid \mathbf{Z})$
- Obtain U, D_m of Σ_ϕ
- D_m is m-dim diagonal matrix with $D_{ii} = i^{th}$ eigenvalue
- FRP: replace **W** with $UD_m^{1/2}\boldsymbol{\delta}$ to approximate SGLMM or
  RRP: replace **W** with $P^\perp UD_m^{1/2}\boldsymbol{\delta}$ to approximate restricted spatial model
- Reduced Model:

$$g\left\{E(Z_i \mid \boldsymbol{\beta}, U, D_m, \boldsymbol{\delta})\right\} = X_i\boldsymbol{\beta} + (P^\perp UD_m^{1/2})_i\boldsymbol{\delta}$$

$$\boldsymbol{\delta} \mid \ldots \overset{approx}{\sim} N_m(\mathbf{0}, \sigma^2 I)$$

Now: inference based on $\pi(\boldsymbol{\beta}, \sigma^2, \phi, \boldsymbol{\delta} \mid \mathbf{Z})$

Reduced Correlations



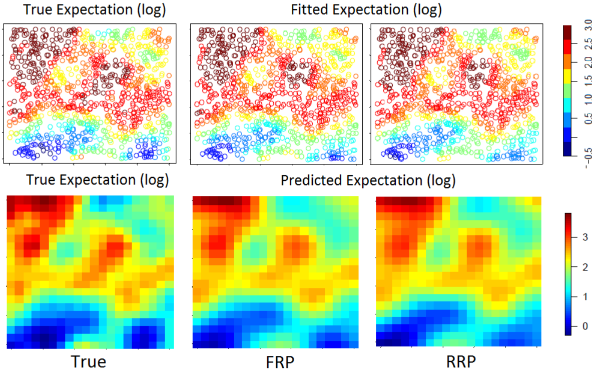
- Reparameterized random effects are approximately independent of each other *and* fixed effects

Computational Speed-up

- Drastic reduction in dimension of random effects, e.g. m = 50 for n = 1,000, or m = 60 for n = 3,000,...
- Reparameterized random effects are approximately independent of each other and fixed effects
- Easy to construct fast-mixing MCMC algorithm
- Eg. 10 to 50 to 300-fold reduction in compute time
- Scale beyond $n > 10,000$?
  - computational cost is of order nm^2
  - discretization of space/pre-computing
  - new decomposition algorithms/parallelization

Prediction Study: Poisson SGLMM

- Simulate n = 1000 spatial count data
- Prediction on 20 x 20 grid using rank = 50



FRP: full model

RRP: restricted model (orthogonalized random effects)

Summary of Projected SGLMM

- reduces dimensions + better MCMC mixing
- adjusts for spatial confounding
- simple to implement, mostly "automated"
- good inference and prediction performance

Summary of Projected SGLMM

- reduces dimensions + better MCMC mixing
- adjusts for spatial confounding
- simple to implement, mostly “automated”
- good inference and prediction performance
- In the context of other reduced-rank approaches
  - Our approach does *not* result in exchangeability between observed and predicted. Predictive process does.
  - But we use optimal (minimal truncation error) projection
  - And prediction is still straightforward
- Other approaches
  - may be better for the basic linear model
  - our approach works better for SGLMMs
  - our approach and predictive process approach: easy for more complex hierarchical settings

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Key References

- Guan and Haran (2017), A Computationally Efficient Projection-Based Approach for Spatial Generalized Linear Mixed Models, *arxiv.org*
- Hughes and Haran (2013), Dimension reduction and alleviation... *Journal of the Royal Statistical Society (B)*
  - R package (CRAN) `ngspatial`
- Banerjee A, Tokdar, S., Dunson, D. (2013) Efficient Gaussian process regression for large datasets, *Biometrika*
- Reich et al. (2006), Effects of residual smoothing on the posterior of the fixed effects... *Biometrics*
- Haran (2011) Gaussian random field models for spatial data, *Handbook of MCMC*

Frequently Asked Questions (FAQs)

- *Q. Why not use nearest neighbor Gaussian processes? (Datta et al., 2016)*
  - Effective way to reduce matrix calculations via composite likelihood. But does not reduce number of random effects
  - Works well for spatial linear mixed models
  - Random effects are of dimension N so not clear how to extend to SGLMMs
- *Q. How does your approach compare to the Gaussian predictive process (Banerjee et al., 2008)?*
  - Applicable to SGLMMs, involves dimension-reduction
  - They provide a process, obvious way to predict (**we do not**)
  - Choice of knots can be non-trivial. (Our low-dimensional representation is easy and also "optimal")
  - In simulated examples, we do better with prediction
  - Does not address spatial confounding

FAQs

- *Q. Is this necessary when we have the Integrated Nested Laplace Approximation (INLA) (Rue et al., 2008)?*
  - INLA is very fast
  - Does not handle spatial confounding
  - No obvious way to handle complications – additional hierarchy, complicated mean structure (e.g. physical model); accuracy of approximation may also be suspect
- *Q. Relationship to fixed rank approaches?*
  - If we fixed covariance parameters, this is a fixed rank approach with fixed eigenvectors/eigenfunctions as basis
  - Eliminating small scale variations can impact SLMMs (Stein, 2014), but less impact in SGLMMs

APPENDIX

Challenges

Challenges posed by spatial generalized linear mixed models (SGLMMs):

(1) Computational challenges
    Rue and Held (2002, 2005), Haran (2011)

(2) Confounding between spatial random effects and fixed effects (covariates)
    Reich, Hodges, Zadnik (2006), Paciorek (2010)

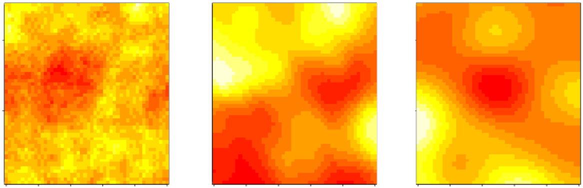
Gaussian Process for Dependence and Interpolation

- A Gaussian process is an infinite-dimensional random process, any finite-dimension of which is a multivariate normal.
- Matérn covariance function describes dependence, e.g.

$\nu = 0.5, \quad C(h) = \sigma^2 \exp(-\frac{|h|}{\phi})$ (Exponential)

$\nu = 2.5, \quad C(h) = \sigma^2 \left(1 + \frac{\sqrt{5}|h|}{\phi} + \frac{5|h|^2}{3\phi^2}\right) \exp(-\frac{\sqrt{5}|h|}{\phi})$

$\nu = \infty, \quad C(h) = \sigma^2 \exp(-\frac{|h|^2}{2\phi^2})$ (Square exponential)



Summary of Sparse Reparameterization for GMRFs

- Regular approach implies unintended/undesirable dependence structure (cf. Wall, 2004)
- Our approach
  - Deletes non-meaningful spatial dependence (weak or negative): “data-based” approach to reduce dimensions
  - Faster inference *and* a better model
- Regression coefficients are easier to interpret
- Automated MCMC is computationally efficient, allowing for routine analysis of large data sets
- Approach takes advantage of the underlying graph

What should we do in continuous-domain settings (in the absence of a graph)?

Our Sparse Reparameterization

- Represent graph $G = (V, E)$ using $\mathbf{A}$, $n \times n$ adjacency matrix with entries $\mathrm{diag}(\mathbf{A}) = \mathbf{0}$ and $\mathbf{A}_{ij} = 1\{(i, j) \in E, i \neq j\}$, with $1\{\cdot\}$ an indicator function
- Basic idea inspired by Griffith (2003): augment a generalized linear model with selected eigenvectors of $(\mathbf{I} - \mathbf{1}\mathbf{1}'/n)\mathbf{A}(\mathbf{I} - \mathbf{1}\mathbf{1}'/n)$. This appears in Moran's I statistic (nonparametric measure of spatial dependence),

$$I(\mathbf{A}) \propto \frac{\mathbf{Z}'(\mathbf{I} - \mathbf{1}\mathbf{1}'/n)\mathbf{A}(\mathbf{I} - \mathbf{1}\mathbf{1}'/n)\mathbf{Z}}{\mathbf{Z}'(\mathbf{I} - \mathbf{1}\mathbf{1}'/n)\mathbf{Z}},$$

Background for Sparse Reparameterization

- Griffith's goal: reveal the structure of missing spatial covariates. Our goal: smoothing orthogonal to $\mathbf{X}$
- Hence, we replace $\mathbf{I} - \mathbf{1}\mathbf{1}'/n$ with $\mathbf{P}^{\perp}$
- $\mathbf{M_X}(\mathbf{A}) = \mathbf{P}^{\perp}\mathbf{A}\mathbf{P}^{\perp}$, Moran operator for $\mathbf{X}$ with respect to the graph G, appears in numerator of generalized Moran's I:

$$I_{\mathbf{X}}(\mathbf{A}) \propto \frac{\mathbf{Z}'\mathbf{P}^{\perp}\mathbf{A}\mathbf{P}^{\perp}\mathbf{Z}}{\mathbf{Z}'\mathbf{P}^{\perp}\mathbf{Z}}.$$

Applying the Sparse Reparameterization

- Replacing **L** with **M** in the RHZ model gives

$$g(\mathbb{E}(Z_i \mid \boldsymbol{\beta}, \boldsymbol{\delta})) = \mathbf{X}_i \boldsymbol{\beta} + \mathbf{M}_i \boldsymbol{\delta}.$$

  And the prior for the random effects is now

$$p(\boldsymbol{\delta} \mid \tau) \propto \tau^{q/2} \exp\left(-\frac{\tau}{2}\boldsymbol{\delta}' \mathbf{Q}^{**} \boldsymbol{\delta}\right),$$

  where $\mathbf{Q}^{**} = \mathbf{M}'\mathbf{Q}\mathbf{M}$.

- Corrects issues due to confounding
- Dimension reduction: if $\mathbf{M}_i$ reduced to q dimensions

  # parameters $q + p + 1 << n + p + 1$ if q is small

Study: Inference for Spatial Binary

30×30 lattice simulated from RHZ model with $\beta_1 = \beta_2 = 1$.

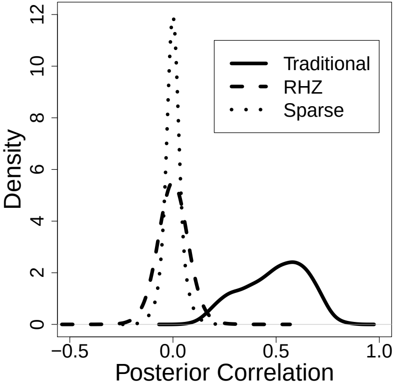
Predictors are the coordinates of unit square.

Model	$\hat{\beta}_1$ CI(β_1)	$\hat{\beta}_2$ CI(β_2)
Sparse	1.080 (0.613, 1.556)	1.130 (0.644, 1.635)
RHZ	1.120 (0.637, 1.606)	1.192 (0.679, 1.713)
Traditional	0.500 (-2.655, 3.616)	-0.605 (-3.698, 2.577)

- Point and interval estimates for Traditional are very poor: 95% interval includes 0
- Sparse and RHZ produce similar (good) results

Similar results for Gaussian (linear) and Poisson

De-correlated Random Effects



Greatly improves efficiency of simple MCMC. No need for elaborate proposals (cf. Held and Rue (2005), Haran et al. (2003), Haran and Tierney (2010)).

Spatial Binary: Computational Efficiency

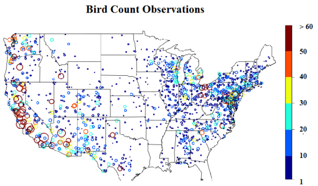
Model	Dimension	Running Time
Sparse	228	2.5 hours
RHZ	901	18.5 hours
Traditional	903	38.5 hours

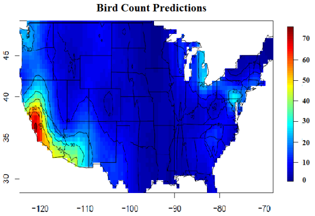
- MCMC algorithm is
  - faster per iteration (far fewer random effects)
  - mixes faster (random effects are "decorrelated")
- Far greater speed-ups with much smaller q, e.g. 25-50 is adequate for our examples (we are also being *extremely* careful by running very long chains!)

  Real data example: 14 days (traditional) versus 2-8 hours

Interpolated Bird Counts

- Approximate the SGLMM with only the intercept term.
- Computation time is about 7 hours,
- Small bird counts in the center and most of the East Coast
- Large counts centered near New York area and the West





Pardieck *et al.* 2015. *North American Breeding Bird Survey Dataset 1966 - 2014*

Outline of Projection-based Approach

1. Fast (approximate) eigendecomposition of Σ_ϕ:
   1.1 Low-distortion embedding of Σ_ϕ,
   1.2 Approximate first m eigenvectors $U = (\mathbf{u}_1, \ldots, \mathbf{u}_m)$ and eigenvalues $D_m = \text{diag}(\lambda_1, ..., \lambda_m)$ via Nyström method. [Banerjee et al., 2012] used a similar algorithm to approximate Σ_ϕ in Gaussian process regression
2. Replace n-dimensional **W** with $UD_m^{1/2}\boldsymbol{\delta}$

   $\boldsymbol{\delta}$: lower dimensional, components $\approx$independent
3. Project $UD_m^{1/2}\boldsymbol{\delta}$ to $C^\perp(X)$
   - Makes random effects orthogonal to fixed effects
4. Fit the reduced model under Bayesian framework.

Gaussian Markov Random Fields

$$W(\mathbf{s}_i) \mid W(\mathbf{s}_{-i}) \sim N\left(\frac{\sum_{j:j\sim i} W(\mathbf{s}_j)}{n_i}, \frac{1}{n_i\tau}\right)$$

where n_i is number of neighbors of ith region and $j \sim i$ means i, j are neighboring regions

- This specifies $Q(\tau)$, a precision matrix

$$(W(\mathbf{s}_1), \ldots W(\mathbf{s}_n))^T \sim N(0, Q^{-1}(\tau))$$

$Q = \text{diag}(A\mathbf{1}) - A$, where adjacency matrix A is such that $A_{ij} = 1$ if locations i and j are neighbors, 0 else

Spatial Confounding: Reparameterization Solution

- Since $\mathbf{K}$ is collinear, delete it from model
- $g(\mathbb{E}(Z_i \mid \boldsymbol{\beta}, \boldsymbol{\delta})) = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{L}_i\boldsymbol{\delta}$. Random effects distribution $\boldsymbol{\delta}$

$$p(\boldsymbol{\delta} \mid \tau) \propto \tau^{(n-p)/2} \exp\left(-\frac{\tau}{2}\boldsymbol{\delta}'\mathbf{Q}^*\boldsymbol{\delta}\right),$$

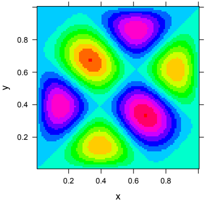
  where $\mathbf{Q}^* = \mathbf{L}'\mathbf{Q}\mathbf{L}$.
- Corrects issues due to confounding
- # of parameters reduced (only slightly) from $n + p + 1$ to $n + 1$. Computational challenge remains.

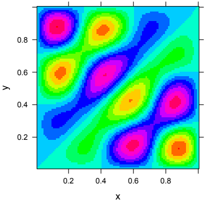
Reich, Hodges, Zadnik (2006)

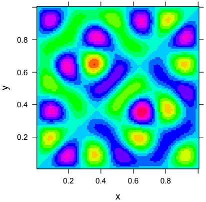
Sketch for Gaussian Markov Random Fields

- "Tailored" to **X** and *G*: eigenvectors comprise all possible patterns of clustering residual to **X** and accounting for *G*

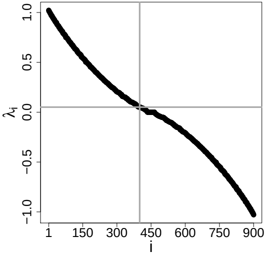
Some selected basis vectors for the 30×30 lattice.







Interpretation: Standardized eigenvalues



- Positive (negative) eigenvalues correspond to degrees of positive (negative) dependence (Boots and Tiefelsdorf, 2000)
- Idea: Remove eigenvectors corresponding to negative (unwanted dependence) or small eigenvalues (noise)

SGLMMs with Latent Gaussian Processes

Recall: example model for count data $Z(\mathbf{s}), s \in \mathcal{D} \subset \mathcal{R}^d$.

1. Data model:

$$Z(\mathbf{s}_i) \mid \boldsymbol{\beta}, W(\mathbf{s}_i) \overset{Indep.}{\sim} \text{Poisson}(\mu(\mathbf{s}_i)), i = 1, \ldots, n$$

$$\log(\mu(\mathbf{s}_i)) = X(\mathbf{s}_i)\boldsymbol{\beta} + W(\mathbf{s}_i),$$

2. Process model: impose dependence via Gaussian process

$$\mathbf{W} \mid \sigma^2, \phi \sim N\left(\mathbf{0}, \sigma^2 \Sigma_\phi\right)$$

3. Priors for $\boldsymbol{\beta}, \sigma^2, \phi$

MCMC Inference based on posterior, $\pi(\boldsymbol{\beta}, \sigma^2, \phi, \mathbf{W} \mid \mathbf{Z})$

Posterior Distribution

$$\pi(\boldsymbol{\beta}, \sigma^2, \phi, \mathbf{W} \mid \mathbf{Z}) \propto$$

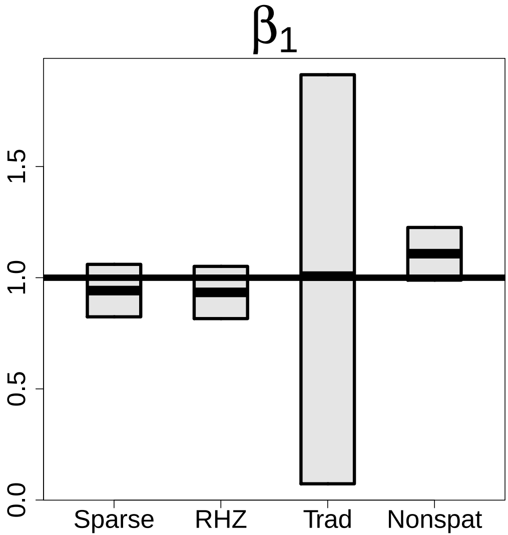
$$\prod_{i}^{n} f\left(Z(\mathbf{s}_i) \mid \boldsymbol{\beta}, W(\mathbf{s}_i)\right) |\sigma^2 \Sigma_\phi|^{-\frac{1}{2}} \exp\left(-\frac{\mathbf{W}' \Sigma_\phi^{-1} \mathbf{W}}{2\sigma^2}\right) p(\boldsymbol{\beta}, \sigma^2, \phi),$$

where the covariance matrix is specified by the covariance function, for example the i, jth element

$$\Sigma_{ij} = \exp(-|\mathbf{s}_i - \mathbf{s}_j| / \phi)$$

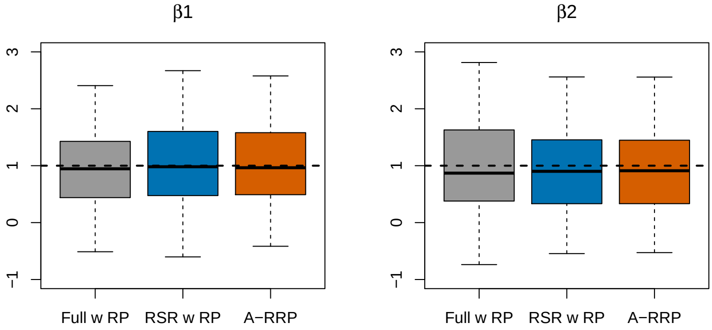
for an exponential covariance function.

Spatial Count Data: Simulation Results



Point Estimation Study: Poisson SGLMM

- Simulate: $\beta = (1, 1)^T$, and Matérn $(\nu, \phi, \sigma^2) = (2.5, 0.2, 1)$



FRP: full model

RRP: restricted model (orthogonalized random effects)

A-RRP: adjusted inference

Summary of Computational Complexity

- matrix multiplication is $n^2 m$, can be parallelized so it is linear in n
- matrix inverse for $m \times m$ matrix, is order m^3
- Eigendecomposition for $n \times m$ matrix is ordernm^2