Bayesian chain point model

Consider the following hierarchical changepoint model for the number of occurrences Y_i of some event during time interval i with change point k.

$$
\begin{aligned}
Y_i|k,\theta,\lambda \sim &\text{Poisson}(\theta) \text{ for } i = 1,\dots,k \\
Y_i|k,\theta,\lambda \sim &\text{Poisson}(\lambda) \text{ for } i = k+1,\dots,n
\end{aligned}
$$

Assume the following prior distributions:

$$
\begin{aligned}
\theta|b_1 \sim &\text{Gamma}(0.5, b_1) \qquad (\text{pdf}=g_1(\theta|b_1)) \\
\lambda|b_2 \sim &\text{Gamma}(0.5, b_2) \qquad (\text{pdf}=g_2(\lambda|b_2)) \\
b_1 \sim &\text{Gamma}(0.01, 100) \qquad (\text{pdf}=h_1(b_1)) \\
b_2 \sim &\text{Gamma}(0.01, 100) \qquad (\text{pdf}=h_2(b_2)) \\
k \sim &\text{Uniform}(1,\dots,n) \qquad (\text{pmf} = u(k))
\end{aligned}
$$

k, θ, λ are conditionally independent and b_1, b_2 are independent.
Assume the Gamma density parameterization $\text{Gamma}(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$

Inference for this model is therefore based on the 5-dimensional **posterior** distribution $f(k,\theta,\lambda,b_1,b_2|\mathbf{Y})$ where $\mathbf{Y}$=$(Y_1,\dots,Y_n)$. The posterior distribution is obtained *upto a constant* by taking the product of all the conditional distributions. Thus we have

$$
\begin{aligned}
f(k,\theta,\lambda,b_1,b_2|\mathbf{Y}) \propto &\prod_{i=1}^{k} f_1(Y_i|\theta,\lambda,k) \prod_{i=k+1}^{n} f_2(Y_i|\theta,\lambda,k) \\
&\times g_1(\theta|b_1) g_2(\lambda|b_2) h_1(b_1) h_2(b_2) u(k) \\
&= \prod_{i=1}^{k} \frac{\theta^{Y_i} e^{-\theta}}{Y_i!} \prod_{i=k+1}^{n} \frac{\lambda^{Y_i} e^{-\lambda}}{Y_i!} \\
&\times \frac{1}{\Gamma(0.5) b_1^{0.5}} \theta^{-0.5} e^{-\theta/b_1} \times \frac{1}{\Gamma(0.5) b_2^{0.5}} \lambda^{-0.5} e^{-\lambda/b_2} \\
&\times FIX e^{-b_1} e^{-b_2} \frac{1}{n}
\end{aligned}
$$

If we are able to draw samples from this distribution, we can answer questions of interest.