Midterm STAT 515, Penn State Statistics
March 6, 2015.

Name:

(1) Show all your work for full credit.
(2) When possible, draw a box around your final answer to each question.
(3) You may leave your answer unsimplified, e.g. $0.7^4 \frac{18!}{4!8!}, 14e^{-23}, \sum_{i=1}^{n} \frac{i}{3^i}$.

1. Consider a Markov chain on a finite state space $\Omega = \{1, 2, \ldots, n\}$. Now assume that the chain has a symmetric transition probability matrix P and that $P_{ij} > 0$ for all $i, j \in \Omega$. Hint: read all parts of this problem before working on it.

   (a) Show that this Markov chain has a unique stationary distribution π that is (discrete) uniform on Ω. [3pts]

   (b) Assume the Markov chain has been running for infinitely long. Is it time reversible? Justify your answers, verifying all necessary properties. [3pts]

   (c) Is π also the limiting distribution of the chain? Why? [2pts]

   Soln: Easy to show that the chain is irreducible. Consider $i, j \in \Omega$ with $i < j$. Paths with positive probability: $i \to i+1 \to \ldots j$, and $j \to j-1 \to \ldots i$.
   Since $P_{ii} > 0$ chain is aperiodic. This is a finite state, irreducible chain so all states are positive recurrent. Hence, chain is irreducible and ergodic so stationary distribution π is unique and also the limiting distribution of chain. Remains to find π. Easy to see that detailed balance condition holds for $\pi = (1/n, 1/n, \ldots, 1/n)$, that is $\pi_i P_{ij} = \pi_j P_{ji} \forall i, j \in \Omega$ when $\pi = (1/n, 1/n, \ldots, 1/n)$. Hence the discrete uniform distribution on Ω is the stationary and limiting distribution of the Markov chain.

2. A professor possesses 2 umbrellas, which she employs in going from her home to office and vice-versa (office to home). If she is at home at the beginning of a day and it is raining, then she will take an umbrella with her to the office, provided there is one to be taken. If it is not raining, then she never takes an umbrella. She does the same thing when she leaves the office. Assume that the probability of rain at the beginning of the day (also at the end of the day), p, is independent of rain on other days.

   (a) Define a Markov chain model for the number of umbrellas she has at her *current* location. E.g. if she has 0 umbrellas now, at the next location she will definitely have 2 umbrellas (since both are at the next location). [3pts]

   (b) In the long run, what fraction of the time will she get wet? (You may assume that the chain is irreducible, positive recurrent and apeiodic.) [2pts]

   Solution: (a) Each state of the Markov chain corresponds to the number of umbrellas she has at her current location so

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix},$$

   (b) Solve ergodic equations to get stationary probability of being in state 0: $\pi_o = \frac{1-p}{3-p}$. Hence, fraction of time she will get wet = probability it rains × probability she doesn't have an umbrella with her = $p\pi_0 = \frac{p(1-p)}{3-p}$.

3. A car mechanic is only able to work on one car at a time. Suppose the amount of time he takes to work on each car is an exponential random variable with mean 30 minutes, and that these times are independent of each other.

   (a) Suppose there are always more cars to work on than the mechanic has time, and that all customers bring in their cars at 9am. What is the expected number of cars the mechanic will have completed work on from 9am to 10am ? [2pts]

   (b) Assume the same set up as in part (a), except now you know that the mechanic completed work on exactly 3 cars between 9am and 10am. What is the expected value of the time he finished work on the first car ? [2pts]

   (c) Now suppose he schedules two appointments, car A at 9am and car B at 9:30am. Assume both cars arrive on time for their appointments. Find the expected amount of time that car B spends at the mechanic's shop. [3pts]

Soln: (a) Number of cars mechanic fixes is a Poisson process with $\lambda = 1/30$. Hence, if he works from 9am to 10am (for 60 minutes), the expected number of cars he will have fixed is the expected value of a Poisson r.v. with expectation $\lambda \times 60 = 2$.
(b) Let times of completion for the three cars be X_1, X_2, X_3. The question is asking what $E(X_1 \mid N(60) = 3)$ is. Now the completion times for these 3 cars is Uniform(0,60) where units are in minutes. Hence X_1 is the minimum order statistics of three $\mathrm{U}(0, 60)$ random variables. It is easy to derive the cdf of X_1 and hence its pdf is $f(x) = \frac{1}{20}(1 - \frac{x}{60})^2$, $x \in (0, 60)$ and $E(X_1 \mid N(60) = 3) = 15$.
(c) Condition on whether car A is done by 9:30am or not. Let completion time for car A be X_A. If $X_A < 30$, expected time for car B to be done is simply 30 minutes. If $X_A > 30$, by memorylessness of exponential r.v.s, expected additional time for car A is 30 minutes and time for completion for car B is therefore 30 minutes + 30 minutes (expected value for car B to be done after mechanic starts working on it.) Hence, expected amount of time car B spends at the mechanic's shop is:

$$30 \times P(X_A < 30) + (30 + 30) \times P(X_A \geq 30) = 30(1 - e^{-1}) + 60e^{-1} = 30 + 30e^{-1}.$$

4. Recall that a stochastic process $\{X(t), t \geq 0\}$ is said to be a compound Poisson process if it can be represented as $X(t) = \sum_{i=1}^{N(t)} Y_i,\ t \geq 0$ where $\{N(t), t \geq 0\}$ is a Poisson process with rate λ and $\{Y_i, i \geq 1\}$ is a family of independent and identically distributed random variables that is also independent of $\{N(t), t \geq 0\}$. Let $\mu_Y = E(Y_i)$ and $\sigma_Y^2 = \text{Var}(Y_i)$ for $i = 1, 2, \ldots$

   (a) Find $\text{Var}(X(t))$. [2pts]

   (b) Find $\text{Cov}(X(t), X(t+s))$, where $t \geq 0,\ s > 0$. Hint: it may be useful to use the fact that this process has independent increments. [3pts]

   Soln: (a) Use conditional variance formula, $V(X(t)) = E(V(X(t) \mid N(t) = n)) + V(E(X(t)) \mid N(t) = n) = \mu_Y^2 \lambda t + \lambda t \sigma_Y^2$.

   (b) To use the independent increments property, rewrite $\text{Cov}(X(t), X(t+s)) = \text{Cov}(X(t), X(t+s) - X(t) + X(t)) = \text{Cov}(X(t), X(t)) + \text{Cov}(X(t), X(t+s) - X(t)) = \text{Var}(X(t)) + 0$ $=$ answer from part (a).