2^{nd} Assignment - Lab Report

Numerical Methods for Finance

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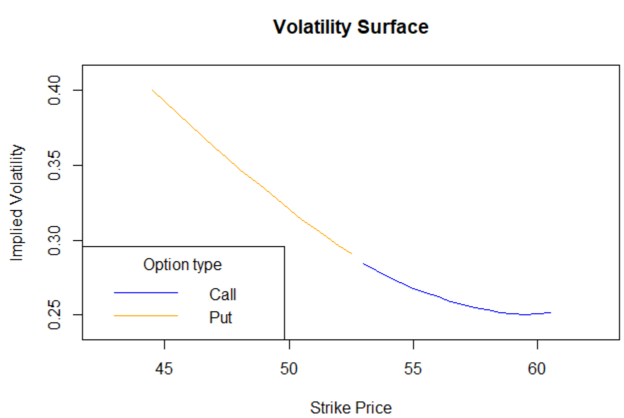
1 Exercise 1

The dataset provided has fixed underlying value S_t, fixed t_0 and maturity T and fixed interest rate r. Varying observed price C_t (Call), P_t (Put) are provided along with varying strike prices K. In the c.c.r, all data satisfies the Merton's constraints:

$$\max\left\{S_0 - Ke^{-r(T-t_0)}, 0\right\} \leq C_{t_0} \leq S_0 \qquad \text{(Call)} \tag{1}$$

$$\max\left\{Ke^{-r(T-t_0)} - S_0, 0\right\} \leq P_{t_0} \leq Ke^{-r(T-t_0)} \qquad \text{(Put)} \tag{2}$$

After having excluded the first five options and the last five ones, which are in-the-money, the volatility surface is computed and drawn in the following picture:



Finally, the volatility σ is calibrated through minimization of MSE, which is defined as:

$$\frac{1}{N}\sum_{i=1}^{N}\left[Opt_i^{Obs} - Opt_i^{B\&S}(\sigma)\right]^2$$

The result is $\hat{\sigma} = 0.2948$.

2 Exercise 2

The Vasicek process has the following SDE:

$$dS_t = \alpha(\mu - S_t)dt + \sigma dW_t \quad , \quad S_{t_0} = S_0 \tag{3}$$

There exists a closed formula for the transition density of a Vasicek process. In particular, we know the conditional distribution:

$$S_{t_i}|S_{t_{i-1}} \;\sim\; N\left(S_{t_{i-1}}e^{-\alpha(t_i - t_{i-1})} + \mu(1 - e^{-\alpha(t_i - t_{i-1})}) \quad , \quad \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(t_i - t_{i-1})})\right) \tag{4}$$

For brevity, we will use $\mathbb{E}[S_{t_i}|S_{t_{i-1}}]$ and $Var[S_{t_i}|S_{t_{i-1}}]$ as the mean and the variance of conditional Vasicek distribution.

2.1 Exact conditional probability

Using $t_{i-1} = t_0$ and $t_i = T$, the probability of the event $\{S_T < a \cup S_T > b\}$ where $a < b$ can be easily computed via the normal distribution function $\Phi(x, \mathbb{E}[S_T|S_{t_0}], [S_T|S_{t_0}])$:

$$\begin{aligned}\mathbb{P}(S_T < a \cup S_T > b|S_0) = \mathbb{P}(S_T < a|S_0) + \mathbb{P}(S_T > b|S_0) = \\ \Phi\big(a, \mathbb{E}[S_T|S_{t_0}], Var[S_T|S_{t_0}]\big) + 1 - \Phi\big(b, \mathbb{E}[S_T|S_{t_0}], Var[S_T|S_{t_0}]\big)\end{aligned}$$

2.2 MC probability

Another way of computing $\mathbb{P}(S_T < a \cup S_T > b|S_0)$ is to approximate it via Monte Carlo simulation. We first note that the probability of each event can be written:

$$\mathbb{P}(S_T < a|S_0) = \mathbb{E}[1_{(-\infty,a)}(S_T)|S_0]$$
$$\mathbb{P}(S_T > b|S_0) = \mathbb{E}[1_{(b,+\infty)}(S_T)|S_0]$$

Then, once we've observed $\{S_T^1, \ldots, S_T^M\}$ (explained at the end of the paragraph), the probabilities can be estimated via sample mean

$$\overline{g}_j = \frac{1}{M}\sum_{i=1}^{M} g_j(S_T^i) \qquad \text{for } g_j \in \{1_{(-\infty,a)}, 1_{(b,+\infty)}\}$$

Finally, via CLT, the preceding estimator is asymptotically normal and a 95% confidence interval for the sample mean can be computed as follows:

$$(\overline{g}_j - \Phi^{-1}(0.975)\frac{\sigma_j}{\sqrt{M}} \quad , \quad \overline{g}_j + \Phi^{-1}(0.975)\frac{\sigma_j}{\sqrt{M}})$$

where σ_j is the sample standard deviation of $\{g_j(S_T^1), \ldots, g_j(S_T^M)\}$.

Coming back to the simulation of $\{S_T^1, \ldots, S_T^M\}$, we have two ways:

- Via Vasicek exact simulation scheme, thanks to the transition density (4), where we put $t_{i-1} = t_0$ and $t_i = T$.
- Via the discretized version of Vasicek SDE (Euler discretization), which leads to a simulation scheme with again a normal random variable (see paragraph 3.3 for more details).

2.3 Numerical results

The exact probability for a set of inputs is shown below along with the two Monte-Carlo simulations: one based on the exact simulation scheme and one based on Euler simulation scheme (the latter with N equally-spaced time intervals from t_0 to T).

Inputs:

$$X_0 = 0, \alpha = 0.1, \mu = 0.04, \sigma = 0.15, t_0 = 0, T = 1$$

$$a = 0.1, b = 0.5$$

$$N = 1000, M = 1000, \text{conf\_level} = 95\%$$

Outputs:

$\mathbb{P}(S_T < a \cup S_T > b | S_0)$

Exact	Monte-Carlo on exact sim. scheme	Monte-Carlo on Euler sim. scheme
0.7500	0.7430 ± 0.02909	0.7500 ± 0.02885

3 Exercise 3

The Vasicek process has the SDE in (3). In this exercise we assume the log-prices of VIX to follow such SDE, calling $S_t = \log(P_t)$, where P_t are the prices.

We create a time grid based on the available observations: $t_0, t_1, \ldots, t_n$, where in our case $n = 502$, and denote $\Delta t_i := t_i - t_{i-1} \quad$ for $i = 1, \ldots, n$ the annualised difference in days (1 delta day = 1/365).

3.1 MLE with log-returns

Calling $X_{t_i} := S_{t_i} - S_{t_{i-1}} = \log\left(\frac{P_{t_i}}{P_{t_{i-1}}}\right)$ the log-returns, we'd like to build the log-likelihood function based on their exact transition density. Starting from the normal r.v. S_{t_i}, the conditional distribution of X_{t_i} given the information at time t_{i-1} is again normally distributed; in particular:

$$X_{t_i}|\mathcal{F}_{t_{i-1}} \;=\; (S_{t_i} - S_{t_{i-1}})|S_{t_{i-1}} \;= S_{t_i}|S_{t_{i-1}} - S_{t_{i-1}}|S_{t_{i-1}} \;\sim N(p_1(i), p_2^2(i))$$

where

$$\begin{aligned} p_1(i) &= (\mu - S_{t_{i-1}})(e^{-\alpha \Delta t_i}) \\ p_2^2(i) &= \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha\Delta t_i}) \end{aligned} \tag{5}$$

Therefore, the log-likelihood function for our sample $X_{t_1}, \ldots, X_{t_n}$ is:

$$I_n(\alpha, \mu, \sigma) = \sum_{i=1}^{n} \log\left[\varphi(X_{t_i}, p_1(i), p_2^2(i))\right]$$

where $\varphi(x, p_1, p_2^2)$ is the normal density with parameters p_1 and p_2^2, evaluated at x.

The Maximum Likelihood Estimation of (α, μ, σ) is performed starting from $(0.01, 0.2, 0.1)$, employing two different methods:

- Nelder-Mead method. This method does not require the computation or the estimation of the gradient.

$$\text{Start: } (1.0, 1.0, 1.0)$$
$$\text{Steps: } 312$$
$$-I_n = -546.4196$$
$$(\hat{\alpha}, \hat{\mu}, \hat{\sigma})_{\text{Nelder-Mead}} = (5.7408, 3.0056, 1.3953)$$

- Limited memory BFGS method (L-BFGS-B). With this method it's possible to set constraints $\alpha, \sigma > 0$.

$$\text{Start: } (1.0, 1.0, 1.0)$$

$$\text{Steps: } 37$$

$$-I_n = -546.4196$$

$$(\hat{\alpha}, \hat{\mu}, \hat{\sigma})_{\text{L-BFGS-B}} = (5.7410, 3.0055, 1.3954)$$

The two methods achieve the same score and approximately the same estimates of the Vasicek parameters. Therefore, we'll make use of only the L-BFGS-B method in the following paragraphs, where variants of ML estimations have been used.

3.2 MLE with log-prices

The likelihood function can be also based on the log-prices. In this case, the transition density is simply the transition density of a Vasicek, as presented in (4). The log-likelihood is again computed as follows:

$$I_n(\alpha, \mu, \sigma) = \sum_{i=1}^{n} \log\left(f\left(S_{t_i}|\mathcal{F}_{t_{i-1}}\right)\right) = \sum_{i=1}^{n} \log\left[\varphi\left(X_{t_i}, \mathbb{E}[S_{t_i}|S_{t_{i-1}}], \mathrm{Var}[S_{t_i}|S_{t_{i-1}}]\right)\right]$$

where, in order to get this result, the term $\log f(S_{t_0})$ was neglected. Here are the L-BFGS-B results:

$$\text{Start: } (1.0, 1.0, 1.0)$$

$$\text{Steps: } 31$$

$$-I_n = -396.64$$

$$(\hat{\alpha}, \hat{\mu}, \hat{\sigma})_{\text{L-BFGS-B}} = (10.7168, 2.9964, 1.8989)$$

The estimated α is substantially different to that obtained in the previous paragraph, while μ and σ resemble the figures previously obtained.

3.3 QMLE with log-returns

Finally, another approach to estimate the Vasicek parameters is to use the quasi-maximum-log-likelihood function of log-returns. We start from the Euler discretization of (3):

$$X_{t_i} = S_{t_i} - S_{t_{i-1}} = \alpha(\mu - S_{t_{i-1}})\Delta t_i + \sigma(W_{t_i} - W_{t_{i-1}}) \tag{6}$$

Then, $X_{t_i}|\mathcal{F}_{t_{i-1}} \ \sim \ N(\alpha(\mu - S_{t_{i-1}})\Delta t_i \ , \ \sigma^2 \Delta t_i)$. The quasi log-likelihood function can be written:

$$I_n(\alpha, \mu, \sigma) = \sum_{i=1}^{n} \log \left[\varphi(X_{t_i} \ , \ \alpha(\mu - S_{t_{i-1}})\Delta t_i \ , \ \sigma^2 \Delta t_i) \right]$$

Here, we get very similar results to those in ¶3.1, namely:

$$(\hat{\alpha}, \hat{\mu}, \hat{\sigma})_{\text{L-BFGS-B}} = (4.9891, 3.0009, 1.3815)$$

4 Exercise 4

4.1 Market volatility estimation

When we assume that S&P500 Index follow a Geometric Brownian Motion we're assuming that, denoting with S_t the value of the index at time t, its SDE is:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad \text{given } S_0$$

For a GBM the transition density is known:

$$S_{t_i}|S_{t_{i-1}} = S_{t_{i-1}} e^{(\mu - \frac{1}{2}\sigma^2)\Delta t_i + \sigma(W_{t_i} - W_{t_{i-1}})} \tag{7}$$

It is easy to show that the log-returns $X_{t_i} = \log(S_{t_i}) - \log(S_{t_{i-1}})$ are normally distributed with mean $\mathbb{E}[X_{t_i}] = (\mu - \frac{1}{2}\sigma^2)\Delta t_i$ and variance $\text{Var}[X_{t_i}] = \sigma^2 \Delta t_i$.

Based on the exact density of log-returns, we built the log-likelihood function

$$I_n(\mu, \sigma) = \sum_{i=1}^{n} \log \left[\varphi \left(X_{t_i} \ , \ \left(\mu - \frac{1}{2}\sigma^2 \right) \Delta t_i \ , \ \sigma^2 \Delta t_i \right) \right]$$

The Maximum Likelihood Estimation of (μ, σ) is performed starting from $(0.5, 0.2)$ through L-BFGS-B method:

$$\text{Start: } (0.5, 0.2)$$

$$\text{Steps: } 30$$

$$-I_n = -1350.758$$

$$(\hat{\mu}, \hat{\sigma})_{\text{L-BFGS-B}} = (0.2379, 0.2795)$$

4.2 Put price in B&S

The exact BS price of an at-the-money European Put Option is:

$$P(t_0, S_0) = K e^{-r(T-t_0)} \Phi(-d_2) - S_0 \Phi(-d_1)$$

$$\text{with} \qquad d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t_0)}{\sigma\sqrt{T - t_0}} \qquad , \qquad d_2 = d_1 - \sigma\sqrt{T - t_0}$$

where our time to maturity and interest rate expressed on yearly basis are, respectively, $T - t_0 = 0.1616$ and $r = 0.015$; The value of S&P500 at t_0 is $S_0 = 3732.04$ and equal to K. Lastly, the volatility σ used is the market one, estimated in the preceding paragraph. The resulting value for the price of the Put Option at t_0 is:

$$P(t_0, S_0) = 162.5238$$