ML

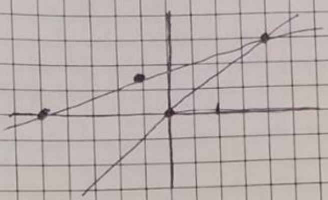
Lineer Algebra:

Lecture 1:

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned} \rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

ilk denklem için orijini temel nokta alıyoruz. Ardından denklemi sağlayan bir diğer nokta belirliyoruz. Örneğin $x = 1, y = 2$.

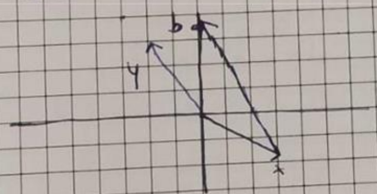
ikinci denklem için 2 nokta belirlemeliyiz. $x = -3, y = 0$ ve $x = -1, y = 2$ noktaları denklemi sağlıyor.



iki denklemi sağlayan noktaları ayrı ayrı birleştirdik. $x = 1, y = 2$ iki denklemi de sağlayan bir nokta.

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

x için $(2, -1)$ ve y için $(-1, 2)$ noktalarını koordinat sistemine ekle.



x yerine 1, y yerine 2 koyarak denklemi sağlayabiliyoruz. bu durumda denklem $\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ oluyor. Bunu da koordinat sistemine ekleyelim. (Aslında vektörel toplam yapıyoruz)

example 2:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \rightarrow \text{where } x = 0, y = 0, z = 1$$

example 3 :

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$ because its equal → $1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Lecture 2 : Elimination

$x + 2y + z = 2$

$3x + 8y + z = 12$

$4y + z = 2$

→

pivot

$$\begin{matrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{matrix}$$

ilk satırı al ve ikinciden çıkar

$$\begin{matrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{matrix}$$

pivot

ikinci satırı al ve üçüncüden çıkar

$$\begin{matrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{matrix}$$

Matrices :

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{matrix} 3x \text{ column } 1 \\ 4 \times \text{ column } 2 \\ 5 \times \text{ column } 3 \end{matrix}$$

matrix x column = column

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = \begin{matrix} 1 \times \text{ row } 1 \\ 2 \times \text{ row } 2 \\ 3 \times \text{ row } 3 \end{matrix}$$

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

A B C → 1 0 0

G H I → 0 0 1

D E F → -3 1 0

Permutation :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Lecture 3 : Matrix Multiplication And Inverse
(Gauss-Jordan, A^{-1}, Inverses)

C_{34} = (row 3 of A) . (column 4 of B)

$$= a_{31} \cdot b_{14} + a_{32} \cdot b_{24} + \dots = \sum_{k=1}^{n} a_{3k} b_{4k}$$

$$A[m \times n] \cdot B[n \times p] \rightarrow C[m \times p]$$

column of A × row of B

$$m \times 1 \quad \times \quad 1 \times p \quad \rightarrow \quad m \times mp$$

AB = sum of (cols of A) × (rows of B)

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} [1 \; 6] + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} [0 \; 0$$

Block Multiplication

$$\underbrace{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]}_{A} \underbrace{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]}_{B} = \underbrace{\left[\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}\right]}_{C}$$

$C_1 = A_1 B_1 + A_2 B_3$

$C_2 = A_1 B_2 + A_2 B_4$

$C_3 = A_3 B_1 + A_4 B_3$

$C_4 = A_3 B_2 + A_4 B_4$

Inverses (Square Matrices)

$A^{-1}.A = I = A.A^{-1}$

Singular case → no inverse

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ → we can find a vector x with $Ax = 0$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Some combination of A can give 0 matrix.

example

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \cdot \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Gauss - Jordan:
Solve 2 equation at once.

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Lecture 4: Factorization into $A = LU$

$A.A^{-1} = I = A^{-1}.A$

$(A.B).(B^{-1}.A^{-1}) = I$ $\qquad (A^{-1})^T . A^T = I$

$(A^{-1})^T = (A^T)^{-1}$

$$\overset{E_{21}}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \overset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \overset{L}{\begin{bmatrix} & \\ & \end{bmatrix}} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

$E_{32} E_{31} E_{21} A = U$ (no row exchanges)

$E_{32} E_{31} E_{21} \rightarrow E_{32} E_{21} E_{31} \rightarrow E_{31} E_{31} A$

$A = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} \cdot U \rightarrow LU$

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix} \rightarrow E A = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L \quad A = LU$$

$A = LU \rightarrow$ If no row exchanges, multipliers go directly into L.

Lecture 5: $PA = LU$

Permutations P: execute rows exchanges

$$A = LU = \begin{bmatrix} 1 & & & 0 \\ - & 1 & & \\ - & - & 1 & \\ - & - & - & 1 \end{bmatrix} \begin{bmatrix} 1 & - & - & - \\ & 1 & - & - \\ & & 1 & - \\ 0 & & & \end{bmatrix}$$

P = identity matrix with reordered rows

$n! = n.(n-1)...(3).(2).(1)$ → counts reorderings counts all $n \times n$ permutations.

$P^{-1} = P^T$ → $P^T . P = I$

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

Vector Spaces :
examples : R^2 = all 2-dimentional • vectors (real)

example :
a vector space inside R^2, Subspaces of R^2

↓

all of R^2

any ~~line~~ through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
line

zero vector only

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

columns in R^3

all their combinations form a subspace called column space $C(A)$

Lecture 6 : Vector Spaces and Subspaces

2 subspaces : P and L

$P \cup L$ → all vectors in P or L

$P \cap L$ → all vectors in both P and L

Note : Subspaces S and T intersection $S \cap T$ is a subspace.

Column space of A is subspace of R^4

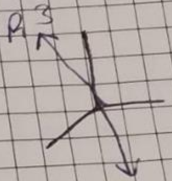
$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \rightarrow \text{all linear combinations of columns.}$$

Question: Does $Ax = b$ have a solution for every b? (No)

$$Ax = \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}}_{3\times 4} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{1\times 3} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}_{1\times 4} \rightarrow b_1 = b_2 = b_3 = b_4 = 0$$

$\rightarrow$ Nullspace of $A \rightarrow$ all solutions x to $Ax = 0$

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



If $Av = 0$ and $Aw = 0$ then $A(w+v) = 0$

$\boxed{\text{Lecture 7 6}}$

If $Av=0$ and $Aw=0$ then $A(w+v)=0$

$\boxed{\text{Lecture 7:}}$

Topics:
Computing the nullspace $(Ax=0)$
Pivot variables - free variables
Special solutions - $\text{ref}(A)=R$

$$A=\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

(*) rank of A = number of pivots

$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$\rightarrow 2x_3 + 4x_4 = 0$

$Ax = 0 \quad Ux = 0$

$x = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

R = reduced row echelon form

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

* zeros above + below pivots = 1

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

ref form

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ pivot rows

pivot cols, $n-r$ (free cols)

when $RN = 0$ $\quad N = \begin{bmatrix} -F \\ I \end{bmatrix}$

$Rx = 0 \rightarrow x_{pivot} = -F x_{(free)}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$r = 2$

$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$\to 2x_3 + 4x_4 = 0$

$Ax = 0 \quad Ux = 0$

$x = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

R = reduced row echelon form

$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

* zeros above + below pivots = 1

$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

ref form

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ pivot rows

pivot cols; $n-r$ (free cols)

when $RN = 0$ $N = \begin{bmatrix} -F \\ I \end{bmatrix}$

$Rx = 0 \to x_{pivot} = -Fx_{(free)}$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$r = 2$

$x_1 + 2x_2 + 3x_3 = 0$

$2x_2 + 2x_3 = 0$

$x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \cdot c = c \underbrace{\begin{bmatrix} -F \\ I \end{bmatrix}}_{N}$

Lecture 8:

Complete Solutions of $\boxed{Ax = b}$

$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$

$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$

$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$

$\left[\begin{array}{cccc:c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$, Augmented Matrix $= [A \ b]$

$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 & \\ 0 & 0 & 2 & 4 & b_2 & -2b_1 \\ 0 & 0 & 0 & 0 & b_3 & -b_2 - b_1 \end{bmatrix}$

$\rightarrow 0 = b_3 - b_2 - b_1 \rightarrow b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$

(*) Solvability Condition on b

$Ax = b$ solvable when b is in $C(A)$, if a combination of rows of A gives zero row, then same combination of entities of b must give 0.

(*) To find complete solution for $Ax = b$

1 $\rightarrow x_{particular} \rightarrow$ Set all free variables to zero. Solve $Ax = b$ for pivot variables.

example $\rightarrow$ $x_1 + 2x_3 = 1$, $2x_3 = 3$; $x_1 = -2$, $x_3 = 3/2$; $x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$

2 $\rightarrow A x_p = b$

$A x_n = 0 \longrightarrow A(x_p + x_n) = b$

$$X_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$\leftrightarrow x_1 + 2x_2 + 2x_3 + 2x_4 = 0$ ✓

Plot all solutions x in R^4

m by n matrix A of rank r (know $r \leq m$, $r \leq n$)

full column rank means $r = n$.

- full row rank means $\boxed{r = m}$

this can solve $Ax = b$ for every b exists.

Left with $n - r$ $(n - m)$ free variables.

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{bmatrix}$$

where $r = m = n$ $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $R = I$

$r = m = n \rightarrow R = I$, is the solution to $Ax = b$

$r = n < m \rightarrow R = \begin{bmatrix} I \\ 0 \end{bmatrix}$, 0 or 1 solution

$r = m < n \rightarrow \infty$ solution

Lecture 9 :

Topics

Linear independence
Spanning a space
Basis and dimension

Suppose A is m by n with m<n. Then there are nonzero solutions $\rightarrow Ax=0$.

$\hookrightarrow$ Reason is free variables

Indepence

Vectors $x_1, \ldots x_n$ are independent if no combination gives zero vector.

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n \neq 0.$$

Columns of A is independent if nullspace of A is zero vector.
they are dependent if $Ac=0$ for some nonzero c.

rank = n $\quad N(A) = \{0\} \rightarrow$ no free variable.

rank < n $\quad$ yes free variables

Vectors $V_1, \ldots V_\ell$ span a space means that $\rightarrow$ The space consists of all combinations of those vectors.

Basis for a space is a sequential of vectors $V_1, V_2 \ldots V_d$ with 2 properties.

1 - They are independent
2 - They span the space

Example

Space is $\mathbb{R}^3$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \quad \Bigg| \quad \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$$

$\mathbb{R}^n$ n vectors give basis if the $n \times n$ matrix with those as columns

⊛ Given a space, every basis for the space has the same number of vectors. DEF, dimension of the space

Space $\rightarrow$ $C(A)$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$N(A)$ 2 $\rightarrow$ rank

! $\dim C(A) = r$, free Variables $\rightarrow n - r$

Lecture 10:

Topics

- ~~Correct error from previous chapter~~ (Bir önceki chapterdaki kısmı tekrar bak, hata var!)
- Four fundamental Subspaces

1) column space $C(A)$
2) null space $N(A)$
3) row space $\rightarrow$ all combinations of rows $C(A^T)$
4) nullspace of A^T

$4 \rightarrow N(A^T) \rightarrow A^T y = 0 \qquad y^T A = 0^T$

$$[y^T]\begin{bmatrix} A \end{bmatrix} = [0]$$

$$E\left[A_{m \times n} \quad I_{m \times m}\right] \rightarrow \left[R_{m \times n} \quad E_{m \times m}\right]$$

$EA = R$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$C(R) \neq C(A)$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow R$$

! Repeat Chapter 10, it's not clear. Farklı kaynaklardan bu konuya bakabilirsin

Chapter 11
Topics:
Bases of new vector spaces
Rank one matrices
Small world graphs

- Bases of new vector spaces

M = all 3 by 3 matrices upper triangular

Symmetric 3x3 should.

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \quad \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \quad \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \quad \cdots \quad \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$S \cap U \rightarrow$ Symmetric and triangular (upper)

$S \cup U \rightarrow S + U \rightarrow$ any element of S + any element of U

$\rightarrow$ all 3×3's.

$\frac{d^2y}{dx^2} + y = 0 \quad (y = \cos x, \sin x, e^x)$, $\underline{\underline{dim = 2}}$

$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$ dim $C(A)$ = rank = dim $C(A^T)$

$r = 1$

$\rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$ $\quad A = U V^T$

$r = 1$

M = all 5×17 matrices subset of rank 1 matrices.

$R^4 \rightarrow V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

$S \rightarrow$ all V in R^4 with $V_1 + V_2 + V_3 + V_4 = 0$.

nullspace $\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \rightarrow$ rank $= 1 = r$

$\underline{n = 4}$ $\quad$ dim $N(A) = n - r$

Basis

$\begin{matrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

- Remember the airplane example for graph!

Lecture 12:

Topics

- Graphs / Networks
- Incidence Matrices
- Kirchhoff's Laws

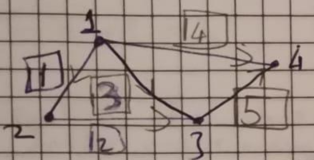
1) Graphs:

Lecture 12 :

Topics

- Graphs & Networks
- Incidence Matrices
- Kirchhoff's Laws

1) Graphs : Just nodes and edges.



n = number of nodes → 4
m = number of edges → 5

• Think this graph like a small network structure. It could be a design of bridge. It's not just a symbol it can be anything in real life!

illustrate this graph to Matrix form

$$
\begin{array}{c}
\begin{array}{cccc} \text{node 1} & \text{node 2} & \text{node 3} & \text{node 4} \end{array} \\
\begin{bmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 0 & -1 & 1
\end{bmatrix}
\end{array}
\begin{array}{l} \\ \text{edge 1} \\ \text{edge 2} \\ \text{edge 3} \\ \text{edge 4} \\ \text{edge 5} \end{array}
$$

• Burada numaralandırma node'dan node'ye olan doğrusal güzergah'a göre yapıldı, örneğin edge 4, node 1 to node 4 olduğu için node 2 ve node 3 boş bırakıldı

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

if $Ax = 0 \rightarrow$
$$x_2 - x_1 = 0$$
$$x_3 - x_2 = 0$$
$$x_3 - x_1 = 0$$
$$x_4 - x_1 = 0$$
$$x_4 - x_3 = 0$$

Potential Points!

$A^T y = 0 \rightarrow \dim N(A^T)$

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \underline{0}$$

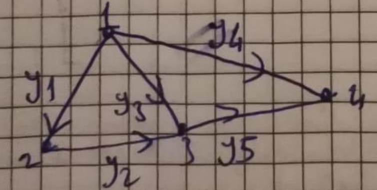
$\rightarrow m - r = 5 - 3 = 2$

$x \rightarrow x_1, x_2, x_3, x_4$ was potential nodes

↓ OHM's LAW

Current y_1, y_2, y_3, y_4, y_5 on edges $\rightarrow A^T y = 0$ (Kirchhoff)

$$-y_1 - y_3 - y_4 = 0 = y_1 - y_2 = y_2 + y_3 - y_5$$



all in all up to now

$\downarrow$

$\dim N(A^T) = m - r$

$\#\text{loops} = \#\text{edges} - (\#\text{nodes} - 1) \rightarrow (\text{rank} = n - 1)$

Number of Nodes − Number of edges + Number of Loops = 1

Euler's Formula

$5 - 7 + 3 = 1$

- Visualize the points and paths on the triangle in your mind for better understanding. All the rules has a logic.

Lecture 13:

Topics

- Exam Review

Question 1) $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (has no solution) $Ax = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ has 1 solution

a) Give all possible information about m and n and the rank of A

$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow r < m$ $Ax = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \rightarrow A = 0,\ r = n,\ m = 3$

$\rightarrow m = 3,\ r = 1 = n$; $m = 3,\ r = 2 = n$

b) Find all solutions to $Ax = 0$

$Ax = 0 \rightarrow x = 0$

c) Write an example of ~~matrix~~ A for part a

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Question 2:

matrix A is 3x3. Matrix $I \to E_{21}$: Sub 4 (row 1) from row 2
$\to E_{31}$: Sub 3 (row 1) from row 3
$\to E_{23}$: Sub row 3 from row 2

a) write A^{-1} in terms of E's. Then compute A^{-1}

$A^{-1} = E_{23} \cdot E_{31} \cdot E_{21}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = A^{-1}$$

b) original A = ?

$A = E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{23}^{-1}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} = A$$

c) Lower triangular factor L in A = LU ?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Question 3:

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{bmatrix} (3 \times 4)$$

a) for each c find a basis for column space of A

elimination $\to$

$$\begin{matrix} 1 & 1 & 2 & 4 \\ 0 & c-3 & -4 & -4 \\ 0 & 0 & 2 & 2 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\hookrightarrow \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ if $c=3$ $U = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

b) if $c \neq 3$ $N(A) = x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$c = 3$, $N(A) = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

c) $x_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ if $c \neq 3 \rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

if $c = 3 \rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

4) a) If A is $[\ ]_{3\times5}$, give info about nullspace of A
$N(A)$ dimension between 2 and 5

b) $A = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ =

Lecture 14:

Topics:

Orthogonal vectors And Subspaces / Nullspaces and row space

$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $x+y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$\|x\|^2 = 14$ $\|y\|^2 = 5$ $\|x+y\|^2 = 19$

$x^Tx + y^Ty \rightarrow (x+y)^T \cdot (x+y)$
$= x^Tx + y^Ty + x^Ty + y^Tx$

- Subspace S is orthogonal to subspace T means: every vector in S is orthogonal to every vector in T.

row space is orthogonal to nullspace (why?!)

$Ax = 0$ $\quad [\quad] \cdot [x] = [0]$

Nullspace and rowspace are orthogonal complements in $\mathbb{R}^n$. Nullspace contains all vectors of row space.

A^TA

$\cancel{n \times n}$ $\cancel{n \times m}$
$n \times m \quad m \times n \rightarrow n \times n$ $\quad (A^TA)^T \rightarrow A^TA$

Lecture 15:
Topics

~~Projection~~:

Least Squares

Projection Matrix

Projection

$a^T a\, x = a^T b \rightarrow x = \frac{a^T b}{a^T a}$ $\quad p = a x \rightarrow p = a \frac{a^T b}{a^T a}$

$$P = \frac{a a^T}{a^T a}$$

- $P^T = P$, $P^2 = P$

main reason is $Ax = b$ can have no solution. Thus, solve $Ax = p$ instead

$p = A\hat{x} \rightarrow \hat{x} = ?$

key: $b - A\hat{x}$ is ~~→~~ $a_1^T(b - A\hat{x}) = 0$ $\quad a_2^T(b - A\hat{x}) = 0$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad A^T(b - A\hat{x}) = 0$$

$$A^T \cdot A \cdot \hat{x} = A^T b$$

$\hat{x} = (A^T A)^{-1} A^T b \rightarrow p = A\hat{x} = A(A^T A)^{-1} A^T b$

matrix $P = A(A^T A)^{-1} A^T$

$\rightarrow A \cdot A^{-1} (A^T)^{-1} \cdot A^T = I$

Lecture 16

Topics

Least Squares and best straight line

$P = A(A^T A)^{-1} A^T$

If b in column space $Pb = b$

~~If~~ b not column space $Pb = 0$

Find $\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$, P

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 5 \\ 6 & 14 & 11 \end{bmatrix}$$

$$3C + 6D = 5$$
$$6C + 14D = 11$$

- If A has indep columns then $A^T A$ is invertible to prove
  suppose $A^T A x = 0$ $\quad x = 0$

$\rightarrow x^T A^T A x = 0 = (Ax)^T (Ax) \longrightarrow Ax = 0 \rightarrow x = 0$

columns definitely independent if they are perp. unit vectors

(means) orthonormal vectors

Lecture 17:

Topics:
- Orthogonal Matrix (cont.)
- Gram - Schmidt ($A \to Q$)

1) Orthogonal Matrix:

orthonormal vectors

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$\boxed{Q^T Q = I}$ Transpozesi ile çarpımı birim matris veriyor.

- what about Q^T and Q^{-1}, are them equal?

$$\text{perm } Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = I$$

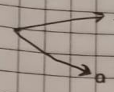
$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad Q = 1/\sqrt{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\to Q = 1/2 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Q has orthonormal columns, project onto its column space

$P = Q(Q^T Q)^{-1} \cdot Q^T \to Q \cdot Q^T$ (if Q is square $= I$)

- Gram - Schmidt



$a, b \to$ orthogonal $\to q_1 = \frac{A}{\|A\|}, q_2 = \frac{B}{\|B\|}, q_3 = \frac{C}{\|C\|}$

$\to B = b - \frac{A^T b}{A^T A} . A$

$A^T B = A^T \underbrace{\left(b - \frac{A^T b}{A^T A} . A \right)}_{B} = 0$

$\to C = c - \frac{A^T c}{A^T A} . A - \frac{B^T c}{B^T B} . B$

$\oplus$ $C \perp A$ and $C \perp B$

$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\to B = b - 2/3 . A = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \to A \perp B$

Lecture 18 :

- Determinants and Properties :

- $\det I = 1$
- exchange rows $\to$ reverse sign of determinant
- $\det P > 1 \to$ even
  $\det P < -1 \to$ odd

$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad \begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

- Subtract l x row doesn't change

logaritma kopti gibi düşün.

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$$

- Row of zeros → $\det A = 0$

$$5 \cdot \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} 5 \cdot 0 & 5 \cdot 0 \\ c & d \end{vmatrix}$$

- $\det U$ = product of pivots
- when A is singular $\det A = 0$
  when A is invertible $\det A \neq 0$

- $\det AB = (\det A) \cdot (\det B)$

Sağlaması

$\det A^{-1} = 1/\det A$

$A^{-1} \cdot A = I \qquad (\det A^{-1}) \cdot (\det A) = 1$

- $\det A^T = \det A$

Sağlaması

$|A^T| = |A|$

$|U^T L^T| = |LU| \quad , \quad |U^T| \, |L^T| = |L| \, |U|$

Lecture 19:

Topics

- Formula for det A
- Cofactor formula
- Tridiagonal matrices

Notes from previous lectures
- $\det I = 1$
- Sign reverse with row exchange
- det is linear in each row seperately

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}$$

$ad - bc$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix}$$

$\rightarrow a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{11} \cancel{a_{21}} a_{33}$

• Formula $\rightarrow$ $\det A = \underset{n!\ \text{terms}}{\sum} \pm\ a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

• cofactor formula:

$\det A = a_{11} C_{11} + a_{12} C_{12} + \cdots a_{1n} C_{1n}$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ - & - & 0 \\ - & - & 0 \end{vmatrix}$$

$a_{12}(a_{21} a_{33} + a_{23} a_{31})$

$$A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$|A_1| = 1 \quad |A_2| = 0 \quad |A_3| = -1 \quad |A_4| = -1$

$|A_4| = |A_3| - |A_2| \qquad |A_5| = 0$

$|A_n| = |A_{n-1}| - |A_{n-2}|$

$|A_6| = 1 \quad |A_7| = 1$

Lecture 20:

Topics

- Formula for A^{-1}
- Cramers Rule for $x = A^{-1}b$
- |Det A| = Volume of box

Formula for A^{-1}

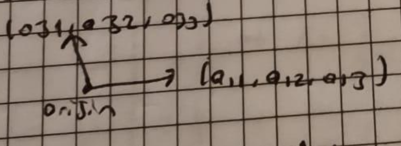
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad A^{-1} = \frac{1}{\det A} \cdot C^{T}$$

Cramer rule was $\rightarrow$ $x_1 = \frac{\det B_1}{\det A}$ $\quad x_J = \frac{\det B_J}{\det A}$

A with column 1 replaced by b

$$B_1 = \begin{bmatrix} b & \begin{matrix} A-1 \\ \text{columns} \\ \text{of } A \end{matrix} \end{bmatrix}, \quad B_J = A \text{ with column } J \text{ replaced by } b.$$

- det A = volume of box



$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

think like its on geographic area.

Final Exam :

Question 1) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix}$

a) Find a lower triangular L and an upper triangular U, that A=LU

b) Find reduced row echelon form R = rref(A).

c) Find basis for the nullspace of A

d) If vector b is sum of 4 columns of A, find solution Ax=b

a) $L \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$ $U \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) $R = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

c) $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

d) $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

Question 2) curves $y = C + D2^t$ which gives the best least squares fit to the points $(t,y) = (0,6), (1,4), (2,0)$.

a) write 3 equations that satisfied if curve went through all 3 points

b) Find coefficients C and D for best curve $y = C + D2^t$

c) What values should y have at times $t = 0,1,2$ so that the best curve is $y=0$?

a) $C + D = 6$ $(c+8d=-2)$

$C + 2D = 4$

$C + 4D = 0$

b)

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix} \quad \begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

c) $p = (0,0,0)$ if $A^Tb = 0$. $\rightarrow$ b = values of $y = c(2, -3, 1)$.

Question 3) Suppose $Av_i = b_i$ for the vectors $v_1 \dots v_n$ and $b_1 \dots b_n$ in R^n. Put the v's into the columns of V and put the b's into the columns of B.

a) Describe the column space of that matrix A in terms of the given vectors.

b) What additional condition on which vectors makes A an invertible matrix?

a) column space of A is all linear combinations of $b \rightarrow b_1 \dots b_n$

b) If b's are independent then B is invertible $\rightarrow$ $A^{-1} = VB^{-1}$

Question 4) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$

a) Find eigenvalues of A^TA and AA^T. Find orthonormal eigenvectors.

b) What is resulting output if you apply Gram-Schmidt process?

c) If A is any $[m \times n]$ matrix with $m > n$, tell why AA^T cannot be positive definite. Is A^TA always this?

Question 5)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

a) find det A ~~and~~

b) Find cofactor C_{11}, then find det B.

c) Find det C for any value of x. (use linearity in row 1)

a) $\det A = 0$ because two rows are equal (row 1 and row 2)

b) $C_{11} = -1$. $\det B = \det A - C_{11} \rightarrow 0 - (-1) = 1$.

c) $\det C = x\,C_{11} + \det B = x \cdot (-1) + 1 \rightarrow 0$