Probabilistic Robotics

Bayesian filtering

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Agenda

Important relations

- Conditional probability
  (the prob that $X = x$ if we know, or assume, that $Y = y$)

$$p(x \mid y) = \frac{p(x,y) \quad \text{joint prob}}{p(y) \quad \text{scaled to fit } y} \qquad \text{note that } p(x,y) \leq p(y)$$

- Joint probability of x and y
  (the prob that both $X = x$ and $Y = y$)

$$p(x,y) = \underbrace{p(x \mid y) \quad p(y)}_{\text{cond prob scaled to fit } \Omega} = p(y \mid x)p(x)$$

- Conditional independence: x and y are independent, given z, iff

$$p(x,y \mid z) = p(x \mid z)p(y \mid z)$$

Causal vs. diagnostic reasoning

- Environment state X: open or closed.
- Robot sensor reading Y: open or closed.
- Assume we know $p(Y = y \mid X = \mathsf{open})$
  (i.e., quality of the sensor — causal knowledge)
- and need to know $p(X = \mathsf{open} \mid Y = y)$
  (i.e., prob that the door is open — diagnostic knowledge)
- Bayes' rule lets us use causal knowledge to infer diagnostic knowledge:

$$p(\mathsf{open} \mid y) = \frac{p(y \mid \mathsf{open})p(\mathsf{open})}{p(y)}$$

- (How to compute $p(y)$? We'll see that later.)

Bayes' formula

$$p(x \mid y) = \frac{p(y \mid x) p(x)}{p(y)}$$

Compare def. of conditional probability:

$$p(x, y) = p(x \mid y) p(y) = p(y \mid x) p(x)$$

Theorem (for discrete RV)

$$p(x \mid y) = \frac{p(y \mid x) p(x)}{p(y)} \overset{\text{law of total prob.}}{=} \frac{p(y \mid x) p(x)}{\sum_{x'} p(y \mid x') p(x')}$$

Bayes' formula, explained

- Prior: $p(x)$ (probability before sensor input)
- Posterior: $p(x \mid y)$ (probability after input = "diagnosis")

- Bayes' rule: probability that x is true given y (the posterior)
  - increases with
    - the prior of x (i.e., prob of x before the test),
    - and the prob of finding y in a world where x is true
  - decreases with
    - the prior prob of finding y (i.e., prob of getting test result y without knowing the state of x)
- The denominator doesn't depend on x, so it's the same for both $p(\text{cancer} \mid \text{pos})$ and $p(\neg\text{cancer} \mid \text{pos})$ and is used to make the posterior $p(x \mid y)$ integrate to 1.

Bayes' formula, robotics example

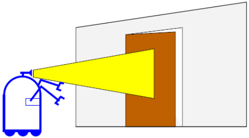
- X: world state, Z: robot measurements.
- Noisy sensors:

$$\left.\begin{aligned} p(Z = \mathsf{open} \mid X = \mathsf{open}) &= 0.6 \\ p(Z = \mathsf{closed} \mid X = \mathsf{open}) &= 0.4 \end{aligned}\right\} \text{hard to sense open door}$$

$$\left.\begin{aligned} p(Z = \mathsf{open} \mid X = \mathsf{closed}) &= 0.2 \\ p(Z = \mathsf{closed} \mid X = \mathsf{closed}) &= 0.8 \end{aligned}\right\} \text{easy to sense closed door}$$

- Prior probabilities

$$p(X = \mathsf{open}) = 0.5$$
$$p(X = \mathsf{closed}) = 0.5$$



State estimation example

- Suppose the robot senses $Z = \text{open}$.
- What is the probability that the door is actually open; that is, $p(X = \text{open} \mid Z = \text{open})$?
- Apply Bayes' formula:

$$
\begin{aligned}
&p(X = \text{open} \mid Z = \text{open}) = \\
&= \frac{p(Z = \text{open} \mid X = \text{open})p(X = \text{open})}{p(Z = \text{open} \mid X = \text{open})p(X = \text{open}) + p(Z = \text{open} \mid X = \text{closed})p(X = \text{closed})} \\
&= \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.2 \cdot 0.5}
\end{aligned}
$$

Law of total probability

Where does the denominator come from? If all y are pairwise disjoint and fill up all of Ω, then

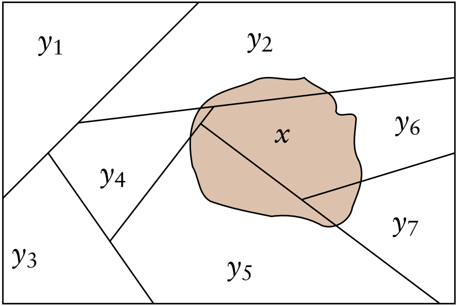
Theorem (Discrete case)

$$p(x) = \sum_y p(x \mid y) p(y) = \sum_y p(x, y)$$

- Follows from the definition of conditional probability and Kolmogorov's axioms.
- Robot state variables fulfil the requirements: Can only be in one state at a time, and all outcomes $= \Omega$.

Law of total probability, illustration

$$p(x) = \sum_{i=1}^{n} p(x, y_i) = \sum_{i=1}^{n} p(x \mid y_i) p(y_i)$$



Law of total probability, proof

- If x occurs, then one of y_i must also occur (since y_i are disjoint and fill Ω).
- So "x occurs" and "both x and one y_i occurs" are equivalent.
- Equivalent to "$\cup_{i=1}^{n}(x \cap y_i)$ occurs".

$$\sum_y p(y) \overset{\text{axiom 1}}{=} 1$$

$$p(x) = \cup_{i=1}^{n}(x \cap y_i) \overset{\text{axiom 3}}{=} \sum_{i=1}^{n} p(x, y_i)$$

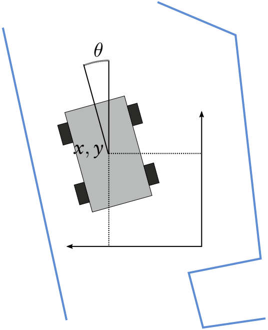
$$p(x) \overset{\text{def. of joint prob.}}{=} \sum_{i=1}^{n} p(x \mid y_i) p(y_i)$$

State

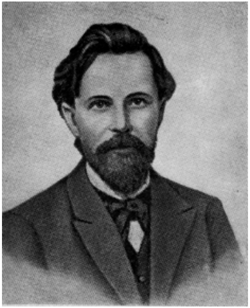
- Description of what the robot needs to know.
- State at time t is denoted x_t.
- State transitions over time:
  $x_0 \rightarrow x_1 \rightarrow \dots$
- The set of all states from time t_1 to time t_2:
  $x_{t_1:t_2} = x_{t_1}, x_{t_1+1}, x_{t_1+1}, \dots, x_{t_2}$

Internal state Typically the pose $[x, y, \theta]$.

External state Map, other agents, etc.



Markov state



The Markov property

The conditional probability distribution of future states depends only upon the present state, not on the sequence of events that preceded it.

In other words, past ($x_{0:t-1}$) and future ($x_{t+1:\infty}$) states are conditionally independent, given the present state x_t.

Markov state, example

Positions of chess pieces is Markov state (complete state), in idealised chess…

…but not in real-world chess!

In reality, complete state descriptions are infeasible.

Measurements

- Sensor input from environment.
- Measurement at time t is denoted z_t.
- Measurements decrease uncertainty.

Actions

- Action at time t is denoted u_t.
- Typical actions:
  - the robot turns its wheels to move,
  - the robot uses its manipulator to grasp an object,
  - do nothing (and let time pass by).
- Note that
  - actions are never carried out with absolute certainty,
  - actions generally increase uncertainty.

Modelling actions

The outcome of an action u is modelled by the conditional probability distribution

$$p(x \mid u, x')$$

That is, the probability that, when in state x', executing action u, changes the state to x.

1. state $x' = [10\,\mathrm{m}, 5\,\mathrm{m}, 0°]$
2. action $u =$ move 1 m forward
3. what is, for example, $p(x = [11\,\mathrm{m}, 5\,\mathrm{m}, 0°])$?
   ($p < 1$ because of wheel slip, etc.)

Belief

- We never know the true state of the robot.
- All we have is the belief.
- Represent belief through conditional probability distribution:
$$\mathrm{bel}(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$$
- A belief distribution assigns a probability density (or mass) to each possible outcome, (given a sequence of actions and measurements).
- Belief distributions are posterior probabilities over state variables, conditioned on the available data.

Prediction vs. belief

Represent belief through conditional probability distribution:

$$\mathrm{bel}(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$$

- Prediction: the belief distribution before incorporating a measurement

$$\overline{\mathrm{bel}}(x_t) = p(x_t \mid z_{1:t-1}, u_{1:t})$$

- Belief: the belief distribution after a measurement

$$\mathrm{bel}(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$$

Bayes filters: framework

- Given:
  1. stream of observations z and action data u

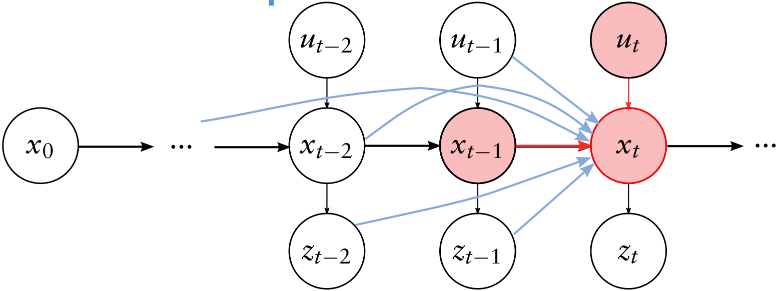
$$\{z_{1:t}, u_{1:t}\} = \{u_1, z_1, \ldots, u_t, z_t\}$$

  2. sensor model $p(z \mid x)$ (how accurate the sensors are)
  3. action model $p(x \mid u, x')$ (how reliable the actuators are)
  4. prior probability of the system state $p(x)$.
- Wanted:
  - estimate of the state x (the belief)

$$\text{bel}(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$$

- Update the belief recursively: $\text{bel}(x_t)$ is computed from $\text{bel}(x_{t-1})$.

Bayes filters: assumptions



Markov assumption implies

- static world
- independent controls
- perfect model — no approximation errors

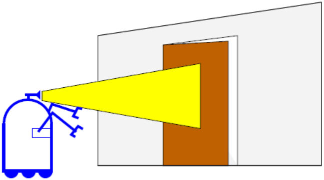
$$p(x_t \mid x_{0:t-1}, z_{1:t}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$ state transition probability

$$p(z_t \mid x_{0:t}, z_{1:t}, u_{1:t}) = p(z_t \mid x_t)$$ measurement probability

State estimation, example

- Robot observing a door
- Given a sensor reading open from the camera, what is the probability that the door is actually open?

$$p(X = \mathsf{open} \mid Z = \mathsf{open})$$



State estimation example, sensor model

- $X_t = \{\text{open}, \text{closed}\}$: world state
- $Z_t = \{\text{open}, \text{closed}\}$: robot measurements.
- Noisy sensors:

$$\left.\begin{aligned} p(Z_t = \text{sense\_open} \mid X_t = \text{open}) &= 0.6 \\ p(Z_t = \text{sense\_closed} \mid X_t = \text{open}) &= 0.4 \end{aligned}\right\} \text{hard to sense open door}$$

$$\left.\begin{aligned} p(Z_t = \text{sense\_open} \mid X_t = \text{closed}) &= 0.2 \\ p(Z_t = \text{sense\_closed} \mid X_t = \text{closed}) &= 0.8 \end{aligned}\right\} \text{easy to sense closed door}$$

State estimation example, actions

Actions $U_t = \{\text{push}, \text{null}\}$

$$\left.\begin{aligned} p(X_t = \text{open} \mid U_t = \text{push}, X_{t-1} = \text{open}) &= 1 \\ p(X_t = \text{closed} \mid U_t = \text{push}, X_{t-1} = \text{open}) &= 0 \end{aligned}\right\} \text{door stays open}$$

$$\left.\begin{aligned} p(X_t = \text{open} \mid U_t = \text{push}, X_{t-1} = \text{closed}) &= 0.8 \\ p(X_t = \text{closed} \mid U_t = \text{push}, X_{t-1} = \text{closed}) &= 0.2 \end{aligned}\right\} \text{can't always open door}$$

$$\left.\begin{aligned} p(X_t = \text{open} \mid U_t = \text{null}, X_{t-1} = \text{open}) &= 1 \\ p(X_t = \text{closed} \mid U_t = \text{null}, X_{t-1} = \text{open}) &= 0 \\ p(X_t = \text{open} \mid U_t = \text{null}, X_{t-1} = \text{closed}) &= 0 \\ p(X_t = \text{closed} \mid U_t = \text{null}, X_{t-1} = \text{closed}) &= 1 \end{aligned}\right\} \text{no other agents}$$

State estimation example, $t = 1$

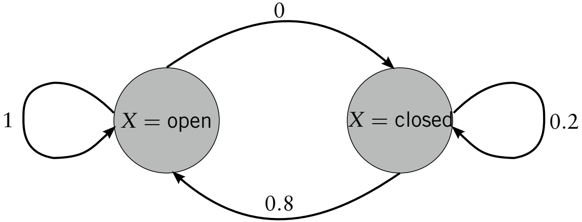
- Suppose at time $t = 1$, the robot takes action $U_1 = \text{null}$ and senses $Z_1 = \text{open}$.
- We want to compute an updated belief distribution $\text{bel}(X_1)$.
- With Bayes' filter, we can do that using the prior belief $\text{bel}(X_0)$.

$$
\begin{aligned}
&\text{bel}(X_1 = \text{open}) \\
&= p(X = \text{open} \mid Z = \text{open}) = \\
&= \frac{p(Z = \text{open} \mid X = \text{open})p(X = \text{open})}{p(Z = \text{open} \mid X = \text{open})p(X = \text{open}) + p(Z = \text{open} \mid X = \text{closed})p(X = \text{closed})} \\
&= \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.2 \cdot 0.5} \\
&= 0.75
\end{aligned}
$$

$$
\text{bel}(X_1 = \text{closed}) = \frac{0.2 \cdot 0.5}{0.6 \cdot 0.5 + 0.2 \cdot 0.5} = 0.25 = 1 - \text{bel}(X_1 = \text{open})
$$

State transisions

$p(x \mid u, x')$ for $u = \mathsf{push}$



- This is a simple two-state Markov chain.
- If the door is closed, the action push succeeds in 80% of the cases.

Integrating the outcome of actions

- We know $p(x \mid u, x')$ (that's our action model).
- How to compute the posterior $p(x \mid u)$? I.e., the resulting belief after the action.
- Integrate over all prior states x'.
- The law of total probability gives us

$$p(x \mid u) = \sum_{x'} p(x \mid u, x') p(x') \qquad \text{discrete case}$$

$$p(x \mid u) = \int p(x \mid u, x') p(x') \, \mathrm{d}x' \qquad \text{continuous case}$$

State estimation example, executing an action

- Suppose at time $t = 2$, the robot takes action $u_2 = \mathsf{push}$.

$p(X = \mathsf{open} \mid u_2)$

$$
\begin{aligned}
&= \sum_{x'} p(X = \mathsf{open} \mid u_2, x') p(x') \\
&= p(X = \mathsf{open} \mid u_2, X = \mathsf{open}) p(X = \mathsf{open}) \\
&+ p(X = \mathsf{open} \mid u_2, X = \mathsf{closed}) p(X = \mathsf{closed}) \\
&= 1 \cdot 0.75 + 0.8 \cdot 0.25 = 0.95
\end{aligned}
$$

$p(X = \mathsf{closed} \mid u_2)$

$$
\begin{aligned}
&= \sum_{x'} p(X = \mathsf{closed} \mid u_2, x') p(x') \\
&= p(X = \mathsf{closed} \mid u_2, X = \mathsf{open}) p(X = \mathsf{open}) \\
&+ p(X = \mathsf{closed} \mid u_2, X = \mathsf{closed}) p(X = \mathsf{closed}) \\
&= 0 \cdot 0.75 + 0.2 \cdot 0.25 = 0.05
\end{aligned}
$$

Combining evidence

- How can we integrate the next observation Z_2?
- More generally, how can we estimate $p(X \mid Z_1, \dots, Z_n)$?

Bayes' rule, with background knowledge

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y)}$$

We can also condition Bayes' rule on additional RVs (background knowledge):

$$p(x \mid y, z) = \frac{p(y \mid x, z)p(x \mid z)}{p(y \mid z)}$$

Recursive Bayesian updating

$$p(x \mid z_1, \ldots, z_t) = \frac{p(z_t \mid x, z_1, \ldots, z_{t-1}) p(x \mid z_1, \ldots, z_{t-1})}{p(z_t \mid z_1, \ldots, z_{t-1})}$$

Markov assumption: z_t is independent of $z_{1:t-1}$ if we know x.
Then we can simplify:

$$p(x \mid z_1, \ldots, z_t) = \frac{\overbrace{p(z_t \mid x)}^{\text{sensor model}} \; \overbrace{p(x \mid z_1, \ldots, z_{t-1})}^{\text{prior}}}{\underbrace{p(z_t \mid z_1, \ldots, z_{t-1})}_{\text{normaliser}}}$$

State estimation example, $t = 2$

After taking action $u_2 = \text{push}$, it senses $z_2 = \text{open}$.

$$
\begin{aligned}
&\mathrm{bel}(X_2 = \text{open}) \\
&= p(X_2 = \text{open} \mid z_1, z_2) = \\
&= \frac{p(z_2 \mid X_1 = \text{open})p(X_1 = \text{open} \mid z_1)}{p(z_2 \mid X_1 = \text{open})p(X_1 = \text{open} \mid z_1) + p(z_2 \mid X_1 = \text{closed})p(X_1 = \text{closed} \mid z_1)} \\
&= \frac{0.6 \cdot 0.75}{0.6 \cdot 0.75 + 0.2 \cdot 0.25} \\
&= 0.90
\end{aligned}
$$

$$
\mathrm{bel}(X_2 = \text{closed}) = \frac{0.2 \cdot 0.25}{0.6 \cdot 0.75 + 0.2 \cdot 0.25} = 0.10 = 1 - \mathrm{bel}(X_2 = \text{open})
$$

The Bayes filter algorithm

- Given
  - the previous belief distribution,
  - the latest action,
  - and the latest sensor measurement,
- compute an updated belief distribution for time t.

function $\textsc{BayesFilter}(\mathrm{bel}(X_{t-1}), u_t, z_t)$
    for all x_t **do**
        $\overline{\mathrm{bel}}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, \mathrm{bel}(x_{t-1})\, \mathrm{d}x_{t-1}$ ▷ control update
        $\mathrm{bel}(x_t) = p(z_t \mid x_t)\, \overline{\mathrm{bel}}(x_t)\, p(z_t)^{-1}$ ▷ measurement update
    end for
    return $\mathrm{bel}(X_t)$
end function

The Bayes filter algorithm explained

- The control update comes from the law of total probability:
  - For all prior states x_{t-1}, sum up (integrate)
    - the product of the prior for x_{t-1}
    - and the prob that u makes the transition from x_{t-1} to x_t.
- The measurement update comes from Bayes rule
  - The prob of getting z_t in x_t
  - times the prior for x_t (after the control update),
  - divided by the prior of z_t, in order to make the total mass of $\mathrm{bel}(x_t) = 1$.

Why can't we use the Bayes filter in reality?

Because we can't compute the update rule for continuous state spaces!

- Because of the integral in the denominator (normaliser) of Bayes' rule
- Because of the integral in the control update

Summary

- Markov assumptions: we don't need history of all previous states.
- Sensor measurements Z decrease uncertainty, robot actions U increase uncertainty.
- Belief is represented as posterior PDF over possible state outcomes, conditioned on sensor data and actions.
- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.
- The Bayes filter cannot be implemented for realistic, continuous, state spaces. (The remainder of the course will discuss approximations.)

Next lecture

Time and space

10.15–12.00, Wednesday April 11
T-111

Reading material

- Thrun et al., Chapters 5 and 6