MTH 299 Portfolio

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1 Statements, Implications, and Quantifiers

1.1 Statements, Examples and Definitions

In this subsection statements are covered. Statements are an integral part of writing in mathematics. A statement is a sentence that must prove to either be true or false. The following are some examples of statements.

- All dogs are beagles. *This is a statement because it can be evaluated to a truth value, false in this case.*
- $15x + 5 = 50$. *This is a conditional statement, since we don't know the value of* x *we don't in fact know whether or not* $15x + 5 = 50$*, but we do know that the two sides either equal each other or do not. So this must either be true or false and therefore it is a statement.*

Examples of Non-Statements:

- Do your math. *This is not a statement, it is a command. It is not true or false. Things similar to this, like questions and explanations are not statements since they can't be determined to be either true or false.*
- $15x+5$. *This is an expression rather than a statement. Above we saw that* $15x+5 = 50$ *is indeed a statement. However here we are missing an equals sign,* $15x+5$ *simply will evaluate to some number based on* x*. This cannot be true or false.*

To detect whether or not a sentence is a statement, determine whether or not the sentence must be either true or false. If it is not known whether or not a statement is true or false, it is still a statement as long as it must be one of the two. Questions, explanations, expressions, commands, and similar sentences are not statements.

1.2 Conditional Statement Example

In this subsection we see a typical misunderstanding of statements and conditional statements. Conditional statements are a type of statement that still conform to the fact that the sentence must be true or false. However with conditional statements we have an unknown included in the statement which makes it so that we cannot actually evaluate the truth value of the statement. Thus the statement is *conditional* because it is based on the value of the unknown.

If we consider the statement, "the 299 quadrillionth digit of π is equal to three," we do not know whether it is true or false, but we do know that the sentence must be one of the two and is therefore a conditional statement. A common misunderstanding with statements is that you must be able to determine the truth value. The necessary part of what makes a statement is that it is either true or false.

1.3 Implication Examples

In this subsection we cover implications. An implication is a statement of the form *if ..., then ...*, where if we have some condition then the following must be true.

One simple example of an implication $P \implies Q$ is the following: *If the sun is shining, then it is not overcast.* One can see that this implication is of course true since the sun cannot be shining unless it is not overcast. The converse, $Q \implies P$, of this statement is: *If it is not overcast, then the sun is shining.* Which is not necessarily true, it may not be overcast but it could be night, in which case the sun is not shining, so the converse implication proves false.

Now we will examine an example where both the original and converse implications prove true. Let our implication, $R \implies S$, be: *If the sun is shining, then it is day and not overcast.* One can again see that this must be true since it must be that if the sun is shining the conditions are that it must be daytime and not overcast. Here however we added in the additional condition that it is daytime. So when we take the converse, $S \implies R$, we have that: *If it is day and it is not overcast, then the sun is shining.* Which must be true since those conditions together guarantee that the sun is out.

1.4 Quantifiers

In this subsection we cover quantifiers. Quantifiers $\forall, \exists$ are used with statements and specify something about a set. $\forall$ means we observe all elements in the specified set. For example $\forall n \in \mathbb{N}, n > 5$ means that for all n in the set of natural numbers, n is greater than 5, which is false. $\exists$ means we consider if something holds for a single element in a set. For example $\exists n \in \mathbb{N}, n < 5$ means that there exists n in the set of natural numbers, such that n is less than 5, which is true.

To negate a statement with a quantifier you do the following:

1. Flip the quantifier, but do not modify the corresponding set. So if you have a $\forall$ quantifier it would become an $\exists$ quantifier in the negation, and vice versa.
2. Then you negate the other parts of the statement like usual.

For example, the statement $\forall x \in X, \exists y \in Y$ such that, $x < 5y$ has the negation $\exists x \in X$ such that, $\forall y \in Y$, $x \geq 5y$.

To avoid mistakes when negating a statement with a quantifier do not do anything besides flip the quantifiers and negate the actual statement itself. The sets should not change. $\exists$ does not mean unique, it means that at least one element satisfies the statement. Similarly, $\forall$ means that for all elements, the statement holds.

Another common source of errors is with the ordering of quantifiers. When mixing $\forall, \exists$ quantifiers make sure that close attention is paid to the order of the quantifiers, as different orders of the quantifiers makes for entirely different statements. In the example $\forall x \in X, \exists y \in Y$ such that, $x < 5y$ means that for all $x \in X$, we can find some y which satisfies $x < 5y$. However if we swap the position of the quantifiers, we have $\exists y \in Y$ such that, $\forall x \in X, x < 5y$. Which says that we can find a y where $x < 5y$ holds for all values of $x \in X$. This difference is extremely subtle but is very important because it completely changes what the statement says.

2 Direct Proof

2.1 Direct Proof Steps And Example

In this subsection we go over the direct proof method. Proving something via direct proof includes a few general steps. They are listed below:

1. Give the assumptions, including such things as what set a variable belongs to.
2. Then write the hypothesis of the implication. For example if we have the implication $P \implies Q$, give P at the start of the proof.
3. Take stock of useful definitions, theorems, and lemmas that you can use to help prove the statement.
4. Start from what you know and work your way to what we want to prove i.e Q.
5. Close the proof by giving a final comment on why this proves the implication.

Proposition 2.1. *Given that $x \in \mathbb{R}$, if $x^2 + 2 < 6$, then $-2 < x < 2$.*

Proof. Assume $x \in \mathbb{R}$ and $x^2 + 2 < 6$.

$$\begin{aligned} x^2 + 2 &< 6 \\ x^2 - 4 &< 0 \\ (x+2)(x-2) &< 0. \end{aligned}$$

So $[(x+2) < 0$ and $(x-2) > 0]$ or $[(x+2) > 0$ and $(x-2) < 0]$.

So ($x < -2$ and $x > 2$) or ($x > -2$ and $x < 2$). The first and statement does not make sense because x cannot be greater than 2 and less than -2.

So $x > -2$ and $x < 2$,

Therefore $-2 < x < 2$. So the proposition is proven.

□

2.2 Direct Proof Of A Statement

In this subsection we cover proving a statement rather than an implication via direct proof. Proving a statement rather than an implication via direct proof is slightly different, as seen in the example below.

Proposition 2.2. *There exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R}, 0 < f(x) < f'(x)$.* ***Source: Homework 3, Questions 1.a and 2.a.***

Proof. Let f be a function where $f : \mathbb{R} \to \mathbb{R}$. Choose $f(x) = e^{2x}$.

$$0 < e^{2x}$$

Since e is a positive number raised to a power, it cannot be negative.

$$f'(x) = 2e^{2x}$$

$$e^{2x} < 2e^{2x}$$

Since $\forall x \in \mathbb{R}, e^{2x} > 0$, then 2 times that term must be greater.

So $0 < f(x) < f'(x)$ holds for this function so there does exist a function that holds for this condition. □

2.3 Direct Proof vs. Indirect Proof Strategies

A direct proof is typically a good idea for proving a statement (not an implication) when we want to prove existence. That is we just want to find as least one variable where the statement holds. Proving that a statement holds for all numbers in a set via direct proof is usually very difficult.

As an example the statement $\exists x, y \in \mathbb{R}$, such that $\forall n \in \mathbb{N}, nx \leq xy+3$ is a good candidate for direct proof since we can choose x and y to easily to prove this statement. However if we were presented with the negation of this statement $\forall x, y \in \mathbb{R}, \exists n \in \mathbb{N}, nx > xy+3$ and asked to prove it, then the direct proof method would not be a good option because we would have to somehow show that for all x and y there exists n that satisfies the statement. This is difficult to do via direct proof, because we can no longer pick an x and y.

2.4 Common Mistakes In Direct Proofs

The following are some typical logical fallacies and mistakes that can occur when doing direct proofs:

1. Assuming the conclusion is true. For example if we want to prove $x+2 < 0 \implies x^2 < 4$ we can't assume $x^2 < 4$ and then show that $x + 2 < 0$.
2. Division by 0 is not allowed.
3. Multiplying by a variable that could be negative, $\frac{1}{x} < 8$ does not imply $1 < 8x$, we would have that $1 < 8x$ or $1 > 8x$.
4. Squaring both sides can lead to errors. For example $\sqrt{2-x} = x$ does not imply $2 - x = x^2$.
5. Be careful with even powers since the sign is not preserved. For example:

   $$(x-2)^2 > 0$$

   does not imply $x - 2 > 0$.

3 Proof by Contradiction and Contrapositive

3.1 Contrapositive Proof Method

This subsection covers the common contrapositive proof method. In a proof by contrapositive we want to start from the negation of the conclusion and prove the negation of the hypothesis. Proving this proves the original since the contrapositive is logically equivalent. The general steps of a proof by the contrapositive:

1. Make a note of what $\neg Q$ means and of any relevant definitions, theorems, and lemmas.
2. Apply any definitions, theorems, propositions, or lemmas you need to get a definition for your hypothesis that allows you to work the problem.
3. Figure out a way to get from $\neg Q$ to $\neg P$.
4. Conclude the proof with something like *"... so the implication is proven by the contrapositive."*

Examples:

1. If $n^2 + 2$ is odd, then n is odd. *This is a good candidate for proof by contrapositive because it allows us to immediately say n is even, which gives us a property of n and we can then construct the number $n^2 + 2$, since we know $n = 2l$, for some $l \in \mathbb{Z}$. Then we can easily show that $n^2 + 2$ would be even.*

2. Given that $x \in \mathbb{R}$, if $2x - 2 > 0$ then $x > 1$. *While we certainly could do a direct proof here, a proof by the contrapositive simplifies matters by giving us a nice property for x by itself rather than having x included in an expression and needing to solve for it.*

3. Suppose that $a, b \in \mathbb{R}$. If ab is irrational, then a or b is irrational. *This is a common example where proof by contrapositive is usually required to prove this statement because we can in no way define ab, and trying to directly prove that a number is irrational is exceedingly difficult.*

We will prove example 1. So we want to start by assuming that n is even, and show that $n^2 + 2$ is even.

Proposition 3.1. *If $n^2 + 2$ is odd, then n is odd.*

Proof. **(Step 1)** Assume n is even.
(Step 2) So $\exists k \in \mathbb{Z}$ such that $n = 2k$.
(Step 3)

$$\begin{aligned} n^2 + 2 &= (2k)^2 + 2 \\ &= 4k^2 + 2 \\ &= 2(2k^2 + 1) \\ &= 2m. \end{aligned}$$

Where $m = 2k^2 + 1 \in \mathbb{Z}$.
(Step 4) So by definition $n^2 + 2$ is even.
So the proposition is proven by the contrapositive. □

3.2 Contrapositive Method vs. Contradiction Method

In this subsection we give comparisons between the contrapositive and contradiction proof methods. The contrapositive method of proof and proof by contradiction are in many ways similar but also vary in a few crucial ways. A proof by contrapositive and proof by contradiction are different in a few ways. Proof by contradiction can be used on either an implication or a statement; however the contrapositive can only be used to prove implications since the contrapositive only makes sense within the context of implications.

The two methods do have something in common, however. In both methods you negate part of the implication or statement in order to acquire information that makes it easier to prove the implication or statement.

If you want to prove a statement via contradiction you negate the statement and try to arrive at a conclusion that makes no sense. Proving an implication via contradiction is slightly different. If you had the implication $P \implies Q$ for example, you start your proof with $P \wedge \neg Q$, so you get to assume the negation of the conclusion and then show that something about these two things won't make sense. To prove an implication, $P \implies Q$, via the contrapositive we instead want to essentially do a direct proof of $\neg Q \implies \neg P$.

Proof by contradiction and proof by contrapositive may be likely to get confused. One thing to remember to keep in mind is that the contradiction method may be used for both statements and implications. However the contrapositive can only be used for implications.

Another point of confusion may focus on the fact that both techniques center on negating statements to gain more information. However the key difference is that we negate a statement or the conclusion of an implication in the contradiction method ultimately because we want to show that some assumption must be violated. We don't have an exact statement that we want to get to.

In the contrapositive method we are negating the two parts of the implication and starting from the conclusion and want to show that we can get to the original hypothesis just like in a direct proof. This is ultimately because the contrapositive and original implication are logically equivalent.

3.3 Proof By Contradiction Example

In this subsection we go over an example where contradiction is the best choice for our method of proof rather than contradiction.

Proposition 3.2. *Let $x, y \in \mathbb{R}$. If x is irrational and $y \neq 0$ and y is rational, then xy is irrational.* ***Source: Class Material***

Proof by contradiction is a far better choice in this case because we can assume that xy is rational and show how this would make no sense. In a contrapositive proof we would struggle to complete the proof because we would assume that xy is rational, but we would have to split xy to show that x is rational and y is irrational and $y = 0$. We have no good way to separate xy however. In the contradiction proof, we get to keep our original hypothesis which allows us to do this.

Proof. Let x be irrational and y be rational where $y \neq 0$. So $\exists p, q \in \mathbb{Z}, p \neq 0, q \neq 0$, such that $y = \frac{p}{q}$. We also assume the negation of the conclusion so we assume xy is rational, and therefore, $\exists r, s \in \mathbb{Z}, s \neq 0$, such that $xy = \frac{r}{s}$.

$$\begin{aligned} xy &= \frac{r}{s} \\ x\frac{p}{q} &= \frac{r}{s} \\ x &= \frac{rq}{sp} \\ x &= \frac{l}{m}. \end{aligned}$$

Where $l = rq \in \mathbb{Z}$ and $m = sp \in \mathbb{Z}, m \neq 0$, since $s \neq 0$ and $p \neq 0$. The multiplication by q and division by p is okay since $p \neq 0$. So by definition x is rational, but this is a contradiction since we assumed x to be irrational. So the proof by contradiction is complete. $\square$

4 Proof by Induction

4.1 Proof By Induction Method

In this subsection we cover the proof methods of induction and strong induction. A proof by induction has a few general steps used in each proof.

1. Always start by stating your statement. For example if you want to prove $\forall n \in \mathbb{N}$, $n \geq 7$, $n^2 \leq 2^{n-1}$. Then start the proof by stating $P(n)$ is the statement $n^2 \leq 2^{n-1}$.

2. Then you must provide a number of base cases which will allow you to do the inductive step. In the case of normal induction it must be proven that $P(n)$ holds for the first value of n. However, if you need to utilize strong induction in which you can use more than one prior statement in the inductive step, you must prove that the base case holds for the number of starting values equal to the number of prior statements you want to use in the inductive step. For example, if you wanted to use the prior statements $P(k-1)$ and $P(k)$ in your inductive step, then you would have to show $P(n)$ holds for the first two values. Continuing with our example from step 1, you would prove that $n^2 \leq 2^{n-1}$ for $n = 7$.

3. Finally, you have to complete the inductive step. This is where you assume $P(k)$ holds and then seek to prove that $P(k+1)$ also holds. You have to use the fact that those prior statements hold in order to show $P(k+1)$ is true. In the case of strong induction, you can use as many prior statements as desired so long as you proved that the first number of such values holds in your base cases. Continuing with our example, we would assume $P(k)$ holds for some $k \in \mathbb{N}, n \geq 7$, we could then use that fact in order to show $P(k+1)$ holds.

Proof by induction works because proving a base case allows you to make a base assumption that lets you prove the general case. For example assume we proved a base case for some integer $n = 1$ where $n \in \mathbb{N}$. Now we want to prove that whatever we are proving holds for all natural numbers. We want to prove $P(k+1)$ and let $k = 1$. We assume $P(1)$ is true which is okay since it is our base case, we can use that information, and then prove that $P(2)$ is true. But that means that we could also try to prove $P(3)$ in a similar way since $P(2)$ was also true. Thus, if we make the statement arbitrary after the base case we can prove all examples at once.

Proof by induction is generally best for proving statements for a lot of integer values in a consistent sequence with a clear base case. For example proving a number divides some function $f : \mathbb{N} \to \mathbb{N}$, $f(n) = 2n$, $\forall n \in \mathbb{N}$. Or proving that a series can be calculated by an expression for a range of integers. Also this works well for proving sequences like the fibonacci sequence.

4.2 Examples Of Induction And Strong Induction

In this subsection we give examples of proofs using induction and strong induction.

Induction:

Proposition 4.1. *Show that $8|9^n - 1, \forall n \in \mathbb{N}$.* ***Source: Class Material***

Proof. Let $P(n)$ be the statement $8|9^n - 1$.

Base Case (n = 1)

$9^1 - 1 = 8$ and $8|8$ so $P(1)$ holds.

Inductive Step

Assume $P(k)$ is true for some $k \in \mathbb{N}$. So $8|9^k - 1$, which implies $\exists l \in \mathbb{Z}$, such that $9^k - 1 = 8l$.

$$\begin{aligned}
9^{k+1} - 1 &= 9(9^k) - 1 \\
&= (8+1)(9^k) - 1 \\
&= 8(9^k) + 9^k - 1 \\
&= 8(9^k) + 8l \\
&= 8(9^k + l) \\
&= 8m.
\end{aligned}$$

Where $m = 9^k + l \in \mathbb{Z}$. And so $8 \mid 9^{k+1} - 1$. So $P(k+1)$ is true. So by induction, $P(n)$ is true $\forall n \in \mathbb{N}$. □

Strong Induction:

Proposition 4.2. *Let $x_1 = 3$, $x_2 = 5$, and $x_n = 3x_{n-1} - 2x_{n-2}$. Prove $x_n = 2^n + 1, \forall n \in \mathbb{N}$.* ***Source: Class Material***

Proof. Let $P(n)$ be the statement $x_n = 2^n + 1$.

Base Cases ($n = 1, 2$)

From the sequence $x_1 = 3$ and from the formula $x_1 = 2^1 + 1 = 3$, so $P(1)$ is true.

From the sequence $x_2 = 5$ and from the formula $x_2 = 2^2 + 1 = 5$, so $P(2)$ is true.

Inductive Step

Assume $P(k-1)$ and $P(k)$ are true for some $k \in \mathbb{N}$ and $k \geq 2$. So $x_{k-1} = 2^{k-1} + 1$ and $x_k = 2^k + 1$.

$$\begin{aligned}
x_{k+1} &= 3x_k - 2x_{k-1} \\
&= 3(2^k + 1) - 2(2^{k-1} + 1) \\
&= 3(2^k) + 3 - 2(2^{k-1}) - 2 \\
&= 3(2^k) - 2^k + 1 \\
&= 2(2^k) + 1 \\
&= 2^{k+1} + 1.
\end{aligned}$$

So $P(k+1)$ is true. By strong induction $P(n)$ is true $\forall n \in \mathbb{N}$. □

4.3 Common Mistakes With Induction

In this subsection we cover common mistakes with the proof by induction method. A common issue with proof by induction is forgetting the base case. By not proving the base case, the proof is starting from a faulty assumption and so it does not make sense that you would be able to assume the prior statement during the inductive step. For example, if you had the statement $8 \mid 8^n, \forall n \in \mathbb{Z}_{\geq 0}$ and forgot the base case you would be able to carry out the inductive step. But by not carrying out the base case a critical step is missed which results in an invalidation of the proof since when $n = 0$, $8^n = 1$, and 8 does not divide 1. So really $8 \mid 8^n, \forall n \in \mathbb{N}$.

Another issue that can arise in inductive proofs, is not proving $P(n)$ for all possible values of n, and instead just checking the statement for many values of n and then saying it is true. For example if we have the statement $9999 - x \geq 0, \forall x \in \mathbb{N}$ and then checked values of x up to 10, it does not prove this statement. Indeed once x reaches the value 10000, the statement proves false. We must assume the general case and use the inductive step. Taking a few examples does not prove the statement.

5 More Mathematics

5.1 Set Operations

In this subsection we define mathematical set operations and give examples. The set operations are defined as follows.

Union, denoted $\cup$, means to take all elements from two input sets and combine them into one set. Examine the following example.

$$A = \{Cat, Dog, Hawk\}$$
$$B = \{Eagle, Hawk, Falcon\}$$

In this case $A \cup B$ is as follows:

$$A \cup B = \{Cat, Dog, Eagle, Hawk, Falcon\}$$

Intersection, denoted $\cap$, means to take the elements that are in common to two input sets and combine them into a set. See the following example.

$$A = \{Cat, Dog, Hawk\}$$
$$B = \{Eagle, Hawk, Falcon\}$$

In which case $A \cap B$ is as follows:

$$A \cap B = \{Hawk\}$$

Difference, denoted $A \backslash B$ where A and B are sets, takes the elements from the first set that are not in the second set and puts them in a new set. See the following example.

$$A = \{Cat, Dog, Hawk\}$$

$$B = \{Eagle, Hawk, Falcon\}$$

In which case $A \backslash B$ is:

$$A \backslash B = \{Cat, Dog\}$$

5.2 Functions

In this subsection we go over functions. The definition of a function is a mapping that maps all values in a set called the domain to a unique value in a set called the co-domain. A function f for example is denoted $f : X \to Y$, where X is the domain and Y is the co-domain.

The following addresses some misconceptions one might have about functions. In order for something to actually satisfy the definition of a function, a few requirements need to be met. Each value of the domain must map to a unique value which must be in the co-domain. Values in the domain cannot map to multiple values. It is okay for multiple values to be mapped to in the co-domain. You do not need to consider whether or not the function is well defined for numbers outside of the domain.

For example $f(x) = 2x$ where $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is well defined and a function with this domain and co-domain, but if $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ then this does not constitute a function since values less than 0 will map to values less than 0 which are not in the co-domain.

5.3 Euclidean Algorithm

In this subsection we cover the Euclidean algorithm which is used to find the greatest common denominator of two numbers. In this example we want to find the greatest common denominator of 786 and 504.

First we want to use the division lemma to find the multiple of 504 that is closest to 786 and less than 786. And then write 786 using the division lemma $y = qx + r$. Where q is the quotient and r is the remainder. We see that the closest multiple of 504 that is less than 786 is 504 itself. We find the remainder of 282 as well and write 786 as a sum of the two numbers.

$$786 = 1(504) + 282$$

Now we take x (504), and r (282) and want to write 504 as y and x as 282 in the division lemma. Again we see that the closest multiple less than 504 of 282 is 282 itself. We find the remainder of 222.

$$504 = 1(282) + 222$$

Then we follow this pattern until we reach a division lemma with a remainder of 0.

$$282 = 1(222) + 60$$

$$222 = 3(60) + 42$$

$$60 = 1(42) + 18$$

$$42 = 2(18) + 6$$

$$18 = 3(6) + 0.$$

We see that we now have 0 as the remainder. We take whatever is in the x place, 6 in our case, as the greatest common denominator.

5.4 Cardinalities And Infinite Sets

In this subsection we cover cardinality of sets, particularly with respect to infinite sets, and some common mistakes regarding this. The concept of cardinality, while appearing to be a simple concept on the surface, becomes a complex concept when the subtle aspects are considered. For example, as cardinality is defined, the cardinality of the sets $\mathbb{N}$ and $\mathbb{N} \cup \{0\}$ are actually the same. This appears strange at first, since $\mathbb{N} \cup \{0\}$ has zero while $\mathbb{N}$ does not. We would expect $\mathbb{N} \cup \{0\}$ to have a cardinality 1 more than that of the cardinality of $\mathbb{N}$. However within the context of infinite sets this is not how we define this.

We can define sets as countably finite, countably infinite, and finally uncountably infinite. Since $\mathbb{N}$ is countably infinite when we add 0 to the set, the cardinality is still the same. As long as a bijection can be defined between the two sets we say the cardinality of the two sets is equal. And when there is a bijection between a set A and the set of natural numbers, then A is countably infinite. Rather than thinking about the actual elements we have in a set we solely focus on creating a bijection.

For equality of the cardinality of two sets A, B we want to define a bijection, $f : A \to B$ between the two sets. This shows $|A| = |B|$. Similarly if we have sets A and B and can define an injection $f : A \to B$, then $|A| \leq |B|$. And if a surjection can be given between two sets $f : A \to B$, then $|A| \geq |B|$.

The core idea with infinite sets is not to try to think that this set or that set seems to have fewer elements or more. In a similar example to the natural numbers idea above, while it may seem that there would be more rational numbers than integers, the sets have the same cardinality. This is a prime example of why cardinality of sets cannot be thought of in the typical way.

Finally, take further caution when dealing with uncountable sets. The set $[0, 1]$ looks to be a small set. However this set still has an infinite cardinality. Always be cautious when thinking about intervals of real numbers.