Cambridge Advanced Subsidiary Level Notes
9709 Mathematics

Abrar Faiyaz Rahim, degrees pending

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1.1 Algebra

Understand the meaning of $|x|$, sketch the graph of $y = |ax+b|$ and use relations such as $|a| = |b| \Leftrightarrow a^2 = b^2$ and $|x-a| < b \Leftrightarrow a-b < x < a+b$ when solving equations and inequalities

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Solutions of Equations

The *modulus* of a number is its magnitude, disregarding its sign, the modulus of x is $|x|$ and hence, $|-a| = a$. The modulus of a number is also called its *absolute value*, defined formally below:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

For $|x| = k$, it is true that $x = \pm k$. This applies for any expression, including, but not limited to:

$$|ax+b| = k \Rightarrow ax+b = \pm k$$

$$|ax+b| = cx+d \Rightarrow ax+b = \pm(cx+d)$$

It can be proven that, $|ax+b| = |cx+d| \Rightarrow ax+b = \pm(cx+d)$

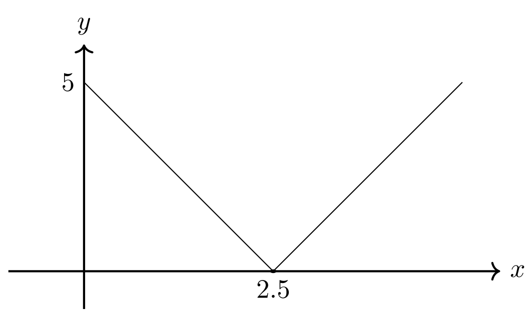
Graphs

The graph $y = |f(x)|$, where $f(x) = ax+b$, (linear), has the following characteristics:

$$y = |b|, \quad \text{for } x = 0$$

$$x = -\frac{b}{a}, \quad \text{for } y = 0$$

For example, the graph of $y = |2x-5|$:



Note that these graphs can be subject to transformations discussed in Section 1.2.

Solutions of Inequalities

Equations of the form $|cx - a| < b$, can be solved algebraically using the fact that:

$$\begin{aligned} &|cx - a| < b \\ \implies &-b < cx - a < b \\ \implies &(a - b)/c < x < (a + b)/c \end{aligned}$$

Graphically, the graphs of the expressions on either side may be drawn, and the ranges for which the condition of the given question stands can be deduced intuitively.

Inequalities where there are modulus functions on both sides of the inequality symbol may be solved using $|a| = |b| \Rightarrow a^2 = b^2$, where the $=$ symbol can be replaced with any inequality symbol. The more general method to solve such inequalities is to draw out the graphs of the two functions, and to to find the range for which the inequality stands.

Divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero)

A polynomial is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_2 x^2 + a_1 x^1 + a_0$$

where x is a variable; n is a non-negative integer; the coefficients $a_n ... a_0$ are constants, of which $a_n \neq 0$, called the *leading coefficient* and a_0 is the *constant term.* The highest power of x in a polynomial is called its *degree.*

Long Division

The division of the polynomial $2x^3 + 3x^3 - 2x + 5$ by $x + 2$ is shown as follows:

$$\begin{array}{r|l} & 2x^2 \quad - x \\ \hline x + 2) & 2x^3 + 3x^2 - 2x + 5 \\ & \underline{-2x^3 - 4x^2} \\ & \qquad\quad -x^2 - 2x \\ & \qquad\quad \underline{x^2 + 2x} \\ & \qquad\qquad\qquad\quad 5 \end{array}$$

The above shows that the quotient of this division is $2x^2 - x$ and the remainder is 5.

Use the factor theorem and the remainder theorem

Observe the following long division:

$$
\begin{array}{rl}
 & \; x^2 \quad + 4x - 3 \\ \cline{2-2}
x-2 & \big) \; x^3 + 2x^2 - 11x + 6 \\
 & \;\; - x^3 + 2x^2 \\ \cline{2-2}
 & \qquad\quad 4x^2 - 11x \\
 & \qquad\;\; - 4x^2 \; + 8x \\ \cline{2-2}
 & \qquad\qquad\qquad - 3x + 6 \\
 & \qquad\qquad\qquad\quad 3x - 6 \\ \cline{2-2}
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

since the remainder here is zero, $x^3 + 2x^2 - 11x + 6$ is divisible by $x - 2$, which we can write as the following:

$$(x-2)(x^2 + 4x - 3) = x^3 + 2x^2 - 11x + 6$$

Now, if we write $f(x) = x^3 + 2x^2 - 11x + 6 = (x-2)(x^2 + 4x - 3)$, we see that $f(2) = 0$. In general, when an expression $f(x)$ is divisible by a linear expression $ax - b$, $f(b/a) = 0$. This is the *factor theorem*.

Observe the following long division:

$$
\begin{array}{rl}
 & \;\; 2x^2 \quad - x \\ \cline{2-2}
x+2 & \big) \; 2x^3 + 3x^2 - 2x + 5 \\
 & \;\; - 2x^3 - 4x^2 \\ \cline{2-2}
 & \qquad\quad - x^2 - 2x \\
 & \qquad\qquad x^2 + 2x \\ \cline{2-2}
 & \qquad\qquad\qquad\qquad 5
\end{array}
$$

the results of the above we may display as

$$(x+2)(2x^2 - x) + 5 = 2x^3 + 3x^2 - 2x + 5$$

same as above, we write $f(x) = 2x^3 + 3x^2 - 2x + 5 = (x+2)(2x^2 - x) + 5$, we find $f(-2) = 5$. In general, an expression $f(x)$ when divided by a linear expression $ax - b$ leaving remainder R has $f(b/a) = R$. This is the *remainder theorem*.

Recall an appropriate form for expressing rational functions in partial fractions, and carry out the decomposition, in cases where the denominator is no more complicated than

- $(ax+b)(cx+d)(ex+f)$
- $(ax+b)(cx+d)^2$
- $(ax+b)(cx^2+d)$

Algebraic improper fractions

An *algebraic improper fraction* is such that $P(x)/Q(x)$ where the degree of P is greater than or equal to that of Q.

We observe the example of $(x^3-3x^2+7)/(x-2)$:

$$\begin{array}{r|l}
 & x^2 \quad - x - 2 \\
\hline
x-2 & x^3 - 3x^2 \qquad + 7 \\
 & -x^3 + 2x^2 \\
\hline
 & -x^2 \\
 & x^2 - 2x \\
\hline
 & -2x + 7 \\
 & 2x - 4 \\
\hline
 & 3
\end{array}$$

thus,

$$x^3-3x^2+7 = (x^2-x-2)(x-2)+3$$
$$\implies \frac{x^3-3x^2+7}{x-2} = x^2-x-2+\frac{3}{x-2}$$

In general, the improper function $P(x)/Q(x)$, with quotient q and remainder R can be written:

$$\boxed{\frac{P(x)}{Q(x)} = q + \frac{R}{Q(x)}}$$

Partial Fractions

For fractions whose denominator consists of two or more distinct linear factors:

$$\boxed{\frac{px+q}{(ax+b)(cx+d)(\ldots)} \equiv \frac{A}{ax+b} + \frac{B}{cx+d} + \ldots}$$

For fractions whose denominator consists of a linear repeated factor:

$$\boxed{\frac{px+q}{(ax+b)^2} \equiv \frac{A}{ax+b} + \frac{B}{(ax+b)^2}}$$

For fractions with a denominator consisting of the product of a linear factor and a depressed quadratic:

$$\boxed{\frac{P(x)}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}}$$

For fractions with an indivisible quadratic denominator:

$$\boxed{\frac{px+q}{(ax+b)(cx^2+d)} \equiv \frac{A}{ax+b} + \frac{Bx+C}{cx^2+d}}$$

The left hand side of the above may be added algebraically, expanded and the denominators cancelled, leading to an equation from which the values of the numerators can be found by means of comparing the coefficients.

Use the expansion of $(1+x)^n$*, where* n *is a rational number and* $|x| < 1$.
