Cambridge Advanced Subsidiary Level Notes
9231 Further Mathematics

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Contents

1 Further Pure Mathematics 1 (for Paper 1)

1.1 Roots of polynomial equations

Recall and use the relations between the roots and coefficients of polynomial equations

Quadratics

The quadratic equation follows,

$$\begin{aligned} ax^2 + bx + c &= 0 \\ \implies x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \end{aligned}$$

which has roots α and β,

$$\begin{aligned} (x-\alpha)(x-\beta) &= 0 \\ \implies x^2 - (\alpha+\beta)x + \alpha\beta &= 0 \end{aligned}$$

hence, considering that, $S_n = \alpha^n + \beta^n$,

$$\boxed{\Sigma\alpha = S_1 = \alpha + \beta = -\frac{b}{a}}$$

$$\boxed{\Sigma\alpha\beta = \alpha\beta = \frac{c}{a}}$$

where $\Sigma\alpha$ and $\Sigma\alpha\beta$ are referred to as *sum of roots* and *product of roots*, respectively.

Furthermore,

$$\boxed{\Sigma\alpha^2 = S_2 = \alpha^2 + \beta^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta}$$

$$\boxed{\Sigma\frac{1}{\alpha} = S_{-1} = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{\Sigma\alpha}{\Sigma\alpha\beta}}$$

$$\boxed{\Sigma\frac{1}{\alpha^2} = S_{-2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2} = \frac{\Sigma\alpha^2}{(\Sigma\alpha\beta)^2}}$$

Using S_n as defined above, we can use known values of S_n to find values required.

> Consider the quadratic equation $ax^2+bx+c=0$ with roots α and β. To find certain values of S_n, we must multiply the equation with a certain power of x, to achieve a power of x in the equation that equals the value of n desired.

For example, when we want S_{-1}, we must first multiply the original equation by x^{-1}, giving us,

$$ax + b + \frac{c}{x} = 0$$

We can now plug $x = \alpha$ and $x = \beta$, since they are roots,

$$\begin{aligned} a\alpha + b + \frac{c}{\alpha} &= 0 && (1) \\ a\beta + b + \frac{c}{\beta} &= 0 && (2) \\ a(\alpha+\beta) + 2b + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) c &= 0 && (1+2) \\ \implies aS_1 + 2b + cS_{-1} &= 0 \end{aligned}$$

here, if we know S_1, we can find S_{-1}.

Note that, the coefficients are multiplied by S_n, where n equals the power of x to which they are a coefficient, and the constant terms are multiplied by 2.

Cubics

The cubic equation follows,

$$\begin{aligned} ax^3 + bx^2 + cx + d &= 0 \\ \implies x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} &= 0 \end{aligned}$$

which has roots α, β, and γ,

$$\begin{aligned} (x-\alpha)(x-\beta)(x-\gamma) &= 0 \\ \implies x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma &= 0 \end{aligned}$$

thus, using $S_n = \alpha^n + \beta^n + \gamma^n$,

$$\boxed{\Sigma\alpha = S_1 = \alpha + \beta + \gamma = -\frac{b}{a}}$$

$$\boxed{\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}}$$

$$\boxed{\Sigma\alpha\beta\gamma = \alpha\beta\gamma = -\frac{d}{a}}$$

Following this,

$$\boxed{\Sigma\alpha^2 = S_2 = (\alpha+\beta+\gamma)^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta}$$

$$\boxed{\Sigma\alpha^3 = (\Sigma\alpha)^3 - 3\Sigma\alpha\beta\Sigma\alpha + 3\Sigma\alpha\beta\gamma}$$

> Here too, values of S_n can be found in the method demonstrated in the **Quadratics** section. Note that, in such cases the constant term is multiplied by 3.

Quartics

The quartic equation

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$$

has roots α, β, γ and δ. Due to the monstrous nature of the equation, it is best to use $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$ notation.

$$\boxed{\Sigma\alpha = -\frac{b}{a}}$$

$$\boxed{\Sigma\alpha\beta = \frac{c}{a}}$$

$$\boxed{\Sigma\alpha\beta\gamma = -\frac{d}{a}}$$

$$\boxed{\Sigma\alpha\beta\gamma\delta = \frac{e}{a}}$$

$$\boxed{S_2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta}$$

$$\boxed{S_{-1} = \frac{\Sigma\alpha\beta\gamma}{\Sigma\alpha\beta\gamma\delta}}$$

Use a substitution to obtain an equation whose roots are related in a simple way to those of the original equation

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If we are given a known polynomial whose roots we also know, we can find any other unknown polynomial given that its roots are in terms of the roots of the first given polynomial. We must first write out the unknown polynomial in factorised form as per its roots and compare with the initial polynomial, finding values of the roots and arriving at the unknown polynomial. This is illustrated in the following:

> *Given the quadratic $x^2 + 3x + 5 = 0$ with roots α and β, find the quadratic that has roots 2α and 2β.*

The unknown quadratic is

$$\begin{aligned} (y-2\alpha)(y-2\beta) &= 0 \\ \implies y^2 - (2\alpha+2\beta)y + 4\alpha\beta &= 0 \end{aligned}$$

comparing with the original, we find

$$\alpha+\beta = -3$$

$$\alpha\beta = 5$$

Solving simultaneously, we find $\boxed{y^2+6y+20=0}$ to be the quadratic asked for in the question.

Alternatively, we can derive a relationship between the roots of both equations. For the same question as above:

The new quadratic has $y = 2x \implies x = y/2$ since each root of the new quadratic is twice that of the given one. Thus, plugging the above into the original equation:

$$\begin{aligned} \left(\frac{y}{2}\right)^2 + 3\left(\frac{y}{2}\right) + 5 &= 0 \\ \implies \boxed{y^2+6y+20=0} \end{aligned}$$

For roots raised to a certain power, we may use the above method or we can modify the original equation such that the above method becomes more convenient.

The cubic $2x^3+7x^2-1=0$ has roots α, β, γ. Find the cubic with roots α^2, β^2 and γ^2.

Here, $y = x^2$. We can manipulate the given equation such that substituting this relationship is made simpler.

$$\begin{aligned} 2x^3+7x^2-1 &= 0 \\ \implies \left(2x^3\right)^2 &= (1-7x)^2 \\ \implies 4x^6 &= 1-14x^2+49x^4 \end{aligned}$$

hence,

$$\boxed{4y^3+49y^2+14y-1=0}$$

High n values fo S_n can be found conveniently using the substitution method. We must formulate another polynomial of the same degree as that given but with roots of a higher power. As such the n values required for the S_n with the new polynomial would be lower for the same result. Observe:

> *Given $x^4 + x^3 - 5 = 0$, find S_4.*
>
> Considering that the given quartic has roots α, β, γ and δ, we consider another such that the roots are $y = x^2$. For this quartic, $S_n = \alpha^{2n} + \beta^{2n} + \gamma^{2n} + \delta^{2n}$.[1] In this case, S_2 of the new polynomial equals S_4 for the given polynomial. So, we find the new polynomial:
>
> $$\begin{aligned} x^4 - 5 &= -x^3 \\ \implies x^8 - 10x^4 + 25 &= x^6 \\ \implies y^4 - y^3 - 10y^2 + 25 &= 0 \end{aligned}$$
>
> Hence, $S_1 = 1$ and $S_2 = 1^2 - 2(-10) = 21$. Thus $\boxed{S_4 = 21}$.

1.2 Rational functions and graphs

Sketch graphs of simple rational functions, including the determination of oblique asymptotes, in cases where the degree of the numerator and the denominator are at most 2

A *rational function* is that which can be defined as an algebraic fraction with polynomials as its numerator *and* denominator. An *asymptote* is generally a line that a curve approaches but never touches. Functions of the form:

$$f(x) = \frac{ax + b}{cx + d}$$

have asymptotes that are vertical and horizontal, which are found as below,

$$\begin{aligned} &cx + d > 0 \\ &\implies x > -\frac{d}{c} \end{aligned}$$

thus, the vertical asymptote is:

$$\boxed{x = -\frac{d}{c}}$$

[1] More generally, for a polynomial with roots $y = x^m$, $S_n = \alpha^{mn} + \beta^{mn}$... and so on.

Let $f(x) = y$

$$
\begin{aligned}
y &= \frac{ax+b}{cx+d} \\
\implies x &= \frac{dy-b}{a-cy}
\end{aligned}
$$

$$
\begin{aligned}
a - cy &> 0 \\
\implies y &< \frac{a}{c}
\end{aligned}
$$

thus, the horizontal asymptote is

$$\boxed{y = \frac{a}{c}}$$

For curves with a quadratic denominator, we observe the following:

Given $y = \dfrac{x}{(x-1)(x-2)}$, *determine its coordinate intercepts, asymptotes, turning points and hence sketch the curve.*

For the coordinate intercepts, we simply plug in $x = 0$ and $y = 0$, getting finding that the curve passes through the origin $\boxed{(0,0)}$.

It is easy to see that the vertical asymptotes are $x = 1$ and $x = 2$, and for $|x| \to \infty$, $y = 0$, the horizontal asymptote.

The turning points of the curve may be found by differentiating and solving the derivative for zero, for which we find $\boxed{(2+\sqrt{2}, 1) \text{ and } (2-\sqrt{2}, 1)}$. We plug x values *slightly* greater and less than that of the turning points to see if the curve increases or decreases on either side.

To find the "gap" in the curve, we cross multiply to find a quadratic equation:

$$yx^2 + (-1-3y)x + 2y = 0$$

whose discriminant is as follows:

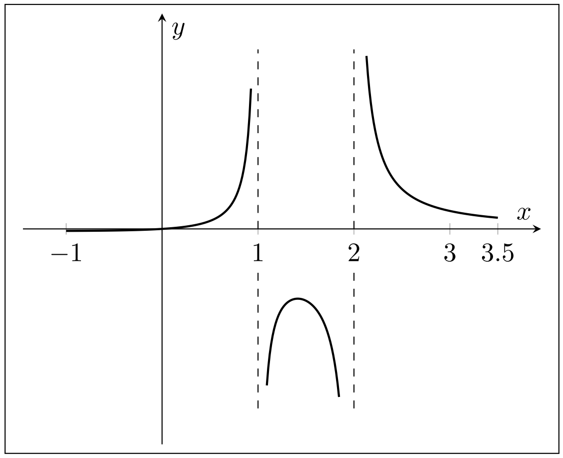
$$(-1-3y)^2 - 4(y)(2y) = y^2 + 6y + 1$$

if we set the discriminant to less than zero, we find an equality in terms of y where the curve does not exist.

$$y^2 + 6y + 1 < 0$$

$$\boxed{-3 - 2\sqrt{2} < y < -3 + 2\sqrt{2}}$$

Thus, for the lower bound of this inequality, we have $x = \sqrt{2}$. Now we may sketch the curve:



For a curve which has quadratics as both numerator and denominator:

$$
\begin{aligned}
y &= \frac{ax^2+bx+c}{dx^2+ex+f} \\
\implies (dy-ay)x^2+(ey-by)x+(fy-cy) &= 0
\end{aligned}
$$

using the discriminant, b^2-4ac and setting it to less than zero gives us values of y such that the curve does not exist.

We may also observe the following:

$$
\begin{aligned}
y &= \frac{ax^2+bx+c}{dx^2+ex+f} \\
&= \frac{(x^2)(a+b/x+c/x^2)}{(x^2)(d+e/x+f/x^2)} \\
&= \frac{a+b/x+c/x^2}{d+e/x+f/x^2}
\end{aligned}
$$

thus, as $x \to \infty$, $y \to a/d$, and hence the horizontal asymtote is:

$$\boxed{y = \frac{a}{d}}$$