

Cambridge Advanced Subsidiary Level Notes
9709 Mathematics

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1 Pure Mathematics 1 (for Paper 1)

1.1 Quadratics

Carry out the process of completing the square for a quadratic polynomial ax^2+bx+c and use a completed square form

The completed square form is

$$ax^2 + bx + c = a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} + c$$

where

$$\left(\frac{-b}{2a}, c - \frac{b^2}{4a^2} \right)$$

is the coordinate of the vertex or stationary point.

Find the discriminant of a quadratic polynomial $ax^2 + bx + c$ and use the discriminant

The discriminant of a quadratic polynomial is the following expression

$$D = b^2 - 4ac$$

where $D > 0$ if solutions are real and $D = 0$ if solutions are real and repeated and $D < 0$ if solutions are imaginary.

Solve quadratic equations, and quadratic inequalities, in one unknown

The quadratic equation has the general solution as follows

$$\begin{aligned} ax^2 + bx + c &= 0 \\ \implies x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

The solutions in case of inequalities are first found in case of inequality, a graph is sketched and the ranges of the solutions are found (intuitively, of course).

Solve by substitution a pair of simultaneous equations of which one is linear and one is quadratic

Intuition.

Recognise and solve equations in x which are quadratic in some function of x

What the following describes is

$$ay^2 + by + c = 0$$

where $y = f(x)$. What is exactly meant by $f(x)$ has to be found by intuition.

1.2 Functions

Understand the terms function, domain, range, one-one function, inverse function and composition of functions

A function is some set of instructions that maps all values in from the domain (the input set) to the range (the output set). Functions are of the following types:

- One-one: A function where each value in the domain is mapped to exactly one value in the range.

For a function $f(x)$, its inverse function $f^{-1}(x)$ maps the range of $f(x)$ to its domain.

Let there be two functions $f(x)$ and $g(x)$. A composed function $fg(x)$ is such that the range of $g(x)$ is mapped as the domain of $f(x)$.

Identify the range of a given function in simple cases, and find the composition of two given functions

Intuition.

Determine whether or not a given function is one-one, and find the inverse of a one-one function in simple cases

Determine it.

Illustrate in graphical terms the relation between a one-one function and its inverse

Functions $f(x)$ and $f^{-1}(x)$ are reflections of each other along $y = x$.

*Understand and use the transformations of the graph of $y = f(x)$ given by
 $y = f(x) + a$, $y = f(x + a)$,
 $y = af(x)$, $y = f(ax)$ and simple combinations of these*

Let there be a function $y = f(x)$. The transformations that can be done unto $y = f(x)$ follow

$$y = f(x) \rightarrow y = f(x) + a$$

is translation of the function $f(x)$ along vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$ and

$$y = f(x) \rightarrow y = f(x + a)$$

is translation of the function $f(x)$ along vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.

$$y = f(x) \rightarrow y = af(x)$$

is the function $f(x)$ scaled parallel to the y -axis by a factor of a and

$$y = f(x) \rightarrow y = f(ax)$$

is the function of $f(x)$ scaled parallel to the x -axis by a factor of $1/a$.

1.3 Coordinate geometry

Find the equation of a straight line given sufficient information

A straight line that passes through coordinates (x_1, y_1) with gradient m has equation of the form

$$y - y_1 = m(x - x_1)$$

Interpret and use any of the forms $y = mx + c$, $y - y_1 = m(x - x_1)$, $ax + by + c = 0$ in solving problems

The equation of a straight line may be in any of the following forms

$$y = mx + c$$

$$y - y_1 = m(x - x_1)$$

$$ax + by + c = 0$$

Rearrangement of all of the above forms will give the general form

$$Y = mX + C$$

where Y is the y -variable, X is the x -variable, m is the gradient and C is the y -intercept.

Understand that the equation $(x - a)^2 + (y - b)^2 = r^2$ represents the circle with centre (a, b) and radius r

Understand it.

Use algebraic methods to solve problems involving lines and circles

Use them.

Understand the relationship between a graph and its associated algebraic equation, and use the relationship between points of intersection of graphs and solutions of equations

For the graphs $y = f(x)$ and $y = g(x)$, the points of their intersection are the solutions to $f(x) = g(x)$.

1.4 Circular measure

Understand the definition of a radian, and use the relationship between radians and degrees

A radian is a measure of the angle of the arc whose length is one radius of the circle.

For an angle θ_r measured in radians, the same angle in degrees, θ_d , is calculated as follows

$$\theta_d = \theta_r \frac{180^\circ}{\pi}$$

which also implies

$$\theta_r = \theta_d \frac{\pi}{180^\circ}$$

Use the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ in solving problems concerning the arc length and sector area of a circle

Use them.

1.5 Trigonometry

Sketch and use graphs of the sine, cosine and tangent functions (for angles of any size, and using either degrees or radians)

Sketch them.

Use the exact values of the sine, cosine and tangent of 30° , 45° , 60° , and related angles

	$\theta = 30^\circ$	$\theta = 45^\circ$	$\theta = 60^\circ$
$\sin \theta$	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$
$\cos \theta$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$
$\tan \theta$	$\sqrt{3}/2$	1	$\sqrt{3}$

Use the notations $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ to denote the principal values of the inverse trigonometric relations

The above described follows

$$\begin{aligned} \sin \theta &= x \\ \implies \theta &= \sin^{-1} x \end{aligned}$$

$$\begin{aligned} \cos \theta &= x \\ \implies \theta &= \cos^{-1} x \end{aligned}$$

$$\begin{aligned} \tan \theta &= x \\ \implies \theta &= \tan^{-1} x \end{aligned}$$

Use the identities $\frac{\sin \theta}{\cos \theta} \equiv \tan \theta$
and $\sin^2 \theta + \cos^2 \theta \equiv 1$

Use 'em

Find all the solutions of simple trigonometrical equations lying in a specified interval (general forms of solution are not included)

Find them.

1.6 Series

Use the expansion of $(a + b)^n$, where n is a positive integer

The expansion of $(a + b)^n$ has $(n + 1)$ terms, where the $(r + 1)$ th term has the form

$$T_{r+1} = \binom{n}{r} (a^{n-r}) (b)^r$$

Recognise arithmetic and geometric progressions

Progressions are terms u_1, u_2, u_3, \dots which may be geometric or arithmetic.

An *arithmetic progression* is where a *common difference* d is added onto an *initial term* a .

A *geometric progression* where a *common ratio* r is multiplied to an *initial term* a .

Use the formulae for the n th term and for the sum of the first n terms to solve problems involving arithmetic or geometric progressions

For an arithmetic progression, the n th term is

$$u_n = a + (n - 1)d$$

and the sum of n terms is

$$S_n = \frac{n}{2} \{2a + (n - 1)d\}$$

For a geometric progression, the n th term is

$$u_n = ar^{n-1}$$

and the sum of n terms is

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Use the condition for the convergence of a geometric progression, and the formula for the sum to infinity of a convergent geometric progression

For the special case that

$$|r| < 1$$

there exists a sum to infinity terms for the geometric progression, which is given by

$$S_\infty = \frac{a}{1 - r}$$

1.7 Differentiation

Understand the gradient of a curve at a point as the limit of the gradients of a suitable sequence of chords, and use the notations $f'(x)$, $f''(x)$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

For a curve

$$y = f(x)$$

we have

$$\begin{aligned} \frac{dy}{dx} &= f'(x) \\ \Rightarrow \frac{d^2y}{dx^2} &= f''(x) \end{aligned}$$

Use the derivative of x^n (for any rational n), together with constant multiples, sums and differences of functions, and of composite functions using the chain rule

For the general equation

$$\begin{aligned} y &= f(x) = ax^n \\ \Rightarrow \frac{dy}{dx} &= f'(x) = nax^{n-1} \\ \Rightarrow \frac{d^2y}{dx^2} &= f''(x) = n(n-1)ax^{n-2} \end{aligned}$$

Apply differentiation to gradients, tangents and normals, increasing and decreasing functions and rates of change

The derivative of a function at a point x is the gradient of the function at that point.

So, for a function $y = f(x)$ which has $dy/dx = f'(x) = m$ at a point (x_1, y_1) the equation of the tangent is

$$y - y_1 = m(x - x_1)$$

and the equation of the normal is

$$y - y_1 = -m^{-1}(x - x_1)$$

In the case of rates of change, I quote from MF19^[1]

If $x = f(t)$ and $y = g(t)$ then

$$\frac{dy}{dx} = \left(\frac{dy}{dt} \right) \left(\frac{dx}{dt} \right)^{-1}$$

Locate stationary points and determine their nature, and use information about stationary points in sketching graphs

For a curve, its stationary points exist where

$$f'(x) = \frac{dy}{dx} = 0$$

and for the coordinates where the stationary points are, if

$$f''(x) = \frac{d^2y}{dx^2} > 0$$

is a minimum point and

$$f''(x) = \frac{d^2y}{dx^2} < 0$$

is a maximum point.

^[1]With minimal alteration as the notation of MF19 is ugly in my opinion.

1.8 Integration

Understand integration as the reverse process of differentiation, and integrate $(ax+b)^n$ (for any rational n except 1), together with constant multiples, sums and differences

The function

$$f(x) = (ax + b)^n$$

is integrated to the form

$$\begin{aligned}\int f(x) \, dx &= \int (ax + b)^n \, dx \\ &= \frac{(ax + b)^{n+1}}{(n+1)(a)} + C\end{aligned}$$

where C is the constant of integration.

Solve problems involving the evaluation of a constant of integration

The constant of integration can be found by plugging in value of x and the value of the integral itself.

Use definite integration to find

- the area of a region bounded by a curve and lines parallel to the axes, or between a curve and a line or between two curves
 - a volume of revolution about one of the axes
-

For the area under a curve from $x = a$ to $x = b$ we integrate definitely

$$\begin{aligned}\int_a^b f(x) \, dx &= [F(x)]_a^b \\ &= F(b) - F(a)\end{aligned}$$

where $F(x) = \int f(x) \, dx$.

For the volume of a curve $f(x)$ rotated about the x -axis from $x = a$ and $x = b$

$$\begin{aligned}V &= \pi \int_a^b f(x)^2 \, dx = \pi [F(x)]_a^b \\ &= \pi (F(b) - F(a))\end{aligned}$$

where $F(x) = \int f(x)^2 \, dx$.

2 Pure Mathematics 2 (for Paper 2)

3 Pure Mathematics 3 (for Paper 3)

3.1 Algebra

Understand and use the meaning of $|x|$, sketch the graph of $y = |ax + b|$ and use relations such as $|a| = |b| \iff a^2 = b^2$ and $|x - a| < b \iff a - b < x < a + b$ when solving equations and inequalities

The modulus of x , written as $|x|$ is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Equation of the following form may be solved as follows

$$\begin{aligned} |cx + d| &= |ex + f| \\ \implies (cx + d)^2 &= (ex + f)^2 \end{aligned}$$

Inequalities, as follows, may be expanded

$$\begin{aligned} |x - a| &< b \\ \implies -b &< x - a < b \\ \implies a - b &< x < a + b \end{aligned}$$

Note that, the above is applicable if the inequality is inclusive as well.

$$|a| > b \implies a < -b \text{ or } a > b$$

Divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero)

A polynomial is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where x is a variable, n is a non-negative integer, the coefficients a_n, a_{n-1} , etc. are constants. a_n is said to be the leading coefficient and a_0 is said to be the constant term.

The highest power of x in a polynomial is called the degree of the polynomial.

The long division of polynomials, I am unable to show here. I am sorry. Open a book. Space is provided for you to record what you learn.

Use the factor theorem and the remainder theorem

Let $P(x)$ be a polynomial, which is exactly divisible by a (linear) polynomial $ax + b$, giving a quotient $Q(x)$. We derive that

$$\begin{aligned} P(x)/(ax + b) &= Q(x) \\ \implies P(x) &= (ax + b)(Q(x)) \end{aligned}$$

we observe that $P(-b/a) = 0$. This is the factor theorem, defined more formally as follows.

If, for a polynomial $P(x)$, $P(-b/a) = 0$ then $ax + b$ is a factor of $P(x)$.

An extension of the factor theorem is the remainder theorem. Let us say the division of $P(x)$ by $ax + b$ gives quotient $Q(x)$ and remainder R . Mathematically, it can be organised as follows

$$\begin{aligned} P(x)/(ax + b) &= Q(x) + R/(ax + b) \\ \implies P(x) &= (ax + b)Q(x) + R \end{aligned}$$

we now observe that $P(-b/a) = R$. This is the remainder theorem, defined formally as

If a polynomial $P(x)$ is divided by $ax + b$, the remainder is $P(-b/a)$.

We may also observe that the factor theorem is a special case of the remainder theorem where $P(-b/a) = 0$.

Recall an appropriate form for expressing rational functions in partial fractions, and carry out the decomposition, in cases where the denominator is no more complicated than

$$\begin{aligned} &-(ax + b)(cx + d)(ex + f) \\ &-(ax + b)(cx + d)^2 \\ &-(ax + b)(cx^2 + d) \end{aligned}$$

We can split proper algebraic fractions into two or more partial fractions by using the following identities.

$$\frac{px + q}{(ax + b)(cx + d)} \equiv \frac{A}{ax + b} + \frac{B}{cx + d}$$

where the above can be extended for any number of linear factors.

$$\frac{px + q}{(ax + b)(cx + d)^2} \equiv \frac{A}{ax + b} + \frac{B}{cx + d} + \frac{C}{(cx + d)^2}$$

$$\frac{px + q}{(ax + b)(cx^2 + d)} \equiv \frac{A}{ax + b} + \frac{Bx + C}{cx^2 + d}$$

Use the expansion of $(1 + x)^n$, where n is a rational number and $|x| < 1$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

3.2 Logarithmic and exponential functions

Understand the relationship between logarithms and indices, and use the laws of logarithms (excluding change of base)

Logarithms are functions that have bases and inputs. In general, for a logarithm with base a and input x , the output of the function is the power a must be raised to to obtain x . Mathematically, what follows is

$$\begin{aligned} \log_a x &= c \\ \implies x &= a^c \end{aligned}$$

The laws of logarithms are as follows

$$\log_a a = 1$$

$$\log_a 1 = 0$$

$$\log_a a^x = x$$

$$a^{\log_a x} = x$$

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a(x/y) = \log_a x - \log_a y$$

$$\log_a x^m = m \log_a x$$

Understand the definition and properties of e^x and $\ln x$, including their relationship as inverse functions and their graphs

If $y = e^x$, then $x = \ln y$. I refuse to show you the graphs because they are inconvenient.

Use logarithms to solve equations and inequalities in which the unknown appears in indices

This requires algebraic talent. I have none. JK I am cramming rn and can't really write this stuff out cuz it's mega elaborate and I already am great at this.

Use logarithms to transform a given relationship to linear form, and hence determine unknown constants by considering the gradient and/or intercept.

Same as above.

3.3 Trigonometry

Understand the relationship of the secant, cosecant and cotangent functions to cosine, sine and tangent, and use properties and graphs of all six trigonometric functions for angles of any magnitude

$$1/\sin x = \operatorname{cosec} x$$

$$1/\cos x = \sec x$$

$$1/\tan x = \cot x$$

Same deal with the graphs.

Use trigonometrical identities for the simplification and exact evaluation of expressions, and in the course of solving equations, and select an identity or identities appropriate to the context, showing familiarity in particular with the use of

- $\sec^2 \theta \equiv \tan^2 \theta$ and $\operatorname{cosec}^2 \theta \equiv 1 + \cot^2 \theta$
 - the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$
 - the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$
 - the expression of $a \sin \theta + b \cos \theta$ in the forms $R \sin(\theta \pm \alpha)$ and $R \cos(\theta \pm \alpha)$.
-

Simple enough, I just need to get to further calculus.

3.4 Numerical solution of equations

3.5 Vectors

4 Mechanics (for Paper 4)

5 Probability & Statistics 1 (for Paper 5)

5.1 Representation of data

Select a suitable way of presenting raw statistical data, and discuss advantages and/ or disadvantages that particular representations may have

Draw and interpret stem-and-leaf diagrams, box-and-whisker plots, histograms and cumulative frequency graphs

Stem-and-leaf diagrams are a way to arrange the raw data.

Box-and-whisker plots (boxplots) are one dimensional diagrams which show the interquartile range (IQR) and the minimum and maximum of the given data, and also outliers when applicable.

Histograms are used where the independent variable comes in the form of ranges. We find frequency density for each range, by finding range width and dividing frequency by it. Then we plot frequency density across the ranges on the x-axis.

Understand and use different measures of central tendency (mean, median, mode) and variation (range, interquartile range, standard deviation)

Measures of Central Tendency

For an ungrouped set of data with n observations and frequencies x_1, x_2, \dots, x_n , the mean is

$$\bar{x} = \frac{\Sigma x}{n}$$

For grouped data where the groups have midvalues x and the groups each have frequencies f , the mean is

$$\bar{x} = \frac{\Sigma xf}{\Sigma f} = \frac{\Sigma fx}{\Sigma f}$$

Coded data is where each observation in the set is offset by a certain value. Here, for ungrouped data,

$$\bar{x} = \frac{\Sigma(x - b)}{n} + b$$

and for grouped data,

$$\bar{x} = \frac{\Sigma(x - b)f}{\Sigma f} + b$$

For ungrouped data, the *median* is at the $(\frac{n+1}{2})$ th value, and it is at the $\frac{n}{2}$ th value on a cumulative frequency graph.

Measures of variation

Interquartile range is difference between upper and lower quartile, $Q_3 - Q_1$ where $Q_n = (n/4)\Sigma f$

For ungrouped data, standard deviation (σ) is as follows

$$\sigma(x) = \sqrt{\text{Var } x} = \sqrt{\frac{(\Sigma x - \bar{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2}$$

and for grouped data,

$$\sigma(x) = \sqrt{\text{Var } x} = \sqrt{\frac{(\Sigma x - \bar{x})^2 f}{\Sigma f}} = \sqrt{\frac{\Sigma x^2 f}{\Sigma f} - \bar{x}^2}$$

5.2 Permutations and combinations

5.3 Probability

Evaluate probabilities in simple cases by means of enumeration of equiprobable elementary events, or by calculation using permutations or combinations

The probability of a certain event is a fraction

$$x/y$$

where x is the number of times the event occurs and y is the number of total events that occur.

Use addition and multiplication of probabilities, as appropriate, in simple cases

Addition is OR, multiplication is AND.

Understand the meaning of exclusive and independent events, including determination of whether events A and B are independent by comparing the values of $P(A \cap B)$ and $P(A) \times P(B)$

5.4 Discrete random variables

Draw up a probability distribution table relating to a given situation involving a discrete random variable X , and calculate $E(X)$ and $\text{Var}(X)$

A probability distribution table displays all possible outcomes with their possibilities. Say, for coin tosses denoted by the discrete random variable X , the probability distribution is as follows where $x = 1$ denotes heads and $x = 0$ denotes tails.

x	0	1
$P(X = x)$	0.5	0.5

The mean (expected) outcome is calculated as follows

$$E(X) = \frac{\sum xf}{\sum f}$$

where x is outcome and f is expected frequency of outcome

The variance and standard deviation give a measure of the spread of values around the mean. The formula, in case of a random variable, follows

$$\begin{aligned}\text{Var } X &= \frac{\sum x^2 f}{\sum f} - \bar{x}^2 \\ &= \frac{\sum x^2 p}{\sum p} - \{E(X)\}^2\end{aligned}$$

since $\sum p = 1$

$$\text{Var}(X) = \sum x^2 p - \{E(X)\}^2$$

Use formulae for probabilities for the binomial and geometric distributions, and recognise practical situations where these distributions are suitable models

Distributions such as the binomial and the geometric may be used to model probabilistic scenarios. The binomial distribution can be used to model the number

of successes in a fixed number of independent trials and the geometric distribution can be used to model the number of trials up to and including the first success in an infinite number of independent trials.

The binomial distribution is denoted as follows

$$X \sim B(n, r)$$

where n is the number of trials and r is number of successes. In this model,

$$P(X = x) = \binom{n}{r} (p)^{n-r} (q)^r$$

The geometric distribution is denoted as follows

$$X \sim \text{Geo}(p)$$

where p is the probability of success. Thus, the probability that the first success occurs on the r th trial is

$$p_r = p(1 - p)^{r-1}$$

Use formulae for the expectation and variance of the binomial distribution and for the expectation of the geometric distribution.

For the binomial distribution,

$$E(X) = \sum xp$$

$$\text{Var}(X) = \sigma^2 = np(1 - p)$$

For the geometric distribution,

$$E(X) = \mu = \sum xp_x = 1/p$$

5.5 The normal distribution

Understand the use of a normal distribution to model a continuous random variable, and use normal distribution tables

A continuous random variable is that which varies infinitely amongst two extremes.

Solve problems concerning a variable X , where $X \sim N(\mu, \sigma^2)$, including

- finding the value of $P(X > x_1)$ or a related probability, given the values of x_1 , μ and σ .
 - finding a relationship between x_1 , μ and σ given the value of $P(X > x_1)$ or a related probability
-

Let us introduce the standard normal variable Z . It has variance 1, and so standard deviation 1. $z = \pm 1, \pm 2$, and ± 3 represents values that are 1, 2 and 3 standard deviations above or below the mean. Any $|z| > 3$ has $\phi(z) \approx 0$.

A vertical line drawn at any value of Z divides the area under the graph into two parts: one representing $P(Z \leq z)$ and the other representing $P(Z > z)$.

The value of $P(Z \leq z)$ is denoted by $\Phi(z)$, the values for which are found from the table provided in MF19.

Coding a continuous random variable X by subtracting its mean μ , brings its mean zero, forming a variable $X - \mu$. Dividing by σ brings the standard deviation and variance to 1. Thus, the coded random variable

$$\frac{X - \mu}{\sigma}$$

is normally distributed with mean 0 and variance 1.

The above transforms the distribution $X \sim N(\mu, \sigma^2)$ to $Z \sim N(0, 1)$. A standardised value

$$z = \frac{x - \mu}{\sigma}$$

tells us exactly how many standard deviations the variable x is from the mean.

Recall conditions under which the normal distribution can be used as an approximation to the binomial distribution, and use this approximation, with a continuity correction, in solving problems

The binomial distribution is approximate to the normal. The approximation becomes more accurate the higher the values of np and $n(1 - p)$. However, approximating a discrete variable to a continuous means that to find $P(X = x)$ we have to find $P(x - 0.5 \leq X < x + 0.5)$.