*Cambridge Advanced Subsidiary Level Notes*
# 9231 Further Mathematics

Abrar Faiyaz Rahim, degrees pending

# Contents

**1 Further Pure Mathematics 1 (for Paper 1)** **2**
1.1 Roots of polynomial equations . . . . 2
1.2 Rational functions and graphs . . . . . 4
1.3 Summation of series . . . . . . . . . . 7
1.4 Matrices . . . . . . . . . . . . . . . . . . 8

**2 Further Mechanics (for Paper 3)** **11**
2.1 Motion of a projectile . . . . . . . . . 11
2.2 Equilibrium of a rigid body . . . . . . 12

# 1 Further Pure Mathematics 1 (for Paper 1)

## 1.1 Roots of polynomial equations

*Recall and use the relations between the roots and coefficients of polynomial equations*

**Quadratics**

The quadratic equation follows,

$$\begin{aligned} ax^2 + bx + c &= 0 \\ \implies x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \end{aligned}$$

which has roots $\alpha$ and $\beta$,

$$\begin{aligned} (x-\alpha)(x-\beta) &= 0 \\ \implies x^2 - (\alpha+\beta)x + \alpha\beta &= 0 \end{aligned}$$

hence, considering that, $S_n = \alpha^n + \beta^n$,

$$\boxed{\Sigma\alpha = S_1 = \alpha + \beta = -\frac{b}{a}}$$

$$\boxed{\Sigma\alpha\beta = \alpha\beta = \frac{c}{a}}$$

where $\Sigma\alpha$ and $\Sigma\alpha\beta$ are referred to as *sum of roots* and *product of roots*, respectively.

Furthermore,

$$\boxed{\Sigma\alpha^2 = S_2 = \alpha^2 + \beta^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta}$$

$$\boxed{\Sigma\frac{1}{\alpha} = S_{-1} = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{\Sigma\alpha}{\Sigma\alpha\beta}}$$

$$\boxed{\Sigma\frac{1}{\alpha^2} = S_{-2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2} = \frac{\Sigma\alpha^2}{(\Sigma\alpha\beta)^2}}$$

Using $S_n$ as defined above, we can use known values of $S_n$ to find values required.

Consider the quadratic equation $ax^2 + bx + c = 0$ with roots $\alpha$ and $\beta$. To find certain values of $S_n$, we must multiply the equation with a certain power of $x$, to achieve a power of $x$ in the equation that equals the value of $n$ desired.

For example, when we want $S_{-1}$, we must first multiply the original equation by $x^{-1}$, giving us,

$$ax + b + \frac{c}{x} = 0$$

We can now plug $x = \alpha$ and $x = \beta$, since they are roots,

$$\begin{aligned} a\alpha + b + \frac{c}{\alpha} &= 0 && (1) \\ a\beta + b + \frac{c}{\beta} &= 0 && (2) \\ a(\alpha+\beta) + 2b + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)c &= 0 && (1 + 2) \\ \implies aS_1 + 2b + cS_{-1} &= 0 \end{aligned}$$

here, if we know $S_1$, we can find $S_{-1}$.

Note that, the coefficients are multiplied by $S_n$, where $n$ equals the power of $x$ to which they are a coefficient, and the constant terms are multiplied by 2.

**Cubics**

The cubic equation follows,

$$\begin{aligned} ax^3 + bx^2 + cx + d &= 0 \\ \implies x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} &= 0 \end{aligned}$$

which has roots $\alpha$, $\beta$, and $\gamma$,

$$\begin{aligned} (x-\alpha)(x-\beta)(x-\gamma) &= 0 \\ \implies x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)x - \alpha\beta\gamma &= 0 \end{aligned}$$

thus, using $S_n = \alpha^n + \beta^n + \gamma^n$,

$$\boxed{\Sigma\alpha = S_1 = \alpha + \beta + \gamma = -\frac{b}{a}}$$

$$\boxed{\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}}$$

$$\boxed{\Sigma\alpha\beta\gamma = \alpha\beta\gamma = -\frac{d}{a}}$$

Following this,

$$\boxed{\Sigma\alpha^2 = S_2 = (\alpha+\beta+\gamma)^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta}$$

$$\boxed{\Sigma\alpha^3 = (\Sigma\alpha)^3 - 3\Sigma\alpha\beta\Sigma\alpha + 3\Sigma\alpha\beta\gamma}$$

> Here too, values of $S_n$ can be found in the method demonstrated in the **Quadratics** section. Note that, in such cases the constant term is multiplied by 3.

### Quartics

The quartic equation

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$$

has roots $\alpha$, $\beta$, $\gamma$ and $\delta$. Due to the monstrous nature of the equation, it is best to use $S_n = \alpha^n+\beta^n+\gamma^n+\delta^n$ notation.

$$\boxed{\Sigma\alpha = -\frac{b}{a}}$$

$$\boxed{\Sigma\alpha\beta = \frac{c}{a}}$$

$$\boxed{\Sigma\alpha\beta\gamma = -\frac{d}{a}}$$

$$\boxed{\Sigma\alpha\beta\gamma\delta = \frac{e}{a}}$$

$$\boxed{S_2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta}$$

$$\boxed{S_{-1} = \frac{\Sigma\alpha\beta\gamma}{\Sigma\alpha\beta\gamma\delta}}$$

---

*Use a substitution to obtain an equation whose roots are related in a simple way to those of the original equation*

---

If we are given a known polynomial whose roots we also know, we can find any other unknown polynomial given that its roots are in terms of the roots of the first given polynomial. We must first write out the unknown polynomial in factorised form as per its roots and compare with the initial polynomial, finding values of the roots and arriving at the unknown polynomial. This is illustrated in the following:

> *Given the quadratic $x^2+3x+5=0$ with roots $\alpha$ and $\beta$, find the quadratic that has roots $2\alpha$ and $2\beta$.*
>
> The unknown quadratic is
>
> $$\begin{aligned} (y-2\alpha)(y-2\beta) &= 0 \\ \implies y^2 - (2\alpha+2\beta)y + 4\alpha\beta &= 0 \end{aligned}$$
>
> comparing with the original, we find
>
> $$\alpha+\beta = -3$$
>
> $$\alpha\beta = 5$$
>
> Solving simultaneously, we find $\boxed{y^2+6y+20=0}$ to be the quadratic asked for in the question.

Alternatively, we can derive a relationship between the roots of both equations. For the same question as above:

> The new quadratic has $y = 2x \implies x = y/2$ since each root of the new quadratic is twice that of the given one. Thus, plugging the above into the original equation:
>
> $$\left(\frac{y}{2}\right)^2 + 3\left(\frac{y}{2}\right) + 5 = 0$$
> $$\implies \boxed{y^2+6y+20=0}$$

For roots raised to a certain power, we may use the above method or we can modify the original equation such that the above method becomes more convenient.

The cubic $2x^3 + 7x^2 - 1 = 0$ has roots $\alpha$, $\beta$, $\gamma$. Find the cubic with roots $\alpha^2$, $\beta^2$ and $\gamma^2$.

Here, $y = x^2$. We can manipulate the given equation such that substituting this relationship is made simpler.

$$\begin{aligned} 2x^3 + 7x^2 - 1 &= 0 \\ \implies \left(2x^3\right)^2 &= (1 - 7x)^2 \\ \implies 4x^6 &= 1 - 14x^2 + 49x^4 \end{aligned}$$

hence,

$$\boxed{4y^3 + 49y^2 + 14y - 1 = 0}$$

High $n$ values fo $S_n$ can be found conveniently using the substitution method. We must formulate another polynomial of the same degree as that given but with roots of a higher power. As such the $n$ values required for the $S_n$ with the new polynomial would be lower for the same result. Observe:

*Given $x^4 + x^3 - 5 = 0$, find $S_4$.*

Considering that the given quartic has roots $\alpha$, $\beta$, $\gamma$ and $\delta$, we consider another such that the roots are $y = x^2$. For this quartic, $S_n = \alpha^{2n} + \beta^{2n} + \gamma^{2n} + \delta^{2n}$.[1] In this case, $S_2$ of the new polynomial equals $S_4$ for the given polynomial. So, we find the new polynomial:

$$\begin{aligned} x^4 - 5 &= -x^3 \\ \implies x^8 - 10x^4 + 25 &= x^6 \\ \implies y^4 - y^3 - 10y^2 + 25 &= 0 \end{aligned}$$

Hence, $S_1 = 1$ and $S_2 = 1^2 - 2(-10) = 21$. Thus $\boxed{S_4 = 21}$.

[1] More generally, for a polynomial with roots $y = x^m$, $S_n = \alpha^{mn} + \beta^{mn}$... and so on.

## 1.2 Rational functions and graphs

*Sketch graphs of simple rational functions, including the determination of oblique asymptotes, in cases where the degree of the numerator and the demoninator are at most 2*

A rational function is that which can be defined as an algebraic fraction with polynomials as its numerator *and* denominator. An asymptote is generally a line that a curve approaches but never touches. Functions of the form:

$$f(x) = \frac{ax + b}{cx + d}$$

have asymptotes that are vertical and horizontal, which are found as below,

$$\begin{aligned} cx + d &> 0 \\ \implies x &> -\frac{d}{c} \end{aligned}$$

thus, the vertical asymptote is:

$$\boxed{x = -\frac{d}{c}}$$

Let $f(x) = y$

$$\begin{aligned} y &= \frac{ax + b}{cx + d} \\ \implies x &= \frac{dy - b}{a - cy} \end{aligned}$$

$$\begin{aligned} a - cy &> 0 \\ \implies y &< \frac{a}{c} \end{aligned}$$

thus, the horizontal asymptote is

$$\boxed{y = \frac{a}{c}}$$

For curves with a quadratic denominator, we observe the following:

*Given $y = \dfrac{x}{(x-1)(x-2)}$, determine its coordinate intercepts, asymptotes, turning points and hence sketch the curve.*

For the coordinate intercepts, we simply plug in $x = 0$ and $y = 0$, getting finding that the curve passes through the origin $\boxed{(0,0)}$.

It is easy to see that the vertical asymptotes are $x = 1$ and $x = 2$, and for $|x| \to \infty$, $y = 0$, the horizontal asymptote.

The turning points of the curve may be found by differentiating and solving the derivative for zero, for which we find $\boxed{(2+\sqrt{2}, 1) \text{ and } (2-\sqrt{2}, 1)}$. We plug $x$ values *slightly* greater and less than that of the turning points to see if the curve increases or decreases on either side.

To find the "gap" in the curve, we cross multiply to find a quadratic equation:

$$yx^2 + (-1-3y)x + 2y = 0$$

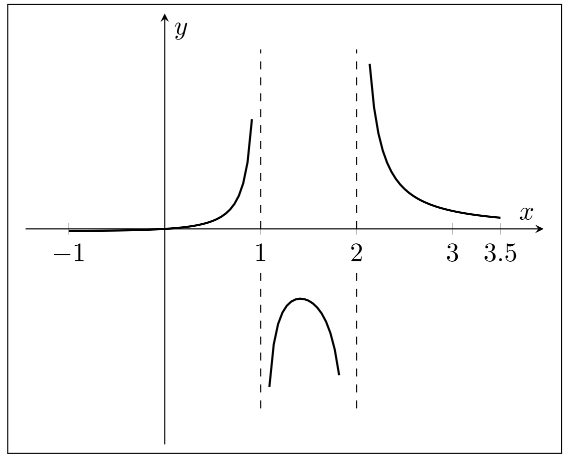
whose discriminant is as follows:

$$(-1-3y)^2 - 4(y)(2y) = y^2 + 6y + 1$$

if we set the discriminant to less than zero, we find an equality in terms of $y$ where the curve does not exist.

$$y^2 + 6y + 1 < 0$$
$$\boxed{-3-2\sqrt{2} < y < -3+2\sqrt{2}}$$

Thus, for the lower bound of this inequality, we have $x = \sqrt{2}$. Now we may sketch the curve:




For a curve which has quadratics as both numerator and denominator:

$$y = \frac{ax^2 + bx + c}{dx^2 + ex + f}$$
$$\implies (dy - ay)x^2 + (ey - by)x + (fy - cy) = 0$$

using the discriminant, $b^2 - 4ac$ and setting it to less than zero gives us values of $y$ such that the curve does not exist.

We may also observe the following:

$$\begin{aligned} y &= \frac{ax^2 + bx + c}{dx^2 + ex + f} \\ &= \frac{(x^2)(a + b/x + c/x^2)}{(x^2)(d + e/x + f/x^2)} \\ &= \frac{a + b/x + c/x^2}{d + e/x + f/x^2} \end{aligned}$$

thus, as $|x| \to \infty$, $y \to a/d$, and hence the horizontal asymptote is:

$$\boxed{y = \frac{a}{d}}$$

**Oblique Asymptotes**

Curves with a quadratic numerator and a linear denominator produce an *oblique asymptote* – which is simply a straight line, along with a veritical asymptote whose equation is easy to determine.

To find the equation of the oblique asymptote, we simply write the given curve in partial fraction form.

*Given* $y = \dfrac{3x^2 + x + 3}{x+1}$, *find its asymptotes and sketch the curve.*

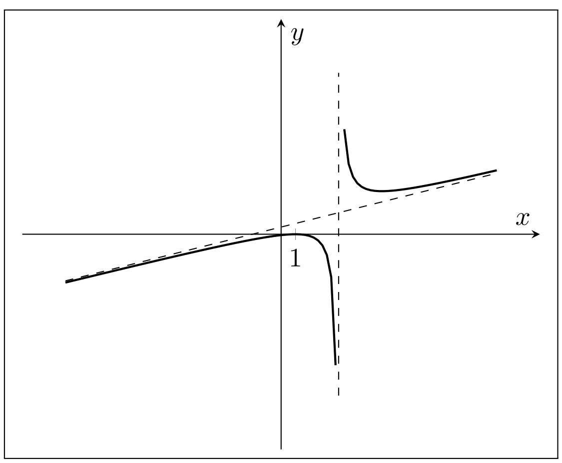
We first perform long division on the given fraction:

$$\begin{array}{r@{}l}
 & \phantom{)}\quad 3x - 2 \\ \cline{2-2}
x+1 & \big) \; 3x^2 \; + x + 3 \\
 & \; - 3x^2 - 3x \\ \cline{2-2}
 & \qquad - 2x + 3 \\
 & \qquad\;\; 2x + 2 \\ \cline{2-2}
 & \qquad\qquad 5
\end{array}$$

therefore,

$$y = 3x - 2 + \frac{5}{x+1}$$

Thus, the asymptotes are $\boxed{x = -1}$ and $\boxed{y = 3x - 2}$. For $y = 0$ the curve gives $x = 1$ and for $x = 0$ the curve gives $y = 3$, thus the intercepts with the coordinate axes are $(1, 0)$ and $(0, 3)$.




In general, a curve

$$y = \frac{ax^2 + bx + c}{dx + e}$$

can be rewritten as

$$y = Ax + B + \frac{C}{dx + e}$$

and hence the asymptotes of such a curve are

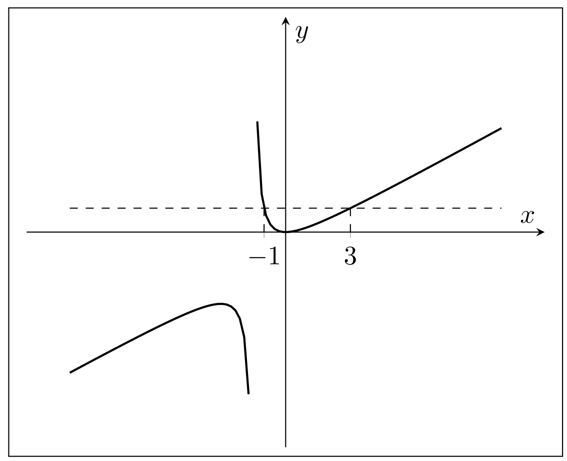
$$\boxed{x = -\frac{e}{d}}$$

$$\boxed{y = Ax + B}$$

**Inequalities**

Given an inequality such that $f(x) < k$, we can solve the inequality but the resulting range may not accurately reflect the true answer.

Consider $y = \dfrac{2x^2}{2x+3}$, such that $y < 2$. We rearrange to get $x^2 - 2x - 3 < 0$, whose solution gives $-1 < x < 3$. We now sketch the curve:




Notice that there is another interval below the $x$-axis, which satisfies the given inequality (the line $y = 2$ is shown by the horizontal dashed line). Thus the correct answer to the question would be $-1 < x < 3$ and $x < -3/2$[2]

So, in general, finding all the information about the curve and sketching it before solving the inequality is the safest method to solve inequalities involving rational functions.

---

*Understand and use relationships between the graphs of* $y = f(x)$, $y^2 = f(x)$, $y = 1/f(x)$, $y = |f(x)|$ *and* $y = f(|x|)$

---

[2] $x = -3/2$ is an asymptote to the curve.

Consider functions of the form,

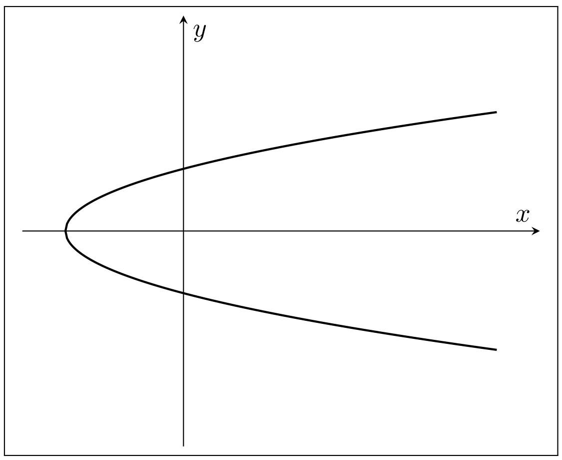
$$y^2 = ax + b$$
$$\implies y = \pm\sqrt{ax+b}$$

therefore, the curves must be such that,

$$ax + b > 0$$

Such curves have an $x$-intercept at $-b/a$ and then turn like a sideways parabola to whichever direction the above inequality points.

Below is the curve of $y^2 = 2x + 3$,




The above has an $x$-intercept of $x = -3/2$.

For curves such that $y = |f(x)|$, the part of the curve below the $x$-axis is simply reflected above the $x$-axis.

For curves such that $y = f(|x|)$, the part to the right of the $y$-axis is reflected to the left of the $y$-axis.

## 1.3 Summation of series

*Use the standard results for $\Sigma r$, $\Sigma r^2$, $\Sigma r^3$ to find related sums*

Below are the standard results of the above summations,

$$\boxed{\sum_{r=1}^{n} r = \frac{n(n+1)}{2} = \frac{n^2+n}{2}}$$

$$\boxed{\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}}$$

$$\boxed{\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2}$$

Thus, for a general expression which may have the form,

$$\sum_{r=1}^{n} ar^3 + br^2 + cr + d$$

where $a$, $b$, $c$ and $d$ are constant terms, the sum is,

$$a\Sigma r^3 + b\Sigma r^2 + c\Sigma r + dn$$[3]

*Use the method of differences to obtain the sum of a finite series*

If the lower limit of a sum is 1, we may plugin whatever the given upper limit is into the standard result. For example,

$$\sum_{r=1}^{2n} r^2 = \frac{(2n)(2n+1)(4n+1)}{6}$$

However, if the lower limit is anything other than 1, suppose $x$, we simply subtract the sum from 1 to $x-1$ from the sum from 1 to the upper limit. That is,

$$\sum_{x}^{n} r = \sum_{1}^{n} r - \sum_{x-1}^{n} r$$

This is the <u>*method of differences*</u>. Note that, the sum being subtracted has a lower limit of $x-1$ because such that the lower limit of the first sum is being included in the sum.

*Recognise, by direct consideration of a sum to $n$ terms, when a series is convergent, and find the sum to infinity in such cases*

[3] $\Sigma r$, when unspecified refers to $\sum_{r=1}^{n} r$

Convergent series usually come down to the sum of a fractional expression. For example, the sum

$$\sum_{r=1}^{n} \frac{1}{r}$$

expands out to

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

We see that, as the fraction gets smaller and smaller, the value by which the sum increases gets smaller and smaller. Thus, this sum is convergent. However, we are not required to find out an expression in terms of $n$.

Consider another sum

$$\sum_{r=1}^{n} \left( \frac{1}{r} - \frac{1}{r+1} \right)$$

which then expands out to

$$\left( \frac{1}{1} - \cancel{\frac{1}{2}} \right) + \left( \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) + \left( \cancel{\frac{1}{3}} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{n-1} - \cancel{\frac{1}{n}} \right) + \left( \cancel{\frac{1}{n}} - \frac{1}{n+1} \right)$$

from which we cancel the terms as above, and we can see that the only terms that remain uncancelled are 1 and $-\frac{1}{n+1}$. Thus the expression comes out to

$$\boxed{\sum_{r=1}^{n} \left( \frac{1}{r} - \frac{1}{r+1} \right) = 1 - \frac{1}{n+1}}$$

where, as $n \to \infty$,

$$1 - \frac{1}{\infty + 1} = 1 - 0 = 1$$

For sums with an fraction with a quadratic function in the denominator, we must factorise and split the fraction into partial fractions, before undergoing the process shown above.

## 1.4 Matrices

*Carry out operations of matrix addition, subtraction and multiplication, and recognise the terms zero matrix and identity (or unit) matrix*

For a matrix $\mathbf{M}$, which has $m$ rows and $n$ columns, we say that the matrix has an *order* or size of $m \times n$. Generally, this matrix is represented mathematically as

$$\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

where $a_{ij}$ is an *element* in the matrix which is in the $i$th column and $j$th row.

A matrix which has the same number of columns and rows is called a *square matrix*.

### Matrix Addition and Subtraction

Two matrices can only be added or subtracted if they are of the same order, where elements in the same column and row are added. Below is a general example for a $3 \times 2$ matrix, which can be extended to a matrix of any order

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

thus

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{pmatrix}$$

and

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \\ a_{31} - b_{31} & a_{32} - b_{32} \end{pmatrix}$$

### Matrix Multiplication

For a scalar multiplied to a matrix, all the elements of the matrix are multiplied by that scalar.

For matrices that are multiplied, the process is significantly more complicated, below is the case for a pair of $2 \times 2$ matrices.

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Note that, for a matrix product to be calculated, the number of columns of the first matrix must be the same as the number of rows of the second matrix. To find each new element in the product matrix, we undergo the following procedure:

1. Take a row from the first matrix.
2. Take a column from the second matrix.
3. Multiply the matching numbers, (1st with 1st, 2nd with 2nd and so on).
4. Add those products up.

So if,

$$\mathbf{E} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

then,

$$\begin{aligned} \mathbf{EF} &= \begin{pmatrix} 1\times 5+2\times 7 & 1\times 6+2\times 8 \\ 3\times 5+4\times 7 & 3\times 6+3\times 8 \end{pmatrix} \\ &= \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix} \end{aligned}$$

Generally, matrix multiplication is said to be *non-commutative*, which means

$$\mathbf{AB} \neq \mathbf{BA}$$

The matrix in which all the elements are zero is known as the <u>*zero matrix*</u>. It can be of any size and is represented

$$0_{mn} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus, $\mathbf{AB} = \mathbf{BA}$ when one of the matrices being multiplied is a zero matrix, since the product is also a zero matrix.

Also, note that

$$\boxed{\mathbf{A}^m\mathbf{A}^n = \mathbf{A}^n\mathbf{A}^m = \mathbf{A}^{m+n}}$$

thus if $\mathbf{A}^m = \mathbf{B}$, $\mathbf{AB} = \mathbf{BA}$.

In a square matrix, the line of elements that goes from the top left corner to the bottom right corner is called the <u>*diagonal*</u>. The line of elements that goes from the top right to the bottom left is called the <u>*off diagonal*</u>.

The square matrix where the diagonal consists of 1 and the rest of the matrix is 0 is called the <u>*identity matrix*</u>. It is represented by $\mathbf{I}$. The property of the identity matrix is that, when it is multiplied to another matrix, the matrix remains unchanged

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

---

*Recall the meaning of the terms 'singular' and 'non-singular' as applied to square matrices and, for $2\times 2$ and $3 \times 3$ matrices, evaluate determinants and find inverses of non-singular matrices*

---

A matrix with repeated rows is called a <u>*singular matrix*</u>. Such matrices cannot have <u>*inverse matrices*</u>. A matrix which can have an inverse is known as a <u>*non-singular matrix*</u>. *These definitions apply for square matrices.*

For a matrix $\mathbf{A}$, its inverse is represented $\mathbf{A}^{-1}$. This is such that

$$\mathbf{AA}^{-1} = \mathbf{I}$$

### Determinants

The determinant of a matrix, written $\det(\mathbf{A})$ for a matrix $\mathbf{A}$ is used to determine whether the inverse of that matrix will exist. If $\det(\mathbf{A}) = 0$, the inverse of the matrix does not exist, otherwise it does. The determinant for a $2 \times 2$ matrix is the product of the elements on the diagonal subtracted by the product of the elements on the off diagonal. That is, for

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant is

$$\det(\mathbf{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

# 2 Further Mechanics (for Paper 3)

## 2.1 Motion of a projectile

*Model the motion of a projectile as a particle moving with constant acceleration and understand any limitations of the model*

A *projectile* is any object that, once it has been thrown, propelled or dropped, continues to move under its own inertia and the force of gravity. From Advanced Level Mechanics, we know that

- $s = ut + \frac{1}{2}at^2$
- $v = u + at$
- $v^2 = u^2 + 2as$
- $s = \frac{1}{2}(u + v)t$
- $s = vt - \frac{1}{2}at^2$

To predict the motion of projectiles, we make certain assumptions. We assume the absence of air resistance, no rotational forces (spin) affect the projectile since we consider it to be a *particle* (no volume) and that force due to gravity is constant. The path travelled by a projectile is known as a *parabolic trajectory*.

*Use horizontal and vertical equations of motion to solve problems on the motion of projectiles, including finding the magnitude and direction of the velocity at a given time or position, the range on a horizontal plane and the greatest height reached*

Consider a particle thrown with initial speed $u$, at an angle $\theta$ above the horizontal direction. Thus the horizontal and vertical components of the velocity is $u\cos\theta$ and $u\sin\theta$, respectively. Thus, from $v = u+at$

$$\boxed{v_x = u\sin\theta}$$

since there is no acceleration in the $x$-direction. The acceleration in the $y$-direction is that of gravity, in the downward direction, $-g$. Hence

$$\boxed{v_y = u\cos\theta - gt}$$

At the peak of the particle's motion,

$$\begin{aligned} v_y &= 0 \\ \implies u\sin\theta &= gt \\ \implies & \boxed{t = \frac{u\sin\theta}{g}} \end{aligned}$$

Since air resistance is ignored, the time for the particle to reach its peak is the same for the particle to drop back down to its original position, thus the total time for the motion is

$$\boxed{t = \frac{2u\sin\theta}{g}}$$

Now, taking $x$ as horizontal displacement and $y$ as vertical displacement, using $s = ut + \frac{1}{2}at^2$, we have

$$\begin{aligned} x &= u\cos\theta\frac{2u\sin\theta}{g} \\ &= \frac{2u^2\sin\theta\cos\theta}{g} \end{aligned}$$

recalling that $2\sin\theta\cos\theta \equiv 2\sin 2\theta$, we have

$$\boxed{x = \frac{u^2\sin 2\theta}{g}}$$

The above is called the *range* of the projectile's motion. This value is at a maximum when $\theta = 45°$, i.e., when $\sin 2\theta = 1$.

The maximum height of the projectile is derived as follows

$$\begin{aligned} y &= (u\sin\theta)\left(\frac{u\sin\theta}{g}\right) \\ & \boxed{= \frac{u^2\sin^2\theta}{g}} \end{aligned}$$

When an object begins its motion from a certain height above the horizontal. We can use $v^2 = u^2 + 2as$, for the $x$ and $y$ directions, which come out to

$$\boxed{v_x^2 = u_x^2}$$

$$\boxed{v_y^2 = u_y^2 - 2gs}$$

Now, if the object is thrown from a height $k$ above the horizontal, to find its velocity when it hits the ground we plug in $s = -k$ into the above. In fact, we can apply any value of $s$ between the maximum height of the projectile's motion.

To find the speed at any instant, we have

$$v = \sqrt{v_x^2 + v_y^2}$$

For a projectile launched below to horizontal, we consider the downward direction as positive for convenience, and we apply into the equations of motion and solve as needed.

---

*Derive and use the Cartesian equation of the trajectory of a projectile, including problems in which the initial speed and/or angle of projection may be unknown*

---

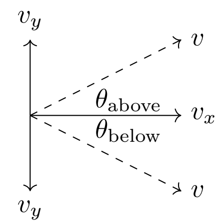
We know that the horizontal part of the motion is $x = ut\cos\theta$, which implies $t = x/u\cos\theta$. If we substitute this into the vertical part of the motion, which is $y = ut\sin\theta - \frac{1}{2}gt^2$, we get

$$\boxed{y = x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta}$$

When the particle is at ground level, $y = 0$, which means

$$x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta = 0$$
$$\implies x\left(\tan\theta - \frac{g}{2u^2}x\sec^2\theta\right) = 0$$
$$x = 0 \text{ and } \boxed{x = \frac{u^2\sin 2\theta}{g}}$$

Across the parabolic motion of the projectile, the angle of its velocity changes. Given that we are asked to find the position of an object when it is at an angle $\theta$ above or below the horizontal, we may use the following




From the above, we see that

$$\tan\theta = \frac{v_x}{v_y}$$

into which, if we plug in the values of $v_x$ and $v_y$, we obtain an equation in which $\theta$ can be solved for.

## 2.2 Equilibrium of a rigid body

---

*Calculate the moment of a force about a point*

---

The <u>*moment*</u> of a force $F$, about a point $O$ is $F \times d$, where $d$ is the perpendicular distance from the point $O$ to the line of action of the force $F$. If the distance between the force and the point $O$ is zero, there is no turning effect. The unit of moment is Newton metres: Nm. Thus,

$$\boxed{\tau = Fd}$$

For a force applied at an angle $\theta$ from the perpendicular of the line drawn from the pivot to the force the moment comes out to be

$$\boxed{\tau = Fd\cos\theta}$$

Here, we are simply finding the component of the force perpendicular to the distance, $F\cos\theta$, and multiplying the distance $d$ to it.

In general, there are two types of moments about a pivot: clockwise and anticlockwise moment. If the totals of the two moments about a pivot comes out to be equal, the object is said to be in *rotatonal <u>equilibrium</u>*. If the clockwise moment is greater, the object turns clockwise and same for if the anticlockwise moment is greater.

---

*Use the result that the effect of gravity on a rigid body is equivalent to a single force acting at the centre of* *mass of the body, and identify the position of the centre of mass of a uniform body using considerations of symmetry*

---

Consider a rod of length 2 m and mass 5 kg. We add a mass of 3 kg to one end. This rod hangs from a point $O$, which is the end opposite to where the mass is added. Thus the total moment, which is clockwise for this rod is

$$\begin{aligned}\tau &= (1)(5g) + (2)(3g) \\ &= 11g\end{aligned}$$

If we now consider the force coming from a point whose distance from $O$ is $\bar{x}$

$$\begin{aligned}\tau &= 11g \\ \implies (5g + 3g)(\bar{x}) &= 11g \\ \implies \bar{x} &= \frac{11}{8}\end{aligned}$$

Thus, the total moment of the rod may be said to be that of one force, 8 N at a distance of 11/8 m from $O$.

Consider a two-dimensional case such as a framework with masses added. This gives us coordinates for the centre of mass.