

Cambridge Advanced Subsidiary Level Notes
9709 Mathematics

Abrar Faiyaz Rahim, degrees pending

Contents

1 Pure Mathematics 3 (for Paper 3)	2
1.1 Algebra	2
2 Probability & Statistics 1	5
2.1 Representation of data	5

1 Pure Mathematics 3 (for Paper 3)

1.1 Algebra

Understand the meaning of $|x|$, sketch the graph of $y = |ax + b|$ and use relations such as $|a| = |b| \Leftrightarrow a^2 = b^2$ and $|x - a| < b \Leftrightarrow a - b < x < a + b$ when solving equations and inequalities

Solutions of Equations

The modulus of a number is its magnitude, disregarding its sign, the modulus of x is $|x|$ and hence, $|-a| = a$. The modulus of a number is also called its absolute value, defined formally below:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

For $|x| = k$, it is true that $x = \pm k$. This applies for any expression, including, but not limited to:

$$|ax + b| = k \Rightarrow ax + b = \pm k$$

$$|ax + b| = cx + d \Rightarrow ax + b = \pm(cx + d)$$

It can be proven that, $|ax+b| = |cx+d| \Rightarrow ax+b = \pm(cx+d)$

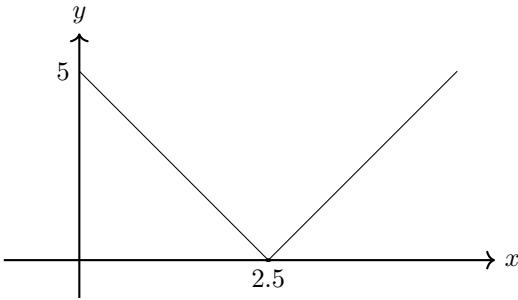
Graphs

The graph $y = |f(x)|$, where $f(x) = ax+b$, (linear), has the following characteristics:

$$y = |b|, \quad \text{for } x = 0$$

$$x = -\frac{b}{a}, \quad \text{for } y = 0$$

For example, the graph of $y = |2x - 5|$:



Note that these graphs can be subject to transformations discussed in Section 1.2.

Solutions of Inequalities

Equations of the form $|cx - a| < b$, can be solved algebraically using the fact that:

$$\begin{aligned} |cx - a| &< b \\ \Rightarrow -b &< cx - a < b \\ \Rightarrow (a - b)/c &< x < (a + b)/c \end{aligned}$$

Graphically, the graphs of the expressions on either side may be drawn, and the ranges for which the condition of the given question stands can be deduced intuitively.

Inequalities where there are modulus functions on both sides of the inequality symbol may be solved using $|a| = |b| \Rightarrow a^2 = b^2$, where the $=$ symbol can be replaced with any inequality symbol. The more general method to solve such inequalities is to draw out the graphs of the two functions, and to find the range for which the inequality stands.

Divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero)

A polynomial is an expression of the form $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x^1 + a_0$ where x is a variable; n is a non-negative integer; the coefficients $a_n \dots a_0$ are constants, of which $a_n \neq 0$, called the leading coefficient and a_0 is the constant term. The highest power of x in a polynomial is called its degree.

Long Division

The division of the polynomial $2x^3 + 3x^2 - 2x + 5$ by $x + 2$ is shown as follows:

$$\begin{array}{r} 2x^2 - x \\ x+2) \overline{2x^3 + 3x^2 - 2x + 5} \\ \underline{-2x^3 - 4x^2} \\ \underline{-x^2 - 2x} \\ \underline{x^2 + 2x} \\ 5 \end{array}$$

The above shows that the quotient of this division is $2x^2 - x$ and the remainder is 5.

Use the factor theorem and the remainder theorem

Observe the following long division:

$$\begin{array}{r} x^2 + 4x - 3 \\ x - 2) \overline{x^3 + 2x^2 - 11x + 6} \\ - x^3 + 2x^2 \\ \hline 4x^2 - 11x \\ - 4x^2 + 8x \\ \hline - 3x + 6 \\ - 3x - 6 \\ \hline 0 \end{array}$$

since the remainder here is zero, $x^3 + 2x^2 - 11x + 6$ is divisible by $x - 2$, which we can write as the following:

$$(x - 2)(x^2 + 4x - 3) = x^3 + 2x^2 - 11x + 6$$

Now, if we write $f(x) = x^3 + 2x^2 - 11x + 6 = (x - 2)(x^2 + 4x - 3)$, we see that $f(2) = 0$. In general, when an expression $f(x)$ is divisible by a linear expression $ax - b$, $f(b/a) = 0$. This is the factor theorem.

Observe the following long division:

$$\begin{array}{r} 2x^2 - x \\ x + 2) \overline{2x^3 + 3x^2 - 2x + 5} \\ - 2x^3 - 4x^2 \\ \hline - x^2 - 2x \\ - x^2 - 2x \\ \hline 5 \end{array}$$

the results of the above we may display as

$$(x + 2)(2x^2 - x) + 5 = 2x^3 + 3x^2 - 2x + 5$$

same as above, we write $f(x) = 2x^3 + 3x^2 - 2x + 5 = (x + 2)(2x^2 - x) + 5$, we find $f(-2) = 5$. In general, an expression $f(x)$ when divided by a linear expression $ax - b$ leaving remainder R has $f(b/a) = R$. This is the remainder theorem.

Recall an appropriate form for expressing rational functions in partial fractions, and carry out the decomposition, in cases where the denominator is no more complicated than

- $(ax + b)(cx + d)(ex + f)$
 - $(ax + b)(cx + d)^2$
 - $(ax + b)(cx^2 + d)$
-

Algebraic improper fractions

An algebraic improper fraction is such that $P(x)/Q(x)$ where the degree of P is greater than or equal to that of Q .

We observe the example of $(x^3 - 3x^2 + 7)/(x - 2)$:

$$\begin{array}{r} x^2 - x - 2 \\ x - 2) \overline{x^3 - 3x^2 + 7} \\ - x^3 + 2x^2 \\ \hline - x^2 \\ \quad x^2 - 2x \\ \quad - 2x + 7 \\ \quad 2x - 4 \\ \hline 3 \end{array}$$

thus,

$$\begin{aligned} x^3 - 3x^2 + 7 &= (x^2 - x - 2)(x - 2) + 3 \\ \Rightarrow \frac{x^3 - 3x^2 + 7}{x - 2} &= x^2 - x - 2 + \frac{3}{x - 2} \end{aligned}$$

In general, the improper function $P(x)/Q(x)$, with quotient q and remainder R can be written:

$$\boxed{\frac{P(x)}{Q(x)} = q + \frac{R}{Q(x)}}$$

Partial Fractions

For fractions whose denominator consists of two or more distinct linear factors:

$$\boxed{\frac{px + q}{(ax + b)(cx + d)(\dots)} \equiv \frac{A}{ax + b} + \frac{B}{cx + d} + \dots}$$

For fractions whose denominator consists of a linear repeated factor:

$$\frac{px + q}{(ax + b)^2} \equiv \frac{A}{ax + b} + \frac{B}{(ax + b)^2}$$

For fractions with a denominator consisting of the product of a linear factor and a depressed quadratic:

$$\frac{P(x)}{(ax + b)(cx + d)^2} = \frac{A}{ax + b} + \frac{B}{cx + d} + \frac{C}{(cx + d)^2}$$

For fractions with an indivisible quadratic denominator:

$$\frac{px + q}{(ax + b)(cx^2 + d)} \equiv \frac{A}{ax + b} + \frac{Bx + C}{cx^2 + d}$$

The left hand side of the above may be added algebraically, expanded and the denominators cancelled, leading to an equation from which the values of the numerators can be found by means of comparing the coefficients.

Use the expansion of $(1 + x)^n$, where n is a rational number and $|x| < 1$.

2 Probability & Statistics 1

2.1 Representation of data

Select a suitable way of presenting raw statistical data, and discuss advantages and/or disadvantages that particular representations may have

Draw and interpret stem-and-leaf diagrams, box-and-whisker plots, histograms and cumulative frequency graphs

Stem-and-leaf diagrams are a way to arrange the raw data.

Box-and-whisker plots (boxplots) are one dimensional diagrams which show the interquartile range (IQR) and the minimum and maximum of the given data, and also outliers when applicable.

Histograms are used where the independent variable comes in the form of ranges. We find frequency density for each range, by finding range width and dividing frequency by it. Then we plot frequency density across the ranges on the x-axis.

Understand and use different measures of central tendency (mean, median, mode) and variation (range, interquartile range, standard deviation)

Measures of Central Tendency

For an ungrouped set of data with n observations and frequencies x_1, x_2, \dots, x_n , the mean is

$$\bar{x} = \frac{\Sigma x}{n}$$

For grouped data where the groups have midvalues x and the groups each have frequencies f , the mean is

$$\bar{x} = \frac{\Sigma xf}{\Sigma f} = \frac{\Sigma fx}{\Sigma f}$$

Coded data is where each observation in the set is offset by a certain value. Here, for ungrouped data,

$$\bar{x} = \frac{\Sigma(x - b)}{n} + b$$

and for grouped data,

$$\bar{x} = \frac{\Sigma(x - b)f}{\Sigma f} + b$$

For ungrouped data, the median is at the $(\frac{n+1}{2})$ th value, and it is at the $\frac{n}{2}$ th value on a cumulative frequency graph.

Measures of variation

Interquartile range is difference between upper and lower quartile, $Q_3 - Q_1$ where $Q_n = (n/4)\Sigma f$

For ungrouped data, standard deviation (σ) is as follows

$$\sigma(x) = \sqrt{\text{Var } x} = \sqrt{\frac{(\Sigma x - \bar{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2}$$

and for grouped data,

$$\sigma(x) = \sqrt{\text{Var } x} = \sqrt{\frac{(\Sigma x - \bar{x})^2 f}{\Sigma f}} = \sqrt{\frac{\Sigma x^2 f}{\Sigma f} - \bar{x}^2}$$