K-Theory

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Contents

Introduction

These notes are based on the course of lectures I gave at Harvard in the fall of 1964. They constitute a self-contained account of vector bundles and K-theory assuming only the rudiments of point-set topology and linear algebra. One of the features of the treatment is that no use is made of ordinary homology or cohomology theory. In fact rational cohomology is defined in terms of K-theory.

The theory is taken as far as the solution of the Hopf invariant problem and a start is made on the J-homomorphism. In addition to the lecture notes proper two papers of mine published since 1964 have been reproduced at the end. The first, dealing with operations, is a natural supplement to the material in Chapter 3. It provides an alternative approach to operations which is less slick but more fundamental than the Grothendieck method of Chapter III and it relates operations and filtration. Actually the lectures deal with compact spaces not cell-complexes and so the skeleton-filtration does not figure in the notes. The second paper provides a new approach to real K-theory and so fills an obvious gap in the lecture notes.

Chapter 1

VECTOR BUNDLES

1.1 Basic definitions. We shall develop the theory of *complex* vector bundles only, though much of the elementary theory is the same for real and symplectic bundles. Therefore, by vector space, we shall always understand complex vector space unless otherwise specified.

Let X be a topological space. A *family of vector spaces over* X is a topological space E, together with:

(i) a continuous map $p : E \to X$

(ii) a finite dimensional vector space structure on each

$$E_x = p^{-1}(x) \qquad \text{for } x \in X,$$

compatible with the topology on E_x induced from E.

The map p is called the projection map, the space E is called the total space of the family, the space X is called the base space of the family, and if $x \in X$, E_x is called the fiber over x.

A *section* of a family $p : E \to X$ is a continuous map $s : X \to E$ such that $ps(x) = x$ for all $x \in X$.

A *homomorphism* from one family $p : E \to X$ to another family $q : F \to X$ is a continuous map $\varphi : E \to F$ such that:

(i) $q\varphi = p$

(ii) for each $x \in X$, $\varphi : E_x \to F_x$ is a linear map of vector spaces.

We say that φ is an *isomorphism* if φ is bijective and φ^{-1} is continuous. If there exists an isomorphism between E and F, we say that they are isomorphic.

Example 1. Let V be a vector space, and let $E = X \times V$, $p : E \to X$ be the projection onto the first factor. E is called the *product family* with fiber V. If F is any family which is isomorphic to some product family, F is said to be a *trivial* family.

If Y is a subspace of X, and if E is a family of vector spaces over X with projection p, $p : p^{-1}(Y) \to Y$ is clearly a family over Y. We call it the *restriction* of E to Y, and denote it by $E|Y$. More generally, if Y is any space, and $f : Y \to X$ is a continuous map, then we define the induced family $f^*(p) : f^*(E) \to Y$ as follows:

$f^*(E)$ is the subspace of $Y \times E$ consisting of all points (y, e) such that $f(y) = p(e)$, together with the obvious projection maps and vector space structures on the fibers. If $g : Z \to Y$, then there is a natural isomorphism $g^* f^*(E) \cong (f g)^*(E)$ given by sending each point of the form (z, e) into the point $(z, g(z), e)$, where $z \in Z$, $e \in E$. If $f : Y \to X$ is an inclusion map, clearly there is an isomorphism $E|Y \cong f^*(E)$ given by sending each $e \in E$ into the corresponding $(p(e), e)$.

A family E of vector spaces over X is said to be *locally trivial* if every $x \in X$ posesses a neighborhood U such that $E|U$ is trivial. A locally trivial family will also be called a *vector bundle*. A trivial family will be called a trivial bundle. If $f : Y \to X$, and if E is a vector bundle over X, it is easy to see that $f^*(E)$ is a vector bundle over Y. We shall call $f^*(E)$ the induced bundle in this case.

Example 2. Let V be a vector space, and let X be its associated projective space. We define $E \subset X \times V$ to be the set of all (x, v) such that $x \in X$, $v \in V$, and v lies in the line determining x. We leave it to the reader to show that E is actually a vector bundle.

Notice that if E is a vector bundle over X, then $\dim(E_x)$ is a locally constant function on X, and hence is a constant on each connected component of X. If $\dim(E_x)$ is a constant on the whole of X, then E is said to have a dimension, and the dimension of E is the common number $\dim(E_x)$ for all x. (Caution: the dimension of E so defined is usually different from the dimension of E as a topological space.)

Since a vector bundle is locally trivial, any section of a vector bundle is locally described by a vector valued function on the base space. If E is a vector bundle, we denote by $\Gamma(E)$ the set of all sections of E. Since the set of functions on a space with values in a fixed vector space is itself a vector space, we see that $\Gamma(E)$ is a vector space in a natural way.

Suppose that V, W are vector spaces, and that $E = X \times V$, $F = X \times W$ are the corresponding product bundles. Then any homomorphism $\varphi : E \to F$ determines a map $\Phi : X \to \mathrm{Hom}(V, W)$ by the formula $\varphi(x, v) = (x, \Phi(x)v)$. Moreover, if we give $\mathrm{Hom}(V, W)$ its usual topology, then Φ is continuous; conversely, any such continuous map $\Phi : X \to \mathrm{Hom}(V, W)$ determines a homomorphism $\varphi : E \to F$. (This is most easily seen by taking bases e_i and f_i for V and W respectively. Then each $\Phi(x)$ is represented by a matrix $\Phi(x)_{i,j}$, where

$$\Phi(x)e_i = \sum_j \Phi(x)_{i,j} f_j$$

The continuity of either φ or Φ is equivalent to the continuity of the functions $\Phi(x)_{i,j}$.)

Let $\mathrm{Iso}(V, W) \subset \mathrm{Hom}(V, W)$ be the subspace of all isomorphisms between V and W. Clearly, $\mathrm{Iso}(V, W)$ is an open set in $\mathrm{Hom}(V, W)$. Further, the inverse map $T \to T^{-1}$ gives us a continuous map $\mathrm{Iso}(V, W) \to \mathrm{Iso}(W, V)$. Suppose that $\varphi : E \to F$ is such that $\varphi_x : E_x \to F_x$ is an isomorphism for all $x \in X$. This is equivalent to the statement that $\Phi(x) \subset \mathrm{Iso}(V, W)$. The map $x \to \Phi(x)^{-1}$ defines $\Psi : X \to \mathrm{Iso}(W, V)$, which is continuous. Thus the corresponding map

$\psi : F \to E$ is continuous. Thus $\varphi : E \to F$ is an isomorphism if and only if it is bijective or, equivalently, φ is an isomorphism if and only if each φ_x is an isomorphism. Further, since $\mathrm{Iso}(V, W)$ is open in $\mathrm{Hom}(V, W)$, we see that for any homomorphism φ, the set of those points $x \in X$ for which φ_x is an isomorphism form an open subset of X. All of these assertions are local in nature, and therefore are valid for vector bundles as well as for trivial families.

Remark: The finite dimensionality of V is basic to the previous argument. If one wants to consider infinite dimensional vector bundles, then one must distinguish between the different operator topologies on $\mathrm{Hom}(V, W)$.

1.2 Operations on vector bundles. Natural operations on vector spaces, such as direct sum and tensor product, can be extended to vector bundles. The only troublesome question is how one should topologize the resulting spaces. We shall give a general method for extending operations from vector spaces to vector bundles which will handle all of these problems uniformly.

Let T be a functor which carries finite dimensional vector spaces into finite dimensional vector spaces. For simplicity, we assume that T is a covariant functor of one variable. Thus, to every vector space V, we have an associated vector space $T(V)$. We shall say that T is a *continuous functor* if for all V and W , the map $T : \mathrm{Hom}(V,W) \to \mathrm{Hom}(T(V), T(W))$ is continuous.

If E is a vector bundle, we define the set $T(E)$ to be the union

$$\bigcup_{x \in X} T(E_x),$$

and, if $\varphi : E \to F$, we define $T(\varphi) : T(E) \to T(F)$ by the maps $T(\varphi_x) : T(E_x) \to T(F_x)$. What we must show is that $T(E)$ has a natural topology, and that, in this topology, $T(\varphi_x)$ is continuous.

We begin by defining $T(E)$ in the case that E is a product bundle. If $E = X \times V$, we define $T(E)$ to be $X \times T(V)$ in the product topology. Suppose that $F = X \times W$, and that $\varphi : E \to F$ is a homomorphism. Let $\Phi : X \to \mathrm{Hom}(V,W)$ be the corresponding map. Since, by hypothesis, $T : \mathrm{Hom}(V,W) \to \mathrm{Hom}(T(V), T(W))$ is continuous, $T\Phi : X \to \mathrm{Hom}(T(V), T(W))$ is continuous. Thus $T(\varphi) : X \times T(V) \to X \times T(W)$ is also continuous. If φ is an isomorphism, then $T\varphi$ will be an isomorphism since it is continuous and an isomorphism on each fiber.

Now suppose that E is trivial, but has no preferred product structure. Choose an isomorphism $\alpha : E \to X \times V$, and topologize $T(E)$ by requiring $T(\alpha) : T(E) \to X \times T(V)$ to be a homeomorphism. If $\beta : E \to X \times W$ is any other isomorphism, by letting $\varphi = \beta\alpha^{-1}$ above, we see that $T(\alpha)$ and $T(\beta)$ induce the same topology on $T(E)$, since $T(\varphi) = T(\beta)T(\alpha)^{-l}$ is a homeomorphism. Thus, the topology on E does not depend on the choice of α. Further, if $Y \subset X$, it is clear that the topology on $T(E)|Y$ is the same as that on $T(E|Y)$. Finally, if $\varphi : E \to F$ is a homomorphism of trivial bundles, we see that $T(\varphi) : T(E) \to T(F)$ is continuous, and therefore is a homomorphism.

Now suppose that E is any vector bundle. Then if $U \subset X$ is such that $E|U$ is trivial, we topologize $T(E|U)$ as above. We topologize $T(E)$ by taking for the open sets, those subsets $V \subset T(E)$ such that $V \cap (T(E)|U)$ is open in $T(E|U)$ for all open $U \subset X$ for which $E|U$ is trivial. The reader can now easily verify that if $Y \subset X$, the topology on $T(E|Y)$ is the same as that on $T(E)|Y$, and that, if $\varphi : E \to F$ is any homomorphism, $T(\varphi) : T(E) \to T(F)$ is also a homomorphism.

If $f : Y \to X$ is a continuous map and E is a vector bundle over X then, for any continuous functor T, we have a natural isomorphism

$$f^*T(E) \cong Tf^*(E).$$

The case when T has several variables both covariant and contravariant, proceeds similarly. Therefore we can define for vector bundles E, F corresponding bundles:

(i) $E \oplus F$, their direct sum

(ii) $E \otimes F$, their tensor product

(iii) $\mathrm{Hom}(E, F)$

(iv) E^*, the dual bundle of E

(v) $\lambda^i(E)$, where λ^i is the i^{th} exterior power.

We also obtain natural isomorphisms

(i) $E \oplus F \cong F \oplus E$

(ii) $E \otimes F \cong F \otimes E$

(iii) $E \otimes (F' \oplus F'') \cong (E \otimes F') \oplus (E \otimes F'')$

(iv) $\mathrm{Hom}(E, F) \cong E^* \otimes F$

(v) $\lambda^k(E \oplus F) \cong \bigoplus_{i+j=k}(\lambda^i(E) \otimes \lambda^j(F))$

Finally, notice that sections of $\mathrm{Hom}(E,F)$ correspond in a 1 - 1 fashion with homomorphisms $\varphi : E \to F$. We therefore define $\mathrm{HOM}(E,F)$ to be the vector space of all homomorphisms from E to F, and make the identification $\mathrm{HOM}(E,F) = \Gamma(\mathrm{Hom}(E,F))$.

1.3 Sub-bundles and quotient bundles. Let E be a vector bundle. A *sub-bundle* of E is a subset of E which is a bundle in the induced structure.

A homomorphism $\varphi : F \to E$ is called a *monomorphism* (respectively *epimorphism*) if each $\varphi_x : F_x \to E_x$ is a monomorphism (respectively epimorphism). Notice that $\varphi : F \to E$ is a monomorphism if and only if $\varphi^* : F^* \to E^*$ is an epimorphism. If F is a sub-bundle of E, and if $\varphi : F \to E$ is the inclusion map, then φ is a monomorphism.

1.3.1 LEMMA *If $\varphi : F \to E$ is a monomorphism, then $\varphi(F)$ is a sub-bundle of E, and $\varphi : F \to \varphi(F)$ is an isomorphism.*

Proof: $\varphi : F \to \varphi(F)$ is a bijection, so if $\varphi(F)$ is a sub-bundle, φ is an isomorphism. Thus we need only show that $\varphi(F)$ is a sub-bundle.

The problem is local, so it suffices to consider the case when E and F are product bundles. Let $E = X \times V$ and let $x \in X$; choose $W_x \subset V$ to be a subspace complementary to $\varphi(F_x)$. $G = X \times W_x$ is a sub-bundle of E. Define $\theta : F \oplus G \to E$ by $\theta(a \oplus b) = \varphi(a) + i(b)$, where $i : G \to E$ is the inclusion. By construction, θ_x is an isomorphism. Thus, there exists an open neighborhood U of x such that $\theta|U$ is an isomorphism. F is a sub-bundle of $F \oplus G$. so $\theta(F) = \varphi(F)$ is a sub-bundle of $\theta(F \oplus G) = E$ on U.

Notice that in our argument, we have shown more than we have stated. We have shown that if $\varphi : F \to E$, then the set of points for which φ_x is a monomorphism form an open set. Also, we have shown that, locally, a sub-bundle is direct summand. This second fact allows us to define quotient bundles.

1.3.2 DEFINITION . If F is a sub-bundle of E, the quotient bundle E/F is the union of all the vector spaces E_x/F_x given the quotient topology.

Since F is locally a direct summand in E, we see that E/F is locally trivial, and thus is a bundle. This justifies the terminology.

If $\varphi : F \to E$ is an arbitrary homomorphism, the function dimension(kernel(φ_x)) need not be constant, or even locally constant.

1.3.3 DEFINITION $\varphi : F \to E$ is said to be a *strict* homomorphism if dimension(kernel(φ_x)) is locally constant.

1.3.4 PROPOSITION . If $\varphi : F \to E$ is strict, then:

(i) $\text{kernel}(\varphi) = \bigcup_x \text{kernel}(\varphi_x)$ is a sub-bundle of F

(ii) $\text{image}(\varphi) = \bigcup_x \text{image}(\varphi_x)$ is a sub-bundle of E

(iii) $\text{cokernel}(\varphi) = \bigcup_x \text{cokernel}(\varphi_x)$ is a bundle in the quotient structure.

Proof: Notice that (ii) implies (iii). We first prove (ii). The problem is local, so we can assume $F = X \times V$ for some V. Given $x \in X$, we choose $W_x \subset V$ complementary to $\ker(\varphi_x)$ in V. Put $G = X \times W_x$; then φ induces, by composition with the inclusion, a homomorphism, $\psi : G \to E$, such that ψ_x is a monomorphism. Thus, ψ is a monomorphism in some neighborhood U of x. Therefore, $\psi(G)|U$ is a sub-bundle of $E|U$. However, $\psi(G) \subset \varphi(F)$, and since $\dim(\varphi(F_y))$ is constant for all y, and $\dim(\psi(G_y)) = \dim(\psi(G_x)) = \dim(\varphi(F_x)) = \dim(\varphi(F_y))$ for all $y \in U$, $\psi(G)|U = \varphi(F)|U$. Thus $\varphi(F)$ is a sub-bundle of E.

Finally, we must prove (i). Clearly, $\varphi^* : E^* \to F^*$ is strict. Since $F^* \to \text{coker}(\varphi^*)$ is an epimorphism, $(\text{coker}(\varphi^*))^* \to F^{**}$ is a monomorphism. However, for each x we have a natural commutative diagram:

$$
\begin{array}{ccc}
\ker(\varphi_x) & \longrightarrow & F_x \\
\downarrow & & \downarrow \\
(\text{coker}\,\varphi_x^*)^* & \longrightarrow & F_x^{**}
\end{array}
$$

in which the vertical arrows are isomorphisms. Thus $\ker(\varphi) \cong (\text{coker}(\varphi^*))^*$ and so, by (1.3.1), is a sub-bundle of F.

Again, we have proved something more than we have stated. Our argument shows that for any $x \in X$, $\dim \varphi_x(F_x) \leq \dim \varphi_y(F_y)$ for all $y \in U$, U some neighborhood of x. Thus, $\text{rank}(\varphi_x)$ *is an upper semi-continuous function of* x.

1.3.5 DEFINITION A projection operator $P : E \to E$ is a homomorphism such that $P^2 = P$.

Notice that $\text{rank}(P_x) + \text{rank}(1 - P_x) = \dim E_x$ so that, since both $\text{rank}(P_x)$ and $\text{rank}(1 - P_x)$ are upper semi-continuous functions of x, they are locally constant. Thus both P and $1-P$ are strict homomorphisms. Since $\ker(P) = (1-P)E$, E is the direct sum of the two sub-bundles PE and $(1-P)E$. Thus any projection operator $P : E \to E$ determines a direct sum decomposition $E = (PE) \oplus ((1-P)E)$.

We now consider metrics on vector bundles. We define a functor Herm which assigns to each vector space V the vector space $\mathrm{Herm}(V)$ of all Hermitian forms on V. By the techniques of section 1.2, this allows us to define a vector bundle $\mathrm{Herm}(E)$ for every bundle E.

1.3.6 DEFINITION A *metric* on a bundle E is any section $h : X \to \mathrm{Herm}(E)$ such that $h(x)$ is positive definite for all $x \in X$. A bundle with a specified metric is called a Hermitian bundle.

Suppose that E is a bundle, F is a sub-bundle of E, and that h is a Hermitian metric on E. Then for each $x \in X$ we consider the orthogonal projection $P_x : E_x \to F_x$ defined by the metric. This defines a map $P : E \to F$ which we shall now check is continuous. The problem being local we may assume F is trivial, so that we have sections $f_1, \ldots, f_n$ of F giving a basis in each fiber. Then for $v \in F_x$ we have

$$P_x(v) = \sum_i h_x(v, f_i(x)) f_i(x)$$

Since h is continuous this implies that P is continuous. Thus P is a projection operator on E. If $F_x^\perp$ is the subspace of E_x which is orthogonal to F under h, we see that $F^\perp = \bigcup_x F_x^\perp$ is the kernel of P, and thus is a sub-bundle of E, and that $E \cong F \oplus F^\perp$. Thus, a metric provides any sub-bundle with a definite complementary sub-bundle.

Remark: So far, most of our arguments have been of a very general nature, and we could have replaced "continuous" with "algebraic", "differentiable", "analytic", etc. without any trouble. In the next section, our arguments become less general.

1.4 Vector bundles on compact spaces. In order to proceed further, we must make some restriction on the sort of base spaces which we consider. We shall assume from now on that our base spaces are *compact Hausdorff*. We leave it to the reader to notice which results hold for more general base spaces.

Recall that if $f : X \to V$ is a continuous vector-valued function the support of f (written $\operatorname{supp} f$) is the closure of $f^{-1}(V - \{0\})$.

We need the following results from point set topology. We state them in vector forms which are clearly equivalent to the usual forms:

Tietze Extension Theorem. Let X be a normal space, $Y \subset X$ a closed subspace, V a real vector space, and $f : Y \to V$ a continuous map. Then there exists a continuous map $g : X \to V$ such that $g|Y = f$.

Existence of Partitions of Unity. Let X be a compact Hausdorff space, $\{U_i\}$ a finite open covering. Then there exist continuous maps $f_i : X \to \mathbb{R}$ such that:

1. $f_i \geq 0 \qquad$ all $x \in X$
2. $\operatorname{supp}(f_i) \subset U_i$
3. $\sum_i f_i(x) = 1 \qquad$ all $x \in X$

Such a collection $\{f_i\}$ is called a *partition of unity*.

We first give a bundle form of the Tietze extension theorem.

1.4.1 LEMMA *Let X be compact Hausdorff, $Y \subset X$ a closed subspace, and E a bundle over X. Then any section $s : Y \to E|Y$ can be extended to X.*

Proof: Let $s \in \Gamma(E|Y)$. Since, locally, s is a vector-valued function, we can apply the Tietze extension theorem to show that for each $x \in X$, there exists an open set U containing x and $t \in \Gamma(E|U)$ such that $t|U \cap Y = s|U \cap Y$. Since X is compact, we can find a finite subcover $\{U_\alpha\}$ by such open sets. Let $t_\alpha \in \Gamma(E|U_\alpha)$ be the corresponding sections and let $\{p_\alpha\}$ be a partition of unity with $\operatorname{supp}(p_\alpha) \in U_\alpha$. We define $S_\alpha \in \Gamma(E)$ by

$$S_\alpha = \begin{cases} p_\alpha(x) t_\alpha(x) & \text{if } x \in U_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum S_\alpha$ is a section of E and its restriction to Y is clearly s.

1.4.2 LEMMA *Let Y be a closed subspace of a compact Hausdorff space X, and let E, F be two vector bundles over X. If $f : E|Y \to F|Y$ is an isomorphism, then there exists an opet set U containing Y and an extension $f : E|U \to F|U$ which is an isomorphism.*

Proof: f is a section of $\mathrm{Hom}(E|Y, F|Y)$, and thus, extends to a section of $\mathrm{Hom}(E, F)$. Let U be the set of those points for which this map is an isomorphism. Then U is open and contains Y.

1.4.3 LEMMA *Let Y be a compact Hausdorff space, $f_t : Y \to X$ $(0 \leq t \leq 1)$ a homotopy and E a vector bundle over X. Then*

$$f_0^* E \cong f_1^* E.$$

Proof: If I denotes the unit interval, let $f : Y \times I \to X$ be the homotopy, so that $f(y,t) = f_t(y)$, and let $\pi : Y \times I \to Y$ denote the projection. Now apply (1.4.2) to the bundles $f^*E, \pi^* f_t^* E$ and the subspace $Y \times \{t\}$ of $Y \times I$, on which the re is an obvious isomorphism s. By the compactness of Y we deduce that f^*E and $\pi^* f_t^* E$ are isomorphic in some strip $Y \times \delta t$ where δt denotes a neighborhood of $\{t\}$ in I. Hence the isomorphism class of $f_t^* E$ is a locally constant function of t. Since I is connected this implies it is constant, whence

$$f_0^* E \cong f_1^* E.$$

We shall use $\mathrm{Vect}(X)$ to denote the set of isomorphism classes of vector bundles on X, and $\mathrm{Vect}_n(X)$ to denote the subset of $\mathrm{Vect}(X)$ given by bundles of dimension n. $\mathrm{Vect}(X)$ is an abelian semi-group under the operation $\oplus$. In $\mathrm{Vect}_n(X)$ we have one naturally distinguished element - the class of the trivial bundle of dimension n.

1.4.4 LEMMA

(1) *If $f : X \to Y$ is a homotopy equivalence, $f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ is bijective.*

(2) *If X is contractible, every bundle over X is trivial and* $\mathrm{Vect}(X)$ *is isomorphic to the non-negative integers.*

1.4.5 LEMMA *If E is a bundle over $X \times I$, and $\pi : X \times I \to X \times \{0\}$ is the projection, E is isomorphic to $\pi^*(E|X \times \{0\})$.*

Both of these lemmas are immediate consequences of (1.4.3).

Suppose now Y is closed in X, E is a vector bundle over X and $\alpha : E|Y \to Y \times V$ is an isomorphism. We refer to α as a *trivialization of E over Y*. Let $\pi : Y \times V \to V$ denote the projection and define an equivalence relation on $E|Y$ by

$$e \sim e' \iff \pi\alpha(e) = \pi\alpha(e')$$

We extend this by the identity on $E|X - Y$ and we let E/α denote the quotient space of E given by this equivalence relation. It has a natural structure of a family of vector spaces over X/Y. We assert that E/α is in fact a vector bundle. To see this we have only to verify the local triviality at the base point Y/Y of X/Y. Now by (1.4.2) we can extend α to an isomorphism $\alpha : E|U \to U \times V$ for some open set U containing Y. Then α induces an isomorphism

$$(E|U)/\alpha \cong (U/Y) \times V$$

which establishes the local triviality of E/α.

Suppose α_0, α_1 are homotopic trivializations of E over Y. This means that we have a trivialization β of $E \times I$ over $Y \times I \subset X \times I$ inducing α_0 and α_1 at the two end points of I. Let $f : (X \times Y) \times I \to (X \times I)/(Y \times I)$ be the natural map. Then $f^*(E \times I/\beta)$ is a bundle on $(X/Y) \times I$ whose restriction to $(X/Y) \times \{i\}$ is $E/\alpha_i \quad (i = 0, 1)$. Hence, by (1.4.3),

$$E/\alpha_0 \cong E/\alpha_1.$$

To summarize we have established

1.4.6 LEMMA *A trivialization α of a bundle E over $Y \subset X$ defines a bundle E/α over X/Y. The isomorphism class of E/α depends only on the homotopy class of α.*

Using this we shall now prove

1.4.7 LEMMA *Let $Y \subset X$ be a closed contractible subspace. Then $f : X \to X/Y$ induces a bijection $f^* : \mathrm{Vect}(X) \to \mathrm{Vect}(X/Y)$.*

Proof: Let E be a bundle on X then by (1.4.4) $E|Y$ is trivial. Thus trivializations $\alpha : E|Y \to Y \times V$ exist. Moreover, two such trivializations differ by an automorphism of $Y \times V$, i.e., by a map $Y \to \mathrm{GL}(V)$. But $\mathrm{GL}(V) = \mathrm{GL}(n, \mathbb{C})$ is connected and V is contractible. Thus α is unique up to homotopy and so the isomorphism class of $E|\alpha$ is uniquely determined by that of E. Thus we have constructed a map

$$\mathrm{Vect}(X) \to \mathrm{Vect}(X/Y)$$

and this is clearly a two-sided inverse for f^*. Hence f^* is bijective as asserted.

Vector bundles are frequently constructed by a glueing or clutching construction which we shall now describe. Let

$$X = X_1 \cup X_2, \qquad A = X_1 \cap X_2,$$

all the spaces being compact. Assume that E_i is a vector bundle over X_i and that $\varphi : E_1|A \to E_2|A$ is an isomorphism. Then we define the vector bundle $E_1 \cup_\varphi E_2$ on X as follows. As a topological space $E_1 \cup_\varphi E_2$ is the quotient of the disjoint sum $E_1 + E_2$ by the equivalence relation which identifies $e_1 \in E_1|A$ with $\varphi(e_1) \in E_2|A$. Identifying X with the corresponding quotient of $X_1 + X_2$ we obtain a natural projection $p : E_1 \cup_\varphi E_2 \to X$, and $p^{-1}(x)$ has a natural vector space structure. It remains to show that $E_1 \cup_\varphi E_2$ is locally trivial. Since $E_1 \cup_\varphi E_2 | X - A = (E_1|X_1 - A) + (E_2|X_2 - A)$ the local triviality at points $x \notin A$ follows from that of E_1 and E_2. Therefore, let $a \in A$ and let V_1 be a closed neighborhood of a in X_1 over which E_1 is trivial. so that we have an isomorphism

$$\theta_1 : E_1|V_1 \to V_1 \times \mathbb{C}^n.$$

Restricting to A we get an isomorphism

$$\theta_1^A : E_1|V_1 \cap A \to (V_1 \cap A) \times \mathbb{C}^n.$$

Let

$$\theta_2^A : E_2|V_1 \cap A \to (V_1 \cap A) \times \mathbb{C}^n.$$

be the isomorphism corresponding to θ_1^A under φ. By (1.4.2) this can be extended to an isomorphism

$$\theta_2 : E_2|V_2 \to V_2 \times \mathbb{C}^n.$$

where V_2 is a neighborhood of a in X_2. The pair θ_1, θ_2 then defines in an obvious way an isomorphism

$$\theta_1 \cup_\varphi \theta_2 : E_1 \cup_\varphi E_2|V_1 \cup V_2 \to (V_1 \cup V_2) \times \mathbb{C}^n.$$

establishing the local triviality of $E_1 \cup_\varphi E_2$.

Elementary properties of this construction are the following:

(i) If E is a bundle over X and $E_i = E|X_i$ then the identity defines an isomorphism $I_A : E_1|A \to E_2|A$. and

$$E_1 \cup_{I_A} E_2 \cong E.$$

(ii) If $\beta_i : E_i \to E_i'$ are isomorphisms on X_i and $\varphi'\beta_1 = \beta_2\varphi$, then

$$E_1 \cup_\varphi E_2 \cong E_1' \cup_{\varphi'} E_2'.$$

(iii) (E_i, φ) and (E_i', φ') are two "clutching data" on the X_i, then

$$(E_1 \cup_\varphi E_2) \oplus (E_1' \cup_{\varphi'} E_2') \cong E_1 \oplus E_1' \cup_{\varphi\oplus\varphi'} E_2 \oplus E_2',$$
$$(E_1 \cup_\varphi E_2) \otimes (E_1' \cup_{\varphi'} E_2') \cong E_1 \otimes E_1' \cup_{\varphi\otimes\varphi'} E_2 \otimes E_2',$$
$$(E_1 \cup_\varphi E_2)^* \cong E_1^* \cup_{(\varphi^*)^{-1}} E_2^*$$

Moreover, we also have

1.4.8 LEMMA *The isomorphism class of* $E_1 \cup_\varphi E_2$ *depends only on the homotopy class of the isomorphism* $\varphi : E_1|A \to E_2|A$.

Proof: A homotopy of isomorphism $E_1|A \to E_2|A$ means an isomorphism

$$\Phi : \pi^* E_1|A \times I \to \pi^* E_2|A \times I$$

where I is the unit interval and $\pi : X \times I \to X$ is the projection.

Let

$$f_t : X \to X \times I$$

be defined by $f_t(x) = x \times \{t\}$ and denote by

$$\varphi_t : E_1|A \to E_2|A$$

the isomorphism induced from Φ by f_t. Then

$$E_1 \cup_{\varphi_t} E_2 \cong f_t^*(\pi^* E_1 \cup_\Phi \pi^* E_2)$$

Since f_0 and f_1 are homotopic it follows from (1.4.3) that

$$E_1 \cup_{\varphi_0} E_2 \cong E_1 \cup_{\varphi_1} E_2$$

as required.

Remark: The "collapsing" and "clutching" constructions for bundles (on X/Y and $X_1 \cup X_2$ respectively) are both special cases of a general process of forming bundles over quotient spaces We leave it as an exercise to the reader to give a precise general formulation.

We shall denote by $[X, Y]$ the set of homotopy classes of maps $X \to Y$.

1.4.9 LEMMA *For any* X, *there is a natural isomorphism* $\mathrm{Vect}_n(S(X)) \cong [X, \mathrm{GL}(n, \mathbb{C})]$.

Proof: Write $S(X)$ as $C^+(X) \cup C^-(X)$, where $C^+(X) = [0, \frac{1}{2}] \times X/\{0\} \times X$, $C^-(X) = [\frac{1}{2}, 1] \times X/\{1\} \times X$. Then $C^+(X) \cap C^-(X) = X$. If E is any n-dimensional bundle over $S(X)$, $E|C^+(x)$ and $E|C^-(x)$ are trivial. Let $\alpha^\pm : E|C^\pm(X) \cong C^\pm(X) \times V$ be such isomorphisms. Then $(\alpha^+|X)(\alpha^-|X)^{-1} : X \times V \to X \times V$ is a bundle map, and thus defines a map α of X into $\mathrm{GL}(n, \mathbb{C}) = \mathrm{Iso}(V)$. Since both $C^+(X)$ and $C^-(X)$ are contractible, the homotopy classes of both α^+ and α^- are well defined, and thus the homotopy class of α is well defined. Thus we have a natural map $\theta : \mathrm{Vect}_n(S(X)) \to [X, \mathrm{GL}(n, \mathbb{C})]$. The clutching construction on the other hand defines by (1.4.8) a map

$$\varphi : [X, \mathrm{GL}(n, \mathbb{C})] \to \mathrm{Vect}_n(S(X))$$

It is clear that θ and φ are inverses of each other and so are bijections.

We have just seen that $\mathrm{Vect}_n(S(X))$ has a homotopy theoretic interpretation. We now give a similar interpretation to $\mathrm{Vect}_n(X)$. First we must establish some simple facts about quotient bundles.

1.4.10 LEMMA *Let E be any bundle over X. Then there exists a (Hermitian) metric on E.*

Proof: A metric on a vector space V defines a metric on the product bundle $X \times V$. Hence metrics exist on trivial bundles. Let $\{U_\alpha\}$ be a finite open covering of X such that $E|U_\alpha$ is trivial and let h_α be a metric for $E|U_\alpha$. Let $\{p_\alpha\}$ be a partition of unity with $\operatorname{supp} p_\alpha \subset U_\alpha$ and define

$$k_\alpha(x) = \begin{cases} p_\alpha(x) h_\alpha(x) & \text{for } x \in U_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

Then k_α is a section of $\mathrm{Herm}(E)$ and is positive semi-definite. But for any $x \in X$ there exists α such that $p_\alpha(x) > 0$ (since $\sum p_\alpha = 1$) and so $x \in U_\alpha$. Hence, for this α, $k_\alpha(x)$ is positive definite. Hence $\sum_\alpha k_\alpha(x)$ is positive definite for all $x \in X$ and so $\sum k_\alpha$ is a metric for E.

A sequence of vector bundle homomorphisms

$$\longrightarrow E \longrightarrow F \longrightarrow \dots$$

is called *exact* if for each $x \in X$ the sequence of vector space homomorphisms

$$\longrightarrow E_x \longrightarrow F_x \longrightarrow \dots$$

is exact.

1.4.11 COROLLARY Suppose that $0 \longrightarrow E' \xrightarrow{\varphi'} E \xrightarrow{\varphi''} E'' \longrightarrow 0$ is an exact sequence of bundles over X. Then there exists an homorphism $E \cong E' \oplus E$".

Proof: Give E a metric. Then $E \cong E' \oplus (E')^{\perp}$. However, $(E')^{\perp} \cong E''$.

A subspace $V \subset \Gamma(E)$ is said to be *ample* if

$$\varphi : X \times V \to E$$

is a surjection, where $\varphi(x,s) = s(x)$.

1.4.12 LEMMA *If E is any bundle over a compact Hausdorff space X, then* $\Gamma(E)$ *contains a finite dimensional ample subspace.*

Proof: Let $\{U_\alpha\}$ be a finite open covering of X so that $E|U_\alpha$ is trivial for each α, and let $\{p_\alpha\}$ be a partition of unity with $\operatorname{supp} p_\alpha \subset U_\alpha$. Since $E|U_\alpha$ is trivial we can find a finite-dimensional ample subspace $V_\alpha \subset \Gamma(E|U_\alpha)$. Now define

$$\theta_\alpha : V_\alpha \to \Gamma(E)$$

by

$$\theta_\alpha v_\alpha(x) = \begin{cases} p_\alpha(x)v_\alpha(x) & \text{if } x \in U_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

The θ_α define a homomorphism

$$\theta : \prod_\alpha V_\alpha \to \Gamma(E)$$

and the image of θ is a finite dimensional subspace of $\Gamma(E)$; in fact, for each

$x \in X$ there exists α with $p_\alpha(x) > 0$ and so the map

$$\theta_\alpha(V_\alpha) \to E_x$$

is surjective.

1.4.13 COROLLARY *If E is any bundle, there exists an epimorphism $\varphi : X \times \mathbb{C}^m \to E$ for some integer m.*

1.4.14 COROLLARY *If E is any bundle, there exists a bundle F such that $E \oplus F$ is trivial.*

We are now in a position to prove the existence of a homotopy theoretic definition for $\mathrm{Vect}_n(X)$. We first introduce Grassmann manifolds. If V is any vector space, and n any integer, the set $G_n(V)$ is the set of all subspaces of V of codimension n. If V is given some Hermitian metric, each subspace of V determines a projection operator. This defines a map $G_n(V) \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ is the set of endomorphisms of V. We give $G_n(V)$ the topology induced by this map.

Suppose that E is a bundle over a space X, V is a vector space, and $\varphi : X \times V \to E$ is an epimorphism. If we map X into $G_n(V)$ by assigining to x the subspace $\ker(\varphi_x)$ this map is continuous for any metric on V (here $n = \dim(E)$). We call the map $X \to G_n(V)$ the map induced by φ.

Let V be a vector space, and let $F \subset G_n(V) \times V$ be the sub-bundle consisting of all points (g, v) such that $v \in g$. Then, if $E = (G_n(V) \times V)/F$ is the quotient bundle, E is called the *classifying bundle* over $G_n(V)$.

Notice that if E' is a bundle over X, and $\varphi : X \times V \to E'$ is an epimorphism, then if $f : X \to G_n(V)$ is the map induced by φ, we have $E' \cong f^*(E)$, where E is the classifying bundle.

Suppose that h is a metric on V. We denote by $G_n(V_h)$ the set $G_n(V)$ with the topology induced by h. If h' is another metric on V, then the epimorphism $G_n(V) \times V \to E$ (where E is the classifying bundle) induces the identity map $G_n(V_h) \to G_n(V_{h'})$. Thus the identity map is continuous. Thus, the topology on $G_n(V)$ does not depend on the metric.

Now consider the natural projections

$$\mathbb{C}^m \to \mathbb{C}^{m-1}$$

given by $(z_1, \ldots, z_m) \to (z_1, \ldots, z_{m-1})$. These induce continuous maps

$$\iota_{m-1} : G_n(\mathbb{C}^m) \to G_n(\mathbb{C}^{m-1}).$$

If $E_{(m)}$ denotes the classifying bundle over $G_n(\mathbb{C}^m)$ it is immediate that

$$\iota_{m-1}^*(E_{(m)}) \cong E_{(m-1)}$$

1.4.15 THEOREM *The map*

$$\lim_{\to m}[X, G_n(\mathbb{C}^m)] \to \mathrm{Vect}_n(X)$$

induced by $f \to f^*(E_{(m)})$ for $f : X \to G_n(\mathbb{C}^m)$, is an isomorphism for all compact Hausdorff spaces X.

Proof: We shall construct an inverse map. If E is a bundle over X, there exists (by (1.4.13)) an epimorphism $\varphi : X \times \mathbb{C}^m \to E$. Let $f : X \to G_n(\mathbb{C}^m)$ be the map induced by φ. If we can show that the homotopy class of f (in $G_n(V^{m'})$ for m' sufficiently large does not depend on the choice of φ, then we construct our inverse map $\mathrm{Vect}_n(X) \to [X, G_n(\mathbb{C}^m)]$ by sending E to the homotopy class of f.

Suppose that $\varphi_i : X \times \mathbb{C}^{m_i} \to E$ are two epimorphisms ($i = 0, 1$). Let $g_i : X \to \mathbb{C}^{m_i}$ be the map induced by φ_i. Define $\psi_t : X \times \mathbb{C}^{m_0} \times \mathbb{C}^{m_1} \to E$ by $\psi(x, v_0, v_1) = (1-t)\varphi_0(x, v_0) + t\varphi_1(x, v_1)$. This is an epimorphism. Let $f_t : X \to G_n(\mathbb{C}^{m_0} \oplus \mathbb{C}^{m_1})$ be the map induced by ψ_t. If we identify $\mathbb{C}^{m_0} \oplus \mathbb{C}^{m_1}$ with $\mathbb{C}^{m_0+m_1}$ by $(z_1, \ldots, z_{m_0}) \oplus (u_1, \ldots, u_{m_1}) \to (z_1, \ldots, z_{m_0}, \ldots, u_{m_1})$ then

$$f_0 = j_0 g_0, \qquad f_1 = T j_1 g_1,$$

where $j_i : G_n(\mathbb{C}^{m_i}) \to G_n(\mathbb{C}^{m_0+m_1})$ is the natural inclusion and

$$T : G_n(\mathbb{C}^{m_0+m_1}) \to G_n(\mathbb{C}^{m_0+m_1})$$

is the map induced by a permutation of coordinates in $\mathbb{C}^{m_0+m_1}$, and so is homotopic to the identity. Hence $j_1 g_1$ is homotopic to f_1 and hence to $j_0 g_0$ as required.

Remark: It is possible to interpret vector bundles as modules in the following way. Let $C(X)$ denote the ring of continuous complex-valued functions on X. If E is a vector bundle over X then $\Gamma(E)$ is a $C(X)$-module under point-wise multiplication, i.e.,

$$fs(x) = f(x)s(x) \qquad f \in C(X), s \in \Gamma(E).$$

Moreover a homomorphism $\varphi : E \to F$ determines a $C(X)$-module homomorphism

$$\Gamma\varphi : \Gamma(E) \to \Gamma(F).$$

Thus Γ is a functor from the category $\mathscr{V}$ of vector bundles over X to the category $\mathscr{M}$ of $C(X)$-modules. If E is trivial of dimension n. then $\Gamma(E)$ is free of rank n. If F is also trivial then

$$\Gamma : \mathrm{HOM}(E,F) \to \mathrm{Hom}_{C(X)}(\Gamma(E), \Gamma(F))$$

is bijective. In fact, choosing isomorphisms $E \cong X \times V$, $F \cong X \times W$ we have

$$\begin{aligned} \mathrm{HOM}(E,F) \cong \mathrm{Hom}_C(V,W)^X &\cong C(X) \otimes \mathrm{Hom}_C(V,W) \\ &\cong \mathrm{Hom}_{C(X)}(\Gamma(E), \Gamma(F)) \end{aligned}$$

Thus Γ induces an equivalence between the category $\mathscr{T}$ of trivial vector bundles to the category $\mathscr{F}$ of free $C(X)$-modules of finite rank. Let $\mathrm{Proj}(\mathscr{T})$ denote the sub-category of $\mathscr{V}$ whose objects are images of projection operators in $\mathscr{T}$, and $\mathrm{Proj}(\mathscr{F}) \subset \mathscr{M}$ be defined similarly. Then it follows at once that Γ induces an equivalence of categories

$$\mathrm{Proj}(\mathscr{T}) \to \mathrm{Proj}(\mathscr{F})$$

But, by (1.4.14), $\mathrm{Proj}(\mathscr{T}) = \mathscr{V}$. By definition $\mathrm{Proj}(\mathscr{F})$ is the category of finitely-generated projective $C(X)$-modules. Thus we have established the following:

PROPOSITION. Γ *induces an equivalence between the category of vector bundles over X and the category of finitely-generated projective modules over $C(X)$.*

1.5 Additional structures. In linear algebra one frequently considers vector spaces with some additional structure, and we can do the same for vector bundles. For example we have already discussed hermitian metrics. The next most obvious example is to consider non-degenerate bilinear forms. Thus if V is a vector bundle, a non-degenerate bilinear form on V means an element T of $\mathrm{HOM}(V \otimes V, 1)$ which induces a non-degenerate element of $\mathrm{Hom}(V_x \otimes V_x, \mathbb{C})$ for all $x \in X$. Equivalently T may be regarded as an element of $\mathrm{ISO}(V, V^*)$. The vector bundle V together with this isomorphism T will be called a *self-dual* bundle.

If T is symmetric, i.e., if T_x is symmetric for all $x \in X$, we shall call (V, T) an *orthogonal* bundle. If T is skew-symmetric, i.e., if T_x is skew-symmetric for all $x \in X$, we shall call (V, T) a *symplectic* bundle.

Alternatively we may consider pairs (V, T) with $T \in \mathrm{ISO}(V, \bar{V})$, where $\bar{V}$ denotes the *complex conjugate bundle* of V (obtained by applying the "complex conjugate functor" to V). Such a (V, T) may be called a *self-conjugate* bundle. The isomorphism T may also be thought of as an anti-linear isomorphism $V \to V$. As such we may form T^2. If $T^2 =$ identity we may call (V, T) a *real* bundle. In fact the subspace $W \subset V$ consisting of all $v \in V$ with $Tv = v$ has the structure of a *real vector bundle* and V may be identified with $W \otimes_{\mathbb{R}} \mathbb{C}$, the complexification of W. If $T^2 =$ —identity then we may call (V, T) a quaternion bundle. In fact, we can define a quaternion vector space structure on each V_x by putting $j(v) = Tv$. The quaternions are generated over $\mathbb{R}$ by i, j with $ij = -ij, i^2 = j^2 = -1$.

Now if V has a hermitian metric h then this gives an isomorphism $\bar{V} \to V^*$ and hence turns a self-conjugate bundle into a self-dual one. We leave it as an exercise to the reader to examine in detail the symmetric[1] and skew-symmetric cases and to show that, up to homotopy, the notions of self-conjugate, orthogonal,symplectic, are essentially equivalent to self-dual, real, quaternion. Thus we may pick which ever alternative is more convenient at any particular stage. For example, the result of the preceeding sections extend immediately to real and quaternion vector bundles although the extension of (1.4.3) for

[1]The point is that $\mathrm{GL}(n, \mathbb{R})$ and $O(n, \mathbb{C})$ have the same maximal compact subgroup $O(n, \mathbb{C})$. Similar remarks apply in the skew case.

example to orthogonal or sympletic bundles is not so immediate. On the other hand the properties of tensor products are more conveniently dealt with in the framework of bilinear forms. Thus the tensor product of (V,T) and (W,S) is $(V \otimes W, T \otimes S)$ and the symmetry properties of $T \otimes S$ follow at once from those of T and S. Note in particular that the tensor product of two sympletic bundles is orthogonal.

A self-conjugate bundle is a special case of a much more general notion. Let F, G be two continuous functors on vector spaces. Then by an $F \to G$ bundle we will mean a pair (V,T) where V is a vector bundle and $T \in \mathrm{ISO}(F(V), G(V))$. Obviously a self-conjugate bundle arises by taking $F =$ identity, $G =^*$. Another example of some importance is to take F and G to be multiplication by a fixed integer m, i.e.,

$$F(V) = G(V) = V \oplus V \oplus \ldots \oplus V \qquad (m \text{ times}).$$

Thus an $m \to m$ bundle (or more briefly an m-bundle) is a pair (V,T) where $T \in \mathrm{Aut}(mV)$. The m-bundle (V,T) is *trivial* if there exists $S \in \mathrm{Aut}(V)$ so that $T = mS$.

In general for $F \to G$ bundles the analogue of (1.4.3) does not hold, i.e. homotopy does not imply isomorphism. Thus the good notion of equivalence must incorporate homotopy. For example, two m-bundles (V_0, T_0) and (V_1, T_1) will be called equivalent if there is an m-bundle (W,S) on $X \times I$ so that

$$(V_i, T_i) \cong (W,S)|X \times \{i\}, \qquad i = 0,1.$$

Remark: An m-bundle over K should be thought of as a "mod m vector bundle" over $S(X)$.

1.6 G-bundles over G-spaces. Suppose that G is a topological group. Then by a *G-space* we mean a topological space X together with a given continuous action of G on X, i.e., G acts on X and the map $G \times X \to X$ is continuous. A *G-map* between G-spaces is a map commuting with the action of G. A G-space E is a G-vector bundle over the G-space X if

(i) E is a vector bundle over X,

(ii) the projection $E \to X$ is a G-maps

(iii) for each $g \in G$ the map $E_x \to E_{g(x)}$ is a vector space homomorphism.

If G is the group of one element then of course every space is a G-space and every vector bundle is a G-vector bundle. At the other extreme if X is a point then X is a G-space for all G and a G-vector bundle over X is just a (finite-dimensional) representation space of G. Thus G-vector bundles form a natural generalization including both ordinary vector bundles and G-modules. Much of the theory of vector bundles over compact spaces generalizes to G-vector bundles provided G is also compact. This however, presupposes the basic facts about representations of compact groups. For the present, therefore we restrict ourselves to *finite groups* where no questions of analysis are involved.

There are two extreme kinds of G-space:

(i) X is a free G-space if $g \neq 1 \implies g(x) \neq x$,

(ii) X is a trivial G-space if $g(x) = x$ for all $x \in X$, $g \in G$,

We shall examine the structure of G-vector bundles in these two extreme cases.

Suppose then that X is a free G-space and let X/G be the orbit space. Then $\pi : X \to X/G$ is a finite covering map. Let E be a G-vector bundle over X. Then E is necessarily a free G-space. The orbit space X/G has a natural vector bundle structure over X/G: in fact $E/G \to X/G$ is locally isomorphic to $E \to X$ and hence the local triviality of E implies that of E/G. Conversely, suppose V is a vector bundle over X/G. Then $\pi^* V$ is a G-vector bundle over X; in fact, $\pi^* V \subset X \times V$ and G acts on $X \times V$ by $g(x,v) = (g(x),v)$. It is clear that $E \to E/G$ and $V \to \pi^* V$ are inverse functors. Thus we have

1.6.1 PROPOSITION *If X is G-free, G-vector bundles over X correspond bijectively to vector bundles over X/G by $E \to E/G$.*

Before discussing trivial G-spaces let us recall the basic fact about representations of finite groups, namely that there exists a finite set $V_1, \dots, V_k$ of irreducible representations of G so that any representation V of G is isomorphic to a unique direct sum $\sum_{i=1}^k n_i V_i$. Now for any two G-modules (i.e., representation spaces) V, W we can define the vector space $\mathrm{Hom}_G(V,W)$ of G-homomorphisms. Then we have

$$\mathrm{Hom}_G(V_i, V_j) \cong \begin{cases} 0 & i \neq j \\ \mathbb{C} & i = j \end{cases}$$

Hence for any V it follows that the natural map

$$\sum V_i \otimes \mathrm{Hom}_G(V_i, V) \to V$$

is a G-isomorphism. In this form we can extend the result to G-bundles over a trivial G-space. In fact, if E is any G-bundle over the trivial G-space X we can define the morphism $\mathrm{Av} \in \mathrm{END}\, E$ by

$$\mathrm{Av}(e) = \frac{1}{|G|} \sum_{g \in G} g(e)$$

where $|G|$ denotes the order of G (This depends on the fact that, X being G-trivial, each $g \in G$ defines an endomorphism of E). It is immediate that Av is a projection operator for E and so its image, the invariant subspace of E, is a vector bundle. We denote this by E^G and call it the invariant *sub-bundle* of E. Thus if E, F are two G-bundles then $\mathrm{Hom}_G(E,F) = (\mathrm{Hom}(E,F))^G$ is again a vector bundle. In particular taking E to be the trivial bundle $\mathbf{V}_i = X \times V_i$ with its natural G-action we can consider the natural bundle map

$$\sum_{i=1}^k \mathbf{V}_i \otimes \mathrm{Hom}_G(\mathbf{V}_i, F) \to F$$

We have already observed that for a G-module F this is a G-isomorphism. In other words for any G-bundle F over X this is a G-isomorphism for all $x \in X$. Hence it is an isomorphism of G-bundles. Thus every G-bundle F is isomorphic to a G-bundle of the form $\sum \mathbf{V}_i \otimes E_i$ Where E_i is a vector bundle with trivial G-action. Moreover the E_i are unique up to isomorphism. In fact we have

$$\begin{aligned} \mathrm{Hom}_G(\mathbf{V}_i, F) &\cong \sum_{j=1}^{k} \mathrm{Hom}_G(\mathbf{V}_i, \mathbf{V}_j \otimes E_j) \\ &\cong \sum_{j=1}^{k} \mathrm{Hom}_G(\mathbf{V}_i, \mathbf{V}_j) \otimes E_j \\ &\cong E_i. \end{aligned}$$

Thus we have established

1.6.2 PROPOSITION *Let X be a trivial G-space, $V_1, \ldots, V_k$ a complete set of irreducible G-modules, $\boldsymbol{V}_i = X \times V_i$, the corresponding G-bundles. Thus every G-bundle F over X is isomorphic to a direct sum $\sum \boldsymbol{V}_i \otimes E_i$ where the E_i are vector bundles with trivial G-action. Moreover the E_i are unique up to isomorphism and are given by $E_i = \mathrm{Hom}_G(\boldsymbol{V}_i, F)$.*

We return now to the case of a general (compact) G-space X and we shall show how to extend the results of section 1.4 to G-bundles.

Observe first that, if E is a G-bundle, G acts naturally on $\Gamma(E)$ by

$$(gs)(x) = g(s(g^{-1}(x)))$$

A section s is invariant if $gs = g$ for all $g \in G$. The set of all invariant sections forms a subspace $\Gamma(E)^G$ of $\Gamma(E)$. The averaging operator

$$\mathrm{Av} = \frac{1}{|G|} \sum g$$

Defines as usual a homomorphism $\Gamma(E) \to \Gamma(E)^G$ which is the identity on $\Gamma(E)^G$.

1.6.3 LEMMA *Let X be a compact G-space $Y \subset X$ a closed sub G-space (i.e., invariant by G) and let E be a G-bundle over X. Then any invariant section $s : Y \to E|Y$ extends to an invariant section over X.*

Proof: by (1.4.1) we can extend s to some section t of E over X. Then $\mathrm{Av}(t)$ is an invariant section of E over X, while over Y we have

$$\mathrm{Av}(t) = \mathrm{Av}(s) = s$$

since s is invariant. Thus $\mathrm{Av}(t)$ is the required extension.

If E, F are two G-bundles then $\mathrm{Hom}(E, F)$ is also a G-bundle and we have

$$\Gamma(\mathrm{Hom}(E, F))^G \cong \mathrm{HOM}_G(E, F)$$

Hence the G-analogues of (1.4.2) and (1.4.3) follow at once from (1.6.3). Thus we have

1.6.4 LEMMA *Let Y be a compact G-space, X a G-space, $f_t : Y \to X \quad (0 \leq t \leq 1)$ a G-homotopy and E a G-vector bundle over X. Then $f_0^* E$ and $f_1^* E$ are isomorphic G-bundles.*

A G-homotopy means of course a G-map $F : Y \times I \to X$ where I is the unit interval with trivial G-action. A G-space is G-contractible if it is G-homotopy equivalent to a point. In particular, the cone over a G-space is always G-contractible. By a *trivial* G-bundle we shall mean a G-bundle isomorphic to a product $X \times V$ where V is a G-module. With these definitions (1.4.4) - (1.4.11) extend without change to G-bundles. We have only to observe that ifh is a metric for E then $\mathrm{Av}(h)$ is an invariant metric.

To extend (1.4.12) we observe that if $V \subset \Gamma(E)$ is ample then $\sum_{g \in G} gV \subset \Gamma(E)$ is ample and invariant. This leads at once to the appropriate extension of (1.4.14).

In extending (1.4.15) we have to consider Grassmannians of G-subspaces of $m \sum_{i=1}^{k} V_i$ for $m \to \infty$, where as before $V_1, \ldots, V_k$ denote a complete set of irreducible G-modules. We leave the formulation to the reader.

Finally, consider the module interpretation of vector bundles. Write $A = C(X)$. Then if X is a G-space G acts on A as a group of algebra automorphisms. If E is a G-vector bundle over X then $\Gamma(E)$ is a projective A-module and G acts on $\Gamma(E)$, the relation between the A- and G- actions being

$$g(as) = g(a)g(s) \qquad a \in A, g \in G, s \in \Gamma(E).$$

We can look at this another way if we introduce the "twisted group algebra" B of G over A, namely elements of B are linear combinations $\sum_{g\in G} a_g g$ with $a \in A$ and the product is defined by

$$(ag)(a'g') = (ag(a'))gg'.$$

In fact, $\Gamma(E)$ is then just a B-module. We leave it as an exercise to the reader to show that the category of G-vector bundles over X is equivalent to the category of B-modules which are finitely generated and projective over A.

Chapter 2

K-THEORY

2.1 Definitions. If X is any space, the set $\mathrm{Vect}(X)$ has the structure of an abelian semigroup, where the additive structure is defined by direct sum. If A is any abelian semigroup, we can associate to A an abelian group $K(A)$ with the following property: there is a semigroup homomorphism $\alpha : A \to K(A)$ such that if G is any group, $\gamma : A \to G$ any semigroup homomorphism there is a unique homomorphism $\chi : K(A) \to G$ such that $\gamma = \chi\alpha$. If such a $K(A)$ exists, it must be unique.

The group $K(A)$ is defined in the usual fashion. Let $F(A)$ be the free abelian group generated by the elements of A, let $E(A)$ be the subgroup of $F(A)$ generated by those elements of the form $a + a' - (a \oplus a')$ where $\oplus$ is the addition in A, $a, a' \in A$. Then $K(A) = F(A)/E(A)$ has the universal property described above, with $\alpha : A \to K(A)$ being the obvious map.

A slightly different construction of $K(A)$ which is sometimes convenient is the following. Let $\Delta : A \to A \times A$ be the diagonal homomorphism of semi-groups, and let $K(A)$ denote the set of cosets of $\Delta(a)$ in $A \times A$. It is, a quotient semi-group, but the interchange of factors in $A \times A$ induces an inverse in $K(A)$ so that $K(A)$ is a group. We then define $\alpha_A : A \to K(A)$ to be the composition of $a \to (a, 0)$ with the natural projection $A \times A \to K(A)$ (we assume A has a zero for simplicity). The pair $(K(A), \alpha_A)$ is a functor of A so that if $\gamma : A \to B$ is a semi-group homomorphism we have a commutative diagram

$$
\begin{tikzcd}
A \arrow[r, "\alpha_A"] \arrow[d, "\gamma"'] & K(A) \arrow[d, "K(\gamma)"] \\
B \arrow[r, "\alpha_B"] & K(B)
\end{tikzcd}
$$

If B is a group α_B is an isomorphism. That shows $K(A)$ has the required universal property.

If A is also a semi-ring (that is, A possesses multiplication which is distributive over the addition of A) then $K(A)$ is clearly a ring.

If X is a space, we write $K(X)$ for the ring $K(\text{Vect}(X))$. No confusion should result from this notation. If $E \in \text{Vect}(X)$, we shall write $[E]$ for the image of E in $K(X)$. Eventually, to avoid excessive notation, we may simply write E instead of $[E]$ when there is no danger of confusion.

Using our second construction of K it follows that, if X is a space, every element of $K(X)$ is of the form $[E]-[F]$, where E, F are bundles over X. Let G be a bundle such that $F \oplus G$ is trivial. We write $\underline{n}$ for the trivial bundle of dimension n . Let $F \oplus G = \underline{n}$. Then $[E]-[F] = [E]+[G]-([F]+[G]) = [E \oplus G]-[\underline{n}]$. Thus, every element of $K(X)$ is of the form $[H]-[\underline{n}]$.

Suppose that E, F are such that $[E] = [F]$, then again from our second construction of K it follows that there is a bundle G such that $E \oplus G \cong F \oplus G$. Let G' be a bundle such that $G \oplus G' \cong \underline{n}$. Then $E \oplus G \oplus G' \cong F \oplus G \oplus G'$, so $E \oplus \underline{n} \cong F \oplus \underline{n}$. If two bundles become equivalent when a suitable trivial bundle is added to each of them, the bundles are said to be *stably equivalent*. Thus, $[E] = [F]$ if and only if E and F are stably equivalent.

Suppose $f : X \to Y$ is a continuous map. Then $f^* : \text{Vect}(Y) \to \text{Vect}(X)$ induces a ring homomorphism $f^* : K(Y) \to K(X)$. By (1.4.3) this homomorphism depends only on the homotopy class of f.

2.2 The periodicity theorem. The fundamental theorem for K-theory is the periodicity theorem. In its simplest form, it states that for any X, there is an isomorphism between $K(X)\otimes K(S^2)$ and $K(X\times S^2)$. This is a special case of a more general theorem which we shall prove.

If E is a vector bundle over a space X, and if $E_0 = E - X$ where X is considered to lie in E as the zero section, the non-zero complex numbers act on E_0 as a group of fiber preserving automorphisms. Let $P(E)$ be the quotient space obtained from E_0 by dividing by the action of the complex number. $P(E)$ is called the projective bundle associated to E. If $p : P(E) \to X$ is the projection map, $p^{-1}(x)$ is a complex projective space for all $x \in X$. If V is a vector space, and W is a vector space of dimension one, V and $V \otimes W$ are isomorphic, but not naturally isomorphic. For any non-zero element $\omega \in W$ the map $v \to v \otimes \omega$ defines an isomorphism between V and $V \otimes W$, and thus defines an isomorphism $P(\omega) : P(V) \to P(V \otimes W)$. However, if ω' is any other non-zero element of W, $\omega' = \lambda\omega$ for some non-zero complex number λ. Thus $P(\omega) = P(\omega')$, so the isomorphism between $P(V)$ and $P(V \otimes W)$ is natural. Thus, if E is any vector bundle, and L is a line bundle, there is a natural isomorphism $P(E) \cong P(E \otimes L)$.

If E is a vector bundle over X then each point $a \in P(E)_x \to P(E_x)$ represents a one-dimensional subspace $H^*_x \subset E_x$. The union of all these defines a subspace $H^* \subset p^*E$, where $p : P(E) \to X$ is the projection. It is easy to check that H^* is a sub-bundle of p^*E. In fact, the problem being local we may assume E is a product and then we are reduced to a special case of the Grassmannian already discussed in section 1.4. We have denoted our line-bundle by H^* because we want its dual H (the choice of convention here is dictated by algebro-geometric considerations which we do not discuss here).

We can now state the periodicity theorem.

2.2.1 THEOREM *Let L be a line bundle over X. Then, as a $K(X)$-algebra $K(P(L \oplus 1))$ is generated by $[H]$, and is subject to the single relation $([H]-1)([L][H]-[1]) = 0$.*

Before we proceed to the proof of this theorem, we would like to point out two corollaries. Notice that $P(1 \oplus 1) = X \times S^2$.

2.2.2 COROLLARY *$K(S^2)$ is generated by $[H]$ as a K (point) module, and $[H]$ is subject to the only single relation $([H]-[1])^2 = 0$.*

2.2.3 COROLLARY *If X is any space, and if $\mu : K(X) \otimes K(S^2) \to K(X \times S^2)$ is defined by $\mu(a \otimes b) = (\pi_1^* a)(\pi_2^* b)$, where π_1, π_2 are the projections onto the two factors, then μ is an isomorphism of rings.*

The proof of the theorem will be broken down into a series of lemmas.

To begin, we notice that for any $x \in X$, there is a natural embedding $L_x \to P(L \oplus 1)_x$ given by the map $y \to (y, 1)$. This map extends to the one point compactification of L_x and gives us a homeomorphism of the one point compactification of L_x onto $P(L \oplus 1)_x$. If we map $X \to P(L \oplus 1)$ by sending x to the image of the "point at infinity" of the one point compactification of L_x, we obtain a section of $P(L \oplus 1)$ which we call the "section at infinity". Similarly, the zero section of L gives us a section of $P(L \oplus 1)$, which we call the zero section of $P(L \oplus 1)$.

We choose a metric on L, and we let $S \subset L$ be the unit circle bundle. We write P^0 for the part of L consisting of vectors of length ≤ 1, and P^∞ for that part of $P(L \oplus 1)$ consisting of the section at infinity, together with all the vectors of length ≤ 1. We denote the projections $S \to X$, $P^0 \to X$, $P^\infty \to X$ by π, π_0, and π_∞ respectively

Since π_0 and π_∞ are homotopy equivalences, every hundle on P^0 is of the form $\pi_0^* E^0$ and every bundle on P^∞ is of the form $\pi_\infty^* E^\infty$, where E^0 and E^∞ are bundles on X. Thus, any bundle E on $P(L \oplus 1)$ is isomorphic to one of the form $(\pi_0^*(E^0), f, \pi_\infty^*(E^\infty))$ where $f \in \mathrm{ISO}(\pi^*(E^0), \pi^*(E^\infty))$ is a clutching function. Moreover, if we insist that the isomorphism

$$E \to (\pi_0^* E^0, f, \pi_\infty^* E^\infty)$$

coincide with the obvious ones over the zero and infinite sections, it follows *that the homotopy class of f is uniquely determined by the isomorphism class of E.* This again uses the fact that the 0-section is a deformation retract of P^0 and the ∞-section a deformation retract of P^∞. We shall simplify our notation slightly by writing (E^0, f, E^∞) for $(\pi_0^*(E^0), f, \pi_\infty^*(E^\infty))$

Our proof will now be devoted to showing that the bundles E^0 and E^∞ and the clutching function f can be taken to have a particularly simple form. In the special case that L is trivial, S is just $X \times S^1$, the projection $S \to S^1$ is a complex-valued function on S which we denote by z (here S^1 is identified with the complex numbers of unit modulus). This allows us to consider functions on S which are finite Laurent series in z whose coefficients are functions on X:

$$\sum_{k=-n}^{n} a_k(x) z^k$$

These finite Laurent series can be used to approximate functions on S in a uniform manner.

If L is not trivial, we have an analogue to finite Laurent series. Here z becomes a section in a bundle rather than a function. Since $\pi^*(L)$ is a subset of $S \times L$, the diagonal map $S \to S \times S \subset S \times L$ gives us a section of $\pi^*(L)$. We denote this section by z. Taking tensor products we obtain, for $k \geq 0$, a section z^k of $(\pi^*(L))^k$, and a section z^{-k} of $(\pi^*(L))^k$. We write L^{-k} for $(L^*)^k$. Then, for any k, k', $L^k \otimes L^{k'} \cong L^{k+k'}$. Suppose that $a_k \in (L^{-k})$. Then $\pi^*(a_k) \otimes z^k \in \Gamma(\pi^*(1))$, and thus $\pi^*(a_k) \otimes z^k$ is a function on S. We write $a_k z^k$ for this function. By a finite Laurent series, we shall understand a sum of functions on S of the form

$$\sum_{k=-n}^{n} a_k(x) z^k$$

where $a_k \in \Gamma(L^{-k})$ for all k.

More generally, if E^0, E^∞ are two vector bundles on X, and $a_k \in \Gamma \operatorname{Hom}(L^k \otimes E^0, E^\infty)$, then if we write $a_k z^k$ for $a_k \otimes z^k$, we see that any finite sum of the form

$$f = \sum_{k=-n}^{n} a_k(x) z^k$$

is an element of $\Gamma(\pi^*(E^0), \pi^*(E^\infty))$. If $f \in \mathrm{ISO}(\pi^*(E^0), \pi^*(E^\infty))$, we call f a *Laurent clutching function* for (E^0, E^∞).

The function z is a clutching function for $(1, L)$. Further, $(1, z, L)$ is just the bundle H^* which we defined earlier. To see this, we first recall that H^* was defined as a sub-bundle of $\pi^*(L \oplus 1)$. For each $y \in P(L \oplus 1)$, H^* is a subspace of $(L \oplus 1)_x$, and

$$H^*_\infty = L_x \oplus 0, \qquad H^*_0 = 0 \oplus 1_x$$

Thus, the composition

$$H^* \to \pi^*(L \oplus 1) \to \pi^*(1)$$

induced by the projection $L \oplus 1 \to 1$ defines an isomorphism:

$$f_0 : H^*|P^0 \to \pi_0^*(1)$$

Likewise, the composition

$$H^* \to \pi^*(L \oplus 1) \to \pi^*(L)$$

induced by the projection $L \oplus 1 \to L$ defines an isomorphism:

$$f_\infty : H^*|P^\infty \to \pi_0^*(L)$$

Hence $f = f_\infty f_0^{-1} : \pi^*(1) \to \pi^*(L)$ is a clutching function for H^*. Clearly, if $y \in S_x$, $f(y)$ is the isomorphism whose graph is H^*_y. Since H^*_y is the subspace of $L_x \oplus 1_x$ spanned by $y \oplus 1$, $\quad (y \in S_x \subset L_x, 1 \in \mathbb{C})$, we see that f is exactly our section z. Thus

$$H^* \cong (1, z, L).$$

Therefore, for any integer k,

$$H^k \cong (1, z^{-k}, L^{-k}).$$

The next step in our classification of the bundles over P is to show that every clutching function can be taken to be a Laurent clutching function. Suppose that $f \in \Gamma\,\mathrm{Hom}(\pi^*E^0, \pi^*E^\infty)$ is any section. We define its Fourier coefficients

$$a_k \in \Gamma\,\mathrm{Hom}(L^k \otimes E^0, E^\infty)$$

by

$$a_k(x) = \frac{1}{2\pi i} \int_{S_x} f_x z_x^{-k-1} dz_x.$$

Here f_x, z_x denote the restrictions of f, z to S_x and dz_x is therefore a differential on S_x with coefficients in L_x. Let S_n be the partial sum

$$S_n = \sum_{-n}^{n} a_k z^k$$

and define the Cesaro means

$$f_n = \frac{1}{n} \sum_{0}^{n-1} S_k.$$

Then the proof of Fejer's theorem on the $(\mathbb{C}, 1)$ summability of Fourier series extends immediately to the present more general case and gives

2.2.4 LEMMA