Numerical Differentiation and Curve Normals for $f(x) = e^{-x}\sin(3x)$

Author: Your Name — Date: November 5, 2025

1. Model

We study the function

$$f(x) = e^{-x}\sin(3x), \qquad f'(x) = e^{-x}\big(3\cos(3x) - \sin(3x)\big),$$

and approximate $f'(x)$ by three finite-difference schemes:

- **Forward difference:** $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$ (*first order*).
- **Backward difference:** $f'(x) \approx \dfrac{f(x) - f(x-h)}{h}$ (*first order*).
- **Central difference:** $f'(x) \approx \dfrac{f(x+h) - f(x-h)}{2h}$ (*second order*).

For geometric interpretation, the curve $y = f(x)$ admits a unit normal vector:

$$\hat{\mathbf{n}}(x) = \frac{(-m, 1)}{\sqrt{1+m^2}}, \qquad m = f'(x),$$

which describes local perpendicular directions to the graph.

2. Approach

We sample $x \in [0, 2\pi]$ uniformly with $n = 1000$ points and compute:

1. the exact derivative $f'(x)$,
2. three numerical approximations using $h = 0.05$,
3. absolute errors $|f'(x) - \tilde{f}'(x)|$ for each scheme,
4. and the unit normal vectors for visualization.

3. Experiment and Results

3.1. Derivative Comparison

Figure 1 shows the exact derivative (black) and three numerical approximations. The central difference closely follows the analytical curve, while forward and backward approximations deviate slightly more due to first-order truncation error.

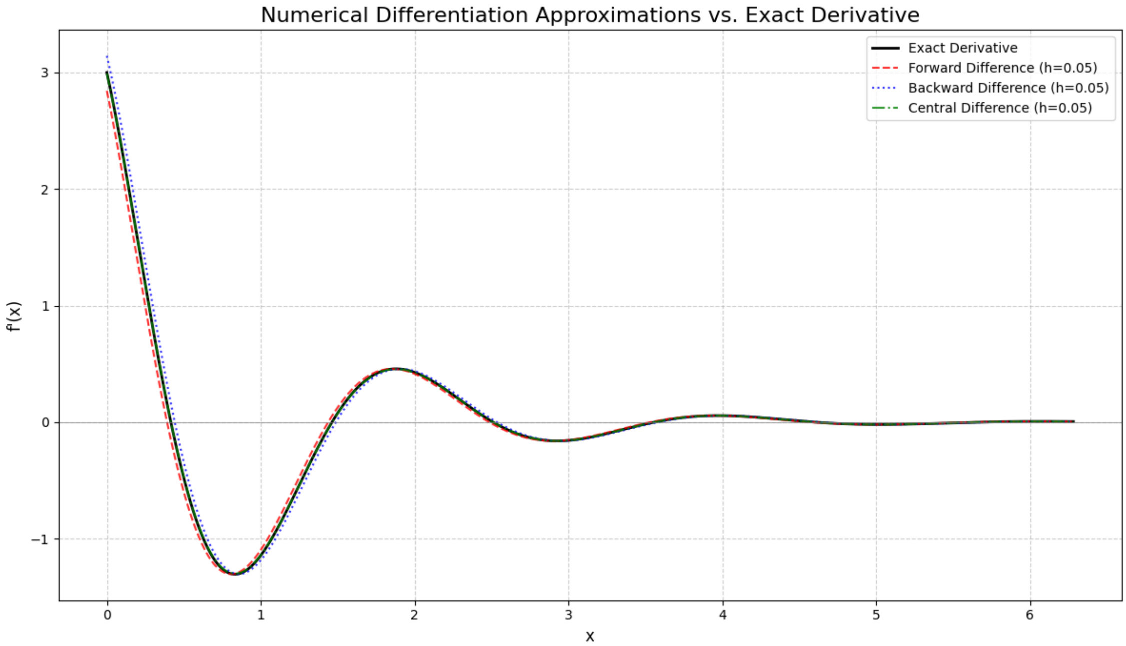

Figure 1: Exact derivative (black) vs. forward (red), backward (blue), and central (green) finite-difference approximations.

3.2. Error Analysis

Figure 2 plots absolute errors in logarithmic scale. The central difference shows the smallest error magnitude, confirming its second-order accuracy, whereas the forward and backward schemes are of first order.

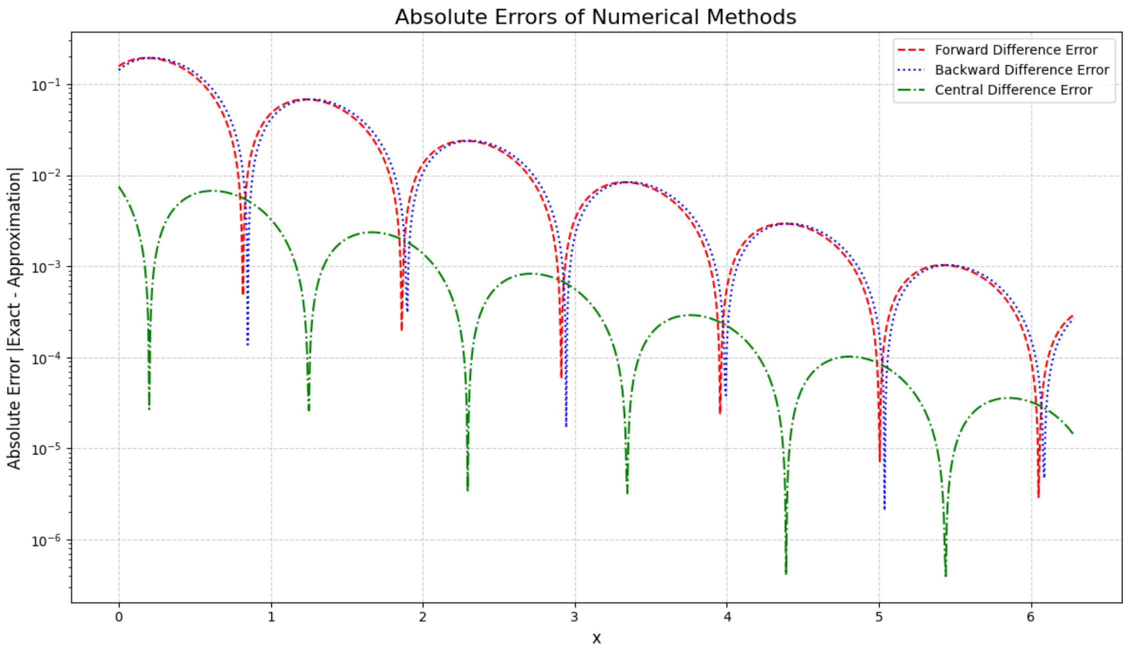

Figure 2: Absolute errors on a log scale. The central difference method exhibits superior precision.

3.3. Curve and Normal Vectors

Finally, Figure 3 visualizes the curve $y = f(x)$ along with unit normal arrows every 20 points. Regions with large slope magnitude show normals leaning more horizontally, while flatter zones display near-vertical normals.

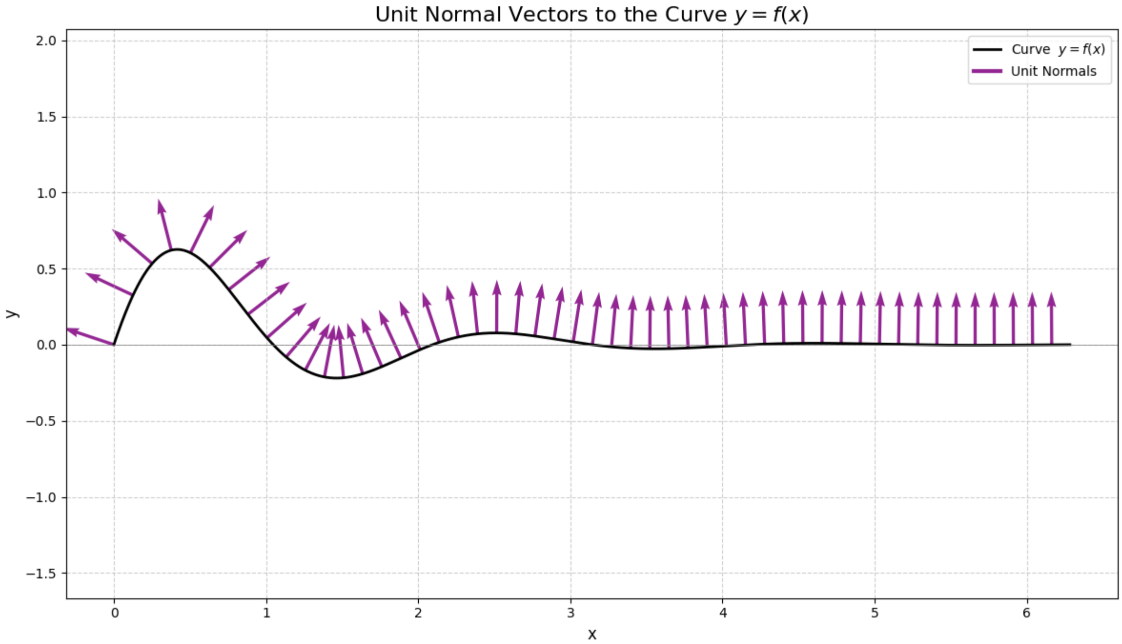

Figure 3: Curve $y = f(x)$ with unit normal vectors computed from the exact derivative.

4. Conclusions

Central differences provide the best accuracy at the same step size due to $O(h^2)$ truncation error. Forward and backward differences, being $O(h)$, yield larger deviations. The unit normal visualization adds geometric insight, connecting numerical differentiation with the curve's local orientation. Overall, this experiment bridges analytic calculus and computational geometry, showing how numerical derivatives and normals can jointly describe function behavior.

Keywords: numerical differentiation, finite difference, error analysis, unit normal vector, curve geometry.