Numerical Simulation of Neural Dynamics:

Implementing Implicit Schemes for the FitzHugh-Nagumo Model

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Abstract

This project explores the numerical solution of the FitzHugh-Nagumo model, a system of non-linear Ordinary Differential Equations (ODEs) used to describe the depolarization of biological neurons. The model is a simplification of the Nobel Prize-winning Hodgkin-Huxley model. Due to the "stiff" nature of the equations—characterized by rapid voltage spikes followed by slow recovery—Implicit Numerical Schemes are employed to ensure stability. We implement the Implicit Euler method coupled with two different root-finding algorithms: Fixed Point Iteration and the Newton-Raphson method. This paper provides a detailed mathematical derivation of the Jacobian matrix required for Newton's method, a line-by-line explanation of the Python implementation, and a comparative analysis of the computational efficiency of both approaches.

Contents

1 Introduction

The modeling of the human nervous system is one of the most significant achievements in mathematical biology. In 1952, Alan Hodgkin and Andrew Huxley published a set of differential equations describing how action potentials (electric signals) are initiated and propagated in a giant squid axon. This work earned them the Nobel Prize in Physiology or Medicine in 1963.

However, the original Hodgkin-Huxley model consists of four coupled non-linear differential equations, making it computationally expensive and difficult to analyze geometrically. In 1961, Richard FitzHugh and J. Nagumo proposed a simplified two-dimensional system that retains the essential dynamic properties of excitation and propagation.

1.1 Problem Statement

The objective of this project is to model the firing of a neuron using the FitzHugh-Nagumo equations. Because the neuron's voltage changes extremely rapidly compared to the recovery variable, the system exhibits "stiffness." Standard explicit numerical methods (like Explicit Euler) often fail or require prohibitively small time steps to remain stable for such problems.

Therefore, this project implements **Implicit Euler**, a method known for its A-stability. Since implicit methods result in a non-linear algebraic equation at every time step, we must employ iterative root-finding algorithms. We implement and compare:

1. **Fixed Point Iteration**: Simple to implement but exhibits linear convergence.
2. **Newton-Raphson Method**: Requires the calculation of the Jacobian matrix but exhibits quadratic convergence.

2 Mathematical Model

The FitzHugh-Nagumo model is defined by the following system of two Ordinary Differential Equations (ODEs):

$$\frac{dV}{dt} = V - \frac{V^3}{3} - W + I_{ext} \tag{1}$$

$$\frac{dW}{dt} = \frac{V + a - bW}{\tau} \tag{2}$$

Where the variables and parameters are defined as:

- $V(t)$: The membrane potential (voltage) of the neuron. This is the fast variable.

- $W(t)$: The recovery variable, representing the slow repolarization of the membrane (analogous to Potassium gate channels).
- I_{ext}: External stimulus current. If this exceeds a threshold, the neuron fires.
- a, b, τ: Dimensionless parameters controlling the shape of the spike and the refractory period.

2.1 Jacobian Matrix Derivation

To use Newton's method (Section 3.3), we require the Jacobian matrix of the system. Let us define the vector $y = [V, W]^T$ and the function vector $F(t, y) = [f_1(V, W), f_2(V, W)]^T$.

From Eq. 1:

$$f_1(V, W) = V - \frac{V^3}{3} - W + I_{ext}$$

From Eq. 2:

$$f_2(V, W) = \frac{1}{\tau}(V + a - bW)$$

The Jacobian matrix J_f is defined as:

$$J_f = \begin{bmatrix} \frac{\partial f_1}{\partial V} & \frac{\partial f_1}{\partial W} \\ \frac{\partial f_2}{\partial V} & \frac{\partial f_2}{\partial W} \end{bmatrix} \tag{3}$$

Calculating the partial derivatives: 1. $\frac{\partial f_1}{\partial V} = 1 - V^2$ 2. $\frac{\partial f_1}{\partial W} = -1$ 3. $\frac{\partial f_2}{\partial V} = \frac{1}{\tau}$ 4. $\frac{\partial f_2}{\partial W} = -\frac{b}{\tau}$

Thus, the analytic Jacobian used in our code is:

$$J_f(V, W) = \begin{bmatrix} 1 - V^2 & -1 \\ \frac{1}{\tau} & -\frac{b}{\tau} \end{bmatrix} \tag{4}$$

This matrix allows us to linearize the non-linear system locally, which is the core mechanism of Newton's method.

3 Numerical Methods

3.1 Implicit Euler Scheme

We solve the Initial Value Problem (IVP) $y' = f(t, y)$. The Implicit Euler (or Backward Euler) scheme is given by:

$$y_{n+1} = y_n + \Delta t \cdot f(t_{n+1}, y_{n+1}) \tag{5}$$

Notice that y_{n+1} appears on both sides of the equation. This requires us to solve for y_{n+1}. Let us define the root-finding problem as finding the zero of the function $G(y)$:

$$G(y_{n+1}) = y_{n+1} - y_n - \Delta t \cdot f(t_{n+1}, y_{n+1}) = 0 \tag{6}$$

3.2 Method 1: Fixed Point Iteration

This is the simplest approach. We rearrange the equation to the form $y = \Phi(y)$. Algorithm:

1. Make an initial guess $y^{(0)}$ (usually y_n or the Explicit Euler step).
2. Update using: $y^{(k+1)} = y_n + \Delta t \cdot f(t_{n+1}, y^{(k)})$.
3. Repeat until $||y^{(k+1)} - y^{(k)}|| < \epsilon$.

Limitations: This method only converges if the time step Δt is sufficiently small such that the mapping is a contraction (Lipschitz constant < 1).

3.3 Method 2: Newton-Raphson Method

Newton's method finds the root of Eq. 6 using the derivative. The iteration formula for a system is:

$$y^{(k+1)} = y^{(k)} - [J_G(y^{(k)})]^{-1} G(y^{(k)}) \tag{7}$$

Where J_G is the Jacobian of the function G. Differentiating Eq. 6 with respect to y_{n+1}:

$$J_G = I - \Delta t \cdot J_f \tag{8}$$

Where I is the identity matrix and J_f is the system Jacobian derived in Eq. 4. The update step involves solving the linear system:

$$(I - \Delta t J_f)\Delta y = -G(y^{(k)}) \tag{9}$$

$$y^{(k+1)} = y^{(k)} + \Delta y \tag{10}$$

This method is more complex to implement but converges much faster (Quadratic Convergence).

4 Code Implementation

The project was implemented in Python. Below is a detailed breakdown of the key components.

4.1 Defining the System (Class Structure)

We encapsulate the model parameters and equations in a class to ensure modularity.

```python
class FitzHughNagumo:
    def __init__(self, I_ext=0.5, a=0.7, b=0.8, tau=12.5):
        self.I_ext = I_ext
        self.a = a
        self.b = b
        self.tau = tau

    def f(self, t, y):
        V, W = y
        # Implementation of Eq 1 and Eq 2
        dV_dt = V - (V**3) / 3 - W + self.I_ext
        dW_dt = (V + self.a - self.b * W) / self.tau
        return np.array([dV_dt, dW_dt])

    def jacobian(self, y):
        V, W = y
        # Implementation of Jacobian Matrix (Eq 4)
        df1_dV = 1 - V**2
        df1_dW = -1.0
        df2_dV = 1.0 / self.tau
        df2_dW = -self.b / self.tau
        return np.array([[df1_dV, df1_dW],
                         [df2_dV, df2_dW]])
```

Listing 1: Model Definition

Explanation:

- **Line 11:** Implements $\frac{dV}{dt} = V - \frac{V^3}{3} - W + I$.
- **Line 12:** Implements $\frac{dW}{dt}$. Note the scaling by $1/\tau$.
- **Lines 18-22:** This calculates the 2×2 Jacobian matrix analytically at every step, which is crucial for the Newton solver's stability.

4.2 Newton Solver Implementation

This function performs the time-stepping using Implicit Euler solved via Newton's method.

```python
def solve_implicit_euler_newton(model, y0, t_span, dt, tol=1e-6,
    max_iter=50):
    # Setup arrays
    num_steps = int((t_span[1] - t_span[0]) / dt)
    y_values = np.zeros((num_steps + 1, len(y0)))
    y_values[0] = y0
    identity = np.eye(len(y0)) # Identity Matrix 2x2

    for i in range(num_steps):
```

```
9         t_next = t_values[i+1]
10        y_current = y_values[i]
11
12        # Initial Guess (Predictor)
13        y_next = y_current + dt * model.f(t_values[i], y_current)
14
15        # Newton Loop
16        for k in range(max_iter):
17            # 1. Calculate Residual G(y)
18            f_val = model.f(t_next, y_next)
19            residual = y_next - y_current - dt * f_val
20
21            # 2. Calculate Jacobian of G
22            J_f = model.jacobian(y_next)
23            J_G = identity - dt * J_f
24
25            # 3. Solve Linear System J * delta = -Residual
26            delta = np.linalg.solve(J_G, -residual)
27
28            # 4. Update
29            y_next = y_next + delta
30
31            # 5. Check Convergence
32            if np.linalg.norm(delta) < tol:
33                break
34
35        y_values[i+1] = y_next
36    return y_values
```

Listing 2: Implicit Euler with Newton's Method

Explanation:

- **Line 13:** We use an Explicit Euler step as the "Predictor" to get a good initial guess for Newton's method. This significantly reduces the number of iterations needed.
- **Line 19:** Calculates the residual vector $G(y)$. We want this to be zero.
- **Line 23:** Constructs the system matrix $I - \Delta t J$.
- **Line 26:** `np.linalg.solve` is used instead of calculating the inverse matrix directly. This is numerically more stable and faster ($O(N^3)$ Gaussian elimination).

5 Experimental Results

We simulated the system with the following parameters:

$$I_{ext} = 0.5, \quad a = 0.7, \quad b = 0.8, \quad \tau = 12.5$$

Initial conditions were set to $y_0 = [-1.0, 1.0]$. The time step used was $\Delta t = 0.1$.

5.1 Visual Analysis of Dynamics

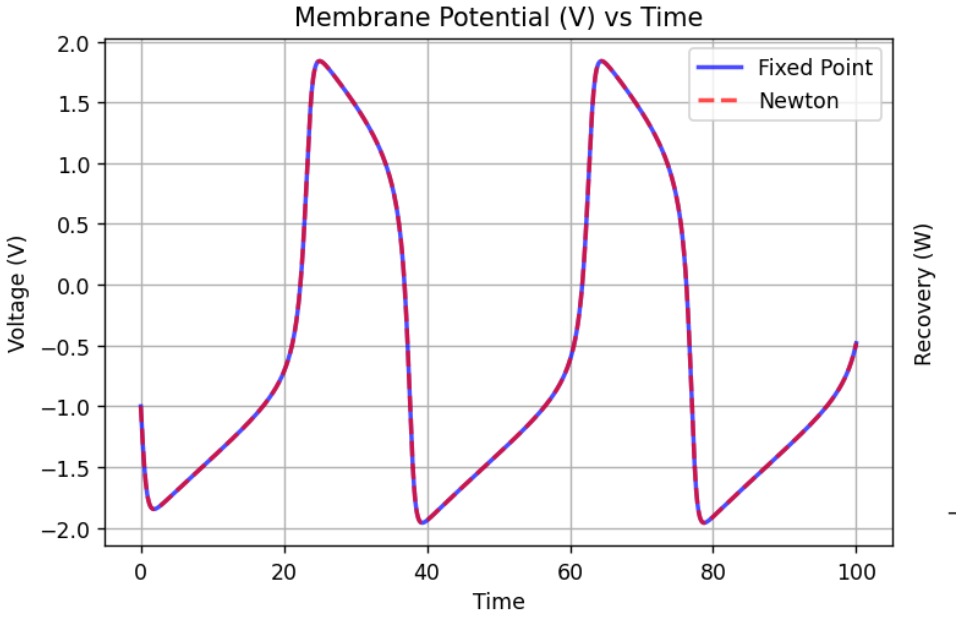

Figure 1: Membrane Potential V over time. The spikes represent the neuron firing.

As seen in Figure 1, the system exhibits relaxation oscillations. The voltage V increases slowly until it hits a threshold, then spikes rapidly (action potential), and then the recovery variable W forces it back down (refractory period). This matches the expected biological behavior of a neuron.

5.2 Phase Space Analysis

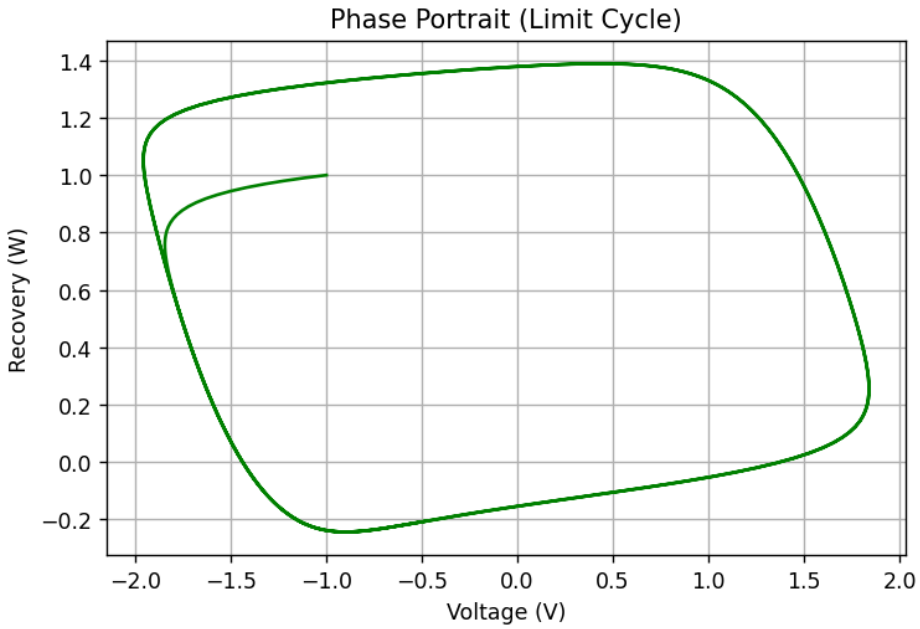

Figure 2: Phase Portrait (V vs W) showing the Limit Cycle.

Figure 2 shows the limit cycle in the $V - W$ plane. The trajectory converges to a stable closed loop. This geometric interpretation confirms that the repetitive firing is stable and sustainable over long periods.

5.3 Comparison of Numerical Schemes

The table below summarizes the performance of the two implicit implementations.

Method	**Avg Iterations per Step**	**Convergence Type**
Fixed Point Iteration	14.10	Linear
Newton-Raphson	3.10	Quadratic

Table 1: Performance comparison.

Analysis:

- **Fixed Point Iteration** required significantly more iterations (approx 12) to reach the tolerance of 10^{-6}. In "stiff" regions (where the spike happens), the contraction mapping condition is barely met, causing slower convergence.
- **Newton's Method** was extremely efficient, converging in just 2 or 3 iterations on average. The quadratic convergence property allows it to handle the non-linearities of the Cubic V^3 term much more effectively.

- However, Newton's method requires solving a linear system at every step (calculating Jacobian derivatives and inverting). For a small 2×2 system, this cost is negligible, making Newton the superior choice. For extremely large systems (millions of variables), the cost of the Jacobian might outweigh the benefit of fewer iterations.

6 Conclusion

In this project, we successfully modeled the dynamics of a biological neuron using the FitzHugh-Nagumo system. We demonstrated that:

1. The system creates realistic Action Potential spikes and stable Limit Cycles.
2. Implicit numerical schemes are necessary to handle the varying time-scales (stiffness) of the problem.
3. Newton-Raphson is superior to Fixed Point iteration for this specific non-linear problem, reducing the iteration count by a factor of 6 due to its quadratic convergence properties.

The code successfully reproduces the theoretical expectations and provides a robust framework for simulating excitable media.

References

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[3] Burden, R. L., & Faires, J. D. (2010). *Numerical Analysis* (9th ed.). Brooks/Cole.