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A title

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Abstract

This will eventually be an abstract.

1. Introduction

1.1. What is a Permutation?

In *The Art of Computer Programming* Donald Knuth defines A *permutation of n objects* is an arrangement of n distinct objects in a row. When considering permutations we can consider them as occuring on the set $[\![n]\!] = \{1, \ldots, n\}$, therefore a permutation is a *bijection* $\pi : [\![n]\!] \mapsto [\![n]\!]$. We can write a permutation π in two line notation

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{pmatrix}$$

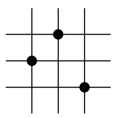
However, the most frequent notation used in computer science is *one-line notation*, in this form we drop the top line of the two line notation so are left with the following.

$$\pi = \pi(1)\pi(2)\ldots\pi(n)$$

Example 1.1. There are 6 permutations on $[\![3]\!]$.

$$123, 132, 213, 231, 312, 321$$

We can display a permutation on a *figure* in order to give a graphical representation of the permutation. In such a figure we let the x-axis denote the index in the permutation, and the y-axis denotes the values of $\pi(x)$. The figure of the permutation $\pi = 231$ is shown below



The class of all permutations of length n is $\mathfrak{S}_n$ and the class has size $n!$. The class of all permutations is $\mathfrak{S} = \bigcup_{i=0}^{\infty} \mathfrak{S}_i$.

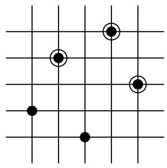
1.2. Classical Permutation Patterns

Classical Permutation Patterns began to be studied as a result of Knuth's statements about stack-sorting in [The Art of Computer Programming].

Definition 1.1. Order isomorphism. Two sequences $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ are said to be *order isomorphic* if they share the same relative order,i.e.,$\alpha_r < \alpha_s$ if and only if $\beta_r < \beta_s$.

A permutation π us said to *contain* the permutation σ of length k as a pattern (denoted $\sigma \preceq \pi$) if there is some an increasing subsequence $i_1, i_2, \ldots, i_n$ such that the sequence $\pi(i_1), \pi(i_2), \ldots, \pi(i_k)$ is order isomorphic to $\sigma(1), \sigma(2), \ldots, \sigma(k)$. If π does not contain σ, we say that π *avoids* σ.

For example the permutation $\pi = 24153$ contains the pattern $\sigma = 231$, since the second, fourth and fifth entries $(4, 5,$ and $3)$ share the same relative order as the entries of σ. This can be seen graphically below, the points order isomorphic to σ are highlighted.



We denote the set of permutations of length n avoiding a pattern σ as $\mathrm{Av}_n(\sigma)$ and $\mathrm{Av}(\sigma) = \bigcup_{i=0}^{\infty} \mathrm{Av}_i(\sigma)$.

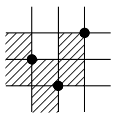
1.3. Mesh Patterns

Mesh Patterns were introduced by Brändén and Claesson to capture explicit expansions for certain permutation statistics. They are a natural extension of Classical permutation patterns. A *mesh-pattern* is a pair

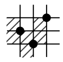
$$p = (\tau, R) \text{ with } \tau \in \mathfrak{S} \text{ and } R \subseteq [0, k] \times [0, k].$$

By this definition the empty permutation ε as a mesh pattern consisting solely of the box $(0, 0)$.

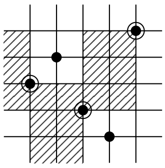
The figure for a mesh pattern looks similar to that for a classical pattern with the addition that we shade the unit square with bottom corner (i, j) for each $(i, j) \in R$:



We define containment, and avoidance, of the pattern p in the permutation π on mesh patterns analogously to classical containment, and avoidance, of τ in π with the additional restrictions on the relative position of the occurence of τ in π. These restrictions say that the shaded regions of the figure above contain no points from π.

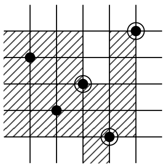
Example 1.2. The pattern $p = (213, \{(0,1), (0,2), (1,0), (1,1), (2,1), (2,2)\}) =$  is contained in $\pi = 34215$ but is not contained in $\sigma = 42315$.

Proof. Let us consider the figure for the permutation π we only need to find one occurence.



We have found an occurence of the pattern p in π and therefore π contains p.

Now we consider the figure for the permutation σ. This permutation avoids p since for every occurence of the classical pattern 213 there is at least one point in one of the shaded boxes. Consider the subsequence 315 in σ, this is an occurence of 213 but not the mesh pattern since the points with values 4 and 2 are in the shaded areas. This is shown in the figure below.



This is true for all occurences of 213 in σ and therefore σ avoids p. □

We denote the avoidance sets for mesh patterns in the same way as for classical patterns.

Note 1.3. Classical patterns are just a subclass of mesh patterns where the mesh set R is empty. The classical pattern π can be represented by a mesh pattern as $(\pi, \emptyset)$.

2. Coincidences amongst families of mesh patterns and classical patterns

One interesting question to ask about permutation patterns considers when a pattern may be avoided by, or contained in, arbitrary permutations. Two patterns π and σ are said to be *coincident* if the set of permutations that avoid π is the same as the set of permutations that avoid σ. This extends to sets of patterns as well as single patterns.

We consider the avoidance sets, $\mathrm{Av}(p, q)$ where p is a classical pattern of length 3 and q is a mesh pattern of length 2 in order to establish some rules about when these two sets give the same avoidance set.

We first define some operations on mesh patterns.

Definition 2.1. Given a pattern p, let $\texttt{add\_point}\,(p,(a,b),D)$ be the operation that returns a mesh pattern equivalent to placing a point in the center of box (a, b) in p, with shading defined by $D \subseteq \{N, E, S, W\}$.

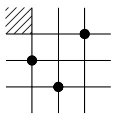
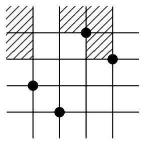
The set D defines the shading by indicating that the boxes in the cardinal directions in D next to the point are shaded in the resulting pattern. Since there is no ambiguity we let $\texttt{add\_point}\,(\varepsilon, D)$ be equivalent to $\texttt{add\_point}\,(\varepsilon,(0,0),D)$. This operation fails if the box (a, b) is in the mesh set of p.

Example 2.1. The result of adding a single point to the empty permutation for each cardinal direction.

$\texttt{add\_point}\,(\varepsilon, \{N\}) =$  $\texttt{add\_point}\,(\varepsilon, \{E\}) =$ 

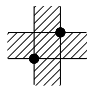
$\texttt{add\_point}\,(\varepsilon, \{S\}) =$  $\texttt{add\_point}\,(\varepsilon, \{W\}) =$ 

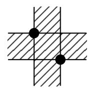
A more complex example for add_point

$\texttt{add\_point}\,\Bigg($  $, (2,3), \{N, E\}\Bigg) =$ 

Definition 2.2. Given a pattern p, define $\texttt{add\_descent}\,(p,(a,b))$, and $\texttt{add\_ascent}\,(p,(a,b))$, as the operations that return a mesh pattern equivalent to placing an decrease, or increase, in the center of box (a,b) in p.

Example 2.2.

$\texttt{add\_ascent}\,(\varepsilon) =$ 

$\texttt{add\_descent}\,(\varepsilon) =$ 

We now attempt to fully classify coincidences in families characterised by avoidance of a classical pattern of length 3 and a mesh pattern of length 2.

It can be easily seen that in order to classify set equivalences one need only consider coincidences within the family of mesh patterns with the same underlying classical pattern, this is due to the fact that $21 \in \mathrm{Av}((12,R))$ and $12 \in \mathrm{Av}((21,R))$ for all mesh-sets R. So $\mathrm{Av}((12,R)) \setminus \mathrm{Av}((21,S)) \neq \emptyset \; \forall R, S \in [0,2] \times [0,2]$, and hence the sets are disjoint.

We know that there are a total of 512 mesh-sets for each underlying classical pattern. By use of the shading lemma, simultaneous shading lemma, and one special case, we can reduce the number of equivalence classes to 220.

2.1. Equivalence classes of Av({321, (21, R)}).

Proposition 2.3 (First Dominating Pattern Rule)**.** Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \emptyset)$ such that $|\pi| \leq |\sigma| + 1$, the sets $\mathrm{Av}(\{\pi, m_1\})$ and $\mathrm{Av}(\{\pi, m_2\})$ are coincident if

1. $R_1 \triangle R_2 = \{(a,b)\}$
2. $\pi \preceq \texttt{add\_point}\,(\sigma, (a,b), \emptyset)$

Proof. □

2.2. Equivalence classes of Av({231, (21, R)}).

Lemma 2.4. Given a mesh pattern $m = (\sigma, R)$, where the box (a, b) is not in R, and a dominating classical pattern $\pi = (\pi, \emptyset)$ if $\pi \preceq \texttt{add\_ascent}\,(\sigma, (a, b))(\pi \preceq \texttt{add\_descent}\,(\sigma, (a, b)))$ then in any occurrence of m in a permutation ϱ the region corresponding to the box (a, b) can only contain an increasing (decreasing) subsequence of ϱ.

The proof is analogous to the proof of Proposition 2.3.

Proposition 2.5 (Second Dominating Pattern Rule)**.** Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \emptyset)$ such that $|\pi| \leq |\sigma| + 2$, the sets $\mathrm{Av}(\{\pi, m_1\})$ and $\mathrm{Av}(\{\pi, m_2\})$ are coincident if

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. (a) $\pi \preceq \texttt{add\_ascent}\,(\sigma, (a, b))$ and
      i. $(a + 1, b) \in \sigma$ and
         $(x, b - 1) \in R \implies (x, b) \in R$ (where $x \neq a, a + 1$) and
         $(a + 1, y) \in R \implies (a, y) \in R$ (where $y \neq b - 1, b$).
      ii. $(a, b + 1) \in \sigma$ and
         $(x, b + 1) \in R \implies (x, b) \in R$ (where $x \neq a - 1, a$) and
         $(a - 1, y) \in R \implies (a, y) \in R$ (where $y \neq b, b + 1$).

   (b) $\pi \preceq \texttt{add\_descent}\,(\sigma, (a, b))$ and
      i. $(a + 1, b + 1) \in \sigma$ and
         $(x, b + 1) \in R \implies (x, b) \in R$ (where $x \neq a, a + 1$) and
         $(a + 1, y) \in R \implies (a, y) \in R$ (where $y \neq b, b + 1$).
      ii. $(a, b) \in \sigma$ and
         $(x, b + 1) \in R \implies (x, b) \in R$ (where $x \neq a - 1, a$) and
         $(a - 1, y) \in R \implies (a, y) \in R$ (where $y \neq b - 1, b$).

Proof. □