

# Equivalence classes of mesh patterns with a Dominating Pattern

Murray Tannock (mtannock@cs.otago.ac.nz)



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# Outline

## 1. Introduction

- ▶ Permutations
- ▶ Classical Permutation Patterns
- ▶ Mesh Patterns

## 2. Coincidence Classes

- ▶ Coincidence
- ▶ Summary of Experimental Results
- ▶ Dominating Pattern Rules
- ▶ Special Cases

# Permutations

A *permutation* is a *bijection*,  $\pi$ , from the set  $\llbracket n \rrbracket = \{1, \dots, n\}$  to itself.

More intuitively “A *permutation of  $n$  objects* is an arrangement of  $n$  distinct objects in a row” (Knuth [1]).

We write permutations in *one-line notation*, writing the entries of the permutation in order

$$\pi = \pi(1)\pi(2) \dots \pi(n)$$

## Example

The 6 permutations on  $\llbracket 3 \rrbracket$  are

123, 132, 213, 231, 312, 321

We can display a permutation in a *plot* to give a graphical representation. We plot the points  $(i, \pi(i))$  in a Cartesian coordinate system.

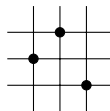


Figure: Plot of the permutation 231

In this setting we call the elements of the permutations *points*.

The set of all permutations of length  $n$  is  $\mathfrak{S}_n$  and has size  $n!$ . The set of all permutations is  $\mathfrak{S} = \bigcup_{i=0}^{\infty} \mathfrak{S}_i$ .

# Classical Permutation Patterns

*Classical permutation patterns* capture many interesting combinatorial objects and properties.

## Definition (Order Isomorphism)

Two sequences  $\alpha_1\alpha_2\cdots\alpha_n$  and  $\beta_1\beta_2\cdots\beta_n$  are said to be *order isomorphic* if  $\alpha_r < \alpha_s$  if and only if  $\beta_r < \beta_s$ .

## Definition

A permutation  $\pi$  is said to *contain* the *classical permutation pattern*  $\sigma$  (denoted  $\sigma \leq \pi$ ) if there is some subsequence  $i_1 i_2 \cdots i_k$  such that the sequence  $\pi(i_1) \pi(i_2) \cdots \pi(i_k)$  is order isomorphic to  $\sigma(1) \sigma(2) \cdots \sigma(k)$ .

If  $\pi$  does not contain  $\sigma$  we say that  $\pi$  *avoids*  $\sigma$ .

The set of permutations of length  $n$  avoiding a pattern  $\sigma$  is denoted as  $\text{Av}_n(\sigma)$  and

$$\text{Av}(\sigma) = \bigcup_{i=0}^{\infty} \text{Av}_i(\sigma)$$

## Example

The permutation  $\pi = 24153$  contains the pattern  $\sigma = 231$

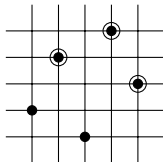


Figure: Plot of the permutation 24153 with an occurrence of 231 indicated

# Mesh Patterns

*Mesh patterns* are a natural extension of classical permutation patterns.

## Definition


A *mesh pattern* is a pair

$$p = (\tau, R) \text{ with } \tau \in \mathfrak{S}_k \text{ and } R \subseteq [0, k] \times [0, k].$$

We say that  $\tau$  is the *underlying classical pattern* of  $p$ .



## Example

The pattern  $p = (213, \{(0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (2, 1), (2, 2)\}) =$   is contained in  $\pi = 34215$ .

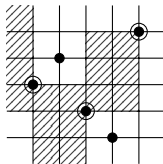




Figure: An occurrence of  $p$  in  $\pi$

## Example

The pattern  $q = (21, \{(0, 1), (0, 2), (1, 0), (1, 1)\}) =$   is contained in  
 $p = (213, \{(0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (2, 1), (2, 2)\}) =$   as a subpattern.

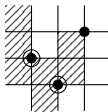


Figure: An occurrence of  $q$  in  $p$

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# Coincidence

## Definition

Two mesh patterns are said to be *coincident* if they are avoided by the same set of permutations at every length.

Distinct classical patterns can never be coincident.

Coincidences of mesh patterns of length 2 are classified.

Aim to establish rules that classify coincidences when we have one mesh pattern and one classical pattern.

We call the classical pattern a *dominating pattern*.

# Experimental Results

	Dominating Pattern			
	231		321	
	12	21	12	21
Starting Size of Classes	512	512	512	512
Removing Symmetries	220	220	220	220
First Dominating rule	85	43	220	29
Second Dominating rule	59	39	220	29
Third Dominating rule	56	39	220	29
Experimental class size	56	39	213	29

**Table:** Coincidence class number reduction by application of Dominating rules

# Experimental Results

	Dominating Pattern			
	231		321	
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Starting Size of Classes	512	512	512	512
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**Table:** Coincidence class number reduction by application of Dominating rules

# First Dominating rule

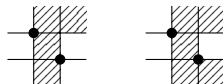
## Proposition: First Dominating rule

*Given two mesh patterns  $m_1 = (\sigma, R_1)$  and  $m_2 = (\sigma, R_2)$ , and a dominating classical pattern  $\pi = (\pi, \emptyset)$  such that  $|\pi| \leq |\sigma| + 1$ , the sets  $\text{Av}(\{\pi, m_1\})$  and  $\text{Av}(\{\pi, m_2\})$  are coincident if*

1.  $R_1 \triangle R_2 = \{(a, b)\}$
2.  $\pi \leq \text{add\_point}(\sigma, (a, b), \emptyset)$

## Example

The following two patterns are coincident in  $Av(321)$



## Corollary

*All coincidences of classes the form  $Av(\{321, (21, R)\})$  are fully explained by the First Dominating rule.*

There are 29 coincidences of mesh patterns of the form  $Av(\{321, (21, R)\})$



# Experimental Results

	Dominating Pattern			
	231		321	
	12	21	12	21
Starting Size of Classes	512	512	512	512
Removing Symmetries	220	220	220	220
First Dominating rule	85	43	220	29
Experimental class size	56	39	213	29

**Table:** Coincidence class number reduction by application of Dominating rules

## Second Dominating rule



The patterns

$$m_1 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \bullet & \text{shaded} \\ \hline \text{shaded} & \bullet \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \bullet & \text{shaded} \\ \hline \text{shaded} & \bullet \\ \hline \end{array}$$

are coincident in  $\text{Av}(231)$ .



## Lemma

*Given a mesh pattern  $m = (\sigma, R)$ , where the box  $(a, b)$  is not in  $R$ , and a dominating classical pattern  $\pi = (\pi, \emptyset)$  if  $\pi \leq \text{add\_ascent}(\sigma, (a, b))$ , then in any occurrence of  $m$  in a permutation  $\varrho$ , the region corresponding to the box  $(a, b)$  can only contain an decreasing subsequence of  $\varrho$ .*

Example ( $m_1 =$  and  $m_2 =$ ) are coincident in  $\text{Av}(231)$ .)



Considering  $m_1$  again



Example ( $m_1 =$  and  $m_2 =$ ) are coincident in  $\text{Av}(231)$ .)



Considering  $m_1$  again



Example ( $m_1 =$  and  $m_2 =$ ) are coincident in  $\text{Av}(231)$ .)

Considering  $m_1$  again



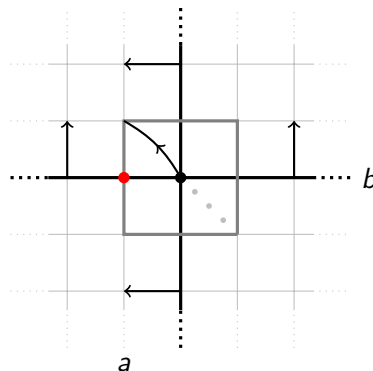
Example ( $m_1 =$  and  $m_2 =$ ) are coincident in  $\text{Av}(231)$ .)

Considering  $m_1$  again



This is  $m_2$ .

## Example (Graphical interpretation of one case of The Second Dominating rule)



**Figure:** If the conditions of The Second Dominating rule are satisfied the box  $(a, b)$  can be shaded.



## Corollary

*All coincidences of classes the form  $\text{Av}(\{231, (21, R)\})$  are fully explained by applying the First Dominating rule, then applying the Second Dominating rule.*

There are 39 coincidences of mesh patterns of the form  $\text{Av}(\{231, (21, R)\})$

# Experimental Results

	Dominating Pattern			
	231		321	
	12	21	12	21
Starting Size of Classes	512	512	512	512
Removing Symmetries	220	220	220	220
First Dominating rule	85	43	220	29
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Experimental class size	56	39	213	29



**Table:** Coincidence class number reduction by application of Dominating rules

# Third Dominating rule



The patterns

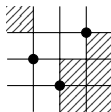
$$m_1 = \begin{array}{|c|c|} \hline \text{shaded} & \bullet \\ \hline \bullet & \text{shaded} \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|} \hline \text{shaded} & \bullet \\ \hline \bullet & \text{shaded} \\ \hline \end{array}$$



are coincident in  $\text{Av}(231)$ . Neither of the previous two rules explain this.

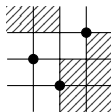
Example ( $m_1 =$  and  $m_2 =$ ) are coincident in  $\text{Av}(231)$ .)





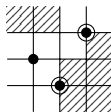
Example ( $m_1 =$   and  $m_2 =$   are coincident in  $\text{Av}(231)$ .)





Example ( $m_1 =$  and  $m_2 =$ ) are coincident in  $\text{Av}(231)$ .)



Example ( $m_1 =$  and  $m_2 =$ ) are coincident in  $\text{Av}(231)$ .)



Example ( $m_1 =$   and  $m_2 =$   are coincident in  $\text{Av}(231)$ .)





### Proposition: Third Dominating rule

*Given two mesh patterns  $m_1 = (\sigma, R_1)$  and  $m_2 = (\sigma, R_2)$ , and a dominating classical pattern  $\pi = (\pi, \emptyset)$ , the sets  $\text{Av}(\{\pi, m_1\})$  and  $\text{Av}(\{\pi, m_2\})$  are coincident if*

1.  $R_1 \triangle R_2 = \{(a, b)\}$
2.  $\text{add\_point}((\sigma, R_1), (a, b), D)$  where  $D \in \{N, E, S, W\}$  is coincident with a mesh pattern containing an occurrence of  $(\sigma, R_2)$  as a subpattern.

## Corollary

*All coincidences of classes the form  $\text{Av}(\{231, (12, R)\})$  are fully explained by applying the First Dominating rule, the Second Dominating rule, and then the Third Dominating rule.*

There are 56 coincidences of mesh patterns of the form  $\text{Av}(\{231, (21, R)\})$

# Experimental Results

	Dominating Pattern			
	231		321	
	12	21	12	21
Starting Size of Classes	512	512	512	512
Removing Symmetries	220	220	220	220
First Dominating rule	85	43	220	29
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Experimental class size	56	39	213	29

**Table:** Coincidence class number reduction by application of Dominating rules



# Special Cases

There are 7 coincidences of the form  $Av(321, m)$  that are not explained by the rules.



## Example

$$m_1 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \bullet & \bullet \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \bullet & \bullet \\ \hline \end{array}$$

This coincidence is explained by mathematical induction on the number of points in the region corresponding to the middle box. We call this number  $n$ .



Example ( $m_1 =$  and  $m_2 =$  are coincident in  $\text{Av}(231)$ .)

Base Case ( $n = 0$ ): The base case holds since we can freely shade the box if it contains no points.

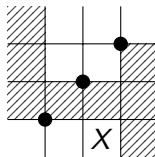
Example ( $m_1 =$  and  $m_2 =$  are coincident in  $\text{Av}(231)$ .)

**Base Case ( $n = 0$ ):** The base case holds since we can freely shade the box if it contains no points.



**Inductive Hypothesis ( $n = k$ ):** Suppose that we can find an occurrence of the second pattern if we have an occurrence of the first with  $k$  points in the middle box.

Example ( $m_1 =$  and  $m_2 =$  are coincident in  $\text{Av}(231)$ .)

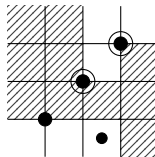
**Inductive Step** ( $n = k + 1$ ) Suppose that we have  $(k + 1)$  points in the middle box. Choose the bottom most point in the middle box, this gives the mesh pattern



Now we need to consider the box labelled  $X$ . If this box is empty then we have an occurrence of  $m_2$  and are done.

Example ( $m_1 =$   and  $m_2 =$   are coincident in  $\text{Av}(231)$ .)

**Inductive Step ( $n = k + 1$ ) (cont.)** If this box contains any points then we gain some extra shading on the mesh pattern due to the dominating pattern



The two highlighted points form an occurrence of  $m_1$  with  $k$  points in the middle box, and thus by the Inductive Hypothesis we are done.





[1] D. Knuth,  
*The Art of Computer Programming: Volume 1*, 1997.