Equivalence classes of mesh patterns with a dominating pattern

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We establish sufficient conditions for coincidence between mesh patterns when considered inside a set of avoiders of a classical permutation pattern, and establish general rules that can be used to construct such coincidences. These rules along with two special cases allow us to completely classify coincidence between all sets of avoiders of a mesh pattern of length 2 and a classical pattern of length 3. We then go on to consider Wilf-equivalence amongst mesh patterns of length 1 and 2 whilst also avoiding the classical pattern 231.

When we consider avoidance of a pattern, p, inside a permutation class we gain additional restrictions on the occurrences of p. We will restrict the set of permutations that form the basis for our permutation class to a single classical pattern, π, which we shall refer to as the *dominating pattern*. In particular, we consider the case where p is mesh pattern of length 2 and π is either of the classical permutation patterns 231 or 321.

By use of previous results of Claesson, Tenner and Ulfarsson[1] we can begin to reduce the number of potential coincidences by only looking for coincidences between patterns that are not coincident outwith the setting of having a dominating pattern. This gives us an initial upper bound of 220 coincidence classes in each of the four cases we consider.

By examining experimental coincidences of non-classical permutation classes, considered up to length 11 we can find rules that allow us to establish coincidences, and thus establish the number of coincidence classes as follows.

	Dominating Pattern			
	Av(231)		Av(321)	
Underlying Classical pattern	12	21	12	21
No Dominating rule	220	220	220	220
First Dominating rule	85	43	220	29
Second Dominating rule	59	39	220	29
Third Dominating rule	56	39	220	29
Experimental class size	56	39	213	29

Table 1: Coincidence class number reduction by application of Dominating Rules

By considering the classical permutation classes Av(231) and Av(321) along with a mesh pattern with underlying classical pattern 21, we gain the following two rules that completely classify the coincidence in theses cases.

Proposition 1 (First Dominating Pattern Rule)**.** *Given two mesh patterns* $m_1 = (\sigma, R_1)$ *and* $m_2 = (\sigma, R_2)$*, and a dominating classical pattern* $\pi = (\pi, \emptyset)$ *such that* $|\pi| \leq |\sigma| + 1$*, the sets* $\mathrm{Av}(\{\pi, m_1\})$ *and* $\mathrm{Av}(\{\pi, m_2\})$ *are coincident if*

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. π *is contained in* `add_point` $(\sigma, (a, b), \emptyset)$

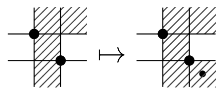


Figure 1: Visual depiction of first dominating pattern rule.

Proposition 2 (Second Dominating Pattern Rule)**.** *Given two mesh patterns* $m_1 = (\sigma, R_1)$ *and* $m_2 = (\sigma, R_2)$*, and a dominating classical pattern* $\pi = (\pi, \emptyset)$ *such that* $|\pi| \leq |\sigma| + 2$*, the sets* $\mathrm{Av}(\{\pi, m_1\})$ *and* $\mathrm{Av}(\{\pi, m_2\})$ *are coincident if*

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. (a) π *is contained in* `add_ascent` $(\sigma, (a, b))$ *and some additional conditions on the mesh are satisfied that allow selection of different points in the occurrence*

   (b) π *is contained in* `add_descent` $(\sigma, (a, b))$ *and some additional conditions on the mesh are satisfied that allow selection of different points in the occurrence*

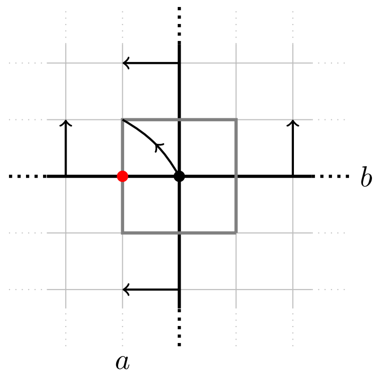


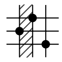
Figure 2: If the conditions of Proposition 2 are satisfied the box (a, b) can be shaded.

We then define the notion of containment of a mesh pattern, p, within another mesh pattern, q, we call this notion *subpattern* containment. This is defined as an extension of regular mesh pattern containment, with the additional condition that if $(i, j) \in R$ then R_{ij} is contained in the mesh set of q. This notion is then used to establish the following rule that fully classifies the coincidences between non-classical permutation classes where we avoid the classical pattern 231 and a mesh pattern with underlying classical pattern 12.

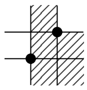
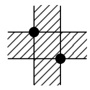
Proposition 3 (Third Dominating Pattern Rule)**.** *Given two mesh patterns* $m_1 = (\sigma, R_1)$ *and* $m_2 = (\sigma, R_2)$*, and a dominating classical pattern* $\pi = (\pi, \emptyset)$*, the sets* $\mathrm{Av}(\{\pi, m_1\})$ *and* $\mathrm{Av}(\{\pi, m_2\})$ *are coincident if*

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. `add_point` $((\sigma, R_1), (a, b), D)$ *where* $D \in \{N, E, S, W\}$ *is coincident with a mesh pattern containing an occurrence of* (σ, R_2) *as a subpattern.*

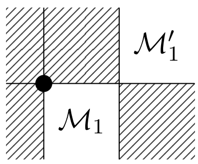
These three rules, along with two special cases, give a complete coincidence classification of the non-classical permutation classes $\mathrm{Av}(\pi, p)$ where π is a fixed classical permutation pattern of length 3 and p is a mesh pattern of length 2.

We then go on to consider Wilf-equivalences between non-classical permutation classes where we avoid the classical permutation pattern 231 and a mesh pattern of length 2. I order to establish these Wilf-equivalences we use the results of the prior section as well as some observations about symmetries in order to reduce the initial upper bound on Wilf-equivalence classes. Using the bijection between avoiders of the vincular pattern  and set partitions presented by Claesson[2], and structural decomposition of avoiders to obtain generating functions we fully classify the Wilf-equivalences. This results in 13 non-trivial (*i.e.* not containing only a single coincidence class) Wilf-equivalence classes. When using generating function methods we also provide enumeration of the non-classical permutation class.

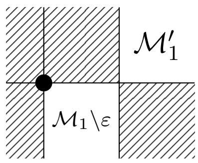
Example 4. *The following patterns are experimentally Wilf-equivalent up to length 10 in* $\mathrm{Av}(231)$

$m_1 =$  *and* $m_2 =$ 

Considering structural decomposition of an avoider of m_1 *in* $\mathrm{Av}(2,3,1)$ *around the leftmost point, we find that the set of avoiders has form*

$\mathcal{M}_1 = \varepsilon \sqcup$ 

Where $\mathcal{M}_1'$ *is a permutation avoiding* $231, m_1$ *and*  *, these permutations have form* $\mathcal{M}_1'$

$\mathcal{M}_1' = \varepsilon \sqcup$ 

The avoiders of m_2 *in* $\mathrm{Av}(2,3,1)$ *can be split in a similar manner around the leftmost point. This gives us that both of these sets have generating function* $M(x)$ *satisfying*

$$M(x) = 1 + xM(x)M'(x) \tag{1}$$
$$M'(x) = 1 + x(M(x) - 1)M'(x) \tag{2}$$

Solving 2 for $M'(x)$ *and substituting into 1 gives us that the generating function for* $M(x)$ *satisfies*

$$M(x) = xM^2(x) - x(M(x) - 1) + 1 \tag{3}$$

Solving $M(x)$ *and evaluating the coefficients gives the sequence*

$$1, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, \ldots \qquad \text{(OEIS: A001006 with offset 1)}$$

This is an offset of the Motzkin numbers.

References

[1] A. Claesson, B. E. Tenner, and H. Ulfarsson, Coincidence among families of mesh patterns, *CoRR*, 2014. `http://arxiv.org/abs/1412.0703`

[2] A. Claesson Generalized pattern avoidance *Eur. J. Comb*, 2001. `http://dx.doi.org/10.1006/eujc.2001.0515`