EQUIVALENCE CLASSES OF MESH PATTERNS WITH A DOMINATING PATTERN

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ABSTRACT. Two mesh patterns are coincident if they are avoided by the same set of permutations, and are Wilf-equivalent if they have the same number of avoiders of each length. We provide sufficient conditions for coincidence of mesh patterns, when only permutations also avoiding a longer classical pattern are considered. Using these conditions we completely classify coincidences between families containing a mesh pattern of length 2 and a classical pattern of length 3. Furthermore, we completely Wilf-classify mesh patterns of length 2 inside the class of 231-avoiding permutations.

Keywords: permutation, pattern, mesh pattern, pattern coincidence

1. INTRODUCTION

The study of permutation patterns began as a result of Knuth's statements on stack sorting in *The Art of Computer Programming* [9, p. 243, Ex. 5,6]. This original concept—a subsequence of symbols having a particular relative order, now known as classical patterns—has been expanded to a variety of definitions. Babson and Steingrímsson [1] considered *vincular* patterns (also known as *generalised* or *dashed* patterns) where two adjacent entries in the pattern can be required to be adjacent in the permutation. Bousquet-Mélou, Claesson, Dukes, *et al.* [3] look at classes of patterns where entries can also be required to be consecutive in value, these are called *bivincular* patterns. *Bruhat-restricted* patterns were studied by Woo and Yong [10] to establish necessary conditions for a Schubert variety to be Gorenstein. These definitions are subsumed under the definition of *mesh patterns*, introduced by Brändén and Claesson [4] to capture explicit expansions for certain permutation statistics.

When considering permutation patterns some of the main questions posed relate to how and when a pattern is avoided by, or contained in, an arbitrary set of permutations. Two patterns π and σ are *Wilf-equivalent* if the number of permutations that avoid π of length n is equal to the number of permutations that avoid σ of length n. A stronger equivalence condition is that of *coincidence*, where the set of permutations avoiding π is exactly equal to the set of permutations avoiding σ. Avoiding pairs of patterns of the same

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2010 *Mathematics Subject Classification.* Primary: 05A05; Secondary: 05A15.

Research partially supported by grant 141761-051 from the Icelandic Research Fund.

length with certain properties has been studied, Claesson and Mansour [5] considered avoiding a pair of vincular patterns of length 3. Bean, Claesson, and Ulfarsson [2] study avoiding a vincular and a covincular pattern simultaneously in order to achieve several counting results. However, little work has been done on avoiding a mesh pattern and a classical pattern simultaneously.

In this work we aim to establish some ground in this field by computing coincidences and Wilf-classes and calculating some of the enumerations of avoiders of a mesh pattern of length 2 and a classical pattern of length 3. We begin by establishing coincidences between mesh patterns of length 2 while avoiding a dominating pattern by computational methods, which are then used to establish three "Dominating Pattern Rules" as well as some special cases that can be used to calculate coincidences.

We then use these coincidence classes to calculate Wilf-equivalence classes showing some of the methods used.

2. Mesh patterns

A *permutation* is a bijection from the set $[\![n]\!] = \{1, \dots, n\}$ to itself. The set of all such bijections is denoted $\mathfrak{S}_n$ and has $n!$ elements. We can denote an individual permutation $\pi \in \mathfrak{S}_n$ in *one-line notation* by writing the entries of the permutation in order, therefore $\pi = \pi(1)\pi(2)\dots\pi(n)$. The set $\mathfrak{S}_0$ has exactly one element, the empty permutation ε.

Definition 2.1. (Order isomorphism.) Two strings of integers $\alpha_1\alpha_2\cdots\alpha_n$ and $\beta_1\beta_2\cdots\beta_n$ are said to be *order isomorphic* if they share the same relative order, *i.e.*, $\alpha_r < \alpha_s$ if and only if $\beta_r < \beta_s$.

The definition of order isomorphism allows us to give the meaning of containment for classical permutation patterns.

Definition 2.2. A permutation $\pi \in \mathfrak{S}_n$ *contains* the permutation $\sigma \in \mathfrak{S}_k$ (denoted $\sigma \preceq \pi$) if there is some sequence $i_1, i_2, \cdots, i_k$ such that $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ and the sequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is order isomorphic to $\sigma(1)\sigma(2)\cdots\sigma(k)$. If this is the case the sequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is called an *occurrence* of σ in π. If π does not contain σ, we say that π *avoids* σ. In this context σ is called a (*classical*) *permutation pattern.*

Example 2.3. The permutation $\pi = 24153$ contains the pattern $\sigma = 231$, since the second, fourth and fifth elements (453) are order isomorphic to 231. The permutation also contains the occurrence 241 of the same pattern. The permutation 24153 avoids the pattern 321.

We denote the set of permutations of length n avoiding a pattern σ as $\mathrm{Av}_n(\sigma)$ and let $\mathrm{Av}(\sigma) = \bigcup_{i=0}^{\infty} \mathrm{Av}_i(\sigma)$.

We can display a permutation graphically in a *plot*, where we display the points $G(\pi) = \{(i, \pi(i)) \mid i \in [\![n]\!]\}$ in a Cartesian coordinate system. The plots of the permutations $\pi = 24153$ and $\sigma = 231$ can be seen in Figure 1. Figure 2 shows the containment of σ in π as in Example 2.3.

The boxes in the plot of a permutation are denoted by $\ulcorner i, j \lrcorner$, where the point (i, j) is the lower left corner of the box.

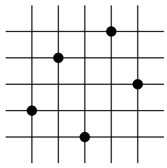
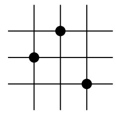
$G(\pi) =$  $G(\sigma) =$ 

FIGURE 1. The plots of the permutations π and σ.

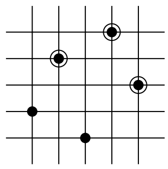


FIGURE 2. The occurrence of 231 in 24153 corresponding to 453.

Definition 2.4. A *mesh pattern* is a pair

$$p = (\tau, R) \text{ with } \tau \in \mathfrak{S}_k \text{ and } R \subseteq [0, k] \times [0, k].$$

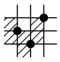
Formally defined by Brändén and Claesson [4], an *occurrence* of p in π is a subset ω of the plot of $\pi, G(\pi) = \{(i, \pi(i) \mid i \in [\![n]\!]\}$ such that there are order-preserving injections $\alpha, \beta : [\![k]\!] \mapsto [\![n]\!]$ satisfying the following two conditions.
Firstly, ω is an occurrence of τ in the classical sense

i. $\omega = \{(\alpha(i), \beta(j)) : (i, j) \in G(\tau)\}$.

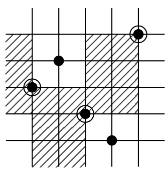
Define $R_{ij} = [\alpha(i) + 1, \alpha(i + 1) - 1] \times [\beta(j) + 1, \beta(j + 1) - 1]$ for $i, j \in [0, k]$ where $\alpha(0) = \beta(0) = 0$ and $\alpha(k + 1) = \beta(k + 1) = n + 1$. Then the second condition is

ii. if $\ulcorner i, j \lrcorner \in R$ then $R_{ij} \cap G(\pi) = \varnothing$.

We call R_{ij} the *region corresponding to* $\ulcorner i, j \lrcorner$.

Example 2.5. The pattern $p = (213, \{(0, 1), (0, 2), (1, 0), (1, 1), (2, 1), (2, 2)\}) =$  is contained in $\pi = 34215$.

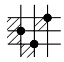
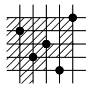
Let us consider the plot for the permutation π. The subsequence 325 is an occurrence of 213 in the classical sense and the remaining points of π are not contained in the regions corresponding to the shaded boxes in p.



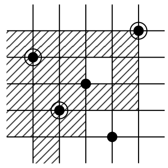
The subsequence 325 is therefore an occurrence of the pattern p in π.

We define containment of a mesh pattern p in another mesh pattern q as above, with the additional condition that if $\ulcorner i, j \lrcorner \in R$ then R_{ij} is contained in the mesh set of q.

Definition 2.6. A mesh pattern $q = (\kappa, T)$ *contains* a mesh pattern $p = (\tau, R)$ as a *subpattern* if κ contains p and $\left(\bigcup_{\ulcorner i,j \lrcorner \in R} R_{ij}\right) \subseteq T$.

Example 2.7. The pattern $p = (213, \{(0,1), (0,2), (1,0), (2,2)\}) =$  is contained in the pattern $r =$  as a subpattern.

The highlighted points form an occurrence of p in r



The permutation 42315 also contains p in the usual sense.

We denote the avoidance sets for mesh patterns in the same way as for classical patterns. Given a mesh pattern $p = (\sigma, R)$ we say that σ is the *underlying classical pattern* of p.

3. Coincidences between Mesh Patterns

Coincidences among small mesh patterns have been considered by Claesson, Tenner, and Ulfarsson [6], in which the authors use the Simultaneous Shading Lemma, a closure result and one worked out special case to fully classify coincidences among mesh patterns of length 2.

Recall that two patterns λ and γ are considered *coincident* if the set of permutations that avoid λ is the same as the set of permutations that avoid γ, *i.e.*, $\mathrm{Av}(\lambda) = \mathrm{Av}(\gamma)$. Equivalently we can say that they have the same set of *containers*, *i.e.*, $\mathrm{Cont}(\lambda) = \mathrm{Cont}(\gamma)$.

We will consider the avoidance sets $\mathrm{Av}(\pi, p)$ where π is a classical pattern of length 3 and p is a mesh pattern of length 2 to establish sufficient conditions for two such sets to be coincident. The classical pattern π will be fixed and called the *dominating pattern*. Given such a dominating pattern π we will write $q_1 \cong_\pi q_2$ if for the mesh patterns, q_1 and q_2, the sets $\mathrm{Av}(p, q_1)$ and $\mathrm{Av}(p, q_2)$ are coincident.

Our first step is to calculate whether $\mathrm{Av}(p, q_1)$ and $\mathrm{Av}(p, q_2)$ are equal up to permutations of length 11 computationally, if they are we write $q_1 \cong_\pi^{\mathrm{comp}} q_2$. If $q_1 \cong_\pi q_2$, it can be seen that $q_1 \cong_\pi^{\mathrm{comp}} q_2$, and therefore the $\cong_\pi$-equivalence classes form partitions of the $\cong_\pi^{\mathrm{comp}}$-coincidence classes.

We then form three Propositions called "Dominating Pattern Rules", if the First Dominating Pattern Rule shows that $q_1 \cong_\pi q_2$ then we write $q_1 \cong_\pi^{(1)} q_2$. If a combination of the First and Second Dominating Pattern Rules show that $\mathrm{Av}(p, q_1) = \mathrm{Av}(p, q_2)$ then we write $q_1 \cong_\pi^{(2)} q_2$; similarly we write $q_1 \cong_\pi^{(3)} q_2$ if a combination of all three rules shows coincidences.

In order to describe the rules it is useful to have a notion for inserting points, ascents, and descents into a mesh pattern.

Definition 3.1. Let $p = (\tau, R)$ be a mesh pattern of length n such that $\ulcorner i, j \lrcorner \notin R$. We define a mesh pattern $p^{\ulcorner i, j \lrcorner} = (\tau', R')$ of length $n+1$ as the pattern where a point is *inserted* into the box $\ulcorner i, j \lrcorner$ in $G(p)$. Formally the

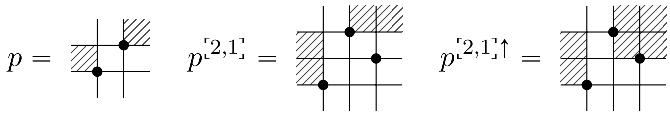


FIGURE 3. The result of inserting a point into $p = (12, \{(0,1),(2,2)\})$

new underlying classical pattern is defined by

$$\tau'(k) = \begin{cases} j+1 & \text{if } k = i+1 \\ \tau(k) & \text{if } \tau(k) \leqslant j \text{ and } k \leqslant i \\ \tau(k)+1 & \text{if } \tau(k) > j \text{ and } k \leqslant i \\ \tau(k-1) & \text{if } \tau(k) \leqslant j \text{ and } k > i+1 \\ \tau(k-1)+1 & \text{if } \tau(k) > j \text{ and } k > i+1 \end{cases}$$

While the mesh becomes

$$\begin{aligned} R' = &\{\ulcorner k,\ell \lrcorner \mid k \leqslant i, \ell \leqslant j, \ulcorner k,\ell \lrcorner \in R\} \cup \\ &\{\ulcorner k,\ell \lrcorner \mid k \leqslant i, \ell > j, \ulcorner k,\ell-1 \lrcorner \in R\} \cup \\ &\{\ulcorner k,\ell \lrcorner \mid k > i, \ell \leqslant j, \ulcorner k-1,\ell \lrcorner \in R\} \cup \\ &\{\ulcorner k,\ell \lrcorner \mid k > i, \ell > j, \ulcorner k-1,\ell-1 \lrcorner \in R\} \end{aligned}$$

In addition, we give the following definitions:

Definition 3.2. Let $p = (\tau, R)$ be a mesh pattern of length n such that $\ulcorner i,j \lrcorner \notin R$ and $p^{\ulcorner i,j \lrcorner} = (\tau', R')$ is as defined in Definition 3.1. We define the following four modifications of $p^{\ulcorner i,j \lrcorner}$.

$$\begin{aligned} p^{\ulcorner i,j \lrcorner\uparrow} &= (\tau', R' \cup \{\ulcorner i,j+1 \lrcorner, \ulcorner i+1,j+1 \lrcorner\}) \\ p^{\overrightarrow{\ulcorner i,j \lrcorner}} &= (\tau', R' \cup \{\ulcorner i+1,j \lrcorner, \ulcorner i+1,j+1 \lrcorner\}) \\ p^{\ulcorner i,j \lrcorner\downarrow} &= (\tau', R' \cup \{\ulcorner i,j \lrcorner, \ulcorner i+1,j \lrcorner\}) \\ p^{\overleftarrow{\ulcorner i,j \lrcorner}} &= (\tau', R' \cup \{\ulcorner i,j \lrcorner, \ulcorner i,j+1 \lrcorner\}) \end{aligned}$$

Informally, these are considering the topmost, rightmost, leftmost, or bottommost point in $\ulcorner i,j \lrcorner$. We collect the resulting mesh patterns in a set

$$p^{\ulcorner i,j \lrcorner\star} = \left\{ p^{\ulcorner i,j \lrcorner}, p^{\overrightarrow{\ulcorner i,j \lrcorner}}, p^{\overleftarrow{\ulcorner i,j \lrcorner}}, p^{\ulcorner i,j \lrcorner\uparrow}, p^{\ulcorner i,j \lrcorner\downarrow} \right\}$$

See Figure 3 for an example of adding a point into a mesh pattern.

Definition 3.3. Let $p = (\tau, R)$ be a mesh pattern of length n such that $\ulcorner i,j \lrcorner \notin R$. We define a mesh pattern $p^{\ulcorner i,j \lrcorner a} = (\tau', R')$ $(p^{\ulcorner i,j \lrcorner d})$ of length $n+2$ as the pattern where an ascent (descent) is *inserted* into the box $\ulcorner i,j \lrcorner$ in $G(p)$. Formally the new underlying classical pattern is defined by

$$\tau'(k) = \begin{cases} j+t & \text{if } k = i+t, t \in \{1,2\} \\ \tau(k) & \text{if } \tau(k) \leqslant j \text{ and } k \leqslant i \\ \tau(k)+2 & \text{if } \tau(k) > j \text{ and } k \leqslant i \\ \tau(k-2) & \text{if } \tau(k) \leqslant j \text{ and } k > i+2 \\ \tau(k-2)+2 & \text{if } \tau(k) > j \text{ and } k > i+2 \end{cases}$$

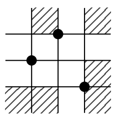
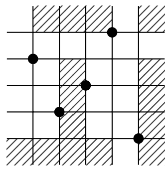
$p =$  $p^{\ulcorner 1,1 \lrcorner_a} =$ 

FIGURE 4. The result of inserting an ascent into $p = (231, \{(0,0),(1,0),(1,3),(3,0),(3,1),(3,3)\})$

The ordering of the top branch determines whether an ascent(or descent) is added. The mesh becomes

$$\begin{aligned} R' = &\{\ulcorner k,\ell \lrcorner \mid k \leqslant i, \ell \leqslant j, \ulcorner k,\ell \lrcorner \in R\} \cup \\ &\{\ulcorner k,\ell \lrcorner \mid k \leqslant i, \ell > j, \ulcorner k,\ell-2 \lrcorner \in R\} \cup \\ &\{\ulcorner k,\ell \lrcorner \mid k > i, \ell \leqslant j, \ulcorner k-2,\ell \lrcorner \in R\} \cup \\ &\{\ulcorner k,\ell \lrcorner \mid k > i, \ell > j, \ulcorner k-2,\ell-2 \lrcorner \in R\} \cup \\ &\{\ulcorner i+1,j \lrcorner, \ulcorner i+1,j+1 \lrcorner, \ulcorner i+1,j+2 \lrcorner\} \end{aligned}$$

An example of adding an ascent to a mesh pattern can be seen in Figure 4.

We now attempt to fully classify coincidences in families characterised by avoidance of a classical pattern of length 3 and a mesh pattern of length 2, that is finding and explaining all coincidences between mesh patterns m and m', $m \cong_\pi m'$, where π is a classical pattern of length 3.

It can be easily seen that in order to classify coincidences one need only consider coincidences within the family of mesh patterns with the same underlying classical pattern, this is due to the fact that $21 \in \mathrm{Av}((12, R))$ and $12 \in \mathrm{Av}(\{(21, R)\})$ for all mesh-sets R.

We know that there are a total of 512 mesh-sets for each underlying classical pattern. By use of the previous results of Claesson, Tenner, and Ulfarsson [6] the number of coincidence classes can be reduced to 220.

3.1. **Coincidence classes of Av({321, (21, R)}).** Through experimentation, considering avoidance of permutations of up to length 11, we discover that there are at least 29 $\cong_{321}^{\mathrm{comp}}$-coincidence classes where the underlying classical pattern of the mesh pattern is 21.

Proposition 3.4 (First Dominating Pattern Rule)**.** Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \varnothing)$ such that $|\pi| \leqslant |\sigma| + 1$, then $m_1 \cong_\pi m_2$ as a consequence of the first dominating pattern rule if

(1) The mesh set $R_2 = R_1 \cup \{(a, b)\}$
(2) $\pi \preceq \sigma^{\ulcorner a,b \lrcorner}$

This rule can be understood in graphical form. In the pattern in Figure 5 we can gain shading in the boxes $(0, 2)$, $(2, 0)$ since if there is a point in either of these boxes there would be an occurrence of the dominating pattern 321.

In order to prove the proposition we must first make the following note.

Note 3.5. Let $R_1 \subseteq R_2$. Then any occurrence of (τ, R_2) in a permutation is an occurrence of (τ, R_1).

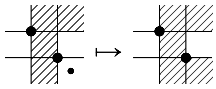


FIGURE 5. Visual depiction of first dominating pattern rule.

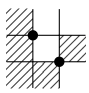
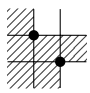
Proof of Proposition 3.4. We need to prove that $m_1 \cong_\pi m_2$.
Assume without meaningful loss of generality that $R_2 = R_1 \cup \{(a,b)\}$. Since R_1 is a subset of R_2, Note 3.5 states that $\mathrm{Av}(\{\pi, m_1\}) \subseteq \mathrm{Av}(\{\pi, m_2\})$.

Now we consider a permutation $\omega \in \mathrm{Av}(\pi)$, containing an occurrence of m_1. If there is a point in the region corresponding to the box (a,b), then that point, along with the points of the occurrence of m_1, form an occurrence of $\sigma^{\ulcorner a,b \urcorner}$. Then condition (2) of the proposition implies an occurrence of π. Therefore there can be no points in this region, which implies the occurrence of m_1 is an occurrence of m_2. Hence every occurrence of m_1 is in fact an occurrence of m_2, and we have that $\mathrm{Av}(\{\pi, m_2\}) \subseteq \mathrm{Av}(\{\pi, m_1\})$.

Taking both directions of the containment we can therefore draw the conclusion that $m_1 \cong_\pi m_2$. $\square$

All coincidence classes of $\mathrm{Av}(\{321, (21, R)\})$ can be explained by application of Proposition 3.4, thus explaining the 29 experimental coincidence classes observed.

3.2. **Coincidence classes of Av({231, (21, *R*)}).** By application of Proposition 3.4 we obtain 43 $\cong^{(1)}_{231}$-coincidence classes between mesh patterns with 21 as an underlying classical pattern. Experimentation shows that there are 39 $\cong^{\mathrm{comp}}_{231}$-coincidence classes between mesh patterns with 21 as an underlying classical pattern, for example the following two patterns seem to be coincident in $\mathrm{Av}(231)$ but this is not explained by Proposition 3.4.

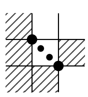
$m_1 =$  and $m_2 =$ 

Consider an occurrence of m_1 in a permutation in $\mathrm{Av}(231)$, consisting of elements x and y. If the region corresponding to the box $(1,1)$ is empty we have an occurrence of m_2. Otherwise, if there is any ascent in this box then we would have an occurrence of 231, however, since we are in $\mathrm{Av}(231)$ this is not possible. This box must therefore contain a (non-empty) decreasing subsequence. This gives rise to the following lemma:

Lemma 3.6. Let $m = (\sigma, R)$ be a mesh pattern, where the box (a,b) is not in R, and $\pi = (\pi, \varnothing)$ be a dominating classical pattern. If $\pi \preceq m^{\ulcorner a,b \urcorner_a}$ ($\pi \preceq m^{\ulcorner a,b \urcorner_d}$), then in any occurrence of m in a permutation ϱ, the region corresponding to the box (a,b) can only contain an decreasing (increasing) subsequence of ϱ.

The proof is analogous to the proof of Proposition 3.4.

Going back to our example mesh patterns



we know that the region corresponding to the box $(1,1)$ contains a decreasing subsequence. If we let z be the topmost point in this decreasing subsequence, then xz is an occurrence of m_2. This shows that our two example patterns are coincident.

This result generalises into the following rule for categorising coincidences of mesh patterns in cases where there is a dominating classical pattern.

Proposition 3.7 (Second Dominating Pattern Rule)**.** Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \varnothing)$ such that $|\pi| \leqslant |\sigma| + 2$, the sets $\mathrm{Av}(\{\pi, m_1\})$ and $\mathrm{Av}(\{\pi, m_2\})$ are coincident as a consequence of the Second Dominating Rule if

1. The mesh set $R_2 = R_1 \cup \{(a,b)\}$
2. Any one of the following four conditions hold
   - (a) $\pi \preceq \sigma^{\ulcorner a,b \urcorner_a}$ and
     - (i) $(a+1, b) \in \sigma$ and $(a+1, b-1) \notin R_1$ and
       $(x, b-1) \in R_1 \implies (x, b) \in R_1$ (where $x \neq a, a+1$) and
       $(a+1, y) \in R_1 \implies (a, y) \in R_1$ (where $y \neq b-1, b$).
     - (ii) $(a, b+1) \in \sigma$ and $(a-1, b+1) \notin R_1$ and
       $(x, b+1) \in R_1 \implies (x, b) \in R_1$ (where $x \neq a-1, a$) and
       $(a-1, y) \in R_1 \implies (a, y) \in R_1$ (where $y \neq b, b+1$).
   - (b) $\pi \preceq \sigma^{\ulcorner a,b \urcorner_d}$ and
     - (i) $(a+1, b+1) \in \sigma$ and $(a+1, b+1) \notin R_1$ and
       $(x, b+1) \in R_1 \implies (x, b) \in R_1$ (where $x \neq a, a+1$) and
       $(a+1, y) \in R_1 \implies (a, y) \in R_1$ (where $y \neq b, b+1$).
     - (ii) $(a, b) \in \sigma$ and $(a-1, b-1) \notin R_1$ and
       $(x, b-1) \in R_1 \implies (x, b) \in R_1$ (where $x \neq a-1, a$) and
       $(a-1, y) \in R_1 \implies (a, y) \in R_1$ (where $y \neq b-1, b$).

Proof. By Note 3.5 we only need to show that an occurrence of m_1 implies an occurrence of m_2. We consider taking the first branch of every choice in condition (2). Now consider a permutation $\omega \in \mathrm{Av}(\pi)$. Suppose ω contains m_1 and consider the region corresponding to (a,b) in R_1. If the region is empty, the occurrence of m_1 is trivially an occurrence of m_2.

If the region is non-empty, then by Lemma 3.6 and condition (2a) of the proposition it must contain a decreasing subsequence. We can choose the topmost point in the region to replace the corresponding point in the mesh pattern and the points from the subsequence are now in the box southeast of the point.

The conditions on the mesh ensure that no elements of the permutation that were inside a region corresponding to an unshaded box in the occurrence of m_1 would be in a region corresponding to a shaded box in an occurrence of m_2.

Hence there are no points in the region corresponding to the box (a,b) in the mesh pattern, and therefore we can shade this region. This implies that every occurrence of m_1 in $\mathrm{Av}(\pi)$ is in fact an occurrence of m_2 so $\mathrm{Av}(\{\pi, m_2\}) \subseteq \mathrm{Av}(\{\pi, m_1\})$.

Similar arguments cover the remainder of the branches. $\square$

This proposition essentially states that we slide all of the points in the box we desire to shade diagonally, and chose the topmost/bottommost point

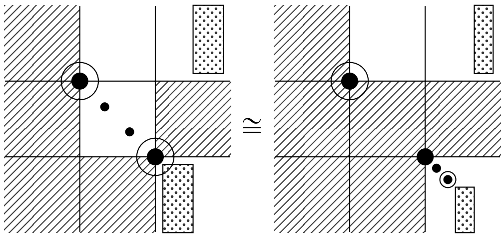


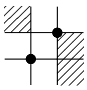
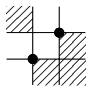
FIGURE 6. A depiction of the second dominating pattern rule using our example patterns.

to replace the original point in the mesh pattern. Figure 6 show how the cases apply to a general container of m_1 to transform it into a container of m_2 , the circled points are the same points in the permutation.

Together the First Dominating Pattern Rule and the Second Dominating Pattern Rule fully explain coincidences of classes of the form $\mathrm{Av}(\{231, (21, R)\})$, obtaining 39 coincidence classes of mesh patterns, confirming experimental observations.

3.3. **Coincidence classes of Av({231, (12, *R*)}).** When considering the coincidence classes of $\mathrm{Av}(\{231, (12, R)\})$ we first apply the two Dominating Pattern rules previously established. Starting from 220 classes, we result in 85 $\cong^{(1)}_{231}$-coincidence classes reducing to 59 $\cong^{(2)}_{231}$-coincidence classes, considering mesh patterns with 12 as the underlying classical patterns. However we know that there are patterns where the coincidences are not explained by the rules given above.

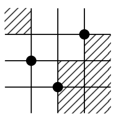
For example the patterns

$m_1 =$  and $m_2 =$ 

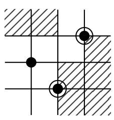
are experimentally coincident.

Consider an occurrence of m_1 in a permutation, if the region corresponding to the box $(1, 0)$ is empty then we have an occurrence of m_2. Now look at the case when this region is not empty, and consider choosing the rightmost point in the region.

This gives us an occurrence of the following mesh pattern.



By application of Proposition 3.4 we then achieve the following mesh pattern



If we look at the highlighted points we see that the subpattern is an occurrence of the mesh pattern that we originally desired. This gives rise to the following rule:

Proposition 3.8 (Third Dominating Pattern Rule)**.** Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \varnothing)$, the sets $\mathrm{Av}(\{\pi, m_1\})$ and $\mathrm{Av}(\{\pi, m_2\})$ are coincident if

(1) The mesh set $R_2 = R_1 \cup \{(a, b)\}$
(2) One of the patterns in $(\sigma, R_1)^{\ulcorner a,b \urcorner \star}$ is coincident with a mesh pattern containing an occurrence of (σ, R_2) as a subpattern.

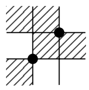
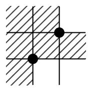
Proof. Assume without meaningful loss of generality that $R_2 = R_1 \cup \{(a, b)\}$. Note 3.5, implies $\mathrm{Av}(\{\pi, m_1\}) \subseteq \mathrm{Av}(\{\pi, m_2\})$ as before. Now consider a permutation σ in $\mathrm{Av}(\pi)$ that contains an occurrence of m_1. If the region corresponding to the box (a, b) is empty then we have an occurrence of m_2. If the region is non-empty then by condition 2 of the proposition there exists a pattern in $(\sigma, R_1)^{\ulcorner a,b \urcorner \star}$ such that there exists an occurrence of a mesh pattern of length one longer than m_1 in this position. This mesh pattern is coincident with another mesh pattern that contains an occurrence of m_2. Hence, every occurrence of m_1 leads to an occurrence of m_2. Thus $\mathrm{Av}(\{\pi, m_2\}) \subseteq \mathrm{Av}(\{\pi, m_1\})$ and the two patterns are coincident. □

This rule reduces the number of classes in $\mathrm{Av}(\{231, (12, R)\})$ to 56, which coincides with the 56 classes observed through experimentation.

3.4. **Coincidence classes of Av({321, (12, *R*)}).** When considering coincidences of mesh patterns with underlying classical pattern 12 in $\mathrm{Av}(321)$ application of the previously established rules give no coincidences. Through experimentation we discover that there are 7 non-trivial coincidence classes (all others are singletons) which can be explained through the use of two different lines of reasoning. Since the number of coincidences is so small we will reason for these coincidences without attempting to generalise into concrete rules.

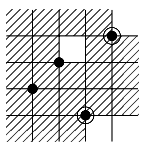
Intuitively it is easy to see why our previous rules have no power here. It is impossible to add a single point to a mesh pattern $(12, R)$ and create an occurrence of $\pi = 321$. It is also impossible to have a position where addition of an ascent, or descent, provides extra shading power.

The patterns

$m_1 =$  and m_2 

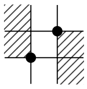
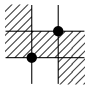
are coincident in $\mathrm{Av}(321)$. (There are 3 symmetries of these patterns that are also coincident to each other by the same reasoning.)

Consider the region corresponding to the box $(0, 1)$ in any occurrence of m_1, in a permutation. By Lemma 3.6 it must contain an increasing subsequence. If the region is empty then we have an occurrence of m_2. If there is only one point in the region we can choose this to replace the 1 in the mesh pattern to get the required shading. If there is more than one point then choosing the two leftmost points gives us the following mesh pattern.



Where the two highlighted points are the original two points. The other two points are an occurrence of the pattern we originally desired, and hence the two patterns are coincident. It is also possible to calculate this coincidence by an extension of the Third Dominating rule, where we allow a sequence of point addition operations, this is discussed further in the future work section.

The other reasoning applies to the patterns

$m_1 =$  and $m_2 =$ 

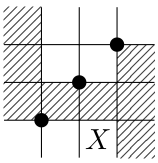
which are coincident by experimentation.

In order to prove this coincidence we will proceed by mathematical induction on the number of points in region corresponding to the middle box. We call this number n.

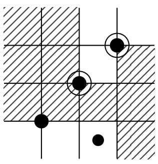
Base Case ($n = 0$)**:** The base case holds since we can freely shade the box if it contains no points.

Inductive Hypothesis ($n = k$)**:** Suppose that we can find an occurrence of the second pattern if we have an occurrence of the first with k points in the middle box.

Inductive Step ($n = k + 1$)**:** Suppose that we have $(k + 1)$ points in the middle box. Choose the bottom most point in the middle box, giving the mesh pattern



Now we need to consider the box labelled X. If this box is empty then we have an occurrence of m_2 and are done. If this box contains any points then we gain some extra shading on the mesh pattern as any points in these boxes would create an occurrence of the dominating pattern 321



The two highlighted points form an occurrence of m_1 with k points in the middle box, and thus by the Inductive Hypothesis we are done.

By induction we have that every occurrence of m_1 leads to an occurrence of m_2 and by Note 3.5 every occurrence of m_2 is an occurrence of m_1 so the two patterns are coincident. This argument applies to another two pairs of classes. Therefore in total in $\mathrm{Av}(\{321, (12, R)\})$ there are 213 coincidence classes.

4. Wilf Equivalences between equivalence classes

Wilf-equivalence is an important aspect to study in the field of permutation patterns.

Definition 4.1. (Wilf Equivalence.) Two patterns π and σ are said to be *Wilf-equivalent* if for all $k \geqslant 0, |\mathrm{Av}_k(\pi)| = |\mathrm{Av}_k(\sigma)|$. Two sets of permutation patterns R and S are *Wilf-equivalent* if for all $k \geqslant 0, |\mathrm{Av}_k(R)| = |\mathrm{Av}_k(S)|$.

Coincident patterns are trivially Wilf-equivalent: if $\mathrm{Av}_k(R) = \mathrm{Av}_k(S)$ then trivially $|\mathrm{Av}_k(R)| = |\mathrm{Av}_k(S)|$. Coincidence is therefore a stronger equivalence condition than Wilf-equivalence.

When examining Wilf-equivalences we can use a number of symmetries to reduce the amount of work required. It can be seen that the reverse, complement and inverse operations (see Figure 7) preserve enumeration, and therefore classes related by these symmetries are trivially Wilf-equivalent.

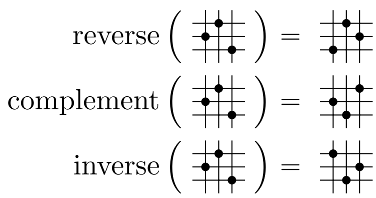


Figure 7. The operations reverse, complement and inverse for the pattern 231

Since we are always considering Wilf-equivalences in a set $\mathrm{Av}(S)$ we must only use symmetries that preserve the dominating pattern(s), if we were to allow other symmetries, then the equivalences calculated in the previous section do not necessarily hold.

Throughout this section we will consider Wilf-equivalences of patterns whilst avoiding the *dominating pattern* 231. We will use $\mathcal{C}$ to denote $\mathrm{Av}(231)$ and $C(x)$ will be the usual Catalan generating function satisfying $C(x) = 1 + xC(x)^2$. The fact that $C(x)$ is the generating function for $\mathcal{C}$ can be seen by structural decomposition around the maximum, as shown in Figure 8.

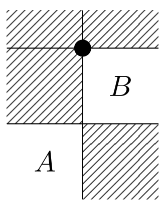


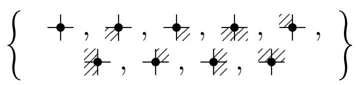
Figure 8. Structural decomposition of a non-empty avoider of 231

The elements to the left of the maximum, A, have the structure of a 231 avoiding permutation, and the elements to the right of the maximum, B, have the structure of a 231 avoiding permutation. Furthermore, all the elements in A lie below all of the elements in B. We call A the *lower-left section* and B the *upper-right section.*

We can also decompose a permutation avoiding 231 around the leftmost-point, giving a similar figure.

4.1. **Wilf-classes with mesh patterns of length 1.** When considering the mesh patterns of length 2 it will be useful to know the Wilf-equivalence classes of the mesh patterns of length 1 inside $\mathrm{Av}(231)$, this means that we are considering the set $\mathrm{Av}(\{231, p\})$ where p is a mesh-pattern of length 1.

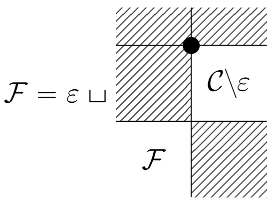
The patterns in the following set are coincident,



due to the fact that every permutation, except the empty permutation, must contain an occurrence of all of these patterns.

The pattern is in its own Wilf-class since the only permutation containing this pattern is the permutation 1. The avoiders of this pattern therefore have generating function $E(x) = C(x) - x$.

The pattern $p =$ is one of the quadrant marked mesh patterns studied by Kitaev, Remmel, and Tiefenbruck [8]. Alternatively we can enumerate avoiders of p by decomposing a non-empty avoider of p around the maximum element in order to give the following structural decomposition.



If the upper-right section was empty the maximum would create an occurrence of the pattern, however no points in this section can create an occurrence since the maximum lies in a region corresponding to the shading in p, so we can use any avoider of 231. The lower-left section however can create occurrences of p and therefore must also avoid p, as well as 231. This gives the generating function of avoiders to be the function $F(x)$ satisfying

$$F(x) = 1 + xF(x)(C(x) - 1)$$

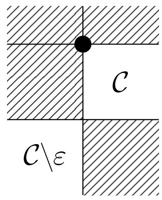
Solving for F gives

$$F(x) = \frac{1}{1 - x(C(x) - 1)}$$

Calculating coefficients given by this generating function gives the Fine numbers.

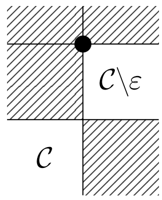
(OEIS: A000957) $\qquad 1, 0, 1, 2, 6, 18, 57, 186, 622, 2120, 7338, \ldots$

It can be shown by use of Proposition 3.7 that the patterns and $q_1 =$ are coincident inside $\mathrm{Av}(231)$. Consider the decomposition of a non-empty avoider of q_1 in Av(231) around the maximum:

$\mathcal{G}_1 = \varepsilon \sqcup$ 

This can be explained succinctly by the fact that a permutation containing q_1 starts with its maximum, and by not allowing the lower-left section of the 231 avoider to be empty we prevent an occurrence from ever happening.

Consider $q_2 =$ . Avoiding this pattern means that a permutation does not end with its maximum. We can perform a similar decomposition as before to get

$\mathcal{G}_2 = \varepsilon \sqcup$ 

The pattern $q_3 =$  is the reverse-complement-inverse of q_2 and hence the avoiders of q_2 and q_3 ($\mathcal{G}_2$ and $\mathcal{G}_3$) are equinumerous. All of these classes have the same generating function, namely

$$G(x) = 1 + xC(x)(C(x) - 1). \tag{4.1}$$

The coefficients of this generating function are

(OEIS: A000245 with offset 1) $1, 0, 1, 3, 9, 28, 90, 297, 1001, 3432, 11934, \ldots$

There is one pattern of length 1 still to consider. The pattern $r =$  is avoided by all permutations that do not end in their minimum. Any avoider of 231 that ends in its minimum must be a decreasing sequence. Therefore this particular class has equation

$$H(0) = 1, H(x) = C(x) - 1$$

Computing these values gives

(OEIS: A141364) $1, 0, 1, 4, 13, 41, 131, 428, 1429, 4861, 16795, \ldots$

5. Wilf-classes with patterns of length 2

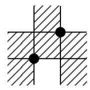
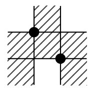
By the use of coincidence classes established in Section 3 we know that there are at most 95 Wilf-equivalence classes.

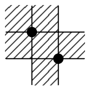
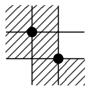
The only symmetry that we are able to consider is *reverse-complement-inverse* as this is the only symmetry that preserves the 231 pattern. Using this symmetry we can find 61 classes of trivial Wilf-equivalence, these equivalences being explained by either the patterns being coincident in Av(231), or by one pattern being the reverse-complement-inverse of some other pattern.

Computing avoiders up to length 10 suggests that there are at least 23 Wilf-classes, of which 13 are non-trivial. Therefore there are Wilf-equivalences that are not explained by coincidences or symmetry.

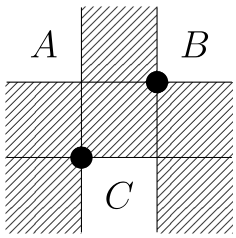
We will use two main methods of establishing Wilf-equivalence between mesh patterns of length 2 in Av(231): the structural decomposition of avoiders, via generating functions; or the structure of a general permutation containing the pattern, looking at a particular occurrence of the pattern in a permutation avoiding 231. Sometimes it will be necessary to use both of these methods to consolidate a single Wilf-class.

5.1. The following patterns are experimentally Wilf-equivalent up to length 10 in Av(231)

(5.1) $m_1 =$  , $m_2 =$  ,

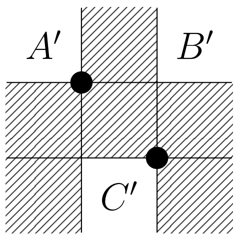
(5.2) $m_3 =$  , and $m_4 =$ 

First we prove the Wilf-equivalence between m_1 and m_2 shown in (5.1), by considering the form of a general permutation containing either of the two patterns. First looking at a general occurrence of m_1 in a permutation in Av(231)



If there is an occurrence of m_1 there must be a lowest occurrence of m_1, *i.e.* the occurrence has the lowest possible values for any occurrence. Now consider the top regions, labelled A and B, the subpermutation contained in these regions must avoid the permutation 231. Also, the subpermutation in the region A must be a decreasing subsequence, otherwise an occurrence of 231 will be created with either of the points in the occurrence. Now, consider the region labelled C, since we specified that we were focused on the lowest possible occurrence of m_1 this region cannot contain an occurrence of either m_1 or 231. We now have a structural decomposition of a container of m_1 inside Av(231).

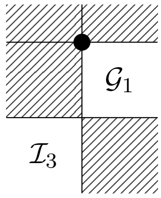
Now consider a general occurrence of m_2 in a permutation in Av(231)



Similarly to a container of m_1 we will look at the lowest possible occurrence of m_2, similar to before the regions A' and B' together contain an avoider of 231 and A contains a decreasing subsequence. Since we are considering the lowest occurrence of m_2 so the region C' does not contain an occurrence of either m_2 or 231.

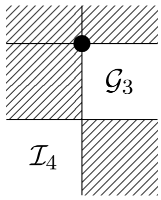
Since all the regions in both of these cases contain the same parts, the classes defined by containment of m_1 and m_2 inside $\mathrm{Av}(231)$ are equinumerous and therefore so are their avoiders.

Consider the class, $\mathcal{I}_3$, of permutations defined by avoiding 231 and m_3. We can decompose a member of this class around the maximum

$\mathcal{I}_3 = \varepsilon \sqcup$ 

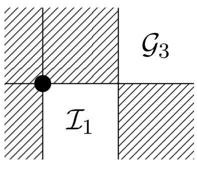
Only the first point in the top right region can create an occurrence of m_3 if and only if it is the element with largest value in this region, therefore the partial permutation in this region must avoid starting with the maximum.

Looking at avoiders of 231 and m_4 we can perform a similar decomposition around the maximum to get

$\mathcal{I}_4 = \varepsilon \sqcup$ 

An occurrence of m_4 can never occur in the top right region. It could only occur between the maximum and the first point in the region, if and only if this first point is the lowest valued element in this region, so this top right region must contain a sub-permutation that does not start with it's minimum, i.e., it is a member of $\mathcal{G}_3$ described in Section 4.1. Since both $\mathcal{G}_1$ and $\mathcal{G}_3$ have the same enumeration, $\mathcal{I}_3$ and $\mathcal{I}_4$ must also have the same enumeration and are therefore Wilf-equivalent.

Now we must consolidate these two subclasses. In order to do this we must consider the decomposition around the leftmost point of a permutation in $\mathrm{Av}(\{231, m_1\})$. We have the following

$\mathcal{I}_1 = \varepsilon \sqcup$ 

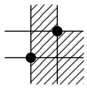
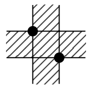
It is therefore obvious that avoiders of m_1 and avoiders of m_4 have the same enumeration, and therefore all four patterns are Wilf-equivalent in $\mathrm{Av}(231)$ with generating function satisfying

$$I(x) = 1 + xI(x)G(x)$$

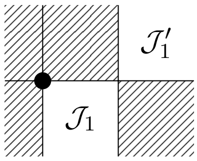
where $G(x)$ is the generating function given in equation (4.1). This can be enumerated to give the sequence

(OEIS: A035929 offset 1) $\quad 1, 1, 1, 2, 6, 19, 61, 200, 670, 2286, 7918, \ldots$

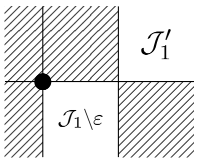
5.2. The following patterns are experimentally Wilf-equivalent up to length 10 in Av(231)

$m_1 =$  and $m_2 =$ 

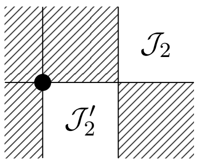
Let $\mathcal{J}_1$ be the set of avoiders of m_1 in Av(231). By structural decomposition around the leftmost point we have

$\mathcal{J}_1 = \varepsilon \sqcup$ 

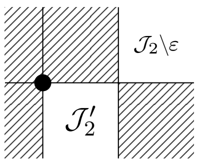
Here $\mathcal{J}_1'$ is a permutation avoiding $231, m_1$ and  . Now consider the decomposition of a permutation in $\mathcal{J}_1'$. It can once again be decomposed around the leftmost point

$\mathcal{J}_1' = \varepsilon \sqcup$ 

This is a complete decomposition of avoiders of m_1. Now we look at an avoider of m_2, decomposed around the leftmost point

$\mathcal{J}_2 = \varepsilon \sqcup$ 

Where $\mathcal{J}_2'$ is a permutation avoiding $231, m_2$ and  . Again we use the same method of decomposition of a permutation in $\mathcal{J}_2'$

$\mathcal{J}_2' = \varepsilon \sqcup$ 

This gives us a generating function $J(x)$ satisfying

$$J(x) = 1 + xJ(x)J'(x) \tag{5.3}$$

$$J'(x) = 1 + x(J(x) - 1)J'(x) \tag{5.4}$$

Solving equation (5.4) for $J'(x)$ and substituting into equation (5.3) gives us that the generating function for $J(x)$ satisfies

$$J(x) = xJ^2(x) - x(J(x) - 1) + 1 \tag{5.5}$$

Evaluating $J(x)$ gives us the sequence

(OEIS: A001006 with offset 1) $\quad 1, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, \ldots$

Which is an offset of the Motzkin numbers.

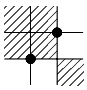
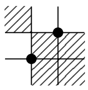
In order to establish the remainder of the Wilf-equivalences of the form $\mathrm{Av}(\{231, p\})$ where p is a mesh pattern we can use similar methods to allow us to consolidate experimental classes into actual classes, these methods allow us to explain all 23 of the observed Wilf-classes seen in experimentation.

6. Conclusions and Future Work

If we consider a similar approach to dominating patterns of length 4 and mesh patterns of length 2, it can be seen that the number of cases required to establish rules increases to a number that is infeasible to compute manually. For an extension of the First Dominating rule alone, we would have to consider placement of points in any pair of unshaded regions. The fact that the rules established do not completely cover the coincidences with a dominating pattern of length 3 shows that this is a difficult task.

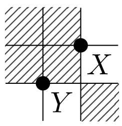
It is interesting to consider the application of the Third Dominating rule, as well as the simple extension of allowing a sequence of point insertions, to mesh patterns without any dominating pattern in order to try to capture some of the coincidences described in Hilmarsson, Jónsdóttir, Sigurðardóttir, *et al.* [7] and Claesson, Tenner, and Ulfarsson [6].

Example 6.1. We can establish the coincidence between the patterns

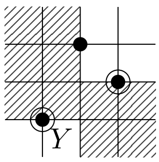
$m_1 =$  , and $m_2 =$ 

That is not explained by the methods presented by Claesson, Tenner, and Ulfarsson [6].

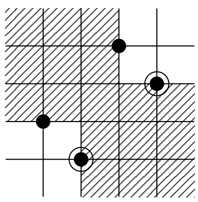
Consider a permutation containing m_1,



If the regions corresponding to both X and Y are empty then we have an occurrence of m_2. If the region corresponding to X is non-empty, we can then choose the lowest valued point in this region



If the region corresponding to Y is empty then we have an occurrence of m_2 with the indicated points. Now if the region corresponding to Y is non-empty, we can choose the rightmost point in this region.



And now the two indicated points form an occurrence of m_2. We have therefore shown that any occurrence of m_1 leads to an occurrence of m_2 and we can easily show the converse by the same reasoning, so m_1 and m_2

are coincident. This is captured by an extension of the Third Dominating rule where we allow multiple steps of adding points before we check for subpattern containment.

It would be interesting to consider a systematic explanation of Wilf-equivalences among classes where 321 is the dominating pattern, possibly using the construction presented in Bean, Claesson, and Ulfarsson [2, Sec. 11], in order to directly reach enumeration and hopefully establish some of the non-trivial Wilf-equivalences between classes with different dominating patterns.

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