Equivalence classes of mesh patterns with a Dominating Pattern

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Outline

1. Introduction
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   - Classical Permutation Patterns
   - Mesh Patterns

Permutations

A *permutation* is a *bijection*, π, from the set $[\![n]\!] = \{1, \ldots, n\}$ to itself. More intuitively "A *permutation of n objects* is an arrangement of n distinct objects in a row" (Knuth).
We write permutations in *one-line notation*, writing the entries of the entries of the permutation in order π

$$\pi = \pi(1)\pi(2)\ldots\pi(n)$$

Example

The 6 permutations on $[\![3]\!]$ are

$$123, 132, 213, 231, 312, 321$$

We can display a permutation in a *plot* to give a graphical represention. We plot the points $(i, \pi(i))$ in a Cartesian coordinate system.

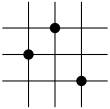


Figure: Plot of the permutation 231

In this setting we call the elements of the permutations *points*.
The set of all permutations of length n is $\mathfrak{S}_n$ and has size $n!$. The set of all permutations is $\mathfrak{S} = \bigcup_{i=0}^{\infty} \mathfrak{S}_i$.

Classical Permutation Patterns

Classical permutation patterns capture many interesting combinatorial objects and properties.

Definition (Order Isomorphism)

Two substrings $\alpha_1\alpha_2\cdots\alpha_n$ and $\beta_1\beta_2\cdots\beta_n$ are said to be *order isomorphic* if $\alpha_r < \alpha_s$ if and only if $\beta_r < \beta_s$.

Definition

A permutation π is said to *contain* the *classical permutation pattern* σ (denoted $\sigma \preceq \pi$) if there is some subsequence $i_1 i_2 \cdots i_k$ such that the sequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is order isomorphic to $\sigma(1)\sigma(2)\cdots\sigma(k)$.

If π does not contain σ we say that π *avoids* σ.
We the set of permutations of length n avoiding a pattern σ is denoted as $\mathrm{Av}_n(\sigma)$ and $\mathrm{Av}(\sigma) = \bigcup_{i=0}^{\infty} \mathrm{Av}_i(\sigma)$.

Example

The permutation $\pi = 24153$ contains the pattern $\sigma = 231$

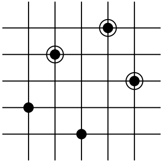


Figure: Plot of the permutation 24153 with an occurrence of 231 indicated

Mesh Patterns

Mesh patterns are a natural extension of classical permutation patterns.

Definition

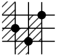
A *mesh pattern* is a pair

$$p = (\tau, R) \text{ with } \tau \in \mathfrak{S}_k \text{ and } R \subseteq [0, k] \times [0, k].$$

We say that τ is the *underlying classical pattern* of p.

Example

The pattern

$p = (213, \{(0,1), (0,2), (0,3), (1,0), (1,1), (2,1), (2,2)\}) =$  is contained in $\pi = 34215$.

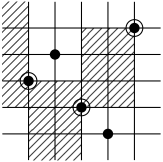
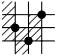


Figure: An occurrence of p in π

Example

The pattern $q = (21, \{(0,1),(0,2),(1,0),(1,1)\}) =$  is contained in $p = (213, \{(0,1),(0,2),(0,3),(1,0),(1,1),(2,1),(2,2)\}) =$  as a subpattern.

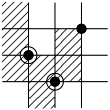


Figure: An occurrence of q in p