Equivalence classes of mesh patterns with a Dominating Pattern

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Outline

Permutations

A *permutation* is a *bijection*, π, from the set $[\![n]\!] = \{1, \ldots, n\}$ to itself.
More intuitively "A *permutation of n objects* is an arrangement of n distinct objects in a row" (Knuth [1]).
We write permutations in *one-line notation*, writing the entries of the permutation in order

$$\pi = \pi(1)\pi(2)\ldots\pi(n)$$

Example

The 6 permutations on $[\![3]\!]$ are

$$123, 132, 213, 231, 312, 321$$

We can display a permutation in a *plot* to give a graphical represention. We plot the points $(i, \pi(i))$ in a Cartesian coordinate system.

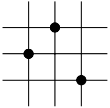


Figure: Plot of the permutation 231

In this setting we call the elements of the permutations *points*.
The set of all permutations of length n is $\mathfrak{S}_n$ and has size $n!$. The set of all permutations is $\mathfrak{S} = \bigcup_{i=0}^{\infty} \mathfrak{S}_i$.

Classical Permutation Patterns

Classical permutation patterns capture many interesting combinatorial objects and properties.

Definition (Order Isomorphism)

Two substrings $\alpha_1\alpha_2\cdots\alpha_n$ and $\beta_1\beta_2\cdots\beta_n$ are said to be *order isomorphic* if $\alpha_r < \alpha_s$ if and only if $\beta_r < \beta_s$.

Definition

A permutation π is said to *contain* the *classical permutation pattern* σ (denoted $\sigma \preceq \pi$) if there is some subsequence $i_1 i_2 \cdots i_k$ such that the sequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is order isomorphic to $\sigma(1)\sigma(2)\cdots\sigma(k)$.

If π does not contain σ we say that π *avoids* σ.
The set of permutations of length n avoiding a pattern σ is denoted as $\mathrm{Av}_n(\sigma)$ and

$$\mathrm{Av}(\sigma) = \bigcup_{i=0}^{\infty} \mathrm{Av}_i(\sigma)$$

Example

The permutation $\pi = 24153$ contains the pattern $\sigma = 231$

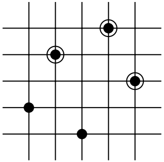


Figure: Plot of the permutation 24153 with an occurrence of 231 indicated

Mesh Patterns

Mesh patterns are a natural extension of classical permutation patterns.

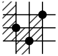
Definition

A *mesh pattern* is a pair

$$p = (\tau, R) \text{ with } \tau \in \mathfrak{S}_k \text{ and } R \subseteq [0, k] \times [0, k].$$

We say that τ is the *underlying classical pattern* of p.

Example

The pattern $p = (213, \{(0,1), (0,2), (0,3), (1,0), (1,1), (2,1), (2,2)\}) =$  is contained in $\pi = 34215$.

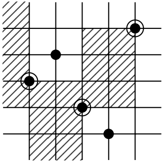
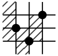


Figure: An occurrence of p in π

Example

The pattern $q = (21, \{(0,1), (0,2), (1,0), (1,1)\}) =$  is contained in $p = (213, \{(0,1), (0,2), (0,3), (1,0), (1,1), (2,1), (2,2)\}) =$  as a subpattern.

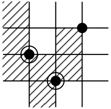


Figure: An occurrence of q in p

Outline

Coincidence

Definition

Two mesh patterns are said to be *coincident* if they are avoided by the same set of permutations at every length.

Distinct classical patterns can never be coincident.
Coincidences of mesh patterns of length 2 are classified.
Aim to establish rules that classify coincidences when we have one mesh pattern and one classical pattern.
We call the classical pattern a *dominating pattern*.

Experimental Results

	Dominating Pattern			
	231		321	
	12	21	12	21
No Dominating rule	220	220	220	220
First Dominating rule	85	43	220	29
Second Dominating rule	59	39	220	29
Third Dominating rule	56	39	220	29
Experimental class size	56	39	213	29

Table: Coincidence class number reduction by application of Dominating rules

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First Dominating rule

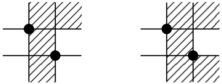
Proposition: First Dominating rule

Given two mesh patterns $m_1 = (\sigma, R_1)$ *and* $m_2 = (\sigma, R_2)$*, and a dominating classical pattern* $\pi = (\pi, \varnothing)$ *such that* $|\pi| \leqslant |\sigma| + 1$*, the sets* $\mathrm{Av}(\{\pi, m_1\})$ *and* $\mathrm{Av}(\{\pi, m_2\})$ *are coincident if*

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. $\pi \preceq \mathtt{add\_point}\,(\sigma, (a, b), \varnothing)$

Example

The following two patterns are coincident in $\mathrm{Av}(321)$



Corollary

All coincidences of classes the form $\mathrm{Av}(\{321, (21, R)\})$ *are fully explained by the First Dominating rule.*

There are 29 coincidences of mesh patterns of the form $\mathrm{Av}(\{321, (21, R)\})$

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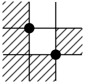
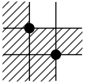
Second Dominating rule

The patterns

$m_1 =$  and $m_2 =$ 

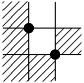
are coincident in $\mathrm{Av}(231)$.

Lemma

Given a mesh pattern $m = (\sigma, R)$, *where the box* (a, b) *is not in* R, *and a dominating classical pattern* $\pi = (\pi, \emptyset)$ *if* $\pi \leq \texttt{add\_ascent}(\sigma, (a, b))$
,then in any occurrence of m *in a permutation* ϱ, *the region corresponding to the box* (a, b) *can only contain an decreasing subsequence of* ϱ.

Example

Considering m_1 again



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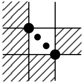

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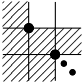
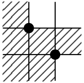


Example

Considering m_1 again



This is m_2.

Proposition: Second Dominating rule

Given two mesh patterns $m_1 = (\sigma, R_1)$ *and* $m_2 = (\sigma, R_2)$*, and a dominating classical pattern* $\pi = (\pi, \varnothing)$ *such that* $|\pi| \leqslant |\sigma| + 2$*, the sets* $\mathrm{Av}(\{\pi, m_1\})$ *and* $\mathrm{Av}(\{\pi, m_2\})$ *are coincident if*

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. 2.1 $\pi \preceq \texttt{add\_ascent}(\sigma, (a, b))$ *and*
      - 2.1.1 $(a+1, b) \in \sigma$ *and* $(a+1, b-1) \notin R$ *and*
        $(x, b-1) \in R \implies (x, b) \in R$ *(where* $x \neq a, a+1$*) and*
        $(a+1, y) \in R \implies (a, y) \in R$ *(where* $y \neq b-1, b$*).*
      - 2.1.2 ...
   - 2.2 ...
      - 2.2.1 ...
      - 2.2.2 ...

Example

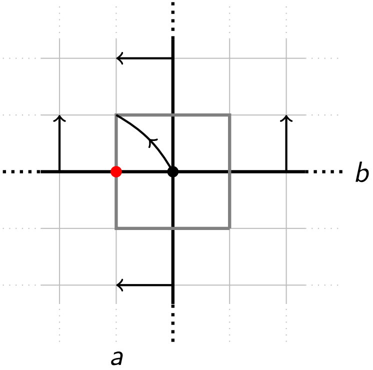


Figure: If the conditions of The Second Dominating rule are satisfied the box (a, b) can be shaded.

Corollary

All coincidences of classes the form $\mathsf{Av}(\{231, (21, R))\}$ *are fully explained by applying the First Dominating rule, then applying the Second Dominating rule.*

There are 39 coincidences of mesh patterns of the form $\mathsf{Av}(\{231, (21, R))\}$

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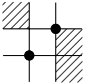
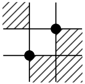
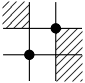
Third Dominating rule

The patterns

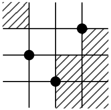
$m_1 =$  and $m_2 =$ 

are coincident in $\mathrm{Av}(231)$. Neither of the previous two rules explain this.

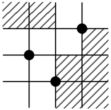
Example



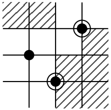
Example



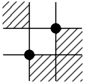
Example



Example



Example



Proposition: Third Dominating rule

Given two mesh patterns $m_1 = (\sigma, R_1)$ *and* $m_2 = (\sigma, R_2)$, *and a dominating classical pattern* $\pi = (\pi, \emptyset)$, *the sets* $\mathrm{Av}(\{\pi, m_1\})$ *and* $\mathrm{Av}(\{\pi, m_2\})$ *are coincident if*

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. $\texttt{add\_point}\left((\sigma, R_1), (a, b), D\right)$ *where* $D \in \{N, E, S, W\}$ *is coincident with a mesh pattern containing an occurrence of* (σ, R_2) *as a subpattern.*

Corollary

All coincidences of classes the form $\mathrm{Av}(\{231, (12, R)\})$ *are fully explained by applying the First Dominating rule, the Second Dominating rule, and then the Third Dominating rule.*

There are 56 coincidences of mesh patterns of the form $\mathrm{Av}(\{231, (21, R)\})$

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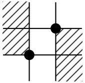
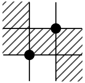
Special Cases

There are 7 coincidences of the form $\mathrm{Av}(321, m)$ that are not explained by the rules.

Example

$m_1 =$  and $m_2 =$ 

This coincidence is explained by mathematical induction on the number of points in the region corresponding to the middle box. We call this number n.

Example

Base Case ($n = 0$): The base case holds since we can freely shade the box if it contains no points.

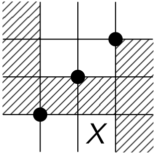
Example

Base Case $(n = 0)$: The base case holds since we can freely shade the box if it contains no points.

Inductive Hypothesis $(n = k)$: Suppose that we can find an occurrence of the second pattern if we have an occurrence of the first with k points in the middle box.

Example

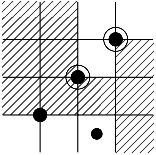
Inductive Step $(n = k + 1)$ Suppose that we have $(k + 1)$ points in the middle box. Choose the bottom most point in the middle box, this gives the mesh pattern



Now we need to consider the box labelled X. If this box is empty then we have an occurrence of m_2 and are done.

Example

Inductive Step ($n = k + 1$) (cont.) If this box contains any points then we gain some extra shading on the mesh pattern due to the dominating pattern



The two highlighted points form an occurrence of m_1 with k points in the middle box, and thus by the Inductive Hypothesis we are done.

[1] D. Knuth,
The Art of Computer Programming: Volume 1, 1997.