Equivalence classes of mesh patterns with a Dominating Pattern

Murray Tannock (murray14@ru.is)

HÁSKÓLINN Í REYKJAVÍK
REYKJAVIK UNIVERSITY

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Outline

Permutations

A *permutation* is a *bijection*, π, from the set $[\![n]\!] = \{1, \ldots, n\}$ to itself.
More intuitively "A *permutation of n objects* is an arrangement of *n* distinct objects in a row" (Knuth [1]).
We write permutations in *one-line notation*, writing the entries of the entries of the permutation in order

$$\pi = \pi(1)\pi(2)\ldots\pi(n)$$

Example

The 6 permutations on $[\![3]\!]$ are

$$123, 132, 213, 231, 312, 321$$

We can display a permutation in a *plot* to give a graphical represention. We plot the points $(i, \pi(i))$ in a Cartesian coordinate system.

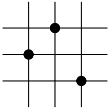


Figure: Plot of the permutation 231

In this setting we call the elements of the permutations *points*.
The set of all permutations of length n is $\mathfrak{S}_n$ and has size $n!$. The set of all permutations is $\mathfrak{S} = \bigcup_{i=0}^{\infty} \mathfrak{S}_i$.

Classical Permutation Patterns

Classical permutation patterns capture many interesting combinatorial objects and properties.

Definition (Order Isomorphism)

Two substrings $\alpha_1\alpha_2\cdots\alpha_n$ and $\beta_1\beta_2\cdots\beta_n$ are said to be *order isomorphic* if $\alpha_r < \alpha_s$ if and only if $\beta_r < \beta_s$.

Definition

A permutation π is said to *contain* the *classical permutation pattern* σ (denoted $\sigma \preceq \pi$) if there is some subsequence $i_1 i_2 \cdots i_k$ such that the sequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is order isomorphic to $\sigma(1)\sigma(2)\cdots\sigma(k)$.

If π does not contain σ we say that π *avoids* σ.
We the set of permutations of length n avoiding a pattern σ is denoted as $\mathrm{Av}_n(\sigma)$ and $\mathrm{Av}(\sigma) = \bigcup_{i=0}^{\infty} \mathrm{Av}_i(\sigma)$.

Example

The permutation $\pi = 24153$ contains the pattern $\sigma = 231$

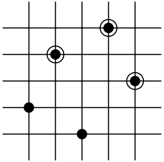


Figure: Plot of the permutation 24153 with an occurrence of 231 indicated

Mesh Patterns

Mesh patterns are a natural extension of classical permutation patterns.

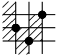
Definition

A *mesh pattern* is a pair

$$p = (\tau, R) \text{ with } \tau \in \mathfrak{S}_k \text{ and } R \subseteq [0, k] \times [0, k].$$

We say that τ is the *underlying classical pattern* of p.

Example

The pattern $p = (213, \{(0,1),(0,2),(0,3),(1,0),(1,1),(2,1),(2,2)\}) =$  is contained in $\pi = 34215$.

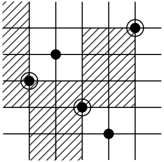
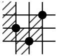


Figure: An occurrence of p in π

Example

The pattern $q = (21, \{(0,1), (0,2), (1,0), (1,1)\}) =$  is contained in $p = (213, \{(0,1), (0,2), (0,3), (1,0), (1,1), (2,1), (2,2)\}) =$  as a subpattern.

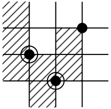


Figure: An occurrence of q in p

Outline

Coincidence

Definition

Two mesh patterns are said to be *coincident* if they avoid the same set of permutations at every length.

Classical patterns can never be coincident.
Aim to establish rules that classify coincidences when we have one mesh pattern and one classical pattern.

Experimental Results

	Dominating Pattern			
	231		321	
	12	21	12	21
No Dominating rule	220	220	220	220
First Dominating rule	85	43	220	29
Second Dominating rule	59	39	220	29
Third Dominating rule	56	39	220	29
Experimental class size	56	39	213	29

Table: Coincidence class number reduction by application of Dominating rules

Experimental Results

	Dominating Pattern			
	231		321	
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First Dominating rule

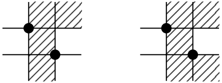
Proposition: First Dominating rule

Given two mesh patterns $m_1 = (\sigma, R_1)$ *and* $m_2 = (\sigma, R_2)$*, and a dominating classical pattern* $\pi = (\pi, \emptyset)$ *such that* $|\pi| \leq |\sigma| + 1$*, the sets* $\mathrm{Av}(\{\pi, m_1\})$ *and* $\mathrm{Av}(\{\pi, m_2\})$ *are coincident if*

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. $\pi \preceq \texttt{add\_point}(\sigma, (a, b), \emptyset)$

Example

The following two patterns are coincident in Av(321)



Corollary

All coincidences of classes the form $\mathrm{Av}(\{321, (21, R)\})$ *are fully explained by the First Dominating rule.*

There are 29 coincidences of mesh patterns of the form $\mathrm{Av}(\{321, (21, R)\})$

Experimental Results

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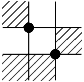
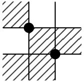
Second Dominating rule

The patterns

$m_1 =$  and $m_2 =$ 

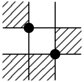
are coincident in Av(231).

Lemma

Given a mesh pattern $m = (\sigma, R)$, *where the box* (a, b) *is not in* R, *and a dominating classical pattern* $\pi = (\pi, \emptyset)$ *if* $\pi \preceq \texttt{add\_ascent}(\sigma, (a, b))$ *(*$\pi \preceq \texttt{add\_descent}(\sigma, (a, b))$*) ,then in any occurrence of* m *in a permutation* ϱ, *the region corresponding to the box* (a, b) *can only contain an increasing (decreasing) subsequence of* ϱ.

Example

Considering m_1 again



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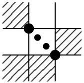

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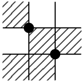

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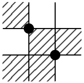
Considering m_1 again



This is m_2.

Proposition: Second Dominating rule

Given two mesh patterns $m_1 = (\sigma, R_1)$ *and* $m_2 = (\sigma, R_2)$*, and a dominating classical pattern* $\pi = (\pi, \emptyset)$ *such that* $|\pi| \leq |\sigma| + 2$*, the sets* $\mathrm{Av}(\{\pi, m_1\})$ *and* $\mathrm{Av}(\{\pi, m_2\})$ *are coincident if*

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. 2.1 $\pi \preceq \texttt{add\_ascent}\,(\sigma, (a, b))$ *and*
    - 2.1.1 $(a+1, b) \in \sigma$ *and* $(a+1, b-1) \notin R$ *and*
      $(x, b-1) \in R \implies (x, b) \in R$ *(where* $x \neq a, a+1$*) and*
      $(a+1, y) \in R \implies (a, y) \in R$ *(where* $y \neq b-1, b$*).*
    - 2.1.2 ...

   2.2 ...
    - 2.2.1 ...
    - 2.2.2 ...

Example

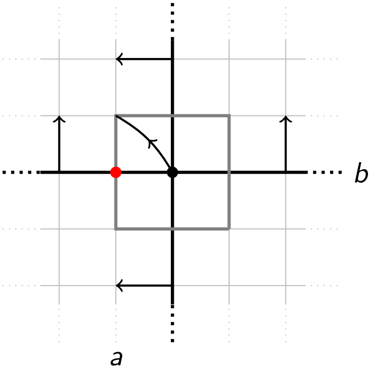


Figure: If the conditions of The Second Dominating rule are satisfied the box (a, b) can be shaded.

Corollary

All coincidences of classes the form $\mathrm{Av}(\{231, (21, R)\})$ *are fully explained by applying the First Dominating rule, then applying the Second Dominating rule.*

There are 39 coincidences of mesh patterns of the form $\mathrm{Av}(\{231, (21, R)\})$

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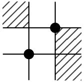
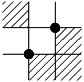
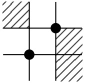
Third Dominating rule

The patterns

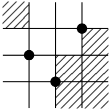
$m_1 =$  and $m_2 =$ 

are coincident in Av(231). Neither of the previous two rules explain this.

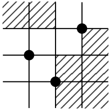
Example



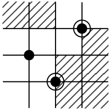
Example



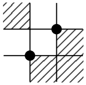
Example



Example



Example



Proposition: Third Dominating rule

Given two mesh patterns $m_1 = (\sigma, R_1)$ *and* $m_2 = (\sigma, R_2)$*, and a dominating classical pattern* $\pi = (\pi, \emptyset)$*, the sets* $\mathrm{Av}(\{\pi, m_1\})$ *and* $\mathrm{Av}(\{\pi, m_2\})$ *are coincident if*

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. $\texttt{add\_point}((\sigma, R_1), (a, b), D)$ *where* $D \in \{N, E, S, W\}$ *is coincident with a mesh pattern containing an occurrence of* (σ, R_2) *as a subpattern.*

Corollary

All coincidences of classes the form $\mathrm{Av}(\{231, (12, R)\})$ *are fully explained by applying the First Dominating rule, the Second Dominating rule, and then the Third Dominating rule.*

There are 56 coincidences of mesh patterns of the form $\mathrm{Av}(\{231, (21, R)\})$

Experimental Results

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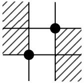
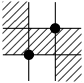
Table: Coincidence class number reduction by application of Dominating rules

Special Cases

There are 7 coincidences of the form $\mathrm{Av}(321, m)$ that are not explained by the rules.

Example

$m_1 =$  and $m_2 =$ 

This coincidence is explained by mathematical induction on the number of points in region corresponding to the middle box. We call this number n.

Example

Base Case ($n = 0$): The base case holds since we can freely shade the box if it contains no points.

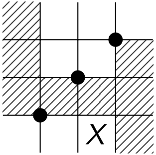
Example

Base Case ($n = 0$): The base case holds since we can freely shade the box if it contains no points.

Inductive Hypothesis ($n = k$): Suppose that we can find an occurrence of the second pattern if we have an occurrence of the first with k points in the middle box.

Example

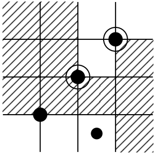
Inductive Step $(n = k + 1)$ Suppose that we have $(k + 1)$ points in the middle box. Choose the bottom most point in the middle box, this gives the mesh pattern



Now we need to consider the box labelled X. If this box is empty then we have an occurrence of m_2 and are done.

Example

Inductive Step ($n = k + 1$) (cont.) If this box contains any points then we gain some extra shading on the mesh pattern due to the dominating pattern



The two highlighted points form an occurrence of m_1 with k points in the middle box, and thus by the Inductive Hypothesis we are done.

Outline

Wilf-equivalence

Definition

Two patterns are said to be *Wilf-equivalent* if they avoid (or contain) the same number of patterns at every length.

All coincident pattern classes are Wilf-equivalent.
The permutations in Av(231) of length n are counted by the Catalan numbers.

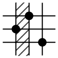
$$C_0 = 1, \text{ and } C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \tag{1}$$

This gives the sequence

$$1, 1, 2, 5, 14, 42, 132, 1430, 4862, \ldots$$

Set Partitions

Note

The avoiders of the pattern $q = (231, \{(1,0),(1,1),(1,2),(1,3)\})$,  , in $\mathfrak{S}_n$ are in one-to-one correspondence with partitions of $[\![n]\!]$. (Claesson)

Example

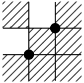
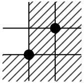
Given the permutation $\pi = 542139687$ this corresponds to the partition $\{\{5,4,2,1\},\{3\},\{9,6\},\{7,8\}\}$.

All permutations in $\mathrm{Av}(231)$ are also in $\mathrm{Av}(q)$.

Set Partitions can be used to explain certain Wilf-equivalences.

Example

The following patterns are experimentally Wilf-equivalent up to length 10 in $\mathrm{Av}(231)$

$m_1 =$  and $m_2 =$ 

Consider containers of the patterns in $\mathrm{Av}(231)$.

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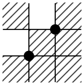
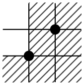
$m_1 =$  and $m_2 =$ 

Consider containers of the patterns in Av(231).
There can only ever be a single occurrence of m_1. The shading shows us that this is a partition of k elements into two non-overlapping parts by the first element and the last element.

Set Partitions can be used to explain certain Wilf-equivalences.

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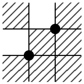
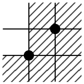
$m_1 =$  and $m_2 =$ 

Consider containers of the patterns in $\mathrm{Av}(231)$.
There can only ever be a single occurrence of m_2. The shading shows us that this is a partition of k elements into two non-overlapping parts, where the split in parts is determined by the value of the last element in the permutation.

Set Partitions can be used to explain certain Wilf-equivalences.

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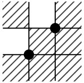
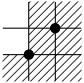
The following patterns are experimentally Wilf-equivalent up to length 10 in Av(231)

$m_1 =$  and $m_2 =$ 

Consider containers of the patterns in Av(231).
Therefore for each of these patterns a container of length k consists of a decreasing sequence of length $k-2$ split into two parts. There are $k-1$ ways to perform such a split. So the number of avoiders of length k is

$$K_k = C_k - (k-1), K_0 = 1$$

Generating FUNctions

" A generating function is a clothesline on which we hang up a sequence of numbers for display." (Herbert Wilf, *generatingfunctionology* [2])

Definition

The *ordinary generating function (OGF)* of a sequence (A_n) is the formal power series

$$A(x) = \sum_{n=0}^{\infty} A_n x^n \tag{2}$$

The right hand side is the Taylor expansion of the left hand side at 0.

Example

The generating function for the constant sequence $\{1, 1, 1, 1, \ldots\}$ is

$$f(x) = \frac{1}{1 - x}$$

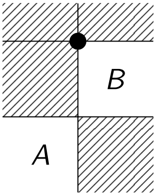
Example

The generating function for the avoiders of the permutation 231 satisfies

$$C(x) = 1 + xC^2(x) \tag{3}$$

This can be seen structurally ($\mathcal{C}$ is the set of 231 avoiders)

$$\mathcal{C} = \varepsilon \sqcup$$



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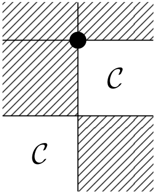
$\mathcal{C} = \varepsilon \sqcup$ 

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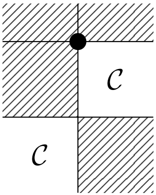
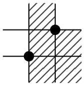
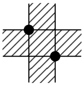
$\mathcal{C} = \varepsilon \sqcup$ 

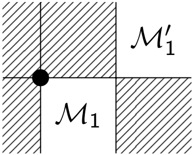
Solving for $C(x)$ and taking the Taylor expansion at 0 gives

$$1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 1430x^7 + 4862x^8 + \cdots$$

Now we consider one of the Wilf-equivalences between mesh patterns in Av(231).

$m_1 =$  and $m_2 =$ 

First we consider the avoiders $\mathcal{M}_1$ of m_1 in $\mathrm{Av}(231)$.

$\mathcal{M}_1 = \varepsilon \sqcup$ 

Where $\mathcal{M}_1'$ is a permutation avoiding $231, m_1$ and  .

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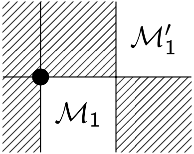

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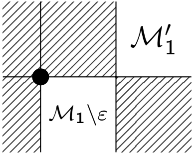
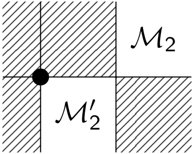


Next we consider the avoiders $\mathcal{M}_2$ of m_2 in $\mathrm{Av}(231)$.

$\mathcal{M}_2 = \varepsilon \sqcup$ 

Where $\mathcal{M}_2'$ is a permutation avoiding $231, m_2$ and  .

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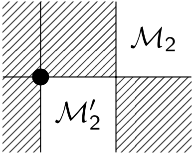
$\mathcal{M}_2 = \varepsilon \sqcup$ 

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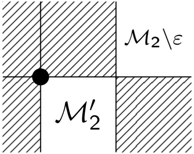
$\mathcal{M}_2' = \varepsilon \sqcup$ 

Both of these sets therefore have generating function $M(x)$ satisfying

$$M(x) = 1 + xM(x)M'(x) \tag{4}$$

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Evaluating coefficients of $M(x)$ then gives the sequence

$$1, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, \ldots$$

This is an offset of the Motzkin numbers.

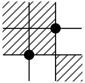
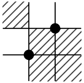
Outline

Conclusions

- Automatic coincidence classification is difficult.
  - Completely classified for length 2 mesh patterns and length 3 classical patterns.
  - Gets harder with larger dominating patterns.
- There are a number of Wilf-classes that give interesting enumerations.
  - Would be interesting to try and connect these sets to different objects.

Future Work: Extensions of rules

One can consider application of the Third Dominating rule, as well as a simple extension, without any dominating pattern. This can capture the special case described by Claesson, Tenner, and Ulfarsson in [3]

$m_1 =$  , and $m_2 =$ 

We cannot apply the first, or second, rule without a dominating pattern.
We can also consider taking sets of mesh patterns, or sets of dominating patterns.

Future Work: Equivalences with different dominating patterns

It would be interesting to consider Wilf-equivalences amongst classes where 321 is the dominating pattern.

It is also interesting to consider Wilf-equivalence when we have different dominating patterns

We can show that the sets $\mathcal{T} = \mathrm{Av}\Big($  $, 231\Big)$ and $\mathcal{U} = \mathrm{Av}\Big($  $, 321\Big)$ are Wilf-equivalent.

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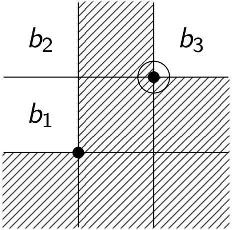

[1] D. Knuth,
The Art of Computer Programming: Volume 1, 1997.

[2] H. Wilf,
generatingfunctionology, 1994. https://www.math.upenn.edu/~wilf/gfology2.pdf

[2] A. Claesson, B. E. Tenner, and H. Ulfarsson
Coincidence among families of mesh patterns
CoRR, 2014. http://arxiv.org/abs/1412.0703