

# Equivalence classes of mesh patterns with a Dominating Pattern

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# Outline

## 1. Introduction

- ▶ Permutations
- ▶ Classical Permutation Patterns
- ▶ Mesh Patterns

## 2. Coincidence Classes

- ▶ Coincidence
- ▶ Summary of Experimental Results
- ▶ Dominating Pattern Rules

# Permutations

A *permutation* is a *bijection*,  $\pi$ , from the set  $\llbracket n \rrbracket = \{1, \dots, n\}$  to itself.

More intuitively “A *permutation of  $n$  objects* is an arrangement of  $n$  distinct objects in a row” (Knuth).

We write permutations in *one-line notation*, writing the entries of the entries of the permutation in order

$$\pi = \pi(1)\pi(2) \dots \pi(n)$$

## Example

The 6 permutations on  $\llbracket 3 \rrbracket$  are

123, 132, 213, 231, 312, 321

We can display a permutation in a *plot* to give a graphical representation. We plot the points  $(i, \pi(i))$  in a Cartesian coordinate system.

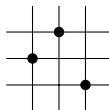


Figure: Plot of the permutation 231

In this setting we call the elements of the permutations *points*.

The set of all permutations of length  $n$  is  $\mathfrak{S}_n$  and has size  $n!$ . The set of all permutations is  $\mathfrak{S} = \bigcup_{i=0}^{\infty} \mathfrak{S}_i$ .

# Classical Permutation Patterns

*Classical permutation patterns* capture many interesting combinatorial objects and properties.

## Definition (Order Isomorphism)

Two substrings  $\alpha_1\alpha_2\cdots\alpha_n$  and  $\beta_1\beta_2\cdots\beta_n$  are said to be *order isomorphic* if  $\alpha_r < \alpha_s$  if and only if  $\beta_r < \beta_s$ .

## Definition

A permutation  $\pi$  is said to *contain* the *classical permutation pattern*  $\sigma$  (denoted  $\sigma \preceq \pi$ ) if there is some subsequence  $i_1 i_2 \cdots i_k$  such that the sequence  $\pi(i_1)\pi(i_2) \cdots \pi(i_k)$  is order isomorphic to  $\sigma(1)\sigma(2) \cdots \sigma(k)$ .

If  $\pi$  does not contain  $\sigma$  we say that  $\pi$  *avoids*  $\sigma$ .

We the set of permutations of length  $n$  avoiding a pattern  $\sigma$  is denoted as  $\text{Av}_n(\sigma)$  and  $\text{Av}(\sigma) = \bigcup_{i=0}^{\infty} \text{Av}_i(\sigma)$ .

## Example

The permutation  $\pi = 24153$  contains the pattern  $\sigma = 231$

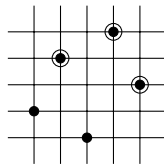


Figure: Plot of the permutation 24153 with an occurrence of 231 indicated

# Mesh Patterns

*Mesh patterns* are a natural extension of classical permutation patterns.

## Definition


A *mesh pattern* is a pair

$$p = (\tau, R) \text{ with } \tau \in \mathfrak{S}_k \text{ and } R \subseteq [0, k] \times [0, k].$$

We say that  $\tau$  is the *underlying classical pattern* of  $p$ .



## Example

The pattern  $p = (213, \{(0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (2, 1), (2, 2)\}) =$   is contained in  $\pi = 34215$ .

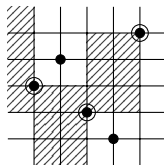




Figure: An occurrence of  $p$  in  $\pi$

## Example

The pattern  $q = (21, \{(0, 1), (0, 2), (1, 0), (1, 1)\}) =$   is contained in  $p = (213, \{(0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (2, 1), (2, 2)\}) =$   as a subpattern.

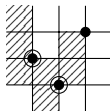


Figure: An occurrence of  $q$  in  $p$

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# Coincidence

## Definition

Two mesh patterns are said to be *coincident* if they avoid the same set of patterns at every length.

Classical patterns can never be coincident.

Aim to establish rules that classify coincidences when we have one mesh pattern and one classical pattern.

# Experimental Results

|                         | Dominating Pattern |     |     |     |
|-------------------------|--------------------|-----|-----|-----|
|                         | 231                |     | 321 |     |
|                         | 12                 | 21  | 12  | 21  |
| No Dominating rule      | 220                | 220 | 220 | 220 |
| First Dominating rule   | 85                 | 43  | 220 | 29  |
| Second Dominating rule  | 59                 | 39  | 220 | 29  |
| Third Dominating rule   | 56                 | 39  | 220 | 29  |
| Experimental class size | 56                 | 39  | 213 | 29  |

**Table:** Coincidence class number reduction by application of Dominating rules

# Experimental Results

|                         | Dominating Pattern |     |     |     |
|-------------------------|--------------------|-----|-----|-----|
|                         | 231                |     | 321 |     |
|                         | 12                 | 21  | 12  | 21  |
| No Dominating rule      | 220                | 220 | 220 | 220 |
|                         |                    |     |     |     |
|                         |                    |     |     |     |
|                         |                    |     |     |     |
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**Table:** Coincidence class number reduction by application of Dominating rules

# First Dominating rule

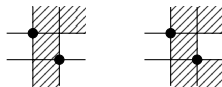
## Proposition: First Dominating rule

Given two mesh patterns  $m_1 = (\sigma, R_1)$  and  $m_2 = (\sigma, R_2)$ , and a dominating classical pattern  $\pi = (\pi, \emptyset)$  such that  $|\pi| \leq |\sigma| + 1$ , the sets  $\text{Av}(\{\pi, m_1\})$  and  $\text{Av}(\{\pi, m_2\})$  are coincident if

1.  $R_1 \triangle R_2 = \{(a, b)\}$
2.  $\pi \preceq \text{add\_point}(\sigma, (a, b), \emptyset)$

## Example

The following two patterns are coincident in  $\text{Av}(321)$



## Corollary

*All coincidences of classes the form  $\text{Av}(\{321, (21, R)\})$  are fully explained by the First Dominating rule.*

There are 29 coincidences of mesh patterns of the form  $\text{Av}(\{321, (21, R)\})$



# Experimental Results

|                         | Dominating Pattern |     |     |     |
|-------------------------|--------------------|-----|-----|-----|
|                         | 231                |     | 321 |     |
|                         | 12                 | 21  | 12  | 21  |
| No Dominating rule      | 220                | 220 | 220 | 220 |
| First Dominating rule   | 85                 | 43  | 220 | 29  |
|                         |                    |     |     |     |
|                         |                    |     |     |     |
| Experimental class size | 56                 | 39  | 213 | 29  |

**Table:** Coincidence class number reduction by application of Dominating rules

## Second Dominating rule

The patterns

$$m_1 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \bullet & \text{shaded} \\ \hline \text{shaded} & \bullet \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \bullet & \text{shaded} \\ \hline \text{shaded} & \bullet \\ \hline \end{array}$$

are coincident in  $\text{Av}(231)$ .

## Lemma

*Given a mesh pattern  $m = (\sigma, R)$ , where the box  $(a, b)$  is not in  $R$ , and a dominating classical pattern  $\pi = (\pi, \emptyset)$  if  $\pi \preceq \text{add\_ascent}(\sigma, (a, b))$  ( $\pi \preceq \text{add\_descent}(\sigma, (a, b))$ ), then in any occurrence of  $m$  in a permutation  $\varrho$ , the region corresponding to the box  $(a, b)$  can only contain an increasing (decreasing) subsequence of  $\varrho$ .*

## Example

Considering  $m_1$  again



## Example

Considering  $m_1$  again



## Example

Considering  $m_1$  again



## Example

Considering  $m_1$  again



This is  $m_2$ .

## Proposition: Second Dominating rule

Given two mesh patterns  $m_1 = (\sigma, R_1)$  and  $m_2 = (\sigma, R_2)$ , and a dominating classical pattern  $\pi = (\pi, \emptyset)$  such that  $|\pi| \leq |\sigma| + 2$ , the sets  $\text{Av}(\{\pi, m_1\})$  and  $\text{Av}(\{\pi, m_2\})$  are coincident if

1.  $R_1 \triangle R_2 = \{(a, b)\}$

2. 2.1  $\pi \preceq \text{add\_ascent}(\sigma, (a, b))$  and

- 2.1.1  $(a+1, b) \in \sigma$  and  $(a+1, b-1) \notin R$  and

- $(x, b-1) \in R \implies (x, b) \in R$  (where  $x \neq a, a+1$ ) and

- $(a+1, y) \in R \implies (a, y) \in R$  (where  $y \neq b-1, b$ ).

- 2.1.2 ...

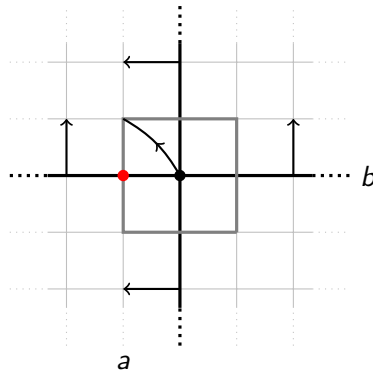
- 2.2 ...

- 2.2.1 ...

- 2.2.2 ...



## Example



**Figure:** If the conditions of The Second Dominating rule are satisfied the box  $(a - 1, b)$  can be shaded.

## Corollary

*All coincidences of classes the form  $\text{Av}(\{231, (21, R)\})$  are fully explained by applying the First Dominating rule, then applying the Second Dominating rule.*

There are 39 coincidences of mesh patterns of the form  $\text{Av}(\{231, (21, R)\})$

# Experimental Results

|                         | Dominating Pattern |     |     |     |
|-------------------------|--------------------|-----|-----|-----|
|                         | 231                |     | 321 |     |
|                         | 12                 | 21  | 12  | 21  |
| No Dominating rule      | 220                | 220 | 220 | 220 |
| First Dominating rule   | 85                 | 43  | 220 | 29  |
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|                         |                    |     |     |     |
| Experimental class size | 56                 | 39  | 213 | 29  |

**Table:** Coincidence class number reduction by application of Dominating rules

# Third Dominating rule

The patterns

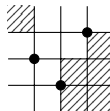
$$m_1 = \begin{array}{|c|c|} \hline \text{shaded} & \bullet \\ \hline \bullet & \text{shaded} \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|} \hline \text{shaded} & \bullet \\ \hline \bullet & \text{shaded} \\ \hline \end{array}$$

are coincident in  $\text{Av}(231)$ . Neither of the previous two rules explain this.

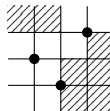
## Example



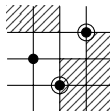
## Example



## Example



## Example





## Example



### Proposition: Third Dominating rule

Given two mesh patterns  $m_1 = (\sigma, R_1)$  and  $m_2 = (\sigma, R_2)$ , and a dominating classical pattern  $\pi = (\pi, \emptyset)$ , the sets  $\text{Av}(\{\pi, m_1\})$  and  $\text{Av}(\{\pi, m_2\})$  are coincident if

1.  $R_1 \triangle R_2 = \{(a, b)\}$
2.  $\text{add\_point}((\sigma, R_1), (a, b), D)$  where  $D \in \{N, E, S, W\}$  is coincident with a mesh pattern containing an occurrence of  $(\sigma, R_2)$  as a subpattern.

## Corollary

*All coincidences of classes the form  $\text{Av}(\{231, (12, R)\})$  are fully explained by applying the First Dominating rule, the Second Dominating rule, and then the Third Dominating rule.*

There are 56 coincidences of mesh patterns of the form  $\text{Av}(\{231, (21, R)\})$

# Experimental Results

|                         | Dominating Pattern |     |     |     |
|-------------------------|--------------------|-----|-----|-----|
|                         | 231                |     | 321 |     |
|                         | 12                 | 21  | 12  | 21  |
| No Dominating rule      | 220                | 220 | 220 | 220 |
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**Table:** Coincidence class number reduction by application of Dominating rules