

Equivalence classes of mesh patterns with a Dominating Pattern

Murray Tannock (murray14@ru.is)



HÁSKÓLINN Í REYKJAVÍK
REYKJAVIK UNIVERSITY

June 28, 2016

Outline

1. Introduction

- ▶ Subpatterns

2. Coincidence Classes

- ▶ Coincidence
- ▶ Summary of Experimental

Results

- ▶ Dominating Pattern Rules
- ▶ Special Cases

3. Wilf-equivalences

- ▶ Wilf-equivalence
- ▶ Set Partitions
- ▶ Generating Functions

4. Conclusions and Future Work



- ▶ Conclusions
- ▶ Future Work
 - ▶ Extensions of rules
 - ▶ Equivalences with different dominating patterns
 - ▶ Other questions

Subpatterns

The notion of *subpatterns* captures containment of mesh patterns within other mesh patterns.

We can think of classical patterns as mesh patterns with no shading, and permutations as completely shaded mesh patterns.

Example

The pattern $q = (21, \{(0, 1), (0, 2), (1, 0), (1, 1)\}) =$  is contained in $p = (213, \{(0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (2, 1), (2, 2)\}) =$  as a subpattern.

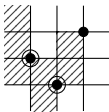


Figure: An occurrence of q in p

Outline

1. Introduction

- ▶ Subpatterns

2. Coincidence Classes

- ▶ Coincidence
- ▶ Summary of Experimental

Results

- ▶ Dominating Pattern Rules
- ▶ Special Cases

3. Wilf-equivalences

- ▶ Wilf-equivalence
- ▶ Set Partitions
- ▶ Generating Functions

4. Conclusions and Future Work

- ▶ Conclusions
- ▶ Future Work
 - ▶ Extensions of rules
 - ▶ Equivalences with different dominating patterns
 - ▶ Other questions

Coincidence

Coincidences of mesh patterns of length 2 are classified.

Aim to establish rules that classify coincidences when we have one mesh pattern and one classical pattern.

We call the classical pattern a *dominating pattern*.

Experimental Results

	Dominating Pattern			
	Av(231)		Av(321)	
	12	21	12	21
No Dominating rule	220	220	220	220
First Dominating rule	85	43	220	29
Second Dominating rule	59	39	220	29
Third Dominating rule	56	39	220	29
Experimental class size	56	39	213	29

Table: Coincidence class number reduction by application of Dominating rules

Experimental Results

	Dominating Pattern			
	Av(231)		Av(321)	
	12	21	12	21
No Dominating rule	220	220	220	220
Experimental class size	56	39	213	29

Table: Coincidence class number reduction by application of Dominating rules

First Dominating rule

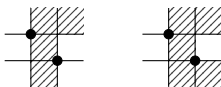
Proposition: First Dominating rule

Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \emptyset)$ such that $|\pi| \leq |\sigma| + 1$, the sets $\text{Av}(\{\pi, m_1\})$ and $\text{Av}(\{\pi, m_2\})$ are coincident if

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. $\pi \preceq \text{add_point}(\sigma, (a, b), \emptyset)$

Example

The following two patterns are coincident in $Av(321)$



Corollary

All coincidences of classes the form $Av(\{321, (21, R)\})$ are fully explained by the First Dominating rule.

There are 29 coincidences of mesh patterns of the form $Av(\{321, (21, R)\})$

Experimental Results

	Dominating Pattern			
	Av(231)		Av(321)	
	12	21	12	21
No Dominating rule	220	220	220	220
First Dominating rule	85	43	220	29
Experimental class size	56	39	213	29

Table: Coincidence class number reduction by application of Dominating rules

Second Dominating rule

The patterns

$$m_1 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \bullet & \text{shaded} \\ \hline \text{shaded} & \bullet \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \bullet & \text{shaded} \\ \hline \text{shaded} & \bullet \\ \hline \end{array}$$

are coincident in $\text{Av}(231)$.

Lemma

Given a mesh pattern $m = (\sigma, R)$, where the box (a, b) is not in R , and a dominating classical pattern $\pi = (\pi, \emptyset)$ if $\pi \preceq \text{add_ascent}(\sigma, (a, b))$, then in any occurrence of m in a permutation ϱ , the region corresponding to the box (a, b) can only contain an decreasing subsequence of ϱ .

Example

Considering m_1 again



Example

Considering m_1 again



Example

Considering m_1 again



Example

Considering m_1 again



This is m_2 .

Proposition: Second Dominating rule

Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \emptyset)$ such that $|\pi| \leq |\sigma| + 2$, the sets $\text{Av}(\{\pi, m_1\})$ and $\text{Av}(\{\pi, m_2\})$ are coincident if

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. 2.1 $\pi \preceq \text{add_ascent}(\sigma, (a, b))$ and
 - 2.1.1 $(a+1, b) \in \sigma$ and $(a+1, b-1) \notin R$ and
 $(x, b-1) \in R \implies (x, b) \in R$ (where $x \neq a, a+1$) and
 $(a+1, y) \in R \implies (a, y) \in R$ (where $y \neq b-1, b$).
 - 2.1.2 ...
- 2.2 ...
 - 2.2.1 ...
 - 2.2.2 ...

Example

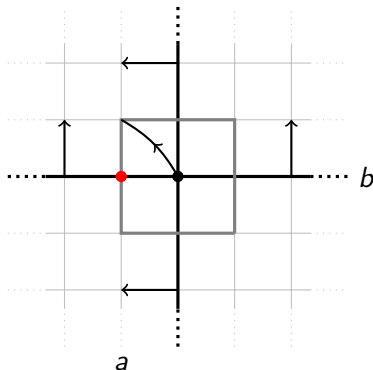


Figure: If the conditions of The Second Dominating rule are satisfied the box (a, b) can be shaded.

Corollary

All coincidences of classes the form $\text{Av}(\{231, (21, R)\})$ are fully explained by applying the First Dominating rule, then applying the Second Dominating rule.

There are 39 coincidences of mesh patterns of the form $\text{Av}(\{231, (21, R)\})$

Experimental Results

	Dominating Pattern			
	Av(231)		Av(321)	
	12	21	12	21
No Dominating rule	220	220	220	220
First Dominating rule	85	43	220	29
Second Dominating rule	59	39	220	29
Experimental class size	56	39	213	29

Table: Coincidence class number reduction by application of Dominating rules

Third Dominating rule

The patterns

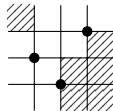
$$m_1 = \begin{array}{|c|c|} \hline \text{shaded} & \bullet \\ \hline \bullet & \text{shaded} \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|} \hline \text{shaded} & \bullet \\ \hline \bullet & \text{shaded} \\ \hline \end{array}$$

are coincident in $\text{Av}(231)$. Neither of the previous two rules explain this.

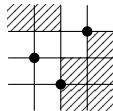
Example



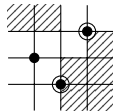
Example



Example



Example



Example



Proposition: Third Dominating rule

Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \emptyset)$, the sets $\text{Av}(\{\pi, m_1\})$ and $\text{Av}(\{\pi, m_2\})$ are coincident if

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. $\text{add_point}((\sigma, R_1), (a, b), D)$ where $D \in \{N, E, S, W\}$ is coincident with a mesh pattern containing an occurrence of (σ, R_2) as a subpattern.

Corollary

All coincidences of classes the form $\text{Av}(\{231, (12, R)\})$ are fully explained by applying the First Dominating rule, the Second Dominating rule, and then the Third Dominating rule.

There are 56 coincidences of mesh patterns of the form $\text{Av}(\{231, (21, R)\})$

Experimental Results

	Dominating Pattern			
	Av(231)		Av(321)	
	12	21	12	21
No Dominating rule	220	220	220	220
First Dominating rule	85	43	220	29
Second Dominating rule	59	39	220	29
Third Dominating rule	56	39	220	29
Experimental class size	56	39	213	29

Table: Coincidence class number reduction by application of Dominating rules

Special Cases

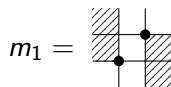
There are 7 coincidences of the form $Av(321, m)$ that are not explained by the rules.

Example

$$m_1 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \text{shaded} & \text{shaded} \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \text{shaded} & \text{shaded} \\ \hline \end{array}$$

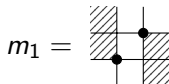
This coincidence is explained by mathematical induction on the number of points in the region corresponding to the middle box. We call this number n .

Example



Base Case ($n = 0$): The base case holds since we can freely shade the box if it contains no points.

Example



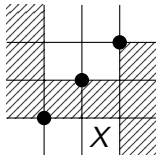
Base Case ($n = 0$): The base case holds since we can freely shade the box if it contains no points.

Inductive Hypothesis ($n = k$): Suppose that we can find an occurrence of the second pattern if we have an occurrence of the first with k points in the middle box.

Example

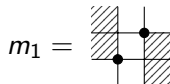
$$m_1 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \bullet & \bullet \\ \hline \end{array}$$

Inductive Step ($n = k + 1$) Suppose that we have $(k + 1)$ points in the middle box. Choose the bottom most point in the middle box, this gives the mesh pattern

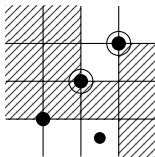


Now we need to consider the box labelled X . If this box is empty then we have an occurrence of m_2 and are done.

Example



Inductive Step ($n = k + 1$) (cont.) If this box contains any points then we gain some extra shading on the mesh pattern due to the dominating pattern



The two highlighted points form an occurrence of m_1 with k points in the middle box, and thus by the Inductive Hypothesis we are done.

Outline

1. Introduction

- ▶ Subpatterns

2. Coincidence Classes

- ▶ Coincidence
- ▶ Summary of Experimental

Results

- ▶ Dominating Pattern Rules
- ▶ Special Cases

3. Wilf-equivalences

- ▶ Wilf-equivalence
- ▶ Set Partitions
- ▶ Generating Functions

4. Conclusions and Future Work

- ▶ Conclusions
- ▶ Future Work
 - ▶ Extensions of rules
 - ▶ Equivalences with different dominating patterns
 - ▶ Other questions


Wilf-equivalence

All coincident pattern classes are Wilf-equivalent.

We Wilf-classify classes of the form $\text{Av}(231, m)$ where m is a mesh pattern.

Set Partitions

Note

The avoiders of the pattern $q = (231, \{(1, 0), (1, 1), (1, 2), (1, 3)\})$, , in \mathfrak{S}_n are in one-to-one correspondence with partitions of $\llbracket n \rrbracket$. (Claesson [2])

Example

Given the permutation $\pi = 542139687$ this corresponds to the partition $\{\{5, 4, 2, 1\}, \{3\}, \{9, 6\}, \{8, 7\}\}$.

All permutations in $\text{Av}(231)$ are also in $\text{Av}(q)$.

Set Partitions can be used to explain certain Wilf-equivalences.

Example

The following patterns are experimentally Wilf-equivalent up to length 10 in $\text{Av}(231)$

$$m_1 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \text{shaded} & \bullet \\ \hline \bullet & \text{shaded} \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \bullet & \text{shaded} \\ \hline \text{shaded} & \bullet \\ \hline \end{array}$$

Consider containers of the patterns in $\text{Av}(231)$.

Set Partitions can be used to explain certain Wilf-equivalences.

Example

The following patterns are experimentally Wilf-equivalent up to length 10 in $\text{Av}(231)$

$$m_1 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \text{shaded} & \bullet \\ \hline \bullet & \text{shaded} \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \text{shaded} & \bullet \\ \hline \bullet & \text{shaded} \\ \hline \end{array}$$

Consider containers of the patterns in $\text{Av}(231)$.

There can only ever be a single occurrence of m_1 . The shading shows us that this is a partition of k elements into two non-overlapping parts by the first element and the last element.

Set Partitions can be used to explain certain Wilf-equivalences.

Example

The following patterns are experimentally Wilf-equivalent up to length 10 in $\text{Av}(231)$

$$m_1 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \text{shaded} & \bullet \\ \hline \bullet & \text{shaded} \\ \hline \text{shaded} & \text{shaded} \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|} \hline \text{shaded} & \text{shaded} \\ \hline \text{shaded} & \bullet \\ \hline \bullet & \text{shaded} \\ \hline \text{shaded} & \text{shaded} \\ \hline \end{array}$$

Consider containers of the patterns in $\text{Av}(231)$.

There can only ever be a single occurrence of m_2 . The shading shows us that this is a partition of k elements into two non-overlapping parts, where the split in parts is determined by the value of the last element in the permutation.

Set Partitions can be used to explain certain Wilf-equivalences.

Example

The following patterns are experimentally Wilf-equivalent up to length 10 in $\text{Av}(231)$

$$m_1 = \begin{array}{|c|c|c|} \hline \text{shaded} & \text{shaded} & \text{shaded} \\ \hline \text{shaded} & \bullet & \text{shaded} \\ \hline \text{shaded} & \bullet & \text{shaded} \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|c|} \hline \text{shaded} & \text{shaded} & \text{shaded} \\ \hline \text{shaded} & \bullet & \text{shaded} \\ \hline \text{shaded} & \bullet & \text{shaded} \\ \hline \end{array}$$

Consider containers of the patterns in $\text{Av}(231)$.

Therefore for each of these patterns a container of length k consists of a decreasing sequence of length $k - 2$ split into two parts. There are $k - 1$ ways to perform such a split. So the number of avoiders of length k is

$$K_k = C_k - (k - 1), K_0 = 1$$

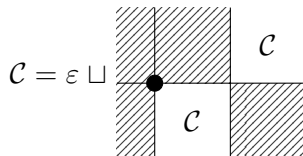
Generating Functions by Structure

Example

The generating function for the avoiders of the permutation 231 satisfies


$$C(x) = 1 + xC^2(x) \quad (1)$$

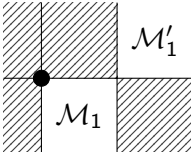
This can be seen structurally (\mathcal{C} is the set of 231 avoiders)





Now we consider one of the Wilf-equivalences between mesh patterns in $\text{Av}(231)$.

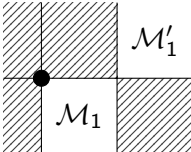
$$m_1 = \begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \quad \text{and} \quad m_2 = \begin{array}{|c|c|} \hline \bullet & \\ \hline & \bullet \\ \hline \end{array}$$


First we consider the avoiders \mathcal{M}_1 of $m_1 =$

in $\text{Av}(231)$.

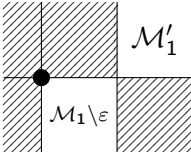
$$\mathcal{M}_1 = \varepsilon \sqcup$$



Where \mathcal{M}'_1 is a permutation avoiding 231, m_1 and .

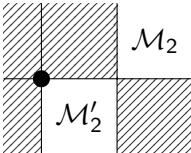
First we consider the avoiders \mathcal{M}_1 of $m_1 =$

in $\text{Av}(231)$.


$$\mathcal{M}_1 = \varepsilon \sqcup$$


Where \mathcal{M}'_1 is a permutation avoiding 231, m_1 and . Now considering a permutation in \mathcal{M}'_1

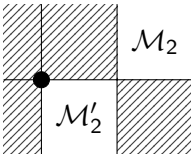
$$\mathcal{M}'_1 = \varepsilon \sqcup$$



Next we consider the avoiders \mathcal{M}_2 of $m_2 =$  in $\text{Av}(231)$.

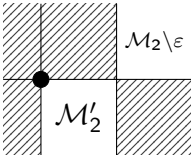
$$\mathcal{M}_2 = \varepsilon \sqcup$$


Where \mathcal{M}'_2 is a permutation avoiding 231, m_2 and .

Next we consider the avoiders \mathcal{M}_2 of $m_2 =$  in $\text{Av}(231)$.

$$\mathcal{M}_2 = \varepsilon \sqcup$$


Where \mathcal{M}'_2 is a permutation avoiding $231, m_2$ and . Now considering a permutation in \mathcal{M}'_2

$$\mathcal{M}'_2 = \varepsilon \sqcup$$


Both of these sets therefore have generating function $M(x)$ satisfying

$$M(x) = 1 + xM(x)M'(x) \quad (2)$$

$$M'(x) = 1 + x(M(x) - 1)M'(x) \quad (3)$$

Both of these sets therefore have generating function $M(x)$ satisfying

$$M(x) = 1 + xM(x)M'(x) \quad (2)$$

$$M'(x) = 1 + x(M(x) - 1)M'(x) \quad (3)$$

Solving 3 for $M'(x)$ and substituting into 2 gives us that the generating function for $M(x)$ satisfies

$$M(x) = xM^2(x) - x(M(x) - 1) + 1 \quad (4)$$

Both of these sets therefore have generating function $M(x)$ satisfying

$$M(x) = 1 + xM(x)M'(x) \quad (2)$$

$$M'(x) = 1 + x(M(x) - 1)M'(x) \quad (3)$$

Solving 3 for $M'(x)$ and substituting into 2 gives us that the generating function for $M(x)$ satisfies

$$M(x) = xM^2(x) - x(M(x) - 1) + 1 \quad (4)$$

Evaluating coefficients of $M(x)$ then gives the sequence

$$1, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, \dots$$

This is an offset of the Motzkin numbers.

Outline

1. Introduction

- ▶ Subpatterns

2. Coincidence Classes

- ▶ Coincidence
- ▶ Summary of Experimental

Results

- ▶ Dominating Pattern Rules
- ▶ Special Cases

3. Wilf-equivalences

- ▶ Wilf-equivalence
- ▶ Set Partitions
- ▶ Generating Functions

4. Conclusions and Future Work

- ▶ Conclusions
- ▶ Future Work
 - ▶ Extensions of rules
 - ▶ Equivalences with different dominating patterns
 - ▶ Other questions

Conclusions

- ▶ Automatic coincidence classification is difficult.
 - ▶ Completely classified for length 2 mesh patterns and length 3 classical patterns.
 - ▶ Gets harder with larger dominating patterns.
- ▶ There are a number of Wilf-classes that give interesting enumerations.
 - ▶ Would be interesting to try and connect these sets to different objects.

Future Work: Extensions of rules

One can consider application of the Third Dominating rule, as well as a simple extension, without any dominating pattern. This can capture the special case described by Claesson, Tenner, and Ulfarsson in [1], that was unexplained by the general results proved in that paper.

$$m_1 = \begin{array}{|c|c|c|} \hline \text{shaded} & \text{shaded} & \text{shaded} \\ \hline \text{shaded} & \bullet & \text{shaded} \\ \hline \text{shaded} & \bullet & \text{shaded} \\ \hline \end{array}, \text{ and } m_2 = \begin{array}{|c|c|c|} \hline \text{shaded} & \text{shaded} & \text{shaded} \\ \hline \text{shaded} & \bullet & \text{shaded} \\ \hline \text{shaded} & \bullet & \text{shaded} \\ \hline \end{array}$$

We cannot apply the first, or second, rule without a dominating pattern. We can also consider taking sets of mesh patterns, or sets of dominating patterns.

Future Work: Equivalences with different dominating patterns

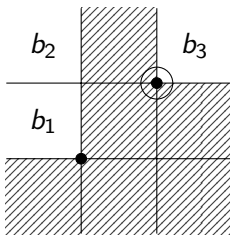
It would be interesting to consider Wilf-equivalences amongst classes where 321 is the dominating pattern.

It is also interesting to consider Wilf-equivalence when we have different dominating patterns. We can show that the sets $\mathcal{T} = \text{Av}\left(\begin{smallmatrix} \text{---} \diagup \text{---} \\ \text{---} \text{---} \end{smallmatrix}, 231\right)$ and $\mathcal{U} = \text{Av}\left(\begin{smallmatrix} \text{---} \diagup \text{---} \\ \text{---} \text{---} \end{smallmatrix}, 321\right)$ are Wilf-equivalent.

Future Work: Equivalences with different dominating patterns

It would be interesting to consider Wilf-equivalences amongst classes where 321 is the dominating pattern.

It is also interesting to consider Wilf-equivalence when we have different dominating patterns. We can show that the sets $\mathcal{T} = \text{Av}\left(\begin{smallmatrix} \bullet & \bullet \\ \bullet & \bullet \end{smallmatrix}, 231\right)$ and $\mathcal{U} = \text{Av}\left(\begin{smallmatrix} \bullet & \bullet \\ \bullet & \bullet \end{smallmatrix}, 321\right)$ are Wilf-equivalent.



Future Work: Other question

- ▶ When can a mesh pattern be contained precisely once in a permutation?
- ▶ What nice properties does the subpattern relation have?



[1] A. Claesson, B. E. Tenner, and H. Ulfarsson

Coincidence among families of mesh patterns

CoRR, 2014. <http://arxiv.org/abs/1412.0703>



[2] A. Claesson

Generalized pattern avoidance

Eur. J. Comb, 2001 <http://dx.doi.org/10.1006/eujc.2001.0515>