*** PROOF ***

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Equivalence classes of mesh patterns with a dominating pattern

Murray Tannock

Thesis of 60 ECTS credits

Master of Science (M.Sc.) in Computer Science

April 2016

Equivalence classes of mesh patterns with a dominating pattern

Thesis of 60 ECTS credits submitted to the School of Science and Engineering
at Reykjavík University in partial fulfillment of
the requirements for the degree of
Master of Science (M.Sc.) in Computer Science

April 2016

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PROOF

Equivalence classes of mesh patterns with a dominating pattern

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Abstract

A permutation is an arrangement of n objects. Permutation classes of classical permutation patterns capture many interesting properties, such as stack-sortability, and have links to many different combinatorial objects. Mesh patterns are an extension of classical patterns that allow additional restrictions to be placed on occurrences of the pattern. Two mesh patterns are coincident if they are avoided by the same set of permutations. We provide sufficient conditions for coincidence among mesh patterns, whilst also avoiding a longer classical pattern. These conditions, along with two special cases are used to completely classify coincidence amongst families containing a mesh pattern of length 2 and a classical pattern of length 3. Two patterns are Wilf-equivalent if they have the same number of avoiders at every length, we completely Wilf-classifies mesh patterns of length 2 when avoiding the classical pattern 231.

Proof

Jafngildisflokkar möskvamynstra með ríkjandi mynstri

Murray Tannock

apríl 2016

Útdráttur

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Acknowledgements

So long, and thanks for all the fish.

Douglas Adams[1]

I would like to thank my advisor, Henning Ulfarsson, for his time, support, guidance and introducing me to a research topic that I greatly enjoy.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 What is a Permutation?

In *The Art of Computer Programming* [2, p. 45] Knuth states, “A *permutation of n objects* is an arrangement of n distinct objects in a row”. When considering permutations we can consider them as occurring on the set $[\![n]\!] = \{1, \dots, n\}$, therefore a permutation is a *bijection* $\pi : [\![n]\!] \mapsto [\![n]\!]$. A permutation π can be written in two-line notation

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$$

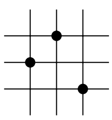
However, This notation is called *one-line notation*. In this form we write the entries of the permutation in order, and get

$$\pi = \pi(1)\pi(2)\dots\pi(n)$$

Example 1.1.1. The 6 permutations on $[\![3]\!]$, in one-line notation, are

$$123, 132, 213, 231, 312, 321$$

We can display a permutation in a *plot* in order to give a graphical representation of the permutation. In such a plot we display the points $(i, \pi(i))$ in a Cartesian coordinate system. The plot of the permutation $\pi = 231$ is shown below



It is convenient to call the elements of the permutation *points* when referring to these plots.

The set of all permutations of length n is $\mathfrak{S}_n$ and has size $n!$. The set of all permutations is $\mathfrak{S} = \bigcup_{i=0}^{\infty} \mathfrak{S}_i$. Note that $\mathfrak{S}_0$ has exactly one element, the empty permutation ε. As a function this is equivalent to the unique bijection $\emptyset \mapsto \emptyset$, and it’s one-line representation is the empty string.

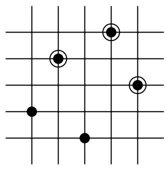
1.2 Classical Permutation Patterns

Classical permutation patterns began to be studied as a result of Knuth’s statements about stack-sorting in *The Art of Computer Programming* [2, p. 243, Ex. 5,6].

Definition 1.2.1. (Order isomorphism.) Two substrings $\alpha_1\alpha_2\cdots\alpha_n$ and $\beta_1\beta_2\cdots\beta_n$ are said to be *order isomorphic* if they share the same relative order, *i.e.*, $\alpha_r < \alpha_s$ if and only if $\beta_r < \beta_s$.

A permutation π is said to *contain* the permutation σ of length k as a *pattern* (denoted $\sigma \leq \pi$) if there is some subsequence $i_1i_2\cdots i_k$ such that the sequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is order isomorphic to $\sigma(1)\sigma(2)\cdots\sigma(k)$. If π does not contain σ, we say that π *avoids* σ.

For example the permutation $\pi = 24153$ contains the pattern $\sigma = 231$, since the second, fourth and fifth elements (453) are order isomorphic to 231. This can be seen graphically below, the subsequence order isomorphic to σ is highlighted.



We denote the set of permutations of length n avoiding a pattern σ as $\mathrm{Av}_n(\sigma)$ and $\mathrm{Av}(\sigma) = \bigcup_{i=0}^{\infty} \mathrm{Av}_i(\sigma)$.

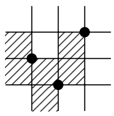
Knuth's statements were exercises in showing that the permutations avoiding the pattern 231 are precisely the permutations that are sortable to the identity permutation using a single stack, and that permutations avoiding the pattern 321 are precisely the permutations that are sortable to the identity permutation using a single queue with bypass.

1.3 Mesh Patterns

Mesh patterns were introduced by Brändén and Claesson [3] to capture explicit expansions for certain permutation statistics. They are a natural extension of classical permutation patterns. A *mesh pattern* is a pair

$$p = (\pi, R) \text{ with } \tau \in \mathfrak{S}_k \text{ and } R \subseteq [0,k] \times [0,k].$$

The set R is called the *mesh set* of the mesh pattern p. The plot for a mesh pattern looks similar to that of a classical pattern with the addition that we shade the unit square with bottom left corner (i, j) for each $(i, j) \in R$:



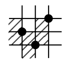
We define containment (denoted $p \leq \pi$), and avoidance, of the pattern p in the permutation τ on mesh patterns analogously to classical containment, and avoidance, of π in τ with the additional restrictions on the relative position of the occurrence of π in τ. These restrictions say that no elements of τ are allowed in the regions of the plot corresponding to shaded boxes in the mesh. Formally defined by Brändén and Claesson [3], an *occurrence* of p in τ is a subset ω of the plot of τ, $G(\tau) = \{(i, \tau(i) \mid i \in [1, n]\}$ such that there are order-preserving injections $\alpha, \beta : [1, k] \mapsto [1, n]$ satisfying the following two conditions.
Firstly, ω is an occurrence of π in the classical sense

i. $\omega = \{(\alpha(i), \beta(j)) : (i, j) \in G(\pi)\}$

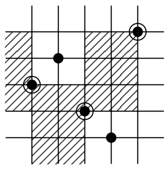
Define $R_{ij} = [\alpha(i) + 1, \alpha(i + 1) - 1] \times [\beta(j) + 1, \beta(j + 1) - 1]$ for $i, j \in [0, k]$ where $\alpha(0) = \beta(0) = 0$ and $\alpha(k + 1) = \beta(k + 1) = n + 1$. Then the second condition is

ii. if $(i, j) \in R$ then $R_{ij} \cap G(\tau) = \emptyset$

We call R_{ij} the region corresponding to (i, j). We define containment of a mesh pattern p in another mesh κ as above, with the additional condition that if $(i, j) \in R$ then R_{ij} is contained in the mesh set of κ, in this case we call p a *subpattern* of κ.

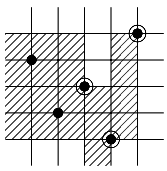
Example 1.3.1. The pattern $p = (213, \{(0, 1), (0, 2), (1, 0), (1, 1), (2, 1), (2, 2)\}) =$  is contained in $\pi = 34215$ but is not contained in $\sigma = 42315$.

Let us consider the plot for the permutation π. The subsequence 325 is an occurrence of 213 in the classical sense and the remaining points of π are not contained in the regions corresponding to the shaded boxes in p.



The subsequence 325 is therefore an occurrence of the pattern p in π and π contains p.

Now we consider the plot for the permutation σ. This permutation avoids p since for every occurrence of the classical pattern 213 there is at least one point in one of the shaded boxes. For example, consider the subsequence 315 in σ, this is an occurrence of 213 but not the mesh pattern since the points with values 4 and 2 are in the regions corresponding to the boxes $(0, 1)$ and $(0, 2)$, which are shaded in p. This is shown in the figure below.



This is true for all occurrences of 213 in σ and therefore σ avoids p.

We denote the avoidance sets for mesh patterns in the same way as for classical patterns. Given a mesh pattern $p = (\sigma, R)$ we say that σ is the *underlying classical pattern* of p.

Note 1.3.2. Classical patterns can be thought of as a subset of mesh patterns: the classical pattern π can be represented by a mesh pattern as $(\pi, \emptyset)$.

Two patterns are said to be *coincident* if they avoid the same set of permutations and *Wilf-equivalent* if they avoid the same number of permutations at every length.

In the past people have studied different classes of permutations that can be described by mesh patterns. Babson and Steingrímsson [4] considered *vincular* patterns (also known as *generalised* or *dashed* patterns), those where two adjacent entries in the pattern must be adjacent in the permutation, *i.e.* R is a union of vertical strips. Bousquet-Mélou, Claesson, Dukes, *et al.* [5] look at classes of pattern where both columns and rows can be shaded, these are called *bivincular* patterns. *Bruhat-restricted* patterns were studied by Woo and Yong [6] in order to establish necessary conditions for a Schubert variety to be Gorenstein. Mesh patterns also encompass a subset of *barred* patterns introduced by West [7], those with only one barred letter.

Avoiding pairs of patterns of the same length with certain properties has also been studied in the past, Claesson and Mansour [8] considered avoiding a pair of vincular patterns of length 3. Bean, Claesson, and Ulfarsson [9] study avoiding a vincular and a covincular pattern simultaneously in order to achieve some interesting counting results. However, very little work has been done on avoiding a mesh pattern and a classical pattern simultaneously. In this work we aim to establish some ground in this field by computing coincidences and Wilf-classes and calculating some of the enumerations of avoiders of a mesh pattern of length 2 and a classical pattern of length 3.

Chapter 2

Coincidences amongst families of mesh patterns and classical patterns

One interesting question to ask about permutation patterns is to consider when a pattern may be avoided by, or contained in, arbitrary permutations. Two patterns π and σ are said to be *coincident* if the set of permutations that avoid π is the same as the set of permutations that avoid σ, *i.e.* $\mathrm{Av}(\pi) = \mathrm{Av}(\sigma)$. This extends to sets of patterns as well as single patterns.

We consider the avoidance sets, $\mathrm{Av}(\{\pi, q\})$ where π is a classical pattern of length 3 and q is a mesh pattern of length 2 in order to establish some rules about when these two sets give the same avoidance set. We fix π in order to define the coincidence and say that π is the *dominating pattern*.

It is useful to be able to modify a mesh pattern by adding points to an already existing mesh pattern. First adding a single point into a pattern.

Definition 2.0.1. Given a mesh pattern $p = (\pi, R)$ `add_point` $(p, (a, b), D)$ gives a mesh pattern $p' = (\pi', R')$ with length $|\pi| + 1$ defined by

$$\pi'(i) = \begin{cases} \pi(i) & \text{if } i \neq a + 1 \text{ and } \pi(i) < b \\ \pi(i) + 1 & \text{if } i \neq a + 1 \text{ and } \pi(i) > b \\ b + 1 & \text{if } i = a + 1 \end{cases}$$

and

$$R' = \bigcup_{(i,j)\in R} r((i, j)),$$

where $r((i, j))$ is defined by

$$r((i, j)) = \begin{cases} \{(i, j)\} & \text{if } i < a, j < b \\ \{(i, j), (i, j + 1)\} & \text{if } i < a, j = b \\ \{(i, j + 1)\} & \text{if } i < a, j > b \\ \{(i, j), (i + 1, j)\} & \text{if } i = a, j < b \\ \{(i, j + 1), (i + 1, j + 1)\} & \text{if } i = a, j > b \\ \{(i + 1, j)\} & \text{if } i > a, j < b \\ \{(i + 1, j), (i + 1, j + 1)\} & \text{if } i > a, j = b \\ \{(i + 1, j + 1)\} & \text{if } i > a, j > b \end{cases}$$

If the shading set D is non-empty we can modify the definition of the directions slightly

$$\begin{aligned} N &= \{(a, b+1), (a+1, b+1)\} \\ E &= \{(a+1, b), (a+1, b+1)\} \\ S &= \{(a, b), (a+1, b)\} \\ W &= \{(a, b), (a, b+1)\} \end{aligned}$$

And we add the union of the sets in D into the mesh set R'.

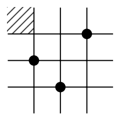
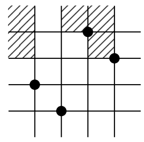
Given a mesh pattern p, `add_point` $(p, (a, b), D)$ is the operation that returns a mesh pattern equivalent to placing a point in the center of box (a, b), where (a, b) is not in the mesh set of p , with shading defined by $D \subseteq \{N, E, S, W\}$. The set D defines the shading by indicating that the boxes in the cardinal directions in D next to the point are shaded in the resulting pattern. Since there is no ambiguity we let `add_point` (ε, D) be equivalent to `add_point` $(\varepsilon, (0, 0), D)$.

Example 2.0.2. The result of adding a single point to the empty permutation for each cardinal direction.

`add_point` $(\varepsilon, \{N\}) =$  `add_point` $(\varepsilon, \{E\}) =$ 

`add_point` $(\varepsilon, \{S\}) =$  `add_point` $(\varepsilon, \{W\}) =$ 

A more complex example for `add_point` could be

`add_point` ( $, (2, 3), \{N, E\}$) = 

It is also useful to think about adding an ascent, or descent, into a pattern

Definition 2.0.3. Considering only adding the ascent, as adding a descent is very similar. Given a mesh pattern $p = (\pi, R)$, `add_ascent` $(p, (a, b))$ gives a mesh pattern $p' = (\pi', R')$ with length $|\pi| + 2$ defined by

$$\pi'(i) = \begin{cases} \pi(i) & \text{if } i \neq a+1, a+2 \text{ and } \pi(i) < b \\ \pi(i) + 2 & \text{if } i \neq a+1, a+2 \text{ and } \pi(i) > b \\ b+j, b+j & \text{if } i = a+j, j \in \{1, 2\} \end{cases}$$

and

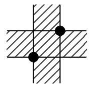
$$R' = \{(a+1, b), (a, b), (a+1, b), (a+2, b), (a+1, b+2)\} \cup \bigcup_{(i,j) \in R} r((i, j)),$$

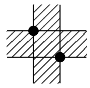
where $r((i, j))$ is defined by

$$r((i, j)) = \begin{cases} \{(i, j)\} & \text{if } i < a, j < b \\ \{(i, j), (i, j+1), (i, j+2)\} & \text{if } i < a, j = b \\ \{(i, j+2)\} & \text{if } i < a, j > b \\ \{(i, j), (i+1, j), (i+2, j)\} & \text{if } i = a, j < b \\ \{(i, j+2), (i+1, j+2), (i+2, j+2)\} & \text{if } i = a, j > b \\ \{(i+2, j)\} & \text{if } i > a, j < b \\ \{(i+2, j), (i+2, j+1), (i+2, j+2)\} & \text{if } i > a, j = b \\ \{(i+2, j+2)\} & \text{if } i > a, j > b \end{cases}$$

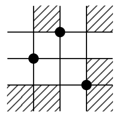
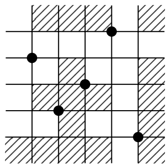
Given a pattern p, `add_descent` $(p, (a, b))$, and `add_ascent` $(p, (a, b))$, are the operations that return a mesh pattern equivalent to placing a decrease, or increase, in the center of box (a, b), where (a, b) is not in the mesh set of p, in p.

Example 2.0.4.

`add_ascent` $(\varepsilon) =$ 

`add_descent` $(\varepsilon) =$ 

A more complex example is

`add_ascent` $\Big($  $, (1, 1)\Big) =$ 

We now attempt to fully classify coincidences in families characterised by avoidance of a classical pattern of length 3 and a mesh pattern of length 2, that is finding and explaining all coincidences where $\mathrm{Av}(\{p, m\}) = \mathrm{Av}(\{p, m'\})$.

It can be easily seen that in order to classify coincidences one need only consider coincidences within the family of mesh patterns with the same underlying classical pattern, this is due to the fact that $21 \in \mathrm{Av}((12, R))$ and $12 \in \mathrm{Av}((21, R))$ for all mesh-sets R.

We know that there are a total of 512 mesh-sets for each underlying classical pattern. By use of the previous results of Claesson, Tenner, and Ulfarsson [10][1] the number of coincidence classes can be reduced to 220.

By discussion of a number of rules we will show that the number of coincidence classes follows the values shown in Table 2.1. The experimental data in the last row of the table is calculated on permutations up to length 11.

	Dominating Pattern			
	231		321	
	12	21	12	21
No Dominating rule	220	220	220	220
First Dominating rule	85	43	220	29
Second Dominating rule	59	39	220	29
Third Dominating rule	56	39	220	29
Experimental class size	56	39	213	29

Table 2.1: Coincidence class number reduction by application of Dominating Rules

From the table it can be seen that the rules established capture almost all coincidences. However, there are still some coincidences that are not able to be explained by the rules. This shows that complete coincidence classification of mesh patterns is a very difficult task, even when we have additional tools available.

[1] The authors use the Simultaneous Shading Lemma, a closure result and one worked out special case.

2.1 Coincidence classes of Av({321, (21, *R*)}).

Through experimentation, up to permutations of length 11, we discover that there are at least 29 coincidence classes of mesh patterns with underlying classical pattern 21.

Proposition 2.1.1 (First Dominating Pattern Rule). Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \emptyset)$ such that $|\pi| \leq |\sigma| + 1$, the sets $\mathrm{Av}(\{\pi, m_1\})$ and $\mathrm{Av}(\{\pi, m_2\})$ are coincident if

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. $\pi \preceq \texttt{add\_point}\,(\sigma, (a, b), \emptyset)$

In order to prove this proposition we must first make the following note.

Note 2.1.2. Let $R' \subseteq R$. Then any occurrence of (τ, R) in a permutation is an occurrence of (τ, R').

Proof of Proposition 2.1.1. We need to prove that $\mathrm{Av}(\{\pi, m_1\}) = \mathrm{Av}(\{\pi, m_2\})$.
Assume without meaningful loss of generality that $R_2 = R_1 \cup \{(a, b)\}$. Since R_1 is a subset of R_2, Note 2.1.2 states that $\mathrm{Av}(\{\pi, m_1\}) \subseteq \mathrm{Av}(\{\pi, m_2\})$

Now we consider a permutation $\omega' \in \mathrm{Av}(\pi)$, containing an occurrence of m_1. Consider placing a point in the region corresponding to the box (a, b), regardless of where in this region we place the point by condition 2 of the Proposition we create an occurrence of π, therefore there can be no points in this region, which could have been represented in the mesh set R_1 by adding the box (a, b). Hence every occurrence of m_1 is in fact an occurrence of m_2, and we have that $\mathrm{Av}(\{\pi, m_2\}) \subseteq \mathrm{Av}(\{\pi, m_1\})$.

Taking both directions of the containment we can therefore draw the conclusion that $\mathrm{Av}(\{\pi, m_1\}) = \mathrm{Av}(\{\pi, m_2\})$. □

All coincidence classes of $\mathrm{Av}(\{321, (21, R)\})$ can be explained by application of Proposition 2.1.1. By experimentation we see that there are at least 29 coincidence classes, and all of these coincidences are explained by this Proposition.

This rule can be understood very in graphical form. In the pattern in Figure 2.1 we can gain shading in three boxes since if there is a point in any of these boxes we would get an occurrence of the dominating pattern 321.

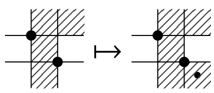
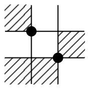
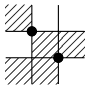


Figure 2.1: Visual depiction of first dominating pattern rule.

Note that there are two natural extensions of this rule. We can replace π with a set of classical patterns, or we can consider π to be a mesh pattern.

2.2 Equivalence classes of Av({231, (21, *R*)}).

By application of Proposition 2.1.1 we obtain 43 coincidence classes. Experimentation shows that there are in fact at least 39 coincidence classes, for example the following two patterns are coincident in Av(231) but this is not explained by Proposition 2.1.1.

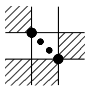
$m_1 =$  and $m_2 =$ 

Consider an occurrence of m_1 in a permutation in Av(231), consisting of elements x and y. If the region corresponding to the box $(1, 1)$ is empty we have an occurrence of m_2. Otherwise, if there is any increase in this box then we would have an occurrence of 231, however, since we are in Av(231) this is not possible. This box must therefore contain a (non-empty) decreasing subsequence. This gives rise to the following lemma:

Lemma 2.2.1. Given a mesh pattern $m = (\sigma, R)$, where the box (a, b) is not in R, and a dominating classical pattern $\pi = (\pi, \emptyset)$ if $\pi \preceq \texttt{add\_ascent}\,(\sigma, (a, b))$ ($\pi \preceq \texttt{add\_descent}\,(\sigma, (a, b))$) then in any occurrence of m in a permutation ϱ the region corresponding to the box (a, b) can only contain an increasing (decreasing) subsequence of ϱ.

The proof is analogous to the proof of Proposition 2.1.1.

Going back to our example mesh patterns



We know that the region corresponding to the box $(1, 1)$ contains a decreasing subsequence. If we let z be the topmost point in this decreasing subsequence, then xz is an occurrence of m_2. This shows that our two example patterns are coincident.

This result generalises into the following rule for categorising coincidences of mesh patterns in cases where there is a dominating classical pattern.

Proposition 2.2.2 (Second Dominating Pattern Rule)**.** Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \emptyset)$ such that $|\pi| \leq |\sigma| + 2$, the sets $\mathrm{Av}(\{\pi, m_1\})$ and $\mathrm{Av}(\{\pi, m_2\})$ are coincident if

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. a) $\pi \preceq \texttt{add\_ascent}\,(\sigma, (a, b))$ and
   - i. $(a + 1, b) \in \sigma$ and $(a + 1, b - 1) \notin R$ and
     $(x, b - 1) \in R \implies (x, b) \in R$ (where $x \neq a, a + 1$) and
     $(a + 1, y) \in R \implies (a, y) \in R$ (where $y \neq b - 1, b$).
   - ii. $(a, b + 1) \in \sigma$ and $(a - 1, b + 1) \notin R$ and
     $(x, b + 1) \in R \implies (x, b) \in R$ (where $x \neq a - 1, a$) and
     $(a - 1, y) \in R \implies (a, y) \in R$ (where $y \neq b, b + 1$).

   b) $\pi \preceq \texttt{add\_descent}\,(\sigma, (a, b))$ and
   - i. $(a + 1, b + 1) \in \sigma$ and $(a + 1, b + 1) \notin R$ and
     $(x, b + 1) \in R \implies (x, b) \in R$ (where $x \neq a, a + 1$) and
     $(a + 1, y) \in R \implies (a, y) \in R$ (where $y \neq b, b + 1$).
   - ii. $(a, b) \in \sigma$ and $(a - 1, b - 1) \notin R$ and
     $(x, b + 1) \in R \implies (x, b) \in R$ (where $x \neq a - 1, a$) and
     $(a - 1, y) \in R \implies (a, y) \in R$ (where $y \neq b - 1, b$).

Proof. We need to prove that $\mathrm{Av}(\{\pi, m_1\}) = \mathrm{Av}(\{\pi, m_2\})$.
Assume without meaningful loss of generality that $R_2 = R_1 \cup \{(a, b)\}$.

Consider a permutation ω that contains an occurrence of m_2. By Note 2.1.2 any of these occurrences is also an occurrence of m_1. This proves that every occurrence of m_2 is also an occurrence of m_1 and therefore $\mathrm{Av}(\{\pi, m_1\}) \subseteq \mathrm{Av}(\{\pi, m_2\})$.

We will consider taking the first branch of every choice. Now consider a permutation $\omega' \in \mathrm{Av}(\pi)$. Suppose ω' contains m_1 and consider the region corresponding to (a, b) in R_1.

If the region is empty, the occurrence of m_1 is trivially an occurrence of m_2.

Now consider if the region is non-empty, by Lemma 2.2.1 and condition 2a of the proposition this region must contain a decreasing subsequence. We can choose the topmost point in the region to replace the corresponding point in the mesh pattern and the points from the subsequence are now in the box southeast of the point. The other conditions allow this to be done without points being present in regions that were shaded. Hence there are no points in the region corresponding to the box (a, b) in the mesh pattern, and therefore we can shade this region. This implies that every occurrence of m_1 in $\mathrm{Av}(\pi)$ is in fact an occurrence of m_2 so $\mathrm{Av}(\{\pi, m_2\}) \subseteq \mathrm{Av}(\{\pi, m_1\})$.

Similar arguments satisfy the remainder of the branches. □

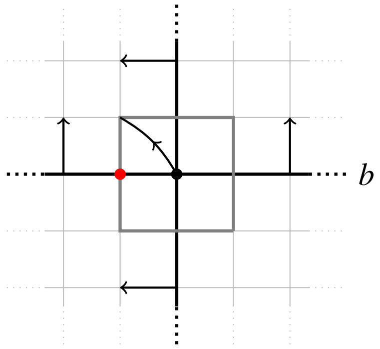


Figure 2.1: If the conditions of Proposition 2.2.2 are satisfied the box $(a - 1, b)$ can be shaded.

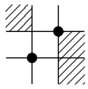
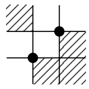
This proposition essentially states that we slide all of the points in the box we desire to shade diagonally, and chose the topmost/bottommost point to replace the original point in the mesh pattern.

By taking the First Dominating Pattern Rule and the Second Dominating Pattern Rule together coincidences of classes of the form $\mathrm{Av}(\{231, (21, R)\})$ are fully explained, obtaining 39 coincidence classes of mesh patterns.

2.3 Equivalence classes of Av({231, (12, *R*)}).

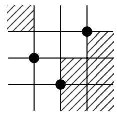
When considering the coincidence classes of $\mathrm{Av}(231, (12, R))$ we first apply the two Dominating Pattern rules previously established. Starting from 220 classes, application of the first Dominating Pattern rule gives 85 classes. Following this with the second Dominating Pattern rule reduces the number of classes to 59. However we know that there are patterns where the coincidences are not explained by the rules given above.

For example the patterns

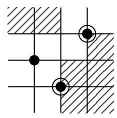
$m_1 =$  and $m_2 =$ 

are experimentally coincident. This coincidence is not explained by our rules, it is necessary to attempt to capture these coincidences by establishing more rules.

Consider an occurrence of m_1 in a permutation, if the region corresponding to the box $(1, 0)$ is empty then we have an occurrence of m_2. Now look at the case when this region is not empty. Consider choosing the rightmost point in region. This gives us an occurrence of the following mesh pattern.



By application of Proposition 2.1.1 we then achieve the following mesh pattern



If we look at the highlighted points we see that the subpattern is an occurrence of the mesh pattern that we originally desired. This gives rise to the following rule:

Proposition 2.3.1 (Third Dominating Pattern Rule)**.** Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \emptyset)$, the sets $\text{Av}(\{\pi, m_1\})$ and $\text{Av}(\{\pi, m_2\})$ are coincident if

1. $R_1 \triangle R_2 = \{(a, b)\}$
2. `add_point` $((\sigma, R_1), (a, b), D)$ where $D \in \{N, E, S, W\}$ is coincident with a mesh pattern containing an occurrence of (σ, R_2) as a subpattern.

Proof. We need to prove that $\text{Av}(\{\pi, m_1\}) = \text{Av}(\{\pi, m_2\})$.
Assume without meaningful loss of generality that $R_2 = R_1 \cup \{(a, b)\}$.

Consider a permutation ω that contains an occurrence of m_2. By Note 2.1.2, $\text{Av}(\{\pi, m_1\}) \subseteq \text{Av}(\{\pi, m_2\})$ as before.

Now consider a permutation ϱ in $\text{Av}(\pi)$ that contains an occurrence of m_1. If the region corresponding to the box (a, b) is empty then we have an occurrence of m_2. If the region is non-empty then by condition 2 of the proposition there exists a direction such that there exists an occurrence of a mesh pattern of length one longer than m_1 in this position. This mesh pattern is coincident with another mesh pattern that contains an occurrence of m_2. Hence, every occurrence of m_1 leads to an occurrence of m_2. Thus $\text{Av}(\{\pi, m_2\}) \subseteq \text{Av}(\{\pi, m_1\})$ and the two patterns are coincident. □

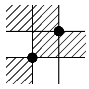
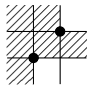
By application of this rule we can reduce the number of classes in $\text{Av}(\{231, (12, R)\})$ to 56.

2.4 Equivalence classes of Av({321, (12, *R*)}).

When considering coincidences of mesh patterns with underlying classical pattern 12 in $\text{Av}(321)$ application of the previously established rules give no coincidences. Through experimentation we discover that there are 7 non-trivial coincidence classes (all others are singletons) which can be explained through the use of two different lines of reasoning. Since the number of coincidences is so small we will reason for these coincidences without attempting to generalise into concrete rules.

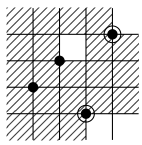
Intuitively it is easy to see why our previous rules have no power here. There is nowhere that it is possible to add a single point to gain an occurrence of $\pi = 321$. It is also impossible to have a position where addition of an increase, or decrease, provides extra shading power.

The patterns

$m_1 =$  and m_2 

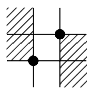
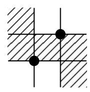
are equivalent in $\mathrm{Av}(321)$. (There are 3 symmetries of these patterns that are also equivalent to each other by the same reasoning.)

Consider the region corresponding to the box $(0, 1)$ in any occurrence of m_1, in a permutation. By Lemma 2.2.1 it must contain an increasing subsequence. If the region is empty then we have an occurrence of m_2. If there is only one point in the region we can choose this to replace the 1 in the mesh pattern to get the required shading. If there is more than one point then choosing the two leftmost points gives us the following mesh pattern.



Where the two highlighted points are the original two points. Now if we take the other two points as the points in our mesh pattern then we get an occurrence of the pattern we originally desired, and hence the two patterns are coincident.

The other reasoning applies to the patterns

$m_1 =$  and $m_2 =$ 

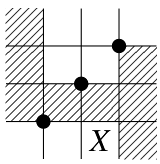
which are coincident by experimentation.

In order to prove this coincidence we will proceed by mathematical induction on the number of points in region corresponding to the middle box. We call this number n.

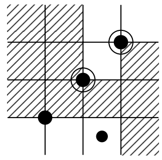
Base Case ($n = 0$)**:** The base case holds since we can freely shade the box if it contains no points.

Inductive Hypothesis ($n = k$)**:** Suppose that we can find an occurrence of the second pattern if we have an occurrence of the first with k points in the middle box.

Inductive Step ($n = k + 1$) Suppose that we have $(k + 1)$ points in the middle box. Choose the bottom most point in the middle box, this gives the mesh pattern



Now we need to consider the box labelled X. If this box is empty then we have an occurrence of m_2 and are done. If this box contains any points then we gain some extra shading on the mesh pattern due to the dominating pattern



The two highlighted points form an occurrence of m_1 with k points in the middle box, and thus by the Inductive Hypothesis we are done.

By induction we have that every occurrence of m_1 leads to an occurrence of m_2 and by Note 2.1.2 every occurrence of m_2 is an occurrence of m_1 so the two patterns are coincident. This argument applies to another two pairs of classes. Therefore in total in $\mathrm{Av}(321, (12, R))$ there are 213 coincidence classes.

Chapter 3

Wilf-equivalences under dominating patterns

Another question often asked in the field of permutation patterns is that of Wilf-equivalence. Two patterns π and σ are said to be Wilf-equivalent if their avoidance sets have the same size at each length. More formally:

Definition 3.0.1 (Wilf-equivalence)**.** Two patterns π and σ are said to be *Wilf-equivalent* if for all $k \geq 0$, $|\mathrm{Av}_k(\pi)| = |\mathrm{Av}_k(\sigma)|$. Two sets of permutation patterns R and S are are *Wilf-equivalent* if for all $k \geq 0$, $|\mathrm{Av}_k(R)| = |\mathrm{Av}_k(S)|$.

Wilf-equivalence is of interest since if two permutation classes are enumerated in the same way then there should exist a bijection between them, and therefore any other combinatorial object that they represent.

Coincident pattern classes are also Wilf-equivalent. This is due to the fact that if $\mathrm{Av}_k(S) = \mathrm{Av}_k(R)$ then obviously $|\mathrm{Av}_k(R)| = |\mathrm{Av}_k(S)|$. Coincidence is therefore a stronger equivalence condition than Wilf-equivalence.

There are a number of symmetries we can use when examining Wilf-equivalences to reduce the amount of work, it can be easily seen that the reverse, complement and inverse operations (see Figure 3.1) preserve enumeration, and therefore these classes are trivially Wilf-equivalent.

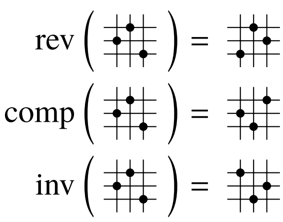


Figure 3.1: The operations reverse, complement and inverse for the pattern 231

The group of symmetries on permutations is isomorphic to the dihedral group of order 8, the group of symmetries of a square. *Reverse-inverse* and *reverse* correspond can be taken as generators of the dihedral group.

Since we are always considering Wilf-equivalences in the set $\mathrm{Av}(S)$ we must only use these symmetries when they preserve the dominating pattern, if we were to allow other symmetries, then the equivalences calculated in the previous section do not necessarily hold.

Throughout this section we will consider Wilf-equivalences of patterns whilst avoiding the *dominating pattern* 231. We will use C to denote $\mathrm{Av}(231)$ and $C(x)$ will be the usual Catalan generating function satisfying $C(x) = 1 + C(x)^2$. This is easy to see by structural decomposition around the maximum, as shown in Figure 3.2.

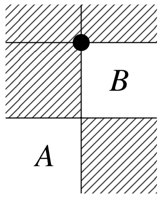


Figure 3.2: Structural decomposition of a non-empty avoider of 231

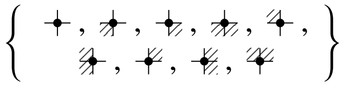
The elements to the left of the maximum, A, have the structure of a 231 avoiding permutation, and the elements to the right of the maximum, B, have the structure of a 231 avoiding permutation. Furthermore, all the elements in A lie below all of the elements in B. We call A the *lower-left section* and B the *upper-right section*.

We can also decompose a permutation avoiding 231 around the leftmost point, giving a similar figure.

3.1 Wilf-classes with patterns of length 1.

When considering the mesh patterns of length 2 it will be useful to know the Wilf-equivalence classes of the mesh patterns of length 1 inside $\mathrm{Av}(231)$, this means that we are considering the set $\mathrm{Av}(\{231, p\})$ where p is a mesh-pattern of length 1.

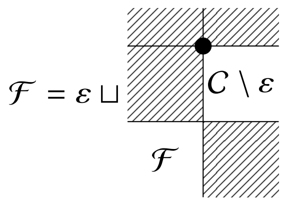
The patterns in the following set are set equivalent,



due to the fact that every permutation, except the empty permutation must contain an occurrence of all of these patterns.

The pattern  is in its own Wilf-class since the only permutation containing this pattern is the permutation 1. The avoiders of this pattern therefore have generating function $E(x) = C(x) - x$.

The pattern $p =$  is one of the quadrant marked mesh patterns studied by Kitaev, Remmel, and Tiefenbruck [11]. Alternatively we can enumerate avoiders by decomposing a non-empty avoider of p_1 around the maximum element in order to give the following structural decomposition.



Since if the upper-right section was empty the maximum would create an occurrence of the pattern, however no points in this section can create an occurrence since the maximum lies in a region corresponding to the shading in p. The lower-left section however can create occurrences of p and therefore must also avoid p, as well as 231. This gives the generating function of avoiders to be the function $F(x)$ satisfying.

$$F(x) = 1 + xF(x)(C(x) - 1)$$

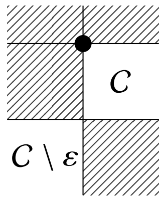
Solving for F gives

$$F(x) = \frac{1}{1 + x - xC(x)}$$

Evaluation of this generating function gives the Fine numbers (OEIS: A000957).

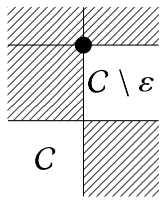
$$1, 0, 1, 2, 6, 18, 57, 186, 622, 2120, 7338, \ldots$$

It can be shown by use of Proposition 2.2.2 that the patterns  and q_1 =  are coincident. Consider the decomposition of a non-empty avoider of q_1 in $\mathrm{Av}(231)$ around the maximum:

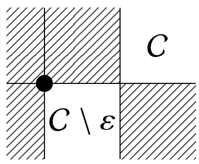
$\mathcal{G}_1 = \varepsilon \sqcup$ 

This can be explained succinctly by the fact that a permutation containing q_1 starts with it's maximum, by not allowing the lower-left section of the 231 avoider to be empty we prevent an occurrence from ever happening.

Consider q_2 = , avoiding this pattern means that a permutation does not end with it's maximum. We can perform a similar decomposition as before to get

$\mathcal{G}_2 = \varepsilon \sqcup$ 

Now consider q_3 = , the avoiders of this pattern are permutations that do not start with their minimum. In this case we perform the decomposition around the leftmost element

$\mathcal{G}_3 = \varepsilon \sqcup$ 

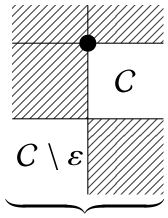
All of these classes have the same generating function, namely

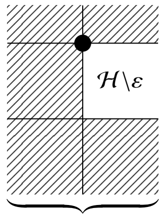
$$G(x) = 1 + xC(x)(C(x) - 1).$$

Enumeration of this generating function gives

$$1, 0, 1, 3, 9, 28, 90, 297, 1001, 3432, 11934, \ldots \qquad \text{(OEIS: A000245 with offset)}$$

There is one pattern of length 1 still to consider. The pattern r =  is avoided by all permutations that do not end in their minimum. Considering the standard decomposition of a 231 avoider around the maximum we can see that an avoider of r must fit into precisely on of the following two forms.

$\mathcal{H} = \varepsilon \sqcup$ 

$\sqcup$ 

Minimum comes before the maximum. Minimum is after the maximum, minimum cannot be last.

Therefore this particular class has generating function $H(x)$ satisfying

$$H(x) = 1 + xC(x)(C(x) - 1) + x(H(x) - 1)$$

This generating function is enumerated to give

$$1, 0, 1, 4, 13, 41, 131, 428, 1429, 4861, 16795, \ldots \qquad \text{(OEIS: A141364)}$$

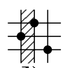
3.2 Wilf-classes with patterns of length 2

By use of the set equivalences from Chapter 2 we know there are at most 95 Wilf-equivalence classes.

In order to consider symmetries we must only take the symmetries that preserve the pattern 231. The only symmetry that preserves the pattern 231 is *reverse-complement-inverse*. Using this symmetry to reduces the number of Wilf-classes gives us 61 classes of trivial Wilf-equivalences, these Wilf-equivalences are explained by patterns being either coincident, or being the reverse-complement-inverse of a pattern.

Computing avoiders up to length 10 suggest that there are at least 23 Wilf-classes, of which 13 are non-trivial.

When considering explanations of Wilf-equivalences we consider how the permutations correspond to set-partitions.

Note 3.2.1. The avoiders of the pattern $q = (231, \{(1,0), (1,1), (1,2), (1,3)\})$,  , in $\mathfrak{S}_n$ are in one-to-one correspondence with partitions of $[\![n]\!]$. (Claesson [12, Prop. 2])

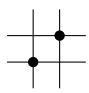
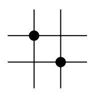
Proof. Let π be a permutation in $\mathrm{Av}_n(q)$ in one-line notation and insert a dash between each ascent in π. This corresponds to set partitions where the blocks are the elements between the dashes, the blocks are listed in increasing order of their least element, with the elements written in each block in descending order. □

Example 3.2.2. Given the permutation $\pi = 542139687$ this corresponds to the partition $\{\{5, 4, 2, 1\}, \{3\}, \{9, 6\}, \{7, 8\}\}$.

We call the least element in each block the *block bottom*

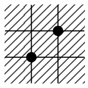
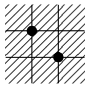
We are looking at permutations in $\mathrm{Av}(231)$, all of these permutations also avoid the mesh pattern in Note 3.2.1, *i.e.* $\mathrm{Av}(231) \subset \mathrm{Av}(q)$.

The classes containing following patterns are experimentally Wilf-equivalent up to length 10 in $\mathrm{Av}(231)$

 and 

This is true since the only avoiders of these patterns are the decreasing sequence and the increasing sequence respectively, and both of these avoid 231 in all cases. There is therefore 1 avoider at every length.

The following patterns are experimentally Wilf-equivalent up to length 10 in Av(231)

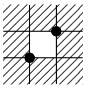
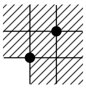
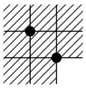
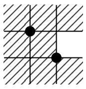
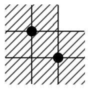
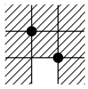
$m_1 =$  and $m_2 =$ 

It is obvious that these two are Wilf-equivalent since the only permutations that contain these patterns are 12 and 21 respectively, therefore the avoiders of these patterns are counted by the Catalan numbers at all lengths except for length 2 where there is precisely 1 avoider. Therefore the generating function is

$$I(x) = C(x) - x^2$$

3.2.1

The following patterns are experimentally Wilf-equivalent up to length 10 in Av(231)

, , , , , 

Consider containers of these patterns. For each of these patterns there is precisely one occurrence in any permutation containing the pattern. Now consider the points in the region corresponding to the unshaded box in each case. Each must contain an avoider of 231 that is of length $n-2$. Therefore these classes are all Wilf-equivalent and the number of length n avoiders is

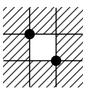
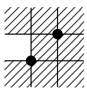
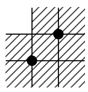
$$J_n = C_n - C_{n-2}$$

for $n \geq 2$ where C_n is the nth Catalan number, the number of 231 avoiders of length n. This gives the sequence

$$1, 1, 1, 4, 12, 37, 118, 387, 1298, 4433, 15366, \ldots \qquad (C_n \text{ - A001453 offset 2})$$

3.2.2

The following patterns are experimentally Wilf-equivalent up to length 10 in Av(231)

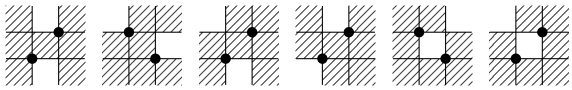
, , 

Consider containers of these patterns. Each of these patterns again occurs precisely once in any containing permutation. However this time when considering the region corresponding to the unshaded box we need to take into consideration Lemma 2.2.1 and so the empty box can only contain a decreasing subsequence. There is precisely one decreasing subsequence at every length, and so there is exactly one container of each pattern at each length. The three patterns are Wilf-equivalent and have $C_n - 1$ avoiders of length n for all $n \geq 2$. This gives the sequence

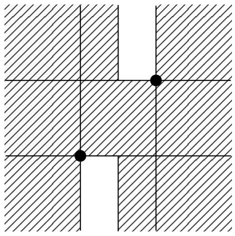
$$1, 1, 1, 4, 13, 41, 131, 428, 1429, 4861, 16795, \ldots \qquad \text{(OEIS: A001453 offset 2)}$$

3.2.3

The following patterns are experimentally Wilf-equivalent up to length 10 in Av(231)



The containers of the patterns have exactly one occurrence. Once again we consider the regions corresponding to the unshaded regions, For each pattern except the first the two regions are independent, and one contains any avoider of 231 and the other must contain a decreasing sequence by Lemma 2.2.1. Let us consider the first pattern separately. In order to avoid 231 across the regions corresponding to the unshaded boxes we can add some additional restrictions, *i.e.* all elements in the top region must be to the right of all elements in the bottom region.



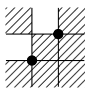
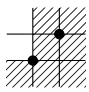
(3.2.1)

Now we can see that the region corresponding to the top free box must contain a decreasing sequence, and the bottom must contain an avoider of 231 and these two do not interact in any manner. The containers of this pattern are counted the same as the other patterns, and due to this they are Wilf-equivalent in Av(231). The containers have generating function $\frac{x^2C(x)}{(1-x)}$. Enumerating avoiders therefore gives us

$$1, 1, 1, 3, 10, 33, 109, 364, 1233, 4236, 14740, \ldots \qquad (C_n\text{– A014137 offset 2})$$

3.2.4

The following patterns are experimentally Wilf-equivalent up to length 10 in Av(231)

$m_1 =$  and $m_2 =$ 

In this case it is better to consider the containers of the patterns instead of the avoiders due to the amount of shadings in the mesh.

We look at the containers of the pattern m_1, there can only ever be one occurrence of this pattern in a permutation corresponding to the last point in the permutation and the minimum. Consider an occurrence of m_1, the points in the two regions corresponding to the the two boxes must form decreasing subsequences. For a permutation of length k if we fix the number of points in one of the boxes the number of points in the other box is determined. Therefore we can have any number of points from $\{0, \ldots, k-2\}$ points in the bottom box. Therefore there are $k-1$ containers of length k. These permutations correspond to set partitions of k points into exactly two non-overlapping parts partitioned by the first element and the minimum.

Now consider the containers of m_2, we know that the unshaded region must contain a decreasing subsequence, with the point corresponding to the 1 in the mesh pattern. This decreasing subsequence has $k-1$ points, we can put the point corresponding to the 2 above any of these points and therefore there are $k-1$ containers of length k.

Therefore these two patterns have been shown to have the same number of avoiders of length k for all k and are Wilf-equivalent. The avoiders have general form

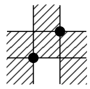
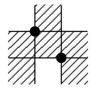
$$K_k = C_k - (k-1), K_0 = 1$$

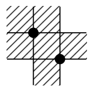
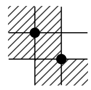
and have enumeration

$$1, 1, 1, 3, 11, 38, 127, 423, 1423, 4854, 16787, \ldots \qquad (C_n\text{– A000027 offset 2})$$

3.2.5

The following patterns are experimentally Wilf-equivalent up to length 10 in $\mathrm{Av}(231)$

$m_1 =$  and $m_2 =$  (3.2.2)

and $m_3 =$  and $m_4 =$  (3.2.3)

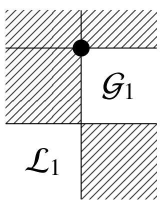
First we prove the Wilf-equivalence between m_1 and m_2 shown in (3.2.2). The easiest way to show that these are equinumerous is to consider the containers as set partitions.

Considering an occurrence of either of these patterns in a permutation we know the following about the points corresponding to the points in the patterns.

- The point corresponding to the first point in both patterns must lie in the first block of the set partition (there are no points southwest from it in the permutation).
- The point corresponding to the second point in both patterns is a block bottom (there are no points southeast of it in the permutation).
- If the region corresponding to box $(2, 2)$ in an occurrence of m_1 is empty, then the point corresponding to the second point is precisely the last block bottom. If the region corresponding to box $(0, 1)$ in an occurrence of m_2 is empty, then the point corresponding to the second point is precisely the first block bottom. If these regions are non-empty then the block containing the point corresponding to the second point in both patterns contains only the point (it is a singleton block).

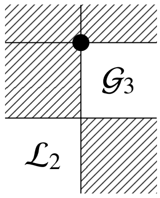
This tells us that an occurrence of the patterns must happen when there is a singleton block occurring after the first block. The difference between the patterns is in the underlying classical pattern. This means that permutations containing m_1 correspond to set partitions with a singleton block with value one higher than some element in the block containing 1. The permutations containing m_2 correspond to the set partitions containing a block with block bottom having value one lower than some element in the block containing 1 and if this block is not the block containing 1 then it is a singleton block. This proves that the containers of both of these patterns in $\mathrm{Av}(231)$ are equinumerous, and therefore so are their avoiders.

Consider an avoider of 231 and m_3. We can perform the decomposition around the maximum

$\mathcal{L}_1 = \varepsilon \sqcup$ 

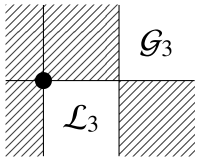
Since only the first point in the top right region can create an occurrence of m_3 if and only if it is the maximum in this region we must avoid starting with the maximum.

Looking at avoiders of 231 and m_4 we can perform a similar decomposition around the maximum to get

$\mathcal{L}_2 = \varepsilon \sqcup$ 

Since an occurrence of m_4 can never occur in the top right region, and could only occur between the first point in the region and the maximum, if and only if this first point is the minimum. Since both $\mathcal{G}_1$ and $\mathcal{G}_3$ have the same enumeration, $\mathcal{L}_1$ and $\mathcal{L}_2$ must also have the same enumeration and are therefore Wilf-equivalent.

Now we must consolidate these two subclasses. In order to do this we must consider the decomposition around the leftmost point of a permutation in $\text{Av}(231, m_1)$ we gain the following.

$\mathcal{L}_3 = \varepsilon \sqcup$ 

It is therefore obvious that avoiders of m_1 and avoiders of m_4 have the same enumeration, and therefore all four patterns are Wilf-equivalent in $\text{Av}(231)$ with generating function satisfying

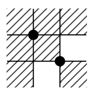
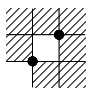
$$L(x) = 1 + xL(x)G(x)$$

This can be enumerated to give the sequence

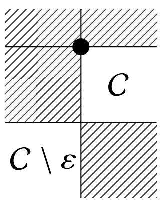
$$1, 1, 1, 2, 6, 19, 61, 200, 670, 2286, 7918, \ldots \qquad \text{(OEIS: A035929 offset 1)}$$

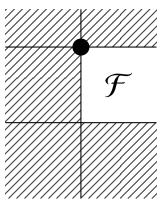
3.2.6

The following patterns are experimentally Wilf-equivalent up to length 10 in $\text{Av}(231)$

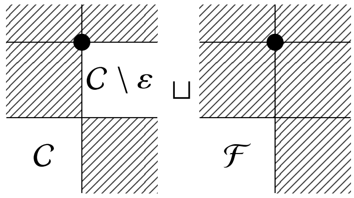
$m_1 =$  and $m_2 =$ 

First consider the structure of an avoider of m_1 in $\text{Av}(231)$. We can perform the usual structural decomposition of an avoider of 231 where we consider decomposition around the maximum. If $\mathcal{M}_1$ is the set $\text{Av}(231, m_1)$ then any permutation in $\mathcal{M}_1$ either starts with a maximum or does not, giving us the decomposition

$\mathcal{M}_1 = \varepsilon \sqcup$ 

$\sqcup$ 

Where $\mathcal{F} = \text{Av}((231,$  $))$. Now consider the decomposition around the maximum of a permutation in $\mathcal{M}_2 = \text{Av}(231, m_2)$, the permutation either ends with the maximum, or it does not, so we get

$\mathcal{M}_2 = \varepsilon \sqcup$ 

Therefore both of these sets of avoiders are enumerated in the same manner having generating function satisfying

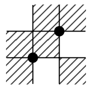
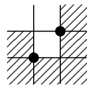
$$M(x) = 1 + xC(x)(C(x) - 1) + xF(x)$$

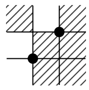
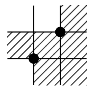
This generating function gives

$$1, 1, 1, 4, 11, 34, 108, 354, 1187, 4054, 14054, \ldots \qquad (C_n\text{−A000958 offset 2})$$

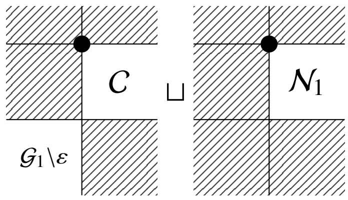
3.2.7

The following patterns are experimentally Wilf-equivalent up to length 10 in $\text{Av}(231)$

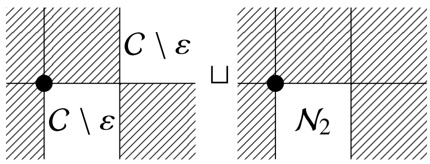
$m_1 =$  and $m_2 =$ 

and $m_3 =$  and $m_4 =$ 

First consider the decomposition of avoiders of m_1 in $\text{Av}(231)$ around the maximum. We have different conditions if we start with the maximum or not.

$\mathcal{N}_1 = \varepsilon \sqcup$ 

Where $\mathcal{G}_1 = \text{Av}((231,$  $))$. Now we decompose the avoiders of m_2 around the leftmost point, we have similar conditions for starting with the maximum

$\mathcal{N}_2 = \varepsilon \sqcup$ 

This gives us two generating functions satisfying the following pair of equations

$$N_1(x) = 1 + xC(x)(G(x) - 1) + xN_1(x) \tag{3.2.4}$$

$$\text{and } N_2(x) = 1 + x(C(x) - 1)^2 + xN_2(x) \tag{3.2.5}$$

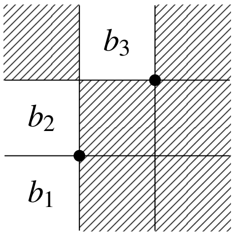
In order for these two functions to give the same value it is necessary to show that (3.2.4) and (3.2.5) are equal, this occurs if $C(x)(G(x) - 1) = (C(x) - 1)^2$.

$$\begin{aligned}
& (C(x) - 1)^2 = C(x)(G(x) - 1) & \\
\Leftrightarrow \quad & x^2C^4(x) = xC(x)(C(x) - 1)C(x) & \text{By definition of } G \text{ and } C \\
\Leftrightarrow \quad & x^2C^4(x) = xC^3(x) - xC^2(x) & \\
\Leftrightarrow \quad & xC^2(x) = C(x) - 1 & \text{Divide by } xC^2(x) \\
\Leftrightarrow \quad & C(x) = 1 + xC^2(x) &
\end{aligned}$$

The final line is always satisfied since it is the form of $C(x)$, and therefore the two generating functions are equal.

Now we look at the other patterns. In particular note that any container of these patterns can contain the pattern precisely once, m_2 specifies the minimum and last point, m_3 and m_4 both use the last point and the previous block bottom (in the set partition context).

Consider an occurrence of of m_3 in Av(231)



The regions b_2 and b_3 must contain a decreasing sequence by Lemma 2.2.1. The box labelled b_1 must contain an avoider of 231. However note that the points in this box can have interaction with any points in box b_2. If there is a point in b_2 then any points in b_1 to the left of this point must be lower than any points to the right of this point. By extension, if b_2 contains a decreasing sequence with k points, there are $k + 1$ non-interacting avoiders of 231 in b_1.

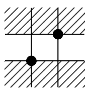
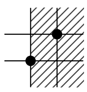
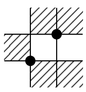
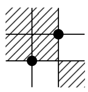
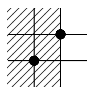
Now in m_2 and m_4 containers we can use the same method as in (3.2.1) to separate the two decreasing sequences in the free regions in the top row, and the mixing happens in the same manner as in a container of m_3. We now have that m_2, m_3 and m_4 have the same number of containers so are Wilf-equivalent, and that m_1 and m_2 have the same generating function so all four classes are Wilf-equivalent.

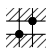
Evaluating the generating function $N(x)$ gives us the enumeration

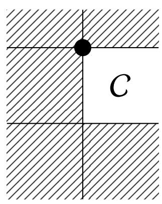
$$1, 1, 1, 2, 6, 20, 68, 233, 805, 2807, 9879, \ldots \qquad (C_n\text{– A014138 offset 1})$$

3.2.8

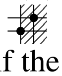
The following patterns are experimentally Wilf-equivalent up to length 10 in Av(231)

, , , , and 

If O_1 is the set of avoiders of , then by the structural decomposition around the maximum we have

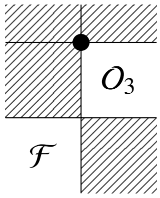
$O_1 = \varepsilon \sqcup$ 

The lower-left section is empty because the minimum must occur after the maximum. These are counted by $O(x) = 1 + xC(x)$

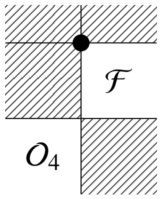
The pattern  occurs if the last element is higher than the penultimate element. This can only occur if the last element is in a single block in the set partition context, In order to construct an avoider of length n we can take any avoider of 231 of length $n - 1$ and insert the new maximum into the last block. This ensures that the last block is never a singleton. This means that these permutations are also counted by $O(x)$.

Considering the last pattern, the only way we can construct an avoider is to take any 231 avoider and add a new minimum at the start of the permutation. Adding a new leftmost point with any other value would either create an occurrence of 231 or the mesh pattern. Therefore these permutations are also counted by $O(x) = 1 + xC(x)$.

The avoiders of the third pattern can be decomposed by the maximum to give

$O_3 = \varepsilon \sqcup$ 

Where $\mathcal{F} = \mathrm{Av}((231,$ $))$. The generating function derived satisfies $O_3(x) = 1 + xF(x)O_3(x)$. The fourth pattern can be decomposed around the maximum in a similar manner.

$O_4 = \varepsilon \sqcup$ 

So clearly $O_4(x) = O_3(x)$. We need to show that the generating function $O_3(x)$ is the same as $O(x)$

$$\begin{aligned}
O_3(x) &= 1 + xF(x)O_3(x) \\
&= \frac{1}{1 - xF(x)} && \text{Solving for } O_3(x) \\
&= \frac{1}{1 - \frac{x}{1+x-xC(x)}} && \text{Substituting for } F(x) \\
&= \frac{1 - xC(x) + x}{1 - xC(x)} \\
&= 1 + xC(x)
\end{aligned}$$

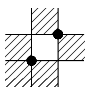
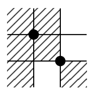
We have that $O_3(x) = 1 + xC(x) = O(x)$ so all four patterns are Wilf-equivalent and have enumeration sequence

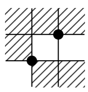
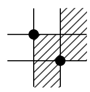
$$1, 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots \qquad \text{(OEIS: A000108 offset 1)}$$

This is an offset of the Catalan numbers.

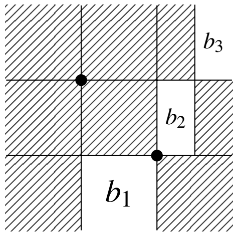
3.2.9

The following patterns are experimentally Wilf-equivalent up to length 10 in Av(231)

$m_1 =$  and $m_2 =$ 

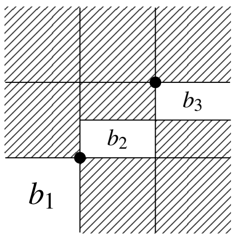
and $m_3 =$  and $m_4 =$ 

First we consider an occurrence of m_2 in a permutation in Av(231)



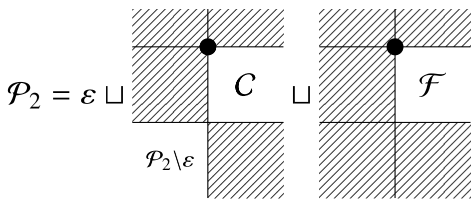
We can choose the lowest occurrence of m_2 in the sense that the region corresponding to b_1 must avoid the pattern as well as 231. The regions corresponding to b_2 and b_3 must now contain avoiders of 231, all points in the region corresponding to b_2 must be to the right of those in the region corresponding to b_3 231. Since we already have an occurrence of m_2 we do not need to care about creating more occurrences so there are no other conditions on these boxes.

Now looking at an occurrence of m_3 in $\pi \in$ Av(231)

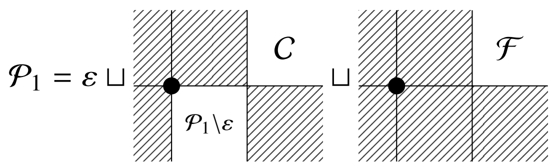


We consider the leftmost occurrence of m_3 in the sense that the region corresponding to b_1 must avoid the pattern as well as 231 once more. The regions corresponding to b_2 and b_3 must avoid 231 and as in a container of m_2 the points in the region corresponding to the box containing b_2 must be lower in value than all of those in the region corresponding to the box containing b_3, as doing so would lead to an occurrence of 231. Therefore both of these sets of containers are enumerated in the same way.

Now we find a structural decomposition for an avoider of m_2. Decomposing around the maximum we see the set of avoiders of m_2 have the form



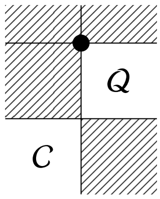
We can decompose an avoider of m_1 in Av(231) around the leftmost point in a similar manner:



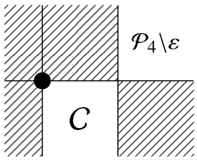
These two decompositions tell us that these two patterns are Wilf-equivalent and have generating function

$$P_1 = 1 + x(P_1(x) - 1)C(x) + xF(x) \tag{3.2.6}$$

Now consider an avoider of m_4 decomposed around the maximum

$\mathcal{P}_4 = \varepsilon \sqcup$ 

Here Q is the permutations avoiding 231, m_4 and $p =$  , since if the subsequence in this box were to start with the maximum then this point and the maximum would create an occurrence of m_4. Now consider decomposition of a permutation in Q around its leftmost point.

$Q = \varepsilon \sqcup$ 

This gives us the generating function

$$Q(x) = 1 + xC(x)(P_4(x) - 1)$$

Now we get the following for P_4

$$P_4(x) = 1 + xC(x)(xC(x)(P_4(x) - 1) + 1) \tag{3.2.7}$$

All that remains to show Wilf-equivalence is to show that equation (3.2.6) and equation (3.2.7) are the same generating function. First solve equation (3.2.7) for $P_4(x)$

$$\begin{aligned}
P_4(x) &= 1 + xC(x)(xC(x)(P_4(x) - 1) + 1) \\
&= 1 + x^2C^2(x)P_4(x) - x^2C^2(x) + xC(x) \\
&= 1 + \frac{xC(x)}{1 - x^2C^2(x)} \\
&= 1 + \frac{xC(x)}{(1 - xC(x))(1 + xC(x))} && \text{Difference of squares} \\
P_4(x) &= 1 + \frac{xC^2(x)}{1 + xC(x)} && C(x) = \frac{1}{1 - xC(x)}
\end{aligned} \tag{3.2.8}$$

Now we solve equation (3.2.6) for $P_1(x)$

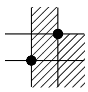
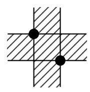
$$\begin{aligned}
P_1(x) &= 1 + x(P_1(x) - 1)C(x) + xF(x) \\
&= 1 + xP_1(x)C(x) - xC(x) + \frac{x}{1 + x - xC(x)} && \text{Substitution of } F(x) \\
P_1(x)(1 - xC(x)) &= \frac{x^2C^2(x) - (x^2 + 2x)C(x) + 2x + 1}{1 + x - xC(x)} \\
P_1(x) &= \frac{x^2C^2(x) - (x^2 + 2x)C(x) + x + 1 + x}{x^2C^2(x) - (x^2 + 2x)C(x) + x + 1} \\
&= 1 + \frac{x}{x^2C^2(x) - (x^2 + 2x)C(x) + x + 1} \\
&= 1 + \frac{x}{1 - x^2C(x) - xC(x)} && xC^2(x) = C(x) - 1 \\
&= 1 + \frac{xC^2(x)}{C^2(x) - xC^3(x)(x + 1)} \\
&= 1 + \frac{xC^2(x)}{C(x) - xC^2(x) + xC(x)} && xC^2(x) = C(x) - 1 \\
P_1(x) &= 1 + \frac{xC^2(x)}{1 + xC(x)} && C(x) = 1 + xC^2(x)
\end{aligned} \tag{3.2.9}$$

We have shown that P_1 and P_4 are indeed the same generating function, and we have that the classes containing these four patterns are Wilf-equivalent. Evaluating the generating function $P(x)$ gives

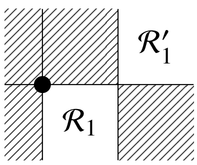
$$1, 1, 1, 3, 8, 24, 75, 243, 808, 2742, 9458, \ldots \qquad \text{(OEIS: A001453)}$$

3.2.10

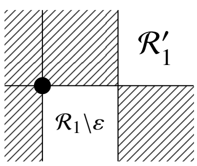
The following patterns are experimentally Wilf-equivalent up to length 10 in $\mathrm{Av}(231)$

$m_1 =$  and $m_2 =$ 

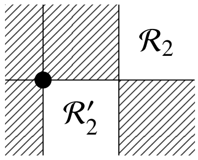
Let $\mathcal{R}_1$ be the set of avoiders of m_1 in $\mathrm{Av}(231)$. By structural decomposition around the leftmost point we have

$\mathcal{R}_1 = \varepsilon \sqcup$ 

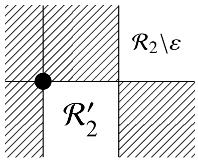
Where $\mathcal{R}_1'$ is a permutation avoiding 231, m_1 and  . Now consider the decomposition of a permutation in $\mathcal{R}_1'$. It can once again be decomposed around the leftmost point

$\mathcal{R}_1' = \varepsilon \sqcup$ 

This is a complete decomposition of avoiders of m_1. Now we look at an avoider of m_2, this time decomposition is around the maximum

$\mathcal{R}_2 = \varepsilon \sqcup$ 

Where $\mathcal{R}'_2$ is a permutation avoiding 231, m_2 and . Again we use the same method of decomposition of a permutation in $\mathcal{R}'_2$

$\mathcal{R}'_2 = \varepsilon \sqcup$ 

This gives us a generating function $R(x)$ satisfying

$$R(x) = 1 + xR(x)R'(x) \tag{3.2.10}$$

$$R'(x) = 1 + x(R(x) - 1)R'(x) \tag{3.2.11}$$

Solving equation (3.2.11) for $R'(x)$ and substituting into equation (3.2.10) gives us that the generating function for $R(x)$ satisfies

$$R(x) = xR^2(x) - x(R(x) - 1) + 1 \tag{3.2.12}$$

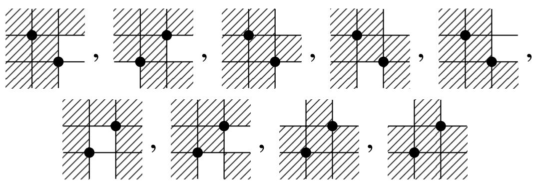
Evaluating $R(x)$ gives us the sequence

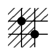
$$1, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, \ldots$$

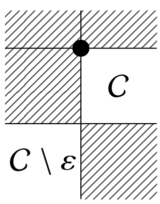
This is an offset of the Motzkin numbers (OEIS: A001006).

3.2.11

The following patterns are experimentally Wilf-equivalent up to length 10 in Av(231)



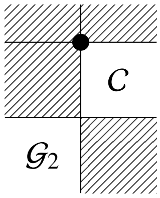
In order to gain enumeration, consider decomposition of avoiders of first pattern, , around the maximum.

$\mathcal{S}_1 = \varepsilon \sqcup \bullet \sqcup$ 

This gives us the following generating function

$$S(x) = 1 + x + xC(x)(C(x) - 1) \tag{3.2.13}$$

Now we consider decomposition of an avoider of the second pattern,  , in Av(231) around the maximum. This avoider has form

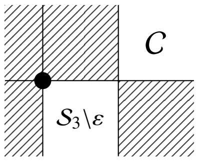
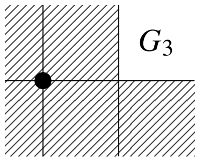
$\mathcal{S}_2 = \varepsilon \sqcup$ 

Where $\mathcal{G}_2 =$ Av$((231,$  $))$. This gives us the generating function

$$
\begin{aligned}
S_2(x) &= 1 + xC(x)G(x) \\
&= 1 + xC(x)(1 + xC(x)(C(x) - 1)) \\
&= 1 + xC(x)(C(x) - xC(x)) \qquad\qquad C(x) = 1 + xC^2(x) \\
&= 1 + x + xC^2(x) - xC(x) \\
&= 1 + x + xC(x)(C(x) - 1)
\end{aligned}
$$

Therefore this generating function is the same as equation (3.2.13). We can decompose  ,  ,  ,  , and  around the leftmost point into an avoider of one of the patterns with generating function $G(x)$ and an avoider of 231.

Now decompose an avoider of  around the leftmost point.

$\mathcal{S}_3 = \varepsilon \sqcup$  $\sqcup$ 

Where $\mathcal{G}_3 =$ Av$((231,$  $))$. This gives generating function $S_3(x)$ satisfying

$$
\begin{aligned}
S_3(x) &= 1 + xC(x)(S_3(x) - 1) + xG(x) \\
&= C(x) - xC^2(x) + xC(x) + x^2C^3(x) - x^2C^2 && \text{Solving for } S_3(x) \\
&= 1 + xC(x) + x^2C^3(x) - xC^2(x) && C = 1 + xC^2(x) \\
&= 1 + x + x^2C^3(x) && xC^2(x) = C(x) - 1 \\
&= 1 + x + xC(x)(C(x) - 1)
\end{aligned}
$$

This is equivalent to equation (3.2.13), and therefore these patterns are Wilf-equivalent. The classes have enumeration

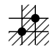
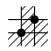
$$1, 1, 1, 3, 9, 28, 90, 297, 1001, 3432, 11934, \ldots \qquad \text{(OEIS: A071724 with offset)}$$

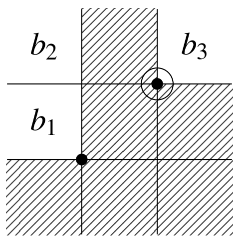
Chapter 4

Conclusions and Future work

From Chapter 2 it can be seen that automatically classifying coincidences of mesh patterns is a difficult task, establishing rules for longer dominating patterns requires many more cases to be taken. It would however be interesting to consider the application of the third Dominating Rule to mesh patterns without any dominating pattern in order to try to capture some of the coincidences described inHilmarsson, Jónsdóttir, Sigurðardóttir, *et al.* [13] and Claesson, Tenner, and Ulfarsson [10]. It is not possible to apply the first and second Dominating Rules to the pattern itself, since when applying the rules we consider containers of the pattern inside the avoiders of the dominating pattern. For example if we were to attempt to apply the first rule to the pattern 12 then we would have to consider containers of 12 inside Av(12), and obviously this can never occur.

It is also possible to take sets of mesh patterns instead of a single mesh pattern when considering dominating rules, and expressing coincidence between these sets. Doing this may give nice results. Coincidences between sets with multiple dominating patterns can also be considered, as this provides even more power to the rules discussed, these methods may be useful for classifying coincidence in large sets of pattern. It is also possible to apply these rules to sets where the dominating pattern is a mesh pattern.

It would be interesting to consider a systematic explanation of Wilf-equivalences amongst classes where 321 is the dominating pattern using the construction presented in [9, Sec. 12], in order to directly reach enumeration and hopefully establish some of the non-trivial Wilf-equivalences between classes with different dominating patterns. For example, it is possible to show that the sets $\mathcal{T} = \mathrm{Av}\big(\big\{$  $, 231\big\}\big)$ and $\mathcal{U} = \mathrm{Av}\big(\big\{$  $, 321\big\}\big)$, are Wilf-equivalent. This can be seen by considering an occurrence of the mesh pattern, $p =$  in a permutation



If we are inside Av(231) then any points in the region corresponding to the box b_1 must be to the right of the points in the region corresponding to box b_2, and must also form a decreasing subsequence by Lemma 2.2.1. Furthermore, the points in the regions corresponding to b_2 and b_3 must form an avoider of 231 with the indicated point. Therefore the containers of p in Av(231) have generating function $T(x) = \frac{x(C(x)-1)}{1-x}$. Now, if we are Av(321) then any points in the region corresponding to the box b_1 must be to the left of the points in the region corresponding to box b_2, and must also form a increasing subsequence by Lemma 2.2.1. Furthermore, the points in the regions corresponding to b_2 and b_3 must form an avoider of 321 with the indicated point. Hence, the containers of p in $\mathrm{Av}(321)$ have generating function $U(x) = \frac{x(C(x)-1)}{1-x}$ and both of these classes have the same enumeration.

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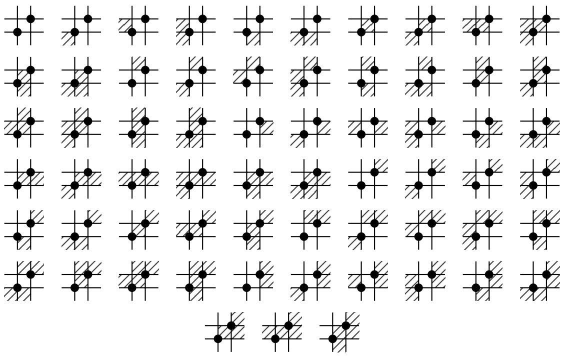
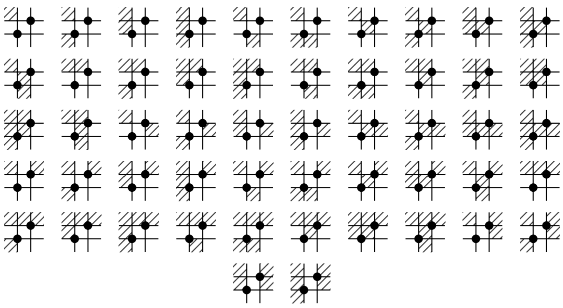
Appendix A

Equivalence classes of mesh patterns

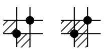
A.1 Coincidence classes with no dominating pattern



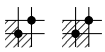
(A.1.1)



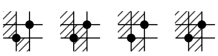
(A.1.2)



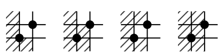
(A.1.3)



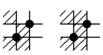
(A.1.4)



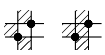
(A.1.5)



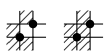
(A.1.6)



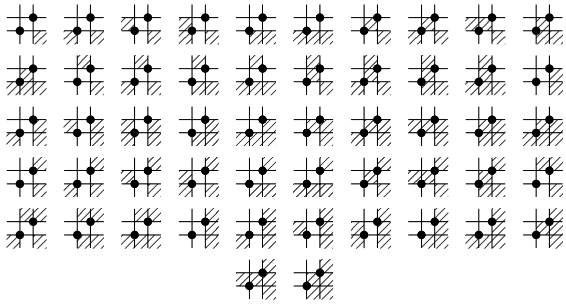
(A.1.7)



(A.1.8)



(A.1.9)



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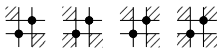

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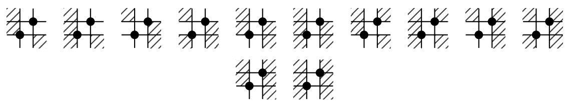

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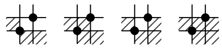

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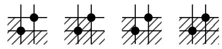

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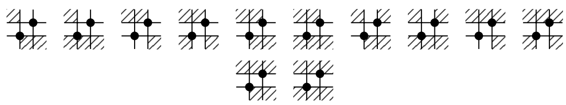

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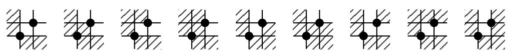

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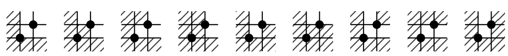

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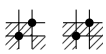

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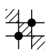

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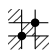

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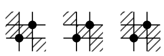

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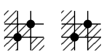

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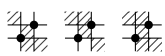

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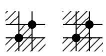

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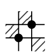

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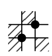

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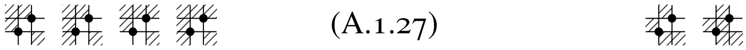
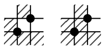
 (A.1.27) (A.1.28)

 (A.1.29)

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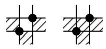
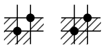
 (A.1.34)

 (A.1.35)

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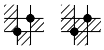
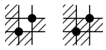
 (A.1.38)

 (A.1.39)

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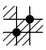
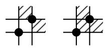
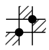
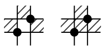
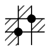
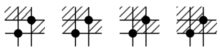
 (A.1.41)

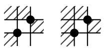
 (A.1.42)

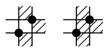
 (A.1.43)

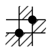
 (A.1.44)

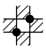
 (A.1.45)

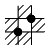
 (A.1.46)

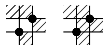
 (A.1.47)

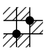
 (A.1.48)

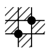
 (A.1.49)

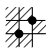
 (A.1.50)

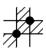
 (A.1.51)

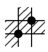
 (A.1.52)

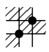
 (A.1.53)

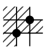
 (A.1.54)

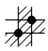
 (A.1.55)

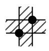
 (A.1.56)

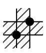
 (A.1.57)

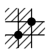
 (A.1.58)

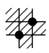
 (A.1.59)

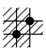
 (A.1.60)

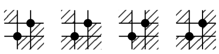
 (A.1.61)

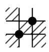
 (A.1.62)

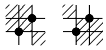
 (A.1.63)

 (A.1.64)

 (A.1.65)

 (A.1.66)

 (A.1.67)

 (A.1.68)

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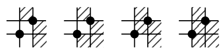
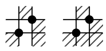
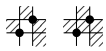
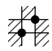
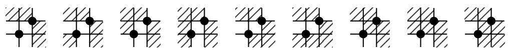
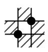
 (A.1.70)

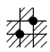
 (A.1.71)

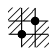
 (A.1.72)

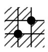
 (A.1.73)

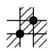
 (A.1.74)

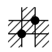
 (A.1.75)

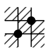
 (A.1.76)

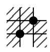
 (A.1.77)

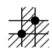
 (A.1.78)

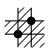
 (A.1.79)

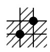
 (A.1.80)

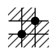
 (A.1.81)

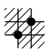
 (A.1.82)

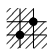
 (A.1.83)

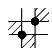
 (A.1.84)

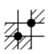
 (A.1.85)

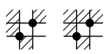
 (A.1.86)

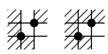
 (A.1.87)

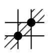
 (A.1.88)

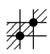
 (A.1.89)

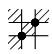
 (A.1.90)

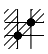
 (A.1.91)

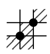
 (A.1.92)

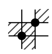
 (A.1.93)

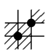
 (A.1.94)

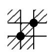
 (A.1.95)

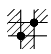
 (A.1.96)

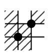
 (A.1.97)

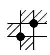
 (A.1.98)

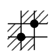
 (A.1.99)

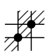
 (A.1.100)

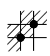
 (A.1.101)

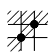
 (A.1.102)

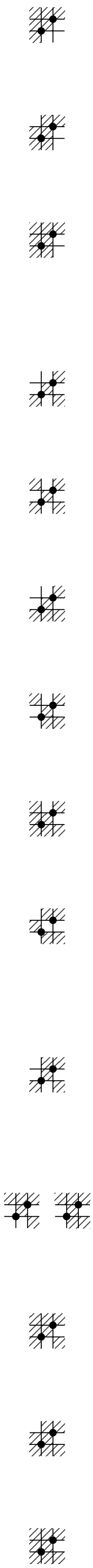
 (A.1.103)

 (A.1.104)

 (A.1.105)

 (A.1.106)

 (A.1.107)



(A.1.108)

(A.1.111)

(A.1.114)

(A.1.117)

(A.1.120)

(A.1.123)

(A.1.126)

(A.1.129)

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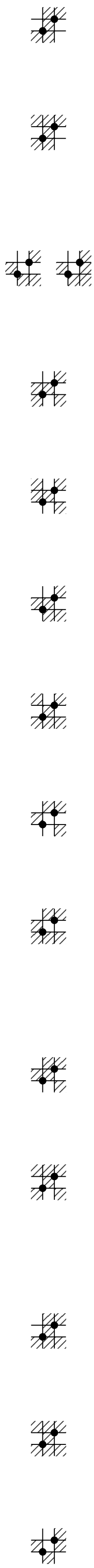
(A.1.135)

(A.1.138)

(A.1.141)

(A.1.144)

(A.1.147)



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(A.1.112)

(A.1.115)

(A.1.118)

(A.1.121)

(A.1.124)

(A.1.127)

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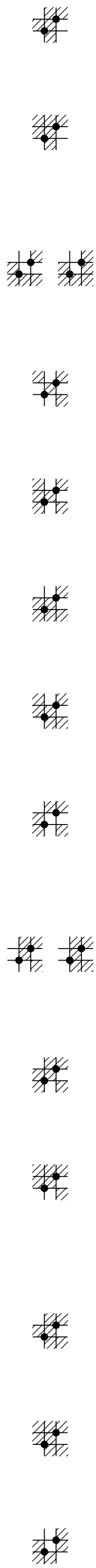
(A.1.142)

(A.1.145)

(A.1.148)



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(A.1.113)

(A.1.116)

(A.1.119)

(A.1.122)

(A.1.125)

(A.1.128)

(A.1.131)

(A.1.134)

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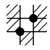
(A.1.143)

(A.1.146)

(A.1.149)

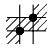
 (A.1.150)

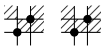
 (A.1.151)

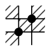
 (A.1.152)

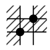
 (A.1.153)

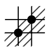
 (A.1.154)

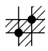
 (A.1.155)

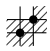
 (A.1.156)

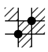
 (A.1.157)

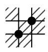
 (A.1.158)

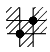
 (A.1.159)

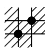
 (A.1.160)

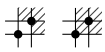
 (A.1.161)

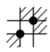
 (A.1.162)

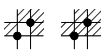
 (A.1.163)

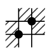
 (A.1.164)

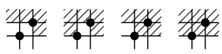
 (A.1.165)

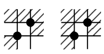
 (A.1.166)

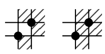
 (A.1.167)

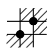
 (A.1.168)

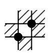
 (A.1.169)

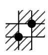
 (A.1.170)

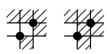
 (A.1.171)

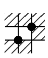
 (A.1.172)

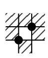
 (A.1.173)

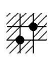
 (A.1.174)

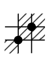
 (A.1.175)

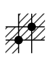
 (A.1.176)

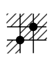
 (A.1.177)

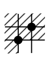
 (A.1.178)

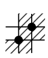
 (A.1.179)

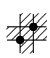
 (A.1.180)

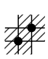
 (A.1.181)

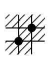
 (A.1.182)

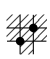
 (A.1.183)

 (A.1.184)

 (A.1.185)

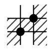
 (A.1.186)

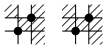
 (A.1.187)

 (A.1.188)

 (A.1.189)

 (A.1.190)

 (A.1.191)

 (A.1.192)

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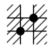 (A.1.194)

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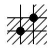
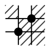
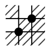
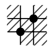
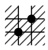
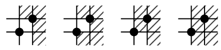
 (A.1.197)

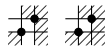
 (A.1.198)

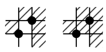
 (A.1.199)

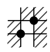
 (A.1.200)

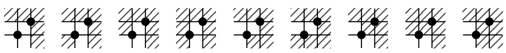
 (A.1.201)

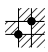
 (A.1.202)

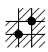
 (A.1.203)

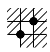
 (A.1.204)

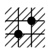
 (A.1.205)

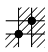
 (A.1.206)

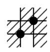
 (A.1.207)

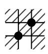
 (A.1.208)

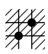
 (A.1.209)

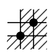
 (A.1.210)

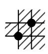
 (A.1.211)

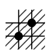
 (A.1.212)

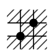
 (A.1.213)

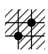
 (A.1.214)

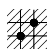
 (A.1.215)

 (A.1.216)

 (A.1.217)

 (A.1.218)

 (A.1.219)

 (A.1.220)

The classes obtained with underlying pattern 21 are obtained by calculating the reverse of each pattern in a class.

A.2 Consolidation of classes by Dominating Pattern rules

Each of the lines in the following tables are the sets of classes that are obtained by successive application of each of the Dominating Rules, only those coincidences that are not already calculated are shown.

A.2.1 First Dominating Rule

A.2.1.1 Dominating pattern 321

Mesh pattern family	
12	21
	1, 93, 94, 97, 105, 106, 109, 154, 155, 159
	2, 7, 95, 96, 100, 107, 108, 112, 156, 157, 158
	3, 4, 89, 90, 98, 99
	5, 6, 91, 92, 101, 102, 113, 114
	8, 9, 103, 104, 110, 111
	10, 18, 117, 118, 123, 134, 135, 142, 190, 191, 195
	11, 19, 20, 119, 120
	12, 21, 22, 67, 121, 122, 192, 193, 194
	13, 14, 115, 116, 124, 125, 196, 197
	15, 23, 24, 32, 126, 127, 138, 139, 145
	16, 17, 128, 129, 146, 147, 200, 201
	25, 26, 30, 31, 130, 131, 136, 137
	27, 33, 66, 68, 69, 140, 141, 198, 199
	28, 29, 132, 133, 143, 144
	34, 35, 148, 149, 160, 161
	36, 37, 40, 41, 150, 151, 162, 163
	38, 39, 152, 153, 164, 165
	42, 43, 56, 166, 167, 180
	44, 45, 57, 168, 169, 181
	46, 47, 58, 59, 170, 171, 182, 183
	48, 49, 60, 172, 173, 184
	50, 51, 61, 62, 174, 175, 185, 186
	52, 53, 63, 176, 177, 187
	54, 55, 64, 65, 178, 179, 188, 189
	70, 71, 79, 83, 202, 203, 211, 215
	72, 73, 80, 204, 205, 212
	74, 81, 82, 86, 206, 213, 214, 218
	75, 76, 84, 85, 207, 208, 216, 217
	77, 78, 87, 88, 209, 210, 219, 220

A.2.1.2 Dominating pattern 231

Mesh pattern family	
12	21
0, 9, 17, 24, 25, 29, 30, 129, 130, 133, 135	0, 41, 42, 43, 44, 47, 48, 55, 56, 59, 165, 166, 167, 168, 171, 172
1, 10, 11, 14, 18, 19, 20, 21, 22, 26, 31, 32, 39, 66, 67, 118, 120, 125, 137, 139	1, 35, 36, 39, 45, 46, 51, 52, 57, 58, 149, 150, 155, 161, 169, 170, 175, 176
2, 12, 33	2, 7, 33, 49, 60
3, 13, 34	3, 8, 34, 50, 61
4, 15, 37, 90, 151	4, 37, 53, 63
5, 16, 38, 91, 152	5, 38, 54, 64
6, 23	6, 40, 62
7, 27	9, 24, 25, 29, 69, 70, 71, 72, 78, 82, 129, 130, 133, 135, 201, 202, 203, 204
8, 28	10, 11, 14, 18, 20, 22, 26, 65, 67, 73, 118, 120, 125, 137, 139, 191, 197, 205
35, 36, 65, 149, 150	12, 27, 74, 83
40, 68	13, 28, 75, 84
41, 47, 69	15, 76, 86, 208
42, 48, 70	16, 77, 87, 209
43, 71	17, 30, 79
44, 72	19, 31, 66, 80
45, 46, 51, 52, 73	21, 32, 81
49, 74	23, 68, 85
50, 75	88, 102, 147, 173
53, 76	89, 103, 148, 174
54, 77	90, 151, 177
55, 78	91, 152, 178
56, 79	92, 104, 153, 179
57, 80	93, 105, 154, 180
58, 81	94, 106, 156, 181
59, 82	95, 107, 157, 182
60, 83	96, 108, 158, 183
61, 84	97, 109, 159, 184
62, 85	98, 110, 160, 185
63, 86	99, 111, 162, 186
64, 87	100, 112, 163, 187
88, 114, 147	101, 113, 164, 188
89, 115, 148	114, 131, 206
92, 116	115, 132, 207
93, 117	116, 134, 189, 210
94, 119	117, 136, 190, 211
95, 121	119, 138, 192, 212
96, 122	121, 140, 193, 213
97, 123	122, 141, 194, 214
98, 124	123, 142, 195, 215
99, 126	124, 143, 196, 216
100, 127	126, 144, 198, 217

101, 128	127, 145, 199, 218
102, 131	128, 146, 200, 219
103, 132	
104, 134	
105, 136	
106, 138	
107, 140	
108, 141	
109, 142	
110, 143	
111, 144	
112, 145	
113, 146	
153, 189	
154, 190	
155, 191	
156, 192	
157, 193	
158, 194	
159, 195	
160, 196	
161, 197	
162, 198	
163, 199	
164, 200	
165, 171, 201	
166, 172, 202	
167, 203	
168, 204	
169, 170, 175, 176, 205	
173, 206	
174, 207	
177, 208	
178, 209	
179, 210	
180, 211	
181, 212	
182, 213	
183, 214	
184, 215	
185, 216	
186, 217	
187, 218	
188, 219	

A.2.2 Second Dominating Rule

A.2.2.1 Dominating pattern 321

There are no new coincidences when the dominating pattern is 321 when applying the second dominating rule.

A.2.2.2 Dominating pattern 231

Mesh pattern family	
12	21
2, 7	1, 6, 155
3, 8	9, 17, 133
12, 27	10, 18, 19, 118
13, 28	11, 20, 21, 66, 120, 191
41, 43	14, 22, 23, 31, 125, 137
42, 44	15, 127, 145, 199
55, 56	25, 30
57, 58	26, 32, 65, 67, 68, 139, 197
69, 71	36, 40
70, 72	52, 62
78, 79	72, 79
80, 81	73, 80, 81, 85, 205
88, 102	76, 86, 208, 218
89, 103	88, 97, 102, 109, 147, 159
92, 93, 104, 105	90, 100, 112, 151, 163
94, 95, 106, 107	114, 123, 131, 142, 195
96, 108	173, 184
97, 109	177, 187
98, 110	206, 215
99, 111	
100, 112	
101, 113	
114, 131	
115, 132	
122, 141	
123, 142	
124, 143	
126, 144	
127, 145	
128, 146	
153, 154	
156, 157	
165, 167	
166, 168	
179, 180	
181, 182	
189, 190	
192, 193	

201, 203	
202, 204	
210, 211	
212, 213	

Appendix B

Wilf-equivalence data

B.1 Sequences with underlying pattern 231

Sequence	Related OEIS entry	Number of patterns in class
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1	A000012	210
1, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835	A001006	32
1, 1, 1, 2, 5, 13, 35, 97, 275, 794, 2327	A086581	8
1, 1, 1, 3, 6, 17, 43, 123, 343, 1004, 2938	A143363	2
1, 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862	A000108	314
1, 1, 1, 3, 7, 19, 53, 153, 453, 1367, 4191	A078481	2
1, 1, 1, 2, 6, 19, 61, 200, 670, 2286, 7918	A035929	32
1, 1, 1, 2, 6, 20, 68, 233, 805, 2807, 9879	A014138	36
1, 1, 1, 3, 8, 24, 75, 243, 808, 2742, 9458	A000958	64
1, 1, 1, 2, 7, 25, 85, 285, 964, 3310, 11527		4
1, 1, 1, 3, 9, 28, 90, 297, 1001, 3432, 11934	A000245	176
1, 1, 1, 4, 10, 31, 97, 316, 1054, 3586, 12394		2
1, 1, 1, 3, 9, 29, 95, 317, 1075, 3699, 12891		4
1, 1, 1, 3, 10, 31, 98, 321, 1078, 3686, 12789	A114487	4
1, 1, 1, 2, 7, 26, 93, 325, 1129, 3935, 13813	A014140	8
1, 1, 1, 4, 11, 33, 105, 343, 1148, 3916, 13563	A127154	2
1, 1, 1, 4, 11, 34, 108, 354, 1187, 4054, 14054	A000958	8
1, 1, 1, 3, 10, 33, 109, 364, 1233, 4236, 14740	A014137	38
1, 1, 1, 4, 12, 37, 118, 387, 1298, 4433, 15366	A00108	46
1, 1, 1, 2, 8, 32, 117, 408, 1402, 4826, 16751	A000217	2
1, 1, 1, 3, 11, 38, 127, 423, 1423, 4854, 16787	A000027	6
1, 1, 1, 4, 13, 41, 131, 428, 1429, 4861, 16795	A001453	18
1, 1, 1, 5, 14, 42, 132, 429, 1430, 4862, 16796	A000108	6

B.2 Sequences with underlying pattern 321

Sequence	Related OEIS entry	Number of patterns in class
1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0		63
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1	A000012	180
1, 1, 1, 2, 3, 4, 5, 6, 7, 8, 9	A000027	5

1, 1, 1, 1, 2, 3, 6, 11, 22, 44, 90	A007477	8
1, 1, 1, 2, 4, 8, 16, 32, 64, 128, 256	A000079	30
1, 1, 1, 1, 3, 6, 13, 28, 60, 129, 277	A002478	2
1, 1, 1, 2, 3, 9, 16, 48, 102, 289, 693		1
1, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835	A001006	17
1, 1, 1, 1, 3, 8, 21, 55, 144, 377, 987	A001906	4
1, 1, 1, 1, 3, 7, 19, 53, 153, 453, 1367	A078481	2
1, 1, 1, 1, 2, 5, 14, 42, 132, 429, 1430	A000108	12
1, 1, 1, 1, 3, 10, 30, 84, 227, 603, 1589		2
1, 1, 1, 2, 5, 13, 34, 89, 233, 610, 1597	A001519	8
1, 1, 1, 2, 3, 7, 19, 56, 174, 561, 1859	A167422	2
1, 1, 1, 2, 5, 13, 36, 103, 303, 910, 2779		8
1, 1, 1, 1, 3, 9, 28, 90, 297, 1001, 3432	A000245	8
1, 1, 1, 3, 6, 18, 47, 139, 405, 1225, 3740		2
1, 1, 1, 3, 7, 19, 53, 153, 453, 1367, 4191	A078481	2
1, 1, 1, 2, 4, 11, 34, 110, 365, 1234, 4237		4
1, 1, 1, 1, 3, 10, 33, 111, 379, 1312, 4596	A001558	2
1, 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862	A000108	170
1, 1, 1, 1, 4, 12, 39, 129, 436, 1498, 5218	A122920	4
1, 1, 1, 2, 5, 15, 48, 159, 538, 1850, 6446		8
1, 1, 1, 1, 4, 14, 48, 165, 572, 2002, 7072	A002057	6
1, 1, 1, 2, 6, 18, 57, 186, 622, 2120, 7338	A000957	4
1, 1, 1, 2, 6, 19, 61, 200, 670, 2286, 7918	A035929	32
1, 1, 1, 1, 5, 17, 57, 193, 662, 2299, 8073		4
1, 1, 1, 2, 6, 19, 61, 202, 683, 2349, 8191		2
1, 1, 1, 2, 6, 19, 62, 207, 704, 2431, 8502	A026012	8
1, 1, 1, 3, 8, 24, 75, 243, 808, 2742, 9458	A000958	18
1, 1, 1, 2, 7, 22, 71, 235, 794, 2728, 9503		2
1, 1, 1, 2, 6, 20, 68, 233, 805, 2807, 9879	A014138	12
1, 1, 1, 3, 8, 25, 80, 264, 890, 3053, 10622		4
1, 1, 1, 1, 4, 16, 63, 239, 880, 3184, 11431		2
1, 1, 1, 2, 8, 26, 85, 283, 959, 3300, 11505		4
1, 1, 1, 1, 5, 20, 74, 265, 937, 3304, 11678		4
1, 1, 1, 2, 6, 21, 75, 266, 938, 3305, 11679		8
1, 1, 1, 2, 7, 25, 86, 292, 995, 3425, 11926		4
1, 1, 1, 3, 9, 28, 90, 297, 1001, 3432, 11934	A000245	86
1, 1, 1, 4, 10, 31, 97, 316, 1054, 3586, 12394		1
1, 1, 1, 3, 10, 31, 98, 321, 1078, 3686, 12789	A114487	8
1, 1, 1, 4, 11, 33, 105, 343, 1148, 3916, 13563	A127154	1
1, 1, 1, 2, 7, 26, 93, 325, 1129, 3935, 13813	A014140	16
1, 1, 1, 4, 11, 34, 108, 354, 1187, 4054, 14054	A000958	2
1, 1, 1, 3, 9, 31, 105, 355, 1210, 4171, 14543		2
1, 1, 1, 3, 10, 33, 109, 364, 1233, 4236, 14740	A014137	36
1, 1, 1, 2, 9, 33, 113, 381, 1291, 4425, 15357	A192480	1
1, 1, 1, 3, 10, 34, 114, 382, 1292, 4426, 15358		2
1, 1, 1, 4, 12, 37, 118, 387, 1298, 4433, 15366	A000108	19
1, 1, 1, 3, 9, 30, 104, 365, 1286, 4542, 16092	A045623	8
1, 1, 1, 1, 6, 22, 91, 349, 1277, 4570, 16235		1

1, 1, 1, 2, 6, 25, 96, 357, 1289, 4587, 16258		2
1, 1, 1, 3, 7, 28, 101, 365, 1301, 4604, 16281		1
1, 1, 1, 1, 7, 25, 102, 377, 1339, 4699, 16496		4
1, 1, 1, 2, 7, 28, 106, 382, 1345, 4706, 16504	A132109	4
1, 1, 1, 1, 8, 28, 108, 387, 1354, 4720, 16524		2
1, 1, 1, 2, 7, 29, 109, 388, 1355, 4721, 16525	A081494	4
1, 1, 1, 2, 8, 31, 112, 392, 1360, 4727, 16532	A006127	4
1, 1, 1, 3, 9, 32, 113, 393, 1361, 4728, 16533	A132736	4
1, 1, 1, 3, 10, 34, 116, 397, 1366, 4734, 16540	A000079	50
1, 1, 1, 1, 8, 31, 116, 407, 1401, 4825, 16750		2
1, 1, 1, 2, 8, 32, 117, 408, 1402, 4826, 16751	A000217	2
1, 1, 1, 2, 9, 34, 122, 417, 1416, 4846, 16778	A003265	2
1, 1, 1, 2, 10, 37, 126, 422, 1422, 4853, 16786		2
1, 1, 1, 3, 10, 37, 126, 422, 1422, 4853, 16786		4
1, 1, 1, 2, 11, 37, 126, 422, 1422, 4853, 16786		4
1, 1, 1, 3, 11, 38, 127, 423, 1423, 4854, 16787	A000027	22
1, 1, 1, 3, 12, 40, 130, 427, 1428, 4860, 16794		4
1, 1, 1, 4, 12, 40, 130, 427, 1428, 4860, 16794		2
1, 1, 1, 3, 13, 40, 130, 427, 1428, 4860, 16794		1
1, 1, 1, 4, 13, 41, 131, 428, 1429, 4861, 16795	A001453	54
1, 1, 1, 5, 14, 42, 132, 429, 1430, 4862, 16796	A000108	9

Appendix C

Code

The exploratory work for this project was done mostly in the python programming language. The code can be obtained from `https://github.com/MurrayT/PRule-MSc-work/tree/master/src` and is split into the following files.

The code relies on the the permuta python package available at `https://github.com/PermutaTriangle/Permuta`

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