EQUIVALENCE CLASSES OF MESH PATTERNS WITH A DOMINATING PATTERN

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ABSTRACT. Two mesh patterns are coincident if they are avoided by the same set of permutations, and are Wilf-equivalent if they have the same number of avoiders at each length. We provide sufficient conditions for coincidence among mesh patterns, whilst also avoiding a longer classical pattern. Using these conditions we completely classify coincidences between families containing a mesh pattern of length 2 and a classical pattern of length 3. Furthermore, we completely Wilf-classify equivalences of mesh patterns of length 2 whilst also avoiding the classical pattern 231.

Keywords: permutation, pattern, mesh pattern, pattern coincidence

1. INTRODUCTION

The study of permutation patterns began as a result of Knuth's statements on stack sorting in *The Art of Computer Programming*[7, p. 243, Ex. 5,6]. This original concept—a subsequence of symbols having a particular relative order, now known as classical patterns—has been expanded to a variety of definitions. Mesh patterns Babson and Steingrímsson [1] considered *vincular* patterns (also known as *generalised* or *dashed* patterns) where two adjacent entries in the pattern must also be adjacent in the permutation. Bousquet-Mélou, Claesson, Dukes, *et al.* [3] look at classes of pattern where both columns and rows can be shaded, these are called *bivincular* patterns. *Bruhat-restricted* patterns were studied by Woo and Yong [8] in order to establish necessary conditions for a Schubert variety to be Gorenstein. All of these definitions are subsumed under the definition of *mesh patterns*, introduced by Brändén and Claesson [4] to capture explicit expansions for certain permutation statistics.

When considering permutation patterns some of the main questions posed relate to how and when a pattern is avoided by, or contained in, a arbitrary set of permutations. Two patterns π and σ are *Wilf-equivalent* if the number of permutations that avoid π of length n is equal to the number of permutations that avoid σ of length n. A stronger equivalence condition is that of *coincidence*, where the set of permutations avoiding π is exactly equal to the set of permutations avoiding σ. Avoiding pairs of patterns of the same length with certain properties has been studied previously, Claesson

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and Mansour [5] considered avoiding a pair of vincular patterns of length 3. Bean, Claesson, and Ulfarsson [2] study avoiding a vincular and a covincular pattern simultaneously in order to achieve some interesting counting results. However, very little work has been done on avoiding a mesh pattern and a classical pattern simultaneously.

In this work we aim to establish some ground in this field by computing coincidences and Wilf-classes and calculating some of the enumerations of avoiders of a mesh pattern of length 2 and a classical pattern of length 3. We begin by establishing coincidences between mesh patterns of length 2 while avoiding a classical pattern of length 3, this is used to establish sufficient conditions for coincidence. We then establish Wilf-equivalence classes of these coincidence classes who avoid the classical pattern 231.

2. Mesh patterns

A *permutation* is a bijection from the set $[\![n]\!] = \{1, \ldots, n\}$ to itself. The set of all such bijections on this set if denoted $\mathfrak{S}_n$ and has size $n!$. We can denote an individual permutation $\pi \in \mathfrak{S}_n$ in *one-line notation* by writing the entries of the permutation in order, therefore $\pi = \pi(1)\pi(2)\ldots\pi(n)$. The set $\mathfrak{S}_n$ has exactly one element, the empty permutation ε.

Definition 2.1. (Order isomorphism.) Two substrings $\alpha_1\alpha_2\cdots\alpha_n$ and $\beta_1\beta_2\cdots\beta_n$ are said to be *order isomorphic* if they share the same relative order, *i.e.*, $\alpha_r < \alpha_s$ if and only if $\beta_r < \beta_s$.

The definition of order isomorphism allows us to give the meaning of containment for classical permutation patterns.

Definition 2.2. A permutation $\pi \in \mathfrak{S}_n$ contains the pattern $\sigma \in \mathfrak{S}_k$ (denoted $\sigma \preceq \pi$) if there is some subsequence of indices of $\pi, i_1 i_2 \cdots i_k$ such that the sequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is order isomorphic to $\sigma(1)\sigma(2)\cdots\sigma(k)$. If π does not contain σ, we say that π *avoids* σ.

Example 2.3. The permutation $\pi = 24153$ contains the pattern $\sigma = 231$, since the second, fourth and fifth elements (453) are order isomorphic to 231, it also contains the occurrence 241. The permutation 24153 avoids the pattern 321.

We denote the set of permutations of length n avoiding a pattern σ as $\mathrm{Av}_n(\sigma)$ and $\mathrm{Av}(\sigma) = \bigcup_{i=0}^{\infty} \mathrm{Av}_i(\sigma)$.

We can display a permutation graphically in a *plot*, in such a plot we display the points $G(\pi) = \{(i, \pi(i)) \mid i \in [1, n]\}$ in a Cartesian coordinate system. The plots of the permutations $\pi = 24153$ and $\sigma = 231$ can be seen in Figure 2.1. Figure 2.2 shows the containment of σ in π as in Example 2.3.

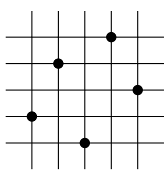
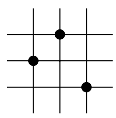
$G(\pi) =$  $G(\sigma) =$ 

Figure 2.1. The plots of the permutations π and σ.

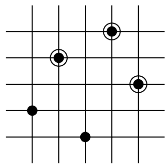


FIGURE 2.2. The occurrence of 231 in 24153 corresponding to 453.

Definition 2.4. A *mesh pattern* is a pair

$$p = (\tau, R) \text{ with } \tau \in \mathfrak{S}_k \text{ and } R \subseteq [0,k] \times [0,k].$$

We define containment (denoted $p \preceq \pi$), and avoidance, of the pattern p in the permutation π on mesh patterns analogously to classical containment, and avoidance, of τ in π with the additional restrictions on the relative position of the occurrence of τ in π. These restrictions say that no elements of π are allowed in the regions of the plot corresponding to shaded boxes in the mesh. These boxes are denoted by $\ulcorner i, j \urcorner$, where the point (i, j) is the lower left corner of the box.

Formally defined by Brändén and Claesson [4], an *occurrence* of p in π is a subset ω of the plot of $\pi, G(\pi) = \{(i, \pi(i) \mid i \in [1, n]\}$ such that there are order-preserving injections $\alpha, \beta : [1, k] \mapsto [1, n]$ satisfying the following two conditions.
Firstly, ω is an occurrence of π in the classical sense

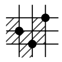
i. $\omega = \{(\alpha(i), \beta(j)) : (i, j) \in G(\tau)\}$

Define $R_{ij} = [\alpha(i) + 1, \alpha(i + 1) - 1] \times [\beta(j) + 1, \beta(j + 1) - 1]$ for $i, j \in [0, k]$ where $\alpha(0) = \beta(0) = 0$ and $\alpha(k + 1) = \beta(k + 1) = n + 1$. Then the second condition is

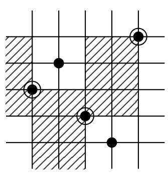
ii. if $\ulcorner i, j \urcorner \in R$ then $R_{ij} \cap G(\pi) = \varnothing$

We call R_{ij} the region corresponding to $\ulcorner i, j \urcorner$. We define containment of a mesh pattern p in another mesh pattern q as above, with the additional condition that if $\ulcorner i, j \urcorner \in R$ then R_{ij} is contained in the mesh set of q, in this case we call p a *subpattern* of q.

Definition 2.5. A mesh pattern $q = (\kappa, T)$ *contains* a mesh pattern $p = (\tau, R)$ as a *subpattern* if κ contains τ and $\left(\bigcup_{\ulcorner i,j \urcorner \in R} R_{ij}\right) \subseteq T$.

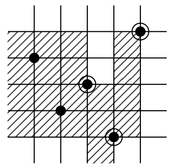
Example 2.6. The pattern $p = (213, \{(0, 1), (0, 2), (1, 0), (1, 1), (2, 1), (2, 2)\}) =$  is contained in $\pi = 34215$ but is not contained in $\sigma = 42315$.

Let us consider the plot for the permutation π. The subsequence 325 is an occurrence of 213 in the classical sense and the remaining points of π are not contained in the regions corresponding to the shaded boxes in p.



The subsequence 325 is therefore an occurrence of the pattern p in π and π contains p.

Now we consider the plot for the permutation σ. This permutation avoids p since for every occurrence of the classical pattern 213 there is at least one point in one of the shaded boxes. For example, consider the subsequence 315 in σ, this is an occurrence of 213 but not the mesh pattern since the points with values 4 and 2 are in the regions corresponding to the boxes $\boxed{0,1}$ and $\boxed{0,2}$, which are shaded in p. This is shown in the plot below.



This is true for all occurrences of 213 in σ and therefore σ avoids p.

We denote the avoidance sets for mesh patterns in the same way as for classical patterns. Given a mesh pattern $p = (\sigma, R)$ we say that σ is the *underlying classical pattern* of p.

3. Coincidences between Mesh Patterns

Coincidences among small mesh patterns have previously been considered by Claesson, Tenner, and Ulfarsson [6], in which the authors use the Simultaneous Shading Lemma, a closure result and one worked out special case to fully classify coincidences among mesh patterns of length 2.

Two patterns λ and γ are considered *coincident* if the set of permutations that avoid λ is the same as the set of permutations that avoid γ, *i.e.* $\mathrm{Av}(\lambda) = \mathrm{Av}(\gamma)$. Equivalently we can say that they have the same set of *containers*,*i.e.* $\mathrm{Cont}(\lambda) = \mathrm{Cont}(\gamma)$.

We will consider the avoidance sets $\mathrm{Av}(\pi, p)$ where π is a classical pattern of length 3 and p is a mesh pattern of length 2 in order to establish sufficient conditions for two such sets to be coincident. We will fix π in order to define these coincidences and say that π is the *dominating pattern.*

In order to describe the rules it is useful to have a notion for inserting points, ascents, and descents into a mesh pattern.

Definition 3.1. Let $p = (\tau, R)$ be a mesh pattern of length n such that $\boxed{i,j} \notin R$. We define a mesh pattern $p^{\boxed{i,j}} = (\tau', R')$ of length $n+1$ as the pattern where a point is *inserted* into the box $\boxed{i,j}$ in $G(p)$. Formally the new underlying classical pattern is defined by

$$\tau'(k) = \begin{cases} j+1 & \text{if } k = i+1 \\ \tau(k) & \text{if } \tau(k) \leqslant j \text{ and } k \leqslant i \\ \tau(k)+1 & \text{if } \tau(k) > j \text{ and } k \leqslant i \\ \tau(k-1) & \text{if } \tau(k) \leqslant j \text{ and } k > i+1 \\ \tau(k-1)+1 & \text{if } \tau(k) > j \text{ and } k > i+1 \end{cases}$$

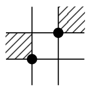
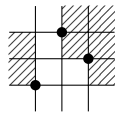
$p =$  $p^{\overrightarrow{\ulcorner 2,1 \lrcorner}\uparrow} =$ 

FIGURE 3.1. The result of inserting a point into $p = (12, \{(0,1),(2,2)\})$

While the mesh becomes

$$\begin{aligned} R' = &\{\ulcorner k,\ell \lrcorner \mid k \leqslant i, \ell \leqslant j, \ulcorner k,\ell \lrcorner \in R\} \cup \\ &\{\ulcorner k,\ell \lrcorner \mid k \leqslant i, \ell > j, \ulcorner k,\ell-1 \lrcorner \in R\} \cup \\ &\{\ulcorner k,\ell \lrcorner \mid k > i, \ell \leqslant j, \ulcorner k-1,\ell \lrcorner \in R\} \cup \\ &\{\ulcorner k,\ell \lrcorner \mid k > i, \ell > j, \ulcorner k-1,\ell-1 \lrcorner \in R\} \end{aligned}$$

In addition, we give the following definitions:

Definition 3.2. Let $p = (\tau, R)$ be a mesh pattern of length n such that $\ulcorner i,j \lrcorner \notin R$ and $p^{\ulcorner i,j \lrcorner} = (\tau', R')$ to be as defined in Definition 3.1. We define the following five modifications of the mesh patterns of the same length as $p^{\ulcorner i,j \lrcorner}$.

$$\begin{aligned} p^{\ulcorner i,j \lrcorner\uparrow} &= (\tau', R' \cup \{\ulcorner i,j+1 \lrcorner, \ulcorner i+1,j+1 \lrcorner\}) \\ p^{\overrightarrow{\ulcorner i,j \lrcorner}} &= (\tau', R' \cup \{\ulcorner i+1,j \lrcorner, \ulcorner i+1,j+1 \lrcorner\}) \\ p^{\ulcorner i,j \lrcorner\downarrow} &= (\tau', R' \cup \{\ulcorner i,j \lrcorner, \ulcorner i+1,j \lrcorner\}) \\ p^{\overleftarrow{\ulcorner i,j \lrcorner}} &= (\tau', R' \cup \{\ulcorner i,j \lrcorner, \ulcorner i,j+1 \lrcorner\}) \end{aligned}$$

Informally, these are considering the topmost, rightmost, leftmost, or bottommost point in $\ulcorner i,j \lrcorner$. We allow composition of these modifications, and collect the resulting mesh patterns in a set

$$p^{\ulcorner i,j \lrcorner\star} = \left\{p^{\ulcorner i,j \lrcorner}, p^{\overrightarrow{\ulcorner i,j \lrcorner}}, p^{\overrightarrow{\ulcorner i,j \lrcorner}\uparrow}, p^{\overrightarrow{\ulcorner i,j \lrcorner}\downarrow}, p^{\overleftarrow{\ulcorner i,j \lrcorner}}, p^{\overleftarrow{\ulcorner i,j \lrcorner}\uparrow}, p^{\overleftarrow{\ulcorner i,j \lrcorner}\downarrow}, p^{\ulcorner i,j \lrcorner\uparrow}, p^{\ulcorner i,j \lrcorner\downarrow}\right\}$$

See Figure 3.1 for an example of adding a point into a mesh pattern.

Definition 3.3. Let $p = (\tau, R)$ be a mesh pattern of length n such that $\ulcorner i,j \lrcorner \notin R$. We define a mesh pattern $p^{\ulcorner i,j \lrcorner_a} = (\tau', R')$ $(p^{\ulcorner i,j \lrcorner_d})$ of length $n+2$ as the pattern where an ascent (descent) is *inserted* into the box $\ulcorner i,j \lrcorner$ in $G(p)$. Formally the new underlying classical pattern is defined by

$$\tau'(k) = \begin{cases} j+t & \text{if } k = i+t, t \in \{1,2\} \\ \tau(k) & \text{if } \tau(k) \leqslant j \text{ and } k \leqslant i \\ \tau(k)+2 & \text{if } \tau(k) > j \text{ and } k \leqslant i \\ \tau(k-2) & \text{if } \tau(k) \leqslant j \text{ and } k > i+2 \\ \tau(k-2)+2 & \text{if } \tau(k) > j \text{ and } k > i+2 \end{cases}$$

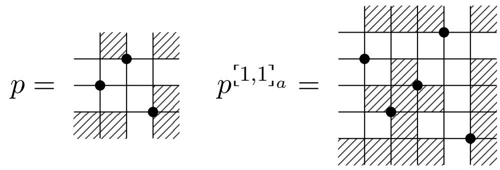


FIGURE 3.2. The result of inserting an ascent into $p = (231, \{(0,0),(1,0),(1,3),(3,0),(3,1),(3,3)\})$

The ordering of the top branch determines whether an ascent(or descent) is added. The mesh becomes

$$\begin{aligned}R' =&\{\ulcorner k,\ell \urcorner \mid k \leqslant i, \ell \leqslant j, \ulcorner k,\ell \urcorner \in R\}\cup\\ &\{\ulcorner k,\ell \urcorner \mid k \leqslant i, \ell > j, \ulcorner k,\ell-2 \urcorner \in R\}\cup\\ &\{\ulcorner k,\ell \urcorner \mid k > i, \ell \leqslant j, \ulcorner k-2,\ell \urcorner \in R\}\cup\\ &\{\ulcorner k,\ell \urcorner \mid k > i, \ell > j, \ulcorner k-2,\ell-2 \urcorner \in R\}\cup\\ &\{\ulcorner i,j+1 \urcorner, \ulcorner i+1,j \urcorner, \ulcorner i+1,j+1 \urcorner, \ulcorner i+1,j+2 \urcorner, \ulcorner i+2,j+1 \urcorner\}\end{aligned}$$

An example of adding an ascent to a mesh pattern can be seen in Figure 3.2.

We now attempt to fully classify coincidences in families characterised by avoidance of a classical pattern of length 3 and a mesh pattern of length 2, that is finding and explaining all coincidences where $\mathrm{Av}(\{p,m\}) = \mathrm{Av}(\{p,m'\})$.

It can be easily seen that in order to classify coincidences one need only consider coincidences within the family of mesh patterns with the same underlying classical pattern, this is due to the fact that $21 \in \mathrm{Av}((12,R))$ and $12 \in \mathrm{Av}((21,R))$ for all mesh-sets R.

We know that there are a total of 512 mesh-sets for each underlying classical pattern. By use of the previous results of Claesson, Tenner, and Ulfarsson [6][1] the number of coincidence classes can be reduced to 220.

3.1. **Coincidence classes of Av({321, (21, R)}).** Through experimentation, considering avoidance of permutations of up to length 11, we discover that there are at least 29 coincidence classes of mesh patterns with underlying classical pattern 21.

Proposition 3.4 (First Dominating Pattern Rule)**.** Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \varnothing)$ such that $|\pi| \leqslant |\sigma| + 1$, the sets $\mathrm{Av}(\{\pi, m_1\})$ and $\mathrm{Av}(\{\pi, m_2\})$ are coincident if

(1) $R_1 \triangle R_2 = \{(a,b)\}$
(2) $\pi \preceq \sigma^{\ulcorner a,b \urcorner}$

In order to prove this proposition we must first make the following note.

Note 3.5. Let $R' \subseteq R$. Then any occurrence of (τ, R) in a permutation is an occurrence of (τ, R').

[1] The authors use the Simultaneous Shading Lemma, a closure result and one worked out special case.

Proof of Proposition 3.4. We need to prove that $\mathrm{Av}(\{\pi, m_1\}) = \mathrm{Av}(\{\pi, m_2\})$. Assume without meaningful loss of generality that $R_2 = R_1 \cup \{(a, b)\}$. Since R_1 is a subset of R_2, Note 3.5 states that $\mathrm{Av}(\{\pi, m_1\}) \subseteq \mathrm{Av}(\{\pi, m_2\})$

Now we consider a permutation $\omega' \in \mathrm{Av}(\pi)$, containing an occurrence of m_1. Consider placing a point in the region corresponding to the box (a, b), regardless of where in this region we place the point by condition 2 of the Proposition we create an occurrence of π, therefore there can be no points in this region, which could have been represented in the mesh set R_1 by adding the box (a, b). Hence every occurrence of m_1 is in fact an occurrence of m_2, and we have that $\mathrm{Av}(\{\pi, m_2\}) \subseteq \mathrm{Av}(\{\pi, m_1\})$.

Taking both directions of the containment we can therefore draw the conclusion that $\mathrm{Av}(\{\pi, m_1\}) = \mathrm{Av}(\{\pi, m_2\})$. □

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