HÁSKÓLINN Í REYKJAVÍK
REYKJAVIK UNIVERSITY

*** DRAFT ***
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Working Title

Murray Tannock

Thesis of 60 ECTS credits
Master of Science (M.Sc.) in Computer Science

December 2015

Working Title

Thesis of 60 ECTS credits submitted to the School of Science and Engineering
at Reykjavík University in partial fulfillment of
the requirements for the degree of
Master of Science (M.Sc.) in Computer Science

December 2015

Supervisor:

Henning Ulfarsson, Supervisor
Professor, Reykjavík University, Iceland

Examiner:

Michael H. Albert, Examiner
Professor, University of Otago, New Zealand

Anders Claesson, Examiner
Docent, University of Iceland, Iceland

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Murray Tannock

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Abstract

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Murray Tannock

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Supervisor:

....

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Examiner:

....

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....

Anders Claesson

date

Murray Tannock
Master of Science

Acknowledgements

> So long, and thanks for all the fish.
>
> Douglas Adams[1]

Acknowledgements are optional; comment this chapter out if they are absent Note that it is important to acknowledge any funding that helped in the work

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Chapter 1

Introduction

1.1 What is a Permutation?

In *The Art of Computer Programming* Donald Knuth defines A *permutation of n objects* is an arrangement of n distinct objects in a row. When considering permutations we can consider them as occuring on the set $\llbracket n \rrbracket = \{1, \ldots, n\}$, therefore a permutation is a *bijection* $\pi : \llbracket n \rrbracket \mapsto \llbracket n \rrbracket$. We can write a permutation π in two line notation

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{pmatrix}$$

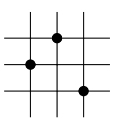
However, the most frequent notation used in computer science is *one-line notation*, in this form we drop the top line of the two line notation so are left with the following.

$$\pi = \pi(1)\pi(2)\ldots\pi(n)$$

Example 1.1.1. There are 6 permutations on $\llbracket 3 \rrbracket$.

$$123, 132, 213, 231, 312, 321$$

We can display a permutation on a *figure* in order to give a graphical representation of the permutation. In such a figure we let the x-axis denote the index in the permutation, and the y-axis denotes the values of $\pi(x)$. The figure of the permutation $\pi = 231$ is shown below



The class of all permutations of length n is $\mathfrak{S}_n$ and the class has size $n!$. The class of all permutations is $\mathfrak{S} = \bigcup_{i=0}^{\infty} \mathfrak{S}_i$.

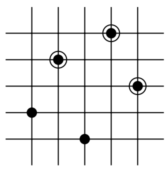
1.2 Classical Permutation Patterns

Classical Permutation Patterns began to be studied as a result of Knuth's statements about stack-sorting in [The Art of Computer Programming].

Definition 1.2.1. Order isomorphism. Two sequences $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ are said to be *order isomorphic* if they share the same relative order,i.e.,$\alpha_r < \alpha_s$ if and only if $\beta_r < \beta_s$.

A permutation π us said to *contain* the permutation σ of length k as a pattern (denoted $\sigma \preceq \pi$) if there is some an increasing subsequence $i_1, i_2, \ldots, i_n$ such that the sequence $\pi(i_1), \pi(i_2), \ldots, \pi(i_k)$ is order isomorphic to $\sigma(1), \sigma(2), \ldots, \sigma(k)$. If π does not contain σ, we say that π *avoids* σ.

For example the permutation $\pi = 24153$ contains the pattern $\sigma = 231$, since the second, fourth and fifth entries (4, 5, and 3) share the same relative order as the entries of σ. This can be seen graphically below, the points order isomorphic to σ are highlighted.



We denote the set of permutations of length n avoiding a pattern σ as $\mathrm{Av}_n(\sigma)$ and $\mathrm{Av}(\sigma) = \bigcup_{i=0}^{\infty} \mathrm{Av}_i(\sigma)$.

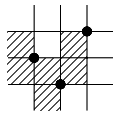
1.3 Mesh Patterns

Mesh Patterns were introduced by Brändén and Claesson to capture explicit expansions for certain permutation statistics. They are a natural extension of Classical permutation patterns. A *mesh-pattern* is a pair

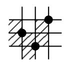
$$p = (\tau, R) \text{ with } \tau \in \mathfrak{S}_k \text{ and } R \subseteq [0, k] \times [0, k].$$

By this definition the empty permutation ε as a mesh pattern consisting solely of the box $(0, 0)$.

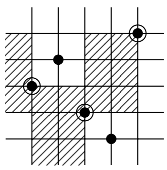
The figure for a mesh pattern looks similar to that for a classical pattern with the addition that we shade the unit square with bottom corner (i, j) for each $(i, j) \in R$:



We define containment, and avoidance, of the pattern p in the permutation π on mesh patterns analogously to classical containment, and avoidance, of τ in π with the additional restrictions on the relative position of the occurence of τ in π. These restrictions say that the shaded regions of the figure above contain no points from π.

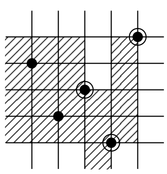
Example 1.3.1. The pattern $p = (213, \{(0, 1), (0, 2), (1, 0), (1, 1), (2, 1), (2, 2)\}) =$  is contained in $\pi = 34215$ but is not contained in $\sigma = 42315$.

Proof. Let us consider the figure for the permutation π we only need to find one occurence.



We have found an occurence of the pattern p in π and therefore π contains p.

Now we consider the figure for the permutation σ. This permutation avoids p since for every occurence of the classical pattern 213 there is at least one point in one of the shaded boxes. Consider the subsequence 315 in σ, this is an occurence of 213 but not the mesh pattern since the points with values 4 and 2 are in the shaded areas. This is shown in the figure below.



This is true for all occurences of 213 in σ and therefore σ avoids p. □

We denote the avoidance sets for mesh patterns in the same way as for classical patterns.

Note 1.3.2. Classical patterns are just a subclass of mesh patterns where the mesh set R is empty. The classical pattern π can be represented by a mesh pattern as $(\pi, \emptyset)$.

Chapter 2

Coincidences amongst families of mesh patterns and classical patterns

One interesting question to ask about permutation patterns considers when a pattern may be avoided by, or contained in, arbitrary permutations. Two patterns π and σ are said to be *coincident* if the set of permutations that avoid π is the same as the set of permutations that avoid σ. This extends to sets of patterns as well as single patterns.

We consider the avoidance sets, $\mathrm{Av}(p, q)$ where p is a classical pattern of length 3 and q is a mesh pattern of length 2 in order to establish some rules about when these two sets give the same avoidance set. We fix p in order to define the equivalences, we say that p is the *Dominating Pattern*. We fix p in order to calculate the coincidences. We never consider occurrences with different *Dominating Patterns*.

We first define some operations on mesh patterns.

Definition 2.0.1. Given a pattern p, let `add_point` $(p, (a, b), D)$ be the operation that returns a mesh pattern equivalent to placing a point in the center of box (a, b) in p, with shading defined by $D \subseteq \{N, E, S, W\}$.

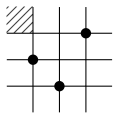
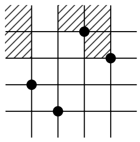
The set D defines the shading by indicating that the boxes in the cardinal directions in D next to the point are shaded in the resulting pattern. Since there is no ambiguity we let `add_point` (ε, D) be equivalent to `add_point` $(\varepsilon, (0, 0), D)$. This operation fails if the box (a, b) is in the mesh set of p.

Example 2.0.2. The result of adding a single point to the empty permutation for each cardinal direction.

`add_point` $(\varepsilon, \{N\}) =$    `add_point` $(\varepsilon, \{E\}) =$ 

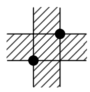
`add_point` $(\varepsilon, \{S\}) =$    `add_point` $(\varepsilon, \{W\}) =$ 

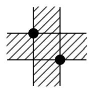
A more complex example for `add_point`

`add_point` ( , $(2, 3), \{N, E\}$) = 

Definition 2.0.3. Given a pattern p, define `add_descent` $(p, (a, b))$, and `add_ascent` $(p, (a, b))$, as the operations that return a mesh pattern equivalent to placing an decrease, or increase, in the center of box (a, b) in p.

Example 2.0.4.

`add_ascent` $(\varepsilon) =$ 

`add_descent` $(\varepsilon) =$ 

We now attempt to fully classify coincidences in families characterised by avoidance of a classical pattern of length 3 and a mesh pattern of length 2.

It can be easily seen that in order to classify set equivalences one need only consider coincidences within the family of mesh patterns with the same underlying classical pattern, this is due to the fact that $21 \in \mathrm{Av}((12, R))$ and $12 \in \mathrm{Av}((21, R))$ for all mesh-sets R. So $\mathrm{Av}((12, R)) \setminus \mathrm{Av}((21, S)) \neq \emptyset \; \forall R, S \in [0, 2] \times [0, 2]$, and hence the sets are disjoint.

We know that there are a total of 512 mesh-sets for each underlying classical pattern. By use of the shading lemma, simultaneous shading lemma, and one special case, we can reduce the number of equivalence classes to 220.

2.1 Equivalence classes of Av({321, (21, R)}).

Through expermentation we discover that there are a total of 29 equivalence classes of mesh patterns with underlying classical pattern 21.

Proposition 2.1.1 (First Dominating Pattern Rule)**.** Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \emptyset)$ such that $|\pi| \leq |\sigma| + 1$, the sets $\mathrm{Av}(\{\pi, m_1\})$ and $\mathrm{Av}(\{\pi, m_2\})$ are coincident if

1. $R_1 \triangle R_2 = \{(a, b)\}$

2. $\pi \leq \texttt{add\_point}\,(\sigma, (a, b), \emptyset)$

In order to prove this proposition we must first make the following note.

Note 2.1.2. Let $R' \subseteq R$. Then any occurence of (τ, R) in a permutation is an occurence of (τ, R').

Proof. Assume without meaningful loss of generality that R' is a proper subset of R.

Consider an occurence of (τ, R) in a permutation σ, obviously this corresponds to an occurrence of τ in σ. Now consider the mesh sets R and R', since $R' \subseteq R$ then there are more restrictions on where points are in an occurrence of (τ, R). Namely, for every shaded box in R the corresponding region in σ must contain no points, since R' has less shading than R there exists a region in the occurence of (τ, R) in σ that is now devoid of restrictions. However, by removing restrictions we cannot make an occurence become not an occurence, and therefore the same occurrence of τ in σ is now an occurence of (τ, R'). □

Proof of Proposition 2.1.1. We need to prove that $\mathrm{Av}(\{\pi, m_1\}) = \mathrm{Av}(\{\pi, m_2\})$.

Assume without meaningful loss of generality that $R_2 = R_1 \cup \{(a, b)\}$.

Consider a permutation ω that contains an occurrence of m_2 by Note 2.1.2 any of these occurrences is also an occurence of m_1. This proves that every occurence of m_2 is also an occurence of m_1 and therefore $\mathrm{Av}(\{\pi, m_1\}) \subseteq \mathrm{Av}(\{\pi, m_2\})$.

Now we consider a permutation $\omega' \in \mathrm{Av}(\pi)$, suppose we have an occurrence of m_1. Consider placing a point in the region corresponding to the box (a, b), regardless of where in this region we place the point by condition 2 of the Proposition we create an occurrence of π, therefore there can be no points in this region, which could have been represented in the mesh set R_1 by adding the box (a, b). Hence every occurrence of m_1 is in fact an occurrence of m_2, and we have that $\mathrm{Av}(\{\pi, m_2\}) \subseteq \mathrm{Av}(\{\pi, m_1\})$.

Taking both directions of the containment we can therefore draw the conclusion that $\text{Av}(\{\pi, m_1\}) = \text{Av}(\{\pi, m_2\})$. $\square$

By using this rule we completely capture the equivalence classes of $\text{Av}(\{321, (21, R)\})$.

This rule is understood very easily by seeing it in graphical form. In the pattern in Figure 2.1 we can gain shading in three boxes since if there is a point in any of these boxes we would gain an occurrence of the dominating pattern 321.

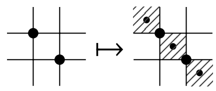
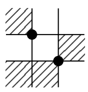
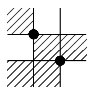


Figure 2.1: Visual Depiction of first dominating pattern rule.

2.2 Equivalence classes of Av({231, (21, *R*)}).

By application of Proposition 2.1.1 we obtain 43 equivalence classes. Experimentation shows that there are in fact 39 equivalence classes, for example the following two patterns are coincident in $\text{Av}(231)$ but this is not explained by Proposition 2.1.1.

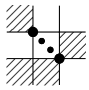
 and 

Consider the box $(1, 1)$, this is the box that we need to find reasoning to allow shading. If this box were to be empty then we could shade it freely, consider what happens if there are points in this box. If there is any increase in this box then we would have an occurrence of 231, however, since we are in $\text{Av}(231)$ this is not possible. This box must contain a decreasing subsequence. This gives rise to the following lemma:

Lemma 2.2.1. Given a mesh pattern $m = (\sigma, R)$, where the box (a, b) is not in R, and a dominating classical pattern $\pi = (\pi, \emptyset)$ if $\pi \leq \texttt{add\_ascent}\,(\sigma, (a, b))$ ($\pi \leq \texttt{add\_descent}\,(\sigma, (a, b))$) then in any occurrence of m in a permutation ϱ the region corresponding to the box (a, b) can only contain an increasing (decreasing) subsequence of ϱ.

The proof is analogous to the proof of Proposition 2.1.1.

Going back to our example mesh patterns



We know that the box $(1, 1)$ contains an decreasing subsequence. The top point of this decrease can be chosen to act as the second point in the mesh pattern, and therefore there are no points between the first point and the new second point. Hence, we can shade this box as it is guaranteed to be empty. This shows that our two example patterns are coincident.

This result generalises into the following rule for categorising equivalences of mesh patterns in cases where there is a dominating classical pattern.

Proposition 2.2.2 (Second Dominating Pattern Rule)**.** Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \emptyset)$ such that $|\pi| \leq |\sigma| + 2$, the sets $\mathrm{Av}(\{\pi, m_1\})$ and $\mathrm{Av}(\{\pi, m_2\})$ are coincident if

1. $R_1 \triangle R_2 = \{(a, b)\}$

2. a) $\pi \leq$ `add_ascent` $(\sigma, (a, b))$ and

   i. $(a + 1, b) \in \sigma$ and $(a + 1, b - 1) \notin R$ and
   $(x, b - 1) \in R \implies (x, b) \in R$ (where $x \neq a, a + 1$) and
   $(a + 1, y) \in R \implies (a, y) \in R$ (where $y \neq b - 1, b$).

   ii. $(a, b + 1) \in \sigma$ and $(a - 1, b + 1) \notin R$ and
   $(x, b + 1) \in R \implies (x, b) \in R$ (where $x \neq a - 1, a$) and
   $(a - 1, y) \in R \implies (a, y) \in R$ (where $y \neq b, b + 1$).

   b) $\pi \leq$ `add_descent` $(\sigma, (a, b))$ and

   i. $(a + 1, b + 1) \in \sigma$ and $(a + 1, b + 1) \notin R$ and
   $(x, b + 1) \in R \implies (x, b) \in R$ (where $x \neq a, a + 1$) and
   $(a + 1, y) \in R \implies (a, y) \in R$ (where $y \neq b, b + 1$).

   ii. $(a, b) \in \sigma$ and $(a - 1, b - 1) \notin R$ and
   $(x, b + 1) \in R \implies (x, b) \in R$ (where $x \neq a - 1, a$) and
   $(a - 1, y) \in R \implies (a, y) \in R$ (where $y \neq b - 1, b$).

Proof. We need to prove that $\mathrm{Av}(\{\pi, m_1\}) = \mathrm{Av}(\{\pi, m_2\})$.

Assume without meaningful loss of generality that $R_2 = R_1 \cup \{(a, b)\}$.

Consider a permutation ω that contains an occurrence of m_2 by Note 2.1.2 any of these occurrences is also an occurence of m_1. This proves that every occurence of m_2 is also an occurence of m_1 and therefore $\mathrm{Av}(\{\pi, m_1\}) \subseteq \mathrm{Av}(\{\pi, m_2\})$.

We will consider taking the first branch of every choice. Now consider a permutation in $\omega' \in \mathrm{Av}(\pi)$. Suppose ω' contains m_1 consider the region corresponding to (a, b) in R_1.

If the region is empty, then we can freely shade the corresponding box (a, b) in m_1 and hence have an occurence of m_2.

Now consider if the region is non-empty, by Lemma 2.2.1 and condition 2a of the proposition this region must contain a decreasing subsequence. We can choose the topmost point in the region to replace the corresponding point in the mesh pattern and the points from the subsequence are now in the box southeast of the point. The other conditions allow this to be done without points being present in regions that were shaded. Hence there are no points in the region corresponding to the box (a, b) in the mesh pattern, and therefore we can shade this region. This implies that every occurrence of m_1 in $\mathrm{Av}(\pi)$ is in fact an occurrence of m_2 so $\mathrm{Av}(\{\pi, m_2\}) \subseteq \mathrm{Av}(\{\pi, m_1\})$.

Similar arguments satisfy the remainder of the branches.

□

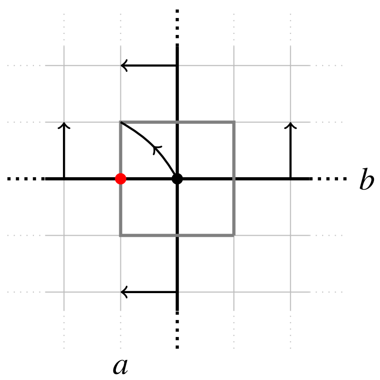


Figure 2.1: An illustration of how lines and points move by this proposition.

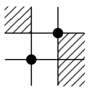
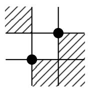
This proposition essentially states that we slide all of the points in the box we desire to shade diagonally, and chose the topmost/bottommost point to replace the original point in the mesh pattern.

By taking the First Dominating Pattern Rule and the Second Dominating Pattern rule together coincidences of classes of the form $\mathrm{Av}(\{132, (21, R)\})$ are fully explained, obtaining 39 equivalence classes of mesh patterns.

2.3 Equivalence classes of Av({231, (12, *R*)}).

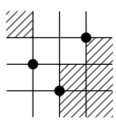
When considering the equivalence classes of $\mathrm{Av}(231, (12, R))$ we first apply the two Dominating Pattern rules previously established. Starting from 220 classes, application of the first Dominating Pattern rule gives 85 classes. Following this with the second Dominating Pattern rule reduces the number of classes to 59. However we know that there are patterns where the coincidences are not explained by the rules given above.

For example the patterns

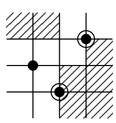
 and 

are experimentally coincident. This coincidence is not explained by our rules, it is necessary to attempt to capture these coincidences by establishing more rules.

In order to rigorously establish this coincidence we need to consider what would happen if we were to choose a point in the box $(1, 0)$ that we would like to shade. In order to get a chance of getting the shading we want consider choosing the rightmost point in the box. This gives us the following mesh pattern.



By application of the Proposition 2.1.1 we then achieve the following mesh pattern



If we look at the highlighted points we see that the subpattern is an occurrence of the mesh pattern that we originally desired. This gives rise to the following rule:

Proposition 2.3.1 (Third Dominating Pattern Rule)**.** Given two mesh patterns $m_1 = (\sigma, R_1)$ and $m_2 = (\sigma, R_2)$, and a dominating classical pattern $\pi = (\pi, \emptyset)$[1], the sets $\mathrm{Av}(\{\pi, m_1\})$ and $\mathrm{Av}(\{\pi, m_2\})$ are coincident if

1. $R_1 \triangle R_2 = \{(a, b)\}$

2. `add_point` $((\sigma, R_1), (a, b), D)$ is coincident with a mesh pattern containing an occurrence of (σ, R_2).

Proof. We need to prove that $\mathrm{Av}(\{\pi, m_1\}) = \mathrm{Av}(\{\pi, m_2\})$.

Assume without meaningful loss of generality that $R_2 = R_1 \cup \{(a, b)\}$.

Consider a permutation ω that contains an occurrence of m_2 by Note 2.1.2 any of these occurrences is also an occurence of m_1. This proves that every occurence of m_2 is also an occurence of m_1 and therefore $\mathrm{Av}(\{\pi, m_1\}) \subseteq \mathrm{Av}(\{\pi, m_2\})$.

Now consider a permutation ϱ in $\mathrm{Av}(\pi)$ that contains an occurrence of m_1. If the region corresponding to the box (a, b) is empty then we have an occurrence of m_2. If the region is non-empty then by condition 2 of the proposition there exists a direction such that there exists an occurrence of a mesh pattern of length one longer than m_1 in this position. This mesh pattern is coincident with another mesh pattern. This mesh pattern contains an occurrence of m_2 so every occurrence of m_1 is also an occurrence of m_2. Thus $\mathrm{Av}(\{\pi, m_2\}) \subseteq \mathrm{Av}(\{\pi, m_1\})$ and the two patterns are coincident. □

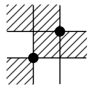
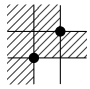
2.4 Equivalence classes of Av({321, (12, *R*)}).

When considering equivalences of mesh patterns with underlying classical pattern 12 in Av(321) application of the previously established rules give no coincidences. Through experimentation we discover that there are 7 equivalence classes which can be explained through the use of two different lines of reasoning. Since the number of coincidences is so small we will reason for these equivalences without attempting to generalise into concrete rules.

[1]The permutation π may be the empty permutation

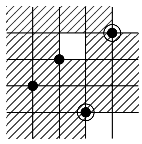
Intuitively it is easy to see why our previous rules have no power here. There is nowhere that it is possible to add a single point to gain an occurrence of $\pi = 321$. It is also impossible to have a position where addition of an increase, or decrease, provides extra shading power.

The patterns

 and 

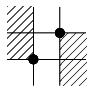
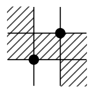
are equivalent in $\mathrm{Av}(321)$. (There are 3 symmetries of these patterns that are also equivalent to each other by the same reasoning.)

Consider the contents of the box $(0, 1)$, by Lemma 2.2.1 it must contain an increasing subsequence. If there is only one point in the box we can choose this to replace the 1 in the mesh pattern to get the required shading. If there is more than one point then choosing the two leftmost points give us the following mesh pattern.



Where the two highlighted points are the original two points. Now if we take the other two points as the points in our mesh permutation then we get an occurrence of the pattern we originally desired, and hence the two patterns are coincident.

The other reasoning applies to the patterns

$m_1 =$  and $m_2 =$ 

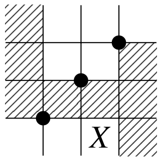
which are coincident by expermentation.

In order to prove this coincidence we will proceed by mathematical induction on the number of points in the middle box we call this number n.

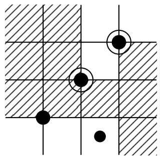
Base Case $(n = 0)$**:** The base case hold since we can freely shade the box if it contains no points.

Inductive Hypothesis $(n = k)$**:** Suppose that the we can find an occurrence of the second pattern if we have an occurrence of the first with k points in the middle box.

Inductive Step ($n = k + 1$) Suppose that we have $(k + 1)$ points in the middle box. Choose the bottom most point in the middle box, this gives a mesh pattern equivalent to



Now we need to consider the box labelled X if this box is empty then we have an occurrence of m_2 and are done. If this box contains any points then we gain some extra shading on the mesh pattern due to the dominating pattern



The two highlighted points form an occurrence of m_1 with k points in the middle box, and thus by the Inductive Hypothesis we are done.

By induction we have that every occurrence of m_1 is an occurrence of m_2 and by Note 2.1.2 every occurrence of m_2 is an occurrence of m_1 so the two patterns are coincident. This argument applies to another two pairs of classes. Therefore in total in $\mathrm{Av}(321, (12, R))$ there are 213 equivalence classes.

Chapter 3

Wilf-equivalences under dominating classical patterns

Another question often asked in thefield of permutation patterns is that of Wilf-equivalence. Two patterns π and σ are said to be Wilf-equivalent if their avoidance sets have the same size at each length. More formally

Definition 3.0.1 (Wilf-equivalence)**.** Two patterns π and σ are said to be *Wilf-equivalent* if for all $k \geq 0$, $|\mathrm{Av}_k(\pi)| = |\mathrm{Av}_k(\sigma)|$.

Two sets of permutation patterns R and S are are *Wilf-equivalent* if for all $k \geq 0$, $|\mathrm{Av}_k(R)| = |\mathrm{Av}_k(S)|$.

Wilf-equivalence is of interest as if two permutation classes are enumerated in the same way then there should exist a bijection between them, and therefore any other combinatorial object that they represent.

There are a number of symmetries we can use when examining Wilf-equivalences to reduce the amount of work, it can be easily seen that the reverse, complement and inverse operations preserve enumeration, and therefore these classes are trivially Wilf-equivalent. Since we are always considering Wilf-equivalences in the set $\mathrm{Av}(\pi)$ we must only use these symmetries when they preserve the dominating pattern.

Bibliography

[1] D. Adams, *So long, and thanks for all the fish*. Harmony Books, 1984.

School of Science and Engineering
Reykjavík University
Menntavegur 1
101 Reykjavík, Iceland
Tel. +354 599 6200
Fax +354 599 6201
www.ru.is