Chapter 2
(Part 3)

Recurrence Relation & Recursive Algorithms

Definition

- A **recurrence relation** is an equation that defines a sequence $\{a_n, a_{n+1}, \ldots\}$ based on a rule that gives the next term as a function of one or more of the previous term in the sequence for all integers n with $n \geq n_0$,

$(\ a_0,\ a_1, \ldots\ )$

- A sequence is called a **solution** of a recurrence relation if it terms satisfy the recurrence relation.

Simple Recurrence Relation

The simplest form of a **recurrence relation** is the case where the next term depends only on the immediately previous term.

$$3, 8, 13, 18, 23, \ldots$$

The above sequence shows a pattern, given initial condition, $a_0 = 3$:

$$3, \; 3+5, \; 8+5, \; 13+5, \; 18+5, \ldots$$

$$a_0, a_1, a_2, a_3, a_4, a_5, \ldots$$

generated from an equation:

$$a_n = a_{n-1} + 5, \qquad n \geq 1$$

n_{th} term of a sequence

Recurrence relation can be used to compute any n-*th* term of the sequence, given the initial condition, a_0:

$$a_n = a_{n-1} + 5, \quad n \geq 1$$

$$3, 8, 13, 18, 23, \textcolor{red}{28,} \; a_{n-1} + 5, \ldots$$

$$a_1 = a_0 + 5, \quad 3 + 5 = 8$$
$$a_2 = a_1 + 5, \quad 8 + 5 = 13$$
$$a_3 = a_2 + 5, \quad 13 + 5 = 18$$
$$\vdots$$
$$a_5 = a_4 + 5, \quad 23 + 5 = 28$$

Example 1

Consider the following sequence:

$$3, 9, 27, 81, 243, \ldots$$

The above sequence shows a pattern:

$$3^1, \ 3^2, \ 3^3, \ 3^4, \ 3^5, \ldots \qquad 3^1, \ 3^2, \ 3^3, \ 3^4, \ 3^5, \ldots$$

$a_1, a_2, a_3, a_4, a_5, \ldots$ **or** $a_0, a_1, a_2, a_3, a_4, a_5, \ldots$

Recurrence relation is defined by:

$$a_n = 3^n, n \geq 1 \qquad \textbf{or} \qquad a_n = 3^{n+1}, n \geq 0$$

Example 2

Given initial condition, $a_0 = 1$ and recurrence relation:

$$a_n = 1 + 2a_{n-1} \ , n \geq 1$$

First few sequence are:

$$a_1 = 1 + 2\,(1) = 3$$
$$a_2 = 1 + 2(3) = 7$$
$$a_3 = 1 + 2(7) = 15$$

1, 3, 7, 15, 31, 63, …

Example 3

Given initial condition, $a_0 = 1$, $a_1 = 2$ and recurrence relation:

$$a_n = 3(a_{n-1} + a_{n-2}), \ n \geq 2$$

First few sequence are:

$$a_2 = 3(2 + 1) = 9$$
$$a_3 = 3(9 + 2) = 33$$
$$a_4 = 3(33 + 9) = 126$$

1, 2, 9, 33, 126, 477, 1809, 6858, 26001,…

Example 4

For a photo shoot, the staff at a company have been arranged such that there are 10 people in the front row and each row has 7 more people than in the row in front of it. Find the recurrence relation and compute number of staff in the first 5 rows.

Notice that the difference between the number of people in successive rows is a constant amount.

This means that the n_{th} term of this sequence can be found using:

$$a_n = a_{n-1} + 7, \quad n \geq 2 \text{ with } a_1 = 10$$

Number of staff in the first 5 rows*:*

$a_1 = 10$,
$a_2 = a_1 + 7$, $10 + 7 = 17$
$a_3 = a_2 + 7$, $17 + 7 = 24$
$a_4 = a_3 + 7$, $24 + 7 = 31$
$a_5 = a_4 + 7$, $31 + 7 = 38$

10, 17, 24, 31, 38

Example 5

Find a recurrence relation and initial condition for

1, 5, 17, 53, 161, 485, …

Solution:

Look at the differences between terms:

4, 12, 36, 108, …

The difference in the sequence is growing by a factor of 3.

4x3, 12x3, 36x3

Solution

However the original sequence is not.

$$1(3) = \textbf{3},\; 5(3) = \textbf{15},\; 17(3) = \textbf{51},\; \dots$$

1, 5, 17, 53, 161, 485, …

It appears that we always end up with **2 less than** the next term. So, the recurrence relation is defined by:

$a_n = 3(\,a_{n-1}\,) + 2,\;\; n \geq 1,$ **with initial condition,** $a_0 = 1$

$$a_1 = 3(\,a_0\,) + 2 = 3(1) + 2 = 5$$
$$a_2 = 3(\,a_1\,) + 2 = 3(5) + 2 = 17$$
$$\vdots$$

Example 6

Example 6

A depositor deposits RM 10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years? Let P_n denote the amount in the account after n years.

Solution

Derive the following **recurrence relation**:

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$$

Where, P_n = Current balance and P_{n-1} = Previous year balance and 0.05 is the compounding interest.

Solution

Initial condition, P_0 = 10,000. Then,

$$P_1 = 1.05P_0$$
$$P_2 = 1.05P_1 = (1.05)^2P_0$$
$$P_3 = 1.05P_2 = (1.05)^3P_0$$
…
$$\mathbf{P_n = 1.05P_{n-1} = (1.05)^nP_0} \; ,$$

now we can use this formula to calculate n_{th} term without iteration

Solution

Let us use this formula to find P_{30} under the initial condition $P_0 = 10{,}000$:

$P_{30} = (1.05)^{30}(10{,}000) = 43{,}219.42$

After 30 years, the account contains **RM 43,219.42**.

Example 7

Consider the following sequence:

1, 5, 9, 13, 17

Find the recurrence relation that defines the above sequence.

Answer:

$$a_n = a_{n-1} + 4 \quad , \; n \geq 1 \text{ with } a_0 = 1$$

Exercise 5

A consumer purchased items costing RM280 with a department store credit card that charges 1.5% interest per month compounded monthly. Write a recurrence relation and initial condition for b_n, the balance of the consumer's account after n months if no further charges occur and the minimum monthly payment of RM25 is made.

Recursive Algorithms

- A recursive procedure is a procedure that invokes itself.

- A recursive algorithm is an algorithm that contains a recursive procedure.

- Recursion is a powerful, elegant and natural way to solve a large class of problems.

example

Factorial problem

- If $n \geq 1$,

$$n! = n\,(n-1)\,\ldots 2.\,1$$

and $\qquad$ $0! = 1$

- Notice that, if $n \geq 2$, n factorial can be written,

$$n! = n\,(n-1)(n-2)\,\ldots 2.\,1$$

$$= n.\,(n-1)!$$

example

- n=5

- 5!

$$5! = 5. 4. 3. 2. 1 = 5. 4!$$

$$4! = 4. 3!$$

$$3! = 3. 2!$$

Algorithm

- Input: n, integer ≥ 0

- Output: n!

- Factorial (n) {

      if (n=0)

          return 1

      return n*factorial(n-1)

  }

example

- Fibonacci sequence, f_n

$$f_1 = 1$$
$$f_2 = 1$$
$$f_n = f_{n-1} + f_{n-2}, \qquad n \geq 3$$

$$1, \ 1, \ 2, \ 3, \ 5, \ 8, \ 13, \ ....$$

Algorithm

- Input: n

- Output: $f(n)$

- $f(n)$ {
    if (n=1 or n=2)
        return 1
    return $f(n$-1) + $f(n$-2)
}

Example

Consider the following arithmetic sequence: 1, 3, 5, 7, 9,...

Suppose a_n is the term sequence. The generating rule is $a_n = a_{n-1} + 2, n > 1$. The relevant recursive algorithm can be written as
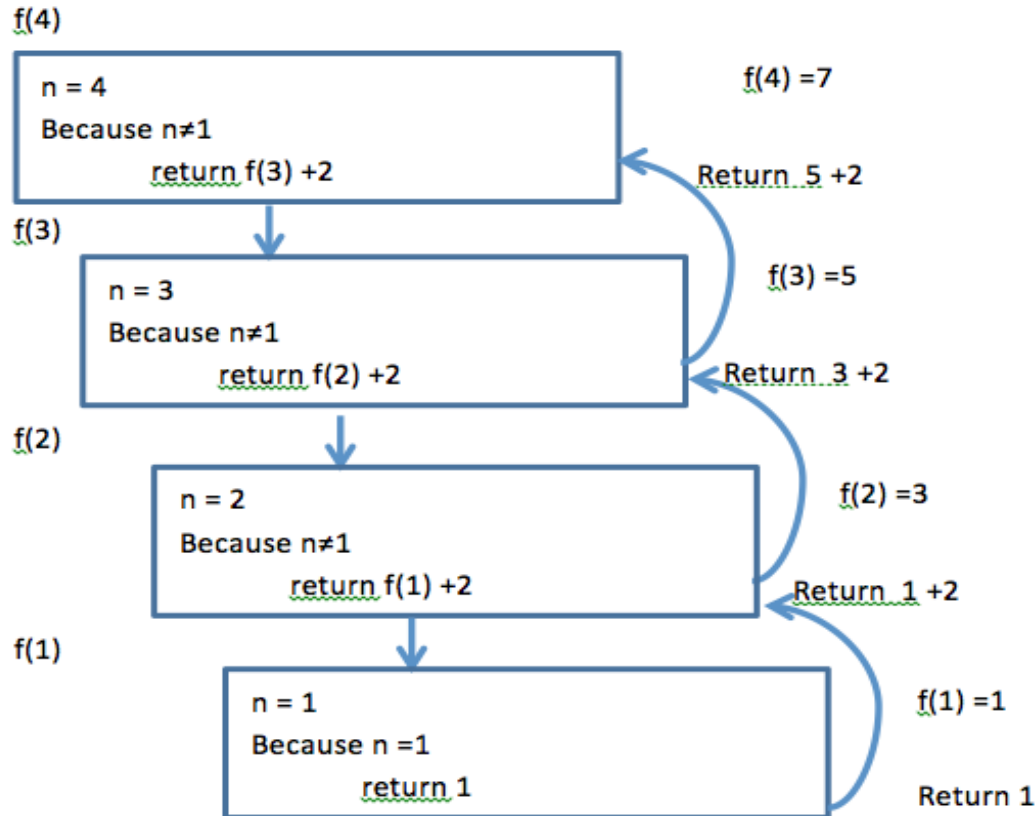
$f(n)$
{   if $(n = 1)$
          return 1
    return $f(n-1) + 2$
}

Use the above recursive algorithm to trace n = 4

Solutions

Trace the output if n = 4

f(4)

| n = 4 |
| Because n≠1 |
| return f(3) +2 |

f(4) =7

Return 5 +2

f(3)

| n = 3 |
| Because n≠1 |
| return f(2) +2 |

f(3) =5

Return 3 +2

f(2)

| n = 2 |
| Because n≠1 |
| return f(1) +2 |

f(2) =3

Return 1 +2

f(1)

| n = 1 |
| Because n =1 |
| return 1 |

f(1) =1

Return 1

Answer = 7

f(4)
n = 1
because n ≠ 1
return f(3) +2

f(4)
return 5+2

f(3)
n = 3
because n ≠ 1
return f(2) +2

f(3) = i
return 3+2

f(2) = 3
return 1+2

F(1)
n = 1
because n = 1
return 1

f(1) = 1
return 1