Chapter 1

Part 4
Quantifiers & Proof Technique

QUANTIFIERS

- Most of the statements in mathematics and computer science are not described properly by the propositions.

- Since most of the statements in mathematics and computer science use **variables**, the system of logic must be extended to include **statements with the variables**.

QUANTIFIERS

- Let *P(x)* is a statement with variable x and *A* is a set.

- *P* a **propositional function** or also known as **predicate** if for each x in *A*, *P(x)* is a proposition.

- Set *A* is the **domain of discourse** of *P*.

- Domain of discourse -> the particular domain of the variable in a propositional function.

QUANTIFIERS

- A **predicate** is a statement that contains variables.

Example:

$P(x) : x > 3$

$Q(x,y) : x = y + 3$

$R(x,y,z) : x + y = z$

Example

$x^2 + 4x$ is an odd integer

(domain of discourse is set of positive numbers).

$x^2 - x - 6 = 0$

(domain of discourse is set of real numbers).

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(domain of discourse is set of university in Malaysia).

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QUANTIFIERS

A predicate becomes a proposition if the variable(s) contained is(are)

- **Assigned specific value(s)**
- **Quantified**

Example

- $P(x) : x > 3$.

  What are the truth values of $P(4)$ and $P(2)$?

- $Q(x,y) : x = y + 3$. (multiple variables)

  What are the truth values of $Q(1,2)$ and $Q(3,0)$?

Propositional functions

- Let P(x) : "x is a multiple of 5"
  - For what values of x is P(x) true?

- Let P(x) : x+1 > x
  - For what values of x is P(x) true?

- Let P(x) : x + 3 =0
  - For what values of x is P(x) true?

QUANTIFIERS

- Two types of quantifiers:
  - **Universal**
  - **Existential**

QUANTIFIERS

- Let *A* be a propositional function with domain of discourse *B*. The statement

$$\text{for every } x, A(x)$$

is **universally quantified statement**

- Symbol ∀ called a **universal quantifier** is used "**for every**".

- Can be read as "**for all**", "**for any**".

Universal quantifiers

Represented by an upside-down A: $\forall$

- It means "for all"

Example

Let P(x) = x+1 > x

We can state the following:

- $\forall$x P(x)
- English translation: "for all values of x, P(x) is true"
- English translation: "for all values of x, x+1>x is true"

- The statement can be written as

$$\forall x\, A(x)$$

- Above statement is true if $A(x)$ is true for every x in B

(**false if $A(x)$ is false for at least one x in B**).

- OR   In order to prove that a universal quantification is **true,**

    it must be shown for **ALL cases**

    In order to prove that a universal quantification is **false**, it must be shown to be false for only **ONE case**

- A value x in the domain of discourse that makes the statement $A(x)$ *false* is called a **counterexample** to the statement.

Example

- Let the universally quantified statement is

$$\forall x\, (x^2 \geq 0)$$

Domain of discourse is the set of real numbers.

- **This statement is true** because for every real number x, it is true that the square of x is positive or zero.

- Let the universally quantified statement is

$$\forall x \, (x^2 \leq 9)$$

Domain of discourse is a set B = {1, 2, 3, 4}

- When x = 4, the statement produce false value.
- Thus, the above statement is false and the **counterexample is 4**.

QUANTIFIERS

- Easy to prove a universally quantified statement is true or false if the domain of discourse is not too large.

- What happen if the domain of discourse contains a large number of elements?

- For example, a set of integer from 1 to 100, the set of positive integers, the set of real numbers or a set of students in School of Computing. It will be hard to show that every element in the set is *true*.

Use existential quantifier!!

QUANTIFIERS

- Let *A* be a propositional function with domain of discourse *B*. The statement

$$\text{There exist } x, A(x)$$

   is **existentially quantified statement**

- Symbol ∃ called an **existential quantifier** is used "**there exist**".

- Can be read as "**for some**", "**for at least one**".

QUANTIFIERS

- The statement can be written as

$$\exists x \, A(x)$$

- Above statement is true if $A(x)$ is true for at least one x in *B* (**false if every x in *B* makes the statement $A(x)$ false**).

- **Just find one x that makes $A(x)$ true!**

Example

- Let the existentially quantified statement is

$$\exists x \left(\frac{x}{x^2 + 1} = \frac{2}{5} \right)$$

Domain of discourse is the set of real numbers.

- **Statement is true** because it is **possible to find at least one real number x** to make the proposition true.

- For example, if $x = 2$, we obtain the true proposition as below

$$\left(\frac{x}{x^2 + 1} = \frac{2}{5} \right) = \left(\frac{2}{2^2 + 1} = \frac{2}{5} \right)$$

Negation of Quantifiers

- Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

$$\neg\,(\forall x\, P(x))\; ;\; \exists x\, \neg\, P(x)$$

$$\neg\,(\exists x\, P(x))\; ;\; \forall x\, \neg\, P(x)$$

- Let $P(x)$: x is taking Discrete Structure course with the domain of discourse is the set of all students.

   $\forall x\, P(x)$

   All students are taking Discrete Structure course.

   $\exists x\, P(x)$

   There is some students who are taking Discrete Structure course.

Example

$$\neg \ (\exists x \ P(x)) \ ; \ \forall x \ \neg \ P(x)$$

¬∃x P(x)

None of the students are taking Discrete Structure course.

∀x ¬ P(x)

All students are not taking Discrete Structure course.

Example

$$\neg \, (\forall x \, P(x)) \; ; \;\; \exists x \, \neg \, P(x)$$

¬∀x P(x)

Not all students are taking Discrete Structure course.

∃x ¬P(x)

There is some students who are not taking Discrete Structure course

- Consider "**For every student in this class, that student has studied calculus**"

- Rephrased: "**For every student x in this class, x has studied calculus**"
  - Let C(x) be "**x has studied calculus**"
  - Let S(x) be "**x is a student in this class**"

    $$\forall x\ C(x)$$

  - True if the universe of discourse is **all students in this class**

- What about if the universe of discourse is all students (or all people?)

  - $\forall x\ (S(x) \wedge C(x))$ X
    - This is wrong! Why? (because this statement says that all people are students in this class and have studied calculus)

  - $\forall x\ (S(x) \rightarrow C(x))$ √

C(x) : "x has studied calculus"
S(x) : "x is a student in this class"

- Consider:
  - "**Some students have visited Mexico**"
  - Rephrasing: "**There exists a student who has visited Mexico**"

- Let:
  - S(x) be "**x is a student**"
  - M(x) be "**x has visited Mexico**"

    $$\exists x\ M(x)$$

  - True if the universe of discourse is all students

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- What about if the universe of discourse is all people?

$$\exists x \, (S(x) \wedge M(x))$$

$\exists x \, (S(x) \rightarrow M(x))$ **This is wrong! Why?**

suppose someone is not student= F->T or F->F, both make the statement true (**refer to truth table *p → q***)

S(x) :"x is a student"
M(x): "x has visited Mexico"

- Consider "**Every student in this class has visited Canada or Mexico**"

- Let, S(x) be "**x is a student in this class**"

    M(x) be "**x has visited Mexico**"

    C(x) be "**x has visited Canada**"

$$\forall x \ (M(x) \lor C(x))$$

(**When the universe of discourse is all students in this class**)

$$\forall x \ (S(x) \rightarrow (M(x) \lor C(x)))$$

(**When the universe of discourse is all people or all students**)

Proof Techniques

Mathematical systems consists:

- ▪ **Axioms**: assumed to be true.
- ▪ **Definitions**: used to create new concepts.
- ▪ **Undefined terms**: some terms that are not explicitly defined.
- ▪ **Theorem**
  - Statement that can be shown to be true (under certain conditions)
  - Typically stated in one of three ways:
    - As Facts
    - As Implications
    - As Bi-implications

Proof Techniques

Direct Proof (Direct Method)

- Proof of those theorems that can be expressed in the form $\forall x\, (P(x) \to Q(x))$, D is the domain of discourse.

- Select a particular, but arbitrarily chosen, member a of the domain D.

- Show that the statement $P(a) \to Q(a)$ is true. (Assume that $P(a)$ is true).

- Show that $Q(a)$ is true.

- By the rule of Universal Generalization (UG),
  $$\forall x\, (P(x) \to Q(x)) \text{ is true.}$$

"**For all integer *x*, if *x* is odd, then *x*² is odd**"

Or   *P(x)* : *x* is an odd integer

          Q(x) : *x²* is an odd integer

$$\forall x (P(x) \rightarrow Q(x))$$

The domain of discourse is set *Z* of all integer.

Can verify the theorem for certain value of *x*.

     x=3, *x²* =9 ; odd

Or show that the square of an odd number is an odd number

Rephrased: "**if *n* is odd, then *n²* is odd**"

Example

- a is an odd integer

$$\implies a = 2n + 1 \quad \rightarrow \text{\textit{for some integer \textbf{n}}}$$

$$\implies a^2 = (2n + 1)^2$$

$$\implies a^2 = 4n^2 + 4n + 1$$

$$\implies a^2 = 2(2n^2 + 2n) + 1$$

$$\implies a^2 = 2m + 1 \quad \rightarrow \text{\textit{where \textbf{m= 2n}}}^2 \text{\textit{+ 2n is an integer}}$$

$$\implies a^2 \quad \rightarrow \text{\textit{is an odd integer}}$$

Indirect Proof

- The implication $p \rightarrow q$ is **equivalent** to the implication $(\neg q \rightarrow \neg p)$ **(contrapositive)**

- Therefore, in order to show that $p \rightarrow q$ is true, one can also **show that the implication $(\neg q \rightarrow \neg p)$ is true**.

- To show that $(\neg q \rightarrow \neg p)$ is true, assume that the negation of q is true and prove that the negation of p is true.

Example

P(n) : *n²+3* is an odd number

Q(n) : *n* is even number

$$\forall n(P(n) \rightarrow Q(n))$$

$$P(n) \rightarrow Q(n) \equiv \neg Q(n) \rightarrow \neg P(n)$$

¬ *Q(n)* is true , *n* is not even (*n* is odd), so *n=2k+1*

$$n^2 + 3 = (2k+1)^2 + 3$$
$$= 4k^2 + 4k + 1 + 3$$
$$= 4k^2 + 4k + 4$$
$$= 2(2k^2 + 2k + 2)$$

Example

$$n^2 + 3 = (2k+1)^2 + 3$$
$$= 4k^2 + 4k + 1 + 3$$
$$= 4k^2 + 4k + 4$$
$$= 2(2k^2 + 2k + 2)$$

$$t = 2k^2 + 2k + 2$$

$$\longrightarrow \quad \boxed{t \text{ is integer}}$$

$$n^2 + 3 = 2t$$

n^2+3 is an even integer, thus ¬ *P(n)* is true

Which to use

- When do you use a direct proof versus an indirect proof?

- If it's not clear from the problem, try direct first, then indirect second
  - If indirect fails, try the other proofs

Example

Prove that "**if n is an integer and n^3+5 is odd, then n is even**"

Via direct proof

- $n^3+5 = 2k+1$ for some integer k (definition of odd numbers)
- $n^3 = 2k - 4$
- $$n = \sqrt[3]{2k-4}$$
- Umm…

So direct proof didn't work out.

Next up: indirect proof

Example

Prove that "**if n is an integer and n^3+5 is odd, then n is even**"

Via indirect proof
- Contrapositive: If n is odd, then n^3+5 is even
- Assume n is odd, and show that n^3+5 is even
- n=2k+1 for some integer k (definition of odd numbers)
- $n^3+5 = (2k+1)^3+5 = 8k^3+12k^2+6k+6 = 2(4k^3+6k^2+3k+3)$
- As $2(4k^3+6k^2+3k+3)$ is 2 times an integer, it is even

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Proof Techniques

Proof by Contradiction

Assume that the hypothesis is true and that the conclusion is false and then, arrive at a contradiction.

Proposition "**if P then Q**"
Proof. **Suppose P and $\sim Q$**

Since we have a contradiction, it must be that Q is true

Prove that "**there are infinitely many prime numbers**".

Proof:

- Assume there are not infinitely many prime numbers, therefore they are can be listed, i.e. $p_1, p_2, ..., p_n$

- Consider the number $q = p_1 \times p_2 \times ... \times p_n + 1$.

- q is either prime or not divisible, but not listed above. Therefore, q is a prime. However, it was not listed.

- **Contradiction**! Therefore, there are infinitely many primes numbers.

Example

- For all real numbers x and y, **if x+y $\geq$2, then either x $\geq$ 1 or y $\geq$ 1**.

Proof

- Suppose that the conclusion is false. Then

    x < 1 and y <1

    Add these inequalities,   x+y < 1+1 = 2 (**x+y <2**)

- **Contradiction**

- Thus we conclude that the statement is true.

Example

Suppose $a \in \mathbb{Z}$. **If a^2 is even, then a is even**

Proof

- Contradiction: Suppose a^2 is even and a is not even.
- Then a^2 is even, and a is odd
- Let, $a = 2c + 1$ (odd)

$$a^2 = (2c+1)^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1 \quad \text{(odd)}$$

- **Contradiction**
- Thus we conclude that the statement is true.

Thank You

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