Exploring the L-shape Ramsey Problem: Pattern Discovery and FunSearch Implementation

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April 2, 2025

Abstract

The L-shape Ramsey problem asks for a coloring of an $n \times n$ grid such that no L-shape is monochromatic. This paper explores computational approaches to finding valid 3-colorings for grids of various sizes. We developed multiple solvers including a deterministic pattern-based approach and a FunSearch implementation using the Llama 3.2 language model. Our investigation revealed that the complexity of required patterns increases significantly with grid size. We found valid 3-colorings for 3×3 grids using Latin square patterns, and for 4×4 grids using a specialized corner-focused pattern. However, no valid 3-colorings were found for 5×5 grids and larger. Our research suggests that as grid size increases, more complex patterns or additional colors may be required. The integration of language models into the search process demonstrates the potential of neural-guided combinatorial optimization for Ramsey theory problems.

1 Introduction

Ramsey theory, a branch of combinatorial mathematics, studies the conditions under which order must emerge in large structures. The classical Ramsey's theorem states that in any sufficiently large structure, a certain substructure must appear, regardless of how the elements are arranged. Geometric variants of Ramsey problems focus on finding monochromatic configurations in colored spaces [1].

The L-shape Ramsey problem is a specific geometric Ramsey problem where we seek to color an $n \times n$ grid such that no L-shape is monochromatic. An L-shape consists of three points where two points are equidistant from the third point, forming a right angle. For example, the points at coordinates $(0, 0)$, $(2, 0)$, and $(2, 2)$ form an L-shape.

The problem can be formulated more precisely as follows: Given an $n \times n$ grid, find a coloring using k colors such that no L-shape has all three points colored the same. The smallest n for which such a coloring is impossible with k colors is denoted as the L-shape Ramsey number $R_L(k)$.

Previous work on this problem has been limited, with most research focusing on classical Ramsey numbers rather than geometric variants. The complexity of the L-shape Ramsey problem increases rapidly with grid size, making it challenging to find valid colorings or prove their non-existence.

In this paper, we explore computational approaches to finding valid 3-colorings for the L-shape Ramsey problem across different grid sizes. We develop deterministic pattern-based solvers and implement a neural-guided search framework called FunSearch, utilizing a large language model (Llama 3.2) to generate candidate solutions.

2 Methods

Our investigation employed multiple computational approaches to find valid 3-colorings for L-shape Ramsey grids of various sizes. We developed three main solvers:

2.1 Simple Deterministic Solver

The simple deterministic solver uses mathematical patterns to generate grid colorings. We implemented several pattern-generating functions:

- Modular arithmetic patterns: $grid[i, j] = (a \cdot i + b \cdot j) \mod 3$
- Latin square patterns for small grids
- Alternating patterns
- Block patterns for larger grids

Each generated grid was verified to ensure no L-shapes were monochromatic. The solver tests multiple patterns and reports the results for each grid size.

2.2 Specialized 4×4 Solver

After observing the difficulty in finding valid 3-colorings for the 4×4 grid using general patterns, we developed a specialized solver that tests various hand-crafted patterns specifically designed for the 4×4 case. These patterns included:

- Corner-focused patterns with strategic diagonal symmetry
- Modified checkerboard patterns
- Spiral and zigzag arrangements

The specialized solver systematically tests each pattern and verifies it against all possible L-shapes in the grid.

2.3 FunSearch Implementation

We implemented a simplified version of the FunSearch algorithm [2], which uses a large language model to generate candidate solutions for mathematical problems. Our implementation utilizes the Llama 3.2 model via Ollama.

The FunSearch process involves:

1. Starting with a baseline solution (Latin square pattern for 3×3)
2. Using the language model to generate Python functions that produce grid colorings
3. Evaluating each generated function by executing it and checking for monochromatic L-shapes
4. Providing feedback to the model through prompting with successful examples
5. Iteratively improving solutions based on discovered patterns

Our scoring function for valid colorings rewards:

- Absence of monochromatic L-shapes (mandatory)
- Diversity of colors in rows and columns
- Patterns that exhibit mathematical regularity

2.4 Verification Process

To ensure consistent verification across all solvers, we implemented a standardized verification process using the `LShapeGrid` class. This verifier:

1. Converts the numerical grid to a colored grid object
2. Checks all possible L-shapes in all orientations
3. Reports whether any L-shape is monochromatic, and if so, which specific L-shape fails

The verification considers all possible L-shapes in the grid, including different orientations (left+up, right+up, left+down, right+down) and distances.

3 Results

Our investigation yielded different results for various grid sizes:

3.1 3×3 Grid Results

For the 3×3 grid, we successfully found multiple valid 3-colorings:

- Latin square pattern emerged as an effective solution with a score of 7.0
- Modular arithmetic pattern $(i + 2 \cdot j) \mod 3$ also produced valid solutions
- Our FunSearch implementation successfully discovered multiple valid patterns

A representative valid 3×3, 3-coloring using the Latin square pattern:

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

This pattern ensures that each row and column contains each color exactly once, creating high diversity while avoiding monochromatic L-shapes.

3.2 4×4 Grid Results

The 4×4 grid proved significantly more challenging:

- Simple patterns like modular arithmetic consistently failed
- Most tested patterns failed with an invalid L-shape (Right+Up) at (0,0) with d=3
- Our specialized solver eventually discovered a valid solution using a corner-focused approach

The valid 4×4, 3-coloring we discovered:

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

This solution exhibits interesting properties:

- Symmetry along the main diagonal
- Strategic placement of duplicate colors to avoid forming L-shapes
- Each corner uses a different color (0, 2, 1, 0 clockwise from top-left)
- The pattern doesn't follow any simple formula, suggesting the need for complex arrangements

3.3 5×5 Grid and Larger

For grids of size 5×5 and larger, our investigation did not yield any valid 3-colorings:

- All tested patterns (modular arithmetic, alternating, block, etc.) failed verification
- FunSearch was unable to discover valid patterns despite multiple iterations

This suggests that either:

- Valid 3-colorings for these sizes require significantly more complex patterns
- More than 3 colors may be required to avoid monochromatic L-shapes
- The L-shape Ramsey number $R_L(3)$ may be 5 or lower

3.4 FunSearch Performance

Our FunSearch implementation successfully discovered valid patterns for the 3×3 grid, demonstrating the capability of language models to assist in combinatorial problem-solving:

- The model discovered multiple valid Latin square variations
- Each iteration improved or maintained the quality of solutions
- The best solution achieved a score of 7.0, representing optimal diversity

However, the FunSearch implementation struggled with larger grid sizes, failing to find valid patterns for 4×4 and 5×5 grids despite multiple iterations and varied temperature settings.

4 Discussion

4.1 Pattern Complexity and Grid Size

Our results demonstrate that the complexity of required patterns increases non-linearly with grid size. The Latin square pattern works effectively for 3×3 grids but fails for larger grids. The 4×4 solution requires a more intricate arrangement that doesn't follow a simple mathematical formula.

This complexity scaling suggests that the L-shape Ramsey problem becomes increasingly challenging as grid size grows, potentially requiring patterns that are difficult to express with simple mathematical formulas.

4.2 Verification Challenges

Our investigation revealed the importance of consistent verification methods. Early in our research, we encountered discrepancies in results due to different interpretations of L-shapes. The standardized verification process using the `LShapeGrid` class provided reliable and consistent results across all solvers.

4.3 FunSearch Capabilities and Limitations

The integration of language models into the search process shows promise for combinatorial optimization problems. Our FunSearch implementation successfully discovered valid patterns for 3×3 grids, demonstrating the capability of neural-guided search to find solutions in complex spaces.

However, the approach showed limitations for larger grids. This may be due to:

- The exponential increase in the search space with grid size
- Limited ability of language models to reason about complex spatial patterns
- The inherent difficulty of expressing complex grid patterns as code

4.4 Implications for L-shape Ramsey Numbers

Our inability to find valid 3-colorings for 5×5 grids and larger, despite extensive search, suggests that the L-shape Ramsey number $R_L(3)$ may be 5 or lower. This would mean that it is impossible to 3-color a 5×5 grid without creating a monochromatic L-shape.

However, without a formal proof, we cannot definitively establish this bound. The successful 4×4 solution demonstrates that valid 3-colorings exist for n=4, establishing that $R_L(3) > 4$.

5 Conclusion

Our investigation of the L-shape Ramsey problem revealed the increasing complexity of patterns required for valid colorings as grid size grows. We successfully found 3-colorings for 3×3 and 4×4 grids but could not discover valid solutions for larger grids.

The specialized 4×4 solution we discovered exhibits properties that suggest larger grids may require increasingly sophisticated, non-systematic patterns. This finding aligns with the general understanding of Ramsey theory, where the complexity of avoiding specific structures grows rapidly with the size of the host structure.

Our FunSearch implementation demonstrates the potential of neural-guided search for combinatorial problems but also highlights the limitations of current language models in reasoning about complex spatial patterns.

Future work could explore:

- Formal mathematical analysis to establish or bound L-shape Ramsey numbers
- Investigation of the minimum number of colors required for larger grid sizes
- Enhanced FunSearch implementations with more sophisticated prompting and evaluation techniques
- Distributed search approaches to explore a wider range of potential solutions
- Analysis of successful patterns to identify principles that might extend to larger grids

This research provides a foundation for further exploration of geometric Ramsey problems and demonstrates the potential of combining traditional deterministic approaches with neural-guided search methods.

References

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