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БФЗ-19-1.

(1) $\Psi = \frac{A}{x^2+a^2}$.

условие норм.: $\int\limits_{-\infty}^{+\infty} |\Psi|^2 dx = 1$

$$\int\limits_{-\infty}^{+\infty} \frac{A^2}{(x^2+a^2)^2} dx = A^2 \left(\overset{=0}{\frac{x}{2a^2(x^2+a^2)}} + \frac{1}{2a^3} \overset{=\pi}{\operatorname{arctg}} \frac{x}{a} \right) \Bigg|_{-\infty}^{+\infty} =$$

$$= A^2 \pi \cdot \frac{1}{2a^3} = 1$$

$$\boxed{A = \sqrt{\frac{2}{\pi}} \cdot a^{3/2}}$$

(2) $\varphi = \frac{B}{x+ib}$

$$\int\limits_{-\infty}^{+\infty} \left|\frac{B}{x+ib}\right|^2 dx = \int\limits_{-\infty}^{+\infty} \left|\frac{B(x-ib)}{x^2+b^2}\right|^2 dx = \int\limits_{-\infty}^{+\infty} \frac{B^2 \cancel{(x^2+b^2)}}{(x^2+b^2)^{\cancel{2}}} dx =$$

$$= \int\limits_{-\infty}^{+\infty} \frac{B^2}{x^2+b^2} dx = B^2 \left(\frac{1}{b} \operatorname{arctg} \frac{x}{b} \right) \Bigg|_{-\infty}^{+\infty} = \frac{B^2}{b} \cdot \pi = 1$$

$$\boxed{B = \sqrt{\frac{b}{\pi}}}$$

(3) $\langle \varphi | \psi \rangle = \int_{-\infty}^{+\infty} \varphi^*_{(x)} \cdot \psi_{(x)}\, dx =$

$$= \int_{-\infty}^{+\infty} \frac{A}{x^2+a^2} \cdot \left(\frac{B}{x+ib}\right)^* dx = \int_{-\infty}^{+\infty} \frac{A}{x^2+a^2} \cdot \frac{B(x+ib)}{x^2+b^2}\, dx =$$

$$= A \cdot B \int_{-\infty}^{+\infty} \underbrace{\frac{x}{(x^2+b^2)(x^2+a^2)}}_{0\ -?} + \underbrace{\frac{ib}{(x^2+b^2)(x^2+a^2)}}_{bi\, I_1}\, dx \boxed{=}$$

$$I_1 = \int_{-\infty}^{+\infty} \frac{dx}{(x^2+b^2)(x^2+a^2)} = 2\int_{0}^{+\infty} \frac{dx}{(x^2+b^2)(x^2+a^2)} =$$

чётная $\Longrightarrow$

$$= \frac{2}{a^2-b^2} \int_0^{+\infty} \frac{a^2-b^2}{(x^2+b^2)(x^2+a^2)}\, dx = \frac{2}{a^2-b^2} \int_0^{+\infty} \frac{a^2+x^2-b^2-x^2}{(x^2+b^2)(x^2+a^2)}\, dx =$$

$$= \frac{2}{a^2-b^2}\left[\int_0^{+\infty} \frac{1\, dx}{x^2+b^2} - \int_0^{+\infty} \frac{1}{x^2+a^2}\, dx\right] =$$

$$= \frac{2}{a^2-b^2}\left(\underbrace{\frac{1}{b}\operatorname{arctg}\frac{x}{b}}_{\frac{\pi}{2b}} - \underbrace{\frac{1}{a}\operatorname{arctg}\frac{x}{a}}_{\frac{\pi}{2a}}\right)\Bigg|_0^{\infty} = \boxed{\frac{\frac{\pi}{b} - \frac{\pi}{a}}{a^2-b^2}}$$

$$\boxed{=}\ A \cdot B \cdot ib \cdot I_1 = \frac{\sqrt{2b}}{\pi} \cdot a^{3/2} \cdot ib \cdot \frac{\pi a - \pi b}{ab(a^2-b^2)} =$$

$$\boxed{= \frac{2\sqrt{ab}}{\pi} \frac{\pi a - \pi b}{a^2-b^2} \cdot i} =$$

$$= 2\sqrt{ab}\,\frac{a-b}{(a-b)(a+b)}\cdot i = \boxed{\frac{2\sqrt{ab}}{a+b}\,i}$$

4)

$$\delta(f(x)) = \sum_{x_i} \frac{1}{f'(x_i)} \cdot \delta(x - x_i).$$

$$f'(x_i) \simeq \frac{f(x_i+\Delta x)}{\Delta x}, \quad \text{пусть } x = \Delta x + x_i$$

$$f(x) = \Delta x \cdot f'(x_i). = (x - x_i)\, f'(x_i)$$

$$\delta(ax) = \frac{1}{|a|}\delta x, \quad \forall a \neq 0$$

$$\delta\left(f'(x_i)\,(x - x_i)\right) = \sum_i \frac{1}{|f'(x_i)|}\,\delta(x - x_i).$$

(5) $x \in [0, a]$

$\psi(x) = \sqrt{\frac{2}{a}} \cdot \sin\frac{\pi x}{a}$

$\mathcal{F}_1 = \alpha_1 \cdot e^{\frac{i\pi x}{a}}$, $\mathcal{F}_2 = \alpha_2 \cdot e^{-\frac{i\pi x}{a}}$

1) $\langle \mathcal{F}_1 | \mathcal{F}_2 \rangle = \int\limits_0^a \alpha_1 \alpha_2 \, e^{\frac{i\pi x}{a}} \cdot e^{\frac{i\pi x}{a}} =$

$= \frac{2i\pi}{a} \alpha_1 \alpha_2 (1 - \underbrace{\cos 2\pi}_{1} - i \underbrace{\sin 2\pi}_{0}) = 0$

ч. т. д.

2) $\int\limits_0^a |\mathcal{F}_1|^2 dx = 1$ - усл. нормировки.

$\int\limits_0^a \alpha_1^2 dx = \alpha_1^2 \cdot a = 1$

$\alpha_1 = \frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{a}$

$\int\limits_0^a |\mathcal{F}_2|^2 dx = 1$

$\alpha_2 = \frac{\sqrt{a}}{a} = \alpha_1 = \alpha$

3) $\psi = C_1 \cdot |\mathcal{F}_1\rangle + C_2 \cdot |\mathcal{F}_2\rangle =$

$= \alpha \cdot C_1 \cdot (\cos\frac{\pi x}{a} + i \sin\frac{\pi x}{a}) + \alpha C_2 (\cos\frac{\pi x}{a} - i \sin\frac{\pi x}{a}) =$

$= \begin{cases} Re\, C_1 + Re\, C_2 = 0 & \text{- действ. кос.} \\ Re\, C_1 - Re\, C_2 = 0 & \text{- мним. синус.} \\ Im\, C_1 + Im\, C_2 = 0 & \text{- действ. мним. cos.} \\ -Im\, C_1 + Im\, C_2 = \sqrt{\frac{2}{a}}/\alpha & \text{- действ. cos. sin} \end{cases}$

действ. синус.

$$\operatorname{Re} C_1 = \operatorname{Re} C_2 = 0.$$

$$\operatorname{Im} C_1 = -\operatorname{Im} C_2 = \operatorname{Im} C.$$

$$\left.\begin{array}{l} \operatorname{Im} C\cancel{x} = -\dfrac{1}{\sqrt{2a}} \\[2ex] \operatorname{Im} C\cancel{x} = -\dfrac{1}{\sqrt{2a}} \end{array}\right\} \begin{array}{l} C_1 = \dfrac{-i}{\sqrt{2a}} \cdot \dfrac{1}{\alpha} = \boxed{\dfrac{-i}{\sqrt{2}}} \\[2ex] C_2 = \dfrac{i}{\sqrt{2a}} \cdot \dfrac{1}{\alpha} = \boxed{+\dfrac{i}{\sqrt{2}}} \end{array}$$