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1.1 $A = \begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix}$

СЗ, СВ: $A|f\rangle = \lambda|f\rangle$.

ХУ: $\left(\begin{array}{cc|c} \cos\theta-\lambda & -i\sin\theta & 0 \\ i\sin\theta & -\cos\theta-\lambda & 0 \end{array}\right)$.

$Det = 0$

$-(\cos\theta-\lambda)(\cos\theta+\lambda) - \sin^2\theta = 0$

$\lambda^2 = 1 \quad \boxed{\lambda = \pm 1}$

1) $\lambda = 1$.

$A|f\rangle = |f\rangle$

$(\cos\theta \quad)f_1 - i\sin\theta f_2 = f_1.$

$f_1 = \dfrac{-i\sin\theta f_2}{-\cos\theta + 1}$ $\qquad |f_{\lambda=1}\rangle = \begin{pmatrix} i\sin\theta \\ \cos\theta - 1 \end{pmatrix}.$

$|f_{\lambda=1}|^2 = \sin^2\theta + \cos^2\theta - 2\cos\theta + 1 \ominus$

$= 2(-\cos\theta + 1).$

$|f^n_{\lambda=1}\rangle = \boxed{\begin{pmatrix} \dfrac{i\sin\theta}{\sqrt{2}(-\cos\theta+1)^{1/2}} \\ -\frac{1}{\sqrt{2}}\sqrt{\cos\theta - 1} \end{pmatrix}} = \begin{pmatrix} \dfrac{i\sin\theta}{\sqrt{2(\cos\theta-1)}} \\ -\frac{1}{\sqrt{2}}\sqrt{\cos\theta-1} \end{pmatrix}$

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2) $\lambda = -1$

$$\cos\theta\, f_1 - i\sin\theta\, f_2 = -f_1.$$

$$f_1 = \frac{+i\sin\theta\, f_2}{\cos\theta + 1} \qquad |f_{\lambda=-1}\rangle = \begin{pmatrix} i\sin\theta \\ \cos\theta + 1 \end{pmatrix}.$$

$$|f_{\lambda=-1}|^2 = \sin^2\theta + \cos^2\theta + 2\cos\theta + 1 = 2(\cos\theta + 1).$$

$$|f^n_{\lambda=-1}\rangle = \boxed{\begin{pmatrix} \dfrac{i\sin\theta}{\sqrt{2(\cos\theta+1)}} \\ \dfrac{1}{\sqrt{2}}\sqrt{\cos\theta+1} \end{pmatrix}}$$

Ортогональность проверяю по ненормир. собственным векторам, так проще:

$$\langle f_{\lambda=1} | f_{\lambda=-1} \rangle = 0 = f^{1*}_{\lambda=1} \cdot f^1_{\lambda=-1} + f^{2*}_{\lambda=1} \cdot f^2_{\lambda=-1} =$$

$$= -1(i\sin\theta)^2 + (\cos\theta - 1)(\cos\theta + 1) = \sin^2\theta + \cos^2\theta - 1 =$$

$$= 0,$$

ч.т.д.

(1.2) $e^{i\alpha A} = \sum_{j=0}^{\infty} \frac{(i\alpha A)^j}{j!} = 1 + \frac{i\alpha A}{1} - \frac{\alpha^2 A^2}{2} - \frac{i\alpha^3 A^3}{6} + \ldots =$

$$= 1 + i\sum_{j=1}^{\infty} \frac{(-1)^{j+1}\alpha^{2j-1}A^{2j-1}}{(2j-1)!} + \sum_{k=1}^{\infty}\frac{(-1)^k \alpha^{2k}A^{2k}}{(2k)!\}}. \boxed{=}$$

$$A^2|f\rangle = A|A|f\rangle =$$

$$= A\left|\begin{pmatrix} f_1\cos\theta - f_2 i\sin\theta \\ f_1 i\sin\theta - f_2\cos\theta \end{pmatrix}\right. =$$

$$= \begin{pmatrix} f_1\cdot\cos^2\theta - f_2 i\sin\theta\cos\theta + f_1\cdot\sin^2\theta + f_2 i\sin\theta\cos\theta \\ f_1 i\sin\theta\cos\theta + f_2\sin^2\theta - f_1 i\sin\theta\cos\theta + f_2\cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = |f\rangle \Rightarrow \boxed{\begin{matrix} A^{2k} = \hat{1} \\ A^{2k+1} = A. \end{matrix}}$$

$$\boxed{=} \; 1 + iA\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\alpha^{2j-1}}{(2j-1)!} + \sum_{k=1}^{\infty}\frac{(-1)^k\alpha^{2k}}{(2k)!} =$$

$$= \boxed{i\cdot A\cdot\sin\alpha + \cos\alpha}$$

2.1 $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$

$A|F\rangle = \lambda|F\rangle.$

$xy: \begin{pmatrix} -\lambda & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & -\lambda & -i/\sqrt{2} \\ 0 & +i/\sqrt{2} & -\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

$-\lambda^3 + \frac{\lambda}{2} + \frac{\lambda}{2} = 0$

$\lambda(\lambda^2 - 1) = 0,$ $\boxed{\begin{matrix}\lambda = \pm 1 \\ \lambda = 0\end{matrix}}$

1) $\lambda = 1.$

$\begin{pmatrix} -\frac{i}{\sqrt{2}}F_2 \\ \frac{i}{\sqrt{2}}F_1 - \frac{i}{\sqrt{2}}F_3 \\ \frac{i}{\sqrt{2}}F_2 \end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix} -F_2 \\ F_1 - F_3 \\ F_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}.$

$F_1 = -\frac{i}{\sqrt{2}} F_2 = -F_3.$

$F_2 = \frac{-\sqrt{2}}{i} F_1 = i\sqrt{2} F_1$

$|F_{\lambda=1}\rangle = \begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}, \quad |F_{\lambda=1}| = \sqrt{4} = 2$

$\boxed{|F^n_{\lambda=1}\rangle = \begin{pmatrix} 1/2 \\ i\sqrt{2}/2 \\ -1/2 \end{pmatrix}}$

2) $\lambda = -1$

$$\frac{i}{\sqrt{2}}\begin{pmatrix} -\mathcal{F}_2 \\ \mathcal{F}_1 - \mathcal{F}_3 \\ \mathcal{F}_2 \end{pmatrix} = -\begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \end{pmatrix}$$

$$\mathcal{F}_1 = -\mathcal{F}_3 = \frac{+i}{\sqrt{2}}\mathcal{F}_2$$

$$\mathcal{F}_2 = -\frac{(-\sqrt{2})}{i}\mathcal{F}_1 = -\sqrt{2}\,i\,\mathcal{F}_1$$

$$|\mathcal{F}_{\lambda=-1}\rangle = \begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix}.$$

$$|\mathcal{F}_{\lambda=-1}| = \sqrt{4} = 2.$$

$$\boxed{|\mathcal{F}^n_{\lambda=-1}\rangle = \begin{pmatrix} 1/2 \\ -\frac{\sqrt{2}}{2}i \\ -1/2 \end{pmatrix}};$$

3) $\lambda = 0$.

$$\frac{i}{\sqrt{2}}\begin{pmatrix} -\mathcal{F}_2 \\ \mathcal{F}_1 - \mathcal{F}_3 \\ \mathcal{F}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathcal{F}_2 = 0$$

$$\mathcal{F}_1 = \mathcal{F}_3 \qquad |\mathcal{F}_{\lambda=0}\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$|\mathcal{F}_{\lambda=0}| = \sqrt{2}$$

$$\boxed{|\mathcal{F}^n_{\lambda=0}\rangle = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{pmatrix}}$$

проверка ортогональности:

$$\langle \mathcal{F}_{\lambda=1} | \mathcal{F}_{\lambda=-1} \rangle = x_1^* x_{-1} + y_1^* y_{-1} + z_1^* z_{-1} =$$

$$= 1 - 2 + 1 = 0.$$

$$\langle \mathcal{F}_{\lambda=1} | \mathcal{F}_{\lambda=0} \rangle = 1 - 1 = 0.$$

ч.т.д.

$$\langle \mathcal{F}_{\lambda=-1} | \mathcal{F}_{\lambda=0} \rangle = 1 - 1 = 0.$$

2.2 $$e^{i\theta A} = \sum_{j=0}^{\infty} \frac{(i\theta A)^n}{n!} = 1 + \frac{i\theta A}{1} - \frac{\theta^2 A^2}{2} - \frac{i\theta^3 A^3}{6} + \ldots$$

$$= \sum_{j=0}^{\infty} \frac{(\theta A)^{2j}}{(2j)!} \cdot (-1)^j + i \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \theta^{2k-1} A^{2k-1}}{(2k-1)!} =$$

Исследуем A^2:

$$A|\mathcal{F}\rangle = \frac{i}{\sqrt{2}} \begin{pmatrix} -\mathcal{F}_2 \\ \mathcal{F}_1 - \mathcal{F}_3 \\ \mathcal{F}_2 \end{pmatrix}$$

$$A^2|\mathcal{F}\rangle = A|A|\mathcal{F}\rangle = \frac{i}{\sqrt{2}} \cdot \frac{i}{\sqrt{2}} \begin{pmatrix} \mathcal{F}_3 - \mathcal{F}_1 \\ -2\mathcal{F}_2 \\ \mathcal{F}_1 - \mathcal{F}_3 \end{pmatrix}$$

теперь A^3:

$$A^3|\mathcal{F}\rangle = \left(\frac{i}{\sqrt{2}}\right)^3 \begin{pmatrix} 2\mathcal{F}_2 \\ 2\mathcal{F}_3 - 2\mathcal{F}_1 \\ -2\mathcal{F}_2 \end{pmatrix} = \frac{+i}{\sqrt{2}} \begin{pmatrix} -\mathcal{F}_2 \\ \mathcal{F}_1 - \mathcal{F}_3 \\ \mathcal{F}_2 \end{pmatrix} = A|\mathcal{F}\rangle$$

т.е.: $A^{2n+1} = A$. $A^{2n} = A^2$

$$\Leftrightarrow A^2 \sum_{j=0}^{\infty} \frac{\Theta^{2j}(-1)^j}{(2j)!} + iA\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\Theta^{2k-1}}{(2k-1)!} \Leftrightarrow$$

$$\Leftrightarrow \boxed{A^2 \cdot \cos\Theta + A \cdot i \sin\Theta}$$

2.3) базис с. в.:

$$|1\rangle = \begin{pmatrix} 1/2 \\ i\frac{\sqrt{2}}{2} \\ -1/2 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 1/2 \\ -i\frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{pmatrix}.$$

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

в новом базисе: $|\psi'\rangle = \begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix}$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ i\frac{\sqrt{2}}{2} & -\frac{i\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\psi_1' = \psi_2'$$

$$\psi_1' = \frac{\sqrt{2}}{2}\psi_3' = 0{,}5 \quad ,$$

$$\boxed{\begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ \sqrt{2}/2 \end{pmatrix}}$$

(3.1) $T_a \psi(x) = \psi(x+a)$

<u>не эрмитов, т.к.:</u>

$$\langle \varphi | \hat{T}_a \psi \rangle = \int_{-\infty}^{+\infty} \varphi^*(x)\, \psi(x+a)\, dx = \int \varphi^*(x-a)\, \psi(x)\, dx \ominus$$

$$\ominus \langle \hat{T}_{-a} \varphi | \psi \rangle, \text{ т.е.:}$$

$$\hat{T}_a^{+} = \hat{T}_{-a}$$

$$\hat{T}_a^{+} \hat{T}_a |\psi\rangle = \hat{T}_a \psi(x+a) = \psi(x)$$

$$\boxed{\hat{T}_a^{+} \hat{T}_a = \hat{1}}$$

ядро: $\hat{T}_a \psi = \int_{-\infty}^{+\infty} T_a(x,x')\, \psi(x')\, dx' = \psi(x+a).$

$$T_a(x,x') = \delta(x' - (x+a)) \cdot \frac{d}{dx'}$$

проверяю:

$$\int_{-\infty}^{+\infty} \delta(x' - (x+a)) \frac{d\psi(x')}{dx'}\, dx' = \psi(x+a)$$

ч.т.д.

④ $\hat{S} = \alpha\left(x^2\frac{d}{dx} - \frac{d}{dx}x^2\right)$

$$\hat{S}|\psi\rangle = \alpha\left(x^2\frac{d\psi}{dx} - 2x\cdot\psi - x^2\frac{d\psi}{dx}\right) =$$

$$= \underline{\underline{-2\alpha\cdot x\,\psi}}$$

проверка на самосопряжённость:

$$\langle\varphi|\hat{S}\psi\rangle = \int_{-\infty}^{+\infty} -2\alpha x\cdot\psi\cdot\varphi^* dx =$$

$$= \langle\hat{S}^+\varphi|\psi\rangle = \int_{-\infty}^{+\infty} (\hat{S}^+\varphi)^*\cdot\psi\, dx = \text{(для } \alpha\in\mathrm{Re})$$

$$= \int_{-\infty}^{+\infty} (-2\alpha x\varphi)^*\cdot\psi\, dx$$

Т.е. для $\forall\,\alpha\in\mathrm{Re}$ данный оператор является эрмитовым.

1) если $\alpha\in\mathrm{Re}$: $\boxed{\hat{S}^+ = S.}$

2) $\alpha \notin \mathrm{Re}$

$$(\hat{S}^+\varphi)^* = -2\alpha x\varphi^* = -2\alpha e^{i(\theta+\pi)} x\varphi^*$$

$\alpha = \alpha\cdot e^{i\theta}$

$$\boxed{\hat{S}^+ = -2\alpha^* x}$$

Ядро: $\hat{S}\Psi = \int\limits_{-\infty}^{+\infty} S(x,x')\Psi(x')dx' = -2\alpha x\Psi(x).$

$$S(x,x') = -2\alpha x\,\delta(x-x')\frac{d}{dx'}$$

проверим:

$$\int\limits_{-\infty}^{+\infty} -2\alpha x\,\delta(x-x')\frac{d\Psi(x')}{dx'}dx' = -2\alpha x\Psi(x), \text{ ч.т.д.}$$

5.1

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$BX = X', \quad B^{-1}X' = X$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$$

$$\begin{pmatrix} a^{-1} & b^{-1} \\ c^{-1} & d^{-1} \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} aa^{-1}x_1 + b\cdot a^{-1}x_2 + b^{-1}cx_1 + b^{-1}dx_2 \\ c^{-1}ax_1 + c^{-1}\cdot bx_2 + d^{-1}\cdot cx_1 + d^{-1}dx_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} a & c & 0 & 0 & 1 \\ b & d & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 \\ 0 & 0 & b & d & 1 \end{array}\right)$$

$$\begin{array}{cc} a^{-1} & b^{-1} \\ \end{array}$$

$$\left(\begin{array}{cc|c} a & c & 1 \\ b & d & 0 \end{array}\right) \to \left(\begin{array}{cc|c} ad & cd & d \\ bc & cd & 0 \end{array}\right) - \quad (ad-bc)a^{-1} = d$$

$$a^{-1} = \frac{d}{ad-bc}$$

$$\left(\begin{array}{cc|c} a & c & 1 \\ b & d & 0 \end{array}\right) \to \left(\begin{array}{cc|c} ab & cb & b \\ ab & ad & 0 \end{array}\right) - (cb-ad)b^{-1} = b$$

$$b^{-1} = \frac{-b}{ad-bc}$$

$$\begin{array}{cc} c^{-1} & d^{-1} \\ \end{array}$$

$$\left(\begin{array}{cc|c} a & c & 0 \\ b & d & 1 \end{array}\right) \to \left(\begin{array}{cc|c} ad & cd & 0 \\ bc & cd & c \end{array}\right) \to (bc-ad)c^{-1} = c$$

$$c^{-1} = \frac{-c}{ad-bc}$$

$$\left(\begin{array}{cc|c} a & c & 0 \\ b & d & 1 \end{array}\right) \to \left(\begin{array}{cc|c} ab & cb & 0 \\ ab & ad & a \end{array}\right) \to (ad-cb)d^{-1} = a$$

$$d^{-1} = \frac{a}{ad-cb}.$$

$$B^{-1} = \begin{pmatrix} a^{-1} & b^{-1} \\ c^{-1} & d^{-1} \end{pmatrix} = \boxed{\frac{1}{ad-cb}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$ Ч.Т.Д.

(5.2) $\mathcal{O} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

$$\langle \varphi | \mathcal{O}\psi \rangle = \langle \mathcal{O}^{+}\varphi | \psi \rangle$$

$$\mathcal{O}\psi = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta\,\psi_1 - \sin\theta\,\psi_2 \\ \sin\theta\,\psi_1 + \cos\theta\,\psi_2 \end{pmatrix}$$

тогда:

$$\langle \varphi | \mathcal{O}\psi \rangle = \varphi_1^* \cdot \psi_1 \cos\theta - \varphi_1^* \psi_2 \sin\theta +$$

$$+ \varphi_2^* \psi_1 \sin\theta + \varphi_2 \psi_2 \cos\theta. =$$

$$= \psi_1 (\varphi_1^* \cos\theta + \varphi_2^* \sin\theta) + \psi_2 (\varphi_2^* \cos\theta - \varphi_1^* \sin\theta). =$$

$$= \langle \mathcal{O}^{+}\varphi | \psi \rangle$$

отсюда: $\boxed{\mathcal{O}^{+} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}$

5.3 $\mathcal{O} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$

$$\mathcal{O}|\psi\rangle = \begin{pmatrix} \psi_1 \cos\theta - \psi_2 \sin\theta \\ \psi_1 \sin\theta + \psi_2 \cos\theta \end{pmatrix} = \psi'$$

$$\mathcal{O}^{-1}\psi' = \psi$$

Воспользуюсь рез-том 5.1

$$\mathcal{O}^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \boxed{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}$$

6. $\underbrace{\langle \varphi | \hat{A}\psi \rangle = \langle \hat{A}^{+}\varphi | \psi \rangle}_{\text{по определению:}}$

$$\langle (\hat{A}\hat{B})^{+}\varphi | \psi \rangle = \langle \varphi | \hat{A}\hat{B}\psi \rangle =$$

$$= \langle \hat{A}^{+}\varphi | \hat{B}\psi \rangle = \langle \hat{B}^{+}\hat{A}^{+}\varphi | \psi \rangle$$

$$\Downarrow$$

$$(\hat{A}\hat{B})^{+} = \hat{B}^{+}\hat{A}^{+}, \text{ ч.т.д.}$$

(7)

$\langle\Psi|\hat{A}|\Psi\rangle \geq 0$ - положительно определенный оператор $\hat{A}$.

$\hat{C}$ - лин. оператор

$$\langle\Psi|\hat{C}^{+}\hat{C}|\Psi\rangle = \langle\hat{C}\Psi|\hat{C}\Psi\rangle \ominus$$

$$\ominus \int_{-\infty}^{+\infty} |\hat{C}\Psi|^2 dx \geq 0. \text{, ч.т.д.}$$