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1) $\psi = \dfrac{A}{x^2+a^2}$

Условие нормировки: $\displaystyle\int_{-\infty}^{+\infty} \psi \, dx = 1 \Leftrightarrow$

$$\Leftrightarrow A \cdot \int_{-\infty}^{+\infty} \frac{dx}{a^2+x^2} = \frac{A}{a}\left(\operatorname{arctg}\frac{x}{a}\Bigg|_{-\infty}^{+\infty}\right) = 2\frac{A}{a} = 1$$

$$\boxed{A = \frac{a}{2}}$$

2) $\varphi = \dfrac{B}{x+ib}$

$$\int_{-\infty}^{+\infty} \frac{B}{x+ib}\,dx = \int_{-\infty}^{+\infty} \frac{B(x-ib)}{x^2+b^2}\,dx = 1 \Leftrightarrow$$

по

$$\Leftrightarrow B\Bigg(\underbrace{\int_{-\infty}^{+\infty}\frac{x}{x^2+b^2}\,dx}_{I_1} - \underbrace{ib\int_{-\infty}^{+\infty}\frac{1}{x^2+b^2}\,dx}_{I_2.}\Bigg) \Leftrightarrow$$

$$I_2 = -ib\int_{-\infty}^{+\infty}\frac{dx}{x^2+b^2} = -ib\left(\operatorname{arctg}\frac{x}{b}\Bigg|_{-\infty}^{+\infty}\right) = \boxed{-2ib}$$

$I_1 = 0$, т.к. $\dfrac{x}{x^2+b^2} = -\left(\dfrac{(-x)}{(-x)^2+b^2}\right) \Rightarrow -2ibB = 1$

$$B = \frac{1}{-2ib} = -\frac{1}{i}\cdot\frac{1}{2b} = \boxed{\frac{i}{2b}}$$

③ $\langle \varphi | \psi \rangle = \int\limits_{-\infty}^{+\infty} \varphi^*(x) \cdot \psi(x)\, dx =$

$$= \int\limits_{-\infty}^{+\infty} \left[B \cdot \frac{i}{2b} \frac{(x-ib)}{x^2+b^2} \right]^* \times \left[\frac{1}{2} \frac{a}{a^2+x^2} \right] dx =$$

$$= \int\limits_{-\infty}^{+\infty} \left[\frac{b \mp ix}{x^2+b^2} \cdot \frac{1}{2b} \right] \times \frac{a}{2} \left[\frac{1}{a^2+x^2} \right] dx =$$

$$= \frac{a}{4b} \int\limits_{-\infty}^{+\infty} \frac{b-ix}{x^4+(a^2+b^2)x^2+a^2b^2}\, dx =$$

$$= \frac{\cancel{\ }a}{4b} \left\{ \underbrace{\int\limits_{-\infty}^{+\infty} \frac{b\, dx}{x^4+(a^2+b^2)x^2+a^2b^2}}_{I_2} - i \underbrace{\int\limits_{-\infty}^{+\infty} \frac{x}{x^4+(a^2+b^2)x^2+a^2b^2}\, dx}_{I_1} \right\} =$$

$I_1 = 0$, т.к. нечетная ф-ия

$I_2 = \cancel{\ }$

$$= b \int\limits_{-\infty}^{+\infty} \frac{dx}{x^4+(a^2+b^2)x^2+a^2b^2} = b \int\limits_{-\infty}^{+\infty} \frac{dx}{x^4+a^2x^2+b^2x^2+a^2b^2} =$$

тут домножу единицу

$$= b \int\limits_{-\infty}^{+\infty} \frac{dx}{(b^2+x^2)(a^2+x^2)} = b \int\limits_{-\infty}^{+\infty} \frac{(a^2-b^2)}{(a^2+x^2)(b^2+x^2)(a^2-b^2)}\, dx =$$

$$= \frac{b}{a^2-b^2} \int\limits_{-\infty}^{+\infty} \left[\frac{a^2}{(a^2+x^2)(b^2+x^2)} - \frac{b^2}{(a^2+x^2)(b^2+x^2)} \right] dx =$$

$$= \frac{b}{a^2-b^2}\int_{-\infty}^{+\infty}\frac{a^2+b^2}{(a^2+x^2)(b^2+x^2)}\,dx =$$

$c(a^2+x^2)=1$

$c=\frac{1}{a^2+x^2}$

$$= \frac{b}{a^2-b^2}\int_{-\infty}^{+\infty}\frac{1}{x^2}\,\cancel{x^2}\,\frac{a^2x^2 + -b^2x^2 + xa^2b^2 + x^4 + a^2b^2 - x^4}{(a^2+x^2)(b^2+x^2)}\,dx =$$

$$= \frac{b}{a^2-b^2}\int_{-\infty}^{+\infty}\frac{1}{x^2}\left[\frac{a^2x^2+a^2b^2+x^4+b^2-}{(a^2+x^2)(b^2+x^2}\right.$$

$$= \frac{b}{a^2-b^2}\int_{-\infty}^{+\infty}\frac{a^2+x^2-b^2-x^2}{(a^2+x^2)(b^2+x^2)}\,dx =$$

$$= \frac{b}{a^2-b^2}\int_{-\infty}^{+\infty}\left[\frac{1}{b^2+x^2}-\frac{1}{a^2+x^2}\right]dx =$$

$$= \frac{b}{a^2-b^2}\left(\arctan\frac{x}{a}-\arctan\frac{x}{b}\right)\Bigg|_{-\infty}^{+\infty} = 0$$

$\boxed{=}$ $\underline{\underline{0}}$ — ответ

(4) $\delta(f(x)) = \sum\limits_{x_i} \frac{1}{f(x_i)} \delta(x - x_i)$. — опечатка???

x_i – нули первого порядка ф-ии $f(x)$.

$f'(x_i) \approx \frac{f(x_i + \Delta x)}{\Delta x}$ $\quad$ ($f(x_i) = 0$ из условия).

$\Downarrow$

$x = \Delta x + x_i \Rightarrow \quad f(x) \approx f'(x_i) \cdot \Delta x = f'(x_i)(x - x_i)$

$\delta(ax) = \frac{1}{|a|} \delta x$, $\forall a \neq 0$ – св-во δ-ф-ии.

$$\delta\left(f'(x_i)(x - x_i)\right) = \boxed{\sum_i \frac{1}{|f'(x_i)|} \delta(x - x_i).}$$

⑤ $x \in [0, a]$

$$\psi(x) = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a}$$

$$\xi_1 = \alpha_1 e^{i\pi x/a}, \quad \xi_2 = \alpha_2 e^{-i\pi x/a}$$

1) $\langle \xi_1 | \xi_2 \rangle = \int_0^a \alpha_1 \alpha_2 e^{\frac{i\pi x}{a}} \cdot e^{\frac{i\pi x}{a}} =$

$= \alpha_1 \alpha_2 \int_0^a e^{\frac{2i\pi x}{a}} = \frac{2i\pi}{a} \alpha_1 \alpha_2 \; e^x \Big|_0^{2i\pi} =$

$= \frac{2i\pi}{a} \alpha_1 \alpha_2 \left(1 - \overset{=1}{\cos 2\pi} - i \overset{=0}{\sin 2\pi}\right) = 0$

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2) $\int_0^a \alpha_1 \cdot e^{\frac{i\pi x}{a}} dx = \alpha_1 \int_0^{i\pi} \frac{i\pi}{a} e^x dx = \frac{i\pi\alpha_1}{a} \left(1 - \underset{=-1}{\cos\pi} - i\underset{=0}{\sin\pi}\right) =$

$= \frac{2i\pi\alpha_1}{a} = 1$

$$\alpha_1 = \frac{a}{2i\pi} = \boxed{\frac{-ia}{2\pi}}$$

$\int_0^a \alpha_2 \cdot e^{-\frac{i\pi x}{a}} = \alpha_2 \cdot \frac{-i\pi}{a} \int_0^{-\pi i} e^x dx = -\frac{i\pi\alpha_2}{a} \left(1 - \overset{=-1}{\cos(-\pi)} - i\overset{=0}{\sin(-\pi)}\right) =$

$= -\frac{2i\pi\alpha_2}{a} = 1$

$$\alpha_2 = \boxed{\frac{ia}{2\pi}}$$

3). $|\psi\rangle = C_1|\mathfrak{F}_1\rangle + C_2|\mathfrak{F}_2\rangle.$

$$|\psi\rangle = C_1\alpha_1\left(\cos\frac{\pi x}{a} + i\sin\frac{\pi x}{a}\right) \oplus$$

$$\oplus\ C_2\alpha_2\left(\cos\frac{\pi x}{a} - i\sin\frac{\pi x}{a}\right) =$$ действительная ~~число.~~ ф-ция.

$\alpha_1 \cdot \mathrm{Re}\,C_1 + \alpha_2 \cdot \mathrm{Re}\,C_2 = 0.$ — косинусы действ.

$\alpha_1 \cdot \mathrm{Re}\,C_1 - \alpha_2\,\mathrm{Re}\,C_2 = 0$ — синусы мнимые

$\mathrm{Re}\,C_1 = \mathrm{Re}\,C_2 = 0$.

$\alpha_1 \cdot \mathrm{Im}\,C_1 + \alpha_2\,\mathrm{Im}\,C_2 = 0$ — косинусы мнимые

$\alpha_1 \cdot \mathrm{Im}\,C_1 - \alpha_2\,\mathrm{Im}\,C_2 = -\sqrt{\frac{2}{a}}$ — синусы действ.

$$\alpha_1 = -\alpha_2 = \alpha. = \frac{-ia}{2\pi}$$

$$\left(\begin{array}{cc|c} \mathrm{Im}\,C_1 & \mathrm{Im}\,C_2 & \cancel{\mathrm{Re}\,C_1\ \mathrm{Re}\,C_2} \\ \alpha & -\alpha & 0 \\ \alpha & +\alpha & -\sqrt{\frac{2}{a}} \end{array}\right) \qquad \left.\begin{array}{l} \mathrm{Im}\,C_1 = \mathrm{Im}\,C_2 \\ \mathrm{Re}\,C_1 = \mathrm{Re}\,C_2 = 0 \end{array}\right\} \underline{C_1 = C_2 = C.}$$

$$2\alpha \overset{\mathrm{Im}}{C} = -\sqrt{\frac{2}{a}}$$

$$+2\frac{ia\overset{\mathrm{Im}}{C}}{2\pi} = +\sqrt{\frac{2}{a}} \Rightarrow \mathrm{Im}\,C = \sqrt{\frac{2}{a}} \cdot \frac{\pi}{ai} = -\sqrt{2a}\,\pi i$$

$$C_1 = C_2 = C = \mathrm{Re}\,C + \mathrm{Im}\,C \cdot i = \boxed{\sqrt{2a}\,\pi}$$