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1.1 $A = \begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix}$

СЗ, СВ: $A|\mathcal{F}\rangle = \lambda|\mathcal{F}\rangle$.

ХУ: $\left(\begin{matrix} \cos\theta-\lambda & -i\sin\theta \\ i\sin\theta & -\cos\theta-\lambda \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right)$.

$Det = 0$

$-(\cos\theta-\lambda)(\cos\theta+\lambda) - \sin^2\theta = 0$

$\lambda^2 = 1$ $\boxed{\lambda = \pm 1}$

1) $\lambda = 1$.

$A|\mathcal{F}\rangle = |\mathcal{F}\rangle$

$(\cos\theta \quad)\mathcal{F}_1 - i\sin\theta\,\mathcal{F}_2 = \mathcal{F}_1$.

$\mathcal{F}_1 = \dfrac{-i\sin\theta\,\mathcal{F}_2}{-\cos\theta+1}$ $\quad |\mathcal{F}_{\lambda=1}\rangle = \begin{pmatrix} i\sin\theta \\ \cos\theta-1 \end{pmatrix}$.

$|\mathcal{F}_{\lambda=1}|^2 = \sin^2\theta + \cos^2\theta - 2\cos\theta + 1 =$

$= 2(-\cos\theta+1)$.

$|\mathcal{F}^n_{\lambda=1}\rangle = \boxed{\begin{pmatrix} \dfrac{i\sin\theta}{\sqrt{2}(-\cos\theta+1)^{1/2}} \\ -\frac{1}{\sqrt{2}}\sqrt{\cos\theta-1} \end{pmatrix}} = \begin{pmatrix} \dfrac{i\sin\theta}{\sqrt{2(\cos\theta-1)}} \\ -\frac{1}{\sqrt{2}}\sqrt{\cos\theta-1} \end{pmatrix}$

2) $\lambda = -1$

$$\cos\theta\, \xi_1 - i\sin\theta\, \xi_2 = -\xi_1.$$

$$\xi_1 = \frac{+i\sin\theta\, \xi_2}{\cos\theta + 1} \qquad |\xi_{\lambda=-1}\rangle = \begin{pmatrix} i\sin\theta \\ \cos\theta + 1 \end{pmatrix}.$$

$$|\xi_{\lambda=-1}|^2 = \sin^2\theta + \cos^2\theta + 2\cos\theta + 1 = 2(\cos\theta + 1).$$

$$|\xi^n_{\lambda=-1}\rangle = \boxed{\begin{pmatrix} \dfrac{i\sin\theta}{\sqrt{2(\cos\theta+1)}} \\ \dfrac{1}{\sqrt{2}}\sqrt{\cos\theta+1} \end{pmatrix}}$$

Ортогональность проверяю по ненормир. собственным векторам, так проще.

$$\langle \xi_{\lambda=1} | \xi_{\lambda=-1} \rangle = 0 = \xi^{1*}_{\lambda=1} \cdot \xi^{1}_{\lambda=-1} + \xi^{2*}_{\lambda=1} \cdot \xi^{2}_{\lambda=-1} =$$

$$= -1(i\sin\theta)^2 + (\cos\theta - 1)(\cos\theta + 1) = \sin^2\theta + \cos^2\theta - 1 =$$

$$= 0,$$

ч.т.д.

(1.2) $e^{i\alpha A} = \sum_{j=0}^{\infty} \frac{(i\alpha A)^j}{j!} = 1 + \frac{i\alpha A}{1} - \frac{\alpha^2 A^2}{2} - \frac{i\alpha^3 A^3}{6} + \ldots =$

$$= 1 + i\sum_{j=1}^{\infty} \frac{(-1)^{j+1} \alpha^{2j-1} A^{2j-1}}{(2j-1)!} + \sum_{k=1}^{\infty} \frac{(-1)^k \alpha^{2k} A^{2k}}{(2k)!} \boxed{=}$$

$$A^2|f\rangle = A|A|f\rangle =$$

$$= A\left|\begin{pmatrix} f_1 \cos\theta - f_2 i \sin\theta \\ f_1 i \sin\theta - f_2 \cos\theta \end{pmatrix}\right. =$$

$$= \begin{pmatrix} f_1 \cdot \cos^2\theta - f_2 i \sin\theta\cos\theta + \cancel{f_1} \cdot \sin^2\theta + f_2 i \sin\theta\cos\theta \\ f_1 i \sin\theta\cos\theta + f_2 \sin^2\theta - f_1 i \sin\theta\cos\theta + f_2 \cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = |f\rangle \implies \boxed{\begin{matrix} A^{2k} = \hat{1} \\ A^{2k+1} = A. \end{matrix}}$$

$$\boxed{=} \ 1 + iA\sum_{j=1}^{\infty} \frac{(-1)^{j+1} \alpha^{2j-1}}{(2j-1)!} + \sum_{k=1}^{\infty} \frac{(-1)^k \alpha^{2k}}{(2k)!} =$$

$$= \boxed{i \cdot A \cdot \sin\alpha + \cos\alpha}$$

(2.1) $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$

$A|\mathcal{F}\rangle = \lambda|\mathcal{F}\rangle.$

$$xy: \begin{pmatrix} -\lambda & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & -\lambda & -i/\sqrt{2} \\ 0 & +i/\sqrt{2} & -\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$-\lambda^3 + \frac{\lambda}{2} + \frac{\lambda}{2} = 0$$

$$\lambda(\lambda^2 - 1) = 0, \quad \boxed{\begin{matrix} \lambda = \pm 1 \\ \lambda = 0 \end{matrix}}$$

1) $\lambda = 1.$

$$\begin{pmatrix} -\frac{i}{\sqrt{2}}\mathcal{F}_2 \\ \frac{i}{\sqrt{2}}\mathcal{F}_1 - \frac{i}{\sqrt{2}}\mathcal{F}_3 \\ \frac{i}{\sqrt{2}}\mathcal{F}_2 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} -\mathcal{F}_2 \\ \mathcal{F}_1 - \mathcal{F}_3 \\ \mathcal{F}_2 \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \end{pmatrix}.$$

$$\mathcal{F}_1 = -\frac{i}{\sqrt{2}} \mathcal{F}_2 = -\mathcal{F}_3.$$

$$\mathcal{F}_2 = \frac{-\sqrt{2}}{i} \mathcal{F}_1 = i\sqrt{2}\,\mathcal{F}_1$$

$$|\mathcal{F}_{\lambda=1}\rangle = \begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}, \quad |\mathcal{F}_{\lambda=1}| = \sqrt{4} = 2$$

$$\boxed{|\mathcal{F}^n_{\lambda=1}\rangle = \begin{pmatrix} 1/2 \\ i\sqrt{2}/2 \\ -1/2 \end{pmatrix}}$$

2) $\lambda = -1$

$$\frac{i}{\sqrt{2}}\begin{pmatrix} -f_2 \\ f_1 - f_3 \\ f_2 \end{pmatrix} = -\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$f_1 = -f_3 = \frac{+i}{\sqrt{2}} f_2$$

$$f_2 = -\frac{(-\sqrt{2})}{i} f_1 = -\sqrt{2}\, i\, f_1$$

$$|f_{\lambda=-1}\rangle = \begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix}.$$

$$|f_{\lambda=-1}| = \sqrt{4} = 2.$$

$$\boxed{|f^n_{\lambda=-1}\rangle = \begin{pmatrix} 1/2 \\ -\frac{\sqrt{2}}{2} i \\ -1/2 \end{pmatrix}} ;$$

3) $\lambda = 0$.

$$\frac{i}{\sqrt{2}}\begin{pmatrix} -f_2 \\ f_1 - f_3 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f_2 = 0$$
$$f_1 = f_3 \qquad |f_{\lambda=0}\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$|f_{\lambda=0}| = \sqrt{2}$$

$$\boxed{|f^n_{\lambda=0}\rangle = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{pmatrix}}$$

проверка ортогональности:

$$\langle \mathcal{F}_{\lambda=1} | \mathcal{F}_{\lambda=-1} \rangle = x_1^* x_{-1} + y_1^* y_{-1} + z_1^* z_{-1} =$$

$$= 1 - 2 + 1 = 0.$$

$$\langle \mathcal{F}_{\lambda=1} | \mathcal{F}_{\lambda=0} \rangle = 1 - 1 = 0.$$

ч.т.д.

$$\langle \mathcal{F}_{\lambda=-1} | \mathcal{F}_{\lambda=0} \rangle = 1 - 1 = 0.$$

2.2 $$e^{i\theta A} = \sum_{j=0}^{\infty} \frac{(i\theta A)^n}{n!} = 1 + \frac{i\theta A}{1} - \frac{\theta^2 A^2}{2} - \frac{i\theta^3 A^3}{6} + \dots$$

$$= \sum_{j=0}^{\infty} \frac{(\theta A)^{2j}}{(2j)!} \cdot (-1)^j + i \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \theta^{2k-1} A^{2k-1}}{(2k-1)!} =$$

Исследуем A^2:

$$A|\mathcal{F}\rangle = \frac{i}{\sqrt{2}} \begin{pmatrix} -\mathcal{F}_2 \\ \mathcal{F}_1 - \mathcal{F}_3 \\ \mathcal{F}_2 \end{pmatrix}$$

$$A^2|\mathcal{F}\rangle = A|A|\mathcal{F}\rangle = \frac{i}{\sqrt{2}} \cdot \frac{i}{\sqrt{2}} \begin{pmatrix} \mathcal{F}_3 - \mathcal{F}_1 \\ -2\mathcal{F}_2 \\ \mathcal{F}_1 - \mathcal{F}_3 \end{pmatrix}$$

теперь A^3:

$$A^3|\mathcal{F}\rangle = \left(\frac{i}{\sqrt{2}}\right)^3 \begin{pmatrix} 2\mathcal{F}_2 \\ 2\mathcal{F}_3 - 2\mathcal{F}_2 \\ -2\mathcal{F}_2 \end{pmatrix} = \frac{+i}{\sqrt{2}} \begin{pmatrix} -\mathcal{F}_2 \\ \mathcal{F}_1 - \mathcal{F}_3 \\ \mathcal{F}_2 \end{pmatrix} = A|\mathcal{F}\rangle$$

т.е.: $A^{2n+1} = A.$ $A^{2n} = A^2$

$$\Leftrightarrow A^2 \sum_{j=0}^{\infty} \frac{\theta^{2j}(-1)^j}{(2j)!} + iA\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,\theta^{2k-1}}{(2k-1)!} \Leftrightarrow$$

$$\Leftrightarrow \boxed{A^2 \cdot \cos\theta + A \cdot i\sin\theta}$$

2.3) базис с. в.:

$$|1\rangle = \begin{pmatrix} 1/2 \\ i\frac{\sqrt{2}}{2} \\ -1/2 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 1/2 \\ -i\frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{pmatrix}.$$

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

в новом базисе: $|\psi'\rangle = \begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix}$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ i\frac{\sqrt{2}}{2} & -\frac{i\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\psi_1' = \psi_2'$$

$$\psi_1' = \frac{\sqrt{2}}{2}\psi_3' = 0{,}5 \quad ,$$

$$\boxed{\begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ \sqrt{2}/2 \end{pmatrix}}$$

3.1 $\hat{T}_a \psi(x) = \psi(x+a)$.

Не эрмитов, т.к.

$$\langle \varphi | T_a \psi \rangle = \int_{-\infty}^{+\infty} \varphi^*(x)\, \psi(x+a)\, |x\rangle\, dx =$$

$x' = x+a$

$$= \int_{-\infty}^{+\infty} \varphi^*(x'-a)\, \psi(x')\, |x'\rangle\, dx' =$$

$$= \langle \hat{T}_{-a} \varphi | \psi \rangle$$

т.е.

$$\boxed{\hat{T}_a^+ = \hat{T}_{-a}}$$

$$\hat{T}_a^+ \hat{T}_a |\psi\rangle = \hat{T}_{-a}\, \psi(x+a) = \psi(x).$$

$$\boxed{\hat{T}_a^+ \hat{T}_a = \hat{1}}$$

ядро: $\ker \hat{T}_a = \{\psi,\ \hat{T}_a \psi = 0\}$.

$\boxed{\psi = a}$ - ядро: