(1) $\hat{S} = \begin{pmatrix} \cos\theta & \sin\theta \cdot e^{-i\varphi} \\ \sin\theta \cdot e^{i\varphi} & -\cos\theta \end{pmatrix}$.

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Моренко С.М.

СЗ и СВ:

$\hat{S} A = \lambda A.$

$$\left(\begin{array}{cc|c} \cos\theta - \lambda & \sin\theta \cdot e^{-i\varphi} & 0 \\ \sin\theta \cdot e^{i\varphi} & -\cos\theta - \lambda & 0 \end{array}\right).$$

Det:

$$-(\cos\theta - \lambda)(\cos\theta + \lambda) - \sin^2\theta = 0$$

$$\lambda^2 - \cos^2\theta - \sin^2\theta = 0.$$

$$\boxed{\lambda = \pm 1}$$

1) $\lambda = 1$.

$$(\cos\theta - 1) x_1 = -\sin\theta \cdot e^{-i\varphi} x_2$$

$$A_1 = \begin{pmatrix} -\sin\theta \cdot e^{-i\varphi} \\ \cos\theta - 1 \end{pmatrix}$$

$$|A_1|^2 = (\cos\theta - 1)^2 + \sin\theta^2 = 1 - 2\cos\theta + 1 = -2\cos\theta + 2$$

$$\boxed{A_1^N = \begin{pmatrix} -\dfrac{\sin\theta \cdot e^{-i\varphi}}{\sqrt{2(-\cos\theta + 1)}} \\ \dfrac{\cos\theta - 1}{\sqrt{2(-\cos\theta + 1)}} \end{pmatrix}}$$

2). $\lambda = -1$

$$(\sin\Theta \cdot e^{i\varphi})x_1 = (\cos\Theta \mp 1)\cdot x_2.$$

$$A_{-1} = \begin{pmatrix} \cos\Theta \mp 1 \\ \sin\Theta \cdot e^{+i\varphi} \end{pmatrix}$$

$$|A_{-1}|^2 = (\cos\Theta \mp 1)^2 + \sin\Theta^2 = 2\cos\Theta + 2 = 2(-\cos\Theta + 1)$$

$$\boxed{A^N_{-1} = \begin{pmatrix} \dfrac{\cos\Theta + 1}{\sqrt{2(-\cos\Theta + 1)}} \\ \dfrac{\sin\Theta \cdot e^{+i\varphi}}{\sqrt{2(\cos\Theta + 1)}} \end{pmatrix}.}$$

Проверка ортогональности:

$$\langle A_1 | A_{-1} \rangle = (-\sin\Theta \cdot e^{+i\varphi})(\cos\Theta \mp 1) + \sin\Theta \cdot e^{+i\varphi} \cdot (\cos\Theta - 1) =$$

$= 0$. ч.т.д.

1.2 Разложим $|\Psi\rangle$ по СВ $\hat{S}$:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \lambda_1 C_1 A_1^N + C_2 A_{-1}^N \cdot \lambda_{-1}.$$

$$\left(\begin{array}{cc|c} -\frac{\sin\theta \cdot e^{-i\varphi}}{\sqrt{\cancel{+}\cos\theta+1}} & \frac{\cos\theta \cancel{+} 1}{\sqrt{-\cancel{2}\cos\theta+1}} & 1. \\ \frac{\cos\theta-1}{\sqrt{1-\cos\theta}} & \frac{\sin\theta\cdot e^{+i\varphi}}{\sqrt{1-\cos\theta}} & 1. \end{array}\right)$$

$$\left(\begin{array}{cc|c} -\sin\theta\cdot e^{-i\varphi} & \cos\theta-1 & \sqrt{1-\cos\theta} \\ \cos\theta-1 & \sin\theta\cdot e^{i\varphi} & \sqrt{1-\cos\theta} \end{array}\right) \Rightarrow \Bigg($$

$$\Rightarrow \left(\begin{array}{cc|c} -\sin^2\theta & (\cos\theta-1)\sin\theta\cdot e^{i\varphi} & \sqrt{1-\cancel{3}\cos\theta}\cdot\sin\theta\cdot e^{-i\varphi} \\ (\cos\theta-1)^2 & (\cos\theta-1)\cdot\sin\theta\cdot e^{i\varphi} & \sqrt{1-\cos\theta}\cdot(\cos\theta-1) \end{array}\right) \Rightarrow$$

$$\Rightarrow \underbrace{\left((\cos\theta-1)^2+\sin^2\theta\right)}_{-2\cos\theta+2} C_1 = \sqrt{1-\cos\theta}\left(\cos\theta-1-\sin\theta\cdot e^{-i\varphi}\right)$$

$$C_1 = \frac{\sqrt{1-\cos\theta}}{2(1-\cos\theta)}\left(\cos\theta-1-\sin\theta\cdot e^{-i\varphi}\right) =$$

$$= \boxed{\frac{1}{2\sqrt{1-\cos\theta}}\left(\cos\theta-\sin\theta\cdot e^{-i\varphi}-1\right)}$$

$$-\frac{\sqrt{1-\cos\theta}}{2}\cdot(\cos\theta-\sin\theta\cdot e^{i\varphi}-1)+\sin\theta\cdot e^{i\varphi}\cdot C_2=\sqrt{1-\cos\theta}$$

$$C_2=\frac{\sqrt{1-\cos\theta}\,(\cos\theta-\sin\theta\cdot e^{i\varphi}+1)}{\sin\theta\cdot e^{i\varphi}}$$

Возможные значения измерения: $\underbrace{1,\ -1}_{\text{С З оператора}}$

Ожидаемая величина измерения A в сост. Ψ:

$$\langle A_\Psi\rangle=\langle\Psi|A|\Psi\rangle=\sum_{i=1}^{2}\lambda_i\underbrace{|\langle\Psi|A_i\rangle|^2}_{\sim\text{ вероятности перехода }(\tilde{P})}$$

$$\tilde{P}_1=|\langle\Psi|A_1\rangle|^2=\frac{1}{2}\left(\left|\frac{-\sin\theta\cdot e^{-i\varphi}}{\sqrt{2(\cos\theta+1)}}+\frac{\cos\theta-1}{\sqrt{2(-\cos\theta+1)}}\right|\right)^2 \ominus$$

$$=\frac{1}{4}\frac{(|\cos\theta-1-\sin\theta\cdot e^{-i\varphi}|)^2}{(1-\cos\theta)}=C_1^2$$ — чего и следовало ожидать

$$\ominus\ \frac{1}{4(1-\cos\theta)}\left[\left|(\cos\theta-1-\sin\theta\cdot\cos\varphi)+(\sin\theta\cdot\sin\varphi\cdot i)\right|\right]^2 \ominus$$

$$\ominus\ \frac{(\cos\theta-1-\sin\theta\cdot\cos\varphi)^2+\sin^2\theta\cdot\sin^2\varphi}{4(1-\cos\theta)}$$

— вероятность измерить 1

$\tilde{P}_2=|\langle\Psi|A_2\rangle$

$$\tilde{P}_2 = |\langle \Psi | A_2 \rangle|^2 = C_2^2 =$$

$$= \cancel{\ldots\cos\theta}\ \frac{1}{4(1-\cos\theta)} \left(|\cos\theta + 1 + \sin\theta \cdot e^{i\varphi}|\right)^2 =$$

$$= \ldots = \tilde{P}_1$$

$$\begin{cases} \tilde{P}_2 + \tilde{P}_1 = 1 \\ \tilde{P}_2 = \tilde{P}_1 \end{cases} \Rightarrow \boxed{P_1 = P_2 = 0{,}5}$$

т.е. вероятность после измерения получить значение 1 или -1 одинаковы.

2.1) $U_{(x,y)} = \frac{m\omega^2}{2}(x^2+y^2).$

$x(0)=a \quad y(0)=0$

$\dot{x}(0)=0 \quad \dot{y}(0)=\upsilon_0.$

$$L = m\left(\frac{\dot{x}^2+\dot{y}^2}{2} - \frac{\omega^2}{2}(x^2+y^2)\right).$$

Ур-ния движения: $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$

$$\begin{cases} \ddot{x} = -\omega^2 x \\ \ddot{y} = -\omega^2 y \end{cases} \Rightarrow \left(\text{с учетом нач. усл.}\right) \Rightarrow \begin{matrix} x = a\cos\omega t \\ y = \frac{\upsilon_0}{\omega}\cdot\sin\omega t \end{matrix}$$

траектория: эллипс:

$$\frac{x^2}{a^2} + \frac{\omega_0^2 y^2}{\upsilon_0^2} = 1$$

2.2) $p_i = \frac{\partial L}{\partial \dot{q}_i} \Rightarrow \begin{matrix} p_x = m\dot{x} \\ p_y = m\dot{y} \end{matrix}$

Гамильтониан: $H = p_x\cdot\dot{x} + p_y\cdot\dot{y} - L =$

$$= \frac{m}{2}\left((\dot{x}^2+\dot{y}^2) + \omega^2(x^2+y^2)\right) = \frac{m}{2}\left(\frac{dx^2+dy^2}{dt^2} + \omega^2(x^2+y^2)\right) = E$$

$$\frac{2E}{m} - \omega^2(x^2+y^2) = \frac{dx^2+dy^2}{dt^2}$$

$$\cancel{dt = \left(\sqrt{\frac{m}{2E}} - \omega^2(x^2+y^2)\right)^{-1}}$$

$$dt = \sqrt{\frac{dx^2+dy^2}{\frac{2E}{m} - \omega^2(x^2+y^2)}}$$

Тогда:

$$P_x = m\frac{dx}{dt} = m\frac{dx}{\sqrt{dx^2+dy^2}} \cdot \sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}$$

$$P_y = m\frac{dy}{dt} = m\frac{dy}{\sqrt{dx^2+dy^2}} \cdot \sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}$$

подставляю в укороченное действие:

$$S_0 = \int m\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}\,\frac{dx^2+dy^2}{\sqrt{dx^2+dy^2}} =$$

$\boxed{\text{Т.К. } S_0 = \int P_x dx + P_y dy}$

$$= \int m\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}\,\sqrt{dx^2+dy^2} = \qquad \left|\; x' = \frac{dx}{dy}\right.$$

$$= \int m\sqrt{\frac{2E}{m} - \omega^2(x^2+y^2)}\,\sqrt{1+x'^2}\,dy$$

тогда y – почти время

x – координата.

Лагранжиан: $L=\sqrt{\left(\frac{2E}{m}-\omega^2(x^2+y^2)\right)(1+x'^2)}\cdot m.$

ур-ие движения (траектории): $\frac{\partial L}{\partial x}=\frac{d}{dy}\frac{\partial L}{\partial x'}$

$$\frac{-\omega^2 x\cdot\sqrt{1+x'^2}}{\sqrt{\frac{2E}{m}-\omega^2(x^2+y^2)}}\cancel{m}=\frac{d}{dy}\left[\cancel{m}\frac{x'\cdot\sqrt{\frac{2E}{m}-\omega^2(x^2+y^2)}}{\sqrt{1+x'^2}}\right]$$

$$-\omega^2 x x'=x'\cdot\sqrt{\frac{\frac{2E}{m}-\omega^2(x^2+y^2)}{1+x'^2}}\cdot\frac{d}{dy}\frac{x'\sqrt{\frac{2E}{m}-\omega^2(x^2+y^2)}}{\sqrt{1+x'^2}}$$

пусть $\phi=x'\cdot\sqrt{\frac{\frac{2E}{m}-\omega^2(x^2+y^2)}{1+x'^2}}$

$$-\omega^2 x\frac{dx}{dy}=\phi\cdot\phi'=\phi\cdot\frac{d\phi}{dy}$$

$$\phi^2=-\omega^2x^2+C.$$

$$x'^2\frac{\frac{2E}{m}-\omega^2(x^2+y^2)}{1+x'^2}=-\omega^2x^2+C.\qquad \left.\frac{dx}{dy}\right|_{\substack{x=a\\y=0}}=0.$$

$$C=\cancel{\frac{2E}{m}}\;\underline{\omega^2a^2}$$

$$\cancel{\frac{2E}{m}-\omega^2a^2=-\omega^2a^2+\frac{2E}{m}.}$$

$$\cancel{\frac{2E}{m}-\omega^2x^2-\omega^2y^2=-\omega^2x^2-\omega^2x^2x'^2+\frac{2E}{m}+\frac{2E}{m}x'^2}$$

$$x'^2 \cdot \frac{2E}{m} - \cancel{\omega^2 x^2 x'^2} - \omega^2 y^2 x'^2 = -\omega^2 x^2 + \omega^2 a^2 - \cancel{\omega^2 x^2 x'^2} + \omega^2 a^2 x'^2$$

$$x'^2 \left(\frac{2E}{m} - \omega^2 (y^2 + a^2) \right) = \omega^2 (a^2 - x^2).$$

$$\frac{x'}{\omega \sqrt{a^2 - x^2}} = \frac{1}{\sqrt{\frac{2E}{m} - \omega^2 (y^2 + a^2)}}$$

$$\frac{2E}{m} = \sigma_0^2 + a^2 \omega^2$$

$$\frac{dx}{\omega \sqrt{a^2 - x^2}} = \frac{dy}{\sqrt{\sigma_0^2 - \omega^2 y^2}}$$

$$\frac{dx}{\sqrt{a^2 - x^2}} = \frac{dy}{\sqrt{\frac{\sigma_0^2}{\omega^2} - y^2}}$$ где-то потерялся минус

$$\boxed{\arcsin \frac{x}{a} = \arccos \frac{y\omega}{\sigma_0}}$$

$$\Updownarrow$$

$$\frac{x^2}{a} + \frac{y^2 \omega^2}{\sigma_0^2} = 1$$

(3.1) $|\psi\rangle(0) = \begin{pmatrix}1\\0\end{pmatrix}$

$$\hat{H} = -\mu B\hat{S} = -\mu B\begin{pmatrix}\cos\theta & \sin\theta\cdot e^{-i\varphi}\\ \sin\theta\cdot e^{i\varphi} & -\cos\theta\end{pmatrix}$$

$$|\psi\rangle(t) = e^{-\frac{\hat{H}t}{i\hbar}}|\psi\rangle(0)$$

$$e^{-\frac{\hat{H}t}{i\hbar}} = \sum_{j=0}^{\infty}\left(\frac{-\hat{H}t}{i\hbar}\right)^j\frac{1}{j!} = 1 + i\frac{\hat{H}t}{\hbar} - \frac{\hat{H}^2t^2}{2\hbar} - i\frac{\hat{H}^3t^3}{6\hbar} + \ldots \ominus$$

$$\ominus \underbrace{\sum_{j=0}^{\infty}\cancel{\frac{\hat{H}t}{(2j)!\hbar}}\left(\frac{\hat{H}t}{\hbar}\right)^{2j}\frac{(-1)^j}{(2j)!}}_{\cos\frac{\hat{H}t}{\hbar}} + i\underbrace{\sum_{k=1}^{\infty}\left(\frac{\hat{H}t}{\hbar}\right)^{2k-1}\frac{(-1)^{k+1}}{(2k-1)!}}_{i\cdot\sin\frac{\hat{H}t}{\hbar}}$$

$$e^{-\frac{\hat{H}t}{i\hbar}}\cdot\begin{pmatrix}1\\0\end{pmatrix} = 1 \mp \frac{ti}{\hbar}\mu B\begin{pmatrix}\cos\theta & \sin\theta\cdot e^{-i\varphi}\\ \sin\theta\cdot e^{i\varphi} & -\cos\theta\end{pmatrix} \mp$$

$$\mp \frac{t^2}{\hbar 2}\mu^2B^2\begin{pmatrix}\cos\theta & \sin\theta\cdot e^{-i\varphi}\\ \sin\theta\cdot e^{i\varphi} & -\cos\theta\end{pmatrix}^2 - \ldots \ominus$$

$$S^2 = \begin{pmatrix}\cos^2\theta + \sin^2\theta & \cos\theta\cdot\sin\theta\cdot e^{-i\varphi} - \cos\theta\cdot\sin\theta\cdot e^{-i\varphi}\\ \sin\theta\cdot e^{i\varphi}\cdot\cos\theta - \cos\theta\cdot\sin\theta\cdot e^{i\varphi} & \cos^2\theta + \sin^2\theta\end{pmatrix} = \hat{1}$$

m.e: $S^{2k} = \hat{1}$, $S^{2k-1} = S$

$$\ominus \sum_{j=0}^{\infty}\frac{\begin{pmatrix}1\\0\end{pmatrix}\left(\frac{\mu Bt}{\hbar}\right)^{2j}(-1)^j}{(2j)!} - i\sum_{k=1}^{\infty}\frac{S\begin{pmatrix}1\\0\end{pmatrix}\left(\frac{t\mu B}{\hbar}\right)^{2k-1}(-1)^{k+1}}{(2k-1)!} \ominus$$

$$\ominus \boxed{\begin{pmatrix}1\\0\end{pmatrix}\cos\frac{\mu Bt}{\hbar} \mp i\begin{pmatrix}\cos\theta\\ \sin\theta\cdot e^{i\varphi}\end{pmatrix}\cdot\sin\frac{t\mu B}{\hbar}}$$

(4.1) $\Psi(x) = \underbrace{const}_{C}\, x^2 \cdot e^{-x/\lambda}$, $x \geq 0$

Условие нормировки:

$$\int_0^{+\infty} |\Psi(x)|^2 dx = 1$$

$$C\int_0^{+\infty} \underset{u}{x^4} \cdot \underset{dv}{e^{-\frac{2x}{\lambda}} dx} = C\cdot\left(\underbrace{-\frac{\lambda}{2} x^4 \cdot e^{-\frac{2x}{\lambda}}\Big|_0^{+\infty}}_{0-0=0} + \int_0^{+\infty} \frac{\lambda}{2} e^{-\frac{2x}{\lambda}} \cdot 4x^3 dx\right) =$$

$$= \cancel{2\lambda}\, 2\lambda \cdot C\left(\underbrace{-\frac{\lambda}{2} x^3 \cdot e^{-\frac{2x}{\lambda}}\Big|_0^{+\infty}}_{0} + \int_0^{+\infty} \frac{\lambda}{2} e^{-\frac{2x}{\lambda}} \cdot 3x^2 dx\right) =$$

$$= 3\lambda^2 \cdot C\left(\underbrace{-\frac{\lambda}{2} x^2 \cdot e^{-\frac{2x}{\lambda}}\Big|_0^{+\infty}}_{0} + \int_0^{+\infty} \frac{\lambda}{2} \cdot e^{-\frac{2x}{\lambda}}\, 2x\, dx\right) =$$

$$= 3\lambda^3 \cdot C\left(\underbrace{-\frac{\lambda}{2} x \cdot e^{-\frac{2x}{\lambda}}\Big|_0^{+\infty}}_{0} + \int_0^{+\infty} \frac{\lambda}{2} \cdot e^{-\frac{2x}{\lambda}} dx\right) =$$

$$= C\frac{3\lambda^4}{2}\int_0^{+\infty} e^{-\frac{2x}{\lambda}} dx = -\frac{3\lambda^5}{4} C \cdot \int_0^{-\infty} e^t dt = \frac{3\lambda^5}{4} \cdot C = 1$$

$$\boxed{Const = \frac{4}{3\lambda^5}}$$

(4.2) $$P([x,x+dx]) = \int_x^{x+dx} |\Psi|^2 dx = |\Psi|^2 dx = \boxed{\frac{4x^4 \cdot e^{-\frac{2x}{\lambda}}}{3\lambda^5} dx}$$

(4.3) $$\langle X_\Psi \rangle = \int_0^{+\infty} x\,|\Psi(x)|^2 dx =$$

$$= C \int_0^{+\infty} x^5 \cdot e^{-\frac{2x}{\lambda}} dx = C\left(\underbrace{-\frac{\lambda}{2} x^5 \cdot e^{-\frac{2x}{\lambda}} \Big|_0^{+\infty}}_{0} + \int_0^{+\infty} \frac{\lambda}{2} e^{-\frac{2x}{\lambda}} \cdot 5x^4 dx\right) =$$

$$= C \cdot \frac{5\lambda}{2} \left(\underbrace{-\frac{\lambda}{2} x^4 \cdot e^{-\frac{2x}{\lambda}} \Big|_0^{+\infty}}_{0} + \int_0^{+\infty} \frac{\lambda}{2} e^{-\frac{2x}{\lambda}} \cdot 4x^3\right) =$$

$$= C \cdot \frac{10\lambda^2}{2} \left(\underbrace{-\frac{\lambda}{2} x^3 \cdot e^{-\frac{2x}{\lambda}}}_{0} \Big|_0^{+\infty} + \int_0^{+\infty} \frac{\lambda}{2} \cdot e^{-\frac{2x}{\lambda}} \cdot 3x^2 dx\right) =$$

$$= C \cdot \frac{15\lambda^3}{2} \left(-\frac{\lambda}{2} x^2 \cdot e^{-\frac{2x}{\lambda}} \Big|_0^{+\infty} + \int_0^{+\infty} \frac{\lambda}{2} \cdot e^{-\frac{2x}{\lambda}} \cdot 2x\, dx\right) =$$

$$= C \cdot \frac{15\lambda^4}{2} \left(\underbrace{-\frac{\lambda}{2} x \cdot e^{-\frac{2x}{\lambda}}}_{0} \Big|_0^{+\infty} + \int_0^{+\infty} \frac{\lambda}{2} \cdot e^{-\frac{2x}{\lambda}} dx\right) =$$

$$= C \cdot \frac{15\lambda^5}{4} \int_0^{+\infty} e^{-\frac{2x}{\lambda}} dx = C \cdot \frac{15\lambda^6}{8} = \frac{4}{3\lambda^5} \cdot \frac{15\lambda^6}{8} = \frac{5}{2}\lambda =$$

$$= 2{,}5\,\lambda$$