Unified musical quantum models for tonal attraction

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Abstract: *Computational models of music cognition generally distinguish between static and dynamic tonal attraction. Starting with the famous Krumhansl-Kessler (KK) pitch class profiles as a particular tonal schema for static attraction, we investigate a one-dimensional quantum system on the continuous line of fifth-width. From a Gaussian mixture model (GMM) fit, we construct its schematic Hamiltonian in the form of a perturbed quantum harmonic oscillator. In the concrete musical application of the C major tonality, we offer a unified approach to static and dynamic tonal attraction as follows. Static attraction is described in terms of the stationary ground state solution of the underlying schematic Schrödinger equation. Dynamic attraction is investigated in terms of the temporal evolution of suitably chosen initial states. As a measure for dynamic attraction among tonal states we calculate the Hilbert space overlaps between the time development of some coherent states of the unperturbed harmonic oscillator in the role of antecedents with some other coherent states in the role of consequents.*

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1. Introduction

Tonal attraction is an important research topic in music psychology and in mathematical musicology. In music psychology, the priming experiments devised by Krumhansl and Kessler [29] for *static attraction* and by Krumhansl and Shepard [30] for *dynamic attraction* that have been further explored in [31, 55], delivered substantial insights into mental processes of music perception and the topology of the underlying configuration spaces [29, 27].

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In a paradigmatic psychological priming experiment, a prime presents a preparing context for the processing of upcoming stimuli. Dependent on the subject's instruction, one distinguishes between static and dynamic tonal attraction as follows. In the experiment of [29] subjects were presented with different musical primes, such as ascending scales, various chords or diverse cadences, all establishing a particular key as context. After each prime, the subjects were presented with one of the twelve tones of the chromatic scale as probes. They were instructed to tell how well the given probe tone fits statically into the established tonal context. By contrast, in the earlier experiment of [30] subjects were presented with either ascending or descending scales as primes, with the instruction to indicate how well a subsequent probe tone completes the given scale. Likewise, in the experiments conducted by [31, 55] subjects were instructed to tell how well short melodic segments [31] or single chords [55] are dynamically resolved by the successive probe tones.

The results of those priming experiments were presented as *tonal attraction profiles* [55] or *anchoring strengths* [32] for minor and major modes, respectively. Bharucha [12, p. 489] remarked:

> As a shorthand, the distortion of psychological tone-space and chord-space by creating focal points around which stable tones or chords locate themselves, along with the inducement of asymmetries, will be described as the *engagement or activation of a tonal schema*. A tonal schema specifies a hierarchy of stability (a tonal hierarchy) of all possible tones and chords [. . .].

Schematic models play an important role in cognitive psychology [3] and musicology [12, 21, 39]. Originally, the concepts of schematism can at least be traced back to Kant's transcendental phenomenology [24, pp. B 176], where schemata provide the interface between the categories of understanding and the pure forms of intuition, i.e. time and space [24, pp. B 177]. As a specific example, Kant referred to *number* as the transcendental schema for the category of *quantity* that is construed in intuition by the operation of counting in time [24, p. B 182].

In cognitive psychology and computational linguistics, schemata are realized, e.g., as frames [3, 25] or attribute-value structures (AVS) [23]. Applied to musical schematism [12, 21, 39], a tonal attraction profile, e.g., assigning to each chromatic tone its corresponding anchoring strength, is represented as an AVS with chromatic tones as attributes and anchoring strengths as their respective values.[1] In his book *Sweet Anticipation: Music and the Psychology of Expectation*, Huron [21] located tonal schemata in the realm of cognition rather than perception, despite of his usage of the attribute "auditory." When Huron [21, pp. 339 – 344] characterized the attitude of listeners, who get disturbed by Schönbergs "contratonal" twelve-tone music as an instance of a "schematic failure," he referred to their reliance on "key schemas," "scale-related schemes" or "tonal schemas."

Several authors assume that musical schemata are acquired through statistical learning in the process of musical enculturation [21, 44, 53]. For the investigations of this paper it is not decisive whether these attraction profiles are real psychological entities or whether they are robust quantitative traces of the tonal qualia that occur in the experience of the musical listener and to which we have no experimental access. When we speak about statistical learning and later about quantum statistical learning we abstain from making ontological assumptions. These are technical terms in the context of computational musicology.

The quotation of Bharucha [12] above also indicates that a particular musical schema induces a specific "distortion of psychological tone-space and chord-space" and hence the emergence of musical forces in the sense of [31]. Those distortions of tonal space have been mathematically modeled by beim Graben and Blutner [8, 9] and beim Graben [7] in terms of musical quantum

[1] It should be noted though, that the term *tonal scheme* has been used quite differently by Noll [41] in extension of the concept of *voice leading schemata* [17, 41].

models (cf. [16, 10, 42, 45] for related developments). This novel research field of *quantum music* can be characterized as an attempt to extend well-established statistical and probabilistic approaches of music cognition [2, 53] through quantum-theoretical models. The prominence of probabilistic and statistical models in the empirical human sciences is—last but not least—influenced by the tremendous success of these methods in quantum theory. In physics these methods form an interface between the physical level of description, i.e. Schrödinger wave functions and experimental observations of accessing the former through measurements. Therefore, the quantum approach chosen here offers itself quite naturally in this wider context. The emphasis is hence on quantum-*theoretical*, as these models are purely motivated by the mathematics of quantum theory, instead by physical interpretations at all.

More specifically, beim Graben and Blutner [9] presented two different quantum models, one quantum deformation model of the circle of fifths accounting for static attraction, and another one that is based on the interval cycle model of Woolhouse [55] for the description of dynamic attraction along the chroma circle. Thus, static and dynamic attraction have been described over two different tonal configuration spaces (cf. [13] for a related approach).

It is the aim of the present study to unify the different accounts of static and dynamic tonal attraction in an overarching quantum model. To this end, one may interpret the schematic attraction profiles as probability density functions over suitable musical configuration spaces in a first step. This is also a common strategy in machine learning approaches for the simulation of musical processes, e.g. through Markov chain models [21, 53]. The second step is the door-opener to our quantum approach, namely the interpretation of the square root of a given probability density function as the ground state solution of a suitable Schrödinger equation as a model for static tonal attraction.

The article provides rigorous mathematical results of more general interest and a detailed investigation of a concrete example. In Sect. 2 we review previous empirical work around the concept of tonal attraction. It serves as a motivation and preparation for our interpretation of the quantum model in Sect. 3. There, after a short review of previous work on quantum models of tonal attraction, we present our mathematical main result: for some Gaussian mixture models over the real numbers, that could be acquired through some classical statistical learning algorithm, there exists an associated perturbation of the Hamiltonian energy operator of a quantum harmonic oscillator, whose stationary ground state renders the prescribed schematic Gaussian mixture. Under this reinterpretation, the focus of investigation shifts to the time-dependent Schrödinger equation affiliated with this schematic Hamiltonian. This defines the dynamic evolution for the quantum mechanical initial value problem. Computing the transition probabilities of the quantum dynamical system by the projection of consequent states onto antecedent states provides a model for dynamic tonal attraction. Section 4 presents the results of our first numerical experiments on static and dynamic attraction. The article concludes with a critical discussion of several peculiarities and open problems in Sect. 5.

2. Empirical Investigations into Tonal Attraction

Musical tones achieve meaning through their position in the context of other tones [21, 38]. In recent quantitative approaches to tonality the term tonal attraction has been proposed to encode the relative importance or salience of the notes of the chromatic scale with respect to the context of a given major or minor key, such as the Krumhansl and Kessler [29] (KK) probe tone profiles (see Sect. 2.1). We refer to this aspect as static tonal attraction. The relative prominence of tone transitions is referred to as dynamic tonal attraction. The term has been used in a computational model by Woolhouse [55].

2.1 Krumhansl-Kessler attraction profiles

In a celebrated study, Krumhansl and Kessler [29] asked listeners to rate how well each note of the chromatic octave fitted with a preceding context, which consisted of short musical sequences in major or minor keys. The empirical rating results of [28, 29] are numerically presented in Tab. 1 and graphically depicted in Fig. 1 after suitable normalization to probabilities $p(x)$. Here, x denotes the note position at the fifth-width axis as outlined in Sect. 2.2 below.

Table 1: *Original attraction ratings $KK(x)$ and normalized ratings $p(x)$ for C major (top) and C minor (bottom) after [28].*

note	$D\flat$	$A\flat$	$E\flat$	$B\flat$	F	C
pos x / fifth-width	-5	-4	-3	-2	-1	0
C major $KK(x)$:	2.23	2.39	2.33	2.28	4.09	6.35
C minor $KK(x)$:	2.68	3.98	5.38	3.34	3.53	6.33
C major: $p(x)$	0.0	0.0203	0.0127	0.0064	0.2365	0.524
C minor: $p(x)$	0.0184	0.1893	0.3733	0.1052	0.1301	0.4982
note	G	D	A	E	B	$F\sharp$
pos x / fifth-width	1	2	3	4	5	6
C major $KK(x)$:	5.19	3.48	3.66	4.38	2.88	2.52
C minor $KK(x)$:	4.75	3.48	3.52	2.6	2.88	2.54
C major: $p(x)$	0.3764	0.159	0.1819	0.2734	0.0827	0.0369
C minor: $p(x)$	0.2905	0.1236	0.1288	0.0079	0.0447	0.0

Figures 1(a, c) plot the results for the C major context and 1(b, d) for the C minor context. We replicate the original rating data from Tab. 1, plotting them over the chromatic line in Figures 1(a, b). Figures 1(c, d) show the same data after rearrangement from the chromatic line to the fifth-width line and after normalization to probability values. Normalization was achieved by the GMM fit explained in Sect. 2.3 below. In Fig. 1(c), the tonic pitch $n = 0$ which is mostly attracting received the highest rating, followed by the pitches completing the tonic triad (fifth $n = 1$ and third $n = 4$), followed by the remaining diatonic scale degrees, and finally the chromatic, nonscale tones. Moreover, in Fig. 1(d), the tonic pitch $n = 0$ received the highest rating, followed by the pitches of the tonic triad (fifth $n = 1$ and third $n = -3$), followed by the remaining diatonic scale degrees, and finally the chromatic, nonscale tones, again.

The note names $D\flat, A\flat, E\flat, B\flat, F, C, G, D, A, E, B, F\sharp$ (ordered by fifths in $\sharp$-ward direction) are assigned to radian angles $x = n\pi/6$ ($n \in \mathbb{Z}$) centered around the note C at the origin $x = 0$. We write $x = \mathrm{pos}(n) \in \mathbb{R}$ for the position of the note n in fifth-width units (“quints”) ($= \pi/6$) at the real width space, equally lifting the values of the measured data for enharmonically equivalent notes $G\flat$ and $F\sharp$.

2.2 Tonal configuration spaces

The concept of tonal attraction hints at the actual mental reality of the (tonal) musical experience. But it does not provide a comprehensive model of this reality. Listeners experience the tonal meanings of musical notes as *qualia* [21]. Whereas these qualia (as phenomena) withstand precise and complete scientific analysis, it is interesting that their seats have been located not in the notes themselves (or not exclusively [47]), but rather in the diatonic scale-height-degrees [4, 5, 20] and in the fifth-width-degrees [19, 40]. From a music-theoretical and music-semiotic point of view, we may consider tonal signification as a function which maps notes to pairs of fifth-width and

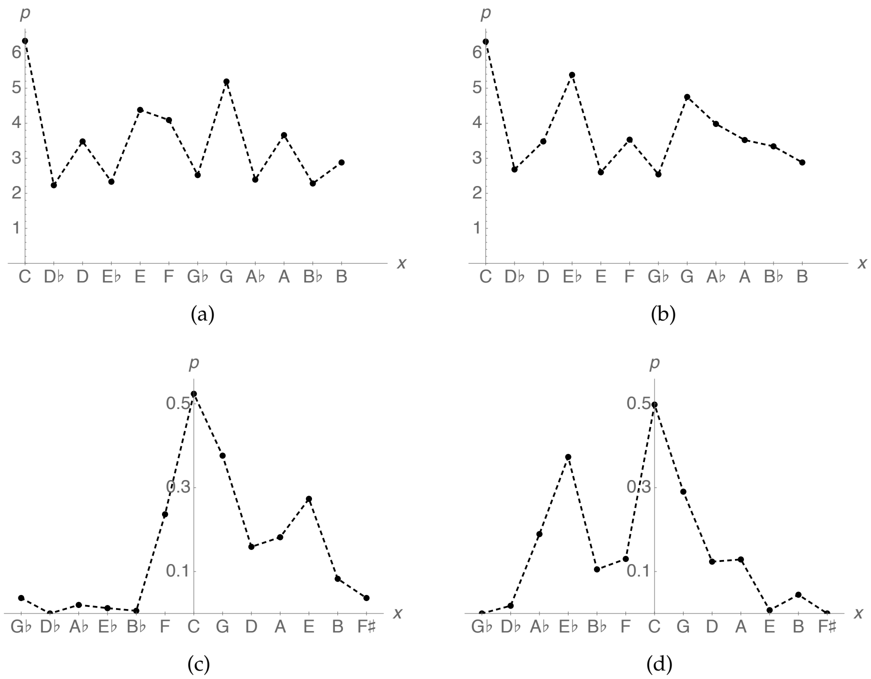

Figure 1: *Results of the Krumhansl and Kessler [29] experiment on static tonal attraction, given in Tab. 1. (a) Original ratings for C major context [28]. (b) Original ratings for C minor context [28]. (c) Rearranged and normalized rating data $p(x)$ in fifth-width ordering for C major context. (d) Rearranged and normalized rating data $p(x)$ in fifth-width ordering for C minor context.*

scale-height degrees. A suitable mathematical model for this kind of tonal signification is the *Regener space*: a convincing unifying space for notes and scale degrees, suggested by Regener [46].[2] The positioning of the notes within the coordinate system of heights and widths allows to describe their tonal meaning. Each tonality corresponds to a particular embedding of the note space into Regener's fifth-width and scale-height degree space. In Fig. 2 we display the C major embedding together with the associated KK attraction values from Tab. 1.

Hence we have the following situation: Tonal attraction deals with an empirical trace of tonal qualia, which themselves—although being not accessible as such—can be anchored in the Regener space. Thus it seems reasonable to define tonal attraction profiles on the Regener space. The fact that Krumhansl and Kessler [29] use the note names $C, C\sharp, D, D\sharp, E, F, F\sharp, G, G\sharp, A, A\sharp, B$ should not be misunderstood in the sense that their attraction profiles are defined on notes. Rather, Regener coordinates are all implied due to the presupposed C major context. One little detail is the interpretation of the sharpened notes $C\sharp, D\sharp, F\sharp, G\sharp, A\sharp$, which stand for their enharmonic classes in the Krumhansl and Kessler [29] study. Here we use the following 13 notes in the Regener space $C = (0,0)$, $D\flat = (7,0)$, $D = (2,1)$, $E\flat = (-3,2)$, $E = (4,2)$, $F = (-1,3)$, $F\sharp = (6,3)$, $G\flat = (-6,4)$, $G = (1,4)$, $A\flat = (-4,5)$, $A = (3,5)$, $B\flat = (-2,6)$, $B = (5,6)$.

In Fig. 2 we use scale degres $\hat{1}, .., \hat{7}$ for the height values, and solmisation syllables for the width

[2] Note intervals are thereby decomposed into scale-degree height intervals on the one hand and fifth-width intervals on the other hand. The elementary fifth-width (Regener called it *quint*) is the difference between the note interval $P5$ of the perfect fifth and the generic fifth (consisting of four scale steps). It corresponds to the elementary horizontal unit of the grid in Fig. 2

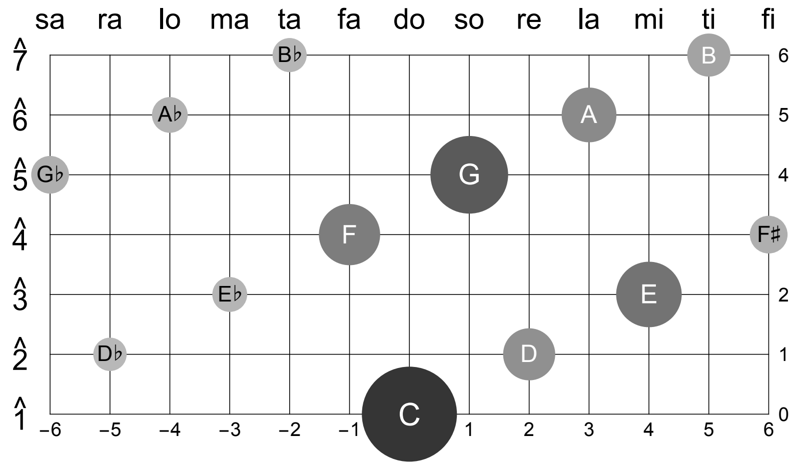

Figure 2: *The Regener space as two-dimensional tonal configuration space. Notes are shown as disks centered at their fifth-width and scale-height coordinates. Their radii and gray scales (from light to dark) correlate with the KK attraction values. The lower and right border lines of the grid are labeled by the respective width and height coordinates. The upper and left border lines are labeled by corresponding music-theoretical symbols: solmization syllables and diatonic scale degrees.*

values, respectively. In addition to Huron's classification of tonal attraction profiles as cognitive schemata we envisage their phenomenological characterisation as *mental gestalts* with reference to Leonard B. Meyer's concept of a *sound term*. Meyer's distinction between *sound terms* and *sound stimuli* is a useful starting point for an answer.

> A sound or group of sounds (whether simultaneous, successive, or both) that indicate, imply, or lead the listener to expect a more or less probable consequent event are a musical gesture or "sound term" within a particular style system. The actual physical stimulus which is the necessary but not sufficient condition for the sound term will be called the "sound stimulus." The same sound stimulus may give rise to different sound terms in different style systems or within one and the same system. [38, p. 45]

(See Sect. 3 for our quantum-theoretical adaption of Meyer's concept).

In a similar vein, Jacques Handschin talked metaphorically about a "society of tones:"

> What I would now like to state—and believe I must state—is that the musical character of the tone is determined precisely by the position it occupies in the series [f-c-g-d-a-e-h], this society of tones. [19, p. 7]

This metaphor resonates with the idea of a *mental gestalt*: Only the wholeness of this society, and not the single notes, creates the musical character of each note. But we think that Handschin underrates the phenomenological challenge of the tonal qualia, when he says:

> What we are dealing with here, with the tone characters, is the actual musical quality of the tone; and it is somehow wonderful that this quality evidently only comes about through the tones' position within the system. Or, better said, it consists in it. [19, p. 24]

Now, looking at the rearranged KK data in Figures 1(c, d), one immediately recognizes that the reordering along Regener's fifth-width line at the abscissa entails a much smoother and thereby much more pregnant gestalt of the tonal attraction schema than its original presentation along the chromatic line. Therefore, we follow Handschin in the emphasis of the line of fifth-widths as a key to an understanding of the tonal qualia with the intention to integrate this with the empirical concept of tonal attraction. The latter can be regarded as an empirical trace of the tonal qualia.

The gestalt idea also motivates our consideration of a continuous quantum model [1]. Quantum wave functions offer themselves as suitable mathematical models for gestalts in the sense that their values along the configuration space and in their time developments form an organic whole thanks to the underlying Schrödinger equation. For the beginning of our investigation we decided to develop a one-dimensional quantum model along the fifth-width axis which musically supports Handschin's position [19] that fifth-ordering corresponds to an ordering of the tonal qualia (*tone character* in Handschin's terminology).

In the concrete investigation of the present article we view Regener's space as being continuous and we practically restrict ourselves to the horizontal width axis. Both decisions deserve further discussion.

First, one has to note that from an acoustical point of view the fifth is merely a fixed frequency ratio and it might seem counterintuitive to assume the line of fifth-widths to be continuous. From a music-phenomenological point of view, however, a modulation in ♯-ward or ♭-ward direction is a transformation of the musical meaning of the note space which could well be said to be experienced as a continuous musical gesture [35]. Furthermore, there is another reason to consider real coordinates as well: The linear Regener transformation

$$R = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

of the note space—converting fifth/fourth coordinates into major/minor-step coordinates [46]—has meaningful eigenvectors [40]. The eigenvector associated with the larger eigenvalue represents the gradient of an intrinsic linear pitch height form. Its coordinates are not integer numbers, though.

Second, one may notice that for the initial formulation of our quantum model it is demanding enough to operate on a one-dimensional configuration space, which becomes the fifth-width axis here. But in order to interpret our findings properly in future research we need to keep an awareness of the entire Regener space.[3]

2.3 Statistical learning of tonal attraction

According to the statistical learning approach to music psychology [21, 44, 53], tonal attraction profiles, degree distributions, and melodic transition probabilities such as those presented in Tab. 1 may result from training statistical models through predictive processing [15, 26].

Therefore, one could fit, e.g., the Krumhansl and Kessler [29] data $p(x)$ from Tab. 1 to Gaussian mixture models (GMM) of the general form

$$p(x) = \sum_{k=1}^{N} \alpha_k \, \nu_k(x) \tag{1}$$

where

$$\nu_k(x) = \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left[-\frac{1}{2}\left(\frac{x - a_k}{\sigma_k}\right)^2\right] \tag{2}$$

[3] For the readers' convenience we will use note names to designate the familiar discrete positions on the fifth-width axis, although it would be theoretically more appropriate to use the syllables as in the left panel of Fig. 2.

are $N \in \mathbb{N}$ normal distribution densities with means $a_k \in \mathbb{R}$ and variances $\sigma_k^2 \in \mathbb{R}$. The coefficients $\alpha_k \in \mathbb{R}$ are the respective weights in the convex linear combination (1) with

$$\sum_{k=1}^{N} \alpha_k = 1 \,.$$

The sum (1) extends over a *tonal context*, e.g. of a given chord as discussed by [9]. In the present case, we identify a context C with a set of note positions

$$C = \{a_k \in \mathbb{R} | 1 \leq k \leq N\} \,. \tag{3}$$

By assuming just three components in the tonal attraction profiles [Fig. 1], the choice $N = 3$ leads to the *most general GMM* with eight free parameters: $a_1, a_2, a_3, \sigma_1, \sigma_2, \sigma_3, \alpha_1, \alpha_2 \in \mathbb{R}$ (while $\alpha_3 = 1 - \alpha_1 - \alpha_2$ through the convexity constraint).

Such a most general GMM is substantially simplified by assuming only one principal *attraction kernel* that is invariant under musical transposition [9],

$$\rho(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right) \,. \tag{4}$$

Now, the kernel is parameterized by its unique variance σ^2, such that the resulting *convolutional GMM* becomes a discrete convolution

$$p(x) = \sum_{k=1}^{N} \alpha_k \, \rho(x - a_k) \tag{5}$$

with six remaining free parameters: $a_1, a_2, a_3, \sigma, \alpha_1, \alpha_2$.[4]

A further constraint identifies the three mean values a_1, a_2, a_3 with the members of the tonic triade in the context $C = \{T, D, M\}$ (*T*onic, *D*ominant, *M*ediant), such that only three parameters $\sigma, \alpha_1, \alpha_2$ of the resulting *tonal GMM* remain.

Another simplification, subsequently called *Lerdahl GMM*, is obtained from the *Lerdahl interpolation*,

$$\rho\left(\frac{\pi}{6}\right) = \frac{1}{2}\rho(0) \,,$$

which is motivated by Lerdahl's hierarchical model [33], whose kernel function assumes half of the attraction of the tonic C for the dominant G and the subdominant F (cf. Fig. 3 below). From this interpolation equation one yields the variance parameter through

$$\sigma = \frac{\pi}{6\sqrt{2 \ln 2}} \,. \tag{6}$$

Hence, only the two independent mixing weights α_1, α_2 remain as free parameters.

We present the results of the Lerdahl GMM in Sect. 4.1 below.

3. Quantum Models of Tonal Attraction

For the motivation of our subsequent quantum model, we consult the respective passage of Meyer [38, p. 52]: "A sound stimulus becomes a sound term by entering into probability relationships with other sound terms within the style. These probability relationships are of different degrees."

[4]The convolution operation in (5) becomes obvious by writing $\alpha_k = \alpha(a_k)$.

Thus, a "sound term" in Meyer's theory is just a pregnant gestalt that is further related to a probability interpretation.

Considering tones as mental gestalts [9, 38] encourages us to formally identify them as quantum wave functions over abstract configuration spaces. Let the configuration space be the continuous fifth-width axis $X = \mathbb{R}$. A *sound term* [38] in the sense of the present study is then a wave function $\psi : \mathbb{R} \to \mathbb{C}$, such that the Hilbert space scalar product $(\psi, \psi) = \int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,\mathrm{d}x = 1$. Hence, a tone is a state in the complex Hilbert space $\mathcal{H} = L^2(X)$. Let $\psi \in L^2(X)$ be a sound term over real configuration space. The squared modulus $p(x) = |\psi(x)|^2 = \psi^*(x)\psi(x)$ is the *attraction rate*—or *anchoring strength* according to [34]—of the probe $x \in \mathbb{R}$ in the *context* ψ. This justifies the proposed terminology, interpreting sound terms as quantum wave functions in our exposition.

As beim Graben and Blutner [8] and beim Graben [7] have shown, it is possible to describe tonal attraction as an eigenvalue problem

$$H\psi(x) = E\psi(x) \tag{7}$$

with real eigenvalue E for a hermitian differential operator H, called the *Hamiltonian* acting on Hilbert space $\mathcal{H} = L^2(X)$. Basically, equation (7) is a stationary Schrödinger equation for the wave function ψ of a single quantum particle [48, 49, 50, 51].

Next, we observe that a Gaussian wave packet

$$\psi(x) = \sqrt[4]{\frac{E_0}{\pi}}\mathrm{e}^{-\frac{E_0 x^2}{2}} \tag{8}$$

with energy

$$E_0 = \frac{1}{2\sigma^2} \tag{9}$$

and variance σ^2 is able to describe the tonal attraction data of [29] equally well, as shown in Fig. 3 [42]. The resulting attraction rate becomes

$$\rho(x) = |\psi(x)|^2 = \sqrt{\frac{E_0}{\pi}}\mathrm{e}^{-E_0 x^2}\,. \tag{10}$$

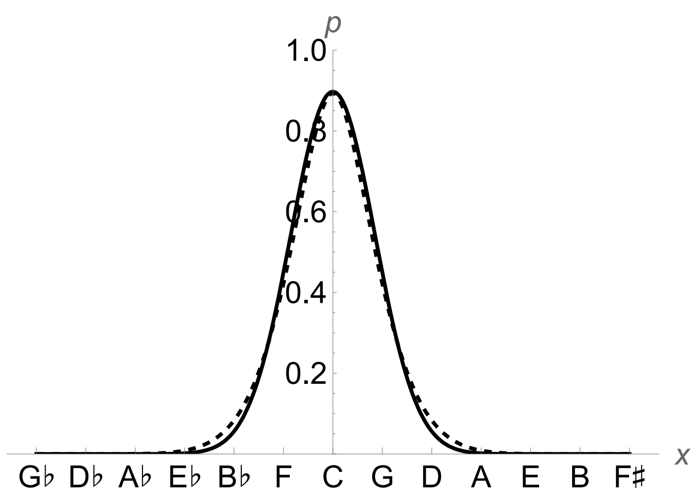

Figure 3: *Attraction kernel $p(x) = |\psi(x)|^2$ of the Krumhansl and Kessler [29] experimental data for C major (centered at tone C of the fifth-width axis). Solid: Gaussian wave function $\psi(x)$ [Eq. (8)], Dashed: deformed cosine similarity kernel of [8], and [7].*

Figure 3 displays the kernel obtained from the Gaussian wave function $\psi(x)$ [Eq. (8)] as solid line and the deformed cosine similarity kernel from the gauge-theoretic approach of [8, 7] as dashed line for comparison.

Interestingly, the Gaussian wave packet (8) solves the Schrödinger equation of the quantum harmonic oscillator

$$-\psi''(x) + E_0^2 x^2 \psi(x) = E\psi(x) \tag{11}$$

for the ground state energy $E = E_0$ [49, 52]. The energy parameter E_0 can be interpolated by comparison with the attraction kernel of Lerdahl's hierarchical model [34]. This leads to

$$E_0 = \frac{36 \ln 2}{\pi^2}\,, \tag{12}$$

proportional to the reciprocal variance (6) in our *Lerdahl interpolation*. The energy of the ground state of the Schrödinger equation (11) is then $E_0 = E$, i.e. the parameter of the Lerdahl interpolation (12) to which we refer as to the *Lerdahl energy* henceforth.[5]

From a mathematical point of view the KK profiles are special instances of *pitch class profiles*, which play a central role also in computational music theory and analysis. Pitch class profiles also comprise single notes (pitch classes) and chords (pitch class sets) as special cases. On the pure mathematical level we can turn pitch class profiles into wave functions as superpositions of Gaussians. But then a phenomenological question arises: Does not such a superposition of waves still represent a kind of "atomistic" model? How does this relate to our idea to consider mental gestalts, where the whole is more than the sum of its constituents?

What Meyer [38] calls a "sound stimulus" would in our concept be an isolated Gaussian wave function (or a superposition of Gaussians) without a tonal context.[6] A "sound term" could, in fact, be also be a single note, but it would be loaded with musical meaning in the context of a tonality. The knowledge about consequent events becomes modeled by the Schrödinger equation in our quantum approach. If there are elementary sound terms in this sense, they are not the Gaussians, modeling single notes, but rather the stationary eigenstates of the Hamilton operator. When we interpret a Gaussian as the initial state of a time development, then it embodies dynamic properties of the tonality and thereby exemplifies an interaction with the tonal attraction as it is modeled by the Hamiltonian below in Sect. 3.1. The key to an understanding of this approach is the insight that the superposition of stationary eigenstates is not stationary. This is precisely the case where the "whole becomes more than its parts."

3.1 Quantum statistical learning of tonal interaction

Quantum statistical learning became recently an important research area in connection with quantum computation and quantum information processing [14, 37]. In the present context of musical statistical learning, we consider the following research problem: Given a statistical model for some empirical data, that could have been acquired by some classical statistical learning algorithm, could one construe a corresponding quantum Hamilton operator that reproduces the statistical model as the ground state of a stationary Schrödinger equation (7)?

Introducing the harmonic oscillator potential

$$V_0(x) = E_0^2 x^2\,, \tag{13}$$

[5] Note that we also confirm Lerdahl's speculation about a Hamiltonian principle of musical least effort [33] in the present framework, as the ground state energy E_0 can be obtained from a corresponding variational principle.

[6] In deviation from Meyer [38] we abstain from a reference to an acoustic level of reality.

entails the Hamilton operator in the form

$$H = -\frac{\partial^2}{\partial x^2} + V_0(x) \,. \tag{14}$$

The correspondingly rewritten Schrödinger equation

$$\psi''(x) = [V_0(x) - E]\psi(x) \tag{15}$$

has the Gaussian wave function (8) as its stationary ground state for eigenenergy $E = E_0$.

Now, given a convolutional GMM $p(x)$ of tonal attraction (5), we want to construe a quantum Hamiltonian H with interaction potential $V(x)$ such that $p(x) = |\psi(x)|^2$ and $\psi(x)$ solves the Schrödinger equation

$$H\psi(x) = E_0\psi(x) \tag{16}$$

as its stationary ground state.

In order to tackle this problem, we first derive a differential equation for the tonal attraction rate. First, we observe that the Gaussian wave function (8) is real-valued and positive. Therefore,

$$p(x) = |\psi(x)|^2 = \psi(x)^2 \,. \tag{17}$$

Differentiating this expression twice, yields

$$\begin{aligned} p'(x) &= 2\psi(x)\psi'(x) && (18) \\ p''(x) &= 2\psi'(x)^2 + 2\psi(x)\psi''(x) \,. && (19) \end{aligned}$$

Squaring (18) gives

$$p'(x)^2 = 4\psi(x)^2\psi'(x)^2 \,. \tag{20}$$

Now, we can eliminate all occurrences of ψ and its derivatives in (19) as follows. First, we insert ψ'' from (15)

$$p''(x) = 2\psi'(x)^2 + 2[V(x) - E_0]\psi(x)^2 \,,$$

multiplication with $2\psi(x)^2$ entails

$$2\psi(x)^2 p''(x) = 4\psi(x)^2\psi'(x)^2 + 4[V(x) - E_0]\psi(x)^4 \,,$$

where we eventually insert (17) and (20)

$$2p(x)p''(x) - p'(x)^2 - 4[V(x) - E_0]p(x)^2 = 0 \,. \tag{21}$$

Equation (21) is a nonlinear differential equation for the quantum probability $p(x)$, corresponding to the linear Schrödinger equation (15) for the quantum probability amplitude $\psi(x)$.

Next, we consider the convolutional GMM (5). Its derivatives are given by the unique derivatives of the convolution kernel (10).

$$\begin{aligned} p'(x) &= \sum_{k=1}^{N} \alpha_k \rho'(x - a_k) && (22) \\ p''(x) &= \sum_{k=1}^{N} \alpha_k \rho''(x - a_k) \,. && (23) \end{aligned}$$

Where we have

$$\begin{aligned} \rho'(x) &= \sqrt{\frac{E_0}{\pi}}(-2E_0 x)\mathrm{e}^{-E_0 x^2} = -2E_0 x\rho(x) && (24) \\ \rho''(x) &= [4E_0^2 x^2 - 2E_0]\rho(x) \,. && (25) \end{aligned}$$

Inserting (22), (23), (24), and (25) into the differential equation (21) yields

$$2\left\{\sum_{k=1}^{N}\alpha_k\,\rho(x-a_k)\right\}\left\{\sum_{k=1}^{N}\alpha_k\,\rho''(x-a_k)\right\}-\left\{\sum_{k=1}^{N}\alpha_k\,\rho'(x-a_k)\right\}^2-4[V(x)-E_0]\left\{\sum_{k=1}^{N}\alpha_k\,\rho(x-a_k)\right\}^2=0$$

$$2\left\{\sum_{k=1}^{N}\alpha_k\,\rho(x-a_k)\right\}\left\{\sum_{k=1}^{N}\alpha_k\,[4E_0^2(x-a_k)^2-2E_0]\rho(x-a_k)\right\}-\left\{\sum_{k=1}^{N}-2E_0\alpha_k\,(x-a_k)\rho(x-a_k)\right\}^2-4[V(x)-E_0]\left\{\sum_{k=1}^{N}\alpha_k\,\rho(x-a_k)\right\}^2=0$$

$$\sum_{k,l=1}^{N}[8E_0^2(x-a_l)^2-4E_0]\,\alpha_k\alpha_l\rho(x-a_k)\,\rho(x-a_l)-\sum_{k,l=1}^{N}4E_0^2\,(x-a_k)(x-a_l)\alpha_k\,\alpha_l\rho(x-a_k)\rho(x-a_l)-4[V(x)-E_0]\sum_{k,l=1}^{N}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)=0$$

$$4[V(x)-E_0]\sum_{k,l=1}^{N}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)=\sum_{k,l=1}^{N}[8E_0^2(x-a_l)^2-4E_0]\,\alpha_k\alpha_l\rho(x-a_k)\,\rho(x-a_l)-\sum_{k,l=1}^{N}4E_0^2\,(x-a_k)(x-a_l)\alpha_k\,\alpha_l\rho(x-a_k)\rho(x-a_l)$$

$$[V(x)-E_0]\sum_{k,l=1}^{N}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)=\sum_{k,l=1}^{N}[2E_0^2(x-a_l)^2-E_0]\,\alpha_k\alpha_l\rho(x-a_k)\,\rho(x-a_l)-\sum_{k,l=1}^{N}E_0^2\,(x-a_k)(x-a_l)\alpha_k\,\alpha_l\rho(x-a_k)\rho(x-a_l)$$

$$V(x)=\frac{\sum_{k,l=1}^{N}[2E_0^2(x-a_l)^2-E_0]\,\alpha_k\alpha_l\rho(x-a_k)\,\rho(x-a_l)-\sum_{k,l=1}^{N}E_0^2\,(x-a_k)(x-a_l)\alpha_k\,\alpha_l\rho(x-a_k)\rho(x-a_l)}{\sum_{k,l=1}^{N}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)}+E_0$$

$$V(x)=\frac{\sum_{k,l=1}^{N}2E_0^2(x-a_l)^2\,\alpha_k\alpha_l\rho(x-a_k)\,\rho(x-a_l)-\sum_{k,l=1}^{N}E_0^2\,(x-a_k)(x-a_l)\alpha_k\,\alpha_l\rho(x-a_k)\rho(x-a_l)}{\sum_{k,l=1}^{N}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)}$$

$$V(x)=E_0^2\frac{\sum_{k,l=1}^{N}[2(x-a_l)^2-(x-a_k)(x-a_l)]\alpha_k\alpha_l\,\rho(x-a_k)\rho(x-a_l)}{\sum_{kl}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)}\,.$$

Note that the tonal interaction potential

$$V(x) = E_0^2 \frac{\sum_{kl}[2(x-a_l)^2 - (x-a_k)(x-a_l)]\alpha_k\alpha_l\,\rho(x-a_k)\rho(x-a_l)}{\sum_{kl}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)} \tag{26}$$

in the presence of N interacting force centers turns out as the center of gravity in the corresponding function space.

Equation (26) can be further simplified, leading to

$$V(x) = E_0^2 \frac{\sum_{kl}[2(x^2 - 2a_l x + a_l^2) - (x^2 - a_k x - a_l x + a_k a_l)]\alpha_k\alpha_l\,\rho(x-a_k)\rho(x-a_l)}{\sum_{kl}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)}$$

$$V(x) = E_0^2 \frac{\sum_{kl}[2x^2 - 4a_l x + 2a_l^2 - x^2 + a_k x + a_l x - a_k a_l]\alpha_k\alpha_l\,\rho(x-a_k)\rho(x-a_l)}{\sum_{kl}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)}$$

$$V(x) = E_0^2 \frac{\sum_{kl}[x^2 - 3a_l x + 2a_l^2 + a_k x - a_k a_l]\alpha_k\alpha_l\,\rho(x-a_k)\rho(x-a_l)}{\sum_{kl}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)}$$

$$V(x) = E_0^2 \frac{\sum_{kl}[x^2 + (a_k - 3a_l)x + a_l(2a_l - a_k)]\alpha_k\alpha_l\,\rho(x-a_k)\rho(x-a_l)}{\sum_{kl}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)}$$

$$V(x) = E_0^2 x^2 + E_0^2 \frac{\sum_{kl}[(a_k - 3a_l)x + a_l(2a_l - a_k)]\alpha_k\alpha_l\,\rho(x-a_k)\rho(x-a_l)}{\sum_{kl}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)}$$

$$V(x) = V_0(x) + W(x) \tag{27}$$

with the harmonic oscillator potential $V_0(x)$ (13) and a perturbation

$$W(x) = E_0^2 \frac{\sum_{kl}[(a_k - 3a_l)x + a_l(2a_l - a_k)]\alpha_k\alpha_l\,\rho(x-a_k)\rho(x-a_l)}{\sum_{kl}\alpha_k\alpha_l\,\rho(x-a_k)\,\rho(x-a_l)} \tag{28}$$

for the interactions between the context tones. The potential (27) belongs to the general class of quantum anharmonic oscillators, as discussed, e.g., in [36, 54, 11, 6]. Henceforth we refer to the differential operator

$$H = -\frac{\partial^2}{\partial x^2} + V(x) \tag{29}$$

with potential (27) as to the *schematic Hamiltonian* of the statistically acquired musical GMM schema. The tonal attraction profile is then obtained as the ground state solution of the stationary Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = E\psi(x) \tag{30}$$

with eigenenergy $E = E_0$.

3.2 Static attraction

In order to solve the stationary Schrödinger equation (30), we exploit the basic idea of quantum perturbation theory [50], by developing the solutions ψ in the complete orthonormal system of the eigenstates of the harmonic oscillator potential V_0. These are known as the Hermite functions

$$\varphi_n(x) = \sqrt[4]{\frac{E_0}{\pi}} \frac{1}{\sqrt{2^n n!}} h_n(\sqrt{E_0}x) \mathrm{e}^{-\frac{E_0 x^2}{2}} \,. \tag{31}$$

with $h_n(x)$ the Hermite polynomials of order n [50, 52]. Then, (31) solves the Schrödinger equation for eigenenergy E_n,

$$-\varphi_n''(x) + V_0(x)\varphi_n(x) = E_n \psi(x) \,, \tag{32}$$

with energy spectrum[7]

$$E_n = 2E_0 \left(n + \frac{1}{2} \right) \,, \tag{33}$$

and orthonormality relations

$$(\varphi_m, \varphi_n) = \int_X \varphi_m(x)^* \varphi_n(x) \, \mathrm{d}x = \delta_{mn} \,. \tag{34}$$

For the anharmonic oscillator potential (27), we make a superposition ansatz for its k-th eigenstate η_k,

$$\eta_k(x) = \sum_l Q_{lk} \varphi_l(x) \tag{35}$$

with the Hermite functions (31).

For the solution of the stationary Schrödinger equation (30), we firstly have to diagonalize the schematic Hamiltonian

$$H = -\frac{\partial^2}{\partial x^2} + V(x) = -\frac{\partial^2}{\partial x^2} + V_0(x) + W(x) \,,$$

which is achieved by solving the associated eigenvalue problem

$$\left[-\frac{\partial^2}{\partial x^2} + V(x) \right] \eta_k(x) = D_k \eta_k(x) \,, \tag{36}$$

i.e. the stationary Schrödinger equation for eigenstate η_k with eigenvalue $D_k \in \mathbb{R}$.

Then,

$$\begin{aligned}
\left[-\frac{\partial^2}{\partial x^2} + V(x) \right] \eta_k(x) &= D_k \eta_k(x) \\
\left[-\frac{\partial^2}{\partial x^2} + V_0(x) + W(x) \right] \eta_k(x) &= D_k \eta_k(x) \\
\left[-\frac{\partial^2}{\partial x^2} + V_0(x) + W(x) \right] \sum_l Q_{lk} \varphi_l(x) &= D_k \sum_l Q_{lk} \varphi_l(x) \\
\sum_l \left[-\frac{\partial^2}{\partial x^2} + V_0(x) + W(x) \right] Q_{lk} \varphi_l(x) &= \sum_l Q_{lk} D_k \varphi_l(x) \\
\sum_l Q_{kl} \left\{ \left[-\frac{\partial^2}{\partial x^2} + V_0(x) \right] \varphi_l(x) + W(x) \varphi_l(x) \right\} &= \sum_l Q_{lk} D_k \varphi_l(x) \\
\sum_l Q_{lk} \left[E_l \varphi_l(x) + W(x) \varphi_l(x) \right] &= \sum_l Q_{lk} D_k \varphi_l(x) \,,
\end{aligned}$$

[7] Note that the results of our quantum statistical learning approach differ from the quantum physical account by a factor 2 in the spectrum (33). This is also reflected by a slightly different normalization of the Hermite polynomials, here.

for φ_l solves the unperturbed Schrödinger equation (15) for the oscillator potential V_0 with eigenvalue E_l given through (33).

Next we multiply with another base function φ_m^* and integrate over configuration space X, exploiting the orthonormality relations (34),

$$\sum_l Q_{lk} E_l \int_X \varphi_m(x)^* \varphi_l(x)\,\mathrm{d}x + \sum_l Q_{lk} \int_X \varphi_m(x)^* W(x) \varphi_l(x)\,\mathrm{d}x = \\ = \sum_l Q_{lk} D_k \int_X \varphi_m(x)^* \varphi_l(x)\,\mathrm{d}x$$

such that

$$\begin{aligned} \sum_l Q_{lk} E_l \delta_{ml} + \sum_l Q_{lk} \int_X \varphi_m(x)^* W(x) \varphi_l(x)\,\mathrm{d}x &= \sum_l Q_{lk} D_k \delta_{ml} \\ Q_{mk} E_m + \sum_l Q_{lk} w_{ml} &= Q_{mk} D_k\,. \end{aligned}$$

Introducing matrix elements

$$w_{mn} = \int_X \varphi_m(x)^* W(x) \varphi_n(x)\,\mathrm{d}x\,, \tag{37}$$

we may write this eigenvalue equation for the D_k as an infinite matrix equation in the coefficient Hilbert space $\ell^2(\mathbb{R})$ by introducing a diagonal matrix

$$e_{mn} = E_n \delta_{mn} \tag{38}$$

with the energy spectrum (33), in compact form writing

$$\boldsymbol{E} = (e_{mn})_{m,n \in \mathbb{N}_0}\,, \tag{39}$$

$$\boldsymbol{Q} = (Q_{mk})_{k,m \in \mathbb{N}_0}\,, \tag{40}$$

and

$$\boldsymbol{\Delta} = (D_k \delta_{km})_{k,m \in \mathbb{N}_0}\,, \tag{41}$$

as

$$(\boldsymbol{E} + \boldsymbol{W}) \cdot \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{\Delta}\,, \tag{42}$$

or, equivalently, as

$$\boldsymbol{Q}^* \cdot (\boldsymbol{E} + \boldsymbol{W}) \cdot \boldsymbol{Q} = \boldsymbol{\Delta}\,, \tag{43}$$

with the diagonalized Hamiltonian $\boldsymbol{\Delta}$ in the eigenvector basis $\boldsymbol{Q}$ with adjoined $\boldsymbol{Q}^*$.

Numerically, we solve (42) for finite-dimensional approximations $\mathbb{C}^d$ of the coefficient Hilbert space $\ell^2(\mathbb{R})$ with $d = 5, 10, 15, 20$ for the present analysis, and for $d = 30$ in the next section [10]. The matrix elements

$$w_{mn} = \int_{-r}^{r} \varphi_m(x)^* W(x) \varphi_n(x)\,\mathrm{d}x \tag{44}$$

of the perturbation potential (28) are numerically estimated with $r = 20$.

3.3 Dynamic attraction

In our quantum music framework, we obtain a concept of *schematic dynamics* in terms of the time-dependent Schrödinger equation [51],

$$H\Psi(x,t) = \mathrm{i}\frac{\partial}{\partial t}\Psi(x,t) \tag{45}$$

for a spatiotemporal wave function $\Psi(x,t) \in L^2(X, \mathbb{R})$.

Now, for the schematic Hamiltonian H is diagonal in its own orthonormal basis (35), it is straightforward to prescribe the general solution algorithm for the musical quantum dynamics. Starting with an arbitrary stationary wave function ψ as an initial condition, we develop it into a series of H eigenstates:

$$\psi(x) = \sum_k R_k \eta_k(x) = \sum_k R_k \sum_l Q_{lk}\varphi_l(x) = \sum_{lk} Q_{lk} R_k \varphi_l(x) = \sum_l (\boldsymbol{Q}\cdot\boldsymbol{r})_l \varphi_l(x) \tag{46}$$

with the product of the eigenvector matrix (40) with the coefficient vector

$$\boldsymbol{r} = (R_k)_{k\in\mathbb{N}_0} \,. \tag{47}$$

The coefficients are obtained by the orthogonal projections

$$R_k = (\eta_k, \psi) = \int_X \eta_k(x)^* \psi(x)\,\mathrm{d}x \,. \tag{48}$$

Then, the solution of the time-dependent Schrödinger equation is given by

$$\Psi(x,t) = \sum_k R_k \mathrm{e}^{-\mathrm{i}D_k t}\eta_k(x) \tag{49}$$

with the eigenenergies D_k solving (42).

In the given framework of a musical quantum model, a single tone could be represented by a *coherent state* of the associated harmonic oscillator with potential $V_0(x)$ and ground state φ_0 [52],

$$\psi_a(x) = \varphi_0(x-a) = \sqrt[4]{\frac{E_0}{\pi}}\mathrm{e}^{-\frac{E_0(x-a)^2}{2}} \tag{50}$$

centered at tone position a at the fifth-width axis. Then, we interpret the tone position a of a coherent initial state ψ_a as an *antecedent state* of schematic dynamics. Letting this state evolve according to (49), leads to

$$\Psi_a(x,t) = \sum_k R_k(a)\mathrm{e}^{-\mathrm{i}D_k t}\eta_k(x) \tag{51}$$

with

$$R_k(a) = (\eta_k, \psi_a) = \int_X \eta_k(x)^* \psi_a(x)\,\mathrm{d}x \,. \tag{52}$$

as the coefficients of the coherent state ψ_a in the schematic orthonormal basis η_k. Then, we compute the transition probability from the antecedent a to any *consequent state* c on the fifth-width axis after some time t as the projection

$$p(c|a;t) = |(\psi_c, \Psi_a(\cdot,t))|^2 \tag{53}$$

of Ψ_a onto ψ_c [9].

4. First Experiments and Results

We present our results for the C major tonality shown in the third row of Tab. 1 for the Krumhansl and Kessler [29] profiles here.

4.1 Gaussian mixture models

Figure 4 displays nonlinear model fits of the Lerdahl Gaussian mixture models (GMM) on the Krumhansl and Kessler [29] probe tone profiles $p(x)$ against the real fifth-width line.

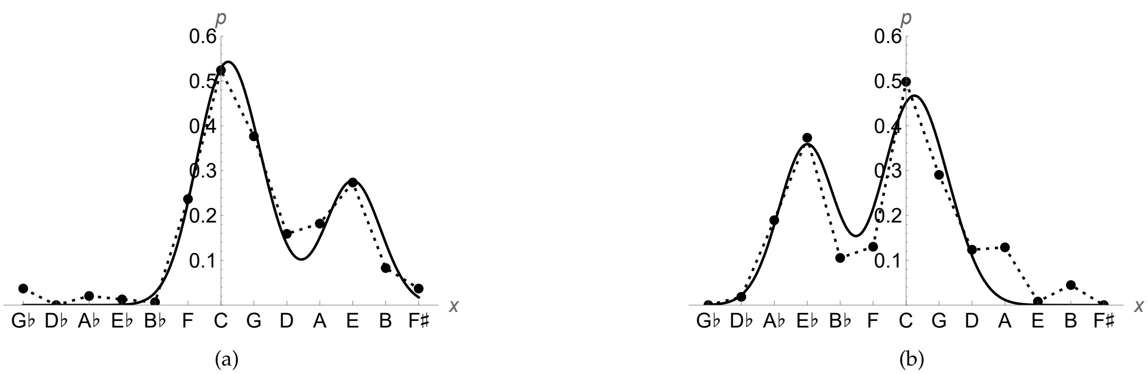

Figure 4: *Lerdahl Gaussian mixture model (GMM) for the Krumhansl and Kessler [29] rating data $p(x)$ against the width axis. Solid: fitted GMM; dashed $p(x)$. (a): For C major context; (b): for C minor context.*

By prescribing the variance of the model through the Lerdahl interpolation (12), yielding (6), we obtain a particular convolution model, dubbed Lerdahl GMM, containing only two parameters α_1, α_2. Figure 4(a, b) shows our results. For C major [Fig. 4(a)] the model is able to cover $r^2 = 0.986$ of the data variance, with parameters $\alpha_1 = 0.49, \alpha_2 = 0.2, \alpha_3 = 0.31$. For C minor [Fig. 4(b)] the covering is a bit lowered to $r^2 = 0.93$, with parameters $\alpha_1 = 0.4, \alpha_2 = 0.19, \alpha_3 = 0.4$.

4.2 Interaction potential

From the Lerdahl GMM in Fig. 4(a), we obtain the musical interaction potential (27) of the C major scale through our quantum statistical learning approach. Figure 5 plots the resulting anharmonic oscillator potential $V(x)$ (solid) in comparison with the unperturbed harmonic oscillator potential $V_0(x)$ (dashed).

As Fig. 5 reveals, the schematic interaction potential $V(x)$ turns out to be a kind of a quantum double-well potential for an anharmonic quantum oscillator [6, 22].

4.3 Static attraction

Solving the stationary Schrödinger equation (7), or (36) respectively, by diagonalizing the schematic Hamiltonian (29), yields the orthonormal eigenfunctions $\eta_n(x)$ together with their associated energy eigenvalues D_n through (42). In finite-dimensional approximations with dimensions $d = 5, 10, 15, 20$, we compute the distance of the eigenvalues D_n from their unperturbed harmonic counterparts E_n, i.e. $D_n - E_n$ as a measure for the quality of the approximation.

Figure 6 plots the spectral differences for the four finite-dimensional approximations.

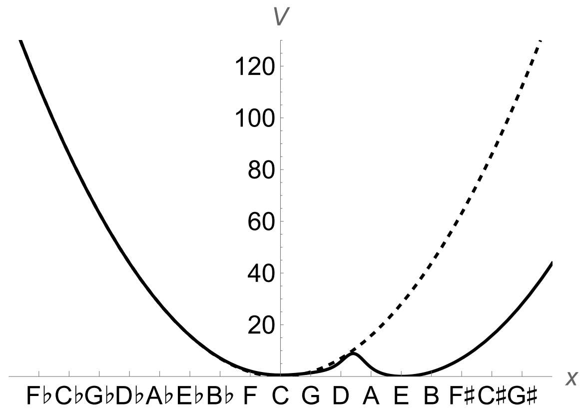

Figure 5: *Musical interaction potential of C major. Solid: $V(x)$ [Eq. (27)]. Dashed: harmonic oscillator potential $V_0(x)$ [Eq. (13)] for comparison.*

As Fig. 6 reveals, the finite-dimensional approximations continuously improve for larger embedding dimensions d. For $d = 5$ [Fig. 6(a)] the anharmonic ground state energy D_0 substantially deviates from its harmonic counterpart E_0. For increasing dimensions, Fig. 6(b,c), the difference $D_0 - E_0$ tends toward zero. Finally, for $d = 20$ [Fig. 6(d)], there is almost no further difference between both ground state energies.

Moreover, the energy spectra show large differences $D_n - E_n$ for intermediary n, while for increasing n these differences become eventually smaller. This is reflected by the fact that the anharmonic schematic interaction potential also converges to the harmonic oscillator potential for large deflections along the fifth-width line, indicating larger deflection energies.

In Figure 7 we present the squared ground state solution $p(x) = |\eta_0(x)|^2$ of the stationary Schrödinger equation (36) in comparison with the Lerdahl GMM from Fig. 4(a) for increasing embedding dimensions.

As already indicated by the behavior of the energy spectra in Fig. 6, we see that also the ground state probability densities $|\eta_0(x)|^2$ converge against the Krumhansl and Kessler [29] profile for C major with increasing dimension. This numerically proves that the ground state of the schematic Hamiltonian is indeed the GMM model of the KK profile.

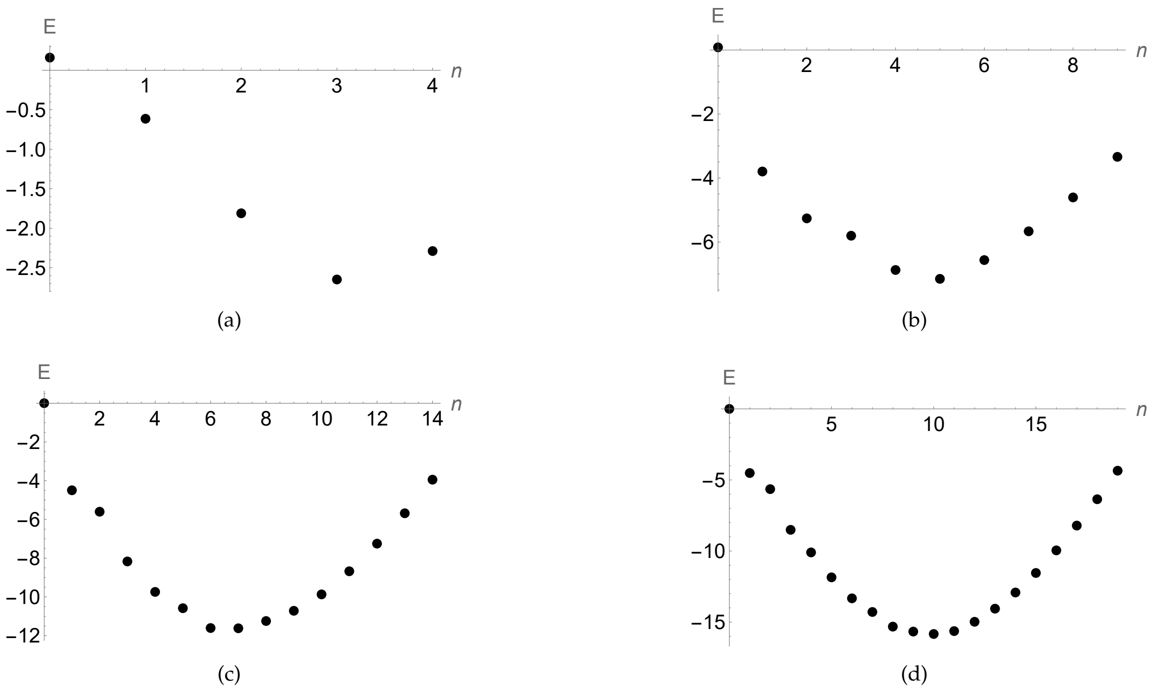

Figure 6: *Energy spectra $D_n - E_n$ of the C major stationary Schrödinger equation (36) for finite-dimensional Hilbert space approximations. Dimensions: (a) $d = 5$, (b) $d = 10$, (c) $d = 15$, (d) $d = 20$.*

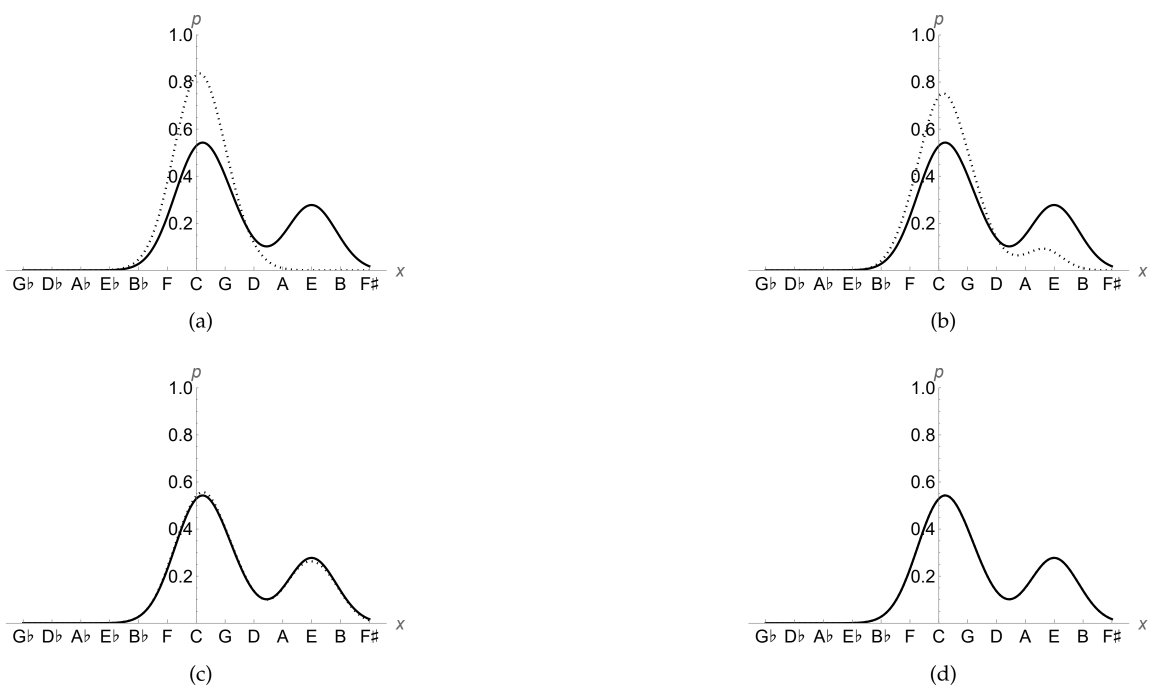

Figure 7: *Ground state probability densities of the C major stationary Schrödinger equation (36) for finite-dimensional Hilbert space approximations. Solid: Lerdahl GMM [Fig. 4(a)], dotted: Schrödinger ground state probability densities $|\eta_0(x)|^2$. Dimensions: (a) $d = 5$, (b) $d = 10$, (c) $d = 15$, (d) $d = 20$.*

4.4 Dynamic attraction

For the simulation of dynamic attraction we use a greater numerical embedding dimension of $d = 30$. Figure 8 displays a temporal snapshot sequence for the dynamics of a Gaussian wave package $p_a(x,t) = |\Psi_a(x,t)|^2$ [Eq. (51)], initially centered at tone $a = \mathrm{pos}(E) = 4\,\mathrm{quint}$ according to the schematic Hamiltonian (29) of C major. Shown are 12 time slices for times $t = 0$ until $t = 5.5$ in steps of $\Delta t = 0.5$.

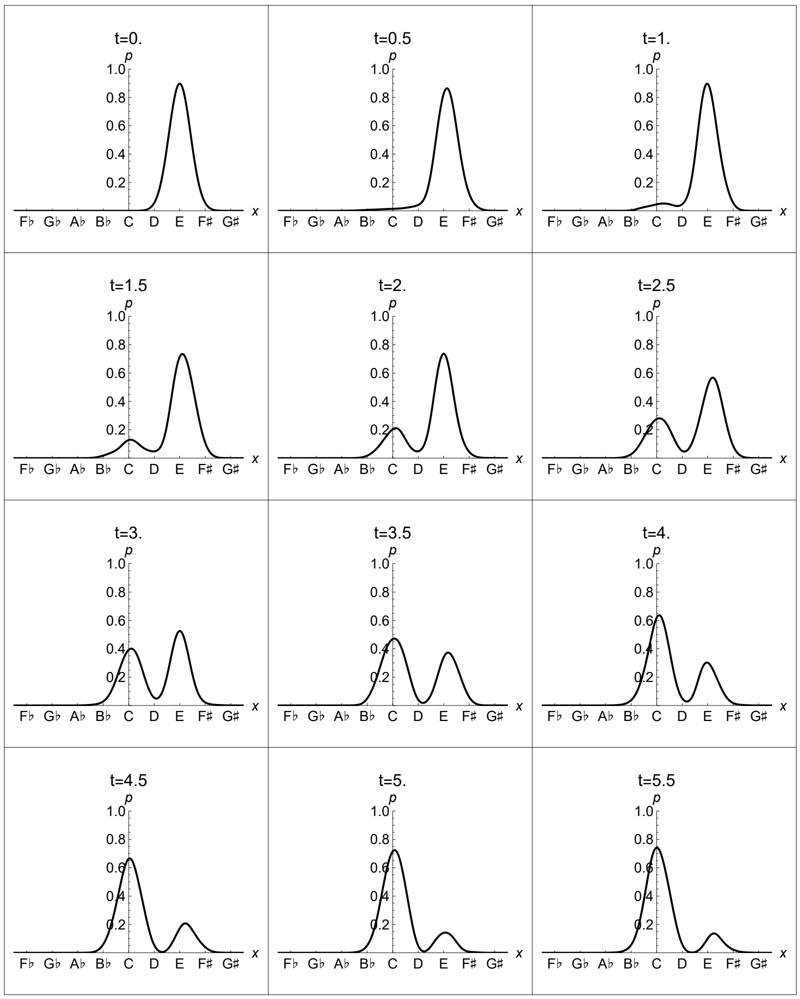

Figure 8: *Temporal development of Gaussian wave package centered at tone* $\mathrm{pos}(E) = 4\,\mathrm{quint}$ *according to schematic C major Hamiltonian* (29).

Figure 8 demonstrates that an initially coherent state with respect to the harmonic oscillator potential $V_0(x)$ disperses under the impact of the schematic Hamiltonian. The Hilbert space overlap (53),

$$p(c|a;t) = |(\psi_c, \Psi_a(\cdot,t))|^2$$

between the temporally evolved antecedent state $\Psi_a(x,t)$ and a tentative consequent wave package $\psi_c(x)$ could be potentially regarded as the continuous transition probability from a to c. Under this interpretation, our dynamic model offers a continuously changing Markov model for transitions between the selected coherent states. Figure 9 illustrates this for selected time points.

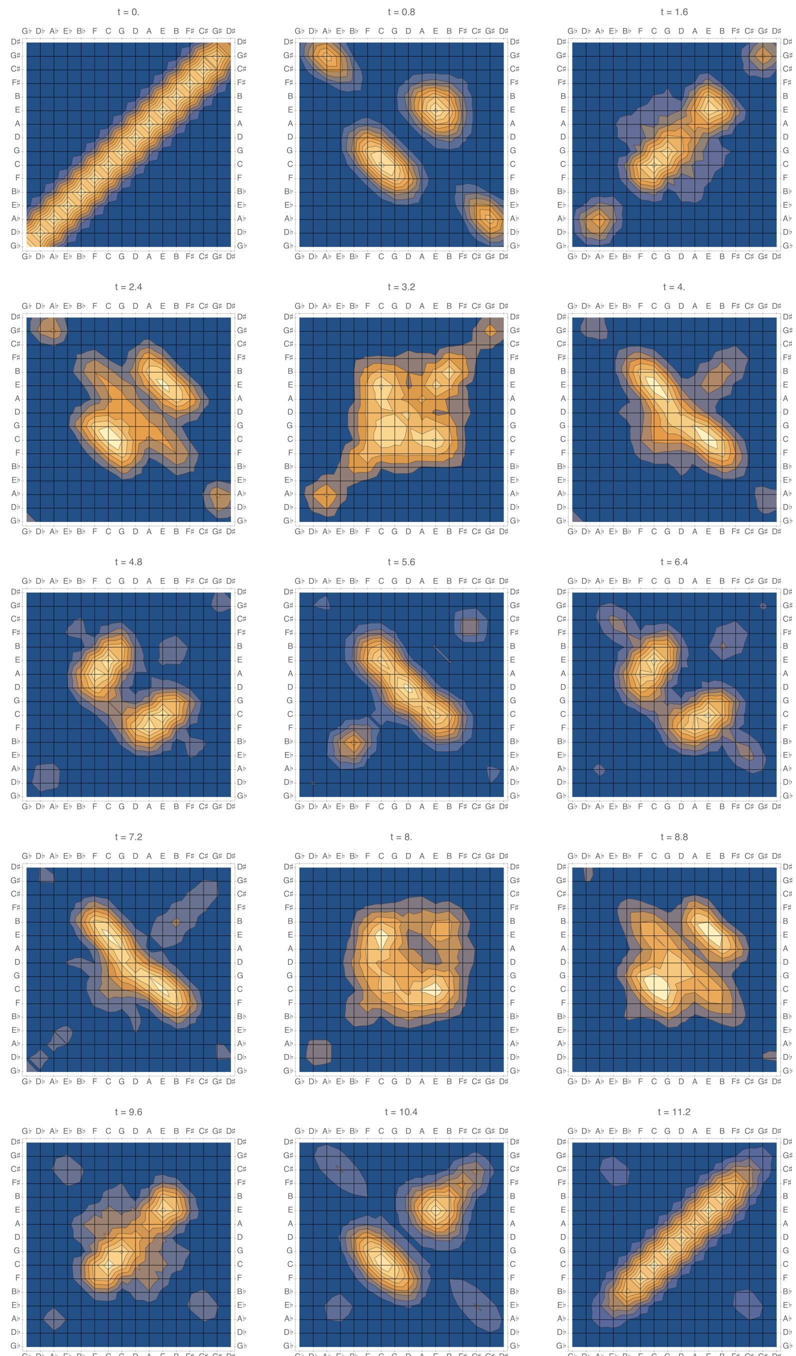

Figure 9: *Contour plots of the time development of the Hilbert space overlaps (53) in fifth-width space for antecedents $a = -6, \ldots, 9$ quint (vertical) and consequents $c = -6, \ldots, 9$ quint (horizontal) for $t = 0, \ldots, 11.2$ in steps of $\Delta t = 0.8$. The respective values are represented by varying degrees of brightness, ranging from dark blue, over blue, yellow, to white.*

Figure 9 shows a snapshot sequence of 12 squared Hilbert space overlaps (53) as Markov transition probabilities from antecedent states centered at tone positions a (vertical) to consequent states that are centered at positions c (horizontal) along the fifth-width line. For $t = 0$ depicted in the first panel, there are obviously only self-transitions as the antecedent state has not evolved at all. Already from a first glance at these contour plots it becomes clear that the choice of the fifth-width as the configuration space is substantial to the resulting dynamics. Pairs with higher Hilbert space overlaps tend to remain in connected regions of the transition landscape. Apart from the perfect primes $P1$, fifth-related pairs are often privileged, in particular in the central region around the pentatonic $\{C, G, D, A, E\}$ at time $t = 2.4$. This observation indicates that none of these momentary *Markov model snapshots* represents the kind of transition probabilities which emerge from the empirical statistics of melodic corpora, where rather transitions between neighbouring diatonic scale degrees are privileged over fifth progressions [21, 53]. It could be interesting, though for a future study, to compare the quantum model to transition probabilities in fundament progressions.

5. Conclusion

In this study we have presented a novel mathematical approach of quantum statistical learning [14, 37], namely the construction of a quantum Hamiltonian and its associated interaction potential from classical statistical models of empirical data. Applied to particular tonal attraction data of music psychology, the famous Krumhansl and Kessler [29] profiles, that are fitted by Gaussian mixture models, we were able to derive a double-well perturbation potential [6, 22] for an anharmonic quantum oscillator [36, 54, 11].

Solving the stationary Schrödinger equation by a finite-dimensional perturbation approximation [50], we have rendered the original KK profile of static tonal attraction as the stationary ground state of the reconstructed schematic Hamiltonian. Additionally, the solutions of the time-dependent Schrödinger equation allow the calculation of first-order Markov chain transition probabilities [21, 53] as a model for dynamic tonal attraction.

Thus, our novel approach allows the principal unification of static and dynamic tonal attraction in one underlying configuration space, namely the fifth-width line as a one-dimensional projection of two-dimensional Regener [46] space. This is in contrast to previous tonal attraction models, distinguishing between the fifth-width line for static attraction and the orthogonal scale-degree line for melodic dynamic attraction as distinct configuration spaces [9, 13]. Future work will reveal possible connections between these different representations, e.g. about the significance of fundament progressions in Regener [46] space, or the separation of characteristic musical time scales for certain types of transitions though a systematic study of the continuous family of Markov-models (c.f. Fig. 9).

A practical application of the investigations of our study has recently been presented by González-Fernández et al. [18]. In our quantum-theoretic interpretation of the concept of heterophony we think of the ensemble, i.e. a family of voices—or a family of tracks in a sequencer—, as a configuration space, serving as the domain of definition for wave functions that change over time. In a more generalized perspective, the unity of a melody can also be the unity of some musical process with other parameters being the source of variation. The premise of this approach is that quantum wave functions exemplify a unity thanks to the underlying schematic Hamilton operator. An experimental environment *QuTE* has been implemented in Max/MSP [18].

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