A Study of Variation in Temporal Structure of Sonata Form

Luka Marohnić
Zagreb University of Applied Sciences
luka.marohnic@tvz.hr

Abstract: *In this paper it is shown that the concept of the Eighteenth-Century Sonata form, under certain conditions, implies exact constraints to its temporal structure, which are essential to keep its inner proportions balanced. The plastic number of Hans van der Laan appears to be closely related to the concordance of lengths of the vital parts of a sonata-form movement of type 3 on Hepokoski and Darcy scale. Furthermore, a probabilistic model of basic variation in the structure of such movement is developed from scratch and empirically justified by analyzing instrumental works of Wolfgang Amadeus Mozart.*

Keywords: *Sonata form. Perception. Plastic number. Morphic number. Lognormal distribution*

I. Introduction

Sonata-form is a simple, yet very potent concept. Besides hundreds of musical pieces written in the last couple of centuries, there also exist dozens of serious books and papers about the subject. We show that it can still be a subject for original research.

In this paper we try to find an exact formulation of certain restrictions[1] inherent to the temporal structure of the Eighteenth-Century Sonata form. More precisely, we ask ourselves what restrictions are necessarily imposed to the ratios between lengths of distinct parts of a sonata-form movement, which we call *inner proportions*. Knowing these restrictions, one would be able to answer questions of type "how long should this part be when compared to the one that follows it?", for example. It would also be possible to explain what exactly means that inner proportions of a sonata-form movement are balanced.

To find the restrictions mentioned above we need to apply a mathematical treatment, presented in Section II, which is similar to the methodology used by Hans van der Laan to develop his unique theory of architectonic space.

Section III is devoted to Sonata form. It begins with a brief introduction to its structure, followed by formulating a set of necessary restrictions applying to it, using the concept of ground ratio formulated in Section II. As we show next, these restrictions imply that the plastic number of Hans van der Laan is tightly related to the temporal organization of a sonata-form movement. These theoretical findings, formulated as a probabilistic model, are justified by performing a simple empirical study based on the set of sonata-form movements from the instrumental opus of W. A. Mozart. Finally, an example of using the plastic number in structural analysis of a sonata-form movement is given.

[1] For an example of such restriction see [11, p. 280].

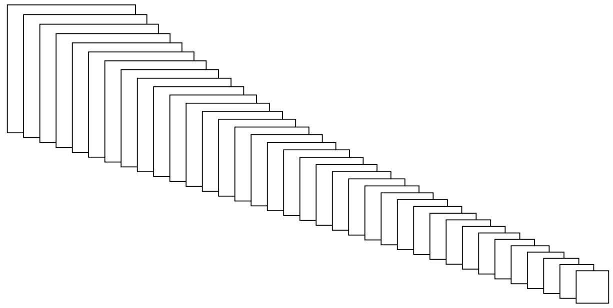

Figure 1: *36 squares forming a geometric sequence*

II. Perception of Spatial Size

Size is one of the fundamental properties of spatial objects. Human sensory system identifies size of an object with one of the available one-dimensional quantities related to that object, namely width, length and breadth. As those are easily compared with one another, the sensory system instinctively assigns the largest of these quantities to represent its size.

Therefore, we may define size of an object to be equal to the length of the longest edge of the smallest cuboid containing it. Size $s > 0$ should be visualized as a thin bar (stick) of length s. Hence we will use the term *length* when referring to size in the rest of this text.

In this section we study how the sensory system interprets relations between different sizes. The goal is to develop a mathematical treatment which is necessary for making assumptions and drawing conclusions later in this text.

i. Automatic Classification of Lengths

When two lengths are exposed simultaneously, the sensory system in our brain automatically attempts to relate them and to generate a valuable information for the conscious sphere of mind. Basically, it decides whether one of the lengths is significantly longer than the other. If that is not the case, they are considered "equal" in a sense of both being (possibly noticeably different) elements of the same class (level) of size. Such a class is called *type of size* [16, p. 55].

For any length ℓ_0 the interval L, consisting of all lengths $\ell \geq \ell_0$ which are "equal" to ℓ_0, represents one type of size. The length $\ell_1 = \sup L$ is the smallest length clearly different than ℓ_0. Dom Hans van der Laan (1904–1991), a Dutch architect, devised a simple and easily reproducible experiment [16, p. 49] to illustrate the concept of type of size. He prepared 36 cardboard squares of different sizes such that the sides of every two consecutive squares differ in length by 4% and thus forming a geometric sequence[2] (see Figure 1). Van der Laan would randomly scatter the squares on the table and ask someone to take out the group of the largest ones. He claimed that it would contain exactly seven squares every time. This action could be repeated until only the smallest square remains, thus dividing the squares into five consecutive groups, each containing seven members and representing one type of size. The smallest square represents an unit; its side is, by design, equal to the difference between sides of the largest members of the first two groups (see Figure 2). The largest members of five groups together with the unit represent six consecutive members of a geometric sequence with quotient $1.04^7 \approx 1.316$.

[2]The relative difference of 4% was obtained as the result of an auxiliary experiment in which a 50 cm long strip of paper needed to be cut in two halves. The experiment was conceived to measure the precision of eye judgement.

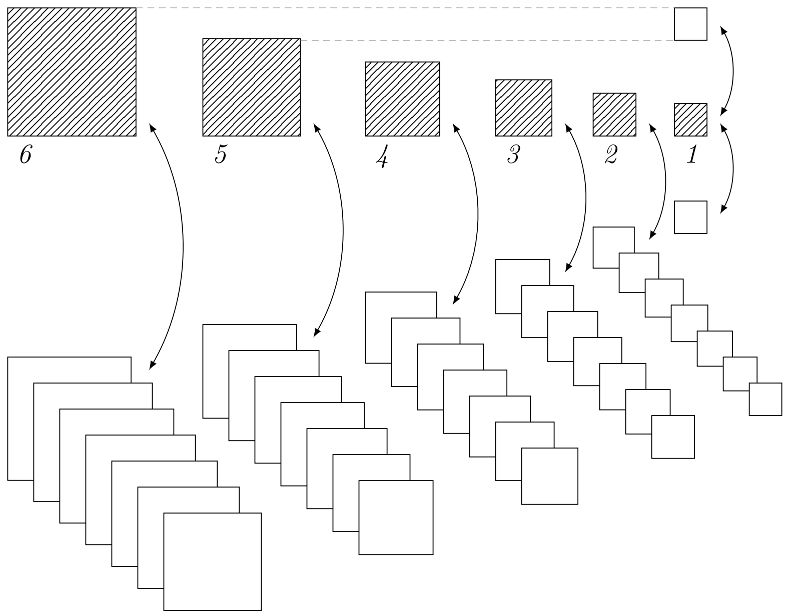

Figure 2: *36 squares sorted by size in six groups*

ii. Ground Ratio

Relating lengths to each other is the only way for the sensory system to "measure" reality. Namely, it does not possess an intrinsic reference value, i.e. an unit, so it cannot do any measuring on its own[3]. However, when two lengths are perceived, the difference between them becomes the source of meaningful information. This is a consequence of Weber–Fechner law [2, p. 83] which states that the sensation s corresponding to a physical stimulus of intensity I is proportional to the logarithm of I [3, p. 90]:

$$s = k \ln I.$$

Here $k > 0$ is a constant, called the Weber fraction [2, p. 83], which is a property of the type of stimulus. Now, the difference d between two sensations s_1 and s_2 of stimuli with intensities I_1 and I_2 is dimensionless, since the unit in the stimulus intensity domain gets cancelled:

$$\delta = |s_2 - s_1| = |k \ln I_2 - k \ln I_1| = k \left| \ln \frac{I_2}{I_1} \right|. \tag{1}$$

In particular, given a positive number r, the relation between any two lengths with ratio equal to r will always appear to be the same since the perceived difference $\delta = k \,|\ln r|$ is the same in each case. Furthermore, there exists a threshold $\delta_0 > 0$ such that two lengths belong to the same type of size if and only if their perceived difference does not exceed δ_0, i.e. the ratio of the longer length to the shorter does not exceed

$$\lambda = \mathrm{e}^{\delta_0 / k},$$

as follows from (1). The constant λ will be called the *ground ratio* in the rest of this text. Clearly, $\lambda > 1$ because δ_0 and k are both positive.

[3]Of course, we are all able to remember some concrete quantities used in everyday life, but resorting to these notions of physical units represents an act of the conscious sphere of mind, the influence of which we tend to ignore in this text.

Definition II.1. Lengths ℓ_1 and ℓ_2 such that $\ell_1 \leq \ell_2$ are *effectively equal* if $\ell_2/\ell_1 \leq \lambda$. When $\ell_2/\ell_1 \geq \lambda$ holds, ℓ_2 is said to be *significantly longer* than ℓ_1.

Remark II.2. In the case when $\ell_2 = \lambda\, \ell_1$, length ℓ_2 is both significantly longer than and effectively equal to ℓ_1 in the sense of Definition II.1. It is the legitimate case characterized by a specific type of balance which emphasizes the clarity of relation between lengths ℓ_1 and ℓ_2.

Given a base length m_0, called the *unit*, the elements of geometric sequence

$$(m_n)_{n \geq 0}, \quad \text{where } m_n = m_0\, \lambda^n \tag{2}$$

are called *measurements*. Every two consecutive measurements delimit a type of size.

iii. Margin

For any fixed length ℓ there is an unique type of size $[\ell_1, \ell_2]$ such that ℓ is perceived as its center. Therefore, ℓ should seem equally distant from ℓ_1 and ℓ_2. The distance between ℓ and ℓ_1 corresponds to the difference $\ell - \ell_1$; but, as the sensory system only interprets the relations between perceived quantities and not the quantities themselves, $\ell - \ell_1$ must be perceived relative to some available reference length. In present case ℓ is the only such length. Analogously, the distance between ℓ and ℓ_2, i.e. the difference $\ell_2 - \ell$, is perceived relative to ℓ, so we have

$$\frac{\ell - \ell_1}{\ell} = \frac{\ell_2 - \ell}{\ell} \implies 1 - \frac{\ell_1}{\ell} = \frac{\ell_2}{\ell} - 1 \implies 2 = \frac{\ell_1 + \ell_2}{\ell} \implies \ell = \frac{\ell_1 + \ell_2}{2}.$$

Hence ℓ is the arithmetic mean [6, p. 4.1] of ℓ_1 and ℓ_2. Now there exists $\Delta\ell > 0$ such that $\ell_1 = \ell - \Delta\ell$ and $\ell_2 = l + \Delta\ell$. Since ℓ_1 and ℓ_2 delimit a type of size, it follows

$$\frac{\ell + \Delta\ell}{\ell - \Delta\ell} = \lambda.$$

Rewriting the above equation, one obtains

$$\frac{\Delta\ell}{\ell} = \frac{\lambda - 1}{\lambda + 1}. \tag{3}$$

Displacement $\Delta\ell$, which obviously depends solely on ℓ, is called the *margin* of ℓ [16, p. 55]. All lengths obtained by changing the given length for values less than or equal to its margin belong to the same type of size and are *practically equal* to ℓ. In other words, the difference between such length and ℓ is *negligible* to ℓ.

The margin of ℓ represents the smallest length which can be related to ℓ. An interval $[\Delta\ell, \ell]$ represents one *order of size* [16, p. 55]. Furthermore, the interval $[m, M]$, where m is equal to the margin of ℓ and ℓ is equal to the margin of M, is called the *scope* of ℓ. It consists of two consecutive orders of size.

The following result characterizes effective equality in terms of negligibility.

Theorem II.3. *Two lengths are effectively equal if and only if their difference is negligible compared to their sum.*

Proof. Let ℓ_1 and ℓ_2 be two arbitrary lengths. As the statement is obviously valid for $\ell_1 = \ell_2$, we can assume that $\ell_2 > \ell_1$ without a loss of generality. First, we assume that $\ell_2 - \ell_1$ is negligible to $\ell_1 + \ell_2$, i.e. that the former does not exceed the margin of the latter:

$$\ell_2 - \ell_1 \leq \Delta(\ell_1 + \ell_2). \tag{4}$$

Dividing the inequality (4) by $\ell_1 + \ell_2$ and using (3), we obtain

$$\frac{\ell_2 - \ell_1}{\ell_1 + \ell_2} \leq \frac{\Delta(\ell_1 + \ell_2)}{\ell_1 + \ell_2} = \frac{\lambda - 1}{\lambda + 1},$$

which readily simplifies down to $\frac{\ell_2}{\ell_1} \leq \lambda$, meaning that ℓ_1 and ℓ_2 are effectively equal.

On the other hand, assuming that (4) is false and reasoning analogously, we conclude that $\frac{\ell_2}{\ell_1} > \lambda$, which means that ℓ_1 and ℓ_2 cannot belong to the same type of size. That completes the proof. □

iv. Derived Measurements

Using (3), for any two consecutive measurements ℓ_1 and $\ell_2 = \lambda\, \ell_1$ we obtain

$$\ell_2 - \Delta\ell_2 = \ell_2 \left(1 - \frac{\Delta\ell_2}{\ell_2}\right) = \lambda\, \ell_1 \left(1 - \frac{\lambda - 1}{\lambda + 1}\right) = \ell_1\, \frac{2\,\lambda}{\lambda + 1} =$$
$$= \ell_1\, \frac{(\lambda + 1) + (\lambda - 1)}{\lambda + 1} = \ell_1 \left(1 + \frac{\lambda - 1}{\lambda + 1}\right) = \ell_1 \left(1 + \frac{\Delta\ell_1}{\ell_1}\right) = \ell_1 + \Delta\ell_1.$$

Hence $\ell_2' = \ell_2 - \Delta\ell_2$ is the only length which is practically equal to both ℓ_1 and ℓ_2, so it may be readily identified with any of the two. It therefore represents the perceived point of balance between the given measurements. Moreover, ℓ_2' coincides with the harmonic mean [6, p. 4.18] of ℓ_1 and ℓ_2 since

$$\ell_2 - \Delta\ell_2 = \ell_2 \left(1 - \frac{\lambda - 1}{\lambda + 1}\right) = \frac{2\,\ell_2}{\lambda + 1} = \frac{2\,\ell_1\,\ell_2}{\ell_1\,(\lambda + 1)} = \frac{2\,\ell_1\,\ell_2}{\ell_1 + \lambda\,\ell_1} = \frac{2\,\ell_1\,\ell_2}{\ell_1 + \ell_2} = \frac{2}{\frac{1}{\ell_1} + \frac{1}{\ell_2}}.$$

Generally, for any two lengths ℓ_1 and ℓ_2 such that ℓ_2 is significantly longer than ℓ_1 there is a unique length ℓ which is perceived as the natural point of balance between ℓ_1 and ℓ_2, i.e. as being equally distant from both lengths. These distances are equal to differences $\ell - \ell_1$ and $\ell_2 - \ell$, which must be taken relative to ℓ_1 and ℓ_2, respectively, as these are the only available reference lengths. Therefore,

$$\frac{\ell - \ell_1}{\ell_1} = \frac{\ell_2 - \ell}{\ell_2} \implies \frac{\ell}{\ell_1} - 1 = 1 - \frac{\ell}{\ell_2} \implies \frac{\ell\,(\ell_1 + \ell_2)}{\ell_1\,\ell_2} = 2 \implies \ell = \frac{2\,\ell_1\,\ell_2}{\ell_1 + \ell_2} = \frac{2}{\frac{1}{\ell_1} + \frac{1}{\ell_2}},$$

i.e. ℓ coincides with the harmonic mean of ℓ_1 and ℓ_2.

The sequence of measurements (2) is naturally interpolated with another geometric sequence with ratio λ, denoted by $(m_n')_{n\geq 1}$, where $m_n' = m_n - \Delta m_n$ is called the *derived measurement* corresponding to m_n. The two sequences, taken together, embody all three Pythagorean means; given $n \geq 1$,

- m_n is the geometric mean [6, p. 4.15] of m_{n-1} and m_{n+1},
- m_{n+1}' is the geometric mean of m_n' and m_{n+2}',
- m_n is the arithmetic mean of m_n' and m_{n+1}',
- m_n' is the harmonic mean of m_{n-1} and m_n.

v. Plastic Number

As the ground ratio is a fixed numeric constant, we ask ourselves what its exact value is. The universality of the concept suggests that its origin is environmental. Indeed, a length can be perceived only in the context of an object whose size it represents; but every perceivable object exists within a certain realm, or environment, which itself has certain properties. While the value of ground ratio may be approximated by conducting the experiment presented in Section i, as van der Laan has demonstrated, its exact value can only be deduced from these properties.

The deductive approach was used by van der Laan in his study of the relations between sizes of physical (spatial) objects [16]. He derived the value of ground ratio by using the fact that the space containing these objects is three-dimensional, showing that λ is equal to the single real root ψ of the trinomial $x^3 - x - 1$. Its exact value is

$$\psi = \frac{\sqrt[3]{108 + 12\sqrt{69}} + \sqrt[3]{108 - 12\sqrt{69}}}{6} \approx 1.324718$$

which is called the *plastic number*. It generates the system of measurements shown in Figure 3, which consists of two consecutive orders of size. The larger order of size contains eight measurements $\mathrm{I}_1, \mathrm{I}_2, \ldots, \mathrm{I}_8$ (blue bars), each one representing a type of size. The smallest measurement I_1, called the *unit*, is equal to the margin of the largest measurement I_8. The other, smaller order of size contains eight measurements $\mathrm{II}_1, \mathrm{II}_2, \ldots, \mathrm{II}_8$ (red bars), where II_k is equal to the margin of I_k: thus $\mathrm{I}_1 = \mathrm{II}_8$. The whole system represents the scope of the unit I_1.

Van der Laan's system of measurements can be used to approximate the Weber fraction corresponding to the visual length stimulus. Since II_1 is the smallest length non-negligible to I_1, which is the margin of I_8, it follows that II_1 approximates the just-noticeable-difference (JND) threshold [2, p. 37–38] associated with I_8. The corresponding Weber fraction is equal to the ratio of II_1 to I_8, i.e. $\psi^{-14} \approx 0.0195$. This is coherent with values, reported by Weber himself, "[...] of about 0.01 or 0.02. The majority of later studies yield similar values [...]" [15, p. 344].

vi. Morphic Numbers

A real number $x > 1$ such that $x - 1 = x^{-m}$ and $x + 1 = x^n$ for some positive integers m and n is called the *morphic number*. A geometric sequence based on morphic number has certain additive properties (their definitions depend on values m and n), i.e. some members of the sequence may be computed by adding/subtracting other members. The above definition may be generally interpreted as follows. Given an unit quantity $x_0 = 1$, assume that x is the smallest quantity significantly greater than x_0. Then x is equal to the corresponding ground ratio. If x is a morphic number, then the difference $x - x_0$ between two quantities, as well as their sum $x + x_0$, belongs to the same system of measurements as x does. This allows the entire system to be reconstructed from any n consecutive measurements using only addition and subtraction.

It can be shown that only two morphic numbers exist [1], namely the plastic number (for $m = 4$ and $n = 3$) and the golden ratio

$$\varphi = \frac{\sqrt{5} - 1}{2} \approx 1.618033$$

(for $m = 1$ and $n = 2$).

Additive properties of the system of measurements based on the plastic number are illustrated in Figure 3. For example, in a sequence of four consecutive measurements the sum of the smallest two is equal to the largest; similarly, in a sequence of six consecutive measurements the difference between the largest two is equal to the smallest. There is also an additive rule which combines authentic and derived measurements: in a sequence of four consecutive measurements, the first one is twice smaller than the derived measurement corresponding to the last.

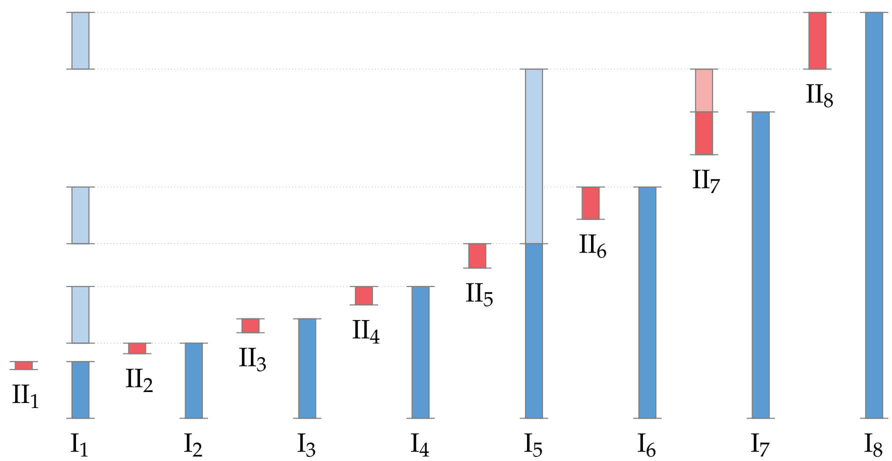

Figure 3: *system of measurements based on the plastic number*

The analogous system based on the golden ratio has similar (albeit not as rich) additive properties. However, the golden ratio is not related to our perception of three-dimensional reality [16, p. 75]. Also, in [18, p. 138] we find:

> For van der Laan, the Golden Section in its application is nothing more than an artificial concept to order matter, as abstract as the discrete quantity of mathematical numbers. Because of its abstract nature, it proves inadequate when brought into relation to concrete and singular reality, since it remains on the level of analysis.

According to van der Laan, the plastic number ratio directly grew from discernment (the human ability to differentiate sizes) and from the necessity of relations [18, p. 138]. As such, it would be an improvement over the golden ratio [19, p. 1].

Nevertheless, many authors still consider φ to be an important proportion in art, architecture and music; see, for example, [11], [12] and [4].

III. Sonata Form

Sonata form is the central musical concept of the Classical period and one of the most important ideas in the history of Western music. During the Classical period it was conceived in two parts: the Exposition (A), in which certain themes are introduced, and the Development and Recapitulation (B) in which the themes are developed and revisited [11], as shown in Figure 4.

Hepokoski and Darcy list five types of sonata-form movements [7]. Type 1 features no Development section and is often used in slow and more peaceful movements. Type 2 is characterized by eliding the end of the Development with the beginning of the Recapitulation, making it difficult to determine a clear bound between them. Types 4 and 5 refer to sonata rondos and concerto sonata movements, respectively. In this paper we focus on type 3, which features full Exposition, Development, and Recapitulation. From now on, the term "sonata-form movement" will refer to that type.

The Exposition is divided in two parts establishing the two different but well-blending tonalities, called the *primary key* and the *secondary key* (the latter usually being either the dominant key for a major primary key or the parallel key for a minor primary key). Often these two tonalities are expressed by mutually contrasting groups of thematic material called the *first subject* and the

Figure 4: *Sonata form*

second subject. We will assume that the two subjects are distinct and disjoint; however, it is not always the case, as they may sometimes blend into each other partially.

The end of the Exposition is marked with a cadence in secondary key, called the *essential expositional closure* (EEC) [7]. Sometimes, however, a short section may follow EEC before the onset of the Development. In this text we consider such a passage to be part of the Exposition.

The end of the Development is marked with a cadence in primary key, called the *essential structural closure* (ESC) [7]. It may be followed by a short passage before the onset of the Recapitulation. We consider such a passage to be part of the Development.

The length of each of three major parts of a sonata-form movement, as well as the length of each subject, is always well-defined with respect to the above conventions. Now we can study how the lengths of different parts are, in general, related to each other. We may use tools developed in the previous section to do so, as the length of each part can be numerically expressed as a number of measures by doing simple counting.

Due to the fact that we are able to memorize, it is not difficult to imagine perceiving a relation between two distinct chunks of music played in succession. However, we may not assume the value of ground ratio *a priori*; we have to obtain it by studying the general properties of Sonata form which organizes the time flow.

From now on, we use the symbols a, b, c, s_1 and s_2 to denote the lengths of parts A and B, the Development and the two subjects, respectively.

i. Inherent Restrictions

The Sonata form itself imposes certain restrictions [11, p. 280]. Hence some of its key aspects may be expressed in form of a set of four structural "rules" (we will call them *propositions*) based on the concept of ground ratio.

The first restriction is related to the shape of Exposition which is determined by the proportion of lengths of two subject groups. The main premise is that the two necessary belong to different keys which are considered equally important in the course of Exposition. Therefore, if the second group was much longer than the first group it would shadow out the importance of the primary key. On the other hand, if it was much smaller, it would be shadowed by the primary key. As the two subject groups have equally demanding tasks of establishing the respective tonalities, effectively equal amounts of time for them to do so should be granted, implying the following statement.

Proposition III.1. *The lengths of two subject groups in the Exposition are effectively equal.*

The Development represents a passage which "renders the established tonal tension[4] more fluid an complex [, . . .] typically [initiating] more active, restless, or frequent tonal shifts—a sense of comparative tonal instability. Here one gets the impression of a series of changing, coloristic

[4]Key displacement at the end of the Exposition results in an unresolved tension.

moods or tonal adventures [. . .] with shadowed, melancholy, or anxious connotations" [7, ch. 2, p. 7]. As the general characteristic of the Development is its tonal instability, it should not dominate over the parts of well-established tonality. Hence the Development should not be significantly longer than both subjects. It should not be significantly shorter than both of them either, since there would be no time for it to build tension, most often by developing thematic material or modulating through distant keys, and to include a retransition before the Recapitulation; it would seem all too tight and undeveloped to the listener who expects a meaningful contrast to the Exposition. This implies the second important inherent restriction of Sonata form, stated below.

Proposition III.2. *The length of the Development is effectively equal to the length of at least one subject group.*

Discussing Mozart's style in context of Sonata form, D. F. Tovey says that "the return to the tonic [the beginning of Recapitulation] always has the effect of being accurately timed" [17, p. 215]. Onset of the Recapitulation is indeed the crucial moment in the course of a sonata-form movement; it is better to say that it is the moment in which the Development ends, i.e. in which its length becomes definitive. The listener's mind therefore has the data required to guess of how long the entire movement should be, which in turn makes possible to determine whether the Development has the "right" length, i.e. is "accurately timed". Namely, as a reflection of the Exposition is expected to follow, the mind naturally assumes that length of the Recapitulation equals that of the Exposition. Hence $b = a + c$, i.e. $c = b - a$. Since Sonata form has a distinct ternary shape, we assume that c is not negligible to the expected length $a + b$ of the whole movement. Now Theorem II.3 implies that b is (expected to be) significantly longer than a. Therefore, to prevent the Development of being too short, we acknowledge the following restriction.

Proposition III.3. *The Exposition with Development is significantly longer than the former.*

The Development can be realized in a myriad of ways once the subject material is presented in the Exposition. A composer should have the highest possible degree of freedom to express his or her ideas within the central section. In particular, it should be possible for the length of Development to vary considerably when compared to the length of Exposition (enough to discourage an educated listener from trying to guess it). In order to measure the variation, we introduce a property called the *central magnitude*:

$$\mu = \frac{\max c}{\min c},$$

where $\max c$ and $\min c$ are equal to the longest and shortest Development possible (for an arbitrary but fixed a) such that conditions stated in propositions III.2 and III.3 hold. The central magnitude measures the "amount of variation" in length of the Development. Its value is an answer to the question "how many times is the longest possible development section longer than the shortest one?". To ensure the maximal amount of variation, we state the following condition, which is naturally imposed as inherent to Sonata form.

Proposition III.4. *The central magnitude has to be as high as possible.*

ii. Parametrization of Shape

The ratio c/a, i.e. the relative length of Development with respect to the length of Exposition, is crucial for the shape of a sonata-form movement. Therefore it will be called the *shape parameter* in the subsequent text. In this section we use propositions III.1, III.2, III.3 and III.4 to compute the exact value of ground ratio and subsequently determine the range of the shape parameter. Without a loss of generality we may assume that $s_2 \geq s_1$.

Lemma III.5. *For $s_2 \geq s_1$ the following inequalities hold:*

$$\frac{s_2}{s_1} \leq \lambda, \quad \frac{s_1}{c} \leq \lambda \quad \textit{and} \quad \frac{c}{s_2} \leq \lambda. \tag{5}$$

Proof. The first inequality readily follows from Proposition III.1. The statement of Proposition III.2 is equivalent to

$$\frac{s_1}{\lambda} \leq c \leq \lambda\, s_1 \quad \text{or} \quad \frac{s_2}{\lambda} \leq c \leq \lambda\, s_2. \tag{6}$$

But, since $\frac{s_2}{s_1} \leq \lambda$ and $\lambda > 1$, we obtain

$$\frac{\frac{s_2}{\lambda}}{\lambda\, s_1} = \frac{1}{\lambda^2} \cdot \frac{s_2}{s_1} \leq \frac{1}{\lambda^2} \cdot \lambda = \frac{1}{\lambda} < 1,$$

implying $\frac{s_2}{\lambda} < \lambda\, s_1$. Because of that, the statement (6) is equivalent to

$$\frac{s_1}{\lambda} \leq c \leq \lambda\, s_2,$$

implying $\frac{s_1}{c} \leq \lambda$ and $\frac{c}{s_2} \leq \lambda$. That completes the proof. □

Theorem III.6. *The ground ratio for Sonata form is equal to the plastic number ψ.*

Proof. Multiplying the first two inequalities in (5) yields $\frac{s_2}{c} \leq \lambda^2$. Hence from Lemma III.5 follows

$$\frac{a}{c} = \frac{s_1}{c} + \frac{s_2}{c} \leq \lambda + \lambda^2. \tag{7}$$

On the other hand, multiplying the first and the third inequality in (5) yields $\frac{c}{s_1} \leq \lambda^2$. Therefore, Lemma III.5 also implies

$$\frac{a}{c} = \frac{s_1}{c} + \frac{s_2}{c} \geq \frac{1}{\lambda} + \frac{1}{\lambda^2} = \frac{\lambda + 1}{\lambda^2}. \tag{8}$$

Inverting the inequalities (7) and (8) yields

$$\frac{1}{\lambda + \lambda^2} \leq \frac{c}{a} \leq \frac{\lambda^2}{\lambda + 1}. \tag{9}$$

Furthermore, Proposition III.3 implies

$$\frac{c}{a} = \frac{a + c - a}{a} = \frac{a + c}{a} - 1 \geq \lambda - 1. \tag{10}$$

Let $m_1(\lambda) = \frac{1}{\lambda + \lambda^2}$, $m_2(\lambda) = \lambda - 1$ and $M(\lambda) = \frac{\lambda^2}{\lambda + 1}$. It is obvious that, for positive λ, the function m_1 is strictly decreasing and the function m_2 is strictly increasing. As both m_1 and m_2 are continuous, the fact that $m_2(1) = 0 < \frac{1}{2} = m_1(1)$ implies that there exists an unique $\lambda_0 > 1$ such that $m_1(\lambda_0) = m_2(\lambda_0)$. Using the equality $1 + \psi = \psi^3$, which follows from the fact that the plastic number is the root of polynomial $x^3 - x - 1$, we readily check that $\lambda_0 = \psi$. Therefore, from (9) and (10) follows

$$m(\lambda) \leq \frac{c}{a} \leq M(\lambda), \quad \text{where } m(\lambda) = \begin{cases} m_1(\lambda), & 1 < \lambda < \psi, \\ m_2(\lambda), & \lambda \geq \psi. \end{cases} \tag{11}$$

Now we can compute the central magnitude:

$$\mu = \frac{\max c}{\min c} = \frac{\frac{1}{a} \max c}{\frac{1}{a} \min c} = \frac{\max \frac{c}{a}}{\min \frac{c}{a}} = \frac{M(\lambda)}{m(\lambda)}.$$

Using (11), it follows

$$\mu = \begin{cases} \frac{M(\lambda)}{m_1(\lambda)} = \lambda^3, & 1 < \lambda < \psi, \\ \frac{M(\lambda)}{m_2(\lambda)} = \frac{\lambda^2}{\lambda^2 - 1}, & \lambda \geq \psi. \end{cases} \tag{12}$$

According to Proposition III.4, μ must have the highest possible value. Differentiating μ with respect to λ using (12) yields $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = 3\lambda^2 > 0$ for $1 < \lambda < \psi$ and $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = -\frac{2\lambda}{(\lambda^2-1)^2} < 0$ for $\lambda > \psi$. Therefore μ is strictly increasing for $1 < \lambda < \psi$ and strictly decreasing for $\lambda > \psi$. It follows that μ, being a continuous function of variable λ on $(1, +\infty)$, attains the maximum value for $\lambda = \psi$. □

We have shown that the essential features of Sonata form imply that the ground ratio must be equal to the plastic number. Therefore, inner proportions of the temporal structure of Sonata form are organized with respect to the same system (imposed by the nature itself) used to organize proportions in a spatial, architectonic structure.

Corollary III.7. *The central magnitude is equal to* ψ^3.

Corollary III.8. *The shape parameter may vary between* ψ^{-4} *and* ψ^{-1}.

Corollary III.9. *The Development is significantly shorter than the Exposition*[5].

The bounds established by Corollary III.8 should not be interpreted as literary as our theoretical deduction suggests. Instead of forcing the shape parameter between some fixed bounds, we should ask ourselves how its value is distributed in probabilistic sense. Hence let us denote $X = c/a$, $X > 0$. Now we use the interval $\mathcal{T} = [\psi^{-4}, \psi^{-1}]$ to deduce the probability distribution of the continuous random variable X.

It should be noted that X may attain any positive value in our model. However, $X \in \mathcal{T}$ has to be much more probable than $X \notin \mathcal{T}$. Therefore it is reasonable to assume that the probability distribution function for X is bell-shaped, peaking somewhere near the center of $\mathcal{T}$, i.e. the average of its endpoints $X_l = \psi^{-4}$ and $X_u = \psi^{-1}$. Since X is dimensionless value represented as a ratio, the appropriate averages are geometric and harmonic mean [6, p. 4.17–4.18]. Indeed, the geometric mean $X_g = \sqrt{X_l\, X_u}$ satisfies

$$\frac{X_g}{X_l} = \frac{X_u}{X_g}.$$

Hence the relation between lengths $a\,X_g$ and $a\,X_l$ is the same as the relation between $a\,X_u$ and $a\,X_g$, where $a > 0$ is arbitrary. In other words, ranges $[X_l, X_g]$ and $[X_g, X_u]$ are perceived as being equally wide, so the possibilities $X_l \leq X \leq X_g$ and $X_g \leq X \leq X_u$ should be equally probable. Now, as $\ln X_g$ is the arithmetic mean of $\ln X_l$ and $\ln X_u$, the probability density curve of $\ln X$ appears to be symmetric, so we simply assume that $\ln X$ is normally distributed with mean $\ln X_g$. Therefore the distribution of X is lognormal [5, p. 1–2.]:

$$X \sim \ln \mathcal{N}(\mu, \sigma). \tag{13}$$

Now it follows $X_g = \mathrm{Med}[X]$, which yields [5, p. 9]

$$\mathrm{e}^{\mu} = X_g \implies \mu = \ln X_g. \tag{14}$$

In Section iv we showed that the naturally perceived middle between two fixed lengths coincides with their harmonic mean. In particular, this is valid for (relative) lengths X_l and X_u,

[5] D. F. Tovey also points this out by saying that the Development in a sonata-form movement of Mozart is generally "short" compared to the Exposition [17, p. 215].

implying that their harmonic mean $X_h = \frac{2\,X_l\,X_u}{X_l+X_u}$ is the natural center of the interval $\mathcal{T}$ and a measure of central tendency of X. In statistics, the main measures of central tendency are the mean (expectation), median and mode [6, p. 4.1]; since it was already stated that $\mathrm{Med}[X] = X_g$, it remains to identify X_h with the mode $\mathrm{Mode}[X]$ or the mean $\mathrm{E}[X]$. But $\mathrm{Mode}[X] < \mathrm{Med}[X] < \mathrm{E}[X]$ holds for any lognormal distribution [5, p. 9] and $X_h < X_g$ [6, p. 4.20], hence we assume $X_h = \mathrm{Mode}[X]$. Because $\mathrm{Mode}[X] = \mathrm{e}^{\mu-\sigma^2}$ [5, p. 9], we have

$$X_h = \mathrm{e}^{\mu-\sigma^2} \implies \ln X_h = \mu - \sigma^2.$$

Using (14) it follows

$$\sigma = \sqrt{\ln \frac{X_g}{X_h}}.$$

As $X_g = \psi^{-5/2}$ and $X_h = \frac{2}{\psi+\psi^4}$, we have

$$\mu = \ln X_g = -\frac{5}{2}\ln\psi = -5\ln\sqrt{\psi}$$

and

$$\frac{X_g}{X_h} = \frac{1+\psi^3}{2\,\psi^{3/2}} = \frac{\psi^{-3/2}+\psi^{3/2}}{2} = \frac{1}{2}\left(\mathrm{e}^{\frac{3}{2}\ln\psi} + \mathrm{e}^{-\frac{3}{2}\ln\psi}\right) = \cosh\left(\frac{3}{2}\ln\psi\right) = \cosh(3\ln\sqrt{\psi}).$$

Letting $\omega = \ln\sqrt{\psi}$, we finally obtain

$$\mu = -5\,\omega \approx -0.7029989, \quad \sigma = \sqrt{\ln(\cosh(3\,\omega))} \approx 0.2940039. \tag{15}$$

Parameters in (15) define the probability distribution of X. Now, using the `plnorm` function from computer software R [13], we compute the probability $P(X_l \leq X \leq X_u) = 0.8486196$, which means that probability for $X \notin \mathcal{T}$ is practically equal to 15%.

Proposition III.10. *Let $\kappa = \frac{1}{2}\sqrt{1+\psi^{-3}}$. Then the expected value and the standard deviation of X are*

$$\mathrm{E}[X] = \frac{\kappa\sqrt{2}}{\psi^{7/4}} \approx 0.5169651 \quad \text{and} \quad \mathrm{SD}[X] = \frac{\kappa\,(\psi^{3/2}-1)}{\psi^{5/2}} \approx 0.1553341.$$

Proof. The expectation and the variance of X are [5, p. 9]

$$\mathrm{E}[X] = \mathrm{e}^{\mu+\frac{1}{2}\sigma^2} \quad \text{and} \quad \mathrm{Var}[X] = \mathrm{e}^{2\mu+\sigma^2}\left(\mathrm{e}^{\sigma^2}-1\right),$$

hence

$$\mathrm{SD}[X] = \sqrt{\mathrm{Var}[X]} = \mathrm{e}^{\mu+\frac{1}{2}\sigma^2}\sqrt{\mathrm{e}^{\sigma^2}-1}.$$

Using (15), we define

$$F_1 = \mathrm{e}^{\mu} = \psi^{-5/2} \quad \text{and} \quad F_2 = \mathrm{e}^{\sigma^2} = \cosh(3\,\omega) = \frac{1}{2}\left(\psi^{3/2}+\psi^{-3/2}\right).$$

Now we have

$$\mathrm{E}[X] = F_1\sqrt{F_2} \quad \text{and} \quad \mathrm{SD}[X] = F_1\sqrt{F_2^2 - F_2}.$$

The statement follows by rewriting the above equations using the equality $1+\psi = \psi^3$. □

iii. An Empirical Study

The probability distribution (13) applies to sonata-form movements of type 3 in general, defining the probability $P(p \leq X \leq q)$ for arbitrary $0 < p < q$. It would be meaningless trying to apply it to a single sonata-form movement. However, given a large number of such movements written by the same composer, a statistical analysis may be performed in order to compare the empirical distribution of X to the theoretical one.

In this section we use a set of sonata-form movements from the instrumental (solo, chamber and orchestral) opus of Wolfgang Amadeus Mozart, which represents a peak of elegance in Western classical music. There are several objective reasons to choose Mozart:

1. a large amount of unified data is at our disposal (see [9]) due to his immense productivity,

2. the quality of his work is uniform; there are no mediocre pieces in his opus,

3. his work is constantly praised for the perfectness of temporal proportions; several distinguished authors are quoted in [11, p. 276]:

> ...the genius of [W. A. Mozart] is manifested in form and balance. His music has been revered, among other things, for its "beautiful and symmetrical proportions" [34, p. 217]. In 1853, Henri Amiel opined that "the balance of the whole is perfect" [I, p. 54]. Hanns Dennerlein described Mozart's music as reflecting the "most exalted proportions," and the composer himself as having "an inborn sense for proportions" [quoted in 7, p. 1], a thought echoed by H. C. Robbins Landon [20, p. 268]. Eric Blom wrote that Mozart had "an infallible taste for saying exactly the right thing at the right time and at the right length" [5, p. 265].

The total of 188 sonata-form movements of type 3 can be extracted from the instrumental part of Mozart's opus as published in [9]; these are listed in Table 1, each of them denoted by the respective Köchel catalog number together with position of the movement in question (expressed in roman numerals) within the corresponding piece.

As noted before, the Exposition and the Development are always well defined in a sense that their lengths can be unambiguously determined. Thus the lengths a (of the Exposition) and c (of the Development) are given for every movement listed in Table 1.

Mathematical expectation of X with respect to the data set $\mathcal{D} = \{c_k/a_k : 1 \leq k \leq 188\}$ with a_k, c_k given in k-th row of Table 1, is

$$\overline{X} = \frac{1}{188} \sum_{k=1}^{188} \frac{c_k}{a_k} \approx 0.5145557.$$

This differs for less than 0.5% from the value $\mathrm{E}[X]$ given in Proposition III.10. In particular, $\mathrm{E}[X]$ represents a statistical measure of Mozart's choice of shape parameter. The important fact is that it was not necessary to include any information from Mozart's music to compute it.

To compare empirical probability distribution of data from $\mathcal{D}$ with the lognormal distribution with parameters given in (15) we apply the Kolmogorov-Smirnov test, available in R as `ks.test` [13]. The test gives p-value of 0.3882 which is significantly greater than the suggested rejection threshold 0.1 (or commonly used 0.05), indicating that there exist strong evidence to support the null hypothesis (that data from $\mathcal{D}$ is log-normally distributed). Next we use the function `fitdistr` available in R [13] to fit a lognormal distribution to data using the method of maximum-likelihood. We obtain the distribution $\ln \mathcal{N}(\mu_f, \sigma_f)$, where

$$\mu_f = -0.71041299 \quad \text{and} \quad \sigma_f = 0.29738620. \tag{16}$$

Table 1: *188 instrumental sonata-form movements composed by W. A. Mozart*

Köchel	*a*	*c*
Piano sonatas		
279 I	38	19
279 II	28	14
279 III	56	30
280 I	56	26
280 II	24	12
280 III	77	30
281 I	40	29
281 II	46	12
282 I	15	6
282 III	39	22
283 I	53	18
283 II	14	9
283 III	102	69
284 I	51	20
309 I	58	35
310 I	49	30
310 II	31	22
311 I	39	39
330 I	58	29
330 III	68	27
332 I	93	39
332 III	90	57
333 I	63	30
333 II	31	19
457 I	74	25
533 I	102	43
533 II	46	26
545 I	28	13
570 I	79	53
576 I	58	40
Piano four-hands		
381 I	30	21
381 III	70	27
358 I	45	11
521 I	84	52
Two pianos		
448 I	80	29
448 II	48	22
Sonatas for organ		
68	26	10
144	27	24
145	37	12
212	30	10
241	35	12
224	45	21
225	50	19
244	46	21
245	35	13
263	35	11
274	32	21
328	41	19
Piano trios		
254 I	81	52
254 II	14	9½
496 I	78	38
502 I	82	35
542 I	101	34
548 I	62	41
548 II	32	23
564 I	41	36
Piano quartets		
478 I	99	41
493 I	95	52
493 II	46	23
Flute quartets		
285 I	65	34
285a I	34	14
285b I	66	44

Köchel	*a*	*c*
370 I	63	34
Violin sonatas		
301 I	84	44
302 I	68	38
304 I	84	28
305 I	73	27
306 I	74	37
306 II	34	23
296 I	68	31
378 I	82	31
376 I	47	26
377 I	51	31
380 I	58	40
454 I	52	24
481 I	92	47
526 I	100	39
526 II	41	15
403 I	24	10
372 I	67	59
547 II	78	37
Bassoon and cello duo		
292 I	46	18
292 II	22	8
Basset horn trios		
439b I/I	51	20
439b III/I	37	20
439b IV/I	28	16
Quintets		
452 I	45	16
452 II	43	30
581 I	79	38
407 I	56	16
407 II	44	25
String Quintets		
174 I	86	34
174 IV	94	68
515 I	151	53
516 I	94	38
593 I	80	43
614 I	86	38
String quartets		
155 I	53	18
156 I	71	38
157 I	52	22
158 I	45	29
159 I	29	15
160 III	54	32
168 I	41	21
169 I	36	36
171 III	10	8
171 IV	71	20
172 I	52	19
172 IV	80	32
173 I	45	19
387 I	56	51
421 I	41	28
458 I	90	47
428 I	68	32
458 I	90	47
458 IV	133	65
464 I	87	74
464 IV	80	64
465 I	84	48
465 IV	136	62
499 I	98	43
575 I	77	39
589 I	71	59
590 I	74	37
590 IV	133	51

Köchel	*a*	*c*
String trio		
563 I	73	37
String duos		
423 I	48	33
424 I	70	43
424 II	80	43
Marches		
189	27	12
237	28	12
Divertimentos		
113 I	27	14
113 IV	50	20
251 I	40	27
213 I	30	10
188 I	14	8
188 II	24	12
240 I	42	22
240 IV	66	24
252 I	18	7
270 I	51	14
136 I	36	28
136 III	58	28
137 II	25	14
137 III	44	20
138 I	35	18
205 I	37	15
334 I	83	42
334 IV	33	16
Serenades		
239 I	25	17
525 I	55	20
388 I	94	35
361 I	76	48
250 I	91	46
203 I	54	35
203 VIII	109	44
185 I	77	22
Symphonies		
48 I	33	26
110 I	69	23
112 I	54	16
112 II	25	12
114 I	59	20
114 IV	73	28
124 I	36	36
128 I	53	31
129 I	46	21
130 I	62	21
199 I	58	28
183 I	82	34
200 I	67	23
201 I	76	30
201 II	38	14
201 IV	61	40
202 I	78	34
202 IV	79	47
204 I	67	21
425 I	103	40
425 II	36	29
425 IV	163	68
504 I	106	65
504 IV	151	64
543 I	117	41
543 IV	104	48
550 I	100	65
550 II	52	21
550 IV	124	82
551 I	120	68
551 IV	157	67

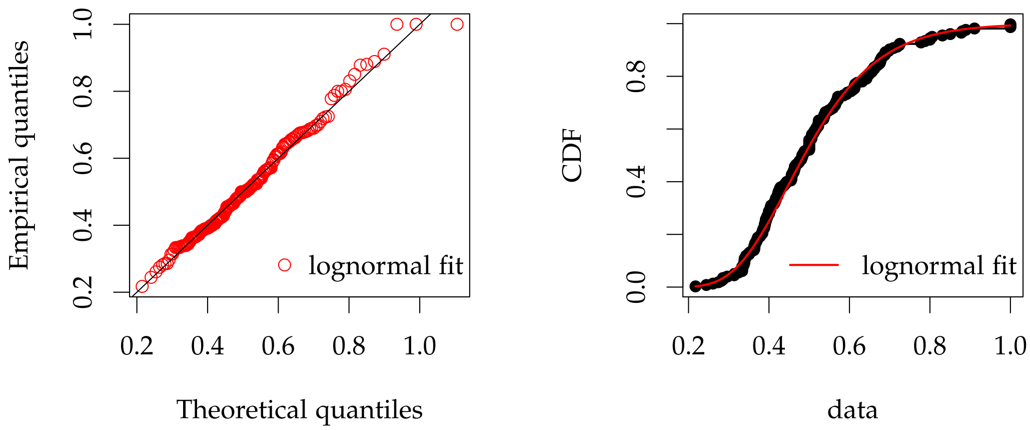

Figure 5: *empirical distribution of X compared to the fitted lognormal: Q–Q plot (left) and CDF plot (right)*

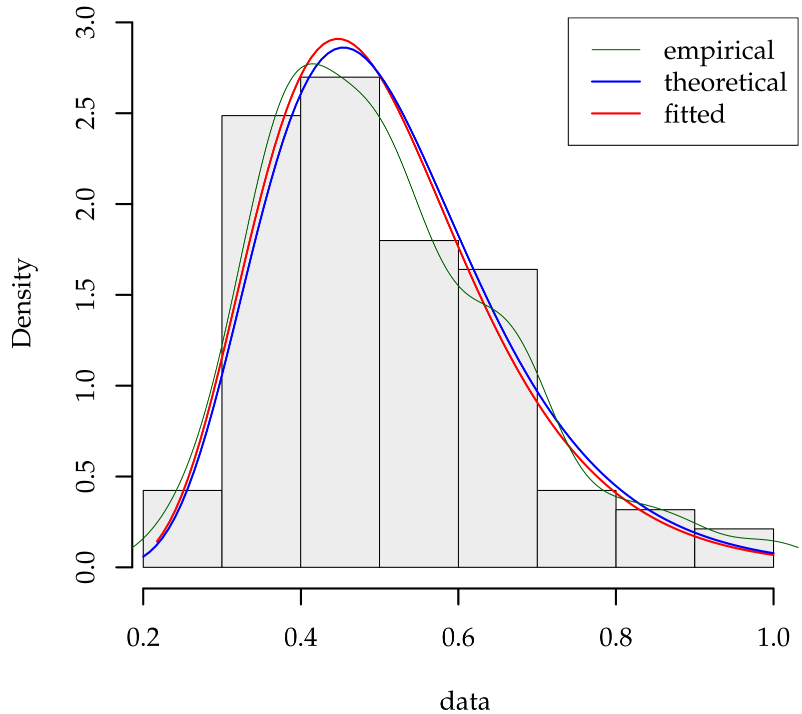

Figure 6: *probability distribution of X*

These values come very close to those given in (15), with relative errors of 1.05% and 1.15%, respectively. The Q–Q and CDF plots shown in Figure 5 present no reason to suspect that data from $\mathcal{D}$ is not log-normally distributed. Figure 6, which includes the histogram and probability density curve corresponding to the empirical data from $\mathcal{D}$, shows that fitted and theoretical distributions of X are practically coincidental.

The results of this simple study suggest that purely theoretical consequences of the inherent restrictions in Sonata form as presented in Section i indeed act as practical necessities, at least in Mozart's work. We can say that this observation justifies our theoretical deductions; since the probability distribution of X depends on two independent parameters, it would be virtually impossible for both values from (15) to coincide with the respective values in (16) so closely simply by chance.

iv. Scope as a Temporal Frame

The notion of scope defined in Section iii may be used to construct a natural time frame for a sonata-form movement. To avoid risking negligibility, the total length of the piece, as well as the length of its opening phrase, should be contained within the scope defined by the length of its first subject, which is the first self-contained part of the movement. In other words, the opening phrase, the first subject and the whole movement should correspond to measurements II_1, I_1 and I_8 from the system shown in Figure 3. These relations are particularly sharp in, for example, first movements of Mozart's piano sonatas K. 279 (C major) and K. 281 (B-flat major). In K. 279 the first subject, ending with half cadence after 15 measures, defines the scope in which the smallest length is 2 measures long and the largest length is 107 measures long. That fits tightly around the span between the length of the opening motive (2 measures) and the total length (100 measures). In K. 281, the first subject is 16 measures long: lengths of the opening motive (2 measures) and the whole movement (109 measures long) again fits within the scope of the first subject, ranging from 2 to 115 measures.

v. An Example of Formal Analysis

The shape of a particular sonata-form movement may be analyzed using the measurements from the system of van der Laan, shown in Figure 3. A remarkable example is the first movement from Mozart's Piano Sonata in B-flat major, KV 333. It is schematically shown in Figure 7 (the black bar).

Let the length of the whole movement be associated with the largest measurement I_8. Then I_1, I_2, I_3 and I_5 approximate lengths of the first subject, the Development, the second subject and the Recapitulation, respectively. The first subject is divided in two parts by the cadence in measure 10; these parts correspond to the margins of I_5 and I_6. The first subject is recapitulated in measures 93–118 and the second subject (including the final closure) in measures 118–165. These two parts correspond to the derived measurements of I_2 and I_4, denoted by Ia_2 and Ia_4, respectively. The measurements are shown as colored bars in Figure 7.

The opening phrase is $3^3/4$ measures long (including the upbeat). It is only slightly larger than II_1, which is approximately equal to $3^1/4$ measures. Therefore the temporal frame of this movement can be practically identified with the scope of I_1.

It should be noted that Proposition III.1 does not apply here. It is due to the fact that there are two consecutive closures in F major contained in measures 22–63. However, it does not matter as the transition between two subjects cannot be unambiguously determined. For example, one could claim that the transition happens between measures 22 and 38, in which case the subject lengths would be treated as being practically equal.

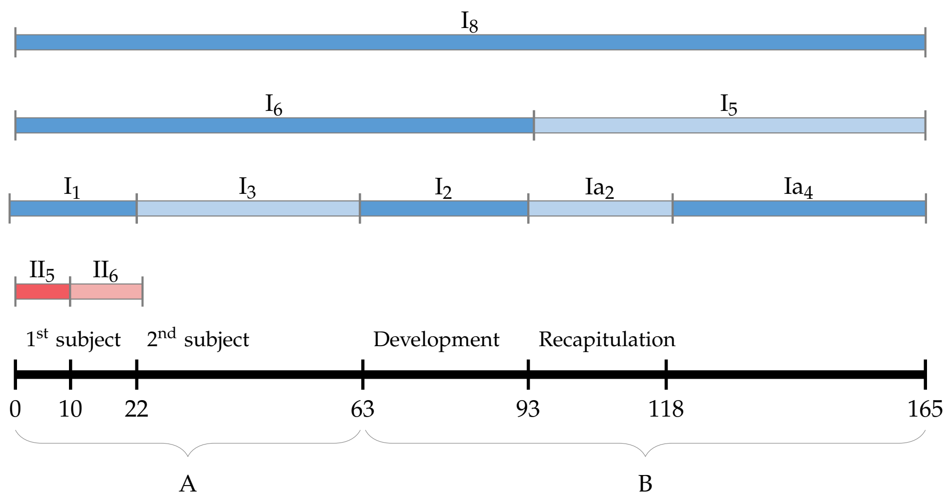

Figure 7: *W. A. Mozart: Piano Sonata in B-flat major KV 333, first movement*

It is obvious that the Recapitulation practically equal to the Exposition, which is coherent to the listener's expectation (forming with the onset of Recapitulation) that the two should be of the same length. It also should be noted that the Recapitulation is prolonged as much as it could be while staying practically equal to the Exposition.

The onset of Recapitulation divides the movement in ratio $^{93}/_{72} \approx \psi$, while the end of Exposition divides it in ratio $^{102}/_{63} \approx \varphi$. Hence both morphic numbers are built in the structure, defining the two most prominent moments in the course of movement. Relative errors of the above approximations are equal to 2.50% and 0.06%, respectively.

Although the structure of the movement illustrated in Figure 7 appears as it was consciously designed by combining measurements based on the plastic number (including the derived ones), chances that it actually happened are next to nothing. Namely, Mozart could not know about the plastic number as it was first discovered in 1928 [10] by Hans van der Laan[6]. As this excludes the possibility of a thoughtful mathematical design, it could only happen as a consequence of the plastic number indeed being a natural necessity [18, p. 138]. Many more examples of analyzing piano sonatas of Mozart in a similar way can be given; plastic number tends to appear frequently in his work.

The appearance of the golden ratio is significant as it indicates that measurements from the system of van der Laan can be combined to approximate it:

$$\frac{\mathrm{I}_2 + \mathrm{Ia}_2 + \mathrm{Ia}_4}{\mathrm{I}_1 + \mathrm{I}_3} \approx \varphi.$$

Relative error of this approximation is equal to 0.16%.

IV. Conclusion

We have shown some exact structural aspects of Sonata form can be deduced from its very concept. Length c of the Development can be chosen by simply using a generator of log-normally

[6] According to some sources, the plastic number was actually discovered in 1924 by a French engineer Gérard Cordonnier, who called it the *radiant number* [14, p. 9].

distributed random variable. Since theoretically any value $c > 0$ could be generated, inherent upper and lower bound ψ^{-4} and ψ^{-1} of the shape parameter $X = c/a$ also have to be taken into consideration. There need to exist strong musical reasons for X not being between these bounds. According to our probabilistic model it should happen in about 15% of cases. Indeed, we find $X \notin \mathcal{T} = [\psi^{-4}, \psi^{-1}]$ in about 13% of movements from Table 1.

A strong relation between Sonata form and the plastic number is established by our findings. The fact that the plastic number takes the role of ground ratio may characterize Sonata form as a "spatial" construction within the temporal domain. This may be one of the reasons for its popularity. Also, the plastic number acts as a bridge between music and architecture, from which it originated.

Results of the empirical study described in Section iii also give additional credit to Mozart (if that is even possible). Namely, an analysis of his work shows that proportions of the movements from Table 1 are coherent with the abstract concept of Sonata form. In fact, his work may serve as an detailed illustration of the concept, exploring all possible combinations of well-balanced proportions. In particular, values of the shape parameter for movements in Table 1 densely cover the whole theoretical range $\mathcal{T}$. Taking only a set of movements from piano sonatas, we find the shortest development section in the first movement of Piano Sonata in G major, KV 283 ($X \approx 0.34$) and the longest one[7] in the first movement of Piano Sonata in B-flat major, KV 281 ($X = 0.725$). The corresponding values of the shape parameter are almost equal to the extreme values $\psi^{-4} \approx \frac{1}{4}$ and $\psi^{-1} \approx \frac{3}{4}$. Although Mozart's treatment of Sonata form may appear rigid and uniform at first (compared to, for example, Beethoven's), we have shown that his choices of the shape parameter are as diverse as possible.

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[7] Actually, the longest development section is found in the first movement of Piano Sonata in D major, KV 311, where it is as long as the Exposition. But a special treatment is needed for that example since it does not feature a traditional Recapitulation, rather one going backwards, recapitulating the second subject before the first.

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