Detailed Solutions to Probability and Statistics Assignments (MA 20205)

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Contents

1 Assignment 1 Solutions **4**
1.1 Question 1 . . . 4
1.2 Question 2 . . . 4
1.3 Question 3 . . . 5
1.4 Question 4 . . . 5
1.5 Question 5 . . . 6
1.6 Question 6 . . . 6
1.7 Question 7 . . . 6
1.8 Question 8 . . . 7
1.9 Question 9 . . . 7
1.10 Question 10 . . . 7
1.11 Question 11 . . . 8
1.12 Question 12 . . . 8
1.13 Question 13 . . . 8
1.14 Question 14 . . . 9
1.15 Question 15 . . . 9
1.16 Question 16 . . . 9
1.17 Question 17 . . . 10
1.18 Question 18 . . . 10
1.19 Question 19 . . . 10
1.20 Question 20 . . . 11
1.21 Question 21 . . . 11
1.22 Question 22 . . . 11
1.23 Question 23 . . . 12

2 Assignment 2 Solutions **13**
2.1 Question 1 . . . 13
2.2 Question 2 . . . 13
2.3 Question 3 . . . 14
2.4 Question 4 . . . 15
2.5 Question 5 . . . 15
2.6 Question 6 . . . 15
2.7 Question 7 . . . 16
2.8 Question 8 . . . 17
2.9 Question 9 . . . 17
2.10 Question 10 . . . 17
2.11 Question 11 . . . 18
2.12 Question 12 . . . 18
2.13 Question 13 . . . 19

3 Assignment 3 Solutions **20**
3.1 Question 1 . . . 20
3.2 Question 2 . . . 20
3.3 Question 3 . . . 21
3.4 Question 4 . . . 21
3.5 Question 5 . . . 22
3.6 Question 6 . . . 22

3.7 Question 7 . . . 23
3.8 Question 8 . . . 23
3.9 Question 9 . . . 24
3.10 Question 10 . . . 24
3.11 Question 11 . . . 25
3.12 Question 12 . . . 25
3.13 Question 13 . . . 26
3.14 Question 14 . . . 26
3.15 Question 15 . . . 26
3.16 Question 16 . . . 27
3.17 Question 17 . . . 28
3.18 Question 18 . . . 28
3.19 Question 19 . . . 29
3.20 Question 20 . . . 29
3.21 Question 21 . . . 30
3.22 Question 22 . . . 30
3.23 Question 23 . . . 31
3.24 Question 24 . . . 31
3.25 Question 25 . . . 32

4 Assignment 4 Solutions **33**

4.1 Question 1 . . . 33
4.2 Question 2 . . . 33
4.3 Question 3 . . . 34
4.4 Question 4 . . . 34
4.5 Question 5 . . . 35
4.6 Question 6 . . . 35
4.7 Question 7 . . . 35
4.8 Question 8 . . . 36
4.9 Question 9 . . . 37
4.10 Question 10 . . . 37
4.11 Question 11 . . . 37
4.12 Question 12 . . . 38
4.13 Question 13 . . . 38
4.14 Question 14 . . . 38

5 Assignment 5 Solutions **40**

5.1 Question 1 . . . 40
5.2 Question 2 . . . 40
5.3 Question 3 . . . 41
5.4 Question 4 . . . 41
5.5 Question 5 . . . 42
5.6 Question 6 . . . 42
5.7 Question 7 . . . 42
5.8 Question 8 . . . 43
5.9 Question 9 . . . 43
5.10 Question 10 . . . 45
5.11 Question 11 . . . 45
5.12 Question 12 . . . 45

6 Formula Sheet **47**

Chapter 1

Assignment 1 Solutions

1.1 Question 1

Problem: If 7 balls are placed at random into 7 cells, find the probability that exactly one cell remains empty.

Solution: First, we determine the total number of ways to place 7 balls into 7 cells. Since each ball can be placed in any of the 7 cells independently, the total number of possible arrangements is 7^7.

Next, we find the number of favorable arrangements where exactly one cell is empty. This implies that one cell must contain two balls, and the other five cells must each contain exactly one ball.

1. **Choose the empty cell:** There are $\binom{7}{1} = 7$ ways to choose which cell will be empty.

2. **Choose the cell with two balls:** From the remaining 6 cells, we must choose one to contain two balls. There are $\binom{6}{1} = 6$ ways.

3. **Choose the two balls:** From the 7 available balls, we need to choose the two that will be placed together. This can be done in $\binom{7}{2} = \frac{7\times 6}{2} = 21$ ways.

4. **Arrange the remaining balls:** We have 5 balls left to be placed in the remaining 5 cells, with one ball per cell. The number of ways to do this is $5! = 120$.

The total number of favorable outcomes is the product of these steps:

$$\text{Favorable ways} = \binom{7}{1} \times \binom{6}{1} \times \binom{7}{2} \times 5! = 7 \times 6 \times 21 \times 120 = 105,840$$

The probability is the ratio of favorable outcomes to the total number of outcomes:

$$P(\text{exactly one empty cell}) = \frac{105,840}{7^7} = \frac{105,840}{823,543} \approx 0.1285$$

$$\boxed{P(\text{exactly one empty cell}) \approx 0.1285}$$

1.2 Question 2

Problem: Let event E be independent of events F, $F \cup G$ and $F \cap G$. Show that E is independent of G.

Solution: We are given the following from the definition of independence:

1. $P(E \cap F) = P(E)P(F)$

2. $P(E \cap (F \cup G)) = P(E)P(F \cup G)$

3. $P(E \cap (F \cap G)) = P(E)P(F \cap G)$

We want to show that $P(E \cap G) = P(E)P(G)$. We start with the second given condition:

$$P(E \cap (F \cup G)) = P(E)P(F \cup G)$$

Using the distributive property for the left side and the addition rule for the right side:

$$P((E \cap F) \cup (E \cap G)) = P(E)[P(F) + P(G) - P(F \cap G)]$$

Applying the addition rule to the left side:

$$P(E \cap F) + P(E \cap G) - P(E \cap F \cap G) = P(E)P(F) + P(E)P(G) - P(E)P(F \cap G)$$

Now, we substitute our known independencies (1) and (3) into this equation:

$$P(E)P(F) + P(E \cap G) - P(E)P(F \cap G) = P(E)P(F) + P(E)P(G) - P(E)P(F \cap G)$$

We can cancel the term $P(E)P(F)$ from both sides, and add the term $P(E)P(F \cap G)$ to both sides, which also cancels out. This leaves us with the desired result:

$$\boxed{P(E \cap G) = P(E)P(G)}$$

Thus, E is independent of G.

1.3 Question 3

Problem: A pair of dice is rolled until a sum of 4 or an odd sum appears. Find the probability that a 4 appears first.

Solution: Let A be the event that the sum is 4. The outcomes for A are $\{(1,3),(2,2),(3,1)\}$. So, $P(A) = \frac{3}{36} = \frac{1}{12}$. Let B be the event that the sum is odd. An odd sum occurs if one die is even and the other is odd. There are $3 \times 3 = 9$ (even, odd) pairs and $3 \times 3 = 9$ (odd, even) pairs. So, $P(B) = \frac{18}{36} = \frac{1}{2}$. The game stops when either A or B occurs. Since A and B are mutually exclusive, the probability of the game stopping on any given roll is $P(A \cup B) = P(A) + P(B) = \frac{1}{12} + \frac{1}{2} = \frac{7}{12}$. The probability that the game does not stop on a roll is $1 - \frac{7}{12} = \frac{5}{12}$. For a 4 to appear first, it must appear on a roll where the game stops. The probability of this is the probability of A given that the game stopped (A or B occurred).

$$P(4 \text{ appears first}) = P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)}$$

$$P(4 \text{ appears first}) = \frac{1/12}{7/12} = \frac{1}{7}$$

$$\boxed{P(4 \text{ appears first}) = \frac{1}{7}}$$

1.4 Question 4

Problem: 50% of faculty own a desktop, 25

Solution: Let D be the event that a faculty member owns a desktop, and L be the event they own a laptop. We are given: $P(D) = 0.50$, $P(L) = 0.25$, and $P(D \cap L) = 0.10$. We want to find the probability of the symmetric difference, which is owning one type of computer but not the other. This can be expressed as $P(D \cup L) - P(D \cap L)$. First, find $P(D \cup L)$:

$$P(D \cup L) = P(D) + P(L) - P(D \cap L) = 0.50 + 0.25 - 0.10 = 0.65$$

The probability of owning a desktop or a laptop but not both is:

$$P(\text{one but not both}) = P(D \cup L) - P(D \cap L) = 0.65 - 0.10 = 0.55$$

Alternatively, this is $P(D) + P(L) - 2P(D \cap L) = 0.50 + 0.25 - 2(0.10) = 0.75 - 0.20 = 0.55$.

$$\boxed{P(\text{owns one but not both}) = 0.55}$$

1.5 Question 5

Problem: 20% of people are smokers. The probability of death from lung cancer for a smoker is 10 times that for a non-smoker. The overall probability of death from lung cancer is 0.006. What is the probability of death from lung cancer for a smoker?

Solution: Let S be the event that a person is a smoker, and C be the event of death due to lung cancer. We are given: $P(S) = 0.20 \implies P(S^c) = 0.80$. $P(C) = 0.006$. $P(C|S) = 10 \times P(C|S^c)$. Let $\alpha = P(C|S^c)$. Then $P(C|S) = 10\alpha$. We need to find 10α. Using the Law of Total Probability:

$$P(C) = P(C|S)P(S) + P(C|S^c)P(S^c)$$

$$0.006 = (10\alpha)(0.20) + (\alpha)(0.80)$$

$$0.006 = 2\alpha + 0.8\alpha = 2.8\alpha$$

$$\alpha = \frac{0.006}{2.8} \approx 0.002143$$

The probability of death due to lung cancer given that a person is a smoker is $P(C|S) = 10\alpha$.

$$P(C|S) = 10 \times \frac{0.006}{2.8} = \frac{0.06}{2.8} \approx 0.0214$$

$$\boxed{P(\text{death from lung cancer — smoker}) \approx 0.0214}$$

1.6 Question 6

Problem: Two balls are drawn with replacement from a box with n balls marked 1 to n. Find the probability that the numbers are consecutive integers (order ignored).

Solution: The total number of outcomes when drawing two balls with replacement is $n \times n = n^2$. We are looking for pairs of draws (a, b) where $|a - b| = 1$. We can count the favorable ordered pairs:

- If the first ball drawn is 1, the second must be 2. (1 pair: (1,2))
- If the first ball drawn is n, the second must be n-1. (1 pair: (n,n-1))
- If the first ball drawn is i, where $1 < i < n$, the second can be $i - 1$ or $i + 1$. There are $n - 2$ such values for i, each providing 2 pairs. This gives $2(n - 2)$ pairs.

The total number of favorable ordered pairs is $1 + 1 + 2(n - 2) = 2 + 2n - 4 = 2n - 2 = 2(n - 1)$. The probability is the ratio of favorable outcomes to the total number of outcomes.

$$P(\text{consecutive}) = \frac{2(n-1)}{n^2}$$

$$\boxed{P(\text{consecutive}) = \frac{2(n-1)}{n^2}}$$

1.7 Question 7

Problem: Given $P(A) < 1$, $P(B) > 0$ and $P(A|B) = 1$. Determine $P(B^C|A^C)$.

Solution: From the definition of conditional probability, $P(A|B) = \frac{P(A\cap B)}{P(B)}$. Given $P(A|B) = 1$, we have $\frac{P(A\cap B)}{P(B)} = 1$, which implies $P(A \cap B) = P(B)$. This means that the event B is a subset of the event A. We want to find $P(B^c|A^c) = \frac{P(B^c\cap A^c)}{P(A^c)}$. By De Morgan's laws, $B^c \cap A^c = (A \cup B)^c$. Since B is a subset of A, $A \cup B = A$. Therefore, $(A \cup B)^c = A^c$. So, $P(B^c \cap A^c) = P(A^c)$. Substituting this into the expression for conditional probability:

$$P(B^c|A^c) = \frac{P(A^c)}{P(A^c)} = 1$$

(This is valid because $P(A) < 1$ implies $P(A^c) > 0$).

$$\boxed{P(B^c|A^c) = 1}$$

1.8 Question 8

Problem: If $P(A^c) = 0.3$, $P(B) = 0.4$, and $P(A \cap B^c) = 0.5$, find $P(B|A \cup B^c)$.

Solution: First, we list the probabilities we know or can derive: $P(A^c) = 0.3 \implies P(A) = 1 - 0.3 = 0.7$. $P(B) = 0.4 \implies P(B^c) = 1 - 0.4 = 0.6$. $P(A \cap B^c) = 0.5$. We need to find $P(B|A \cup B^c) = \frac{P(B \cap (A \cup B^c))}{P(A \cup B^c)}$. Let's evaluate the numerator and denominator separately. **Numerator:** Using the distributive property: $P(B \cap (A \cup B^c)) = P((B \cap A) \cup (B \cap B^c)) = P(A \cap B) + P(\emptyset) = P(A \cap B)$. We can find $P(A \cap B)$ from the relation $P(A) = P(A \cap B) + P(A \cap B^c)$. $0.7 = P(A \cap B) + 0.5 \implies P(A \cap B) = 0.2$. **Denominator:** Using the addition rule: $P(A \cup B^c) = P(A) + P(B^c) - P(A \cap B^c) = 0.7 + 0.6 - 0.5 = 0.8$. Finally, we can calculate the conditional probability:

$$P(B|A \cup B^c) = \frac{0.2}{0.8} = \frac{1}{4} = 0.25$$

$$\boxed{P(B|A \cup B^c) = 0.25}$$

1.9 Question 9

Problem: Given a trinary communication channel, find certain probabilities.

Solution: Let T_i be the event "Digit i is transmitted" and R_i be the event "Digit i is received". **Priors:** We are given $P(T_3) = 3P(T_1)$ and $P(T_2) = 2P(T_1)$. Since $P(T_1) + P(T_2) + P(T_3) = 1$, we have $P(T_1) + 2P(T_1) + 3P(T_1) = 1 \implies 6P(T_1) = 1$. So, $P(T_1) = 1/6, P(T_2) = 2/6, P(T_3) = 3/6$. **Channel Probabilities (from diagram):** $P(R_1|T_1) = 1 - \alpha$, $P(R_2|T_1) = \alpha/2$, $P(R_3|T_1) = \alpha/2$. $P(R_2|T_2) = 1 - \beta$, $P(R_1|T_2) = \beta/2$, $P(R_3|T_2) = \beta/2$. $P(R_3|T_3) = 1 - \gamma$, $P(R_1|T_3) = \gamma/2$, $P(R_2|T_3) = \gamma/2$.

(i) **Probability $P(T_1|R_1)$:** Using Bayes' theorem: $P(T_1|R_1) = \frac{P(R_1|T_1)P(T_1)}{P(R_1)}$. $P(R_1) = \sum_{i=1}^{3} P(R_1|T_i)P(T_i) = (1-\alpha)\frac{1}{6} + (\frac{\beta}{2})\frac{2}{6} + (\frac{\gamma}{2})\frac{3}{6} = \frac{1-\alpha+\beta+1.5\gamma}{6} = \frac{2-2\alpha+2\beta+3\gamma}{12}$. $P(T_1|R_1) = \frac{(1-\alpha)/6}{(2-2\alpha+2\beta+3\gamma)/12} = \frac{2(1-\alpha)}{2-2\alpha+2\beta+3\gamma}$.

$$\boxed{P(T_1|R_1) = \frac{2(1-\alpha)}{2(1-\alpha) + 2\beta + 3\gamma}}$$

(ii) **Probability of transmission error:** $P(\text{error}) = 1 - P(\text{correct}) = 1 - \sum_{i=1}^{3} P(R_i, T_i) = 1 - \sum_{i=1}^{3} P(R_i|T_i)P(T_i)$. $P(\text{correct}) = (1-\alpha)\frac{1}{6} + (1-\beta)\frac{2}{6} + (1-\gamma)\frac{3}{6} = \frac{1-\alpha+2-2\beta+3-3\gamma}{6} = \frac{6-\alpha-2\beta-3\gamma}{6}$. $P(\text{error}) = 1 - \frac{6-\alpha-2\beta-3\gamma}{6} = \frac{\alpha+2\beta+3\gamma}{6}$.

$$\boxed{P(\text{error}) = \frac{\alpha + 2\beta + 3\gamma}{6}}$$

(iii) **Probability of receiving digit i:** $P(R_1)$ was calculated in part (i): $\frac{2-2\alpha+2\beta+3\gamma}{12}$. $P(R_2) = \sum P(R_2|T_i)P(T_i) = (\frac{\alpha}{2})\frac{1}{6} + (1-\beta)\frac{2}{6} + (\frac{\gamma}{2})\frac{3}{6} = \frac{\alpha+4-4\beta+3\gamma}{12}$. $P(R_3) = \sum P(R_3|T_i)P(T_i) = (\frac{\alpha}{2})\frac{1}{6} + (\frac{\beta}{2})\frac{2}{6} + (1-\gamma)\frac{3}{6} = \frac{\alpha+2\beta+6-6\gamma}{12}$.

$$\boxed{P(R_1) = \frac{2 - 2\alpha + 2\beta + 3\gamma}{12}, P(R_2) = \frac{\alpha + 4 - 4\beta + 3\gamma}{12}, P(R_3) = \frac{\alpha + 2\beta + 6 - 6\gamma}{12}}$$

1.10 Question 10

Problem: Which of the given statements is true?

Solution: (i) $P(A) = 0.3, P(B) = 0.7, P(A \cap B) = 0.5$. This is impossible, since $A \cap B \subseteq A$, which means $P(A \cap B)$ cannot be greater than $P(A)$. Here $0.5 > 0.3$. **False**.

(ii) $P(A) = 0.5, P(A \cup B) = 0.7$, A and B are independent, $P(B) = 0.4$. If A and B are independent, $P(A \cup B) = P(A) + P(B) - P(A)P(B)$. Let's check: $0.7 = 0.5 + 0.4 - (0.5)(0.4) \implies 0.7 = 0.9 - 0.2 \implies 0.7 = 0.7$. This is consistent. **True**.

(iii) $P(A) = 0.2, P(A \cup B) = 0.9$, A and B are disjoint, $P(B) = 0.6$. If A and B are disjoint, $P(A \cup B) = P(A) + P(B)$. Let's check: $0.9 = 0.2 + 0.6 \implies 0.9 = 0.8$. This is a contradiction. **False**.

$$\boxed{\text{Statement (ii) is true.}}$$

1.11 Question 11

Problem: Thirteen cards are distributed to each of four players. What is the probability that player C has all four kings?

Solution: We consider the hand of player C. The total number of possible 13-card hands for player C is $\binom{52}{13}$. For the event to be favorable, player C's hand must contain all 4 kings and 9 other cards from the remaining $52 - 4 = 48$ non-king cards. The number of ways to form such a hand is $\binom{4}{4} \times \binom{48}{9}$. The probability is the ratio of favorable hands to total possible hands:

$$P(\text{C has 4 kings}) = \frac{\binom{4}{4}\binom{48}{9}}{\binom{52}{13}} = \frac{1 \cdot \frac{48!}{9!39!}}{\frac{52!}{13!39!}} = \frac{48! \cdot 13!}{52! \cdot 9!} = \frac{13 \cdot 12 \cdot 11 \cdot 10}{52 \cdot 51 \cdot 50 \cdot 49}$$

$$= \frac{17160}{2082500} = \frac{11}{4165} \approx 0.00264$$

$$\boxed{P(\text{C has 4 kings}) = \frac{11}{4165}}$$

1.12 Question 12

Problem: A student takes 5 courses per semester for 4 semesters. Probability of getting at least an 'A' in any course is 0.5. Find the probability of getting all 'A's in at least one semester.

Solution: Let S be the event of success in a semester, which means getting at least an 'A' in all 5 courses. The probability of getting at least an 'A' in one course is $p = 0.5$. Since grades are independent, the probability of success in one semester is $P(S) = (0.5)^5 = \frac{1}{32}$. The probability of failure in one semester is $P(S^c) = 1 - \frac{1}{32} = \frac{31}{32}$. We have 4 semesters (4 independent trials). We want the probability of at least one success. It is easier to calculate the complement: the probability of zero successes (failure in all 4 semesters).

$$P(\text{failure in all 4 semesters}) = (P(S^c))^4 = \left(\frac{31}{32}\right)^4$$

The probability of at least one successful semester is:

$$P(\text{at least one S}) = 1 - P(\text{no S}) = 1 - \left(\frac{31}{32}\right)^4 \approx 1 - 0.8807 = 0.1193$$

$$\boxed{P(\text{at least one semester with all A's}) = 1 - \left(\frac{31}{32}\right)^4 \approx 0.1193}$$

1.13 Question 13

Problem: If $P(A) > 0$, show that $P(A \cap B|A) \geq P(A \cap B|A \cup B)$.

Solution: We evaluate the left-hand side (LHS) and right-hand side (RHS) separately. LHS: $P(A \cap B|A) = \frac{P((A\cap B)\cap A)}{P(A)} = \frac{P(A\cap B)}{P(A)}$. RHS: $P(A \cap B|A \cup B) = \frac{P((A\cap B)\cap(A\cup B))}{P(A\cup B)} = \frac{P(A\cap B)}{P(A\cup B)}$. The inequality we need to prove is:

$$\frac{P(A \cap B)}{P(A)} \geq \frac{P(A \cap B)}{P(A \cup B)}$$

This inequality holds if $P(A \cap B) = 0$. If $P(A \cap B) > 0$, we can divide by it, and the inequality becomes:

$$\frac{1}{P(A)} \geq \frac{1}{P(A \cup B)}$$

This is equivalent to $P(A \cup B) \geq P(A)$. This is always true, because the event A is a subset of the event $A \cup B$. Thus, the original statement is proven.

$$\boxed{\text{The statement is proven as } P(A \cup B) \geq P(A) \text{ is always true.}}$$

1.14 Question 14

Problem: Let A and B be random subsets of S=1,...,n. Find $P(|B|=i)$, $P(A \subset B||B|=i)$, $P(A \subset B)$ and deduce $P(A \cap B = \emptyset)$.

Solution: Total number of subsets of S is 2^n. $P(|B|=i)$: The number of subsets of size i is $\binom{n}{i}$. Since B is chosen equally likely, $P(|B|=i) = \frac{\binom{n}{i}}{2^n}$. $P(A \subset B||B|=i)$: Given that B is a specific subset of size i, A must be one of the 2^i subsets of B. Since there are 2^n total choices for A, the probability is $\frac{2^i}{2^n} = 2^{i-n}$. $P(A \subset B)$: Using the law of total probability:

$$P(A \subset B) = \sum_{i=0}^{n} P(A \subset B||B|=i)P(|B|=i) = \sum_{i=0}^{n} \frac{2^i}{2^n}\frac{\binom{n}{i}}{2^n} = \frac{1}{4^n}\sum_{i=0}^{n}\binom{n}{i}2^i$$

Using the binomial theorem, $\sum_{i=0}^{n}\binom{n}{i}2^i = (1+2)^n = 3^n$.

$$P(A \subset B) = \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$$

$P(A \cap B = \emptyset)$: This event is equivalent to $A \subseteq B^c$. Since the choice of B is random, the choice of its complement B^c is also random over all possible subsets. Therefore, the problem is identical to finding $P(A \subset C)$ where C is a random subset, which we just found.

$$\boxed{P(|B|=i) = \frac{\binom{n}{i}}{2^n}, P(A \subset B||B|=i) = \frac{2^i}{2^n}, P(A \subset B) = \left(\frac{3}{4}\right)^n, P(A \cap B = \emptyset) = \left(\frac{3}{4}\right)^n}$$

1.15 Question 15

Problem: A test has 6 T/F and 4 multiple-choice questions. An unprepared student guesses all answers. Find the probability of scoring at least 8 marks.

Solution: Let X be the score from the 6 T/F questions ($p = 1/2$) and Y be the score from the 4 MC questions ($p = 1/4$). We want $P(X+Y \geq 8)$. The possible pairs (X,Y) that sum to at least 8 are: (4,4), (5,3), (5,4), (6,2), (6,3), (6,4). We sum the probabilities of these independent events:
$P(X=k) = \binom{6}{k}(1/2)^6$, $P(Y=k) = \binom{4}{k}(1/4)^k(3/4)^{4-k}$.
$P(X+Y \geq 8) = P(X=4)P(Y=4) + P(X=5)P(Y=3) + P(X=5)P(Y=4) + P(X=6)P(Y=2) + P(X=6)P(Y=3) + P(X=6)P(Y=4)$
$= [\binom{6}{4}(\frac{1}{64})][\binom{4}{4}(\frac{1}{256})] + [\binom{6}{5}(\frac{1}{64})][\binom{4}{3}(\frac{12}{256})] + [\binom{6}{5}(\frac{1}{64})][\binom{4}{4}(\frac{1}{256})] + [\binom{6}{6}(\frac{1}{64})][\binom{4}{2}(\frac{54}{256})] + [\binom{6}{6}(\frac{1}{64})][\binom{4}{3}(\frac{12}{256})] + [\binom{6}{6}(\frac{1}{64})][\binom{4}{4}(\frac{1}{256})]$
$= \frac{1}{16384}[(15)(1) + (6)(12) + (6)(1) + (1)(54) + (1)(12) + (1)(1)]$
$= \frac{1}{16384}[15 + 72 + 6 + 54 + 12 + 1] = \frac{160}{16384} = \frac{5}{512}$.

$$\boxed{P(\text{score} \geq 8) = \frac{5}{512} \approx 0.00977}$$

1.16 Question 16

Problem: If 2n students qualify, and boys and girls are equally likely, what is the probability that more girls qualify than boys?

Solution: Let X be the number of girls who qualify among the $2n$ students. Each student is a girl with probability 0.5. Thus, $X \sim \text{Binomial}(2n, 0.5)$. We want to find $P(X > n)$. Let $p = P(X > n)$, $q = P(X < n)$, and $r = P(X = n)$. We know $p + q + r = 1$. For a binomial distribution with $p = 0.5$, the distribution is symmetric: $P(X = k) = P(X = 2n - k)$. Therefore, $q = P(X < n) = \sum_{k=0}^{n-1} P(X = k) = \sum_{k=0}^{n-1} P(X = 2n - k)$. Let $j = 2n - k$. The sum becomes $\sum_{j=n+1}^{2n} P(X = j) = P(X > n) = p$. Since $p = q$, we have $2p + r = 1$, which gives $p = \frac{1-r}{2}$. The term r is $P(X = n) = \binom{2n}{n}(0.5)^{2n}$.

$$p = \frac{1 - \binom{2n}{n}(0.5)^{2n}}{2}$$

$$\boxed{P(\text{more girls than boys}) = \frac{1}{2}\left(1 - \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right)}$$

1.17 Question 17

Problem: The hat-check problem (derangements). Find the probability that no one gets their own hat, and the probability that at least one person does.

Solution: This is a classic derangement problem. Let D_n be the number of ways n items can be arranged so that none are in their original position. The total number of arrangements is $n!$. The probability that no one gets their own hat is $P_0 = \frac{D_n}{n!}$. The formula for D_n can be derived using the principle of inclusion-exclusion, and the probability is:

$$P(\text{no one gets own hat}) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}$$

As $n \to \infty$, this series converges to e^{-1}. The probability that at least one person gets their hat back is the complement of the above event:

$$P(\text{at least one gets own hat}) = 1 - P(\text{no one gets own hat}) = 1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

As $n \to \infty$, this probability converges to $1 - e^{-1}$.

$$\boxed{P(\text{no match}) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \xrightarrow{n\to\infty} e^{-1}, \quad P(\text{at least one match}) = 1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!} \xrightarrow{n\to\infty} 1 - e^{-1}}$$

1.18 Question 18

Problem: n balls are placed randomly in R boxes. What is the probability that exactly k balls are placed in the first r boxes?

Solution: This scenario can be modeled as a sequence of n independent trials. For each ball (trial), there are two outcomes: it lands in one of the first r boxes, or it does not. The probability of a single ball landing in one of the first r boxes is $p = \frac{r}{R}$. The probability of it landing in one of the remaining $R - r$ boxes is $1 - p = 1 - \frac{r}{R}$. Let X be the number of balls that land in the first r boxes. Then X follows a Binomial distribution with parameters n and $p = r/R$. We want to find the probability that $X = k$. Using the Binomial PMF:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} \left(\frac{r}{R}\right)^k \left(1 - \frac{r}{R}\right)^{n-k}$$

$$\boxed{P(k \text{ balls in first } r \text{ boxes}) = \binom{n}{k} \left(\frac{r}{R}\right)^k \left(1 - \frac{r}{R}\right)^{n-k}}$$

1.19 Question 19

Problem: Choose 13 cards from a standard deck. What is the probability of getting 3 red cards?

Solution: The total number of ways to choose 13 cards from a 52-card deck is $\binom{52}{13}$. The deck has 26 red cards and 26 black cards. A favorable hand consists of 3 red cards and $13 - 3 = 10$ black cards. The number of ways to choose 3 red cards from 26 is $\binom{26}{3}$. The number of ways to choose 10 black cards from 26 is $\binom{26}{10}$. The total number of favorable hands is the product $\binom{26}{3}\binom{26}{10}$. The probability is the ratio:

$$P(3 \text{ red, } 10 \text{ black}) = \frac{\binom{26}{3}\binom{26}{10}}{\binom{52}{13}}$$

$$\boxed{P(3 \text{ red cards}) = \frac{\binom{26}{3}\binom{26}{10}}{\binom{52}{13}} \approx 0.2117}$$

1.20 Question 20

Problem: Choose 13 cards. Find the probability of getting 3 clubs, 4 diamonds, 4 hearts, and 2 spades.

Solution: The total number of ways to choose 13 cards from 52 is $\binom{52}{13}$. Each suit has 13 cards. We want a specific composition for our hand. Number of ways to choose 3 clubs from 13: $\binom{13}{3}$. Number of ways to choose 4 diamonds from 13: $\binom{13}{4}$. Number of ways to choose 4 hearts from 13: $\binom{13}{4}$. Number of ways to choose 2 spades from 13: $\binom{13}{2}$. The total number of favorable hands is the product of these combinations.

$$P(\text{event}) = \frac{\binom{13}{3}\binom{13}{4}\binom{13}{4}\binom{13}{2}}{\binom{52}{13}}$$

$$\boxed{P(\text{event}) = \frac{\binom{13}{3}\binom{13}{4}^2\binom{13}{2}}{\binom{52}{13}} \approx 0.00538}$$

1.21 Question 21

Problem: Choose 4 cards with replacement. Find the probability of drawing (a) four distinct kings, (b) a queen each time.

Solution: The total number of outcomes when drawing 4 cards with replacement is 52^4. (a) **Four distinct kings:** This means drawing the King of Spades, King of Hearts, King of Diamonds, and King of Clubs in some order. There are 4 such cards. The number of ways to arrange these 4 distinct cards is $4! = 24$.

$$P(4 \text{ distinct kings}) = \frac{4!}{52^4} = \frac{24}{7,311,616} \approx 3.28 \times 10^{-6}$$

(b) **A queen each time:** There are 4 queens in the deck. The probability of drawing a queen on any single draw is $\frac{4}{52} = \frac{1}{13}$. Since the draws are independent (with replacement), the probability of this happening 4 times in a row is:

$$P(4 \text{ queens}) = \left(\frac{4}{52}\right)^4 = \left(\frac{1}{13}\right)^4 = \frac{1}{28,561} \approx 3.5 \times 10^{-5}$$

$$\boxed{\text{(a)}\frac{4!}{52^4} \quad \text{(b)}\left(\frac{4}{52}\right)^4}$$

1.22 Question 22

Problem: n balls are distributed in n numbered boxes. What is the probability that only the 1st box will remain empty?

Solution: Total number of arrangements is n^n. For the 1st box to be the only empty box, all n balls must be placed in the other $n-1$ boxes, with none of these $n-1$ boxes being empty. This implies that one of these $n-1$ boxes must contain exactly 2 balls, while the other $n-2$ boxes contain 1 ball each.

1. Choose which of the $n-1$ boxes (box 2 to n) will receive 2 balls: $\binom{n-1}{1}$ ways.

2. Choose which 2 of the n balls will go into that box: $\binom{n}{2}$ ways.

3. Arrange the remaining $n-2$ balls in the remaining $n-2$ boxes (one per box): $(n-2)!$ ways.

Total favorable arrangements: $\binom{n-1}{1}\binom{n}{2}(n-2)! = (n-1) \cdot \frac{n(n-1)}{2} \cdot (n-2)!$. The probability is the ratio:

$$P(\text{only 1st empty}) = \frac{(n-1)\binom{n}{2}(n-2)!}{n^n} = \frac{(n-1)\frac{n(n-1)}{2}(n-2)!}{n^n}$$

$$\boxed{P(\text{only 1st empty}) = \frac{(n-1)\binom{n}{2}(n-2)!}{n^n}}$$

1.23 Question 23

Problem: A rumor spreads in a village of $n+1$ people. Find the probability (a) it does not return to the originator, and (b) it is not told to anyone who already knows it.

Solution: Assume that at each step, a person tells the rumor to one of the n other people at random. There is a sequence of r transmissions. Total number of rumor paths: The originator has n choices. Each subsequent person has n choices. Total paths $= n \times n^{r-1} = n^r$.

(a) **Rumor does not return to the originator:** The originator tells one of n people. For the subsequent $r-1$ steps, each person must choose from the $n-1$ people who are not the originator. Favorable paths $= n \times (n-1)^{r-1}$.

$$P(\text{not back to originator}) = \frac{n(n-1)^{r-1}}{n^r} = \left(\frac{n-1}{n}\right)^{r-1}$$

$$\boxed{P(\text{a}) = \left(\frac{n-1}{n}\right)^{r-1}}$$

(b) **Rumor is not repeated to anyone:** This means the sequence of $r+1$ people involved (originator + r others) must all be distinct. The originator tells person 1 (n choices). Person 1 must tell a new person (n-1 choices available). Person 2 must tell a new person (n-2 choices available). ... Person r-1 must tell a new person (n-(r-1) choices available). Favorable paths $= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) = P(n,r)$.

$$P(\text{no repeats}) = \frac{P(n,r)}{n^r} = \frac{n!/(n-r)!}{n^r}$$

$$\boxed{P(\text{b}) = \frac{n(n-1)\dots(n-r+1)}{n^r} = \frac{{}_nP_r}{n^r}}$$

Chapter 2

Assignment 2 Solutions

2.1 Question 1

Problem: Let X be a continuous random variable with PDF $f(x) = x^2$ for $0 < x \leq 1$ and $f(x) = c/x^2$ for $1 < x < \infty$. Determine c, the CDF, E(X), Var(X), the median, $P(0.5 < X < 2)$, and $P(X > 3)$.

Solution: Find c: For $f(x)$ to be a valid PDF, its total integral must be 1.

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 x^2 dx + \int_1^{\infty} \frac{c}{x^2} dx = 1$$

$$\left[\frac{x^3}{3}\right]_0^1 + c\left[-\frac{1}{x}\right]_1^{\infty} = \frac{1}{3} + c[0-(-1)] = \frac{1}{3} + c = 1 \implies c = \frac{2}{3}$$

CDF of X: We integrate the PDF from $-\infty$ to x. For $x \leq 0$: $F(x) = 0$. For $0 < x \leq 1$: $F(x) = \int_0^x t^2 dt = \frac{x^3}{3}$. For $x > 1$: $F(x) = \int_0^1 t^2 dt + \int_1^x \frac{2/3}{t^2} dt = \frac{1}{3} + \frac{2}{3}\left[-\frac{1}{t}\right]_1^x = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{x} - (-1)\right) = 1 - \frac{2}{3x}$.

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x^3}{3}, & 0 < x \leq 1 \\ 1 - \frac{2}{3x}, & x > 1 \end{cases}$$

E(X) and Var(X):

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x^3 dx + \int_1^{\infty} x\frac{2/3}{x^2} dx = \left[\frac{x^4}{4}\right]_0^1 + \frac{2}{3}\int_1^{\infty} \frac{1}{x} dx$$

The second integral is $\frac{2}{3}[\ln(x)]_1^{\infty}$, which diverges. Thus, $E(X)$ does not exist. Since $E(X)$ does not exist, $Var(X)$ also does not exist. **Median of X:** We solve $F(M) = 0.5$. Since $F(1) = 1/3 < 0.5$, the median must be in the range $x > 1$.

$$1 - \frac{2}{3M} = 0.5 \implies \frac{2}{3M} = 0.5 \implies 3M = 4 \implies M = \frac{4}{3}$$

Probabilities: We use the CDF. $P(0.5 < X < 2) = F(2) - F(0.5) = (1 - \frac{2}{3(2)}) - \frac{(0.5)^3}{3} = (1 - \frac{1}{3}) - \frac{0.125}{3} = \frac{2}{3} - \frac{1}{24} = \frac{16-1}{24} = \frac{15}{24} = \frac{5}{8}$. $P(X > 3) = 1 - F(3) = 1 - (1 - \frac{2}{3(3)}) = \frac{2}{9}$.

$$\boxed{c = \frac{2}{3}, \text{Median} = \frac{4}{3}, P(0.5 < X < 2) = \frac{5}{8}, P(X > 3) = \frac{2}{9}, \text{E(X) and Var(X) do not exist.}}$$

2.2 Question 2

Problem: Given a CDF for a mixed random variable X, find probabilities, E(X), Var(X), and the median.

Solution: Probabilities from CDF: $P(1/2 < X < 5/2) = F(5/2^-) - F(1/2)$. Since $5/2 = 2.5$ is in $[2, 3)$, $F(5/2^-) = 11/12$. Since $1/2$ is in $[0, 1)$, $F(1/2) = 1/2/4 = 1/8$. $P(1/2 < X < 5/2) = \frac{11}{12} - \frac{1}{8} = \frac{22-3}{24} = \frac{19}{24}$. $P(1 < X < 3) = F(3^-) - F(1) = \frac{11}{12} - \frac{1+1}{4} = \frac{11}{12} - \frac{1}{2} = \frac{5}{12}$.

E(X) and Var(X): This is a mixed random variable. **Nice Trick: Identifying Mixed Components** The jumps in the CDF indicate discrete probability masses. The derivative in smooth regions gives the continuous density. $P(X=1) = F(1) - F(1^-) = \frac{1+1}{4} - \frac{1}{4} = \frac{1}{4}$.
$P(X=2) = F(2) - F(2^-) = \frac{11}{12} - \frac{2+1}{4} = \frac{11-9}{12} = \frac{2}{12} = \frac{1}{6}$.
$P(X=3) = F(3) - F(3^-) = 1 - \frac{11}{12} = \frac{1}{12}$.
For $x \in (0,1) \cup (1,2)$, the density is $f(x) = F'(x) = 1/4$. $E(X) = \int_0^1 x\frac{1}{4}dx + \int_1^2 x\frac{1}{4}dx + 1 \cdot P(X = 1) + 2 \cdot P(X=2) + 3 \cdot P(X=3)$
$E(X) = \frac{1}{4}[\frac{x^2}{2}]_0^1 + \frac{1}{4}[\frac{x^2}{2}]_1^2 + 1(\frac{1}{4}) + 2(\frac{1}{6}) + 3(\frac{1}{12}) = \frac{1}{8} + \frac{3}{8} + \frac{1}{4} + \frac{1}{3} + \frac{1}{4} = \frac{4}{3}$.
$E(X^2) = \int_0^1 x^2\frac{1}{4}dx + \int_1^2 x^2\frac{1}{4}dx + 1^2(\frac{1}{4}) + 2^2(\frac{1}{6}) + 3^2(\frac{1}{12}) = \frac{1}{12} + \frac{7}{12} + \frac{1}{4} + \frac{4}{6} + \frac{9}{12} = \frac{34}{12} = \frac{17}{6}$.
$V(X) = E(X^2) - [E(X)]^2 = \frac{17}{6} - (\frac{4}{3})^2 = \frac{17}{6} - \frac{16}{9} = \frac{51-32}{18} = \frac{19}{18}$. **Median:** We need $F(M) = 0.5$. $F(1) = (1+1)/4 = 1/2$. So the median is 1.

$$\boxed{P(\frac{1}{2} < X < \frac{5}{2}) = \frac{19}{24}, P(1 < X < 3) = \frac{5}{12}, E(X) = \frac{4}{3}, V(X) = \frac{19}{18}, \text{Median} = 1}$$

2.3 Question 3

Problem: Find the probability distributions of the number of survivals (X) and deaths (Y) in a multi-stage experiment with guinea pigs.

Solution: Let $P(S) = 2/3$ and $P(D) = 1/3$. The sample space and probabilities of outcomes are:

- SS: $(2/3)^2 = 4/9$. (X=2, Y=0)
- SD...: (1S, 1D) then one more trial.
- SDS: $(2/3)(1/3)(2/3) = 4/27$. (X=2, Y=1)
- SDD: $(2/3)(1/3)(1/3) = 2/27$. (X=1, Y=2)
- DS...: (1S, 1D) then one more trial.
- DSS: $(1/3)(2/3)(2/3) = 4/27$. (X=2, Y=1)
- DSD: $(1/3)(2/3)(1/3) = 2/27$. (X=1, Y=2)
- DD...: (0S, 2D) then two more trials.
- DDSS: $(1/3)^2(2/3)^2 = 4/81$. (X=2, Y=2)
- DDSD: $(1/3)^2(2/3)(1/3) = 2/81$. (X=1, Y=3)
- DDDS: $(1/3)^2(1/3)(2/3) = 2/81$. (X=1, Y=3)
- DDDD: $(1/3)^4 = 1/81$. (X=0, Y=4)

Distribution of X (Survivors):
$P(X=2) = P(SS)+P(SDS)+P(DSS)+P(DDSS) = 4/9+4/27+4/27+4/81 = (36+12+12+4)/81 = 64/81$.
$P(X=1) = P(SDD) + P(DSD) + P(DDSD) + P(DDDS) = 2/27 + 2/27 + 2/81 + 2/81 = (6+6+2+2)/81 = 16/81$.
$P(X=0) = P(DDDD) = 1/81$.
Distribution of Y (Deaths): $P(Y=0) = P(SS) = 4/9$.
$P(Y=1) = P(SDS) + P(DSS) = 4/27 + 4/27 = 8/27$.
$P(Y=2) = P(SDD) + P(DSD) + P(DDSS) = 2/27 + 2/27 + 4/81 = (6+6+4)/81 = 16/81$.
$P(Y=3) = P(DDSD) + P(DDDS) = 2/81 + 2/81 = 4/81$.
$P(Y=4) = P(DDDD) = 1/81$.

PMF of X: $P(X=0) = 1/81, P(X=1) = 16/81, P(X=2) = 64/81$
PMF of Y: $P(Y=0) = 4/9, P(Y=1) = 8/27, P(Y=2) = 16/81, P(Y=3) = 4/81, P(Y=4) = 1/81$

2.4 Question 4

Problem: Find the PMF of the number of third-generation particles in a branching process.

Solution: Let X be the number of 2nd gen particles, $P(X = k) = 1/3$ for $k = 1, 2, 3$. Let Z_i be the number of particles from the i-th 2nd gen particle, $P(Z_i = j) = 1/3$ for $j = 1, 2, 3$. Let Y be the total number of 3rd gen particles, $Y = \sum_{i=1}^{X} Z_i$. We use the Law of Total Probability: $P(Y = j) = \sum_k P(Y = j|X = k)P(X = k)$.

- $P(Y = 1) = P(Y = 1|X = 1)\frac{1}{3} = P(Z_1 = 1)\frac{1}{3} = \frac{1}{3}\frac{1}{3} = 1/9$.
- $P(Y = 2) = [P(Z_1 = 2) + P(Z_1 + Z_2 = 2)]\frac{1}{3} = [\frac{1}{3} + \frac{1}{9}]\frac{1}{3} = 4/27$.
- $P(Y = 3) = [P(Z_1 = 3) + P(Z_1 + Z_2 = 3) + P(Z_1 + Z_2 + Z_3 = 3)]\frac{1}{3} = [\frac{1}{3} + \frac{2}{9} + \frac{1}{27}]\frac{1}{3} = 16/81$.
- $P(Y = 4) = [P(Z_1 + Z_2 = 4) + P(Z_1 + Z_2 + Z_3 = 4)]\frac{1}{3} = [\frac{3}{9} + \frac{3}{27}]\frac{1}{3} = 12/81 = 4/27$.
- $P(Y = 5) = [P(Z_1 + Z_2 = 5) + P(Z_1 + Z_2 + Z_3 = 5)]\frac{1}{3} = [\frac{2}{9} + \frac{6}{27}]\frac{1}{3} = 12/81 = 4/27$.
- $P(Y = 6) = [P(Z_1 + Z_2 = 6) + P(Z_1 + Z_2 + Z_3 = 6)]\frac{1}{3} = [\frac{1}{9} + \frac{7}{27}]\frac{1}{3} = 10/81$.
- $P(Y = 7) = [P(Z_1 + Z_2 + Z_3 = 7)]\frac{1}{3} = [\frac{6}{27}]\frac{1}{3} = 6/81 = 2/27$.
- $P(Y = 8) = [P(Z_1 + Z_2 + Z_3 = 8)]\frac{1}{3} = [\frac{3}{27}]\frac{1}{3} = 3/81 = 1/27$.
- $P(Y = 9) = [P(Z_1 + Z_2 + Z_3 = 9)]\frac{1}{3} = [\frac{1}{27}]\frac{1}{3} = 1/81$.

$$\boxed{\begin{array}{l} P(Y = 1) = 1/9, P(Y = 2) = 4/27, P(Y = 3) = 16/81, P(Y = 4) = 4/27, P(Y = 5) = 4/27, \\ P(Y = 6) = 10/81, P(Y = 7) = 2/27, P(Y = 8) = 1/27, P(Y = 9) = 1/81 \end{array}}$$

2.5 Question 5

Problem: Use Chebyshev's inequality to bound the probability of an IQ score being above 148 or below 52, given mean 100 and standard deviation 16.

Solution: Concept: Chebyshev's Inequality This inequality provides a lower bound on the probability that a random variable falls within a certain distance of its mean, regardless of the specific distribution. The form is $P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}$. Here, X is the IQ score, $\mu = 100$, and $\sigma = 16$. The event is $X < 52$ or $X > 148$. This can be written as $|X - 100| > 48$. So, we set $c = 48$. Applying Chebyshev's inequality:

$$P(|X - 100| > 48) \leq \frac{\sigma^2}{c^2} = \frac{16^2}{48^2} = \left(\frac{16}{48}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$\boxed{P(X < 52 \text{ or } X > 148) \leq \frac{1}{9}}$$

2.6 Question 6

Problem: Find the cdf, mean, variance, and median for a continuous random variable with a given piecewise PDF.

Solution: The PDF is symmetric about $x = 1.5$. **CDF:** We integrate the PDF. For $x < 0$, $F(x) = 0$.
For $0 \leq x \leq 1$, $F(x) = \int_0^x t/2dt = x^2/4$.
For $1 < x \leq 2$, $F(x) = F(1) + \int_1^x 1/2dt = 1/4 + (x - 1)/2 = (2x - 1)/4$.
For $2 < x \leq 3$, $F(x) = F(2) + \int_2^x (3 - t)/2dt = 3/4 + [3t - t^2/2]_2^x/2 = (-x^2 + 6x - 5)/4$.
For $x > 3$, $F(x) = 1$.
Mean: Due to symmetry about 1.5, $E(X) = 1.5$.
Variance: We calculate $E(X^2) = \int_0^1 \frac{x^3}{2}dx + \int_1^2 \frac{x^2}{2}dx + \int_2^3 \frac{x^2(3-x)}{2}dx$
$= \frac{1}{8} + \frac{7}{6} + \frac{11}{8} = \frac{8}{3}$.
$Var(X) = E(X^2) - [E(X)]^2 = 8/3 - (3/2)^2 = 8/3 - 9/4 = 5/12$.
Median: Due to symmetry, Median = Mean = 1.5. Check: $F(1.5) = (2(1.5) - 1)/4 = 2/4 = 0.5$.

$$\boxed{E(X) = 1.5, Var(X) = 5/12, \text{Median} = 1.5}$$

2.7 Question 7

Problem: Let X be a continuous random variable with PDF $f(x) = k/4$ for $0 < x < 1$, $x^2/4$ for $1 \leq x \leq 2$, and $(1-k)/4$ for $2 < x < 3$. Determine the values of k for which f(x) is a density function. Find the CDF, the median M, and show that $\sqrt{2} \leq M \leq \sqrt{3}$.

Solution: 1. Determine valid range for k: For $f(x)$ to be a valid PDF, two conditions must be met: (a) $f(x) \geq 0$ for all x. (b) $\int_{-\infty}^{\infty} f(x)dx = 1$.

Condition (a):

- For $0 < x < 1$, we need $k/4 \geq 0 \implies k \geq 0$.
- For $1 \leq x \leq 2$, $x^2/4$ is always non-negative.
- For $2 < x < 3$, we need $(1-k)/4 \geq 0 \implies 1-k \geq 0 \implies k \leq 1$.

Combining these, we get the constraint $0 \leq k \leq 1$.

Condition (b):

$$\int_0^1 \frac{k}{4}dx + \int_1^2 \frac{x^2}{4}dx + \int_2^3 \frac{1-k}{4}dx = 1$$

$$\frac{k}{4}[x]_0^1 + \frac{1}{4}\left[\frac{x^3}{3}\right]_1^2 + \frac{1-k}{4}[x]_2^3 = 1$$

$$\frac{k}{4}(1) + \frac{1}{4}\left(\frac{8}{3} - \frac{1}{3}\right) + \frac{1-k}{4}(1) = 1$$

$$\frac{k}{4} + \frac{7}{12} + \frac{1-k}{4} = 1$$

$$\frac{k+1-k}{4} + \frac{7}{12} = 1 \implies \frac{1}{4} + \frac{7}{12} = 1 \implies \frac{3+7}{12} = \frac{10}{12} = 1$$

This is a contradiction, as $10/12 \neq 1$. There appears to be a typo in the problem statement as given, because the integral does not sum to 1. Let's assume there is a typo in the second interval and the PDF is $f(x) = kx^2/4$ for $1 \leq x \leq 2$. Let's re-calculate:

$$\frac{k}{4} + \frac{k}{4}\left(\frac{7}{3}\right) + \frac{1-k}{4} = 1 \implies \frac{1}{4} + \frac{7k}{12} = 1 \implies \frac{7k}{12} = \frac{3}{4} \implies k = \frac{9}{7}$$

This violates $k \leq 1$. Let's adhere strictly to the problem as written in the PDF, which means no value of k makes it a valid density. However, to proceed with the structure of the question as intended, we will assume the calculation error in the provided solution key implies a different function. Based on the solution key leading to $M = \sqrt{3-k}$, the intended CDF in the second interval was likely $\frac{k+x^2-1}{4}$. This corresponds to a PDF of $f(x) = x/2$ for $1 < x \leq 2$. Let's solve with this assumption.

2. Find the CDF and Median (assuming $f(x) = x/2$ on $[1, 2]$):

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{kx}{4} & 0 < x \leq 1 \\ \frac{k}{4} + \frac{x^2-1}{4} & 1 < x \leq 2 \\ \frac{k+3}{4} + \frac{(1-k)(x-2)}{4} & 2 < x < 3 \\ 1 & x \geq 3 \end{cases}$$

The median M satisfies $F(M) = 1/2$. The value of $F(2) = (k+3)/4$. Since $0 \leq k \leq 1$, $F(2)$ ranges from 3/4 to 1. And $F(1) = k/4$ ranges from 0 to 1/4. Thus, the median must lie in the interval $(1, 2]$.

$$\frac{k+M^2-1}{4} = \frac{1}{2} \implies k + M^2 - 1 = 2 \implies M^2 = 3-k \implies M = \sqrt{3-k}$$

Since $0 \leq k \leq 1$, we can find the bounds for M: When $k = 1$, $M = \sqrt{3-1} = \sqrt{2}$. When $k = 0$, $M = \sqrt{3-0} = \sqrt{3}$. Thus, $\sqrt{2} \leq M \leq \sqrt{3}$.

$$\boxed{\text{Valid k is } [0,1], \text{Median} M = \sqrt{3-k}, \text{ which implies } \sqrt{2} \leq M \leq \sqrt{3}}$$

2.8 Question 8

Problem: Find the expected number of blood tests for a batch of 10 people, using a pooling strategy.

Solution: Concept: Pooled Testing Efficiency This is a classic problem demonstrating how pooling samples can save resources. The expectation is calculated over the two possible outcomes for the pooled sample. Let X be the number of tests for a batch of 10. $p = 0.01$ is the incidence rate.

- **Outcome 1:** The pooled test is negative. This happens if all 10 people are healthy. The probability of one person being healthy is $1 - p = 0.99$. The probability of all 10 being healthy is $(0.99)^{10}$. In this case, $X = 1$.

- **Outcome 2:** The pooled test is positive. This happens if at least one person has the disease. The probability is $1 - (0.99)^{10}$. In this case, the initial pooled test is followed by 10 individual tests, so $X = 1 + 10 = 11$.

The expected number of tests is $E(X) = 1 \cdot P(X = 1) + 11 \cdot P(X = 11)$. $P(X = 1) = (0.99)^{10} \approx 0.9044$. $P(X = 11) = 1 - (0.99)^{10} \approx 0.0956$. $E(X) = 1(0.9044) + 11(0.0956) = 0.9044 + 1.0516 = 1.956$.

$$\boxed{E(\text{tests per batch}) = 11 - 10(0.99)^{10} \approx 1.956}$$

2.9 Question 9

Problem: Show that the expected number of children in a randomly selected child's family, E(Y), is greater than or equal to the expected number of children in a randomly selected family, E(X).

Solution: Nice Trick: Jensen's Inequality / Variance Property The core of this proof lies in the fact that variance is non-negative.
$Var(U) = E(U^2) - (E(U))^2 \geq 0$.
Let U be a random variable for the number of children in a randomly chosen family.
$P(U = i) = n_i/m$, so $E(U) = \sum i \frac{n_i}{m} = E(X)$.
And $E(U^2) = \sum i^2 \frac{n_i}{m}$.
Now, consider Y, the number of children in a randomly selected child's family. The probability of selecting a child from a family of size i is proportional to $i \cdot n_i$.
$P(Y = i) = \frac{in_i}{\sum in_i} = \frac{in_i}{m \cdot E(U)}$.
$E(Y) = \sum iP(Y = i) = \sum i \frac{in_i}{mE(U)} = \frac{\sum i^2 n_i}{mE(U)} = \frac{E(U^2)}{E(U)}$.
We need to show $E(Y) \geq E(X)$, which is $\frac{E(U^2)}{E(U)} \geq E(U)$. Since $E(U) > 0$, this is equivalent to showing $E(U^2) \geq [E(U)]^2$. This is always true because $Var(U) = E(U^2) - [E(U)]^2 \geq 0$.

$$\boxed{\text{The result is proven, as it is equivalent to the property } E(U^2) \geq [E(U)]^2.}$$

2.10 Question 10

Problem: For a discrete random variable X, find the values of d for which $p(x)$ is a valid PMF, and find the value of d that minimizes Var(X).

Solution: 1. Find valid range for d: For a valid PMF, we need $p(x) \geq 0$ for all x and $\sum p(x) = 1$. The sum is $\frac{1+3d}{4} + \frac{1-d}{4} + \frac{1+2d}{4} + \frac{1-4d}{4} = \frac{4+0d}{4} = 1$. This condition is always met. Now, we enforce $p(x) \geq 0$:

- $p(1) = 1 + 3d \geq 0 \implies d \geq -1/3$.

- $p(2) = 1 - d \geq 0 \implies d \leq 1$.

- $p(3) = 1 + 2d \geq 0 \implies d \geq -1/2$.

- $p(4) = 1 - 4d \geq 0 \implies d \leq 1/4$.

Combining all these constraints, the valid range for d is $[-1/3, 1/4]$.

2. Minimize Var(X): Nice Trick: Variance as a Quadratic Variance is often a quadratic function of the parameter. We can find the minimum by finding the vertex of the parabola. First, find E(X) and E(X^2): $E(X) = 1(\frac{1+3d}{4}) + 2(\frac{1-d}{4}) + 3(\frac{1+2d}{4}) + 4(\frac{1-4d}{4}) = \frac{1+3d+2-2d+3+6d+4-16d}{4} = \frac{10-9d}{4}$. $E(X^2) = 1^2(\frac{1+3d}{4}) + 2^2(\frac{1-d}{4}) + 3^2(\frac{1+2d}{4}) + 4^2(\frac{1-4d}{4}) = \frac{1+3d+4-4d+9+18d+16-64d}{4} = \frac{30-47d}{4}$.

$Var(X) = E(X^2) - [E(X)]^2 = \frac{30-47d}{4} - \left(\frac{10-9d}{4}\right)^2 = \frac{4(30-47d)-(100-180d+81d^2)}{16} = \frac{120-188d-100+180d-81d^2}{16}$. $Var(X) = \frac{20-8d-81d^2}{16}$. This is a downward-opening parabola in d. The minimum value will occur at the boundaries of the valid interval $[-1/3, 1/4]$. Let's evaluate the variance at the endpoints: $V(d = -1/3) = \frac{20-8(-1/3)-81(-1/3)^2}{16} = \frac{20+8/3-9}{16} = \frac{11+8/3}{16} = \frac{41/3}{16} = \frac{41}{48}$. $V(d = 1/4) = \frac{20-8(1/4)-81(1/4)^2}{16} = \frac{20-2-81/16}{16} = \frac{18-5.0625}{16} = \frac{12.9375}{16}$. The vertex of the parabola $ad^2 + bd + c$ is at $d = -b/2a$. Here, $a = -81, b = -8$. Vertex at $d = -(-8)/(2 \cdot -81) = -4/81$. This is within our valid range. This point corresponds to a maximum. Therefore, the minimum must be at one of the endpoints. Comparing the values, the variance is smaller when $d = -1/3$. There seems to be an error in the provided solution key; the minimum occurs at an endpoint, not the value given. Let's recheck the calculation of $V(d = 1/4)$. $V(1/4) = (20 - 2 - 81/16)/16 = (18 - 5.0625)/16 = 12.9375/16 \approx 0.808$. $V(-1/3) = 41/48 \approx 0.854$. The minimum is indeed at $d = 1/4$.

$$\boxed{\text{Valid d is } [-\frac{1}{3}, \frac{1}{4}], \text{ Variance is minimized at } d = \frac{1}{4}}$$

2.11 Question 11

Problem: For the PMF $p(x) = k/((x + 1)(x + 2))$ for $x = 0, 1, 2, ...$, find k, the CDF, E(X) and the median.

Solution:

Nice Trick: Telescoping Series

We use partial fraction decomposition: $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$.

Find k: $\sum_{x=0}^{\infty} p(x) = 1$.

$\sum_{x=0}^{\infty} k \left(\frac{1}{x+1} - \frac{1}{x+2}\right) = k \left[(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots\right] = k(1) = 1 \implies k = 1.$

CDF: $F(x) = P(X \le x) = \sum_{i=0}^{\lfloor x \rfloor}(\frac{1}{i+1} - \frac{1}{i+2}) = 1 - \frac{1}{\lfloor x \rfloor+2}$.

E(X): $E(X) = \sum_{x=0}^{\infty} \frac{x}{(x+1)(x+2)}$. Using partial fractions: $\frac{x}{(x+1)(x+2)} = \frac{2}{x+2} - \frac{1}{x+1}$.

$E(X) = \sum_{x=0}^{\infty}(\frac{2}{x+2} - \frac{1}{x+1}) = (\frac{2}{2} - \frac{1}{1}) + (\frac{2}{3} - \frac{1}{2}) + (\frac{2}{4} - \frac{1}{3}) + \dots$.

This series diverges (related to the harmonic series). So E(X) does not exist.

Median: We need $F(M) \ge 0.5$.

$F(0) = P(X \le 0) = p(0) = \frac{1}{1 \cdot 2} = 0.5$.

Since $F(0) = 0.5$, any value M in the interval $[0, 1)$ is a median.

$$\boxed{k = 1, CDF = 1 - \frac{1}{\lfloor x \rfloor + 2}, \text{E(X) does not exist, Median} \in [0, 1)}$$

2.12 Question 12

Problem: The number of items produced in a factory during a week is a random variable with mean 50 and standard deviation 5. Using Chebyshev's inequality find the minimum probability that this week's production will be between 40 and 60?

Solution: Concept: Chebyshev's Inequality (Lower Bound for Probability) Chebyshev's inequality gives a lower bound for the probability that a random variable with finite variance will be within a certain distance of its mean. It is powerful because it makes no assumptions about the underlying distribution. The relevant form for this problem is:

$$P(|X - \mu| < c) \ge 1 - \frac{\sigma^2}{c^2}$$

where μ is the mean and σ is the standard deviation.

1. Identify Parameters and the Event: We are given:

- Mean, $\mu = 50$.
- Standard deviation, $\sigma = 5$.

The event is that the production is between 40 and 60, which we can write as an inequality:

$$40 < X < 60$$

2. Express the Event in the Form of Chebyshev's Inequality: We need to express the event in the form $|X - \mu| < c$.

$$40 - 50 < X - 50 < 60 - 50$$
$$-10 < X - 50 < 10$$
$$|X - 50| < 10$$

From this, we can see that the distance from the mean is $c = 10$.

3. Apply the Inequality: Now we substitute our values into the formula:

$$P(|X - 50| < 10) \geq 1 - \frac{5^2}{10^2}$$
$$P(40 < X < 60) \geq 1 - \frac{25}{100}$$
$$P(40 < X < 60) \geq 1 - 0.25 = 0.75$$

The minimum probability that the production will be between 40 and 60 is 0.75.

$$\boxed{P(40 < X < 60) = 0.75}$$

2.13 Question 13

Problem: Given MGF $M_X(t) = \frac{(3+2e^t)^4}{625}$, find the mean, variance, and $P(X \leq 1)$. Is it skewed?

Solution: Concept: Recognizing MGFs

Identifying the form of an MGF allows you to immediately know the distribution and its properties without calculus.

The MGF can be rewritten as $M_X(t) = \left(\frac{3}{5} + \frac{2}{5}e^t\right)^4$. This is the MGF of a Binomial distribution, $(q + pe^t)^n$, with $n = 4$ and $p = 2/5$.

So, $X \sim \text{Bin}(4, 0.4)$.

Mean and Variance: For a Binomial, $E(X) = np = 4(0.4) = 1.6$. $Var(X) = np(1-p) = 4(0.4)(0.6) = 0.96$.

P(X ≤ 1): $P(X \leq 1) = P(X = 0) + P(X = 1)$.

$P(X = 0) = \binom{4}{0}(0.4)^0(0.6)^4 = 0.1296$.

$P(X = 1) = \binom{4}{1}(0.4)^1(0.6)^3 = 4(0.4)(0.216) = 0.3456$.

$P(X \leq 1) = 0.1296 + 0.3456 = 0.4752$.

Skewness: For a binomial distribution, the skewness is determined by p. Since $p = 0.4 < 0.5$, the distribution is positively skewed (skewed to the right).

$$\boxed{E(X) = 1.6, Var(X) = 0.96, P(X \leq 1) = 0.4752, \text{Positively Skewed}}$$

Chapter 3

Assignment 3 Solutions

3.1 Question 1

Distribution Used: Hypergeometric Distribution.

Reasoning: This problem involves sampling without replacement from a finite population. The population consists of 7 judges, which is divided into two distinct groups (4 who favor Ruby, 3 who favor Mini). We are selecting a sample of 3 judges and are interested in the probability of getting a specific number of judges from the first group.

Solution: Let X be the number of judges in the panel of three who favor Ruby. Ruby wins if she gets a majority of votes, which means X must be 2 or 3. The total number of ways to choose 3 judges from a group of 7 is $\binom{7}{3}$.

The number of ways to choose 2 judges who favor Ruby (from the 4 available) and 1 judge who favors Mini (from the 3 available) is:

$$\binom{4}{2}\binom{3}{1} = \frac{4!}{2!2!} \cdot \frac{3!}{1!2!} = 6 \cdot 3 = 18$$

The number of ways to choose 3 judges who favor Ruby (from the 4 available) and 0 who favor Mini is:

$$\binom{4}{3}\binom{3}{0} = \frac{4!}{3!1!} \cdot 1 = 4$$

The total number of ways to form a panel of 3 from 7 is:

$$\binom{7}{3} = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

The probability that Ruby wins is the sum of the probabilities of these favorable outcomes:

$$P(\text{Ruby wins}) = P(X = 2) + P(X = 3) = \frac{\binom{4}{2}\binom{3}{1} + \binom{4}{3}}{\binom{7}{3}} = \frac{18 + 4}{35} = \frac{22}{35} \approx \boxed{0.6286}$$

3.2 Question 2

Distribution Used: Binomial Distribution.

Reasoning: This scenario involves a fixed number of independent trials (n bombs dropped). Each trial has two possible outcomes (hit or miss), and the probability of success (a hit) is constant for each trial ($p = 0.5$). We are interested in the number of successes.

Solution: Let n be the number of bombs that must be dropped. Let X be the number of successful hits. Given the probability of a single hit is $p = 0.5$. The target is destroyed if there are at least 2 hits. The random variable X follows a Binomial distribution, $X \sim \text{Bin}(n, 0.5)$. We want to find the smallest n such that the probability of destroying the target is at least 99%, i.e., $P(X \geq 2) \geq 0.99$.

This is equivalent to $1 - P(X < 2) \geq 0.99$, which simplifies to $P(X = 0) + P(X = 1) \leq 0.01$. Using the Binomial probability mass function $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$:

$$\binom{n}{0}(0.5)^0(0.5)^n + \binom{n}{1}(0.5)^1(0.5)^{n-1} \leq 0.01$$

$$(1)(0.5)^n + n(0.5)^n \leq 0.01$$
$$(n+1)(0.5)^n \leq 0.01$$

We can test values of n:

- For $n = 8 : (8+1)(0.5)^8 = 9 \cdot 0.00390625 = 0.035 > 0.01$
- For $n = 9 : (9+1)(0.5)^9 = 10 \cdot 0.001953125 = 0.0195 > 0.01$
- For $n = 10 : (10+1)(0.5)^{10} = 11 \cdot 0.0009765625 = 0.0107 > 0.01$
- For $n = 11 : (11+1)(0.5)^{11} = 12 \cdot 0.00048828125 = 0.00586 \leq 0.01$

The smallest integer value of n that satisfies the condition is $\boxed{n = 11}$.

3.3 Question 3

Distribution Used: Binomial Distribution.

Reasoning: We have a fixed number of independent items (3). Each item can be either defective or not (two outcomes), with a constant probability of being defective ($p = 0.1$).

Solution: Let X be the number of defective items in a sample of three. X follows a Binomial distribution with parameters $n = 3$ and $p = 0.1$, i.e., $X \sim \text{Bin}(3, 0.1)$. We want to find the probability that at most one item is defective, which is $P(X \leq 1)$.

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

Using the Binomial PMF:

$$P(X = 0) = \binom{3}{0}(0.1)^0(0.9)^3 = 1 \cdot 1 \cdot (0.9)^3 = 0.729$$
$$P(X = 1) = \binom{3}{1}(0.1)^1(0.9)^2 = 3 \cdot 0.1 \cdot 0.81 = 0.243$$
$$P(X \leq 1) = 0.729 + 0.243 = \boxed{0.972}$$

3.4 Question 4

Distribution Used: Binomial Distribution.

Reasoning: The question explicitly states that X follows a Binomial distribution. We use the properties of the mean and variance of this distribution to find its parameters.

Solution: For a Binomial distribution, the mean is $\mu = np$ and the variance is $\sigma^2 = npq$, where $q = 1 - p$. We are given: Mean $np = 8$. Standard deviation $\sqrt{npq} = 2$, which implies variance $npq = 4$.

We can solve for the parameters:

$$q = \frac{npq}{np} = \frac{4}{8} = 0.5$$

Since $p + q = 1$, we have $p = 1 - 0.5 = 0.5$. Now we find n:

$$np = 8 \implies n(0.5) = 8 \implies n = 16$$

So, $X \sim \text{Bin}(16, 0.5)$. We need to find $P(X \geq 3)$.

$$P(X \geq 3) = 1 - P(X < 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

Calculate the individual probabilities:

$$P(X = 0) = \binom{16}{0}(0.5)^0(0.5)^{16} = (\frac{1}{2})^{16}$$
$$P(X = 1) = \binom{16}{1}(0.5)^1(0.5)^{15} = 16(\frac{1}{2})^{16}$$
$$P(X = 2) = \binom{16}{2}(0.5)^2(0.5)^{14} = \frac{16 \cdot 15}{2}(\frac{1}{2})^{16} = 120(\frac{1}{2})^{16}$$
$$P(X \geq 3) = 1 - \frac{1 + 16 + 120}{2^{16}} = 1 - \frac{137}{65536} = \frac{65399}{65536} \approx \boxed{0.9979}$$

3.5 Question 5

Distribution Used: Binomial Distribution.

Reasoning: For an n-component system, each component functions independently with probability p. The number of functioning components follows a Binomial distribution.

Solution: Let P_n be the probability that an n-component system operates effectively. For a 3-component system, it operates if at least $\lceil 3/2 \rceil = 2$ components function. Let $X_3 \sim \text{Bin}(3, p)$.

$$P_3 = P(X_3 \geq 2) = P(X_3 = 2) + P(X_3 = 3) = \binom{3}{2} p^2(1-p) + \binom{3}{3} p^3 = 3p^2(1-p) + p^3 = 3p^2 - 2p^3$$

For a 5-component system, it operates if at least $\lceil 5/2 \rceil = 3$ components function. Let $X_5 \sim \text{Bin}(5, p)$.

$$\begin{aligned}
P_5 &= P(X_5 \geq 3) = P(X_5 = 3) + P(X_5 = 4) + P(X_5 = 5) \\
&= \binom{5}{3} p^3(1-p)^2 + \binom{5}{4} p^4(1-p) + \binom{5}{5} p^5 \\
&= 10p^3(1 - 2p + p^2) + 5p^4(1-p) + p^5 \\
&= 10p^3 - 20p^4 + 10p^5 + 5p^4 - 5p^5 + p^5 \\
&= 6p^5 - 15p^4 + 10p^3
\end{aligned}$$

We want to find the values of p for which $P_5 > P_3$.

$$6p^5 - 15p^4 + 10p^3 > 3p^2 - 2p^3$$

Since $p \neq 0$, we can divide by p^2 (as $p = 0$ makes both probabilities 0).

$$6p^3 - 15p^2 + 10p > 3 - 2p$$

$$6p^3 - 15p^2 + 12p - 3 > 0$$

$$2p^3 - 5p^2 + 4p - 1 > 0$$

We can factor this polynomial. Notice that $p = 1/2$ is a root: $2(1/8) - 5(1/4) + 4(1/2) - 1 = 1/4 - 5/4 + 2 - 1 = -1 + 1 = 0$. So $(p - 1/2)$ or $(2p - 1)$ is a factor.

$$(2p - 1)(p^2 - 2p + 1) > 0$$

$$(2p - 1)(p - 1)^2 > 0$$

Since $(p-1)^2 \geq 0$ for all p and we require $0 < p < 1$, the term $(p-1)^2$ is always positive. Therefore, the inequality holds if $2p - 1 > 0$, which means $\boxed{p > 1/2}$. So, for $p \in (1/2, 1)$, the 5-component system is more reliable.

3.6 Question 6

Distribution Used: Geometric Distribution.

Reasoning: The standard moment generating function (MGF) for a geometric random variable X (number of trials for the first success) with success probability p is $M_X(t) = \frac{pe^t}{1-qe^t}$, where $q = 1 - p$. The given MGF can be rearranged into this form.

Solution: The given MGF is $M_X(t) = \frac{2e^t}{7-5e^t}$. To match the standard form, we divide the numerator and denominator by 7:

$$M_X(t) = \frac{(2/7)e^t}{1 - (5/7)e^t}$$

By comparison, we see this is the MGF of a geometric distribution with success probability $p = 2/7$. The probability of failure is $q = 1 - p = 5/7$. We need to find $P(X > 7 | X > 5)$. The geometric distribution has the memoryless property, which states that for any integers $a, b > 0$:

$$P(X > a + b | X > a) = P(X > b)$$

Applying this property with $a = 5$ and $b = 2$:

$$P(X > 7|X > 5) = P(X > 5 + 2|X > 5) = P(X > 2)$$

The probability that more than k trials are needed is $P(X > k) = q^k$.

$$P(X > 2) = q^2 = (\frac{5}{7})^2 = \boxed{\frac{25}{49}}$$

3.7 Question 7

Distribution Used: Binomial Distribution (twice).

Reasoning: First, the number of defective diskettes in a single pack of 10 is modeled by a binomial distribution. Second, the number of packs returned out of 3 is also modeled by a binomial distribution, where a "success" is the event that a pack is returned.

Solution: Step 1: Find the probability that a single pack is returned. Let X be the number of defective diskettes in a pack of 10. The probability of a diskette being defective is 0.01. So, $X \sim \text{Bin}(10, 0.01)$. A pack is returned if it contains more than one defective diskette, i.e., if $X > 1$. Let p_{return} be the probability of returning a pack.

$$p_{return} = P(X > 1) = 1 - P(X \leq 1) = 1 - [P(X = 0) + P(X = 1)]$$

$$P(X = 0) = \binom{10}{0}(0.01)^0(0.99)^{10} = (0.99)^{10} \approx 0.90438$$

$$P(X = 1) = \binom{10}{1}(0.01)^1(0.99)^9 = 10(0.01)(0.99)^9 \approx 0.09135$$

$$p_{return} = 1 - (0.90438 + 0.09135) = 1 - 0.99573 = \boxed{0.00427}$$

Step 2: Find the probability that at most one of three packs is returned. Let Y be the number of packs returned out of 3. Now we have a new binomial experiment with $n = 3$ trials and success probability $p = p_{return} = 0.00427$. So, $Y \sim \text{Bin}(3, 0.00427)$. We need to find $P(Y \leq 1) = P(Y = 0) + P(Y = 1)$.

$$P(Y = 0) = \binom{3}{0}(0.00427)^0(1 - 0.00427)^3 = (0.99573)^3 \approx 0.98726$$

$$P(Y = 1) = \binom{3}{1}(0.00427)^1(1 - 0.00427)^2 = 3(0.00427)(0.99573)^2 \approx 0.01270$$

$$P(Y \leq 1) = 0.98726 + 0.01270 = \boxed{0.99996}$$

3.8 Question 8

Distribution Used: Negative Binomial Distribution.

Reasoning: This problem involves repeating independent Bernoulli trials until a fixed number of successes ($r = 3$) is achieved. The random variable X is the total number of trials required. This is the definition of a Negative Binomial distribution.

Solution: Let X be the number of repetitions required to achieve $r = 3$ successful outcomes. The probability of success in a single trial is $p = 0.8$. Thus, X follows a Negative Binomial distribution, $X \sim \text{NB}(r = 3, p = 0.8)$. The probability mass function is $P(X = k) = \binom{k-1}{r-1}p^r(1-p)^{k-r}$ for $k = r, r+1, \ldots$. We need to find the probability that at least 5 repetitions are required, i.e., $P(X \geq 5)$.

$$P(X \geq 5) = 1 - P(X < 5) = 1 - [P(X = 3) + P(X = 4)]$$

Calculate the probabilities:

$$P(X = 3) = \binom{3-1}{3-1}(0.8)^3(0.2)^{3-3} = \binom{2}{2}(0.8)^3 = 1 \cdot 0.512 = 0.512$$

$$P(X = 4) = \binom{4-1}{3-1}(0.8)^3(0.2)^{4-3} = \binom{3}{2}(0.8)^3(0.2)^1 = 3 \cdot 0.512 \cdot 0.2 = 0.3072$$

$$P(X \geq 5) = 1 - (0.512 + 0.3072) = 1 - 0.8192 = \boxed{0.1808}$$

3.9 Question 9

Distribution Used: Hypergeometric Distribution.

Reasoning: This is another example of sampling without replacement from a finite population that contains two types of items (defective and non-defective TVs).

Solution: Let X be the number of defective TV sets in the sample. The parameters for the Hypergeometric distribution are:

- Population size, $N = 20$
- Number of successes in the population (defective TVs), $K = 5$
- Sample size, $n = 4$

We want to find the probability that the sample has exactly one defective, $P(X = 1)$. The Hypergeometric probability formula is $P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$.

$$P(X = 1) = \frac{\binom{5}{1}\binom{20-5}{4-1}}{\binom{20}{4}} = \frac{\binom{5}{1}\binom{15}{3}}{\binom{20}{4}}$$

$$\binom{5}{1} = 5$$

$$\binom{15}{3} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = 5 \cdot 7 \cdot 13 = 455$$

$$\binom{20}{4} = \frac{20 \cdot 19 \cdot 18 \cdot 17}{4 \cdot 3 \cdot 2 \cdot 1} = 5 \cdot 19 \cdot 3 \cdot 17 = 4845$$

$$P(X = 1) = \frac{5 \cdot 455}{4845} = \frac{2275}{4845} = \boxed{\frac{455}{969} \approx 0.4696}$$

3.10 Question 10

Distribution Used: Poisson Distribution.

Reasoning: The number of computers sold, X, is given to follow a Poisson distribution. The profit, Y, is a transformation of this random variable X.

Solution: The number of computers the store sells in a week is $X \sim \text{Poisson}(\lambda = 2)$. The profit is $Y = 2000 \times \min(X, 10)$, since there are only 10 computers in stock. The possible values for Y are $2000j$ for $j \in \{0, 1, 2, \ldots, 10\}$. The probability mass function of X is $P(X = k) = \frac{e^{-2}2^k}{k!}$.

For $j \in \{0, 1, \ldots, 9\}$, the profit is $Y = 2000j$ if and only if the number of computers sold is $X = j$.

$$P(Y = 2000j) = P(X = j) = \frac{e^{-2}2^j}{j!} \quad \text{for } j = 0, 1, \ldots, 9$$

The profit is $Y = 20000$ if the number of potential sales is 10 or more, but the store can only sell its entire stock of 10.

$$P(Y = 20000) = P(X \geq 10) = \sum_{k=10}^{\infty} P(X = k) = 1 - \sum_{k=0}^{9} P(X = k)$$

So the probability distribution of Y is:

$$P(Y = y) = \begin{cases} \frac{e^{-2}2^{y/2000}}{(y/2000)!} & \text{if } y \in \{0, 2000, 4000, \ldots, 18000\} \\ 1 - \sum_{k=0}^{9} \frac{e^{-2}2^k}{k!} & \text{if } y = 20000 \\ 0 & \text{otherwise} \end{cases}$$

3.11 Question 11

Distribution Used: Poisson Distribution and Bayes' Theorem.

Reasoning: The number of colds follows a Poisson distribution with one of two possible parameters. We are given prior probabilities for which parameter is correct. After observing new data (no colds), we use Bayes' Theorem to update our belief about which parameter is correct.

Solution: Let B be the event that the drug is beneficial for an individual. Let B^C be the event that the drug has no effect. Let X be the number of colds the individual contracts in a year.

We are given the following information:

- Prior probabilities: $P(B) = 0.75$, $P(B^C) = 0.25$.
- Conditional distributions: $X|B \sim \text{Poisson}(\lambda = 2)$, and $X|B^C \sim \text{Poisson}(\lambda = 3)$.

The individual has no cold, which is the event $X = 0$. We want to find the probability that the drug is beneficial given this evidence, i.e., $P(B|X = 0)$.

Using Bayes' Theorem:

$$P(B|X=0) = \frac{P(X=0|B)P(B)}{P(X=0)} = \frac{P(X=0|B)P(B)}{P(X=0|B)P(B) + P(X=0|B^C)P(B^C)}$$

First, calculate the likelihoods from the Poisson PMF $P(X = k|\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$:

$$P(X=0|B) = \frac{e^{-2}2^0}{0!} = e^{-2} \approx 0.1353$$

$$P(X=0|B^C) = \frac{e^{-3}3^0}{0!} = e^{-3} \approx 0.0498$$

Now, substitute all values into Bayes' formula:

$$P(B|X=0) = \frac{(e^{-2})(0.75)}{(e^{-2})(0.75) + (e^{-3})(0.25)} = \frac{0.1353 \cdot 0.75}{0.1353 \cdot 0.75 + 0.0498 \cdot 0.25}$$

$$P(B|X=0) = \frac{0.1015}{0.1015 + 0.01245} = \frac{0.1015}{0.11395} \approx \boxed{0.8907}$$

It is highly likely (about 89% probability) that the drug is beneficial for this individual.

3.12 Question 12

Distribution Used: Poisson Distribution.

Reasoning: The Poisson distribution is used to model the number of events (defects) occurring in a fixed interval of space (or time). The average rate of defects is given for the whole area, and we can scale this rate down to find the average rate for a smaller sub-area.

Solution: Let X_{total} be the number of defects on the entire chip. The average is given as 300. We can model this with a Poisson distribution where the rate parameter for the whole chip is $\lambda_{total} = 300$. We are interested in a randomly selected area that comprises 2% of the total surface area. Assuming the defects are uniformly distributed over the chip's surface, the rate for this smaller area will be proportional to its size. Let Y be the number of defects in the 2% area. The new rate parameter is:

$$\lambda_{new} = \lambda_{total} \times 0.02 = 300 \times 0.02 = 6$$

So, $Y \sim \text{Poisson}(\lambda = 6)$. We want to find the probability that no more than 4 defects are found in this area, i.e., $P(Y \leq 4)$.

$$P(Y \leq 4) = P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3) + P(Y=4)$$

$$P(Y \leq 4) = \sum_{k=0}^{4} \frac{e^{-6}6^k}{k!} = e^{-6}\left(\frac{6^0}{0!} + \frac{6^1}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!}\right)$$

$$P(Y \leq 4) = e^{-6}\left(1 + 6 + \frac{36}{2} + \frac{216}{6} + \frac{1296}{24}\right) = e^{-6}(1 + 6 + 18 + 36 + 54) = 115e^{-6}$$

Using $e^{-6} \approx 0.00247875$:

$$P(Y \leq 4) \approx 115 \times 0.00247875 \approx \boxed{0.2851}$$

3.13 Question 13

Distribution Used: Continuous Uniform Distribution (Geometric Probability).

Reasoning: The arrival times are independent and uniformly distributed over a fixed time interval. This is a classic geometric probability problem where probabilities are found by comparing areas.

Solution: Let the time interval from 5 p.m. to 6 p.m. be represented by $[0, 60]$ minutes. Let X be the arrival time of the boy and Y be the arrival time of the girl. We have $X \sim U(0, 60)$ and $Y \sim U(0, 60)$. The joint sample space is a square in the Cartesian plane with vertices at (0,0), (60,0), (60,60), and (0,60). The area of this sample space is $A_{total} = 60 \times 60 = 3600$.

They will meet if the absolute difference between their arrival times is no more than 20 minutes.

$$|X - Y| \leq 20$$

This is equivalent to the pair of inequalities:

$$-20 \leq X - Y \leq 20 \implies Y \leq X + 20 \quad \text{and} \quad Y \geq X - 20$$

It is easier to calculate the area of the region where they do *not* meet and subtract it from the total area. They do not meet if $|X - Y| > 20$, which means:

$$Y > X + 20 \quad \text{or} \quad Y < X - 20$$

These two inequalities define two triangular regions at the corners of our sample space square.

1. The region $Y > X + 20$ is an upper-left triangle with vertices (0,20), (0,60), and (40,60). The base and height are both $60 - 20 = 40$. Area $= \frac{1}{2} \times 40 \times 40 = 800$.
2. The region $Y < X - 20$ is a lower-right triangle with vertices (20,0), (60,0), and (60,40). The base and height are both $60 - 20 = 40$. Area $= \frac{1}{2} \times 40 \times 40 = 800$.

The total area where they do not meet is $A_{no-meet} = 800 + 800 = 1600$. The area of the favorable region where they do meet is $A_{meet} = A_{total} - A_{no-meet} = 3600 - 1600 = 2000$. The probability that they will meet is the ratio of the areas:

$$P(\text{meet}) = \frac{A_{meet}}{A_{total}} = \frac{2000}{3600} = \frac{20}{36} = \boxed{\frac{5}{9}}$$

3.14 Question 14

Distribution Used: Discrete Uniform Distribution.

Reasoning: We need to identify the distribution by matching its MGF to a known form. The MGF of a discrete uniform distribution on the integers $\{1, 2, \ldots, N\}$ is $M_X(t) = \frac{e^t(e^{Nt}-1)}{N(e^t-1)}$.

Solution: The given MGF is:

$$M_X(t) = \frac{e^t(e^{10t} - 1)}{10(e^t - 1)}$$

This perfectly matches the MGF of a discrete uniform distribution on $\{1, 2, \ldots, N\}$ with $N = 10$. The variance of a discrete uniform random variable on the first N integers is given by the formula:

$$\text{Var}(X) = \frac{N^2 - 1}{12}$$

Substituting $N = 10$:

$$\text{Var}(X) = \frac{10^2 - 1}{12} = \frac{99}{12} = \frac{33}{4} = \boxed{8.25}$$

3.15 Question 15

Distribution Used: Continuous Uniform Distribution.

Reasoning: The low bid is explicitly defined as a continuous uniform random variable. The problem requires maximizing an expected value, which involves integrating the profit function against the probability density function.

Solution: Let X be the low bid for the job, with $X \sim U(\frac{3}{4}C, 2C)$. The PDF of X is $f(x) = \frac{1}{2C - \frac{3}{4}C} = \frac{1}{\frac{5}{4}C} = \frac{4}{5C}$ for $x \in [\frac{3}{4}C, 2C]$. Let b be the contractor's bid. The contractor wins the job if his bid is the lowest, which means $b < X$ (assuming X represents the lowest competing bid). The profit, $P(b, X)$, is:

$$P(b, X) = \begin{cases} b - C & \text{if } b < X \text{ (wins the job)} \\ 0 & \text{if } b \geq X \text{ (loses the job)} \end{cases}$$

We want to maximize the expected profit, $g(b) = E[P(b, X)]$. The contractor should only bid in the range where winning is possible, so $b \in [\frac{3}{4}C, 2C]$.

$$g(b) = \int_{3C/4}^{2C} P(b, x) f(x) dx = \int_{b}^{2C} (b - C) \frac{4}{5C} dx$$

$$g(b) = (b - C) \frac{4}{5C} \int_{b}^{2C} dx = (b - C) \frac{4}{5C} [x]_b^{2C} = (b - C) \frac{4}{5C} (2C - b)$$

To find the maximum, we take the derivative with respect to b and set it to zero.

$$g(b) = \frac{4}{5C}(2Cb - b^2 - 2C^2 + Cb) = \frac{4}{5C}(-b^2 + 3Cb - 2C^2)$$

$$g'(b) = \frac{d}{db}\left[\frac{4}{5C}(-b^2 + 3Cb - 2C^2)\right] = \frac{4}{5C}(-2b + 3C)$$

Set $g'(b) = 0$:

$$-2b + 3C = 0 \implies b = \frac{3}{2}C$$

The second derivative is $g''(b) = -\frac{8}{5C} < 0$, which confirms this is a maximum. The bid that maximizes expected profit is $\boxed{b = 1.5C}$.

3.16 Question 16

Distributions Used: Exponential, then Binomial.

Reasoning: The lifetime of a single bulb is modeled by an exponential distribution. The number of bulbs from a set of 10 that are still working after a certain time is a count of successes in 10 independent Bernoulli trials, which follows a Binomial distribution.

Solution: Step 1: Find the probability a single bulb is working after 100 hours. Let X be the lifetime of a bulb. X is exponentially distributed with a mean of 50 hours. Mean $= 1/\lambda = 50$, so the rate parameter is $\lambda = 1/50$. The probability that a single bulb works for more than 100 hours is $p = P(X > 100)$. For an exponential distribution, the survival function is $P(X > t) = e^{-\lambda t}$.

$$p = P(X > 100) = e^{-(1/50) \cdot 100} = e^{-2}$$

Step 2: Use the Binomial distribution for the 10 bulbs. Let Y be the number of bulbs working after 100 hours. We have 10 independent bulbs, so Y follows a Binomial distribution with $n = 10$ and success probability $p = e^{-2}$. $Y \sim \text{Bin}(10, e^{-2})$. We want the probability that at least two bulbs are working, $P(Y \geq 2)$.

$$P(Y \geq 2) = 1 - P(Y < 2) = 1 - [P(Y = 0) + P(Y = 1)]$$

Using $p = e^{-2} \approx 0.1353$ and $q = 1 - p \approx 0.8647$:

$$P(Y = 0) = \binom{10}{0} p^0 q^{10} = (1 - e^{-2})^{10} \approx (0.8647)^{10} \approx 0.2335$$

$$P(Y = 1) = \binom{10}{1} p^1 q^9 = 10(e^{-2})(1 - e^{-2})^9 \approx 10(0.1353)(0.8647)^9 \approx 0.3627$$

$$P(Y \geq 2) = 1 - (0.2335 + 0.3627) = 1 - 0.5962 = \boxed{0.4038}$$

3.17 Question 17

Distribution Used: Exponential Distribution (in a mixture model).

Reasoning: The lifetime of a bulb follows one of two different exponential distributions depending on its origin. We use the Law of Total Probability to find the overall probability of survival for a randomly selected bulb.

Solution: Let A be the event that a bulb is from plant A, and B be the event it is from plant B. Plant B produces three times as many bulbs as A. If plant A produces k bulbs, plant B produces $3k$ bulbs, for a total of $4k$. The prior probabilities are: $P(A) = \frac{k}{4k} = \frac{1}{4}$ and $P(B) = \frac{3k}{4k} = \frac{3}{4}$.

Let X be the lifetime of a bulb. For plant A: mean is 5 months, so $\lambda_A = 1/5$. The survival function is $P(X > t|A) = e^{-t/5}$. For plant B: mean is 2 months, so $\lambda_B = 1/2$. The survival function is $P(X > t|B) = e^{-t/2}$.

We want to find the probability that a bulb purchased at random will burn for at least 5 months, $P(X > 5)$. Using the Law of Total Probability:

$$P(X > 5) = P(X > 5|A)P(A) + P(X > 5|B)P(B)$$

$$P(X > 5) = (e^{-5/5}) \cdot (\frac{1}{4}) + (e^{-5/2}) \cdot (\frac{3}{4})$$

$$P(X > 5) = \frac{1}{4}e^{-1} + \frac{3}{4}e^{-2.5}$$

Using $e^{-1} \approx 0.3679$ and $e^{-2.5} \approx 0.0821$:

$$P(X > 5) \approx \frac{1}{4}(0.3679) + \frac{3}{4}(0.0821) = 0.091975 + 0.061575 = \boxed{0.15355}$$

3.18 Question 18

Distribution Used: Exponential Distribution.

Reasoning: The motherboard's lifetime is modeled by an exponential distribution. The profit is a piecewise function of this random lifetime. The expected profit is found by integrating this profit function against the lifetime's PDF.

Solution: Let X be the lifetime (time to failure) of a motherboard in years. The mean life is 2 years, so the rate parameter is $\lambda = 1/2$ per year. The PDF of X is $f(x) = \frac{1}{2}e^{-x/2}$ for $x > 0$. The guarantee period is 6 months, which is 0.5 years.

The profit, let's call it $H(X)$, is a random variable that depends on X:

$$H(x) = \begin{cases} 5000 - 2000 = 3000 & \text{if } X < 0.5 \text{ (fails within guarantee)} \\ 5000 & \text{if } X \geq 0.5 \text{ (does not fail within guarantee)} \end{cases}$$

The expected profit is $E[H(X)]$:

$$E[H(X)] = 3000 \cdot P(X < 0.5) + 5000 \cdot P(X \geq 0.5)$$

First, we find the probabilities. For an exponential distribution, the CDF is $P(X \leq t) = 1 - e^{-\lambda t}$.

$$P(X < 0.5) = 1 - e^{-(1/2)\cdot 0.5} = 1 - e^{-1/4}$$

$$P(X \geq 0.5) = 1 - P(X < 0.5) = e^{-1/4}$$

Now, calculate the expected profit:

$$E[H(X)] = 3000(1 - e^{-1/4}) + 5000(e^{-1/4}) = 3000 - 3000e^{-1/4} + 5000e^{-1/4} = 3000 + 2000e^{-1/4}$$

Using $e^{-1/4} \approx 0.7788$:

$$E[H(X)] \approx 3000 + 2000(0.7788) = 3000 + 1557.60 = \boxed{4557.60}$$

The expected profit is Rs. 4557.60.

3.19 Question 19

Distribution Used: Exponential Distribution.

Reasoning: This is a problem about competing risks. The system fails as soon as the *first* component fails. The minimum of independent exponential random variables is itself an exponential random variable. We need to find a conditional probability.

Solution: Let X_i be the lifetime of component i, with $X_i \sim \text{Exp}(\lambda_i)$. The system is a series system, so its lifetime X_{sys} is the minimum of the component lifetimes: $X_{sys} = \min(X_1, X_2, \ldots, X_n)$. The lifetime of the system, X_{sys}, is exponentially distributed with a rate parameter $\lambda_{sys} = \sum_{i=1}^{n} \lambda_i$. The probability that the system fails before time t is $P(X_{sys} \leq t) = 1 - e^{-t\sum \lambda_i}$.

We want to find the probability that the failure was caused *only* by component j, given that the system failed before time t. The event "failure caused only by component j" is the event that component j fails first, i.e., $X_j < X_i$ for all $i \neq j$. The probability that component j is the one that fails first is given by $P(X_j = \min_i X_i) = \frac{\lambda_j}{\sum_i \lambda_i}$. This result is independent of time. The question asks for $P(\text{only component j fails before t}|\text{system fails before t})$. The event "only component j fails before t" means $X_j \leq t$ and $X_i > t$ for all $i \neq j$. This interpretation seems incorrect for a series system, as the system fails once the first component fails. A better interpretation is: $P(X_j < \min_{i\neq j} X_i | X_{sys} \leq t)$. Since the event $X_j < \min_{i\neq j} X_i$ is independent of the time of failure, this conditional probability is simply $P(X_j < \min_{i\neq j} X_i)$.

$$P(\text{failure caused by j}) = \frac{\lambda_j}{\sum_{i=1}^{n} \lambda_i}$$

Note: The solution provided seems to calculate a different quantity. Let's analyze it: $\frac{(1-e^{-\lambda_j t})e^{-t\sum_{i\neq j}\lambda_i}}{1-e^{-t\sum \lambda_i}}$. The numerator represents $P(X_j \leq t \text{ and } X_i > t \text{ for } i \neq j)$. This is the probability that component j fails before t AND all other components survive past t. This is not the definition of a series system failure. However, if asked to follow the provided solution's logic, that would be the answer. Based on standard theory, the time-independent result is correct.

3.20 Question 20

Distributions Used: Exponential, then Binomial.

Reasoning: This is analogous to question 16. The lifetime of a single AC follows an exponential distribution. The number of ACs still working after 100 hours, out of a set of 5, follows a Binomial distribution.

Solution: Step 1: Find the probability a single AC is working after 100 hours. Let X be the lifetime of an AC. X is exponentially distributed with a mean of 100 hours. Mean $= 1/\lambda = 100$, so the rate parameter is $\lambda = 1/100$. The probability that a single AC works for more than 100 hours is $p = P(X > 100)$.

$$p = P(X > 100) = e^{-\lambda t} = e^{-(1/100)\cdot 100} = e^{-1}$$

Step 2: Use the Binomial distribution for the 5 ACs. Let Y be the number of ACs working after 100 hours. We have 5 independent ACs, so Y follows a Binomial distribution with $n = 5$ and success probability $p = e^{-1}$. $Y \sim \text{Bin}(5, e^{-1})$. We want the probability that at least two ACs are in working condition, $P(Y \geq 2)$.

$$P(Y \geq 2) = 1 - P(Y < 2) = 1 - [P(Y = 0) + P(Y = 1)]$$

Using $p = e^{-1} \approx 0.3679$ and $q = 1 - p \approx 0.6321$:

$$P(Y = 0) = \binom{5}{0} p^0 q^5 = (1 - e^{-1})^5 \approx (0.6321)^5 \approx 0.1003$$

$$P(Y = 1) = \binom{5}{1} p^1 q^4 = 5(e^{-1})(1 - e^{-1})^4 \approx 5(0.3679)(0.6321)^4 \approx 0.2952$$

$$P(Y \geq 2) = 1 - (0.1003 + 0.2952) = 1 - 0.3955 = \boxed{0.6045}$$

3.21 Question 21

Distribution Used: Poisson Process / Gamma Distribution.

Reasoning: The time *between* arrivals is exponential, which implies the *number* of arrivals in a given time period follows a Poisson distribution. The time *until the k-th arrival* follows a Gamma distribution. Let T_k be the time of the k-th arrival. $T_k = X_1 + X_2 + \cdots + X_k$, where X_i are the independent inter-arrival times. Since $X_i \sim \text{Exp}(\lambda)$, their sum T_k follows a Gamma distribution with shape k and rate λ.

Solution: The time between arrivals, X, is exponential with mean 10 minutes. Mean $= 1/\lambda = 10$, so the rate of arrivals is $\lambda = 1/10$ customers per minute. The time until the 3rd customer arrives, T_3, is the sum of three independent exponential random variables, each with rate $\lambda = 1/10$. Thus, T_3 follows a Gamma distribution with shape parameter $r = 3$ and rate parameter $\lambda = 1/10$. The PDF of T_3 is $f(t) = \frac{\lambda^r t^{r-1} e^{-\lambda t}}{\Gamma(r)} = \frac{(1/10)^3 t^2 e^{-t/10}}{2!}$ for $t > 0$. We want to find the probability that the third customer arrives within 15 minutes, $P(T_3 \leq 15)$.

$$P(T_3 \leq 15) = \int_0^{15} \frac{1}{2000} t^2 e^{-t/10} dt$$

An alternative approach uses the relationship between the Gamma CDF and the Poisson PMF: $P(T_r \leq t) = P(N_t \geq r)$, where N_t is the number of arrivals in time t. Here, $N_t \sim \text{Poisson}(\lambda t)$. In our case, $t = 15$ minutes. The parameter for the Poisson distribution is $\mu = \lambda t = (1/10) \cdot 15 = 1.5$. So we need to calculate $P(N_{15} \geq 3)$, where $N_{15} \sim \text{Poisson}(1.5)$.

$$P(N_{15} \geq 3) = 1 - P(N_{15} < 3) = 1 - [P(N_{15} = 0) + P(N_{15} = 1) + P(N_{15} = 2)]$$

$$P(N_{15} = 0) = \frac{e^{-1.5}(1.5)^0}{0!} = e^{-1.5} \approx 0.2231$$

$$P(N_{15} = 1) = \frac{e^{-1.5}(1.5)^1}{1!} = 1.5e^{-1.5} \approx 0.3347$$

$$P(N_{15} = 2) = \frac{e^{-1.5}(1.5)^2}{2!} = 1.125e^{-1.5} \approx 0.2510$$

$$P(N_{15} \geq 3) = 1 - (0.2231 + 0.3347 + 0.2510) = 1 - 0.8088 = \boxed{0.1912}$$

Note: The solution provided gives 0.7769, which is $1 - e^{-1.5}$. This would be the probability of the FIRST customer arriving in 15 minutes, not the third. There seems to be an error in the provided solution key.

3.22 Question 22

Distribution Used: Gamma Distribution.

Reasoning: The question explicitly states that the lead time follows a Gamma distribution. We use the given mean and variance to determine the distribution's parameters.

Solution: Let X be the lead time. X follows a Gamma distribution with shape parameter r and rate parameter λ. The mean is $E[X] = r/\lambda$ and the variance is $\text{Var}(X) = r/\lambda^2$. We are given: $E[X] = 20$ $\text{Var}(X) = 100$ (since standard deviation is 10). From these, we can find the parameters:

$$\lambda = \frac{E[X]}{\text{Var}(X)} = \frac{20}{100} = \frac{1}{5}$$

$$r = \lambda \cdot E[X] = \frac{1}{5} \cdot 20 = 4$$

So, $X \sim \text{Gamma}(r = 4, \lambda = 1/5)$. We want to find the probability of receiving an order within 15 days, $P(X \leq 15)$. We use the relationship $P(X \leq t) = P(N_t \geq r)$, where $N_t \sim \text{Poisson}(\lambda t)$. Here $t = 15$, so the Poisson parameter is $\mu = \lambda t = (1/5) \cdot 15 = 3$. We need to calculate $P(N_{15} \geq 4)$, where $N_{15} \sim \text{Poisson}(3)$.

$$P(N_{15} \geq 4) = 1 - [P(N_{15} = 0) + P(N_{15} = 1) + P(N_{15} = 2) + P(N_{15} = 3)]$$

$$P(N_{15} \geq 4) = 1 - e^{-3}\left(\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right)$$

$$P(N_{15} \geq 4) = 1 - e^{-3}\left(1 + 3 + 4.5 + 4.5\right) = 1 - 13e^{-3}$$

Using $e^{-3} \approx 0.049787$:

$$P(X \leq 15) \approx 1 - 13(0.049787) = 1 - 0.64723 = \boxed{0.35277}$$

3.23 Question 23

Distribution Used: Gamma Distribution and Bayes' Theorem.

Reasoning: The equipment's life follows one of two Gamma distributions depending on the manufacturer. We use the mean and variance to find the parameters for each. Then, given that a unit has survived for 12 years, we use Bayes' Theorem to find the posterior probability that it came from manufacturer A.

Solution: Let A be the event the equipment is from manufacturer A, and B from B. Priors: $P(A) = 0.75$, $P(B) = 0.25$. Let X be the lifetime. For manufacturer A: $E[X] = r_A/\lambda_A = 4$, $\text{Var}(X) = r_A/\lambda_A^2 = 8$. $\lambda_A = 4/8 = 1/2$. $r_A = 4 \cdot \lambda_A = 2$. So, $X|A \sim \text{Gamma}(r = 2, \lambda = 1/2)$.

For manufacturer B: $E[X] = r_B/\lambda_B = 2$, $\text{Var}(X) = r_B/\lambda_B^2 = 4$. $\lambda_B = 2/4 = 1/2$. $r_B = 2 \cdot \lambda_B = 1$. So, $X|B \sim \text{Gamma}(r = 1, \lambda = 1/2)$, which is an Exponential distribution.

We are given the event E that a unit is working after 12 years, i.e., $X > 12$. We want to find $P(A|X > 12)$. Using Bayes' Theorem:

$$P(A|X > 12) = \frac{P(X > 12|A)P(A)}{P(X > 12|A)P(A) + P(X > 12|B)P(B)}$$

We need to calculate the survival probabilities $P(X > 12)$ for each case. For a Gamma distribution, $P(X > t) = \sum_{k=0}^{r-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$. $P(X > 12|A)$: Here $r = 2, \lambda = 1/2, t = 12$. $\lambda t = 6$.

$$P(X > 12|A) = e^{-6}\left(\frac{6^0}{0!} + \frac{6^1}{1!}\right) = e^{-6}(1 + 6) = 7e^{-6}$$

$P(X > 12|B)$: Here $r = 1, \lambda = 1/2, t = 12$. $\lambda t = 6$.

$$P(X > 12|B) = e^{-6}\left(\frac{6^0}{0!}\right) = e^{-6}$$

Now substitute into Bayes' formula:

$$P(A|X > 12) = \frac{(7e^{-6})(0.75)}{(7e^{-6})(0.75) + (e^{-6})(0.25)} = \frac{7 \cdot 0.75}{7 \cdot 0.75 + 0.25} = \frac{5.25}{5.25 + 0.25} = \frac{5.25}{5.5} = \frac{21}{22} \approx \boxed{0.9545}$$

Note: The provided solution seems to have made an error in the final calculation, resulting in $2e^{-6}$. The Bayesian probability should be a value between 0 and 1.

3.24 Question 24

Distributions Used: Exponential, Gamma, and Uniform (Mixture Model).

Reasoning: The printing time follows one of three different distributions based on which printer is selected. We use the Law of Total Probability to find the overall probability that a print job takes less than one minute.

Solution: Let I, II, III be the events that Printer I, II, or III is used. Priors: $P(I) = 0.3, P(II) = 0.3, P(III) = 0.4$. Let X be the printing time. We want to find $P(X < 1)$. Using the Law of Total Probability:

$$P(X < 1) = P(X < 1|I)P(I) + P(X < 1|II)P(II) + P(X < 1|III)P(III)$$

Printer I: $X|I \sim \text{Exp}(\text{mean} = 3)$. So $\lambda_I = 1/3$.

$$P(X < 1|I) = 1 - e^{-(1/3) \cdot 1} = 1 - e^{-1/3} \approx 1 - 0.7165 = 0.2835$$

Printer II: $X|II \sim \text{Gamma}(\text{mean} = 2, \text{variance} = 2)$. $E[X] = r/\lambda = 2$. $\text{Var}(X) = r/\lambda^2 = 2$. $\lambda = 2/2 = 1$. $r = 2 \cdot \lambda = 2$. So $X|II \sim \text{Gamma}(r = 2, \lambda = 1)$. $P(X < 1|II) = P(N_1 \geq 2)$ where $N_1 \sim \text{Poisson}(\lambda t = 1 \cdot 1 = 1)$. $P(N_1 \geq 2) = 1 - [P(N_1 = 0) + P(N_1 = 1)] = 1 - e^{-1}(\frac{1^0}{0!} + \frac{1^1}{1!}) = 1 - 2e^{-1} \approx 1 - 2(0.3679) = 1 - 0.7358 = 0.2642$. **Printer III:** $X|III \sim U(0, 4)$. The PDF is $f(x) = 1/4$ for $x \in [0, 4]$.

$$P(X < 1|III) = \int_0^1 \frac{1}{4} dx = \frac{1}{4} = 0.25$$

Now, combine everything:

$$P(X < 1) = (0.2835)(0.3) + (0.2642)(0.3) + (0.25)(0.4)$$

$$P(X < 1) = 0.08505 + 0.07926 + 0.10 = \boxed{0.26431}$$

3.25 Question 25

Distribution Used: Weibull Distribution.

Reasoning: The lifetime is explicitly modeled by a Weibull distribution. We are given information about a conditional probability of failure, which allows us to solve for the unknown parameter α.

Solution: Let X be the lifetime of a component. $X \sim \text{Weibull}(\alpha, \beta)$ with $\beta = 2$. The survival function for a Weibull distribution is $R(t) = P(X > t) = e^{-(\alpha t)^\beta}$. With $\beta = 2$, this is $R(t) = e^{-\alpha^2 t^2}$. Let's rename α^2 to just α for simplicity as in the solution hint, so $R(t) = e^{-\alpha t^2}$.

We are given that 15% of components that have lasted 90 hours fail before 100 hours. This is a conditional probability:

$$P(X < 100|X > 90) = 0.15$$

Using the formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$:

$$P(X < 100|X > 90) = \frac{P(90 < X < 100)}{P(X > 90)} = \frac{P(X > 90) - P(X > 100)}{P(X > 90)} = 1 - \frac{P(X > 100)}{P(X > 90)}$$

$$0.15 = 1 - \frac{e^{-\alpha(100)^2}}{e^{-\alpha(90)^2}} = 1 - e^{-\alpha(10000-8100)} = 1 - e^{-1900\alpha}$$

Now, we solve for α:

$$e^{-1900\alpha} = 1 - 0.15 = 0.85$$

$$-1900\alpha = \ln(0.85) \approx -0.1625$$

$$\alpha = \frac{-0.1625}{-1900} \approx 0.0000855$$

Now we need to determine the probability that a component is working after 80 hours, which is $P(X > 80)$.

$$P(X > 80) = R(80) = e^{-\alpha(80)^2} = e^{-0.0000855 \cdot 6400}$$

$$P(X > 80) = e^{-0.5472} \approx \boxed{0.5786}$$

Chapter 4

Assignment 4 Solutions

4.1 Question 1

Distribution/Concept Used: Survival Analysis (Hazard Rate).
Reasoning: The problem provides a mortality rate, which is a hazard rate function $Z(t)$. The probability of survival is described by the survival function $R(t)$, which can be derived from the hazard rate. Conditional survival probability is then calculated from the survival function.

Solution: The survival function $R(t) = P(\text{survival time} > t)$ is related to the hazard rate $Z(t)$ by:

$$R(t) = \exp\left(-\int_0^t Z(u)du\right)$$

Given $Z(t) = 0.5 + 2t$, we first compute the integral:

$$\int_0^t (0.5 + 2u)du = \left[0.5u + u^2\right]_0^t = 0.5t + t^2$$

So, the survival function is $R(t) = e^{-(0.5t+t^2)}$. We need to find the probability that the child will survive to age 2, given that they have survived to age 1. This is the conditional probability $P(T > 2|T > 1)$.

$$P(T > 2|T > 1) = \frac{P(T > 2 \cap T > 1)}{P(T > 1)} = \frac{P(T > 2)}{P(T > 1)} = \frac{R(2)}{R(1)}$$

Now, we calculate $R(1)$ and $R(2)$:

$$R(1) = e^{-(0.5(1)+1^2)} = e^{-1.5}$$

$$R(2) = e^{-(0.5(2)+2^2)} = e^{-(1+4)} = e^{-5}$$

The conditional probability is:

$$\frac{R(2)}{R(1)} = \frac{e^{-5}}{e^{-1.5}} = e^{-5-(-1.5)} = e^{-3.5}$$

$$\boxed{P(T > 2|T > 1) = e^{-3.5} \approx 0.0302}$$

4.2 Question 2

Solution: Let X be the marks, with $X \sim N(\mu, \sigma^2)$. Let $Z = (X - \mu)/\sigma \sim N(0, 1)$. We are given: 1. $P(X < 45) = 0.10 \implies P(Z < \frac{45-\mu}{\sigma}) = 0.10$. From the Z-table, $\Phi(z) = 0.10$ gives $z \approx -1.28$.

$$\frac{45 - \mu}{\sigma} = -1.28 \quad (1)$$

2. $P(X > 75) = 0.05 \implies P(Z > \frac{75-\mu}{\sigma}) = 0.05 \implies P(Z \leq \frac{75-\mu}{\sigma}) = 0.95$. From the Z-table, $\Phi(z) = 0.95$ gives $z \approx 1.645$.

$$\frac{75 - \mu}{\sigma} = 1.645 \quad (2)$$

We have a system of two linear equations: $45 - \mu = -1.28\sigma$ $75 - \mu = 1.645\sigma$ Subtracting the first from the second: $(75 - \mu) - (45 - \mu) = 1.645\sigma - (-1.28\sigma) \implies 30 = 2.925\sigma \implies \sigma = \frac{30}{2.925} \approx 10.256$. Substituting σ back into (1): $45 - \mu = -1.28(10.256) \implies 45 - \mu = -13.128 \implies \mu = 58.128$.

Now we find the percentages for first and second class. **First Class (60% to 75%):** $P(60 < X \leq 75) = P(\frac{60-58.128}{10.256} < Z \leq \frac{75-58.128}{10.256}) = P(0.18 < Z \leq 1.645)$ $P(0.18 < Z \leq 1.645) = \Phi(1.645) - \Phi(0.18) = 0.95 - 0.5714 = 0.3786$. **Second Class (45% to 60%):** $P(45 < X \leq 60) = P(\frac{45-58.128}{10.256} < Z \leq \frac{60-58.128}{10.256}) = P(-1.28 < Z \leq 0.18)$ $P(-1.28 < Z \leq 0.18) = \Phi(0.18) - \Phi(-1.28) = 0.5714 - 0.10 = 0.4714$.

$$\boxed{\text{First Class: 37.86\%, Second Class: 47.14\%}}$$

4.3 Question 3

Solution: Let X be the diameter. We are given $X \sim N(\mu = 3, \sigma^2 = 0.005^2)$. The specifications are 3.0 ± 0.01 cm, so the acceptable range is $(2.99, 3.01)$. A ball is scrapped if its diameter is outside this range, i.e., $X < 2.99$ or $X > 3.01$. First, find the probability that a ball is *not* scrapped:

$$P(2.99 < X < 3.01) = P\left(\frac{2.99 - 3}{0.005} < Z < \frac{3.01 - 3}{0.005}\right)$$

$$= P\left(\frac{-0.01}{0.005} < Z < \frac{0.01}{0.005}\right) = P(-2 < Z < 2)$$

$$= \Phi(2) - \Phi(-2) = \Phi(2) - (1 - \Phi(2)) = 2\Phi(2) - 1$$

From the Z-table, $\Phi(2) \approx 0.97725$.

$$P(\text{not scrapped}) = 2(0.97725) - 1 = 1.9545 - 1 = 0.9545$$

The probability that a ball is scrapped is $1 - P(\text{not scrapped})$.

$$P(\text{scrapped}) = 1 - 0.9545 = 0.0455$$

On average, 4.55% of manufactured balls will be scrapped.

$$\boxed{\text{4.55\% of balls will be scrapped}}$$

4.4 Question 4

Solution: Let X be the width, with $X \sim N(\mu = 0.9, \sigma^2 = 0.003^2)$. Specification limits are 0.9 ± 0.005, i.e., $(0.895, 0.905)$. **Part 1: Percentage of defective forgings** A forging is defective if $X < 0.895$ or $X > 0.905$.

$$P(\text{defective}) = 1 - P(0.895 < X < 0.905)$$

$$= 1 - P\left(\frac{0.895 - 0.9}{0.003} < Z < \frac{0.905 - 0.9}{0.003}\right) = 1 - P(-1.67 < Z < 1.67)$$

$$= 1 - (2\Phi(1.67) - 1) = 2 - 2\Phi(1.67) = 2(1 - \Phi(1.67))$$

From Z-table, $\Phi(1.67) \approx 0.9525$.

$$P(\text{defective}) = 2(1 - 0.9525) = 2(0.0475) = 0.095$$

The percentage of defective forgings is 9.5%.

Part 2: Maximum allowable value of σ We need no more than 1 defective in 100, so $P(\text{defective}) \leq 0.01$. This means $P(\text{acceptable}) = P(0.895 < X < 0.905) \geq 0.99$.

$$P\left(\frac{0.895 - 0.9}{\sigma} < Z < \frac{0.905 - 0.9}{\sigma}\right) \geq 0.99$$

$$P\left(-\frac{0.005}{\sigma} < Z < \frac{0.005}{\sigma}\right) \geq 0.99$$

This implies $\Phi\left(\frac{0.005}{\sigma}\right) - \Phi\left(-\frac{0.005}{\sigma}\right) \geq 0.99$, which means $2\Phi\left(\frac{0.005}{\sigma}\right) - 1 \geq 0.99$.

$$\Phi\left(\frac{0.005}{\sigma}\right) \geq \frac{1.99}{2} = 0.995$$

From the Z-table, we find the z-value corresponding to a cumulative probability of 0.995, which is $z \approx 2.575$.

$$\frac{0.005}{\sigma} \geq 2.575 \implies \sigma \leq \frac{0.005}{2.575} \approx 0.00194$$

$$\boxed{\text{Percentage defective: 9.5\%. Maximum allowable } \sigma: 0.00194}$$

4.5 Question 5

Solution: Let X be the height in cm. $X \sim N(\mu = 200, \sigma^2 = 10^2)$. **Part 1: Greatest height jumped with probability 0.95** We want to find a height h_1 such that $P(X \leq h_1) = 0.95$.

$$P\left(Z \leq \frac{h_1 - 200}{10}\right) = 0.95$$

From the Z-table, the z-value for a 0.95 cumulative probability is $z \approx 1.645$.

$$\frac{h_1 - 200}{10} = 1.645 \implies h_1 - 200 = 16.45 \implies h_1 = 216.45 \text{ cm}$$

Part 2: Height cleared only 10% of the time We want to find a height h_2 such that $P(X > h_2) = 0.10$. This is equivalent to $P(X \leq h_2) = 0.90$.

$$P\left(Z \leq \frac{h_2 - 200}{10}\right) = 0.90$$

From the Z-table, the z-value for a 0.90 cumulative probability is $z \approx 1.28$.

$$\frac{h_2 - 200}{10} = 1.28 \implies h_2 - 200 = 12.8 \implies h_2 = 212.8 \text{ cm}$$

$$\boxed{\text{Greatest height (0.95 prob): 216.45 cm. Height cleared 10\% of time: 212.8 cm}}$$

4.6 Question 6

Solution: Let X be the marks. $X \sim N(\mu = 74, \sigma^2 = 62.41)$. The standard deviation is $\sigma = \sqrt{62.41} = 7.9$. a) **Lowest passing grade (lowest 10% fail):** Find grade g_a such that $P(X \leq g_a) = 0.10$. $P(Z \leq \frac{g_a - 74}{7.9}) = 0.10$. The z-score is $z \approx -1.28$. $\frac{g_a - 74}{7.9} = -1.28 \implies g_a = 74 - 1.28(7.9) = 74 - 10.112 = 63.888$.

b) **Highest B (top 5% get A's):** Find grade g_b such that $P(X > g_b) = 0.05$. This is the minimum score for an A. The highest B is just below this. $P(Z > \frac{g_b - 74}{7.9}) = 0.05 \implies P(Z \leq \frac{g_b - 74}{7.9}) = 0.95$. The z-score is $z \approx 1.645$. $\frac{g_b - 74}{7.9} = 1.645 \implies g_b = 74 + 1.645(7.9) = 74 + 12.9955 = 86.9955$.

c) **Lowest B (top 10% A's, next 25% B's):** The B range starts where the top 35% of students begin. Find g_c such that $P(X > g_c) = 0.35$. $P(Z > \frac{g_c - 74}{7.9}) = 0.35 \implies P(Z \leq \frac{g_c - 74}{7.9}) = 0.65$. The z-score is $z \approx 0.385$. $\frac{g_c - 74}{7.9} = 0.385 \implies g_c = 74 + 0.385(7.9) = 74 + 3.0415 = 77.0415$.

$$\boxed{\text{a) 63.89 \quad b) 87.00 \quad c) 77.04}}$$

4.7 Question 7

Solution: Let $X \sim N(\mu, \sigma^2 = 1)$. The profit function $G(X)$ is:

$$G(X) = \begin{cases} C_0 & 6 \leq X \leq 8 \\ -C_1 & X < 6 \\ -C_2 & X > 8 \end{cases}$$

The expected profit $E[G(X)]$ is:

$$E[G(X)] = C_0P(6 \le X \le 8) - C_1P(X < 6) - C_2P(X > 8)$$

In terms of the standard normal CDF Φ:

$$E(\mu) = C_0[\Phi(8-\mu) - \Phi(6-\mu)] - C_1\Phi(6-\mu) - C_2[1 - \Phi(8-\mu)]$$

$$E(\mu) = (C_0 + C_2)\Phi(8-\mu) - (C_0 + C_1)\Phi(6-\mu) - C_2$$

To maximize, we take the derivative with respect to μ and set to zero. Let $\phi(z)$ be the standard normal PDF. $\frac{d}{d\mu}\Phi(z-\mu) = -\phi(z-\mu)$.

$$\frac{dE}{d\mu} = -(C_0 + C_2)\phi(8-\mu) + (C_0 + C_1)\phi(6-\mu) = 0$$

$$(C_0 + C_1)\phi(6-\mu) = (C_0 + C_2)\phi(8-\mu)$$

$$(C_0 + C_1)\frac{1}{\sqrt{2\pi}}e^{-(6-\mu)^2/2} = (C_0 + C_2)\frac{1}{\sqrt{2\pi}}e^{-(8-\mu)^2/2}$$

Taking the natural logarithm of both sides:

$$\ln(C_0 + C_1) - \frac{(6-\mu)^2}{2} = \ln(C_0 + C_2) - \frac{(8-\mu)^2}{2}$$

$$2\ln\left(\frac{C_0 + C_1}{C_0 + C_2}\right) = (6-\mu)^2 - (8-\mu)^2 = (36 - 12\mu + \mu^2) - (64 - 16\mu + \mu^2) = -28 + 4\mu$$

$$4\mu = 28 + 2\ln\left(\frac{C_0 + C_1}{C_0 + C_2}\right) \implies \mu = 7 + \frac{1}{2}\ln\left(\frac{C_0 + C_1}{C_0 + C_2}\right)$$

$$\boxed{\mu = 7 + \frac{1}{2}\ln\left(\frac{C_1 + C_0}{C_2 + C_0}\right)}$$

4.8 Question 8

Distributions Used: Normal, then Binomial. **Reasoning:** First, we use the normal distribution to find the probability (p) that a single randomly selected candidate meets the IQ criteria. Then, since we are selecting four candidates independently, the number of candidates who meet the criteria follows a binomial distribution.

Solution: Let X be the IQ level, $X \sim N(\mu = 90, \sigma^2 = 5^2)$. **Part 1: Probability for one candidate** Find $p = P(85 < X < 95)$.

$$p = P\left(\frac{85-90}{5} < Z < \frac{95-90}{5}\right) = P(-1 < Z < 1) = 2\Phi(1) - 1$$

Using $\Phi(1) \approx 0.8413$, we get $p = 2(0.8413) - 1 = 0.6826$.

Part 2: Probability for four candidates Let Y be the number of candidates (out of 4) with IQ between 85 and 95. $Y \sim \text{Bin}(n = 4, p = 0.6826)$. We want to find $P(Y \ge 2)$.

$$P(Y \ge 2) = 1 - P(Y < 2) = 1 - [P(Y = 0) + P(Y = 1)]$$

Let $q = 1 - p = 0.3174$.

$$P(Y = 0) = \binom{4}{0}p^0q^4 = (0.3174)^4 \approx 0.0101$$

$$P(Y = 1) = \binom{4}{1}p^1q^3 = 4(0.6826)(0.3174)^3 \approx 0.0872$$

$$P(Y \ge 2) = 1 - (0.0101 + 0.0872) = 1 - 0.0973 = 0.9027$$

$$\boxed{P(Y \ge 2) \approx 0.9027}$$

4.9 Question 9

Distributions Used: Binomial and Normal Approximation. **Reasoning:** First, we calculate the probability (p) that a single machine works for more than two years by integrating its PDF. The number of machines working for more than two years (Y) follows a Binomial distribution. Since the number of trials ($n = 200$) is large, we can use the normal approximation to the binomial distribution to calculate the final probability.

Solution: Let X be the time to failure. PDF is $f(x) = 2/x^3$ for $x > 1$. **Step 1: Find p** $p = P(X > 2) = \int_2^\infty \frac{2}{x^3} dx = 2\left[-\frac{1}{2x^2}\right]_2^\infty = -\left[\frac{1}{x^2}\right]_2^\infty = -(0 - \frac{1}{4}) = \frac{1}{4} = 0.25$.

Step 2: Binomial to Normal Approximation Let Y be the number of machines working for more than 2 years. $Y \sim \text{Bin}(n = 200, p = 0.25)$. Mean of Y: $\mu = np = 200(0.25) = 50$. Variance of Y: $\sigma^2 = np(1-p) = 200(0.25)(0.75) = 37.5$. Standard deviation: $\sigma = \sqrt{37.5} \approx 6.124$. We use normal approximation $Y \approx N(50, 37.5)$ to find $P(Y \geq 60)$. We apply continuity correction.

$$P(Y \geq 60) \approx P(Y_{norm} > 59.5) = P\left(Z > \frac{59.5 - 50}{6.124}\right) = P(Z > 1.55)$$

$$= 1 - \Phi(1.55) = 1 - 0.9394 = 0.0606$$

$$\boxed{P(Y \geq 60) \approx 0.0606}$$

4.10 Question 10

Distribution Used: Binomial and Normal Approximation. **Reasoning:** The number of students who cannot solve the question is a binomial random variable. Since the number of students ($n = 200$) is large, we can use the normal approximation with continuity correction.

Solution: Let X be the number of students who *cannot* solve the question. The probability of a student not solving it is $p = 0.5$. So, $X \sim \text{Bin}(n = 200, p = 0.5)$. Mean: $\mu = np = 200(0.5) = 100$. Variance: $\sigma^2 = np(1-p) = 200(0.5)(0.5) = 50$. Standard deviation: $\sigma = \sqrt{50} \approx 7.071$. We need to find $P(X \geq 110)$. Using normal approximation with continuity correction:

$$P(X \geq 110) \approx P(X_{norm} > 109.5) = P\left(Z > \frac{109.5 - 100}{7.071}\right) = P(Z > 1.34)$$

$$= 1 - \Phi(1.34) = 1 - 0.9099 = 0.0901$$

$$\boxed{P(X \geq 110) \approx 0.0901}$$

4.11 Question 11

Distribution Used: Poisson and Normal Approximation. **Reasoning:** The number of arrivals in a fixed time interval in a Poisson process follows a Poisson distribution. Since the rate parameter (λ) is large, we can approximate this Poisson distribution with a normal distribution, using continuity correction.

Solution: The rate of arrivals is 1 per 3 minutes, so $\lambda_{min} = 1/3$ trains/minute. The time interval is from 2:00 p.m. to 3:00 p.m., which is 60 minutes. The rate for this interval is $\lambda = 60 \times (1/3) = 20$. Let X be the number of trains in this hour. $X \sim \text{Poisson}(\lambda = 20)$. Since $\lambda = 20$ is large, we can use a normal approximation. For a Poisson, $\mu = \lambda = 20$ and $\sigma^2 = \lambda = 20$. So, $\sigma = \sqrt{20} \approx 4.472$. We need to find $P(17 \leq X \leq 25)$. Applying continuity correction:

$$P(17 \leq X \leq 25) \approx P(16.5 < X_{norm} < 25.5)$$

$$= P\left(\frac{16.5 - 20}{4.472} < Z < \frac{25.5 - 20}{4.472}\right) = P(-0.78 < Z < 1.23)$$

$$= \Phi(1.23) - \Phi(-0.78) = 0.8907 - (1 - \Phi(0.78)) = 0.8907 - (1 - 0.7823) = 0.8907 - 0.2177 = 0.673$$

$$\boxed{P(17 \leq X \leq 25) \approx 0.673}$$

4.12 Question 12

Distribution Used: Log-normal Distribution. **Reasoning:** A variable Y is log-normally distributed if $\ln(Y)$ is normally distributed. We use this property to transform the problem into a standard normal distribution problem.

Solution: Given Y has a log-normal distribution with parameters $\mu = 0.8$ and $\sigma = 0.1$. This means $X = \ln(Y)$ is normally distributed with mean $\mu_X = 0.8$ and standard deviation $\sigma_X = 0.1$. So, $X \sim N(0.8, 0.1^2)$. **Part 1: Probability** We want to find $P(Y > 2.7)$.

$$P(Y > 2.7) = P(\ln(Y) > \ln(2.7)) = P(X > 0.99326)$$

Standardize this:

$$P\left(Z > \frac{0.99326 - 0.8}{0.1}\right) = P(Z > 1.9326) \approx P(Z > 1.93)$$
$$= 1 - \Phi(1.93) = 1 - 0.9732 = 0.0268$$

Part 2: Interval We want to find values y_1, y_2 such that $P(y_1 < Y < y_2) = 0.95$. This is equivalent to $P(\ln(y_1) < \ln(Y) < \ln(y_2)) = 0.95$, or $P(\ln(y_1) < X < \ln(y_2)) = 0.95$. For a normal distribution, the symmetric interval containing 95% of the probability is $(\mu_X - 1.96\sigma_X, \mu_X + 1.96\sigma_X)$. Lower bound for X: $0.8 - 1.96(0.1) = 0.8 - 0.196 = 0.604$. Upper bound for X: $0.8 + 1.96(0.1) = 0.8 + 0.196 = 0.996$. So, $\ln(y_1) = 0.604 \implies y_1 = e^{0.604} \approx 1.8294$. And, $\ln(y_2) = 0.996 \implies y_2 = e^{0.996} \approx 2.7074$.

$$\boxed{P(Y > 2.7) \approx 0.0268. \quad \text{95\% interval for Y is (1.829, 2.707)}}$$

4.13 Question 13

Distribution Used: Beta Distribution. **Reasoning:** We are given the mean and variance of a Beta distributed random variable. We can use the formulas for the mean and variance to solve for the distribution's parameters, α and β. Once the PDF is fully specified, we can integrate it to find the required probability.

Solution: Let $X \sim \text{Beta}(\alpha, \beta)$. The PDF is $f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$. Mean: $E[X] = \frac{\alpha}{\alpha+\beta} = \frac{2}{3}$. Variance: $\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{18}$. From the mean, $3\alpha = 2(\alpha + \beta) \implies \alpha = 2\beta$. Substitute this into the variance formula:

$$\frac{(2\beta)\beta}{(2\beta+\beta)^2(2\beta+\beta+1)} = \frac{2\beta^2}{(3\beta)^2(3\beta+1)} = \frac{2\beta^2}{9\beta^2(3\beta+1)} = \frac{2}{9(3\beta+1)} = \frac{1}{18}$$
$$36 = 9(3\beta+1) \implies 4 = 3\beta + 1 \implies 3\beta = 3 \implies \beta = 1$$

Since $\alpha = 2\beta$, we have $\alpha = 2$. So, $X \sim \text{Beta}(2, 1)$. The PDF is $f(x) = \frac{1}{B(2,1)} x^{2-1}(1-x)^{1-1} = \frac{x}{1/2} = 2x$ for $0 < x < 1$. We need to find $P(0.2 < X < 0.5)$.

$$P(0.2 < X < 0.5) = \int_{0.2}^{0.5} 2x dx = [x^2]_{0.2}^{0.5} = (0.5)^2 - (0.2)^2 = 0.25 - 0.04 = 0.21$$

$$\boxed{P(0.2 < X < 0.5) = 0.21}$$

4.14 Question 14

Distribution Used: Zero-Truncated Poisson Distribution. **Reasoning:** We need to compute the expected value of a function of a discrete random variable, $g(X) = 1/(1+X)$. This is done by summing $g(x)P(X = x)$ over all possible values of x.

Solution: The PMF is $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!(1-e^{-\lambda})}$ for $x = 1, 2, 3, \ldots$. We want to find $E\left[\frac{1}{1+X}\right]$.

$$E\left[\frac{1}{1+X}\right] = \sum_{x=1}^{\infty} \frac{1}{1+x} P(X = x) = \sum_{x=1}^{\infty} \frac{1}{1+x} \frac{e^{-\lambda}\lambda^x}{x!(1-e^{-\lambda})}$$
$$= \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x+1)!}$$

To evaluate the sum, we manipulate it to resemble the Taylor series for $e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$. Let $k = x + 1$. When $x = 1, k = 2$. The sum becomes:

$$\sum_{k=2}^{\infty} \frac{\lambda^{k-1}}{k!} = \frac{1}{\lambda} \sum_{k=2}^{\infty} \frac{\lambda^k}{k!}$$

The full series is $e^\lambda = \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \sum_{k=2}^{\infty} \frac{\lambda^k}{k!} = 1 + \lambda + \sum_{k=2}^{\infty} \frac{\lambda^k}{k!}$. So, $\sum_{k=2}^{\infty} \frac{\lambda^k}{k!} = e^\lambda - 1 - \lambda$. Substituting this back:

$$E\left[\frac{1}{1+X}\right] = \frac{e^{-\lambda}}{1-e^{-\lambda}} \cdot \frac{1}{\lambda}(e^\lambda - 1 - \lambda) = \frac{e^{-\lambda}(e^\lambda - 1 - \lambda)}{\lambda(1-e^{-\lambda})} = \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{\lambda(1-e^{-\lambda})}$$

$$\boxed{E\left[\frac{1}{1+X}\right] = \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{\lambda(1-e^{-\lambda})}}$$

Chapter 5

Assignment 5 Solutions

5.1 Question 1

Problem: Let X be a continuous random variable with the density function $f_X(x) = \frac{2(x+1)}{9}$ for $-1 < x < 2$. Find the density function of $Y = X^2$.

Solution: We use the transformation method. The support of X is $(-1, 2)$, so the support of $Y = X^2$ is $[0, 4)$. We consider two cases for the value of y.

Case 1: $0 \le y < 1$ For a given value of y in this range, there are two inverse images from the transformation $y = x^2$: $x_1 = -\sqrt{y}$ and $x_2 = \sqrt{y}$. Both of these values lie within the support of X. The derivative of the inverse function is $|\frac{dx}{dy}| = \frac{1}{2\sqrt{y}}$. The PDF of Y is given by:

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right|$$

$$f_Y(y) = \frac{2(-\sqrt{y}+1)}{9} \cdot \frac{1}{2\sqrt{y}} + \frac{2(\sqrt{y}+1)}{9} \cdot \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = \frac{1}{9\sqrt{y}}(-\sqrt{y}+1+\sqrt{y}+1) = \frac{2}{9\sqrt{y}}$$

Case 2: $1 \le y < 4$ For a value of y in this range, there is only one inverse image that lies in the support of X: $x = \sqrt{y}$ (since $-\sqrt{y}$ would be less than -1).

$$f_Y(y) = f_X(\sqrt{y})\left|\frac{dx}{dy}\right| = \frac{2(\sqrt{y}+1)}{9} \cdot \frac{1}{2\sqrt{y}} = \frac{\sqrt{y}+1}{9\sqrt{y}} = \frac{1}{9}\left(1 + \frac{1}{\sqrt{y}}\right)$$

Combining these results, the PDF of Y is:

$$\boxed{f_Y(y) = \begin{cases} \frac{2}{9\sqrt{y}} & 0 \le y < 1 \\ \frac{1}{9}\left(1 + \frac{1}{\sqrt{y}}\right) & 1 \le y < 4 \\ 0 & \text{otherwise} \end{cases}}$$

5.2 Question 2

Problem: Let X be a continuous random variable with the density $f_X(x) = \begin{cases} \frac{x}{2} & 0 < x \le 1 \\ \frac{1}{2} & 1 < x \le 2 \\ \frac{3-x}{2} & 2 < x < 3 \end{cases}$. Find the density of $Y = (X - \frac{3}{2})^2$.

Solution: The transformation is $Y = (X - 3/2)^2$. The support of X is $(0, 3)$, so the support of $(X - 3/2)$ is $(-3/2, 3/2)$, and the support of Y is $[0, 9/4)$. The inverse transformation gives $X - 3/2 = \pm\sqrt{Y}$, so $x = \frac{3}{2} \pm \sqrt{y}$. The Jacobian is $|\frac{dx}{dy}| = \frac{1}{2\sqrt{y}}$.

Case 1: $0 \le y \le 1/4$ The inverse images are $x_1 = \frac{3}{2} - \sqrt{y}$ and $x_2 = \frac{3}{2} + \sqrt{y}$. For $y \in [0, 1/4]$, both x_1 and x_2 fall in the interval $[1, 2]$, where $f_X(x) = 1/2$.

$$f_Y(y) = f_X(x_1)\left|\frac{dx}{dy}\right| + f_X(x_2)\left|\frac{dx}{dy}\right| = \frac{1}{2} \cdot \frac{1}{2\sqrt{y}} + \frac{1}{2} \cdot \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}$$

Case 2: $1/4 < y < 9/4$ The inverse images $x_1 = \frac{3}{2} - \sqrt{y}$ and $x_2 = \frac{3}{2} + \sqrt{y}$ fall outside the interval $[1, 2]$. Specifically, $x_1 \in (0, 1)$ and $x_2 \in (2, 3)$. For x_1, $f_X(x_1) = x_1/2 = \frac{1}{2}(\frac{3}{2} - \sqrt{y})$. For x_2, $f_X(x_2) = (3 - x_2)/2 = \frac{1}{2}(3 - (\frac{3}{2} + \sqrt{y})) = \frac{1}{2}(\frac{3}{2} - \sqrt{y})$.

$$f_Y(y) = \left[\frac{1}{2}\left(\frac{3}{2} - \sqrt{y}\right)\right]\frac{1}{2\sqrt{y}} + \left[\frac{1}{2}\left(\frac{3}{2} - \sqrt{y}\right)\right]\frac{1}{2\sqrt{y}} = 2 \cdot \frac{\frac{3}{2} - \sqrt{y}}{4\sqrt{y}} = \frac{3}{4\sqrt{y}} - \frac{1}{2}$$

The complete PDF for Y is:

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 \le y \le 1/4 \\ \frac{3}{4\sqrt{y}} - \frac{1}{2} & 1/4 < y < 9/4 \\ 0 & \text{otherwise} \end{cases}}$$

5.3 Question 3

Problem: Let X be a random variable with the density function $f_X(x) = \frac{2x}{\pi^2}$ for $0 < x < \pi$. Find the distribution of $Y = \sin(X)$.

Solution: The transformation is $Y = \sin(X)$. Since $X \in (0, \pi)$, the support of Y is $(0, 1]$. For any $y \in (0, 1)$, there are two inverse images in the support of X: $x_1 = \arcsin(y)$ and $x_2 = \pi - \arcsin(y)$. The derivative is $\frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}}$. So, $|\frac{dx_1}{dy}| = \frac{1}{\sqrt{1-y^2}}$ and $|\frac{dx_2}{dy}| = |-\frac{1}{\sqrt{1-y^2}}| = \frac{1}{\sqrt{1-y^2}}$. The PDF of Y is:

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right|$$

$$f_Y(y) = \frac{2(\arcsin y)}{\pi^2} \cdot \frac{1}{\sqrt{1-y^2}} + \frac{2(\pi - \arcsin y)}{\pi^2} \cdot \frac{1}{\sqrt{1-y^2}}$$

$$f_Y(y) = \frac{2}{\pi^2\sqrt{1-y^2}}(\arcsin y + \pi - \arcsin y) = \frac{2\pi}{\pi^2\sqrt{1-y^2}}$$

$$\boxed{f_Y(y) = \begin{cases} \frac{2}{\pi\sqrt{1-y^2}} & 0 < y \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

5.4 Question 4

Problem: Let $X \sim Bin(n, p)$. Find the p.m.f. of (a) $Y_1 = 3X + 4$; (b) $Y_2 = X - 3$; (c) $Y_3 = X^2 + 2$; (d) $Y_4 = \sqrt{X}$.

Solution: The PMF of X is $P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$ for $k \in \{0, 1, \ldots, n\}$. We use the direct method for discrete transformations.

(a) $Y_1 = 3X + 4$: The inverse is $X = (Y_1 - 4)/3$. The support for Y_1 is $\{4, 7, 10, \ldots, 3n + 4\}$.

$$\boxed{P(Y_1 = y) = \binom{n}{(y-4)/3}p^{(y-4)/3}(1-p)^{n-(y-4)/3}, \quad y \in \{4, 7, \ldots, 3n + 4\}}$$

(b) $Y_2 = X - 3$: The inverse is $X = Y_2 + 3$. The support for Y_2 is $\{-3, -2, \ldots, n - 3\}$.

$$\boxed{P(Y_2 = y) = \binom{n}{y+3}p^{y+3}(1-p)^{n-y-3}, \quad y \in \{-3, -2, \ldots, n - 3\}}$$

(c) $Y_3 = X^2+2$: The inverse is $X = \sqrt{Y_3 - 2}$ (since $X \geq 0$). The support for Y_3 is $\{2, 3, 6, \ldots, n^2+2\}$.

$$\boxed{P(Y_3 = y) = \binom{n}{\sqrt{y-2}} p^{\sqrt{y-2}}(1-p)^{n-\sqrt{y-2}}, \quad y \in \{k^2+2 | k = 0, \ldots, n\}}$$

(d) $Y_4 = \sqrt{X}$: The inverse is $X = Y_4^2$. The support for Y_4 is $\{0, 1, \sqrt{2}, \ldots, \sqrt{n}\}$.

$$\boxed{P(Y_4 = y) = \binom{n}{y^2} p^{y^2}(1-p)^{n-y^2}, \quad y \in \{\sqrt{k} | k = 0, \ldots, n\}}$$

5.5 Question 5

Problem: Let $X \sim \text{Beta}(a, b)$. Find the distributions of $Y_1 = 1 - X$ and $Y_2 = \frac{1}{1+X}$.

Solution: The PDF of X is $f_X(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$ for $0 < x < 1$.

For $Y_1 = 1 - X$: The support for Y_1 is $(0, 1)$. The inverse is $X = 1 - Y_1$. The Jacobian is $|\frac{dx}{dy_1}| = |-1| = 1$.

$$f_{Y_1}(y) = f_X(1-y) \cdot 1 = \frac{1}{B(a,b)}(1-y)^{a-1}(1-(1-y))^{b-1} = \frac{1}{B(a,b)} y^{b-1}(1-y)^{a-1}$$

This is the PDF of a Beta distribution with parameters swapped.

$$\boxed{Y_1 \sim \text{Beta}(b, a)}$$

For $Y_2 = \frac{1}{1+X}$: The support for Y_2 is $(1/2, 1)$. The inverse is $X = \frac{1}{Y_2} - 1 = \frac{1-Y_2}{Y_2}$. The Jacobian is $|\frac{dx}{dy_2}| = |-\frac{1}{y_2^2}| = \frac{1}{y_2^2}$.

$$f_{Y_2}(y) = f_X\left(\frac{1-y}{y}\right)\left|\frac{dx}{dy}\right| = \frac{1}{B(a,b)}\left(\frac{1-y}{y}\right)^{a-1}\left(1 - \frac{1-y}{y}\right)^{b-1}\frac{1}{y^2}$$

$$f_{Y_2}(y) = \frac{1}{B(a,b)}\frac{(1-y)^{a-1}}{y^{a-1}}\left(\frac{y-(1-y)}{y}\right)^{b-1}\frac{1}{y^2} = \frac{1}{B(a,b)}\frac{(1-y)^{a-1}(2y-1)^{b-1}}{y^{a-1}y^{b-1}y^2}$$

$$\boxed{f_{Y_2}(y) = \frac{(1-y)^{a-1}(2y-1)^{b-1}}{B(a,b)y^{a+b}}, \quad \frac{1}{2} < y < 1}$$

5.6 Question 6

Problem: Let C be uniformly distributed over $(15, 21)$. Let $F = \frac{9}{5}C + 32$. Find the density of F.

Solution: The PDF of C is $f_C(c) = \frac{1}{21-15} = \frac{1}{6}$ for $15 < c < 21$. This is a linear transformation. First, find the support of F. When $c = 15, F = \frac{9}{5}(15) + 32 = 59$. When $c = 21, F = \frac{9}{5}(21) + 32 = 69.8$. So, $F \in (59, 69.8)$. The inverse transformation is $C = \frac{5}{9}(F - 32)$. The Jacobian is $|\frac{dC}{dF}| = \frac{5}{9}$. The PDF of F is:

$$f_F(f) = f_C\left(\frac{5}{9}(f-32)\right)\left|\frac{dC}{dF}\right| = \frac{1}{6}\cdot\frac{5}{9} = \frac{5}{54}$$

Thus, F is also uniformly distributed.

$$\boxed{f_F(f) = \frac{5}{54}, \quad 59 < f < 69.8}$$

5.7 Question 7

Problem: Let X have density $f_X(x) = cx^2e^{-bx^2}$ for $x > 0$. Find the density of kinetic energy $Y = \frac{1}{2}mX^2$.

Solution: The support of X is $(0,\infty)$, so the support of Y is also $(0,\infty)$. The transformation is one-to-one on the support of X. Inverse transformation: $X^2 = \frac{2Y}{m} \implies X = \sqrt{\frac{2Y}{m}}$ (since $x > 0$). Jacobian: $\frac{dX}{dY} = \sqrt{\frac{2}{m}} \cdot \frac{1}{2\sqrt{Y}} = \frac{1}{\sqrt{2mY}}$. The PDF of Y is:

$$f_Y(y) = f_X\left(\sqrt{\frac{2y}{m}}\right)\left|\frac{dX}{dY}\right| = c\left(\sqrt{\frac{2y}{m}}\right)^2 e^{-b(\sqrt{2y/m})^2} \cdot \frac{1}{\sqrt{2mY}}$$

$$f_Y(y) = c\left(\frac{2y}{m}\right) e^{-2by/m}\frac{1}{\sqrt{2m}\sqrt{y}} = \frac{2c}{m\sqrt{2m}} y^{1/2} e^{-2by/m}$$

$$\boxed{f_Y(y) = \frac{\sqrt{2}c}{m^{3/2}}\sqrt{y}e^{-2by/m}, \quad y > 0}$$

This is a Gamma distribution, $Y \sim \text{Gamma}(k = 3/2, \theta = m/2b)$.

5.8 Question 8

Problem: Let X have the pdf $f_X(x) = \begin{cases} \frac{x+1}{4} & -1 < x \leq 1 \\ \frac{3-x}{4} & 1 < x < 3 \end{cases}$. Find the distribution of $Y = |X|$.

Solution: The support of X is $(-1,3)$, so the support for $Y = |X|$ is $[0,3)$. We use the CDF method: $F_Y(y) = P(Y \leq y) = P(-y \leq X \leq y) = \int_{-y}^{y} f_X(x)dx$.

Case 1: $0 \leq y \leq 1$ The interval $(-y,y)$ is contained in $(-1,1)$, where $f_X(x) = (x+1)/4$.

$$F_Y(y) = \int_{-y}^{y} \frac{x+1}{4}dx = \frac{1}{4}\left[\frac{x^2}{2} + x\right]_{-y}^{y} = \frac{1}{4}\left[\left(\frac{y^2}{2} + y\right) - \left(\frac{y^2}{2} - y\right)\right] = \frac{2y}{4} = \frac{y}{2}$$

Differentiating gives the PDF: $f_Y(y) = \frac{d}{dy}(\frac{y}{2}) = \frac{1}{2}$.

Case 2: $1 < y < 3$ The interval is $(-y,y)$. Since X cannot be less than -1, the integral is from -1 to y.

$$F_Y(y) = \int_{-1}^{y} f_X(x)dx = \int_{-1}^{1} \frac{x+1}{4}dx + \int_{1}^{y} \frac{3-x}{4}dx$$

$$F_Y(y) = \frac{1}{4}\left[\frac{x^2}{2} + x\right]_{-1}^{1} + \frac{1}{4}\left[3x - \frac{x^2}{2}\right]_{1}^{y} = \frac{1}{2} + \frac{1}{4}\left[(3y - \frac{y^2}{2}) - (3 - \frac{1}{2})\right] = \frac{3y}{4} - \frac{y^2}{8} - \frac{1}{8}$$

Differentiating gives the PDF: $f_Y(y) = \frac{d}{dy}\left(\frac{3y}{4} - \frac{y^2}{8} - \frac{1}{8}\right) = \frac{3}{4} - \frac{y}{4} = \frac{3-y}{4}$.

The complete PDF for Y is:

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{2} & 0 \leq y \leq 1 \\ \frac{3-y}{4} & 1 < y < 3 \\ 0 & \text{otherwise} \end{cases}}$$

5.9 Question 9

Problem: Let X be a standard normal random variable and $Y = \begin{cases} \frac{\sqrt{X}}{2} & X \geq 0 \\ -\sqrt{|X|} & X < 0 \end{cases}$. Find the pdf of Y.

Solution: Concept: Transformation of Variables (Piecewise Function) We need to find the probability density function (PDF) of Y, which is defined as a piecewise function of a standard normal random variable X. We will use the change of variable method, applying it separately to the two cases defined by the transformation. The general formula for a one-to-one transformation $Y = g(X)$ is $f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|$.

1. Define the PDF of X: X is a standard normal random variable, so its PDF is:

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad -\infty < x < \infty$$

2. Analyze the Transformation in Two Cases: We handle the positive and negative ranges of Y separately, as they correspond to different parts of the transformation.

Case 1: $y \geq 0$ A non-negative value of y can only be produced by the first part of the transformation, where $X \geq 0$. The transformation is $Y = \frac{\sqrt{X}}{2}$.

- **Inverse Transformation:** We solve for x in terms of y.

$$y = \frac{\sqrt{x}}{2} \implies 2y = \sqrt{x} \implies x = (2y)^2 = 4y^2$$

The inverse function is $x = g_1^{-1}(y) = 4y^2$.

- **Jacobian:** We find the derivative of the inverse function.

$$\frac{dx}{dy} = \frac{d}{dy}(4y^2) = 8y$$

The absolute value of the Jacobian is $|\frac{dx}{dy}| = |8y| = 8y$ (since $y \geq 0$).

- **Apply Formula:** We substitute $x = 4y^2$ into the PDF of X and multiply by the Jacobian.

$$f_Y(y) = f_X(4y^2)\left|\frac{dx}{dy}\right| = \frac{1}{\sqrt{2\pi}}e^{-(4y^2)^2/2} \cdot (8y) = \frac{8y}{\sqrt{2\pi}}e^{-16y^4/2}$$

$$f_Y(y) = \frac{8y}{\sqrt{2\pi}}e^{-8y^4}, \quad \text{for } y \geq 0$$

Case 2: $y < 0$ A negative value of y can only be produced by the second part of the transformation, where $X < 0$. The transformation is $Y = -\sqrt{|X|}$. Since $X < 0$, $|X| = -X$. So, $Y = -\sqrt{-X}$.

- **Inverse Transformation:** We solve for x in terms of y.

$$y = -\sqrt{-x} \implies -y = \sqrt{-x} \quad (\text{Note: } -y > 0, \text{ so this is valid})$$

$$(-y)^2 = -x \implies y^2 = -x \implies x = -y^2$$

The inverse function is $x = g_2^{-1}(y) = -y^2$.

- **Jacobian:** We find the derivative of the inverse function.

$$\frac{dx}{dy} = \frac{d}{dy}(-y^2) = -2y$$

The absolute value of the Jacobian is $|\frac{dx}{dy}| = |-2y| = -2y$ (since $y < 0$, $-2y$ is positive).

- **Apply Formula:** We substitute $x = -y^2$ into the PDF of X and multiply by the Jacobian.

$$f_Y(y) = f_X(-y^2)\left|\frac{dx}{dy}\right| = \frac{1}{\sqrt{2\pi}}e^{-(-y^2)^2/2} \cdot (-2y)$$

$$f_Y(y) = \frac{-2y}{\sqrt{2\pi}}e^{-y^4/2}, \quad \text{for } y < 0$$

3. Combine the Results: We combine the two pieces to form the complete probability density function for Y.

$$\boxed{f_Y(y) = \begin{cases} \frac{-2y}{\sqrt{2\pi}}e^{-y^4/2} & y < 0 \\ \\ \frac{8y}{\sqrt{2\pi}}e^{-8y^4} & y \geq 0 \end{cases}}$$

5.10 Question 10

Problem: Let X be a discrete random variable. Find the distribution of $Y = X^2$.

Solution: To find the PMF of $Y = X^2$, we sum the probabilities of the values of X that map to each value of y. The PMF of X, $p_X(k) = P(X = k)$, is not clearly specified in the problem statement. However, the general method is as follows: The possible values for Y are the squares of the possible values of X. The PMF of Y, $p_Y(y)$, is found by:

$$P(Y = y) = P(X^2 = y) = P(X = \sqrt{y}) + P(X = -\sqrt{y}) \quad \text{for } y > 0$$

$$P(Y = 0) = P(X = 0)$$

For any value y in the support of Y, its probability is:

$$\boxed{p_Y(y) = \begin{cases} p_X(0) & \text{if } y = 0 \\ p_X(\sqrt{y}) + p_X(-\sqrt{y}) & \text{if } y = k^2 \text{ for integer } k > 0 \\ 0 & \text{otherwise} \end{cases}}$$

The median of Y is the value m such that $P(Y \leq m) \geq 0.5$ and $P(Y \geq m) \geq 0.5$. This would be found by calculating the CDF of Y from its PMF and finding the point where it crosses 0.5.

5.11 Question 11

Problem: Let X have p.d.f. $f_X(x) = k\sqrt{x}$ for $0 < x < 1$. Find the pdf of $Y = \sqrt[4]{X}$.

Solution: First, we find the constant k by ensuring the PDF integrates to 1.

$$\int_0^1 k\sqrt{x}dx = k\left[\frac{x^{3/2}}{3/2}\right]_0^1 = k\frac{2}{3} = 1 \implies k = \frac{3}{2}$$

So, $f_X(x) = \frac{3}{2}\sqrt{x}$ for $0 < x < 1$. The transformation is $Y = X^{1/4}$. The support of Y is $(0, 1)$. The inverse is $X = Y^4$. The Jacobian is $|\frac{dX}{dY}| = |4y^3| = 4y^3$. The PDF of Y is:

$$f_Y(y) = f_X(y^4)\left|\frac{dX}{dY}\right| = \left(\frac{3}{2}\sqrt{y^4}\right)(4y^3) = \frac{3}{2}y^2 \cdot 4y^3 = 6y^5$$

$$\boxed{f_Y(y) = \begin{cases} 6y^5 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}}$$

5.12 Question 12

Problem: Let X follow a uniform distribution on $(-1, 1)$. Let $Y = -2\log_e |X|$. Find $E(Y)$.

Solution: The PDF of X is $f_X(x) = 1/2$ for $-1 < x < 1$. We find the expected value of the function of X using the definition $E[g(X)] = \int g(x)f_X(x)dx$.

$$E[Y] = E[-2\ln|X|] = \int_{-1}^1 (-2\ln|x|) \cdot \frac{1}{2}dx = -\int_{-1}^1 \ln|x|dx$$

Since $\ln|x|$ is an even function, we can write the integral as:

$$E[Y] = -2\int_0^1 \ln(x)dx$$

We use integration by parts, $\int udv = uv - \int vdu$. Let $u = \ln(x)$ and $dv = dx$. Then $du = (1/x)dx$ and $v = x$.

$$\int \ln(x)dx = x\ln(x) - \int x \cdot \frac{1}{x}dx = x\ln(x) - x$$

Now we evaluate the definite integral:

$$E[Y] = -2[x\ln(x) - x]_0^1 = -2\left((1\ln(1) - 1) - \lim_{x\to 0^+}(x\ln(x) - x)\right)$$

The limit $\lim_{x\to 0^+} x\ln(x)$ is 0 (can be shown with L'Hopital's rule). So the expression evaluates to:

$$E[Y] = -2((0-1) - (0-0)) = -2(-1) = 2$$

$$\boxed{E[Y] = 2}$$

Chapter 6

Formula Sheet

Figure 6.1: If this helped you, send money for more such projects.

Table is on last page

Relationships Between Distributions

- **Binomial → Poisson:** If n is large and p is small such that $np = \lambda$ (fixed), then

$$\text{Binomial}(n, p) \approx \text{Poisson}(\lambda).$$

- **Poisson → Normal:** If λ is large, then

$$\text{Poisson}(\lambda) \approx N(\mu = \lambda, \sigma^2 = \lambda).$$

- **Binomial → Normal:** If n is large, then

$$\text{Binomial}(n, p) \approx N(np, np(1-p)).$$

- **Geometric and Negative Binomial:** Geometric(p) is a special case of Negative Binomial with $r = 1$.
- **Exponential and Gamma:** Sum of k independent Exponential(λ) random variables follows

$$\text{Gamma}(k, 1/\lambda).$$

- **Normal and χ^2:** If $Z \sim N(0, 1)$, then $Z^2 \sim \chi^2(1)$. More generally, the sum of squares of k independent $N(0, 1)$ random variables gives $\chi^2(k)$.
- **Gamma and Exponential:** Exponential(λ) is a special case of Gamma with shape $k = 1$.
- **Beta and Binomial:** The Beta distribution is the *conjugate prior* of the Binomial distribution in Bayesian statistics.
- **Central Limit Theorem (CLT):** The sum (or average) of i.i.d. random variables with finite variance converges in distribution to a Normal distribution as $n \to \infty$.
- **Poisson process links:**
  - Counts in fixed time → Poisson.
  - Time between events → Exponential (Negative Exponential).
  - Waiting time for k events → Gamma.

Distribution	**PMF / PDF**	$\mathbf{E[X]}$	$\mathbf{Var(X)}$	**MGF** $M_X(t)$	**Standard case / Example**
Bernoulli(p)	$P(X=1)=p,\ P(X=0)=1-p$	p	$p(1-p)$	$1-p+pe^t$	Single trial with success probability p (e.g. coin flip).
Binomial(n,p)	$\binom{n}{x}p^x(1-p)^{n-x},\ x=0,\dots,n$	np	$np(1-p)$	$(1-p+pe^t)^n$	Number of successes in n independent Bernoulli trials.
Hypergeometric(N,K,n)	$\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$	$n\frac{K}{N}$	$n\frac{K}{N}\left(1-\frac{K}{N}\right)\frac{N-n}{N-1}$	—	Number of successes in n draws *without replacement.*
Negative Binomial(r,p)	$\binom{k+r-1}{k}(1-p)^k p^r,\ k=0,1,\dots$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{p}{1-(1-p)e^t}\right)^r$	Failures before r successes.
Geometric(p)	$p(1-p)^{k-1},\ k=1,2,\dots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$	Trials until first success.
Poisson(λ)	$\frac{e^{-\lambda}\lambda^x}{x!},\ x=0,1,\dots$	λ	λ	$\exp(\lambda(e^t-1))$	Counts of rare events in time/space.
Uniform(a,b) (cont.)	$\frac{1}{b-a},\ a<x<b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$	Continuous uniform on interval $[a,b]$.
Exponential(λ)	$\lambda e^{-\lambda x},\ x\geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$	Time between Poisson arrivals.
Normal(μ,σ^2)	$\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2	$\exp(\mu t+\frac{1}{2}\sigma^2t^2)$	Central Limit Theorem, measurement error.
Gamma(α,β)	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x},\ x>0$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\left(\frac{\beta}{\beta-t}\right)^\alpha$	Waiting time for α-th Poisson event.
Beta(α,β)	$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)},\ 0<x<1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	—	Models random probabilities; conjugate prior to Binomial.

Table 6.1: Distributions with mean, variance, MGF, and examples