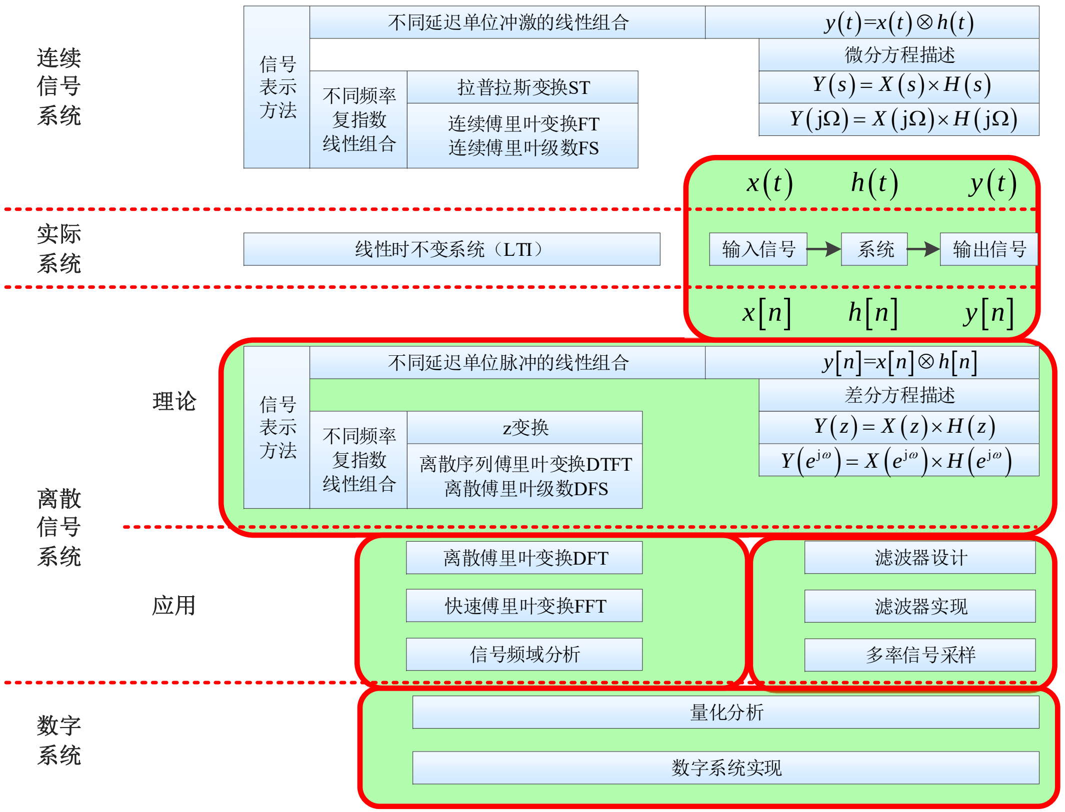
离散傅里叶变换（DFT）

- 0 引言
- 1 离散傅里叶变换DFT
- 2 DFT的性质和定理
- 3 DFT完成线性卷积
- 4 小结

信号与系统


连续信号系统
信号表示方法
不同延迟单位冲激的线性组合
y(t)=x(t)⊗h(t)
微分方程描述
不同频率复指数线性组合
拉普拉斯变换ST
连续傅里叶变换FT
连续傅里叶级数FS
Y(s) = X(s)×H(s)
Y(jΩ) = X(jΩ)×H(jΩ)
x(t) h(t) y(t)
实际系统
线性时不变系统（LTI）
输入信号
系统
输出信号
x[n] h[n] y[n]
离散信号系统
理论
信号表示方法
不同延迟单位脉冲的线性组合
y[n]=x[n]⊗h[n]
差分方程描述
不同频率复指数线性组合
z变换
离散序列傅里叶变换DTFT
离散傅里叶级数DFS
Y(z) = X(z)×H(z)
Y(e^jω) = X(e^jω)×H(e^jω)
应用
离散傅里叶变换DFT
快速傅里叶变换FFT
信号频域分析
滤波器设计
滤波器实现
多率信号采样
数字系统
量化分析
数字系统实现

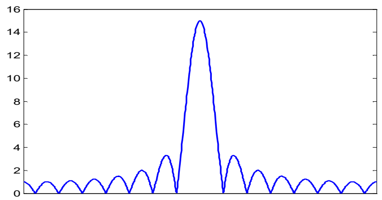
0 引言

连续信号

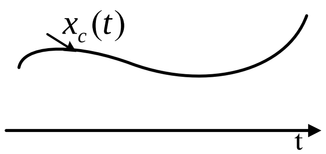
非周期信号

其他信号

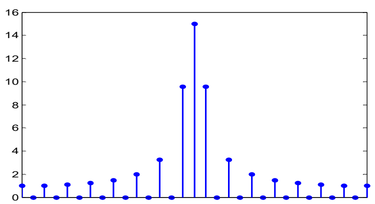
狄利赫里条件



傅里叶变换FT



周期信号

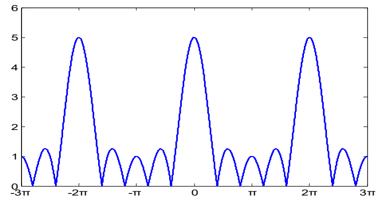


傅里叶级数FS

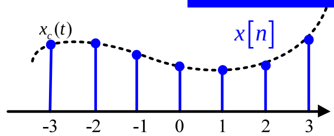
其他信号

离散信号

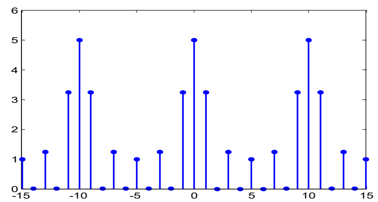
3类非周期序列



离散时间序列傅里叶变换DTFT

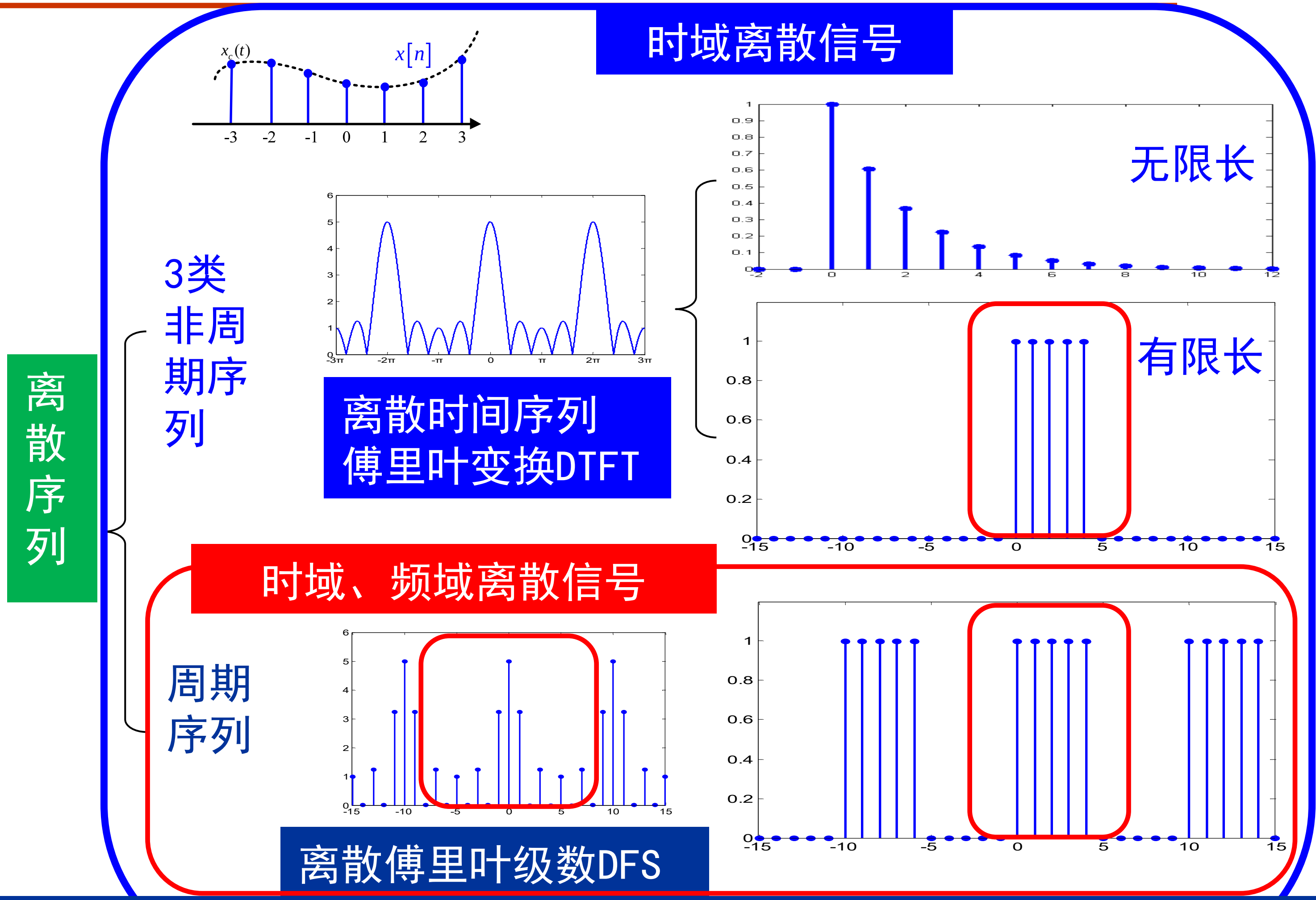


周期序列



离散傅里叶级数DFS

0 引言



周期序列时域、频域均为离散的。可由数字系统操作
但周期序列无论时域、频域取值都是无限长

离散傅里叶变换

- 0 引言
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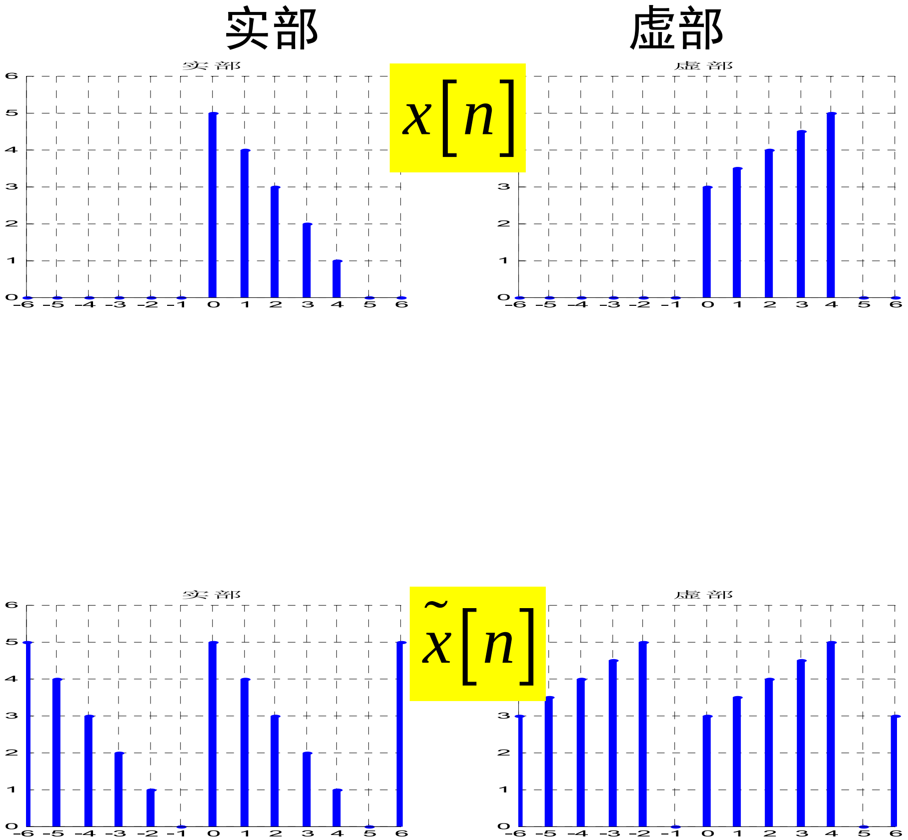
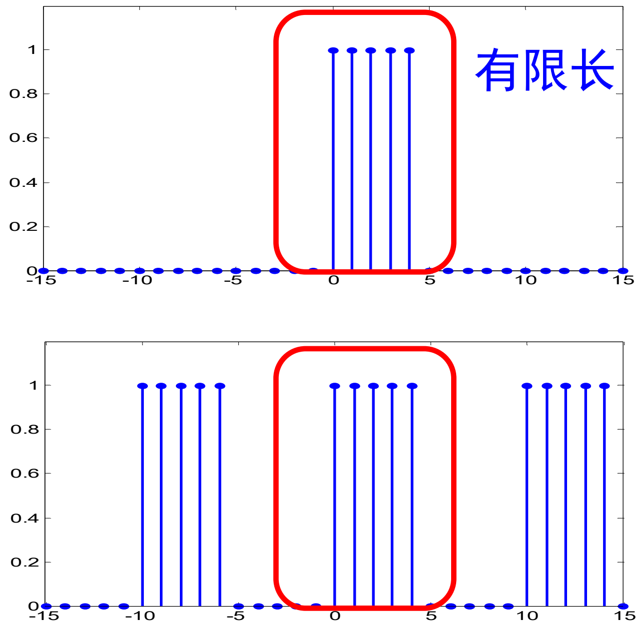
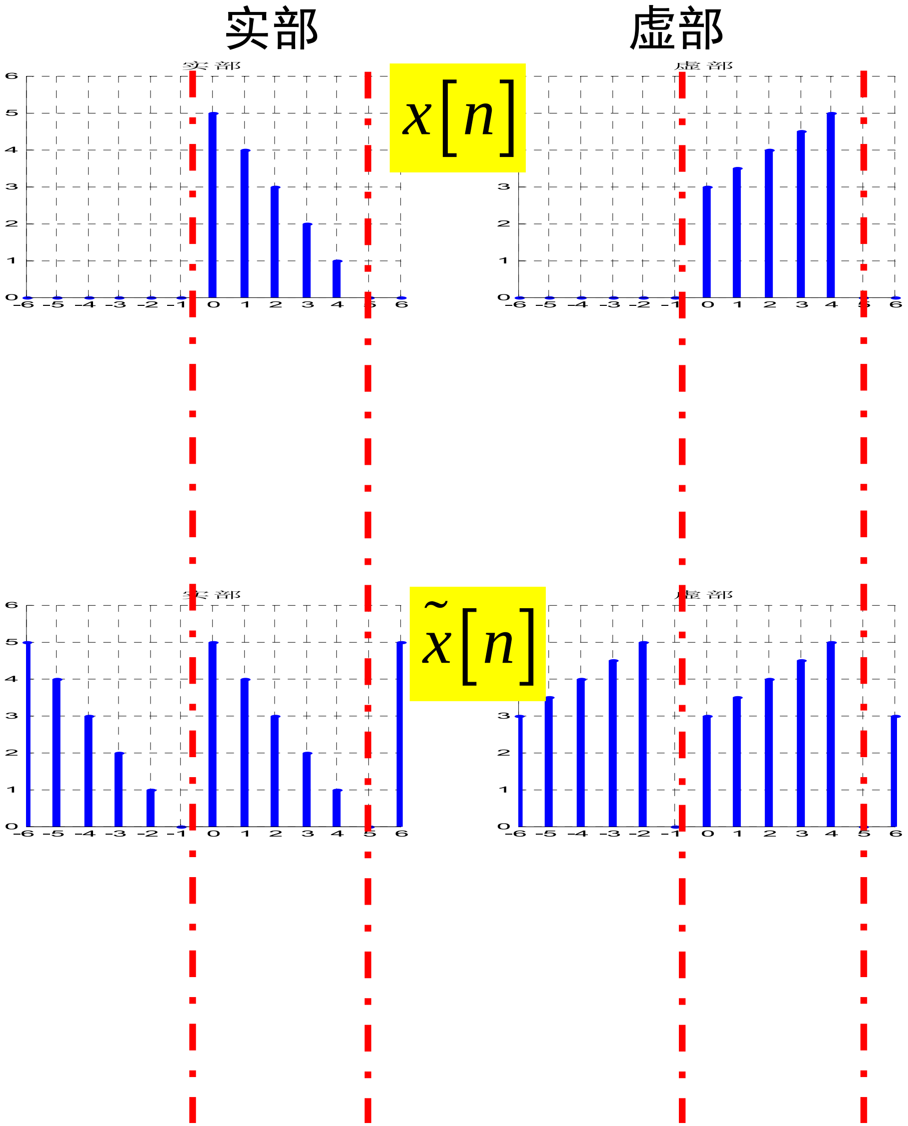
1 离散傅里叶变换DFT



- 有限长序列与周期序列的关系



1 离散傅里叶变换DFT



- 有限长序列与周期序列的关系

$$R_N[n]=\begin{cases}1, & 0\le n\le N-1\\ 0, & \text{其它}\end{cases}$$

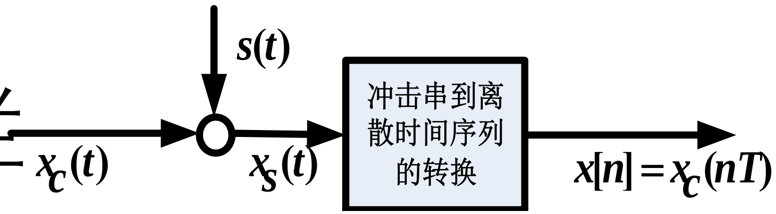
$$\boxed{x[n]=\tilde{x}[n]R_N[n]}$$

$$\tilde{x}[n]=\sum_{r=-\infty}^{\infty}x[n-rN]$$

$$\tilde{x}[n]=x[n\text{以}N\text{为模}]$$

$$\boxed{\tilde{x}[n]=x[((n))_N]}$$

2 连续时间信号的理想采样



$$x_s(t)=x_c(t)s(t)$$

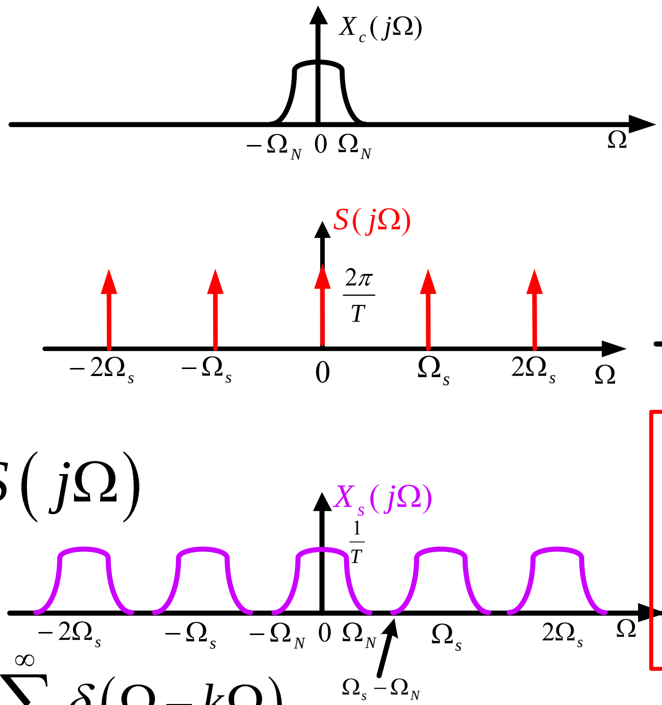
$$S(j\Omega)=\frac{2\pi}{T_S}\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_s)$$

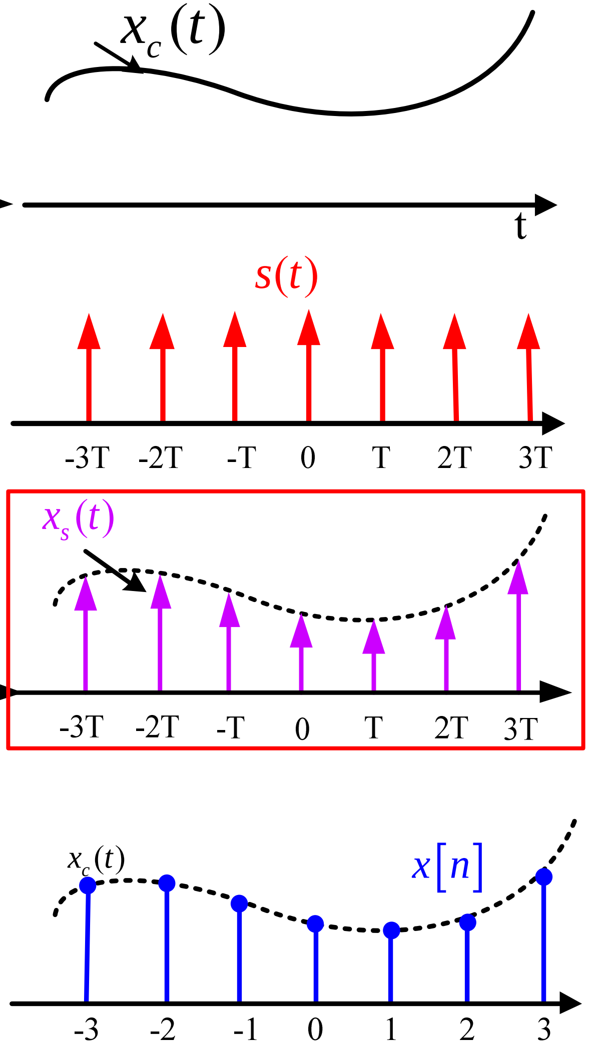
$$X_S(j\Omega)=\frac{1}{2\pi}X_C(j\Omega)\otimes S(j\Omega)$$

$$X_S(j\Omega)=\frac{1}{2\pi}X_C(j\Omega)\otimes\frac{2\pi}{T_S}\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega)_s$$

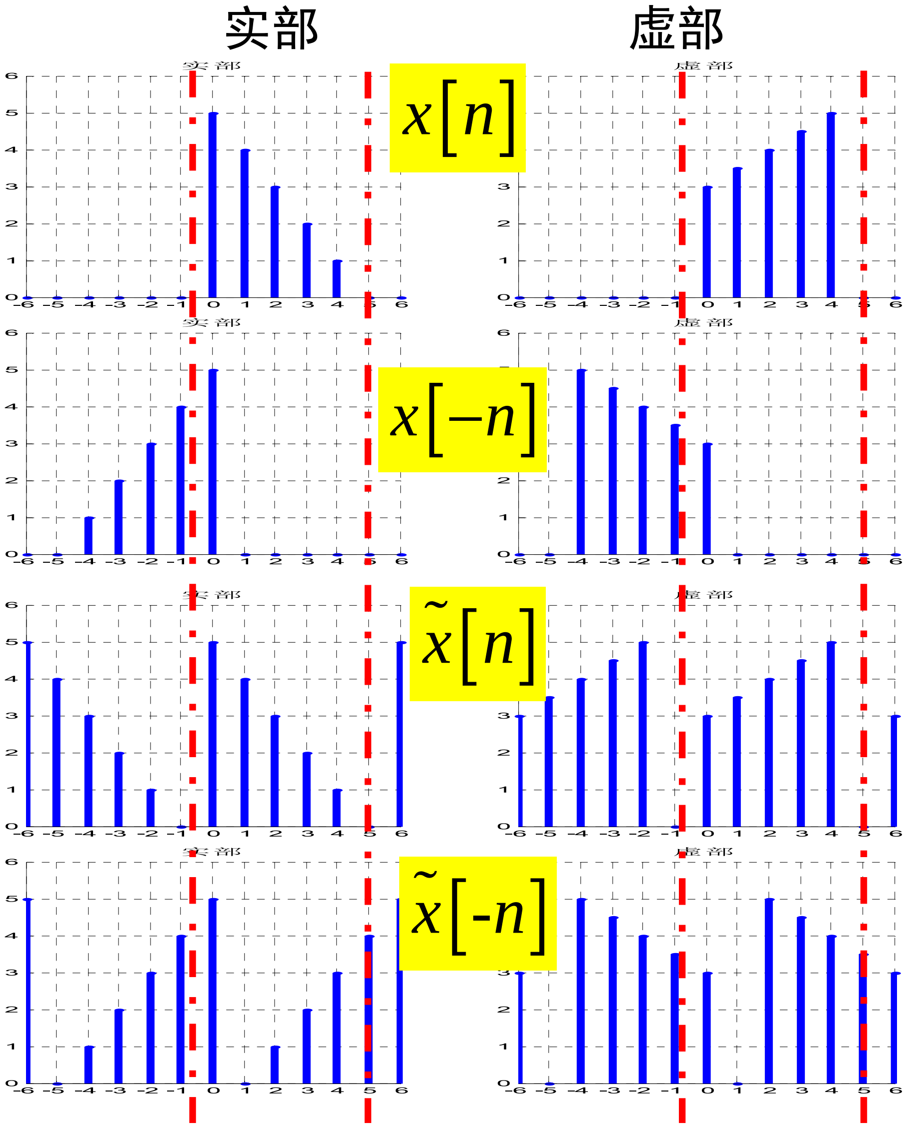
$$=\frac{1}{2\pi}\frac{2\pi}{T_S}\sum_{k=-\infty}^{\infty}X_C(j\Omega)\otimes\delta(\Omega-k\Omega_s)$$

$$\boxed{=\frac{1}{T_S}\sum_{k=-\infty}^{\infty}X_C(j\Omega-jk\Omega_s)}$$





1 离散傅里叶变换DFT



- 有限长序列与周期序列的关系

$$R_N[n]=\begin{cases}1, & 0\le n\le N-1\\ 0, & \text{其它}\end{cases}$$

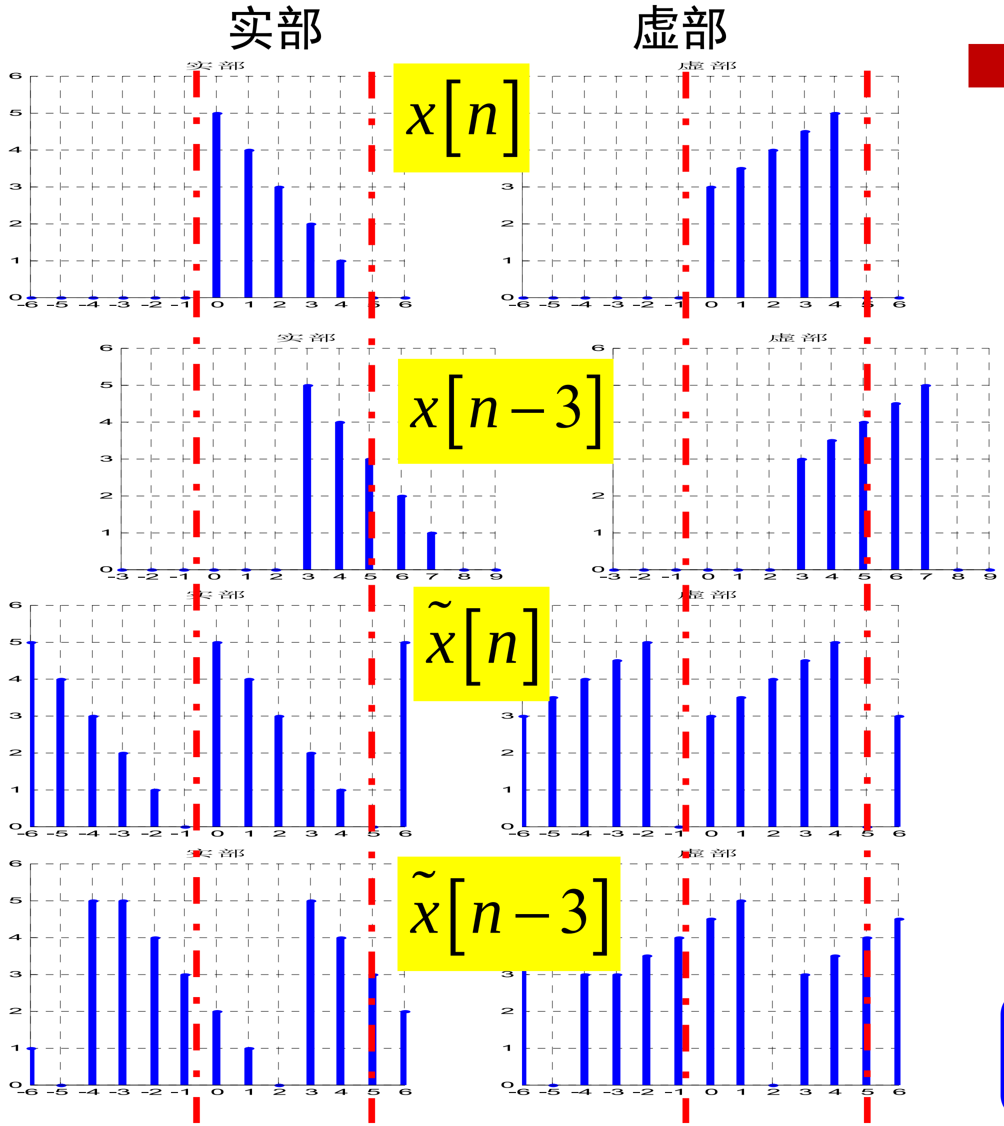
$$\boxed{x[n]=\tilde{x}[n]R_N[n]}$$

$$\boxed{x[-n]=\tilde{x}[-n]R_N[-n]}$$

$$\boxed{\tilde{x}[n]=x\left[((n))_N\right]}$$

$$\boxed{\tilde{x}[-n]=x\left[((-n))_N\right]}$$

1 离散傅里叶变换DFT



- 有限长序列与周期序列的关系

$$R_N[n]=\begin{cases}1, & 0\le n\le N-1\\ 0, & \text{其它}\end{cases}$$

$$\boxed{x[n]=\tilde{x}[n]R_N[n]}$$

$$\boxed{x[n\text{-}k]=\tilde{x}[n\text{-}k]R_N[n\text{-}k]}$$

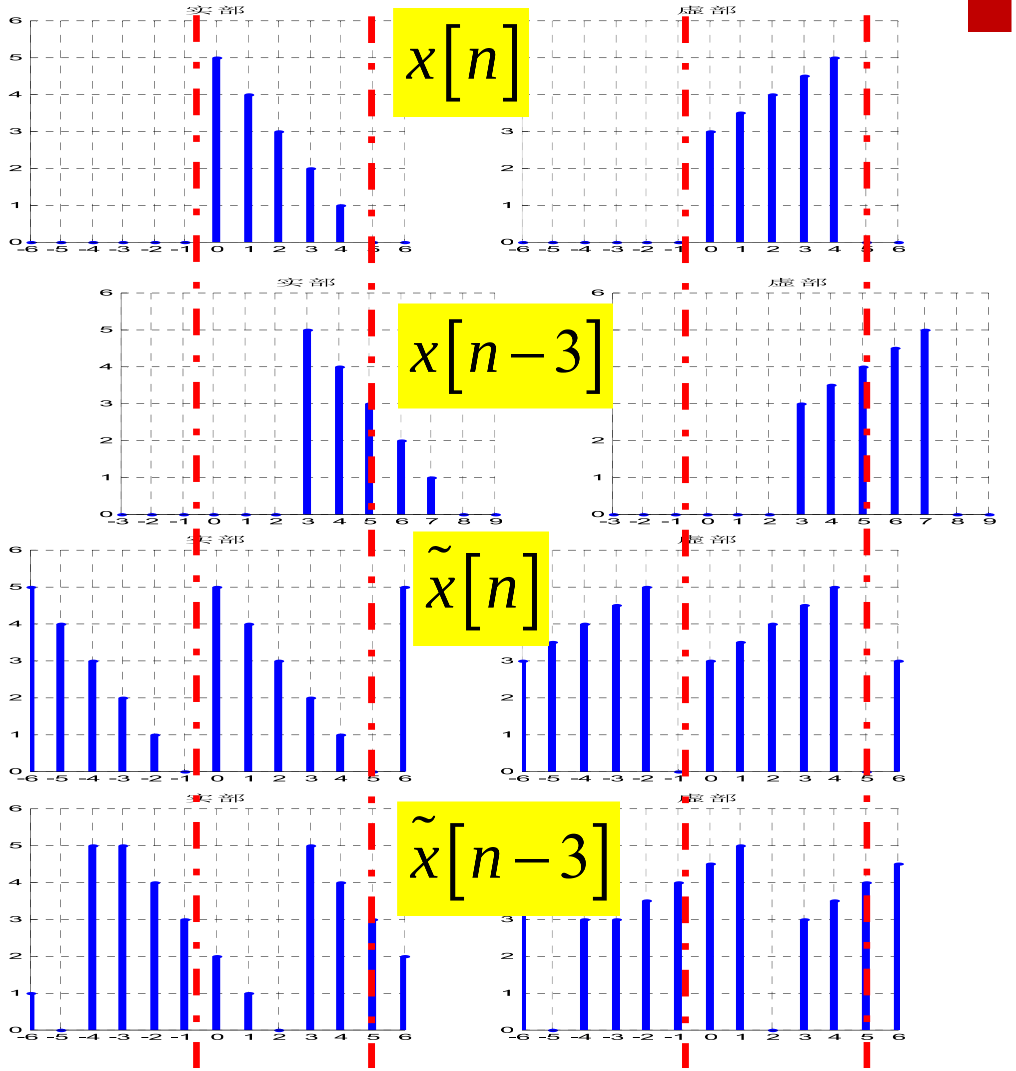
$$\boxed{\tilde{x}[n]=x\left[((n))_N\right]}$$

$$\boxed{\tilde{x}[n\text{-}k]=x\left[((n\text{-}k))_N\right]}$$

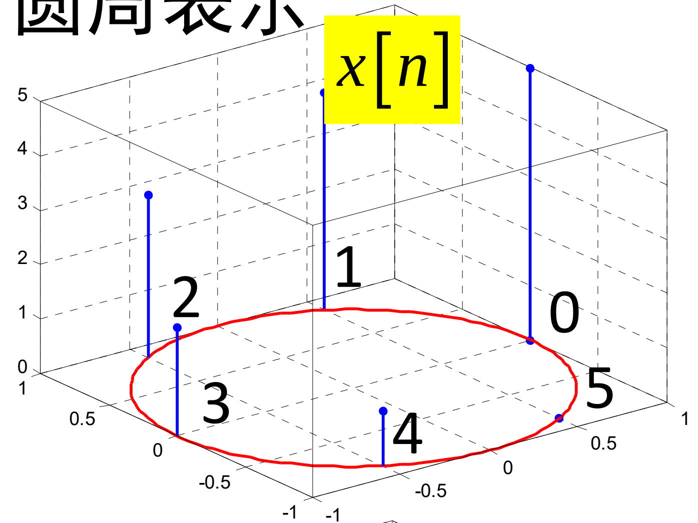
1 离散傅里叶变换DFT

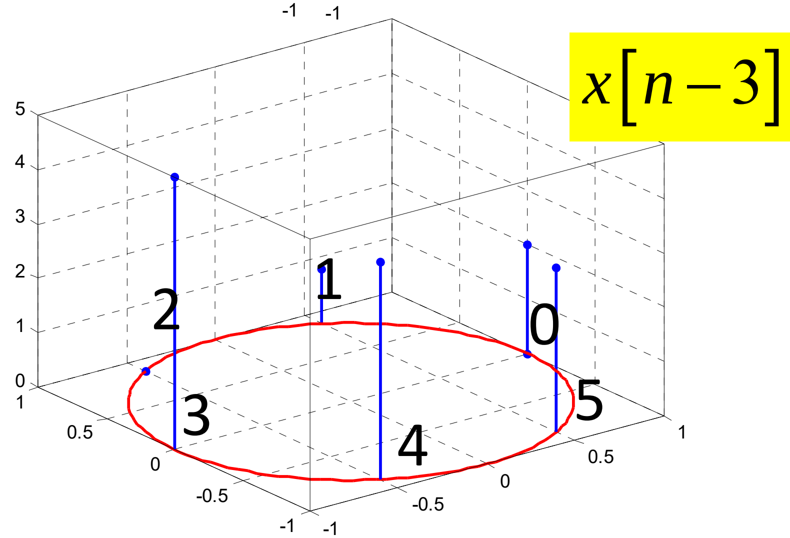
$$\tilde{x}[n]=x\left[((n))_N\right]$$

实部　　虚部



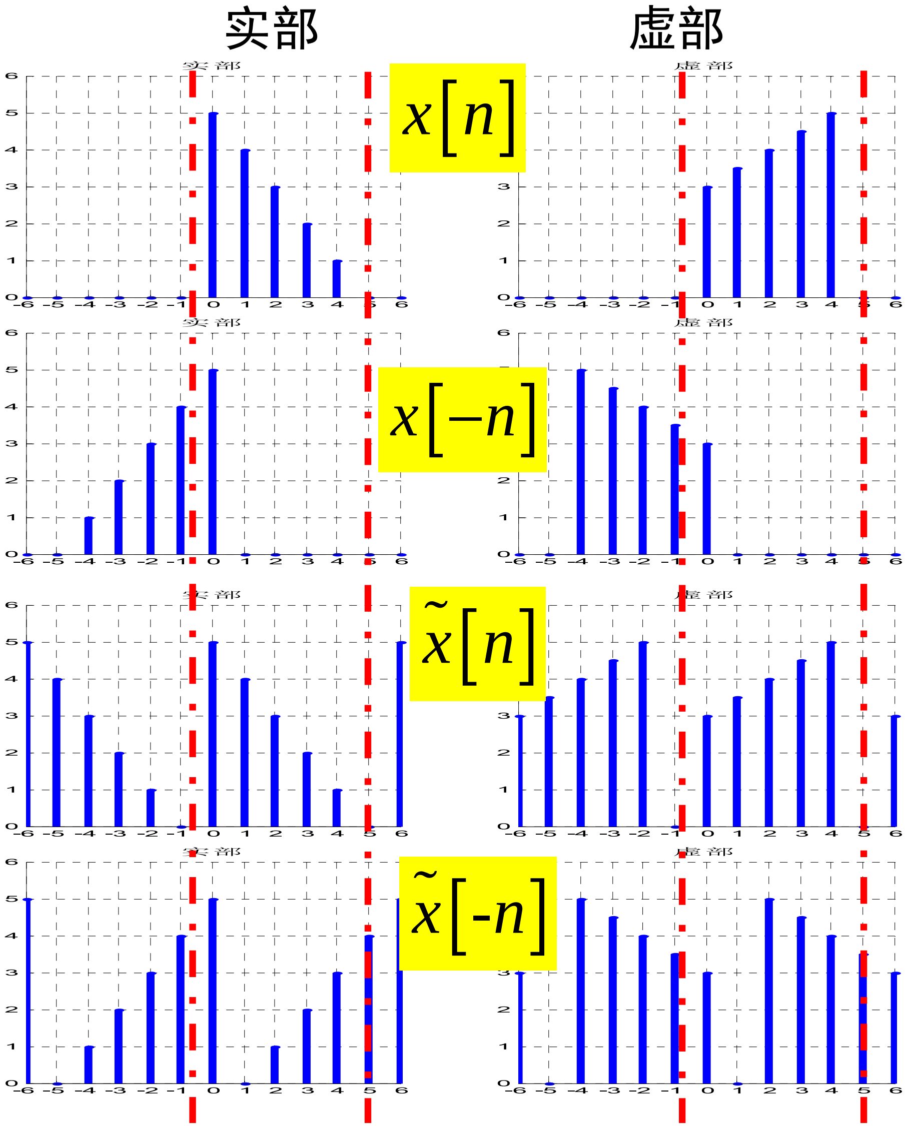
■ 圆周表示



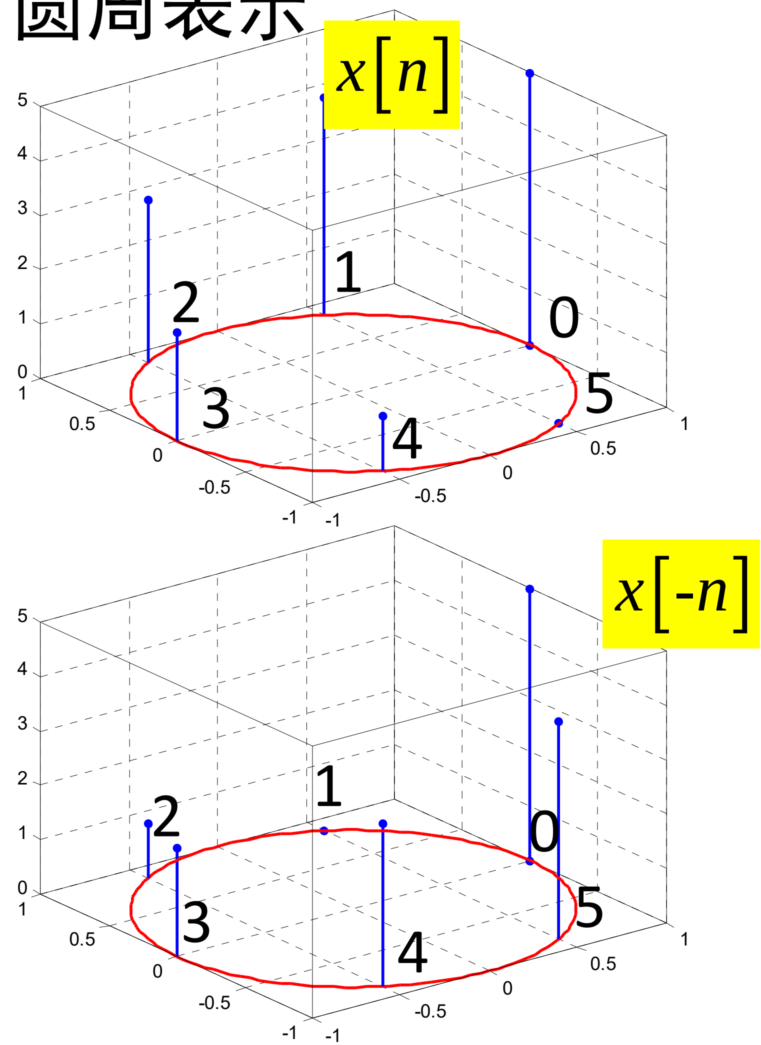


1 离散傅里叶变换DFT

$$\tilde{x}[n] = x\left[((n))_N\right]$$



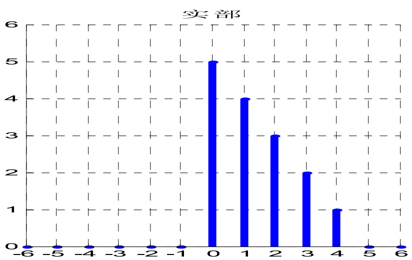
■ 圆周表示

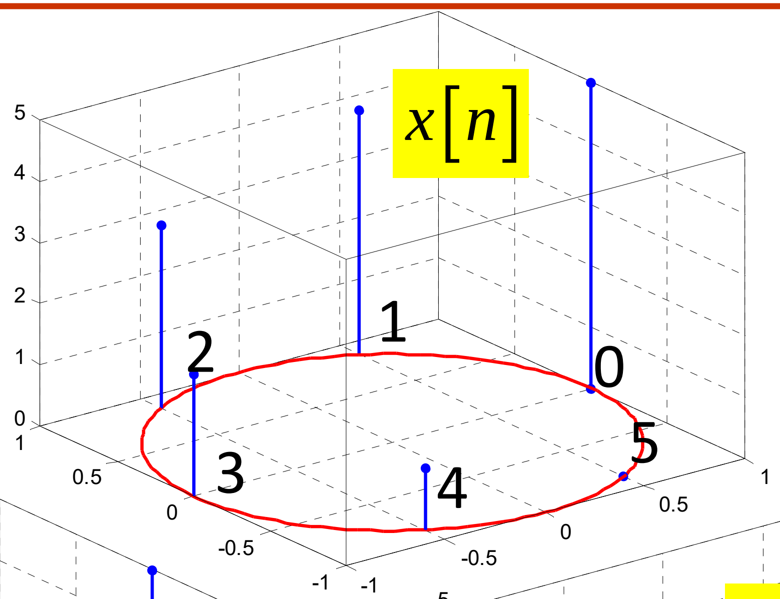


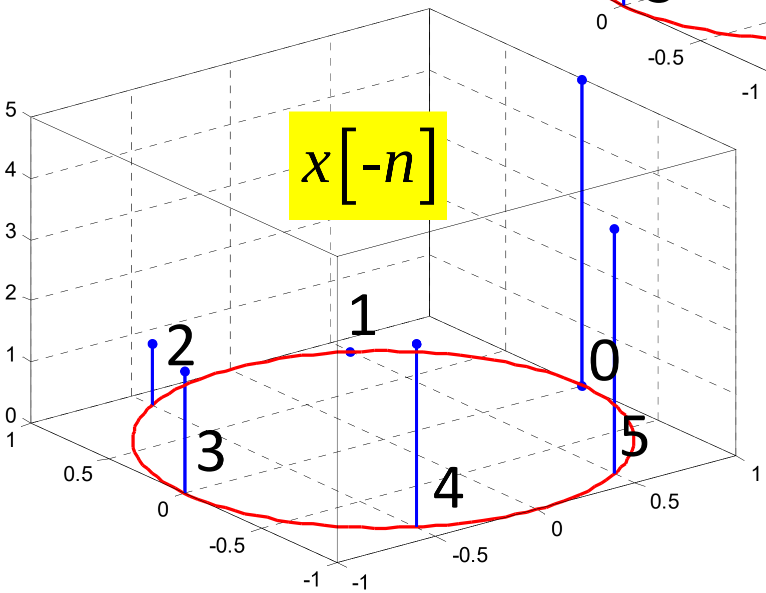
$$\begin{cases} \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-jk(2\pi/N)n} \\ \tilde{x}[\mathrm{n}] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{jk(2\pi/N)n} \end{cases}$$

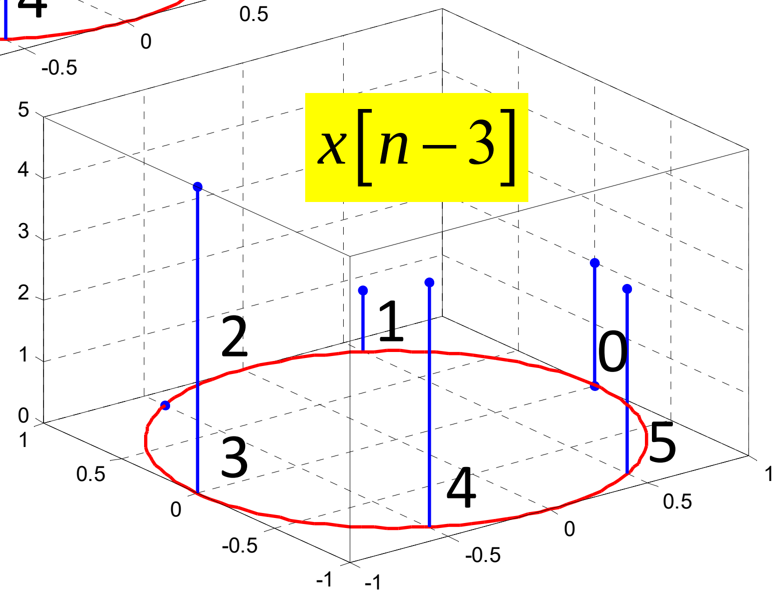
1 离散傅里叶变换DFT

- 圆周表示





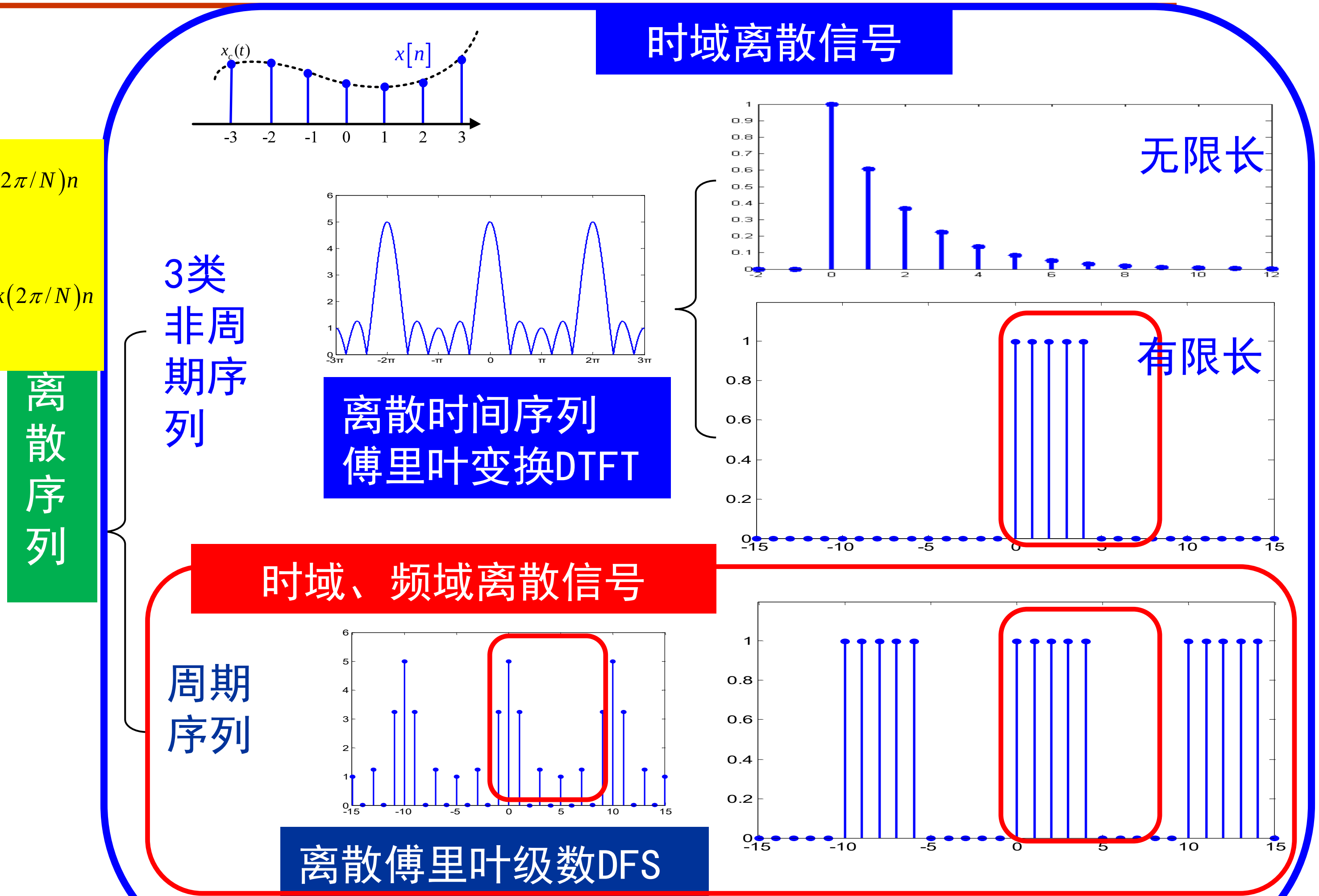




$$\begin{cases}\tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]e^{-jk(2\pi/N)n}\\ \tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{jk(2\pi/N)n}\end{cases}$$

1 离散傅里叶变换DFT

$$\begin{cases}X[k]=\sum_{n=0}^{N-1}x[n]e^{-jk(2\pi/N)n}\\ x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{jk(2\pi/N)n}\end{cases}$$



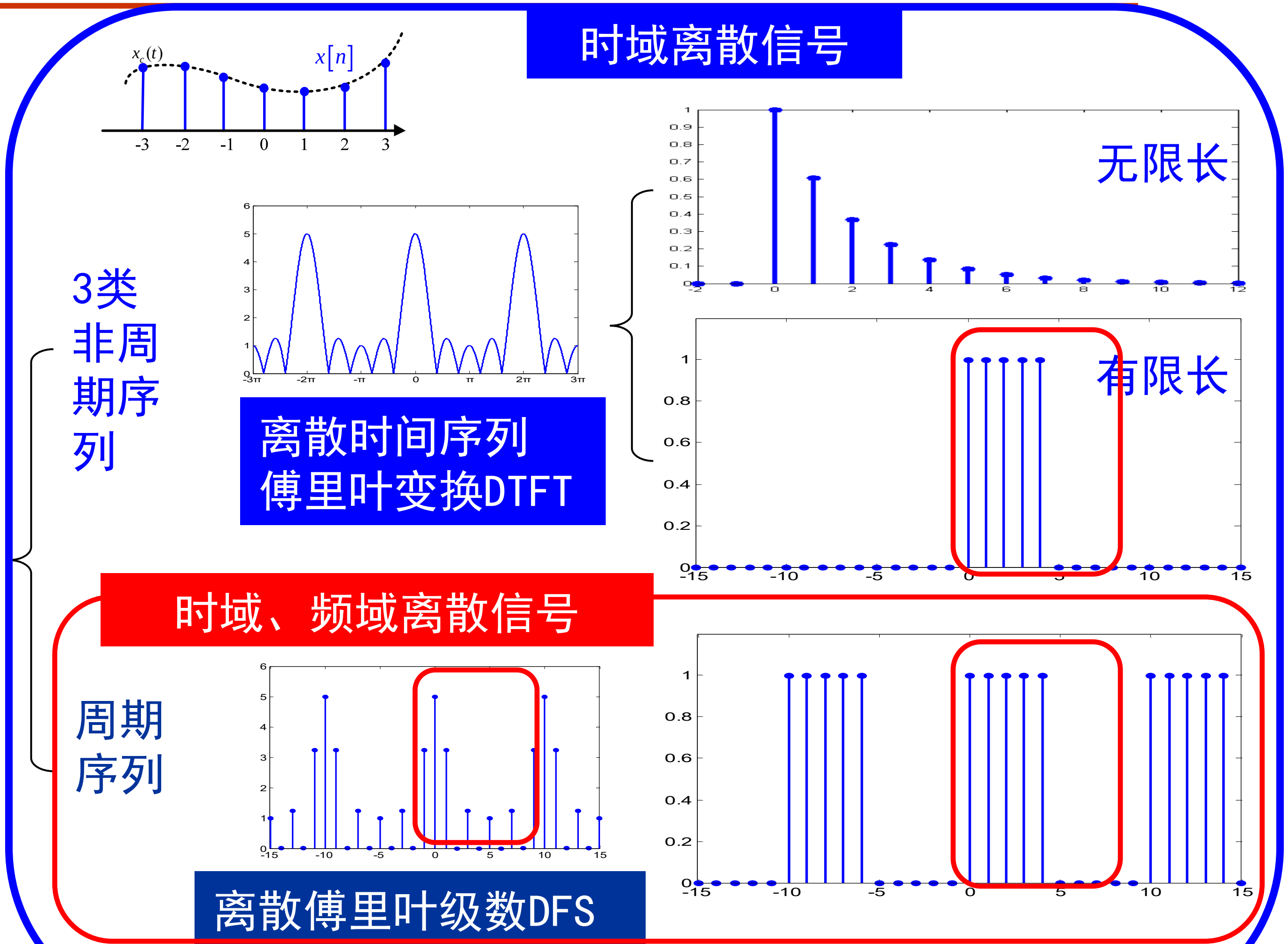
频域，在区间[0，N-1]外，$X[k]=0$

时域，在区间[0，N-1]外，$x[n]=0$

$$\begin{cases} X[k]=\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n} \\ x[n]=\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jk(2\pi/N)n} \end{cases}$$

$$\begin{cases} \tilde{X}[k]=\sum_{n=0}^{N-1} \tilde{x}[n]e^{-jk(2\pi/N)n} \\ \tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{jk(2\pi/N)n} \end{cases}$$

1 离散傅里叶变换DFT



$$\tilde{x}[n-k]=x\left[((n-k))_N\right]$$

$$\tilde{x}[-n]=x\left[((-n))_N\right]$$

$$\tilde{X}[k-l]=X\left[((k-l))_N\right]$$

$$\tilde{X}[-k]=X\left[((-k))_N\right]$$

DFT是DFS的一个周期

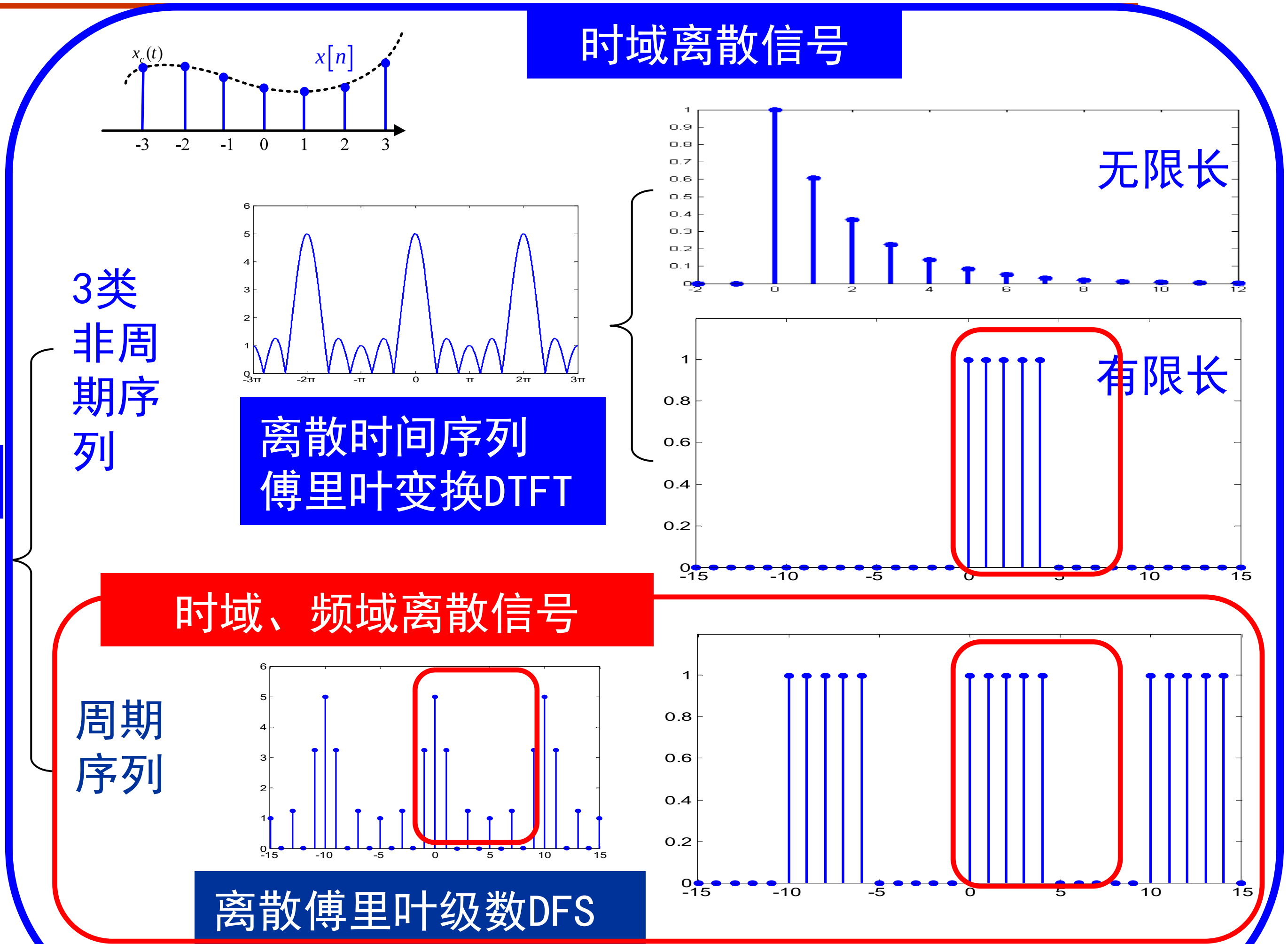
$$\begin{cases} X[k]=\sum_{n=0}^{N-1}x[n]e^{-jk(2\pi/N)n} \\ x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{jk(2\pi/N)n} \end{cases}$$

$$\begin{cases} \tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]e^{-jk(2\pi/N)n} \\ \tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{jk(2\pi/N)n} \end{cases}$$

1 离散傅里叶变换DFT

- DFT、
  DTFT、
  z变换

DFS与DTFT的关系



DFT是DFS的一个周期

复习2.1 DFS定义

$e^{j\frac{2\pi kn}{N}}$ 正交性、周期性

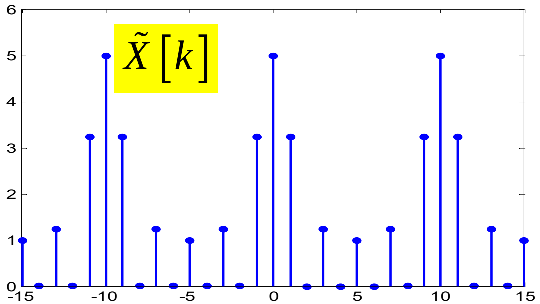
$$\boxed{\sum_{n=0}^{N-1} e^{j\frac{2\pi kn}{N}} e^{j\frac{-2\pi ln}{N}} = \begin{cases} N & k=l \\ 0 & k \neq l \end{cases} = N\delta[k-l]}$$

$$\boxed{\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi kn}{N}} e^{j\frac{-2\pi ln}{N}} = \begin{cases} 1 & k-l=rN \\ 0 & k-l \neq rN \end{cases} = \sum_{r=-\infty}^{\infty} \delta[k-l-rN]}$$

1 离散傅里叶变换DFT $\tilde{X}[k]=X(e^{j\omega})\big|_{\omega=2\pi k/N}$

DFS

$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\mathrm{e}^{\mathrm{j}\frac{2\pi kn}{N}}$$



$$=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty}x[m]\mathrm{e}^{-\mathrm{j}\frac{2\pi k}{N}m}\mathrm{e}^{\mathrm{j}\frac{2\pi kn}{N}}=\frac{1}{N}\sum_{m=-\infty}^{\infty}x[m]\boxed{\sum_{k=0}^{N-1}\mathrm{e}^{-\mathrm{j}\frac{2\pi k}{N}m}\mathrm{e}^{\mathrm{j}\frac{2\pi kn}{N}}}=\sum_{r=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}x[m]\delta[n\text{-}m\text{-}rN]$$

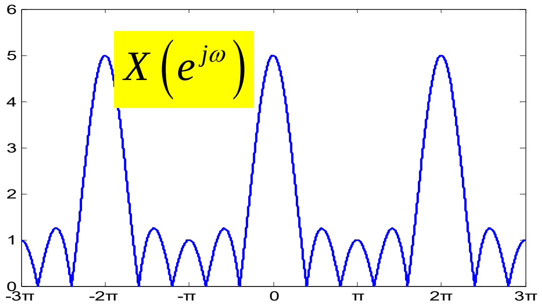
$$=\sum_{r=-\infty}^{\infty}x[n\text{-}rN]$$

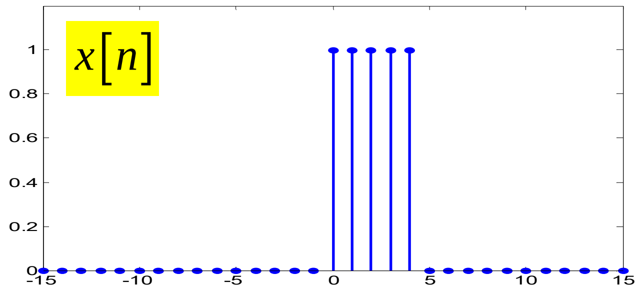
$$\tilde{X}[k]=\sum_{n=-\infty}^{\infty}x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi k}{N}n}$$

$$\boxed{\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{e}^{\mathrm{j}\frac{2\pi kn}{N}}\mathrm{e}^{\mathrm{j}\frac{-2\pi ln}{N}}=\sum_{r=-\infty}^{\infty}\delta[k\text{-}l\text{-}rN]}$$

DTFT

$$X(\mathrm{e}^{\mathrm{j}\omega})=\sum_{n=-\infty}^{\infty}x[n]\mathrm{e}^{-\mathrm{j}\omega n}$$





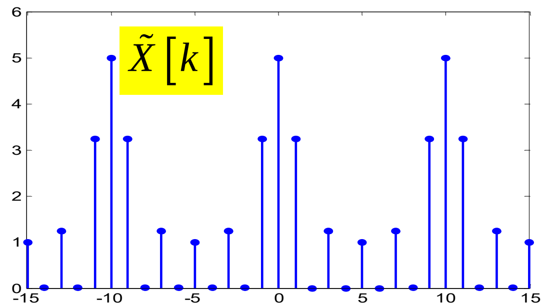
频域采样

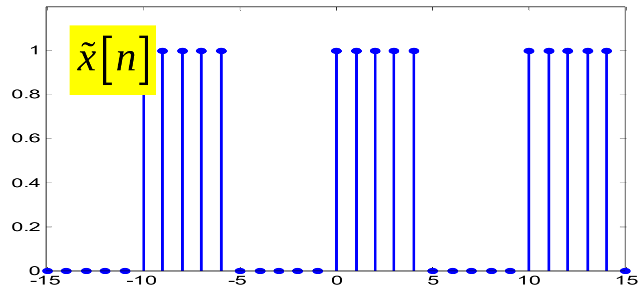
1 离散傅里叶变换DFT

$$\tilde{X}[k]=X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N}$$

DFS

$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\mathrm{e}^{\mathrm{j}\frac{2\pi kn}{N}}$$





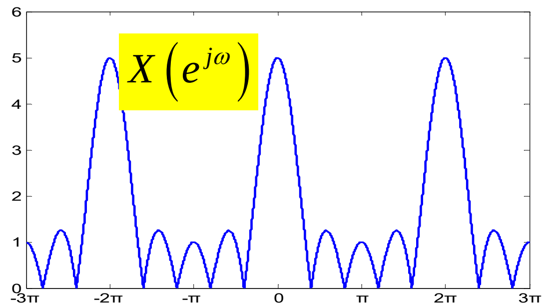
$$\tilde{x}[n]=\sum_{r=-\infty}^{\infty}x[n-rN]$$

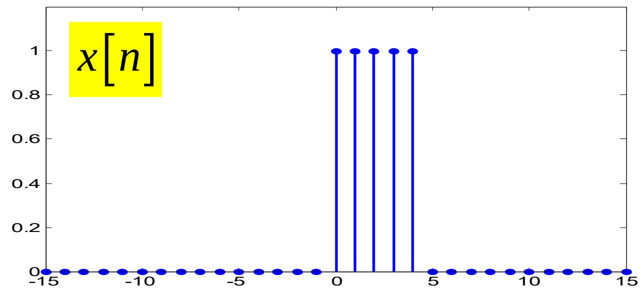
$$=\ldots+x[n+N]+x[n]+x[n-N]+\ldots$$

$$\tilde{X}[k]=\sum_{n=-\infty}^{\infty}x[n]\mathrm{e}^{-\mathrm{j}\frac{2\pi k}{N}n}$$

DTFT

$$X\left(\mathrm{e}^{\mathrm{j}\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]\mathrm{e}^{-\mathrm{j}\omega n}$$



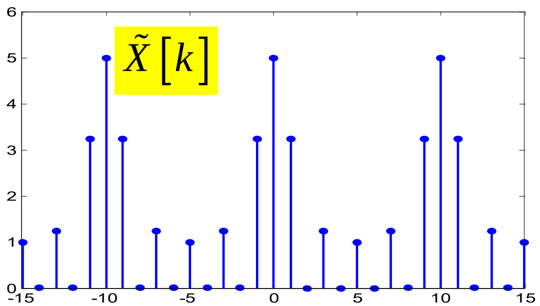


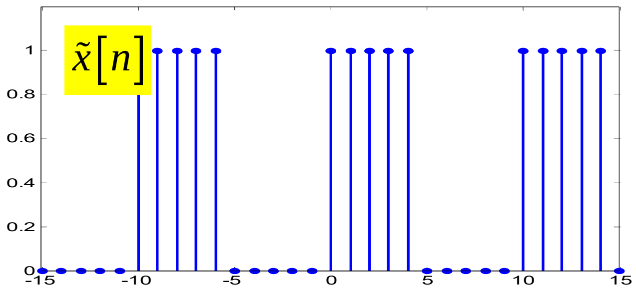
频域采样、时域周期延拓

1 离散傅里叶变换DFT $\tilde{X}[k]=X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N}$

DFS

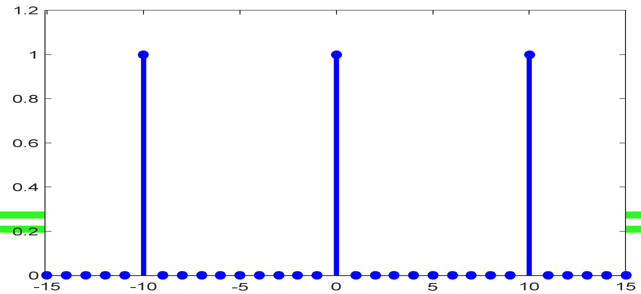
$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi kn}{N}}$$





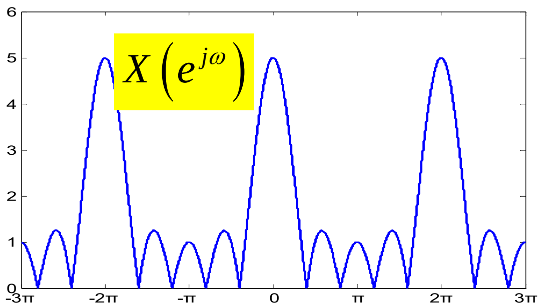
$$=\sum_{r=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}x[m]\delta[n-m-rN]=\sum_{m=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}x[m]\delta[n-m-rN]$$

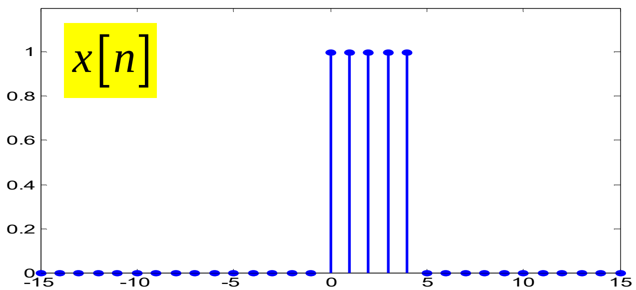
$$=x[n]\otimes\sum_{r=-\infty}^{\infty}\delta[n-rN]$$



$$\tilde{X}[k]=\sum_{n=-\infty}^{\infty}x[n]e^{-j\frac{2\pi k}{N}n}$$

DTFT





$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$

频域采样

1 DFT

$$X\left[((k))_N\right]=\tilde{X}[k]=X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N}$$

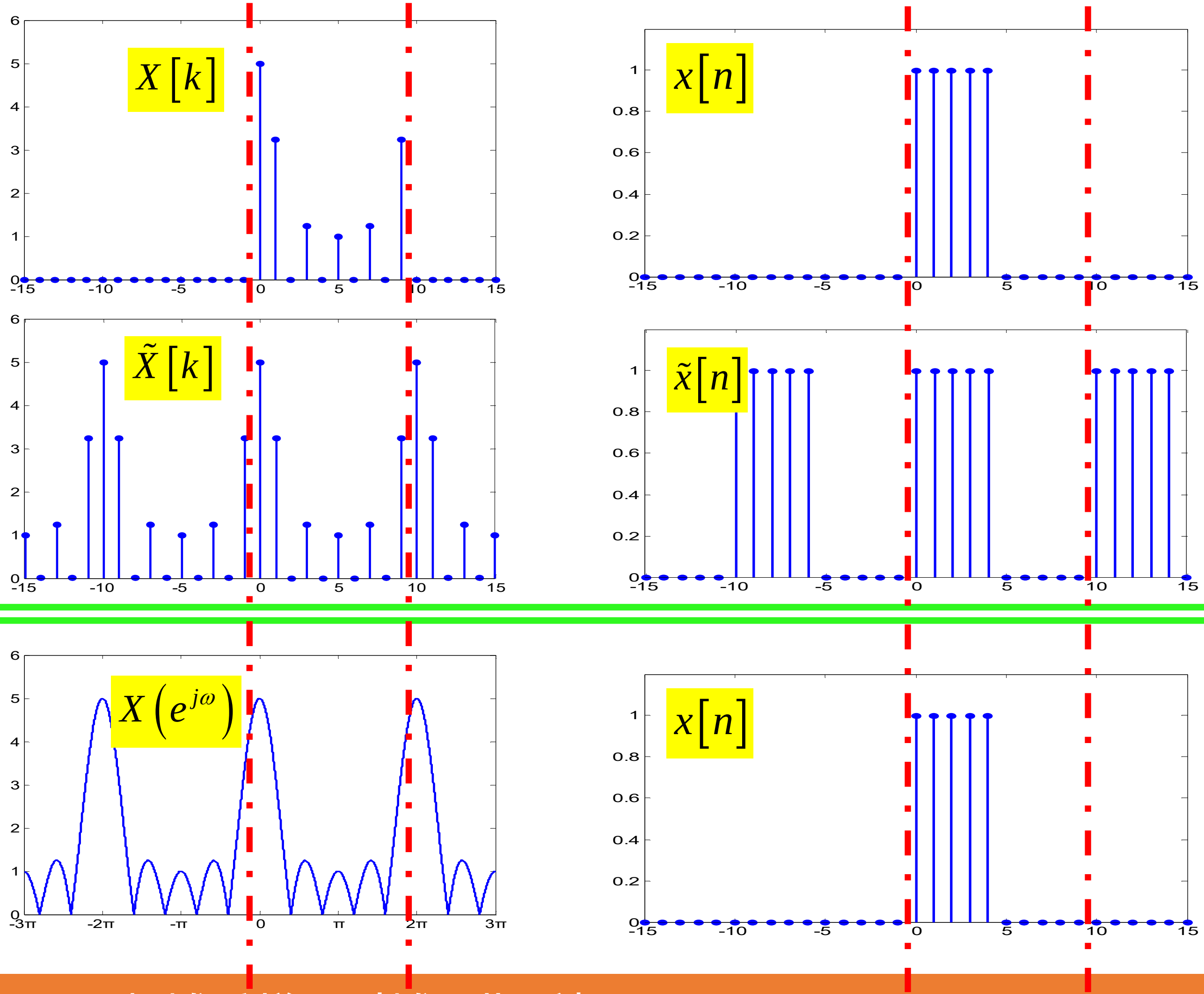
$$x\left[((n))_N\right]=\tilde{x}[n]=\sum_{r=-\infty}^{\infty}x[n-rN]$$

DFT

DFT是DFS的一个周期

DFS

DTFT



频域采样、时域周期延拓

1 DFT

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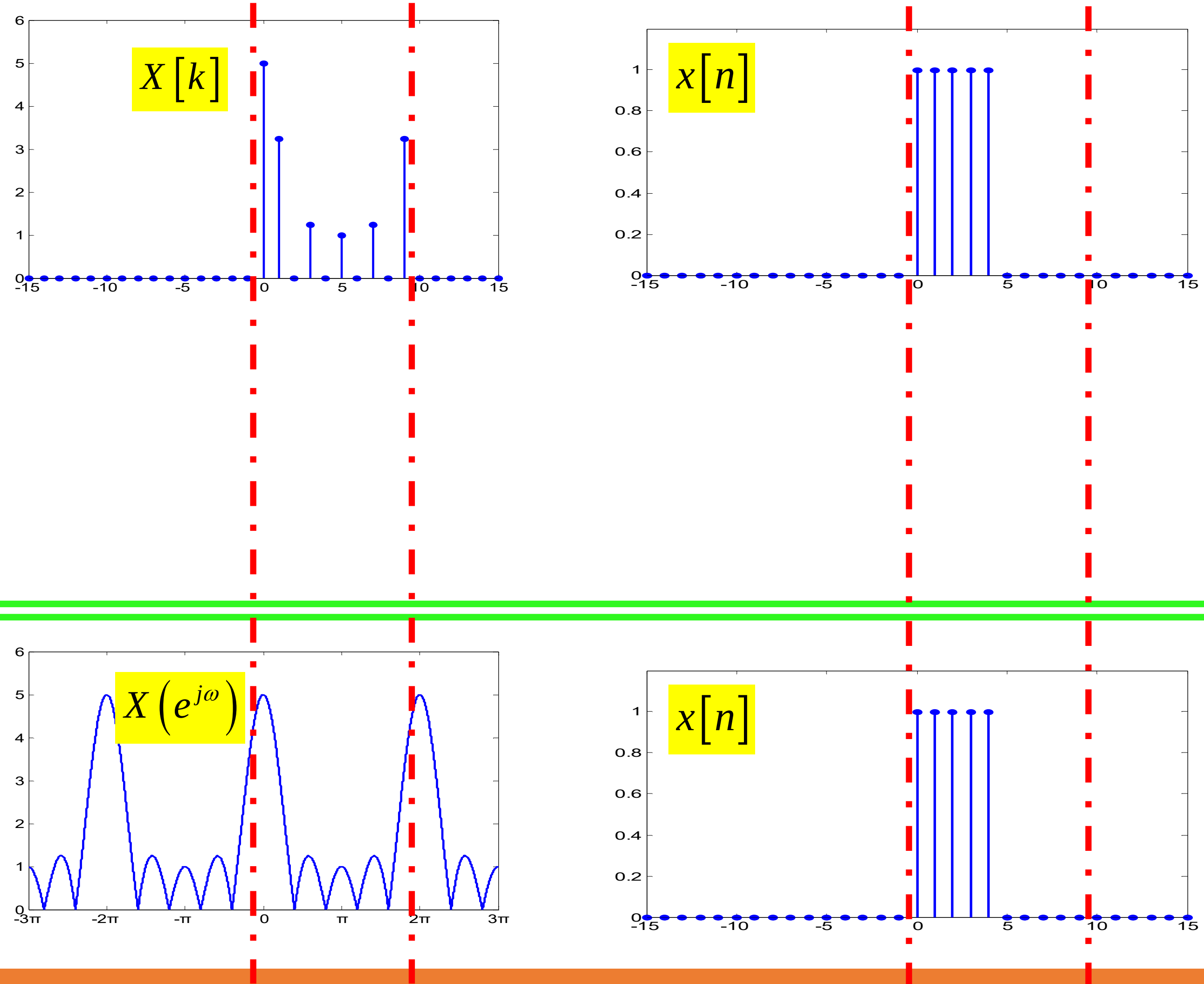
DFT

DTFT

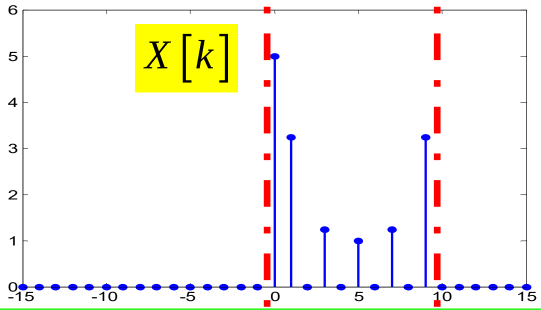


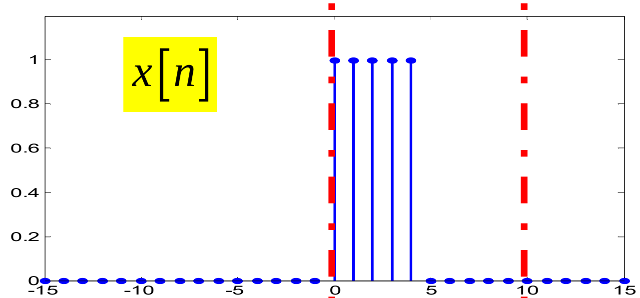
频域采样、时域周期延拓

1 离散傅里叶变换DFT

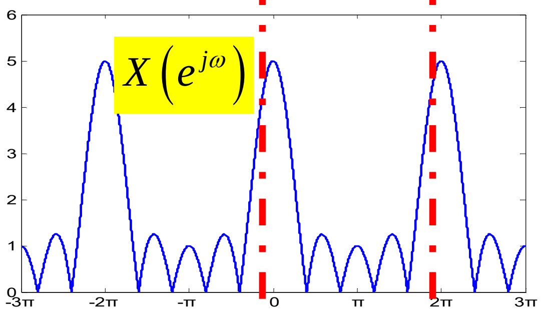
$$X\left[((k))_N\right]=\tilde{X}[k]=X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N}$$

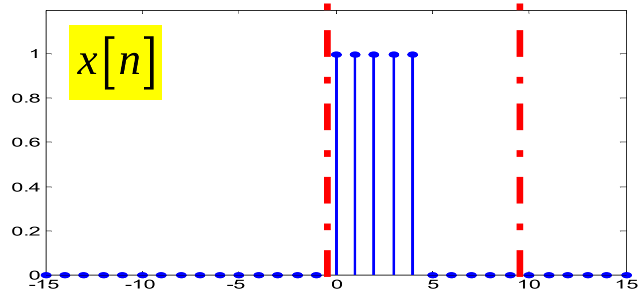
DFT





DTFT

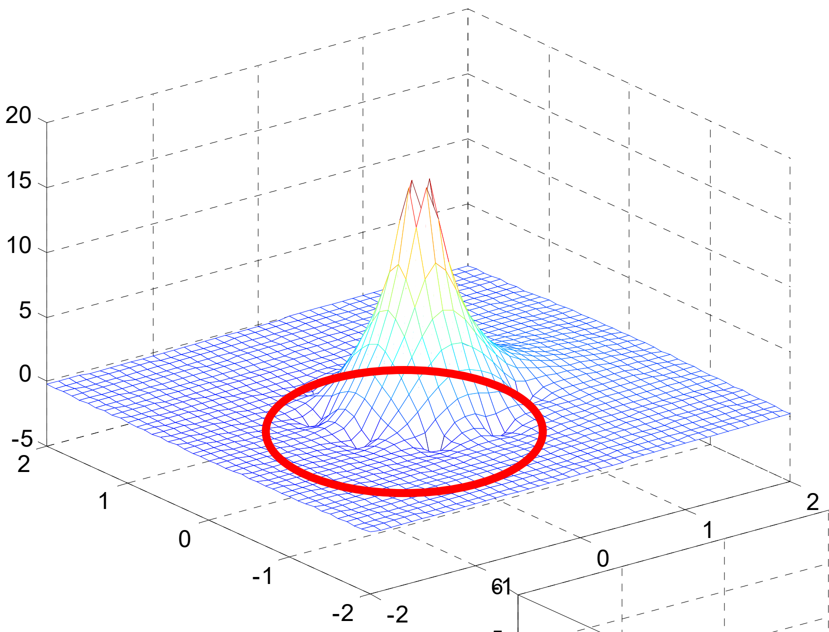


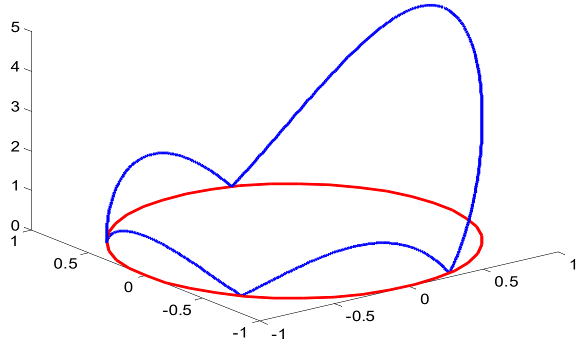


频域采样、时域周期延拓

1离散傅里叶变换DFT

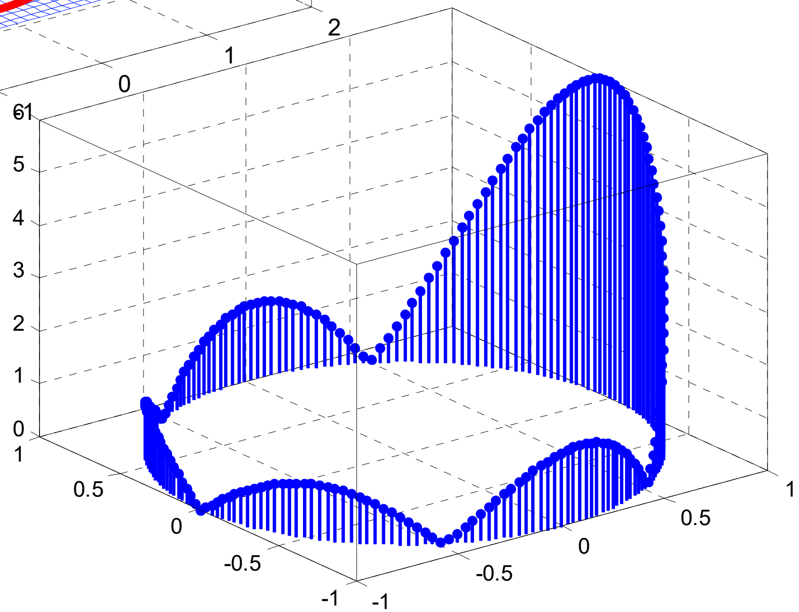
$$X[k] = X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N}$$





$$X(z)\Big|_{z=e^{j\omega}} = X\left(e^{j\omega}\right)$$

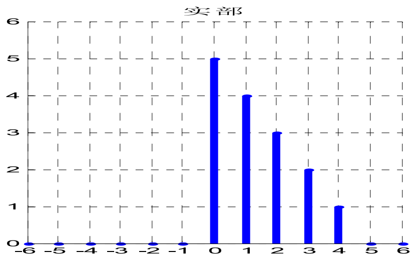
$$X[k] = X(z)\Big|_{z=e^{j2\pi k/N}}$$

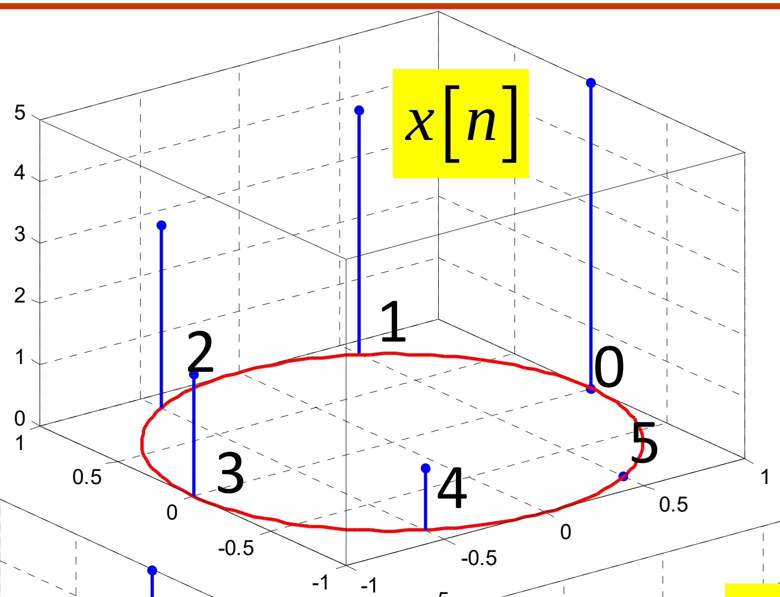


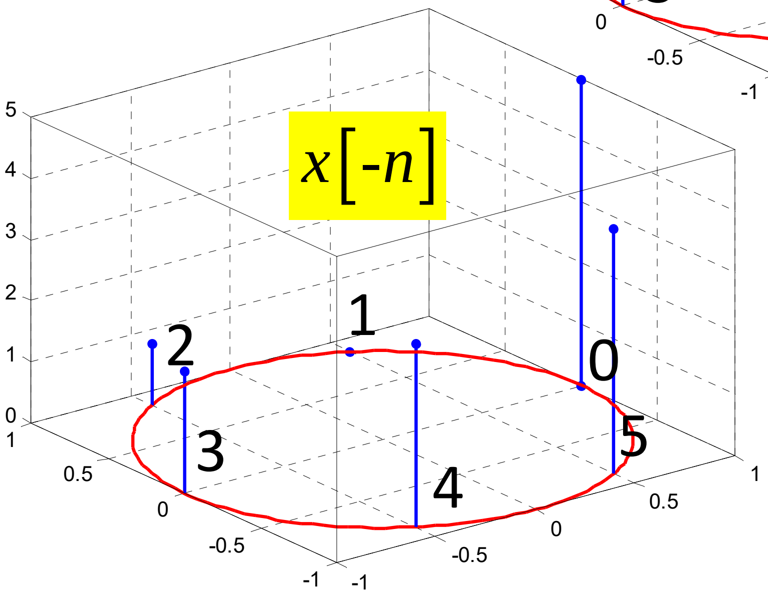
$$\begin{cases}\tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]e^{-jk(2\pi/N)n}\\ \tilde{x}[\mathrm{n}]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{jk(2\pi/N)n}\end{cases}$$

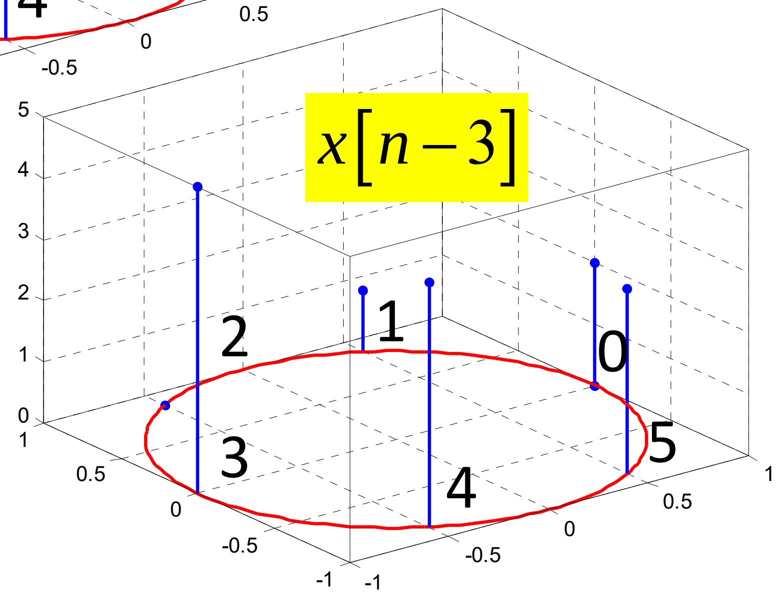
1 离散傅里叶变换DFT

■ 圆周表示









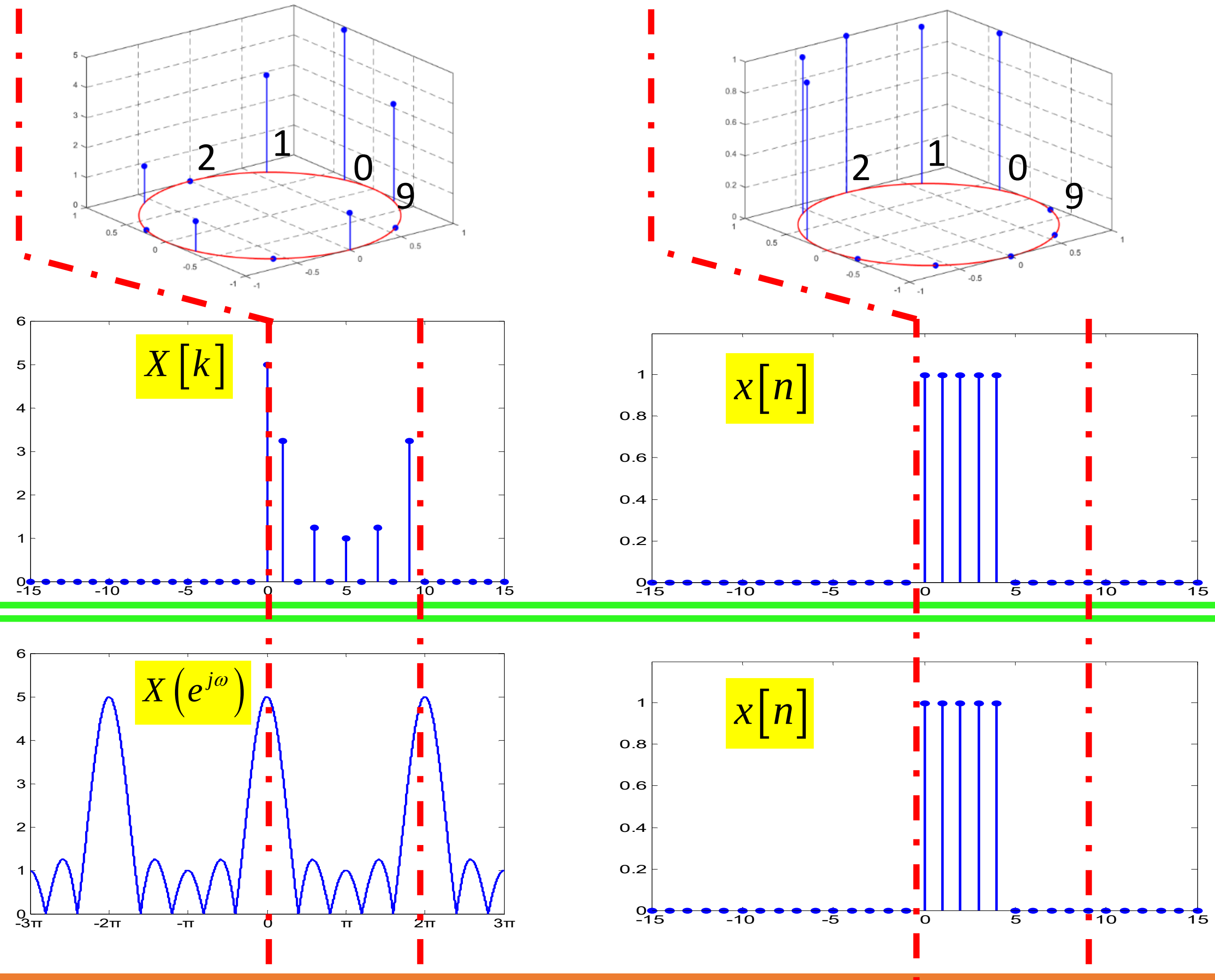
1 DFT

$$X\left[((k))_N\right] = \tilde{X}[k] = X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N}$$

$$x\left[((n))_N\right] = \tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$$

DFT

DTFT



频域采样、时域周期延拓

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$$X\left[((k))_N\right] = \tilde{X}[k] = X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N}$$

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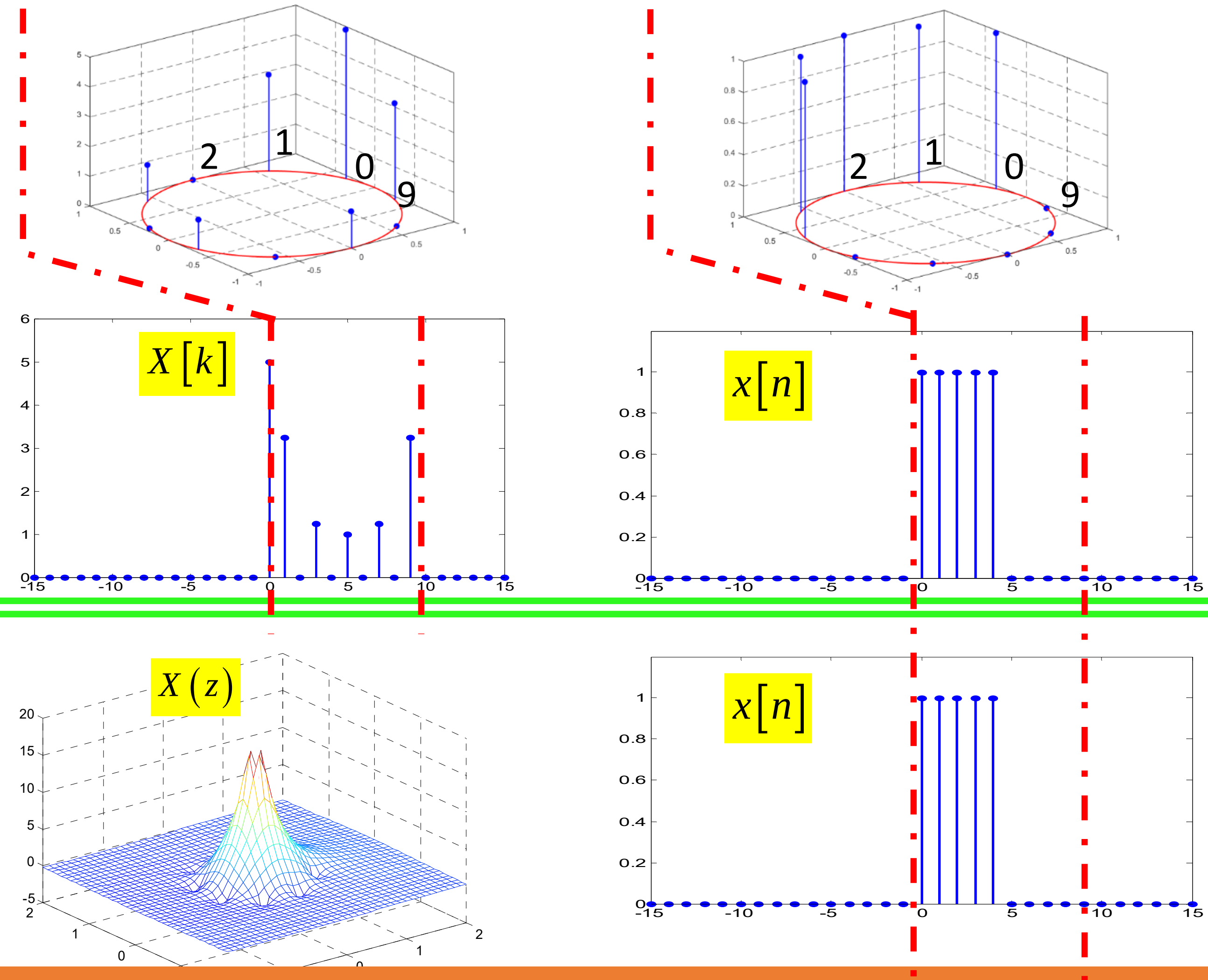
DFT

DTFT

Z变换

$$X[k] = X(z)\Big|_{z=e^{j2\pi k/N}}$$



频域采样、时域周期延拓

1 DFT

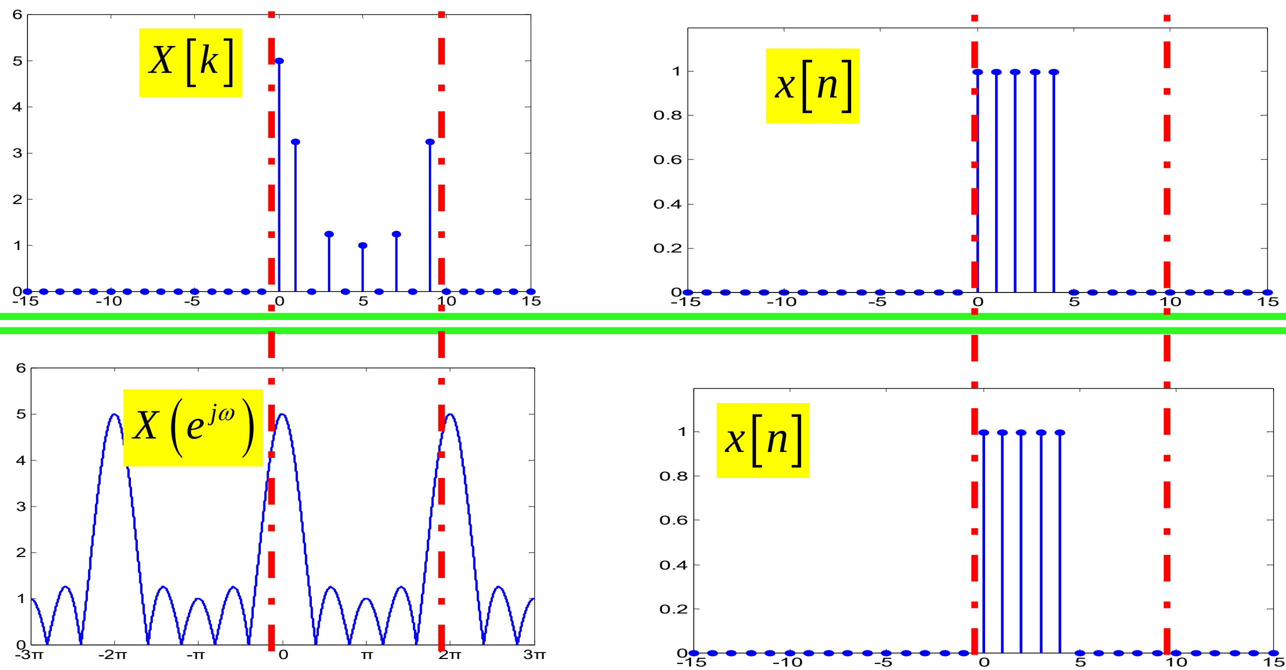
$$X\left[((k))_N\right] = \tilde{X}[k] = X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N}$$

$$x\left[((n))_N\right] = \tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$$

- 长度为L的序列计算N点长的DFT
- N与L之间的关系

DFT

DTFT



频域采样、时域周期延拓

1 DFT

$$X\left[((k))_N\right]=\tilde{X}[k]=X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N}$$

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DFT

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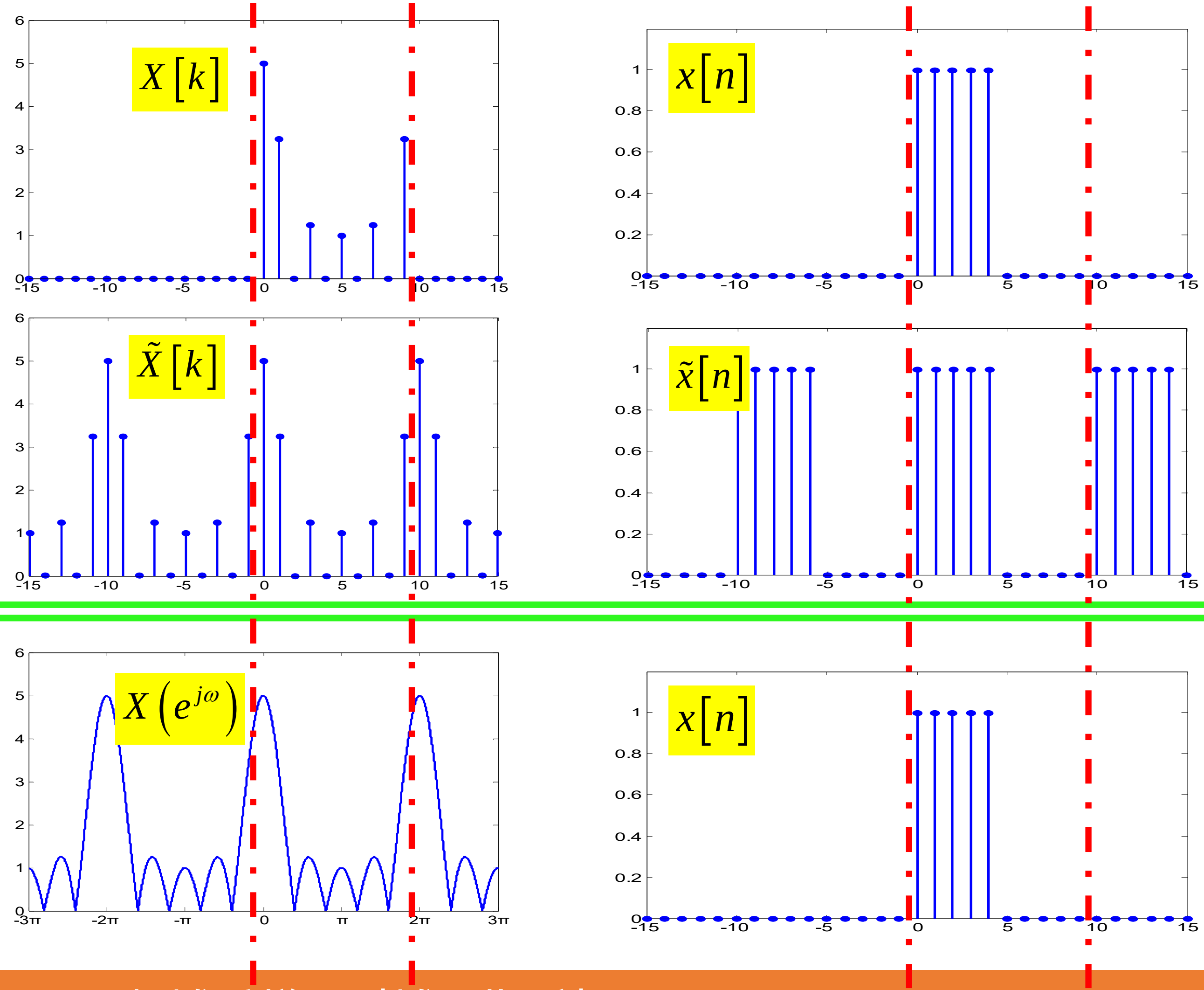
DFS

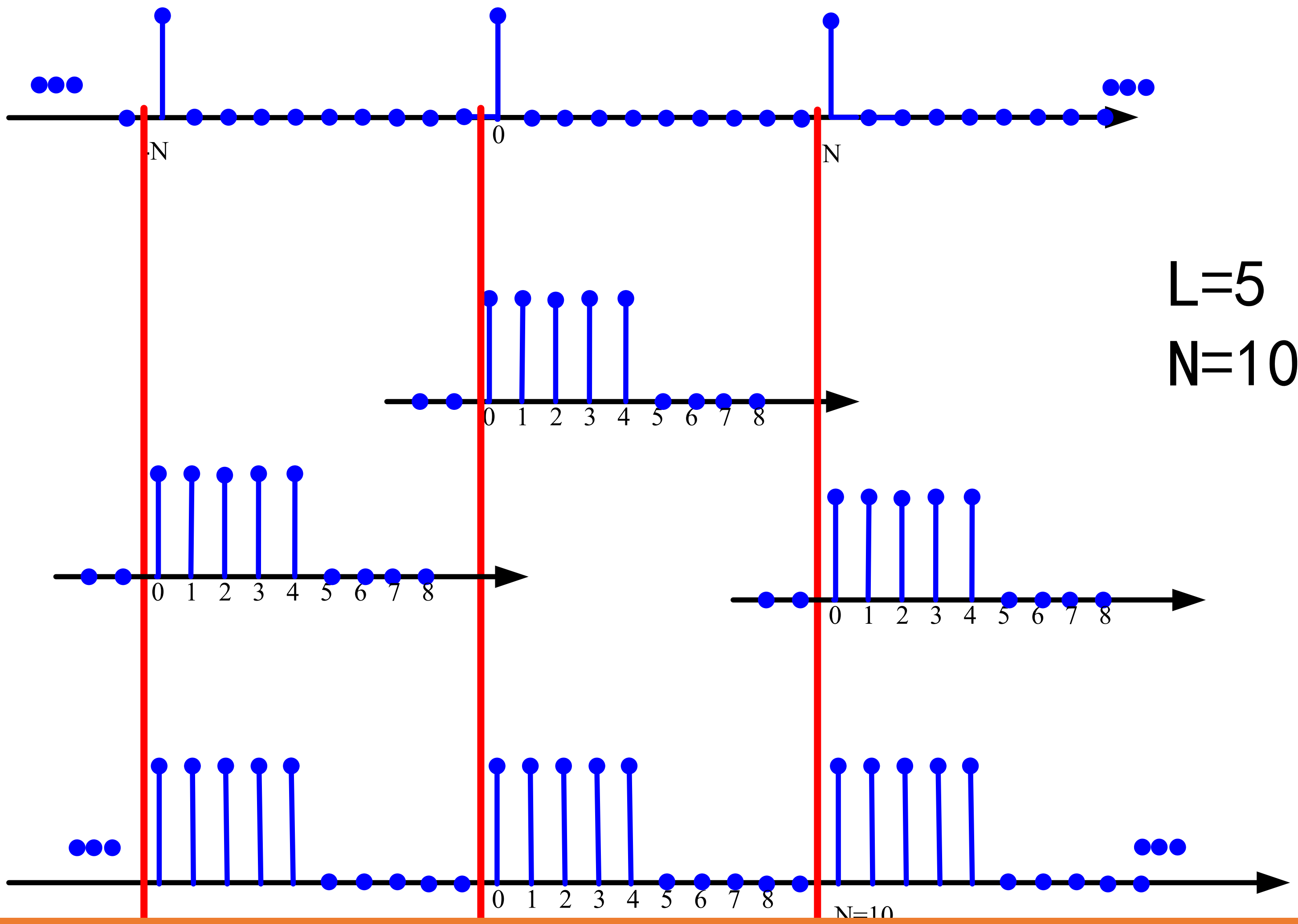
DTFT



频域采样、时域周期延拓

1 DFT

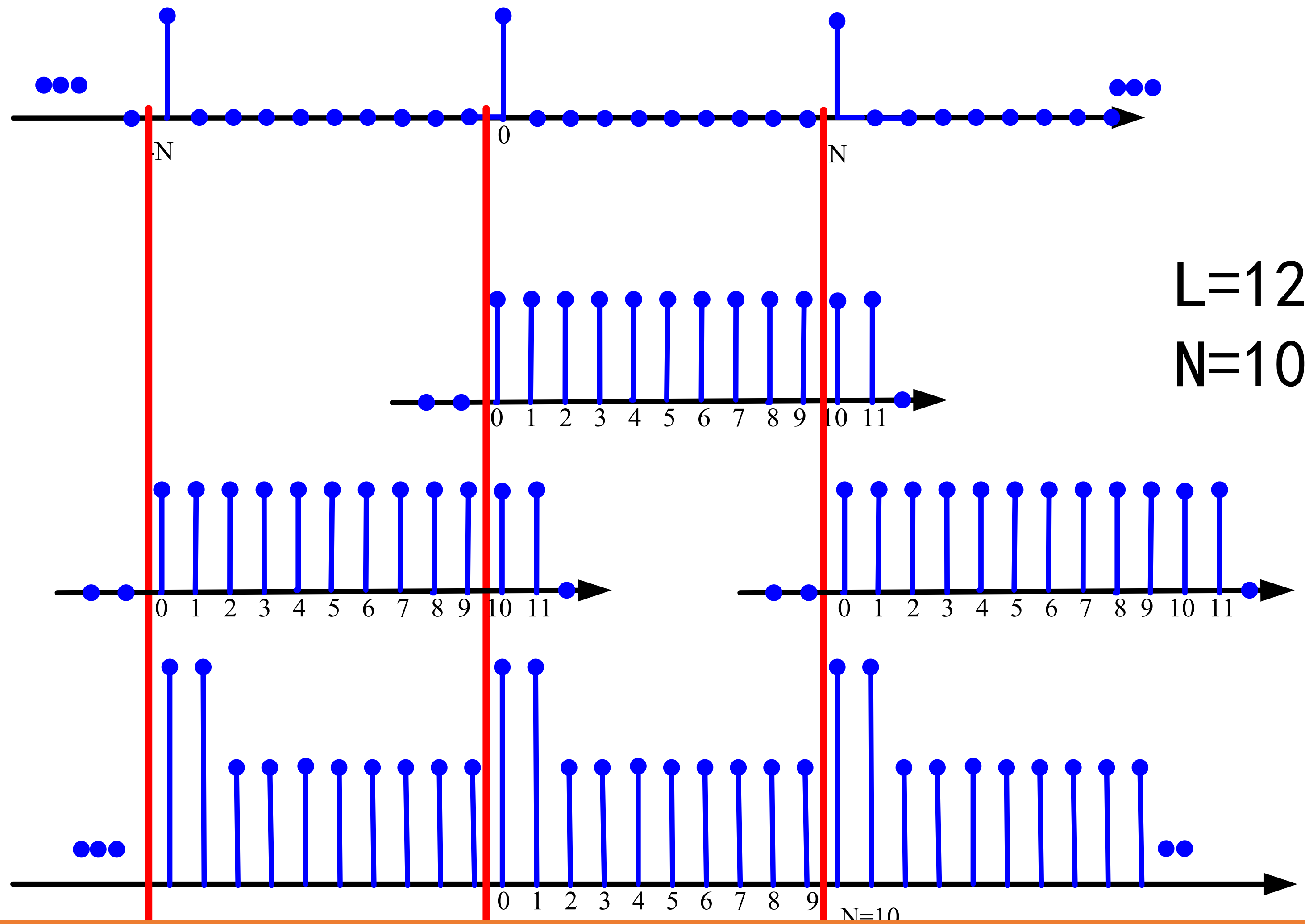
$$x\left[((n))_N\right] = \tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$$



当N>=L时，频域采样不会造成时域混叠

1 DFT

$$x\left[((n))_N\right] = \tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$$



L=12

N=10

当N>=L时，频域采样不会造成时域混叠

1 DFT

$$X[((k))_N] = \tilde{X}[k] = X(e^{j\omega})\Big|_{\omega=2\pi k/N}$$

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DFT是DFS的一个周期

DFS

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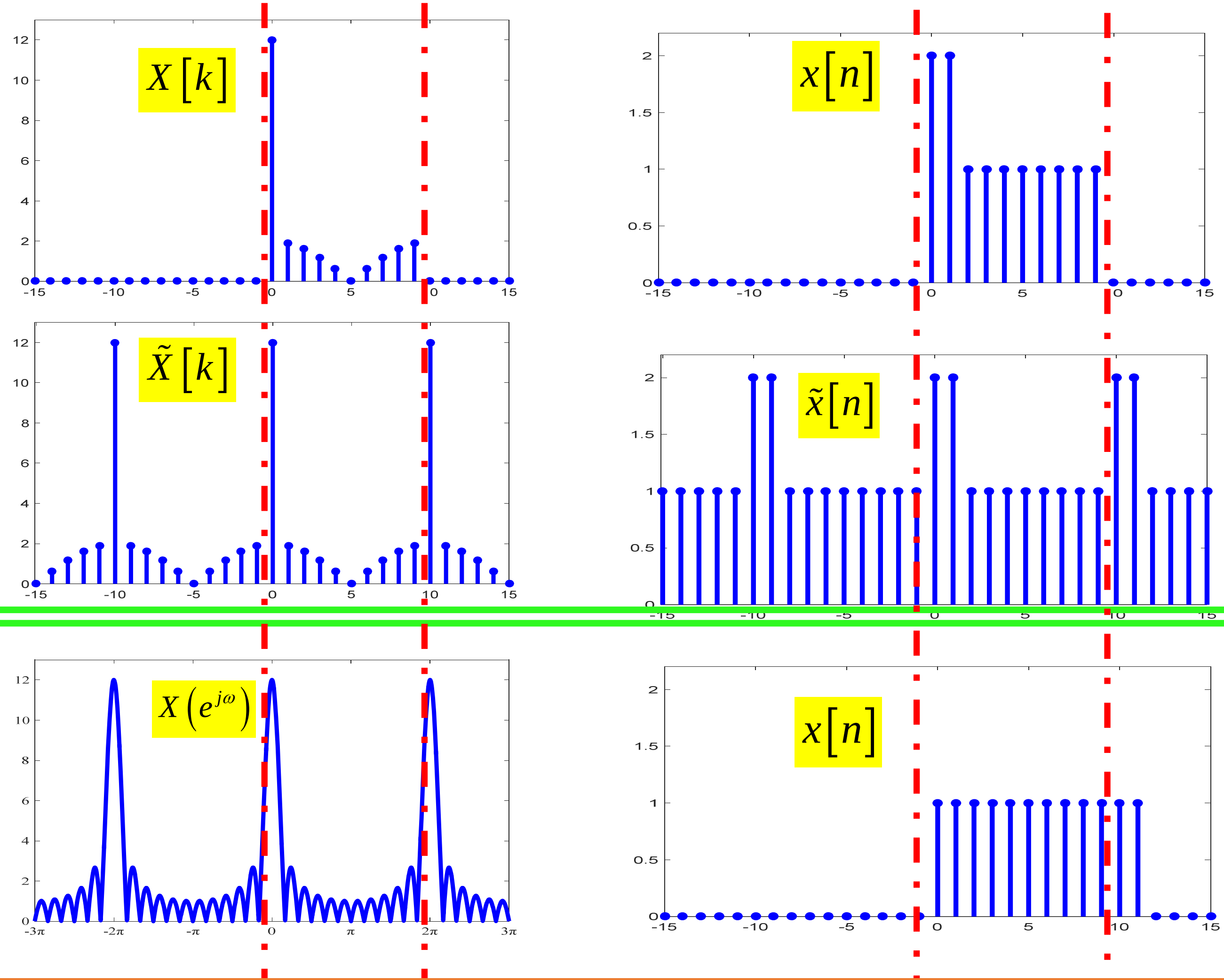

当N<L时，频域采样造成时域混叠

1 DFT

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DFT

DFT是DFS的
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DFS

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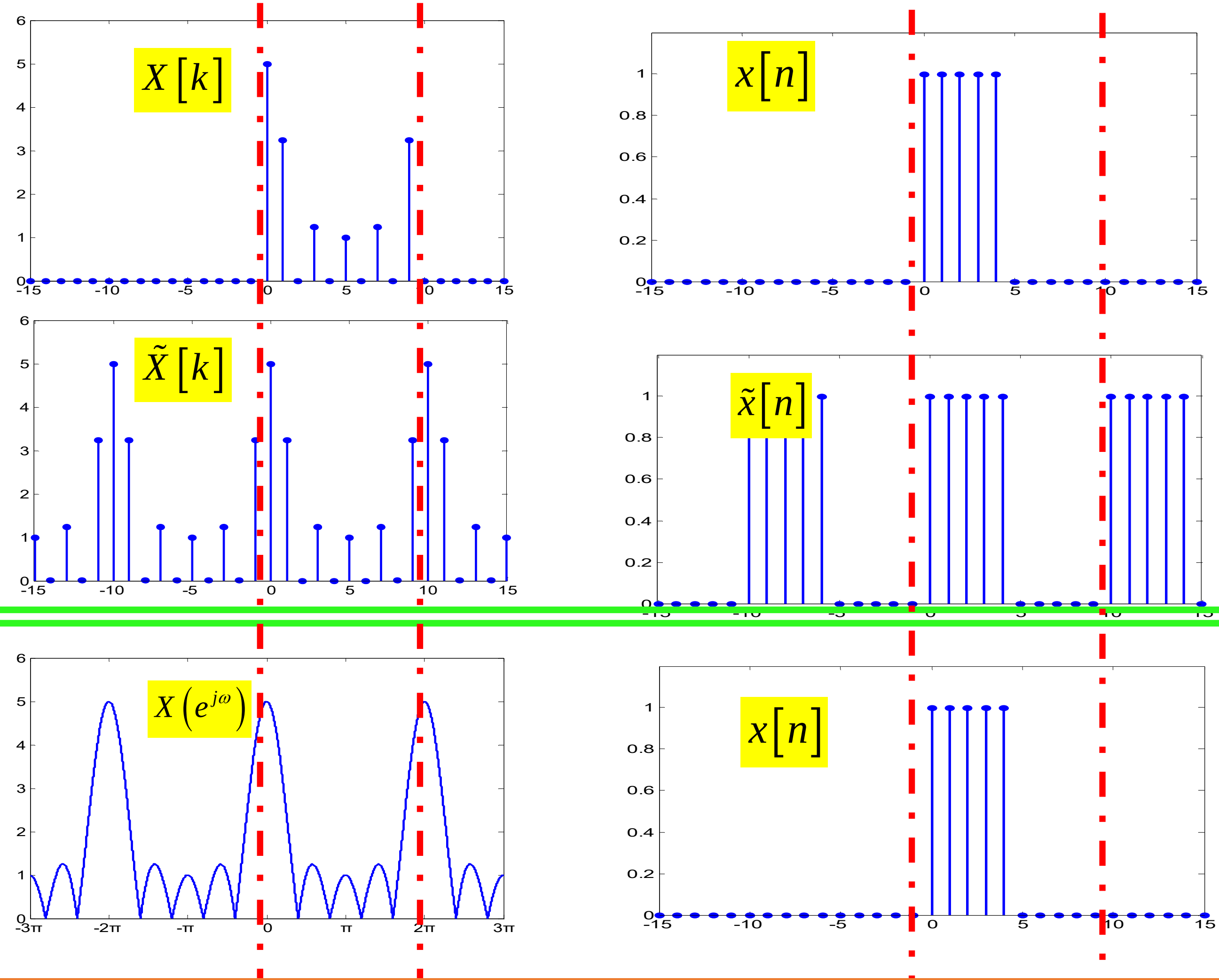

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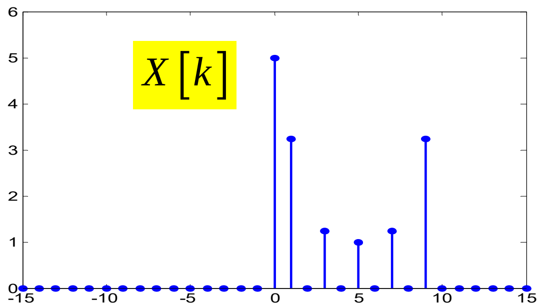
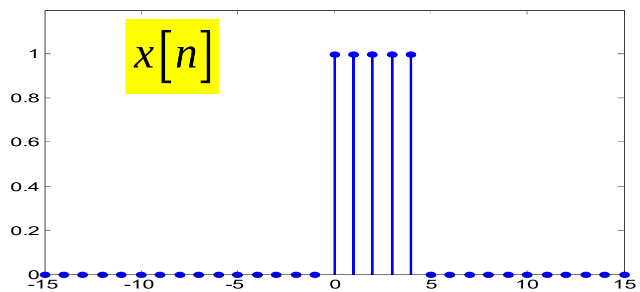
$$\boxed{X\left[((k))_N\right]=\tilde{X}[k]=X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N}}$$

$$\boxed{x\left[((n))_N\right]=\tilde{x}[n]=\sum_{r=-\infty}^{\infty}x[n-rN]}$$

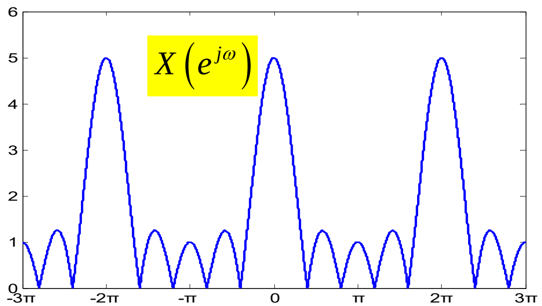
DFT

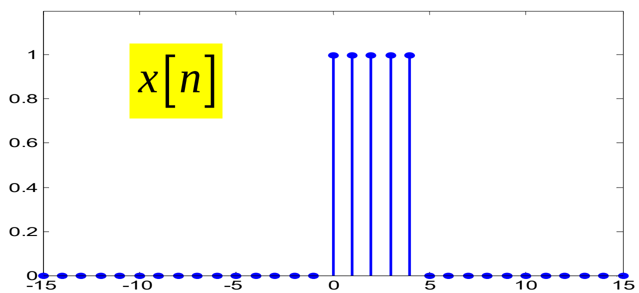




$$X\left(e^{j\omega}\right)=\frac{1}{N}\sum_{k=0}^{N-1}X[k]\frac{\sin\left(\frac{N\omega-2\pi k}{2}\right)}{\sin\left(\frac{N\omega-2\pi k}{2N}\right)}e^{-j\left(\omega-\frac{2\pi k}{N}\right)\frac{N-1}{2}}$$

DTFT





当N>=L时，频域采样不会造成时域混叠

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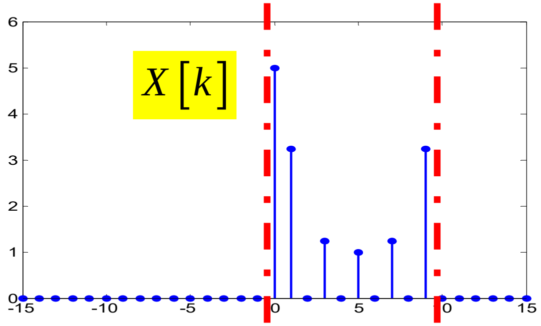
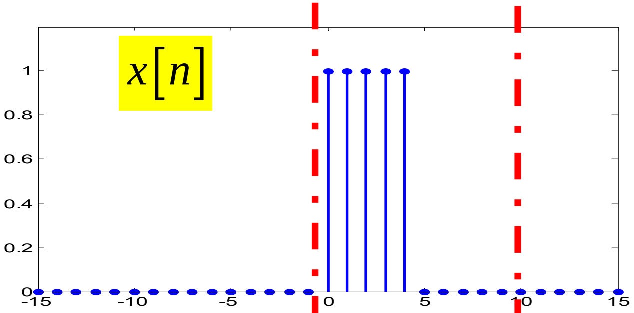
DFT



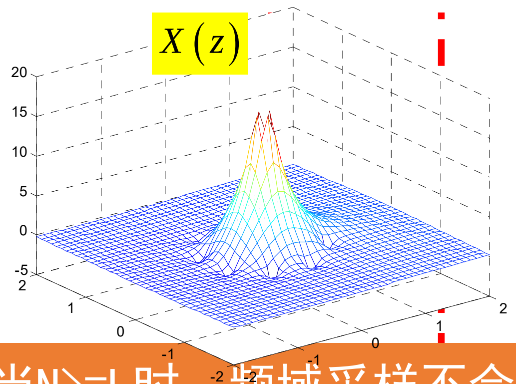


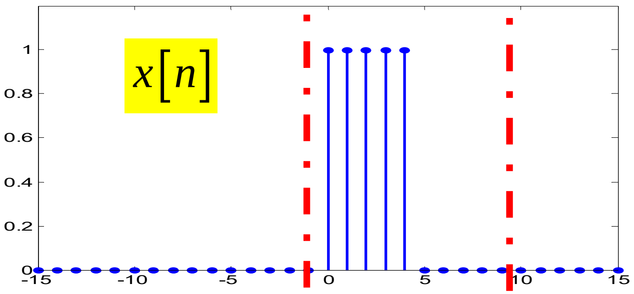
$$X(z)=\frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{X[k]}{1-e^{j\frac{2\pi k}{N}}z^{-1}}$$

DTFT

Z变换

$$X[k]=X(z)\Big|_{z=e^{j2\pi k/N}}$$





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1 离散傅里叶变换DFT