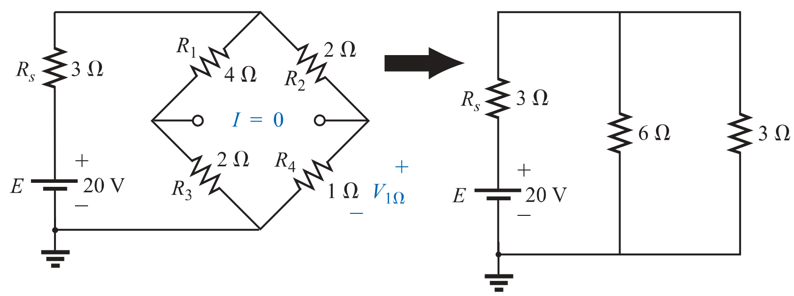


(a) (b)

FIG. 8.69

Substituting the open-circuit equivalent for the balance arm of a balanced bridge.

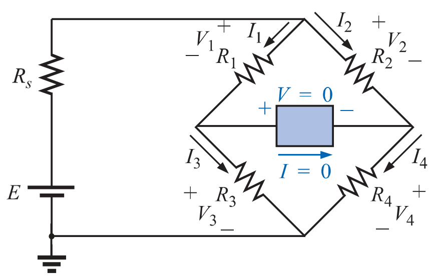


FIG. 8.70

Establishing the balance criteria for a bridge network.

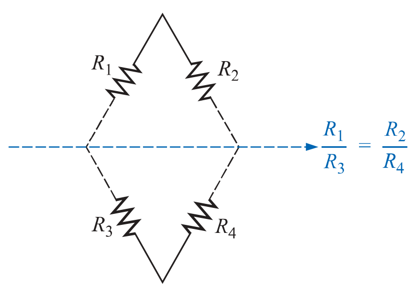


FIG. 8.71

A visual approach to remembering the balance condition.

and
$$I_1R_1 = I_2R_2$$

or
$$I_1 = \frac{I_2R_2}{R_1}$$

In addition, when $V = 0$ V,
$$V_3 = V_4$$

and
$$I_3R_3 = I_4R_4$$

If we set $I = 0$ A, then $I_3 = I_1$ and $I_4 = I_2$, with the result that the above equation becomes
$$I_1R_3 = I_2R_4$$

Substituting for I_1 from above yields
$$\left(\frac{I_2R_2}{R_1}\right)R_3 = I_2R_4$$

or, rearranging, we have
$$\boxed{\frac{R_1}{R_3} = \frac{R_2}{R_4}} \qquad \textbf{(8.4)}$$

This conclusion states that if the ratio of R_1 to R_3 is equal to that of R_2 to R_4, the bridge will be balanced, and $I = 0$ A or $V = 0$ V. A method of memorizing this form is indicated in Fig. 8.71.

For the example above, $R_1 = 4\ \Omega$, $R_2 = 2\ \Omega$, $R_3 = 2\ \Omega$, $R_4 = 1\ \Omega$, and
$$\frac{R_1}{R_3} = \frac{R_2}{R_4} \rightarrow \frac{4\ \Omega}{2\ \Omega} = \frac{2\ \Omega}{1\ \Omega} = 2$$

The emphasis in this section has been on the balanced situation. Understand that if the ratio is not satisfied, there will be a potential drop across the balance arm and a current through it. The methods just described (mesh and nodal analysis) will yield any and all potentials or currents desired, just as they did for the balanced situation.

8.12 Y-Δ (T-π) AND Δ-Y (π-T) CONVERSIONS

Circuit configurations are often encountered in which the resistors do not appear to be in series or parallel. Under these conditions, it may be necessary to convert the circuit from one form to another to solve for any unknown quantities if mesh or nodal analysis is not applied. Two circuit configurations that often account for these difficulties are the **wye (Y)** and **delta (Δ) configurations,** depicted in Fig. 8.72(a). They are also referred to as the **tee (T)** and **pi (π),** respectively, as indicated in Fig. 8.72(b). Note that the pi is actually an inverted delta.

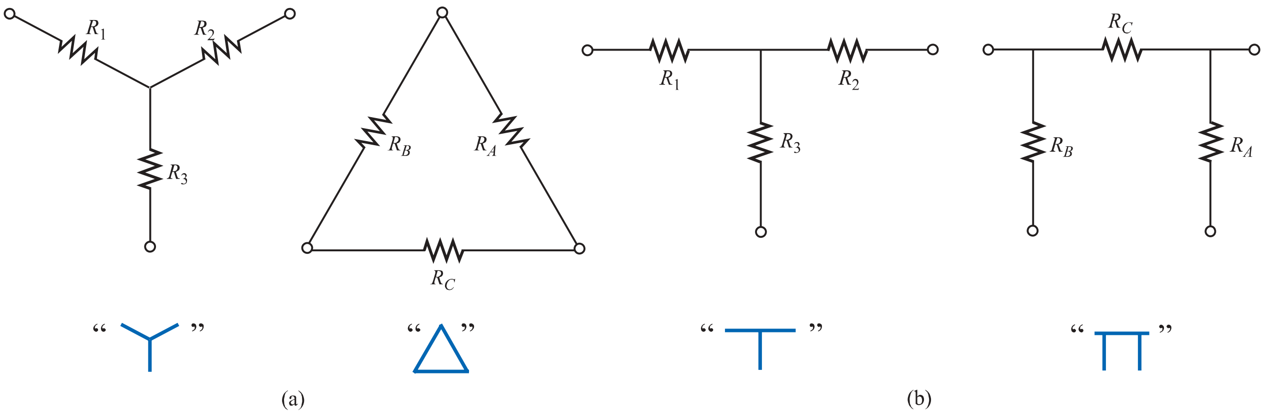


FIG. 8.72
The Y (T) and Δ (π) configurations.

The purpose of this section is to develop the equations for converting from Δ to Y, or vice versa. This type of conversion will normally lead to a network that can be solved using techniques such as those described in Chapter 7. In other words, in Fig. 8.73, with terminals *a, b,* and *c* held fast, if the wye (Y) configuration were desired *instead of* the inverted delta (Δ) configuration, all that would be necessary is a direct application of the equations to be derived. The phrase *instead of* is emphasized to ensure that it is understood that only one of these configurations is to appear at one time between the indicated terminals.

It is our purpose (referring to Fig. 8.73) to find some expression for R_1, R_2, and R_3 in terms of R_A, R_B, and R_C, and vice versa, that will ensure that the resistance between any two terminals of the Y configuration will be the same with the Δ configuration inserted in place of the Y configuration (and vice versa). If the two circuits are to be equivalent, the total resistance between any two terminals must be the same. Consider terminals *a-c* in the Δ-Y configurations of Fig. 8.74.

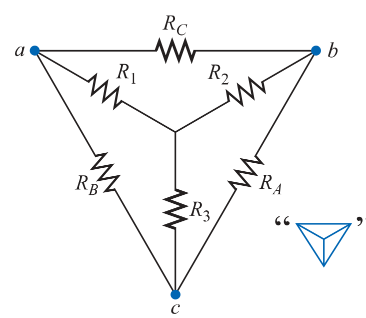


FIG. 8.73
Introducing the concept of Δ-Y or Y-Δ conversions.

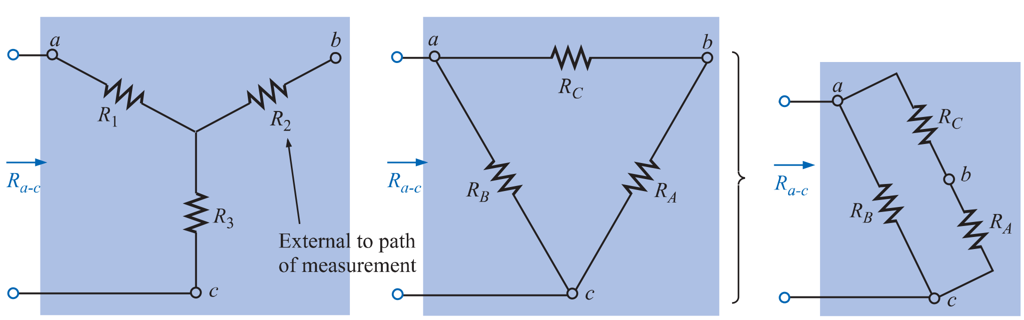


FIG. 8.74
Finding the resistance $R_{a\text{-}c}$ for the Y and Δ configurations.

Let us first assume that we want to convert the Δ (R_A, R_B, R_C) to the Y (R_1, R_2, R_3). This requires that we have a relationship for R_1, R_2, and R_3 in terms of R_A, R_B, and R_C. If the resistance is to be the same between terminals *a-c* for both the Δ and the Y, the following must be true:

$$R_{a\text{-}c}\,(\mathrm{Y}) = R_{a\text{-}c}\,(\Delta)$$

so that

$$\boxed{R_{a\text{-}c} = R_1 + R_3 = \frac{R_B(R_A + R_C)}{R_B + (R_A + R_C)}} \qquad \textbf{(8.5a)}$$

Using the same approach for *a-b* and *b-c*, we obtain the following relationships:

$$\boxed{R_{a\text{-}b} = R_1 + R_2 = \frac{R_C(R_A + R_B)}{R_C + (R_A + R_B)}} \qquad \textbf{(8.5b)}$$

and

$$\boxed{R_{b\text{-}c} = R_2 + R_3 = \frac{R_A(R_B + R_C)}{R_A + (R_B + R_C)}} \qquad \textbf{(8.5c)}$$

Subtracting Eq. (8.5a) from Eq. (8.5b), we have

$$(R_1 + R_2) - (R_1 + R_3) = \left(\frac{R_C R_B + R_C R_A}{R_A + R_B + R_C}\right) - \left(\frac{R_B R_A + R_B R_C}{R_A + R_B + R_C}\right)$$

so that

$$\boxed{R_2 - R_3 = \frac{R_A R_C - R_B R_A}{R_A + R_B + R_C}} \qquad \textbf{(8.5d)}$$

Subtracting Eq. (8.5d) from Eq. (8.5c) yields

$$(R_2 + R_3) - (R_2 - R_3) = \left(\frac{R_A R_B + R_A R_C}{R_A + R_B + R_C}\right) - \left(\frac{R_A R_C - R_B R_A}{R_A + R_B + R_C}\right)$$

so that

$$2R_3 = \frac{2R_B R_A}{R_A + R_B + R_C}$$

resulting in the following expression for R_3 in terms of R_A, R_B, and R_C:

$$\boxed{R_3 = \frac{R_A R_B}{R_A + R_B + R_C}} \qquad \textbf{(8.6a)}$$

Following the same procedure for R_1 and R_2, we have

$$\boxed{R_1 = \frac{R_B R_C}{R_A + R_B + R_C}} \qquad \textbf{(8.6b)}$$

and

$$\boxed{R_2 = \frac{R_A R_C}{R_A + R_B + R_C}} \qquad \textbf{(8.6c)}$$

Note that each resistor of the Y is equal to the product of the resistors in the two closest branches of the Δ divided by the sum of the resistors in the Δ.

To obtain the relationships necessary to convert from a Y to a Δ, first divide Eq. (8.6a) by Eq. (8.6b):

$$\frac{R_3}{R_1} = \frac{(R_A R_B)/(R_A + R_B + R_C)}{(R_B R_C)/(R_A + R_B + R_C)} = \frac{R_A}{R_C}$$

or
$$R_A = \frac{R_C R_3}{R_1}$$

Then divide Eq. (8.6a) by Eq. (8.6c):

$$\frac{R_3}{R_2} = \frac{(R_A R_B)/(R_A + R_B + R_C)}{(R_A R_C)/(R_A + R_B + R_C)} = \frac{R_B}{R_C}$$

or
$$R_B = \frac{R_3 R_C}{R_2}$$

Substituting for R_A and R_B in Eq. (8.6c) yields

$$R_2 = \frac{(R_C R_3/R_1)R_C}{(R_3 R_C/R_2) + (R_C R_3/R_1) + R_C}$$
$$= \frac{(R_3/R_1)R_C}{(R_3/R_2) + (R_3/R_1) + 1}$$

Placing these over a common denominator, we obtain

$$R_2 = \frac{(R_3 R_C/R_1)}{(R_1 R_2 + R_1 R_3 + R_2 R_3)/(R_1 R_2)}$$
$$= \frac{R_2 R_3 R_C}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

and
$$\boxed{R_C = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_3}} \qquad \textbf{(8.7a)}$$

We follow the same procedure for R_B and R_A:

$$\boxed{R_A = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_1}} \qquad \textbf{(8.7b)}$$

and
$$\boxed{R_B = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_2}} \qquad \textbf{(8.7c)}$$

Note that the value of each resistor of the Δ is equal to the sum of the possible product combinations of the resistances of the Y divided by the resistance of the Y farthest from the resistor to be determined.

Let us consider what would occur if all the values of a Δ or Y were the same. If $R_A = R_B = R_C$, Equation (8.6a) would become (using R_A only) the following:

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C} = \frac{R_A R_A}{R_A + R_A + R_A} = \frac{R_A^2}{3R_A} = \frac{R_A}{3}$$

and, following the same procedure,

$$R_1 = \frac{R_A}{3} \qquad R_2 = \frac{R_A}{3}$$

In general, therefore,

$$\boxed{R_Y = \frac{R_\Delta}{3}} \tag{8.8a}$$

or

$$\boxed{R_\Delta = 3R_Y} \tag{8.8b}$$

which indicates that *for a* Y *of three equal resistors, the value of each resistor of the* Δ *is equal to three times the value of any resistor of the* Y. If only two elements of a Y or a Δ are the same, the corresponding Δ or Y of each will also have two equal elements. The converting of equations will be left as an exercise for the reader.

The Y and the Δ will often appear as shown in Fig. 8.75. They are then referred to as a **tee (T)** and a **pi (π)** network, respectively. The equations used to convert from one form to the other are exactly the same as those developed for the Y and Δ transformation.

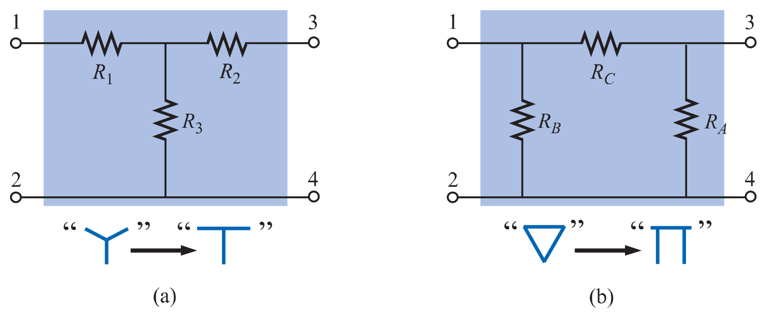


FIG. 8.75
The relationship between the Y and T configurations and the Δ and π configurations.

EXAMPLE 8.27 Convert the Δ of Fig. 8.76 to a Y.

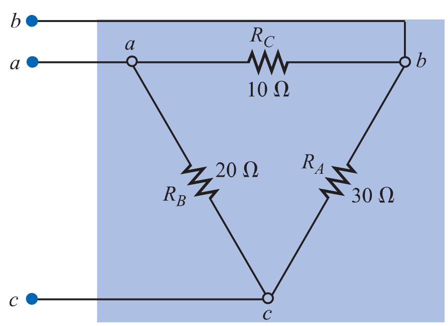


FIG. 8.76
Example 8.27.

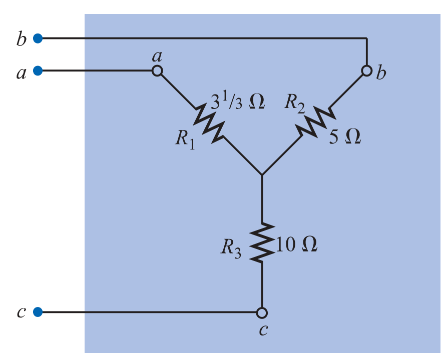


FIG. 8.77
The Y equivalent for the Δ of Fig. 8.76.

Solution:

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} = \frac{(20\ \Omega)(10\ \Omega)}{30\ \Omega + 20\ \Omega + 10\ \Omega} = \frac{200\ \Omega}{60} = \mathbf{3\tfrac{1}{3}\ \Omega}$$

$$R_2 = \frac{R_A R_C}{R_A + R_B + R_C} = \frac{(30\ \Omega)(10\ \Omega)}{60\ \Omega} = \frac{300\ \Omega}{60} = \mathbf{5\ \Omega}$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C} = \frac{(20\ \Omega)(30\ \Omega)}{60\ \Omega} = \frac{600\ \Omega}{60} = \mathbf{10\ \Omega}$$

The equivalent network is shown in Fig. 8.77 (page 298).

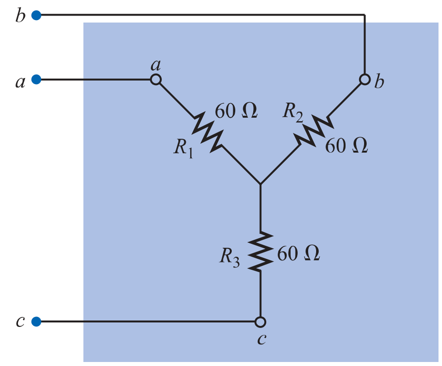


FIG. 8.78
Example 8.28.

EXAMPLE 8.28 Convert the Y of Fig. 8.78 to a Δ.

Solution:

$$R_A = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_1}$$

$$= \frac{(60\ \Omega)(60\ \Omega) + (60\ \Omega)(60\ \Omega) + (60\ \Omega)(60\ \Omega)}{60\ \Omega}$$

$$= \frac{3600\ \Omega + 3600\ \Omega + 3600\ \Omega}{60} = \frac{10{,}800\ \Omega}{60}$$

$$R_A = \mathbf{180\ \Omega}$$

However, the three resistors for the Y are equal, permitting the use of Eq. (8.8) and yielding

$$R_\Delta = 3R_Y = 3(60\ \Omega) = 180\ \Omega$$

and
$$R_B = R_C = \mathbf{180\ \Omega}$$

The equivalent network is shown in Fig. 8.79.

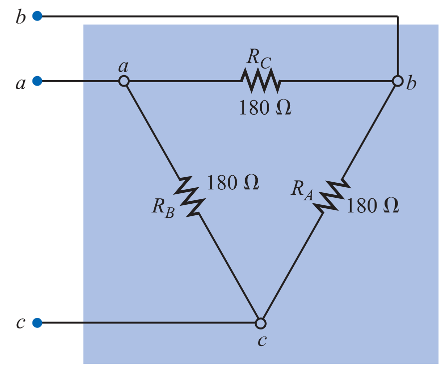


FIG. 8.79
The Δ equivalent for the Y of Fig. 8.78.

EXAMPLE 8.29 Find the total resistance of the network of Fig. 8.80, where $R_A = 3\ \Omega$, $R_B = 3\ \Omega$, and $R_C = 6\ \Omega$.

Solution:

Two resistors of the Δ were equal; therefore, two resistors of the Y will be equal.

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} = \frac{(3\ \Omega)(6\ \Omega)}{3\ \Omega + 3\ \Omega + 6\ \Omega} = \frac{18\ \Omega}{12} = \mathbf{1.5\ \Omega}$$

$$R_2 = \frac{R_A R_C}{R_A + R_B + R_C} = \frac{(3\ \Omega)(6\ \Omega)}{12\ \Omega} = \frac{18\ \Omega}{12} = \mathbf{1.5\ \Omega}$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C} = \frac{(3\ \Omega)(3\ \Omega)}{12\ \Omega} = \frac{9\ \Omega}{12} = \mathbf{0.75\ \Omega}$$

Replacing the Δ by the Y, as shown in Fig. 8.81, yields

$$R_T = 0.75\ \Omega + \frac{(4\ \Omega + 1.5\ \Omega)(2\ \Omega + 1.5\ \Omega)}{(4\ \Omega + 1.5\ \Omega) + (2\ \Omega + 1.5\ \Omega)}$$

$$= 0.75\ \Omega + \frac{(5.5\ \Omega)(3.5\ \Omega)}{5.5\ \Omega + 3.5\ \Omega}$$

$$= 0.75\ \Omega + 2.139\ \Omega$$

$$R_T = \mathbf{2.889\ \Omega}$$

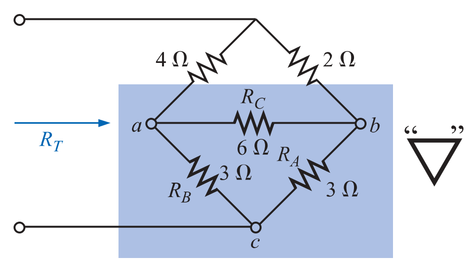


FIG. 8.80
Example 8.29.

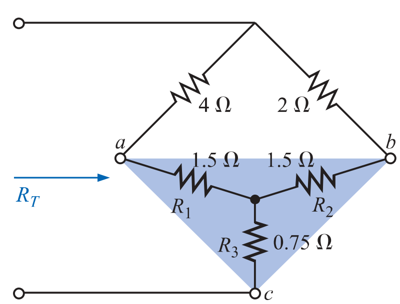


FIG. 8.81
Substituting the Y equivalent for the bottom Δ of Fig. 8.80.

9

Network Theorems

9.1 INTRODUCTION

This chapter will introduce the important fundamental theorems of network analysis. Included are the **superposition, Thévenin's, Norton's, maximum power transfer, substitution, Millman's,** and **reciprocity theorems.** We will consider a number of areas of application for each. A thorough understanding of each theorem is important because a number of the theorems will be applied repeatedly in the material to follow.

9.2 SUPERPOSITION THEOREM

The **superposition theorem,** like the methods of the last chapter, can be used to find the solution to networks with two or more sources that are not in series or parallel. The most obvious advantage of this method is that it does not require the use of a mathematical technique such as determinants to find the required voltages or currents. Instead, each source is treated independently, and the algebraic sum is found to determine a particular unknown quantity of the network.

The superposition theorem states the following:

The current through, or voltage across, an element in a linear bilateral network is equal to the algebraic sum of the currents or voltages produced independently by each source.

When one is applying the theorem, it is possible to consider the effects of two sources at the same time and reduce the number of networks that have to be analyzed, but, in general,

$$\boxed{\begin{matrix}\text{Number of networks}\\ \text{to be analyzed}\end{matrix} = \begin{matrix}\text{Number of}\\ \text{independent sources}\end{matrix}} \qquad \textbf{(9.1)}$$

To consider the effects of each source independently requires that sources be removed and replaced without affecting the final result. To

This results because 2 A + 6 A = 8 A, but

$$(2\text{ A})^2 + (6\text{ A})^2 \neq (8\text{ A})^2$$

As mentioned previously, the superposition principle is not applicable to power effects since power is proportional to the square of the current or voltage (I^2R or V^2/R).

Figure 9.14 is a plot of the power delivered to the 6-Ω resistor versus current.

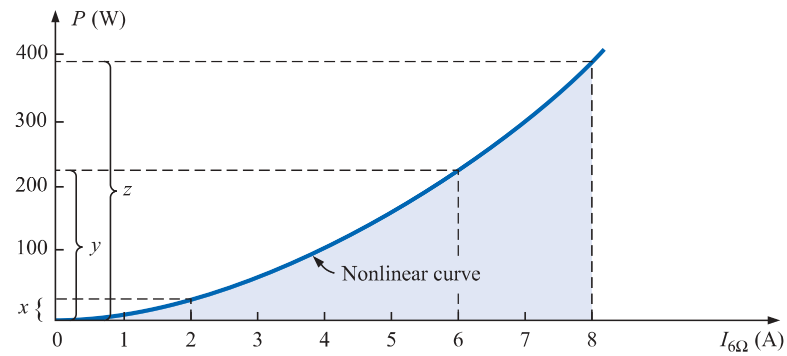


FIG. 9.14
Plotting the power delivered to the 6-Ω resistor versus current through the resistor.

Obviously, $x + y \neq z$, or 24 W + 216 W ≠ 384 W, and superposition does not hold. However, for a linear relationship, such as that between the voltage and current of the fixed-type 6-Ω resistor, superposition can be applied, as demonstrated by the graph of Fig. 9.15, where $a + b = c$, or 2 A + 6 A = 8 A.

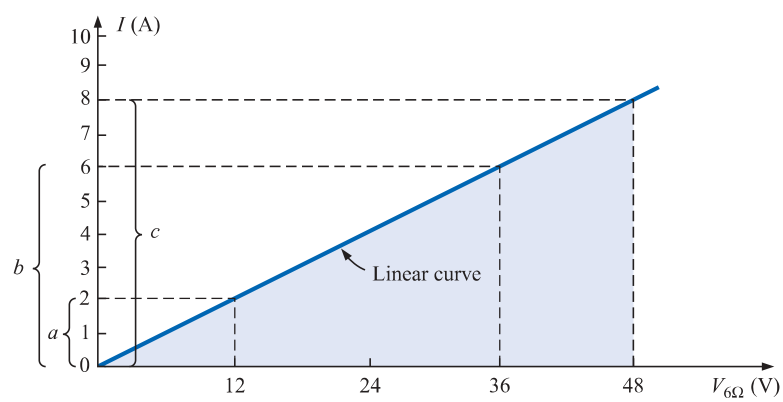


FIG. 9.15
Plotting I versus V for the 6-Ω resistor.

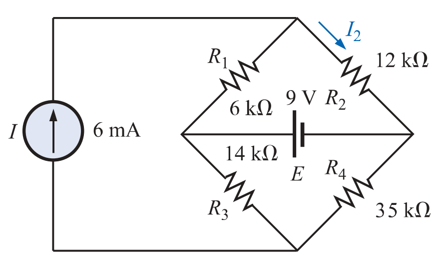


FIG. 9.16
Example 9.4.

EXAMPLE 9.4 Using the principle of superposition, find the current I_2 through the 12-kΩ resistor of Fig. 9.16.

Solution: *Considering the effect of the 6-mA current source (Fig. 9.17):*

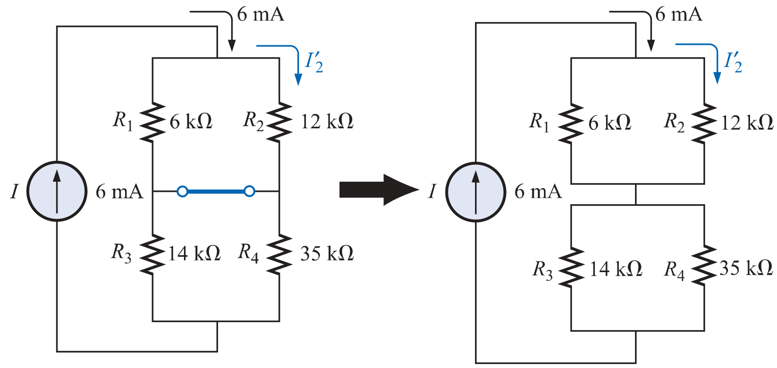


FIG. 9.17
The effect of the current source I on the current I_2.

Current divider rule:

$$I'_2 = \frac{R_1 I}{R_1 + R_2} = \frac{(6\ \text{k}\Omega)(6\ \text{mA})}{6\ \text{k}\Omega + 12\ \text{k}\Omega} = 2\ \text{mA}$$

Considering the effect of the 9-V voltage source (Fig. 9.18):

$$I''_2 = \frac{E}{R_1 + R_2} = \frac{9\ \text{V}}{6\ \text{k}\Omega + 12\ \text{k}\Omega} = 0.5\ \text{mA}$$

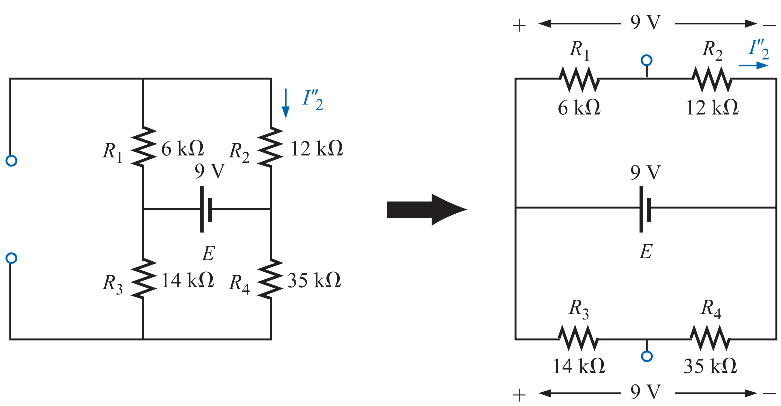


FIG. 9.18
The effect of the voltage source E on the current I_2.

Since I'_2 and I''_2 have the same direction through R_2, the desired current is the sum of the two:

$$\begin{aligned} I_2 &= I'_2 + I''_2 \\ &= 2\ \text{mA} + 0.5\ \text{mA} \\ &= \mathbf{2.5\ mA} \end{aligned}$$

EXAMPLE 9.5 Find the current through the 2-Ω resistor of the network of Fig. 9.19. The presence of three sources will result in three different networks to be analyzed.

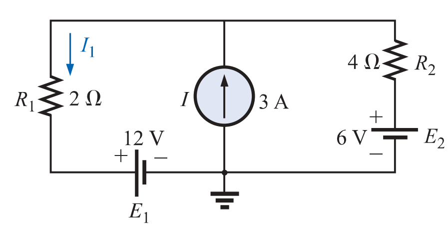


FIG. 9.19
Example 9.5.

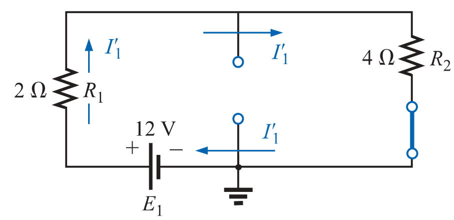


FIG. 9.20
The effect of E_1 on the current I.

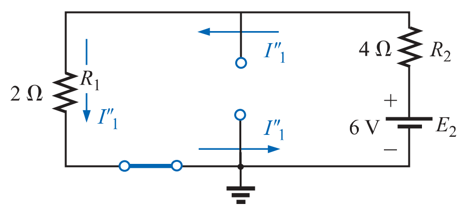


FIG. 9.21
The effect of E_2 on the current I_1.

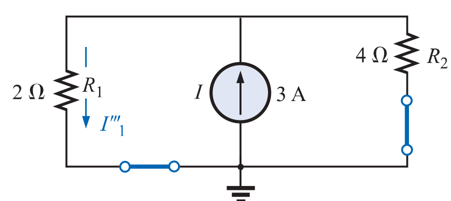


FIG. 9.22
The effect of I on the current I_1.

Solution: *Considering the effect of the 12-V source (Fig. 9.20):*

$$I'_1 = \frac{E_1}{R_1 + R_2} = \frac{12\text{ V}}{2\ \Omega + 4\ \Omega} = \frac{12\text{ V}}{6\ \Omega} = 2\text{ A}$$

Considering the effect of the 6-V source (Fig. 9.21):

$$I''_1 = \frac{E_2}{R_1 + R_2} = \frac{6\text{ V}}{2\ \Omega + 4\ \Omega} = \frac{6\text{ V}}{6\ \Omega} = 1\text{ A}$$

Considering the effect of the 3-A source (Fig. 9.22):
Applying the current divider rule,

$$I'''_1 = \frac{R_2 I}{R_1 + R_2} = \frac{(4\ \Omega)(3\text{ A})}{2\ \Omega + 4\ \Omega} = \frac{12\text{ A}}{6} = 2\text{ A}$$

The total current through the 2-Ω resistor appears in Fig. 9.23, and

$$I_1 = \overbrace{I''_1 + I'''_1}^{\text{Same direction as } I_1 \text{ in Fig. 9.19}} - \overset{\text{Opposite direction to } I_1 \text{ in Fig. 9.19}}{I'_1}$$
$$= 1\text{A} + 2\text{A} - 2\text{A} = 1\textbf{A}$$

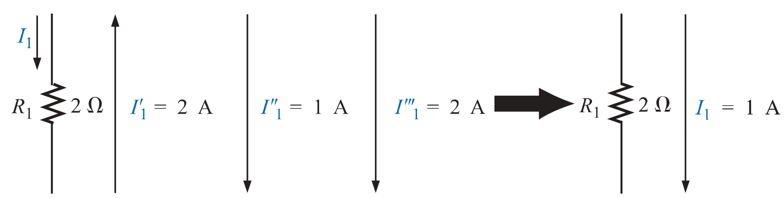


FIG. 9.23
The resultant current I_1.

9.3 THÉVENIN'S THEOREM

Thévenin's theorem states the following:

Any two-terminal, linear bilateral dc network can be replaced by an equivalent circuit consisting of a voltage source and a series resistor, as shown in Fig. 9.24.

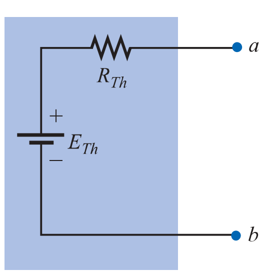
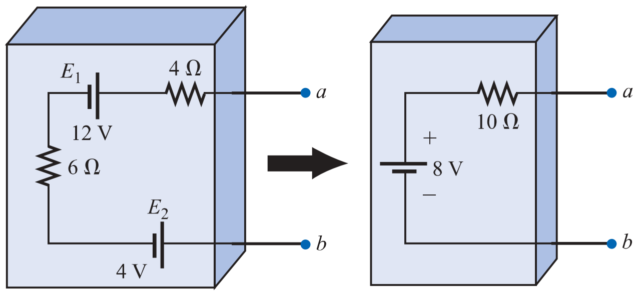


FIG. 9.24
Thévenin equivalent circuit.

In Fig. 9.25(a), for example, the network within the container has only two terminals available to the outside world, labeled *a* and *b*. It is possible using Thévenin's theorem to replace everything in the container with one source and one resistor, as shown in Fig. 9.25(b), and maintain the same terminal characteristics at terminals *a* and *b*. That is, any load connected to terminals *a* and *b* will not know whether it is hooked up to the network of Fig. 9.25(a) or Fig. 9.25(b). The load will receive the same current, voltage, and power from either configuration of Fig. 9.25. Throughout the discussion to follow, however, always keep in mind that

the Thévenin equivalent circuit provides an equivalence at the terminals only—the internal construction and characteristics of the original network and the Thévenin equivalent are usually quite different.



(a) (b)

FIG. 9.25
The effect of applying Thévenin's theorem.

For the network of Fig. 9.25(a), the Thévenin equivalent circuit can be found quite directly by simply combining the series batteries and resistors. Note the exact similarity of the network of Fig. 9.25(b) to the Thévenin configuration of Fig. 9.24. The method described below will allow us to extend the procedure just applied to more complex configurations and still end up with the relatively simple network of Fig. 9.24.

In most cases, other elements will be connected to the right of terminals *a* and *b* in Fig. 9.25. To apply the theorem, however, the network to be reduced to the Thévenin equivalent form must be isolated as shown in Fig. 9.25, and the two "holding" terminals identified. Once the proper Thévenin equivalent circuit has been determined, the voltage, current, or resistance readings between the two "holding" terminals will be the same whether the original or the Thévenin equivalent circuit is connected to the left of terminals *a* and *b* in Fig. 9.25. Any load connected to the right of terminals *a* and *b* of Fig. 9.25 will receive the same voltage or current with either network.

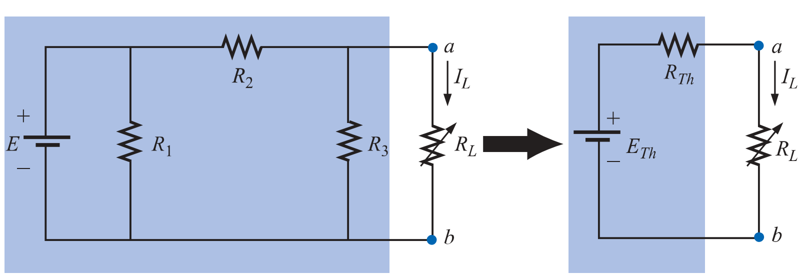
This theorem achieves two important objectives. First, as was true for all the methods previously described, it allows us to find any particular voltage or current in a linear network with one, two, or any other number of sources. Second, we can concentrate on a specific portion of a network by replacing the remaining network with an equivalent circuit. In Fig. 9.26, for example, by finding the Thévenin equivalent circuit for the network in the shaded area, we can quickly calculate the change in current through or voltage across the variable resistor R_L for the various values that it may assume. This is demonstrated in Example 9.6.

French (Meaux, Paris)
(1857–1927)
Telegraph Engineer, Commandant and Educator
École Polytechnique and École Supérieure de Télégraphie

Courtesy of the Bibliothèque École Polytechnique, Paris, France

Although active in the study and design of telegraphic systems (including underground transmission), cylindrical condensers (capacitors), and electromagnetism, he is best known for a theorem first presented in the French *Journal of Physics—Theory and Applications* in 1883. It appeared under the heading of "Sur un nouveau théorème d'électricité dynamique" ("On a new theorem of dynamic electricity") and was originally referred to as the *equivalent generator theorem.* There is some evidence that a similar theorem was introduced by Hermann von Helmholtz in 1853. However, Professor Helmholtz applied the theorem to animal physiology and not to communication or generator systems, and therefore he has not received the credit in this field that he might deserve. In the early 1920s AT&T did some pioneering work using the equivalent circuit and may have initiated the reference to the theorem as simply Thévenin's theorem. In fact, Edward L. Norton, an engineer at AT&T at the time, introduced a current source equivalent of the Thévenin equivalent currently referred to as the Norton equivalent circuit. As an aside, Commandant Thévenin was an avid skier and in fact was commissioner of an international ski competition in Chamonix, France, in 1912.

LEON-CHARLES THÉVENIN



(a) (b)

FIG. 9.26
Substituting the Thévenin equivalent circuit for a complex network.

Before we examine the steps involved in applying this theorem, it is important that an additional word be included here to ensure that the implications of the Thévenin equivalent circuit are clear. In Fig. 9.26, the entire network, except R_L, is to be replaced by a single series resistor and battery as shown in Fig. 9.24. The values of these two elements of the Thévenin equivalent circuit must be chosen to ensure that the resistor R_L will react to the network of Fig. 9.26(a) in the same manner as to the network of Fig. 9.26(b). In other words, the current through or voltage across R_L must be the same for either network for any value of R_L.

The following sequence of steps will lead to the proper value of R_{Th} and E_{Th}.

Preliminary:

1. ***Remove that portion of the network across which the Thévenin equivalent circuit is to be found. In Fig. 9.26(a), this requires that the load resistor R_L be temporarily removed from the network.***
2. ***Mark the terminals of the remaining two-terminal network. (The importance of this step will become obvious as we progress through some complex networks.)***

R_{Th}:

3. ***Calculate R_{Th} by first setting all sources to zero (voltage sources are replaced by short circuits, and current sources by open circuits) and then finding the resultant resistance between the two marked terminals. (If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero.)***

E_{Th}:

4. ***Calculate E_{Th} by first returning all sources to their original position and finding the open-circuit voltage between the marked terminals. (This step is invariably the one that will lead to the most confusion and errors. In all cases, keep in mind that it is the open-circuit potential between the two terminals marked in step 2.)***

Conclusion:

5. ***Draw the Thévenin equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the equivalent circuit. This step is indicated by the placement of the resistor R_L between the terminals of the Thévenin equivalent circuit as shown in Fig. 9.26(b).***

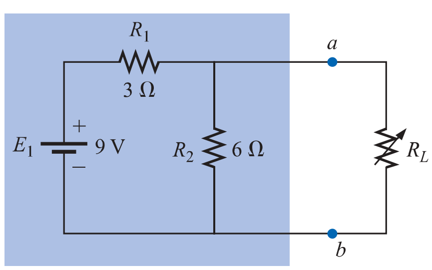


FIG. 9.27
Example 9.6.

EXAMPLE 9.6 Find the Thévenin equivalent circuit for the network in the shaded area of the network of Fig. 9.27. Then find the current through R_L for values of 2 Ω, 10 Ω, and 100 Ω.

Solution:

Steps 1 and 2 produce the network of Fig. 9.28. Note that the load resistor R_L has been removed and the two "holding" terminals have been defined as *a* and *b*.

Step 3: Replacing the voltage source E_1 with a short-circuit equivalent yields the network of Fig. 9.29(a), where

$$R_{Th} = R_1 \parallel R_2 = \frac{(3\ \Omega)(6\ \Omega)}{3\ \Omega + 6\ \Omega} = \mathbf{2\ \Omega}$$

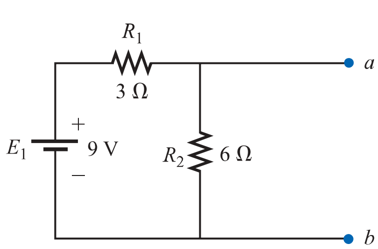


FIG. 9.28
Identifying the terminals of particular importance when applying Thévenin's theorem.

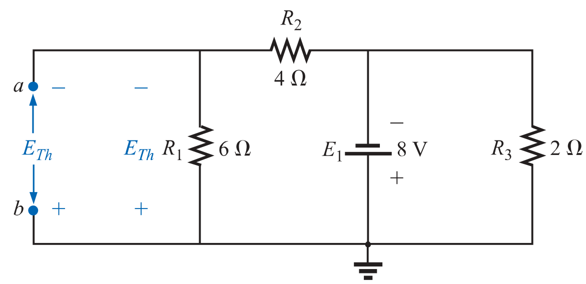


FIG. 9.41
Determining E_{Th} for the network of Fig. 9.39.

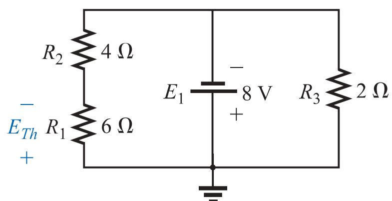


FIG. 9.42
Network of Fig. 9.41 redrawn.

Step 4: See Fig. 9.41. In this case, the network can be redrawn as shown in Fig. 9.42, and since the voltage is the same across parallel elements, the voltage across the series resistors R_1 and R_2 is E_1, or 8 V. Applying the voltage divider rule,

$$E_{Th} = \frac{R_1 E_1}{R_1 + R_2} = \frac{(6\ \Omega)(8\ \text{V})}{6\ \Omega + 4\ \Omega} = \frac{48\ \text{V}}{10} = \mathbf{4.8\ V}$$

Step 5: See Fig. 9.43.

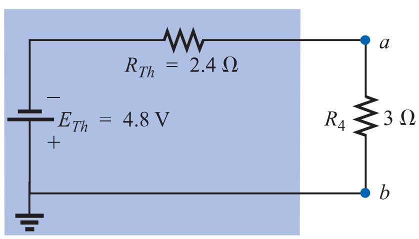


FIG. 9.43
Substituting the Thévenin equivalent circuit for the network external to the resistor R_4 of Fig. 9.38.

The importance of marking the terminals should be obvious from Example 9.8. Note that there is no requirement that the Thévenin voltage have the same polarity as the equivalent circuit originally introduced.

EXAMPLE 9.9 Find the Thévenin equivalent circuit for the network in the shaded area of the bridge network of Fig. 9.44.

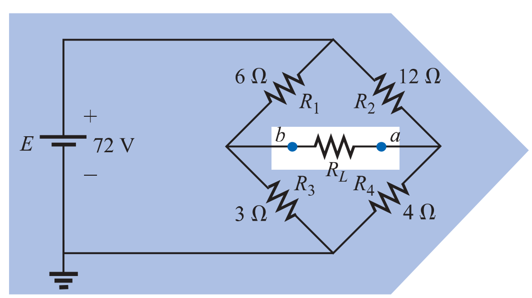


FIG. 9.44
Example 9.9.

Solution:

Steps 1 and 2 are shown in Fig. 9.45.

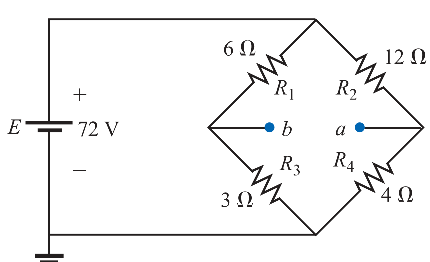


FIG. 9.45
Identifying the terminals of particular interest for the network of Fig. 9.44.

Step 3: See Fig. 9.46. In this case, the short-circuit replacement of the voltage source E provides a direct connection between c and c' of Fig. 9.46(a), permitting a "folding" of the network around the horizontal line of a-b to produce the configuration of Fig. 9.46(b).

$$\begin{aligned} R_{Th} = R_{a\text{-}b} &= R_1 \parallel R_3 + R_2 \parallel R_4 \\ &= 6\ \Omega \parallel 3\ \Omega + 4\ \Omega \parallel 12\ \Omega \\ &= 2\ \Omega + 3\ \Omega = \mathbf{5\ \Omega} \end{aligned}$$

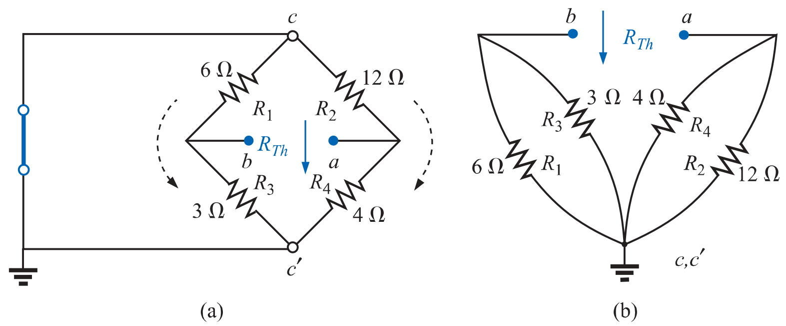


FIG. 9.46
Solving for R_{Th} for the network of Fig. 9.45.

Step 4: The circuit is redrawn in Fig. 9.47. The absence of a direct connection between *a* and *b* results in a network with three parallel branches. The voltages V_1 and V_2 can therefore be determined using the voltage divider rule:

$$V_1 = \frac{R_1 E}{R_1 + R_3} = \frac{(6\ \Omega)(72\ \text{V})}{6\ \Omega + 3\ \Omega} = \frac{432\ \text{V}}{9} = 48\ \text{V}$$

$$V_2 = \frac{R_2 E}{R_2 + R_4} = \frac{(12\ \Omega)(72\ \text{V})}{12\ \Omega + 4\ \Omega} = \frac{864\ \text{V}}{16} = 54\ \text{V}$$

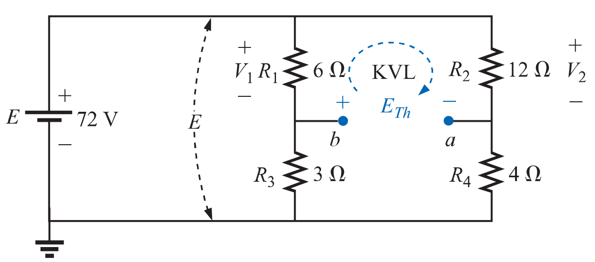


FIG. 9.47
Determining E_{Th} for the network of Fig. 9.45.

Assuming the polarity shown for E_{Th} and applying Kirchhoff's voltage law to the top loop in the clockwise direction will result in

$$\Sigma_{\circlearrowright} V = +E_{Th} + V_1 - V_2 = 0$$

and $$E_{Th} = V_2 - V_1 = 54\ \text{V} - 48\ \text{V} = \mathbf{6\ V}$$

Step 5 is shown in Fig. 9.48.

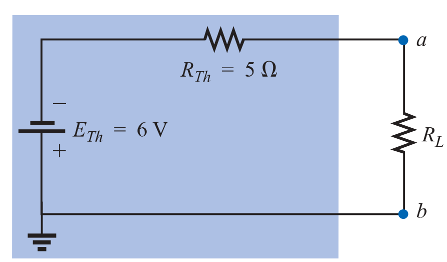


FIG. 9.48
Substituting the Thévenin equivalent circuit for the network external to the resistor R_L of Fig. 9.44.

Thévenin's theorem is not restricted to a single passive element, as shown in the preceding examples, but can be applied across sources, whole branches, portions of networks, or any circuit configuration, as shown in the following example. It is also possible that one of the methods previously described, such as mesh analysis or superposition, may have to be used to find the Thévenin equivalent circuit.

EXAMPLE 9.10 (Two sources) Find the Thévenin circuit for the network within the shaded area of Fig. 9.49.

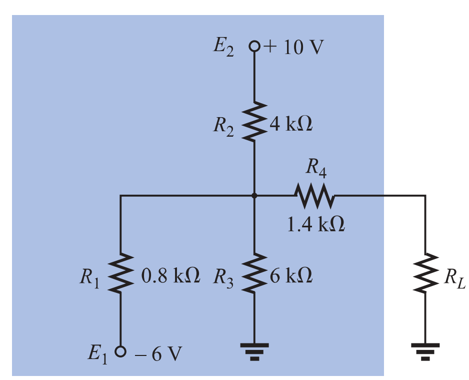


FIG. 9.49
Example 9.10.

Solution: The network is redrawn and *steps 1 and 2* are applied as shown in Fig. 9.50.

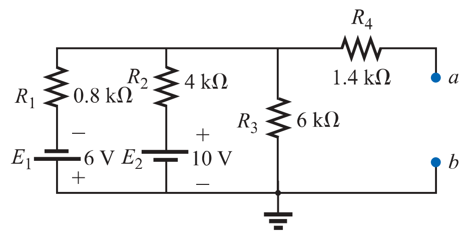


FIG. 9.50
Identifying the terminals of particular interest for the network of Fig. 9.49.

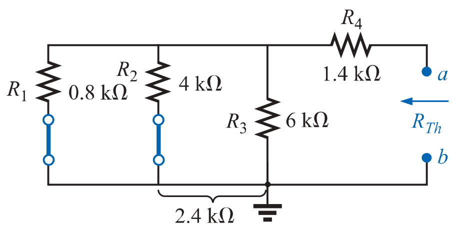


FIG. 9.51
Determining R_{Th} for the network of Fig. 9.50.

Step 3: See Fig. 9.51.

$$
\begin{aligned}
R_{Th} &= R_4 + R_1 \parallel R_2 \parallel R_3 \\
&= 1.4\ \text{k}\Omega + 0.8\ \text{k}\Omega \parallel 4\ \text{k}\Omega \parallel 6\ \text{k}\Omega \\
&= 1.4\ \text{k}\Omega + 0.8\ \text{k}\Omega \parallel 2.4\ \text{k}\Omega \\
&= 1.4\ \text{k}\Omega + 0.6\ \text{k}\Omega \\
&= \mathbf{2\ k\Omega}
\end{aligned}
$$

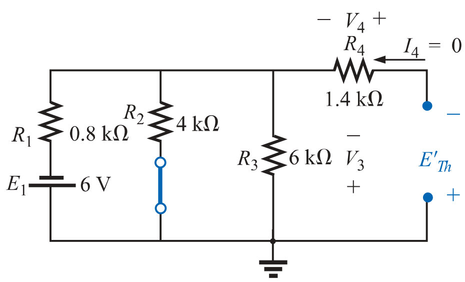


FIG. 9.52
Determining the contribution to E_{Th} from the source E_1 for the network of Fig. 9.50.

Step 4: Applying superposition, we will consider the effects of the voltage source E_1 first. Note Fig. 9.52. The open circuit requires that $V_4 = I_4R_4 = (0)R_4 = 0$ V, and

$$E'_{Th} = V_3$$
$$R'_T = R_2 \parallel R_3 = 4\ \text{k}\Omega \parallel 6\ \text{k}\Omega = 2.4\ \text{k}\Omega$$

Applying the voltage divider rule,

$$V_3 = \frac{R'_T E_1}{R'_T + R_1} = \frac{(2.4\ \text{k}\Omega)(6\ \text{V})}{2.4\ \text{k}\Omega + 0.8\ \text{k}\Omega} = \frac{14.4\ \text{V}}{3.2} = 4.5\ \text{V}$$

$$E'_{Th} = V_3 = 4.5\ \text{V}$$

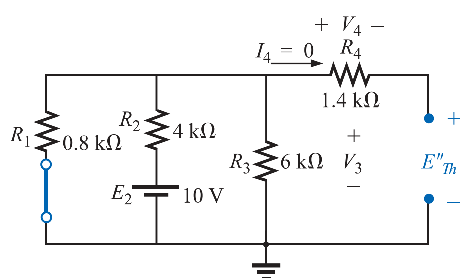


FIG. 9.53
Determining the contribution to E_{Th} from the source E_2 for the network of Fig. 9.50.

For the source E_2, the network of Fig. 9.53 will result. Again, $V_4 = I_4R_4 = (0)R_4 = 0$ V, and

$$E''_{Th} = V_3$$
$$R'_T = R_1 \parallel R_3 = 0.8\ \text{k}\Omega \parallel 6\ \text{k}\Omega = 0.706\ \text{k}\Omega$$

and $$V_3 = \frac{R'_T E_2}{R'_T + R_2} = \frac{(0.706\ \text{k}\Omega)(10\ \text{V})}{0.706\ \text{k}\Omega + 4\ \text{k}\Omega} = \frac{7.06\ \text{V}}{4.706} = 1.5\ \text{V}$$

$$E''_{Th} = V_3 = 1.5\ \text{V}$$

Since E'_{Th} and E''_{Th} have opposite polarities,

$$
\begin{aligned}
E_{Th} &= E'_{Th} - E''_{Th} \\
&= 4.5\ \text{V} - 1.5\ \text{V} \\
&= \mathbf{3\ V} \qquad (\text{polarity of } E'_{Th})
\end{aligned}
$$

Step 5: See Fig. 9.54.

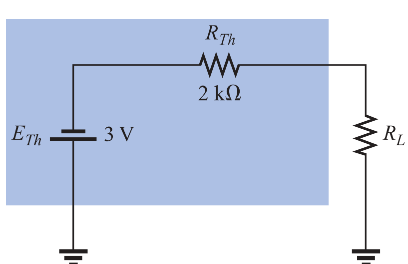


FIG. 9.54
Substituting the Thévenin equivalent circuit for the network external to the resistor R_L of Fig. 9.49.

Experimental Procedures

There are two popular experimental procedures for determining the parameters of a Thévenin equivalent network. The procedure for measuring the Thévenin voltage is the same for each, but the approach for determining the Thévenin resistance is quite different for each.

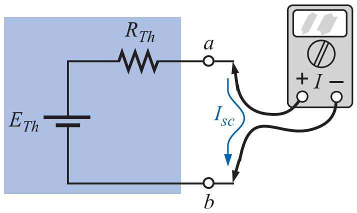


FIG. 9.57
Measuring I_{sc}.

American (Rockland, Maine; Summit, New Jersey)
(1898–1983)
Electrical Engineer, Scientist, Inventor
Department Head: Bell Laboratories
Fellow: Acoustical Society and Institute of Radio Engineers

Courtesy of AT&T Archives

Although interested primarily in communications circuit theory and the transmission of data at high speeds over telephone lines, Edward L. Norton is best remembered for development of the dual of Thévenin's equivalent circuit, currently referred to as *Norton's equivalent circuit*. In fact, Norton and his associates at AT&T in the early 1920s are recognized as some of the first to perform pioneering work applying Thévenin's equivalent circuit and who referred to this concept simply as Thévenin's theorem. In 1926 he proposed the equivalent circuit using a current source and parallel resistor to assist in the design of recording instrumentation that was primarily current driven. He began his telephone career in 1922 with the Western Electric Company's Engineering Department, which later became Bell Laboratories. His areas of active research included network theory, acoustical systems, electromagnetic apparatus, and data transmission. A graduate of MIT and Columbia University, he held nineteen patents on his work.

EDWARD L. NORTON

that is, $E_{Th} = V_{oc}$. To determine R_{Th}, a short-circuit condition is established across the terminals of interest, as shown in Fig. 9.57, and the current through the short circuit is measured with an ammeter. Using Ohm's law, we find that the short-circuit current is determined by

$$I_{sc} = \frac{E_{Th}}{R_{Th}}$$

and the Thévenin resistance by

$$R_{Th} = \frac{E_{Th}}{I_{sc}}$$

However, $E_{Th} = V_{oc}$ resulting in the following equation for R_{Th}:

$$\boxed{R_{Th} = \frac{V_{oc}}{I_{sc}}} \qquad \textbf{(9.2)}$$

9.4 NORTON'S THEOREM

It was demonstrated in Section 8.3 that every voltage source with a series internal resistance has a current source equivalent. The current source equivalent of the Thévenin network (which, you will note, satisfies the above conditions), as shown in Fig. 9.58, can be determined by **Norton's theorem.** It can also be found through the conversions of Section 8.3.

The theorem states the following:

Any two-terminal linear bilateral dc network can be replaced by an equivalent circuit consisting of a current source and a parallel resistor, as shown in Fig. 9.58.

The discussion of Thévenin's theorem with respect to the equivalent circuit can also be applied to the Norton equivalent circuit. The steps leading to the proper values of I_N and R_N are now listed.

Preliminary:

1. ***Remove that portion of the network across which the Norton equivalent circuit is found.***
2. ***Mark the terminals of the remaining two-terminal network.***

R_N:

3. ***Calculate R_N by first setting all sources to zero (voltage sources are replaced with short circuits, and current sources with open circuits) and then finding the resultant resistance between the two marked terminals. (If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero.) Since $R_N = R_{Th}$, the procedure and value obtained using the approach described for Thévenin's theorem will determine the proper value of R_N.***

I_N:

4. ***Calculate I_N by first returning all sources to their original position and then finding the short-circuit current between the marked terminals. It is the same current that would be measured by an ammeter placed between the marked terminals.***

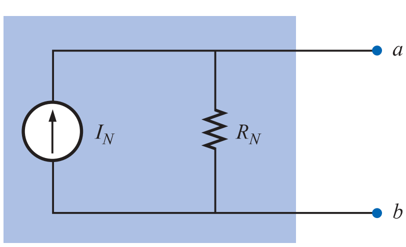


FIG. 9.58
Norton equivalent circuit.

Conclusion:

5. Draw the Norton equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the equivalent circuit.

The Norton and Thévenin equivalent circuits can also be found from each other by using the source transformation discussed earlier in this chapter and reproduced in Fig. 9.59.

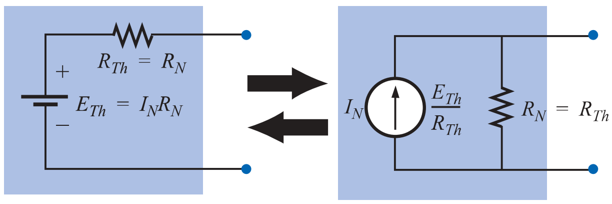


FIG. 9.59
Converting between Thévenin and Norton equivalent circuits.

EXAMPLE 9.11 Find the Norton equivalent circuit for the network in the shaded area of Fig. 9.60.

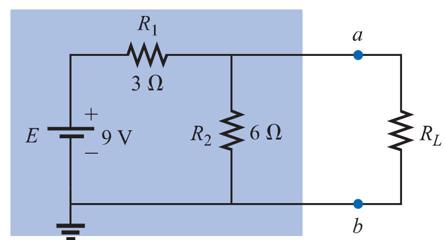


FIG. 9.60
Example 9.11.

Solution:

Steps 1 and 2 are shown in Fig. 9.61.

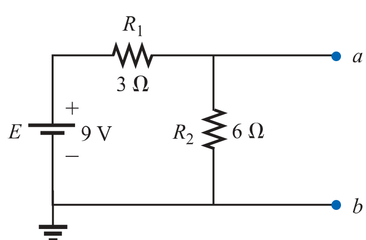


FIG. 9.61
Identifying the terminals of particular interest for the network of Fig. 9.60.

Step 3 is shown in Fig. 9.62, and

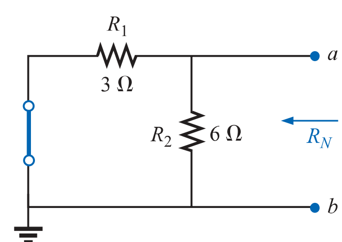


FIG. 9.62
Determining R_N for the network of Fig. 9.61.

$$R_N = R_1 \parallel R_2 = 3\ \Omega \parallel 6\ \Omega = \frac{(3\ \Omega)(6\ \Omega)}{3\ \Omega + 6\ \Omega} = \frac{18\ \Omega}{9} = \mathbf{2\ \Omega}$$

Step 4 is shown in Fig. 9.63, clearly indicating that the short-circuit connection between terminals *a* and *b* is in parallel with R_2 and eliminates its effect. I_N is therefore the same as through R_1, and the full battery voltage appears across R_1 since

$$V_2 = I_2 R_2 = (0)6\ \Omega = 0\ \text{V}$$

Therefore,

$$I_N = \frac{E}{R_1} = \frac{9\ \text{V}}{3\ \Omega} = \mathbf{3\ A}$$

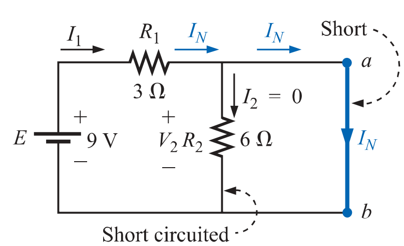


FIG. 9.63
Determining I_N for the network of Fig. 9.61.

Step 5: See Fig. 9.64. This circuit is the same as the first one considered in the development of Thévenin's theorem. A simple conversion indicates that the Thévenin circuits are, in fact, the same (Fig. 9.65).

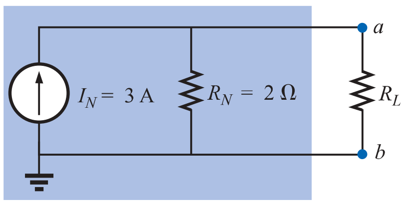


FIG. 9.64
Substituting the Norton equivalent circuit for the network external to the resistor R_L of Fig. 9.60.

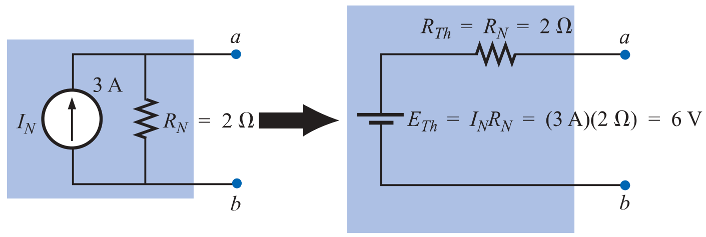


FIG. 9.65
Converting the Norton equivalent circuit of Fig. 9.64 to a Thévenin equivalent circuit.

EXAMPLE 9.12 Find the Norton equivalent circuit for the network external to the 9-Ω resistor in Fig. 9.66.

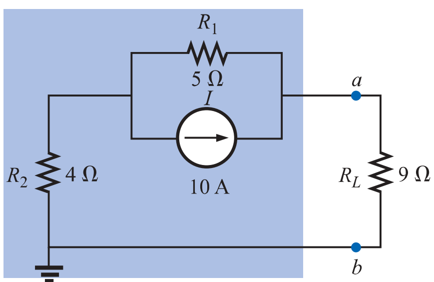


FIG. 9.66
Example 9.12.

Solution:

Steps 1 and 2: See Fig. 9.67.

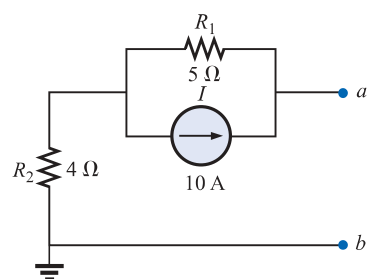


FIG. 9.67
Identifying the terminals of particular interest for the network of Fig. 9.66.

Step 3: See Fig. 9.68, and

$$R_N = R_1 + R_2 = 5\ \Omega + 4\ \Omega = \mathbf{9\ \Omega}$$

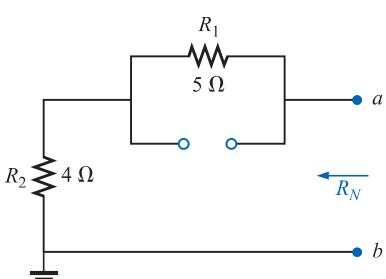


FIG. 9.68
Determining R_N for the network of Fig. 9.67.

Step 4: As shown in Fig. 9.69, the Norton current is the same as the current through the 4-Ω resistor. Applying the current divider rule,

$$I_N = \frac{R_1 I}{R_1 + R_2} = \frac{(5\ \Omega)(10\ \text{A})}{5\ \Omega + 4\ \Omega} = \frac{50\ \text{A}}{9} = \mathbf{5.556\ A}$$

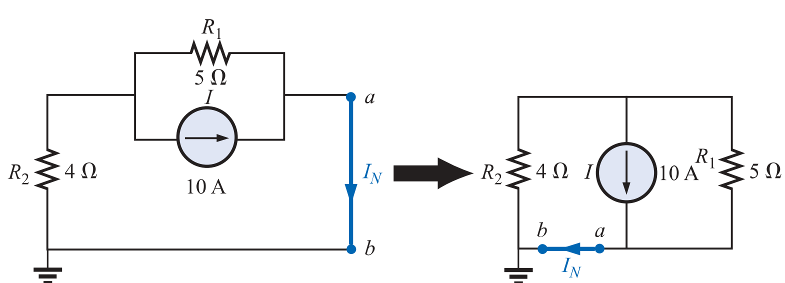


FIG. 9.69
Determining I_N for the network of Fig. 9.67.

Step 5: See Fig. 9.70.

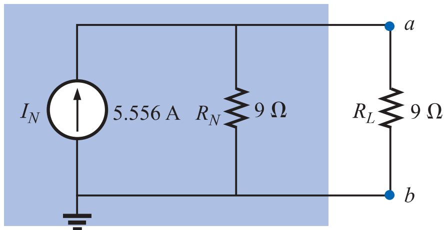


FIG. 9.70
Substituting the Norton equivalent circuit for the network external to the resistor R_L of Fig. 9.66.

EXAMPLE 9.13 (Two sources) Find the Norton equivalent circuit for the portion of the network to the left of *a-b* in Fig. 9.71.

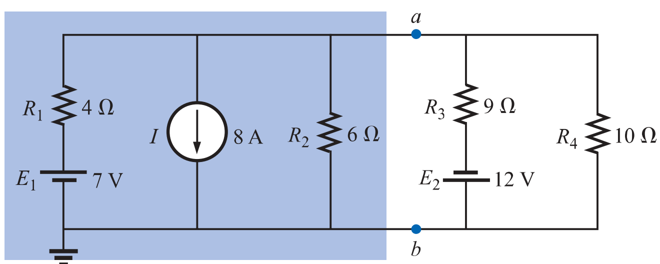


FIG. 9.71
Example 9.13.

Solution:

Steps 1 and 2: See Fig. 9.72.

Step 3 is shown in Fig. 9.73, and

$$R_N = R_1 \parallel R_2 = 4\ \Omega \parallel 6\ \Omega = \frac{(4\ \Omega)(6\ \Omega)}{4\ \Omega + 6\ \Omega} = \frac{24\ \Omega}{10} = \mathbf{2.4\ \Omega}$$

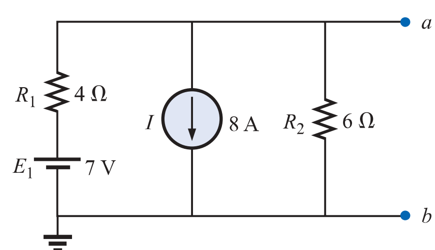


FIG. 9.72
Identifying the terminals of particular interest for the network of Fig. 9.71.

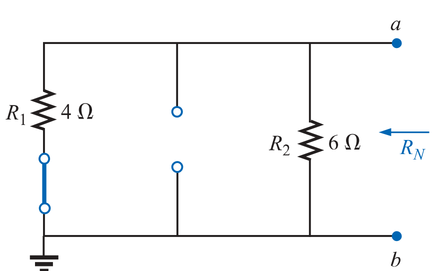


FIG. 9.73
Determining R_N for the network of Fig. 9.72.

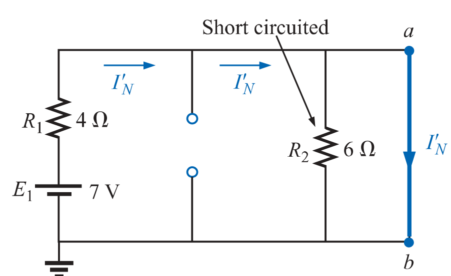


FIG. 9.74
Determining the contribution to I_N from the voltage source E_1.

Step 4: (Using superposition) For the 7-V battery (Fig. 9.74), Note Fig. 9.91, where

$$V_1 = V_3 = 0\text{ V}$$

and $$V_2 = I_2R_2 = IR_2 = (6\text{ A})(10\ \Omega) = 60\text{ V}$$

Applying Kirchhoff's voltage law,

$$\Sigma_{\circlearrowright} V = -V_2 - E_1 + E_{Th} = 0$$

and $$E_{Th} = V_2 + E_1 = 60\text{ V} + 68\text{ V} = 128\text{ V}$$

Thus, $$P_{L_{max}} = \frac{E_{Th}^2}{4R_{Th}} = \frac{(128\text{ V})^2}{4(15\ \Omega)} = \mathbf{273.07\ W}$$

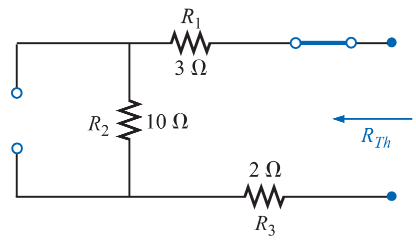


FIG. 9.90
Determining R_{Th} for the network external to the resistor R_L of Fig. 9.89.

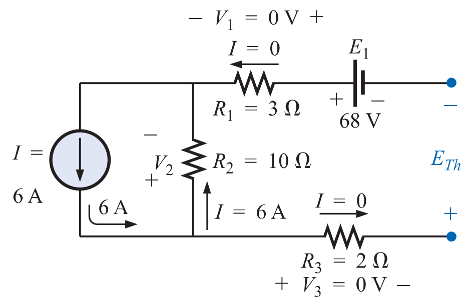


FIG. 9.91
Determining E_{Th} for the network external to the resistor R_L of Fig. 9.89.

9.6 MILLMAN'S THEOREM

Through the application of **Millman's theorem,** any number of parallel voltage sources can be reduced to one. In Fig. 9.92, for example, the three voltage sources can be reduced to one. This would permit finding the current through or voltage across R_L without having to apply a method such as mesh analysis, nodal analysis, superposition, and so on. The theorem can best be described by applying it to the network of Fig. 9.92. Basically, three steps are included in its application.

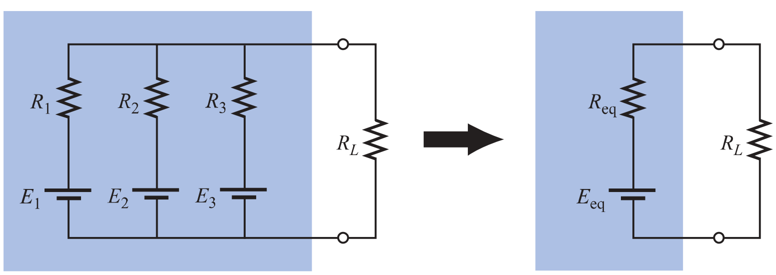


FIG. 9.92
Demonstrating the effect of applying Millman's theorem.

Step 1: Convert all voltage sources to current sources as outlined in Section 8.3. This is performed in Fig. 9.93 for the network of Fig. 9.92.

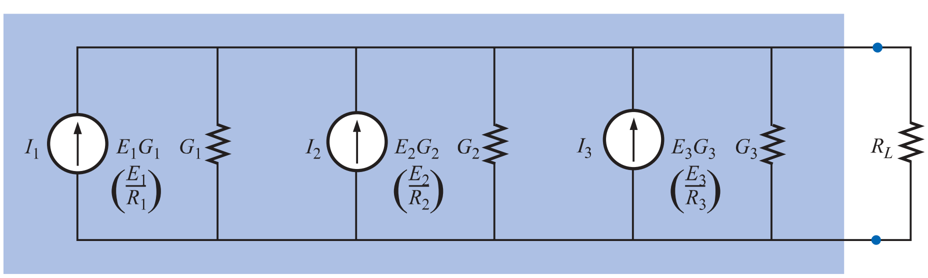


FIG. 9.93
Converting all the sources of Fig. 9.92 to current sources.

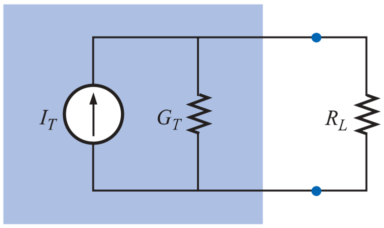


FIG. 9.94
Reducing all the current sources of Fig. 9.93 to a single current source.

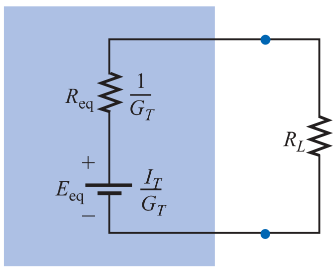


FIG. 9.95
Converting the current source of Fig. 9.94 to a voltage source.

Step 2: Combine parallel current sources as described in Section 8.4. The resulting network is shown in Fig. 9.94, where

$$I_T = I_1 + I_2 + I_3 \quad \text{and} \quad G_T = G_1 + G_2 + G_3$$

Step 3: Convert the resulting current source to a voltage source, and the desired single-source network is obtained, as shown in Fig. 9.95.

In general, Millman's theorem states that for any number of parallel voltage sources,

$$E_{eq} = \frac{I_T}{G_T} = \frac{\pm I_1 \pm I_2 \pm I_3 \pm \cdots \pm I_N}{G_1 + G_2 + G_3 + \cdots + G_N}$$

or

$$\boxed{E_{eq} = \frac{\pm E_1G_1 \pm E_2G_2 \pm E_3G_3 \pm \cdots \pm E_NG_N}{G_1 + G_2 + G_3 + \cdots + G_N}} \qquad \textbf{(9.10)}$$

The plus-and-minus signs appear in Eq. (9.10) to include those cases where the sources may not be supplying energy in the same direction. (Note Example 9.18.)

The equivalent resistance is

$$\boxed{R_{eq} = \frac{1}{G_T} = \frac{1}{G_1 + G_2 + G_3 + \cdots + G_N}} \qquad \textbf{(9.11)}$$

In terms of the resistance values,

$$\boxed{E_{eq} = \frac{\pm\dfrac{E_1}{R_1} \pm \dfrac{E_2}{R_2} \pm \dfrac{E_3}{R_3} \pm \cdots \pm \dfrac{E_N}{R_N}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \cdots + \dfrac{1}{R_N}}} \qquad \textbf{(9.12)}$$

and

$$\boxed{R_{eq} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \cdots + \dfrac{1}{R_N}}} \qquad \textbf{(9.13)}$$

The relatively few direct steps required may result in the student's applying each step rather than memorizing and employing Eqs. (9.10) through (9.13).

EXAMPLE 9.18 Using Millman's theorem, find the current through and voltage across the resistor R_L of Fig. 9.96.

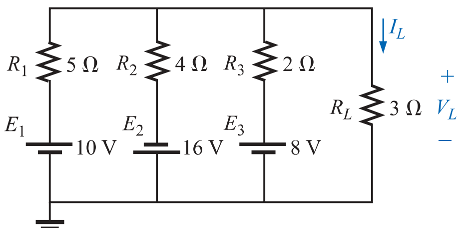


FIG. 9.96
Example 9.18.

Solution: By Eq. (9.12),

$$E_{eq} = \frac{+\dfrac{E_1}{R_1} - \dfrac{E_2}{R_2} + \dfrac{E_3}{R_3}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}}$$

The minus sign is used for E_2/R_2 because that supply has the opposite polarity of the other two. The chosen reference direction is therefore that of E_1 and E_3. The total conductance is unaffected by the direction, and

$$E_{eq} = \frac{+\dfrac{10\text{ V}}{5\ \Omega} - \dfrac{16\text{ V}}{4\ \Omega} + \dfrac{8\text{ V}}{2\ \Omega}}{\dfrac{1}{5\ \Omega} + \dfrac{1}{4\ \Omega} + \dfrac{1}{2\ \Omega}} = \frac{2\text{ A} - 4\text{ A} + 4\text{ A}}{0.2\text{ S} + 0.25\text{ S} + 0.5\text{ S}}$$

$$= \frac{2\text{ A}}{0.95\text{ S}} = \mathbf{2.105\ V}$$

with $$R_{eq} = \frac{1}{\dfrac{1}{5\ \Omega} + \dfrac{1}{4\ \Omega} + \dfrac{1}{2\ \Omega}} = \frac{1}{0.95\text{ S}} = \mathbf{1.053\ \Omega}$$

The resultant source is shown in Fig. 9.97, and

$$I_L = \frac{2.105\text{ V}}{1.053\ \Omega + 3\ \Omega} = \frac{2.105\text{ V}}{4.053\ \Omega} = \mathbf{0.519\ A}$$

with $$V_L = I_L R_L = (0.519\text{ A})(3\ \Omega) = \mathbf{1.557\ V}$$

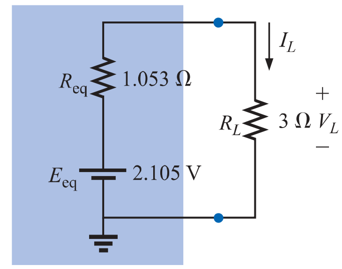


FIG. 9.97
The result of applying Millman's theorem to the network of Fig. 9.96.

EXAMPLE 9.19 Let us now consider the type of problem encountered in the introduction to mesh and nodal analysis in Chapter 8. Mesh analysis was applied to the network of Fig. 9.98 (Example 8.12). Let us now use Millman's theorem to find the current through the 2-Ω resistor and compare the results.

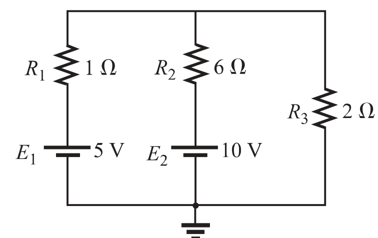


FIG. 9.98
Example 9.19.

Solutions:

a. Let us first apply each step and, in the (b) solution, Eq. (9.12). Converting sources yields Fig. 9.99. Combining sources and parallel conductance branches (Fig. 9.100) yields

$$I_T = I_1 + I_2 = 5\text{ A} + \frac{5}{3}\text{ A} = \frac{15}{3}\text{ A} + \frac{5}{3}\text{ A} = \frac{20}{3}\text{ A}$$

$$G_T = G_1 + G_2 = 1\text{ S} + \frac{1}{6}\text{ S} = \frac{6}{6}\text{ S} + \frac{1}{6}\text{ S} = \frac{7}{6}\text{ S}$$

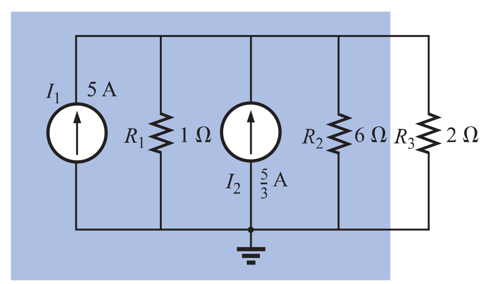


FIG. 9.99
Converting the sources of Fig. 9.98 to current sources.

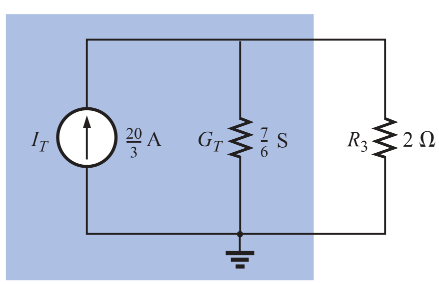


FIG. 9.100
Reducing the current sources of Fig. 9.99 to a single source.

Converting the current source to a voltage source (Fig. 9.101), we obtain

$$E_{eq} = \frac{I_T}{G_T} = \frac{\dfrac{20}{3}\text{ A}}{\dfrac{7}{6}\text{ S}} = \frac{(6)(20)}{(3)(7)}\text{ V} = \mathbf{\frac{40}{7}\ V}$$

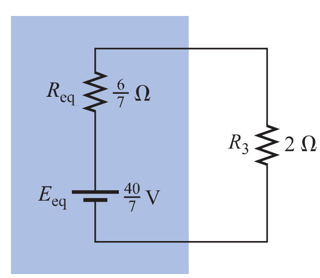


FIG. 9.101
Converting the current source of Fig. 9.100 to a voltage source.

and $$R_{eq} = \frac{1}{G_T} = \frac{1}{\frac{7}{6}\,\text{S}} = \mathbf{\frac{6}{7}\,\Omega}$$

so that

$$I_{2\Omega} = \frac{E_{eq}}{R_{eq} + R_3} = \frac{\frac{40}{7}\,\text{V}}{\frac{6}{7}\,\Omega + 2\,\Omega} = \frac{\frac{40}{7}\,\text{V}}{\frac{6}{7}\,\Omega + \frac{14}{7}\,\Omega} = \frac{40\,\text{V}}{20\,\Omega} = \mathbf{2\,A}$$

which agrees with the result obtained in Example 8.18.

b. Let us now simply apply the proper equation, Eq. (9.12):

$$E_{eq} = \frac{+\frac{5\,\text{V}}{1\,\Omega} + \frac{10\,\text{V}}{6\,\Omega}}{\frac{1}{1\,\Omega} + \frac{1}{6\,\Omega}} = \frac{\frac{30\,\text{V}}{6\,\Omega} + \frac{10\,\text{V}}{6\,\Omega}}{\frac{6}{6\,\Omega} + \frac{1}{6\,\Omega}} = \mathbf{\frac{40}{7}\,V}$$

and

$$R_{eq} = \frac{1}{\frac{1}{1\,\Omega} + \frac{1}{6\,\Omega}} = \frac{1}{\frac{6}{6\,\Omega} + \frac{1}{6\,\Omega}} = \frac{1}{\frac{7}{6}\,\text{S}} = \mathbf{\frac{6}{7}\,\Omega}$$

which are the same values obtained above.

The dual of Millman's theorem (Fig. 9.92) appears in Fig. 9.102. It can be shown that I_{eq} and R_{eq}, as in Fig. 9.102, are given by

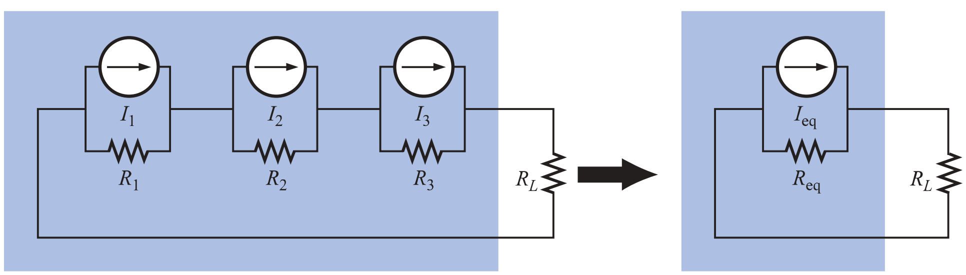


FIG. 9.102
The dual effect of Millman's theorem.

$$\boxed{I_{eq} = \frac{\pm I_1R_1 \pm I_2R_2 \pm I_3R_3}{R_1 + R_2 + R_3}} \qquad \mathbf{(9.14)}$$

and $$\boxed{R_{eq} = R_1 + R_2 + R_3} \qquad \mathbf{(9.15)}$$

The derivation will appear as a problem at the end of the chapter.

9.7 SUBSTITUTION THEOREM

The **substitution theorem** states the following:

If the voltage across and the current through any branch of a dc bilateral network are known, this branch can be replaced by any

combination of elements that will maintain the same voltage across and current through the chosen branch.

More simply, the theorem states that for branch equivalence, the terminal voltage and current must be the same. Consider the circuit of Fig. 9.103, in which the voltage across and current through the branch *a-b* are determined. Through the use of the substitution theorem, a number of equivalent *a-a′* branches are shown in Fig. 9.104.

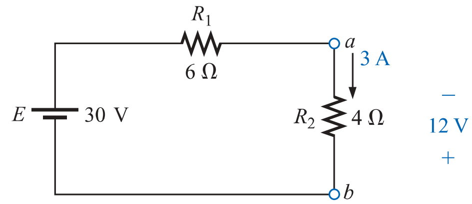


FIG. 9.103
Demonstrating the effect of the substitution theorem.

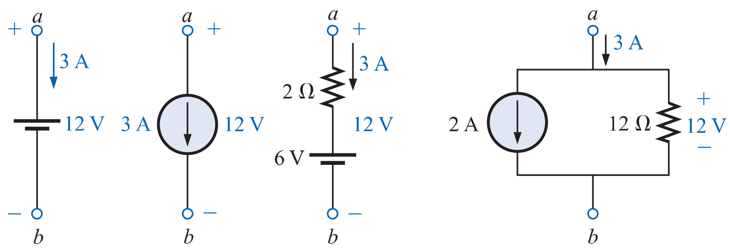


FIG. 9.104
Equivalent branches for the branch a-b of Fig. 9.103.

Note that for each equivalent, the terminal voltage and current are the same. Also consider that the response of the remainder of the circuit of Fig. 9.103 is unchanged by substituting any one of the equivalent branches. As demonstrated by the single-source equivalents of Fig. 9.104, *a known potential difference and current in a network can be replaced by an ideal voltage source and current source, respectively.*

Understand that this theorem cannot be used to *solve* networks with two or more sources that are not in series or parallel. For it to be applied, a potential difference or current value must be known or found using one of the techniques discussed earlier. One application of the theorem is shown in Fig. 9.105. Note that in the figure the known potential difference V was replaced by a voltage source, permitting the isolation of the portion of the network including R_3, R_4, and R_5. Recall that this was basically the approach employed in the analysis of the ladder network as we worked our way back toward the terminal resistance R_5.

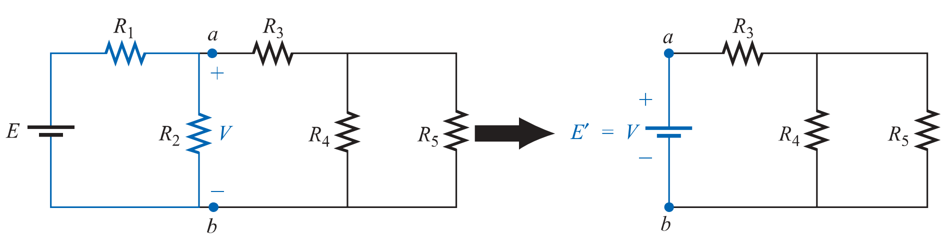


FIG. 9.105
Demonstrating the effect of knowing a voltage at some point in a complex network.

The current source equivalence of the above is shown in Fig. 9.106, where a known current is replaced by an ideal current source, permitting the isolation of R_4 and R_5.

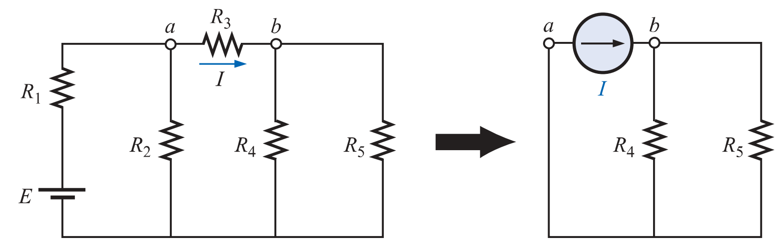


FIG. 9.106
Demonstrating the effect of knowing a current at some point in a complex network.

You will also recall from the discussion of bridge networks that $V = 0$ and $I = 0$ were replaced by a short circuit and an open circuit, respectively. This substitution is a very specific application of the substitution theorem.

9.8 RECIPROCITY THEOREM

The **reciprocity theorem** is applicable only to single-source networks. It is, therefore, not a theorem employed in the analysis of multisource networks described thus far. The theorem states the following:

The current I in any branch of a network, due to a single voltage source E anywhere else in the network, will equal the current through the branch in which the source was originally located if the source is placed in the branch in which the current I was originally measured.

In other words, the location of the voltage source and the resulting current may be interchanged without a change in current. The theorem requires that the polarity of the voltage source have the same correspondence with the direction of the branch current in each position.

In the representative network of Fig. 9.107(a), the current I due to the voltage source E was determined. If the position of each is interchanged as shown in Fig. 9.107(b), the current I will be the same value as indicated. To demonstrate the validity of this statement and the theorem, consider the network of Fig. 9.108, in which values for the elements of Fig. 9.107(a) have been assigned.

The total resistance is

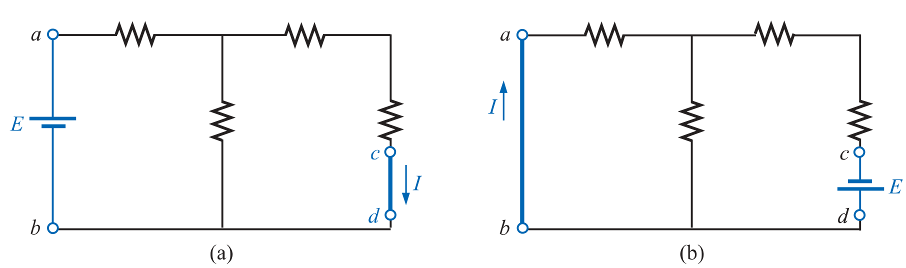


FIG. 9.107
Demonstrating the impact of the reciprocity theorem.

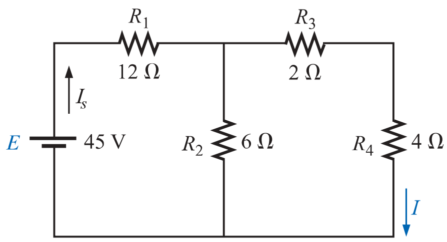


FIG. 9.108
Finding the current I due to a source E.

$$R_T = R_1 + R_2 \parallel (R_3 + R_4) = 12\ \Omega + 6\ \Omega \parallel (2\ \Omega + 4\ \Omega)$$
$$= 12\ \Omega + 6\ \Omega \parallel 6\ \Omega = 12\ \Omega + 3\ \Omega = 15\ \Omega$$

and
$$I_s = \frac{E}{R_T} = \frac{45\text{ V}}{15\ \Omega} = 3\text{ A}$$

with
$$I = \frac{3\text{ A}}{2} = \mathbf{1.5\ A}$$

For the network of Fig. 9.109, which corresponds to that of Fig. 9.107(b), we find

$$R_T = R_4 + R_3 + R_1 \parallel R_2$$
$$= 4\ \Omega + 2\ \Omega + 12\ \Omega \parallel 6\ \Omega = 10\ \Omega$$

and
$$I_s = \frac{E}{R_T} = \frac{45\text{ V}}{10\ \Omega} = 4.5\text{ A}$$

so that
$$I = \frac{(6\ \Omega)(4.5\text{ A})}{12\ \Omega + 6\ \Omega} = \frac{4.5\text{ A}}{3} = \mathbf{1.5\ A}$$

which agrees with the above.

The uniqueness and power of such a theorem can best be demonstrated by considering a complex, single-source network such as the one shown in Fig. 9.110.

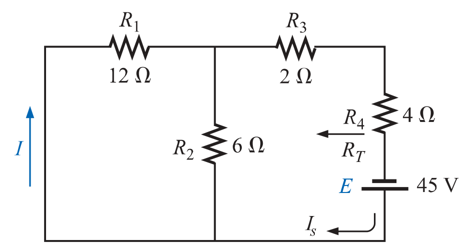


FIG. 9.109
Interchanging the location of E and I of Fig. 9.108 to demonstrate the validity of the reciprocity theorem.

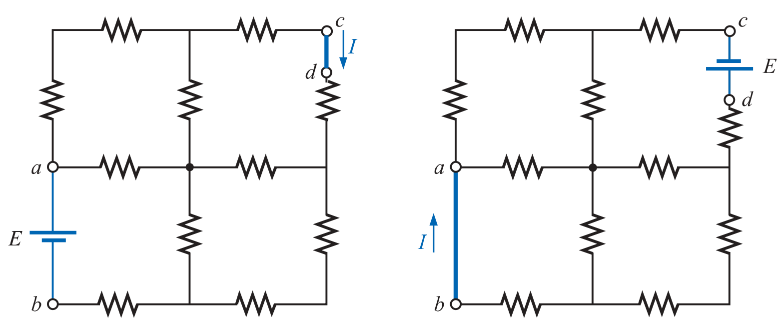


FIG. 9.110
Demonstrating the power and uniqueness of the reciprocity theorem.

9.9 APPLICATION

Speaker System

One of the most common applications of the maximum power transfer theorem introduced in this chapter is to speaker systems. An audio amplifier (amplifier with a frequency range matching the typical range

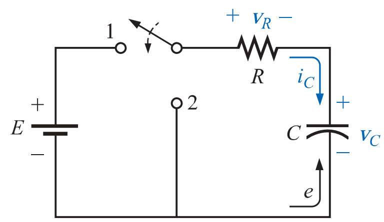


FIG. 10.24
Basic charging network.

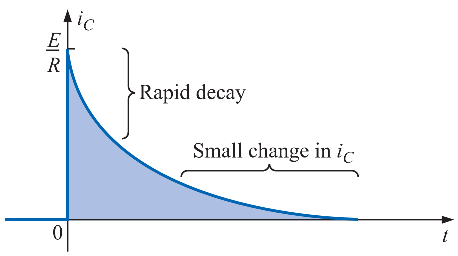


FIG. 10.25
i_C during the charging phase.

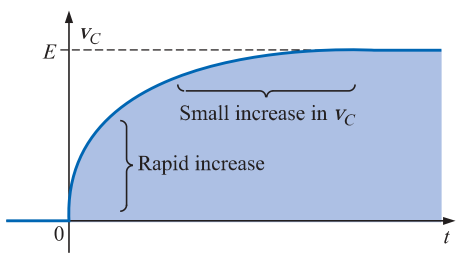


FIG. 10.26
v_C during the charging phase.

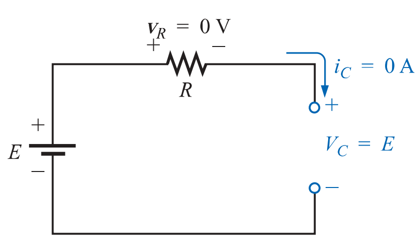


FIG. 10.27
Open-circuit equivalent for a capacitor following the charging phase.

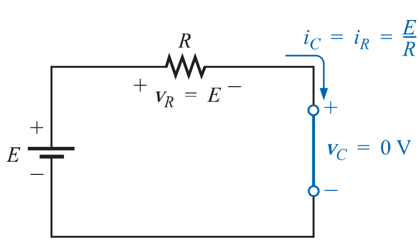


FIG. 10.28
Short-circuit equivalent for a capacitor (switch closed, $t = 0$).

gest the multiplier level. The J represents a ±5% tolerance level. For capacitors such as appearing in Fig. 10.23(c), the first two numbers are actual digits of the value, while the third number is the power of a multiplier (or number of zeros to be added). The F represents a ±1% tolerance level. Multipliers of 0.01 use an 8, while 9 is used for 0.1 as shown for the capacitor of Fig. 10.23(d) where the M represents a ±20% tolerance level.

10.7 TRANSIENTS IN CAPACITIVE NETWORKS: CHARGING PHASE

Section 10.3 described how a capacitor acquires its charge. Let us now extend this discussion to include the potentials and current developed within the network of Fig. 10.24 following the closing of the switch (to position 1).

You will recall that the instant the switch is closed, electrons are drawn from the top plate and deposited on the bottom plate by the battery, resulting in a net positive charge on the top plate and a negative charge on the bottom plate. The transfer of electrons is very rapid at first, slowing down as the potential across the capacitor approaches the applied voltage of the battery. When the voltage across the capacitor equals the battery voltage, the transfer of electrons will cease and the plates will have a net charge determined by $Q = CV_C = CE$.

Plots of the changing current and voltage appear in Figs. 10.25 and 10.26, respectively. When the switch is closed at $t = 0$ s, the current jumps to a value limited only by the resistance of the network and then decays to zero as the plates are charged. Note the rapid decay in current level, revealing that the amount of charge deposited on the plates per unit time is rapidly decaying also. Since the voltage across the plates is directly related to the charge on the plates by $v_C = q/C$, the rapid rate with which charge is initially deposited on the plates will result in a rapid increase in v_C. Obviously, as the rate of flow of charge (I) decreases, the rate of change in voltage will follow suit. Eventually, the flow of charge will stop, the current I will be zero, and the voltage will cease to change in magnitude—the *charging phase* has passed. At this point the capacitor takes on the characteristics of an open circuit: a voltage drop across the plates without a flow of charge "between" the plates. As demonstrated in Fig. 10.27, the voltage across the capacitor is the source voltage since $i = i_C = i_R = 0$ A and $v_R = i_R R = (0)R = 0$ V. For all future analysis:

A capacitor can be replaced by an open-circuit equivalent once the charging phase in a dc network has passed.

Looking back at the instant the switch is closed, we can also surmise that a capacitor behaves as a short circuit the moment the switch is closed in a dc charging network, as shown in Fig. 10.28. The current $i = i_C = i_R = E/R$, and the voltage $v_C = E - v_R = E - i_R R = E - (E/R)R = E - E = 0$ V at $t = 0$ s.

Through the use of calculus, the following mathematical equation for the charging current i_C can be obtained:

$$\boxed{i_C = \frac{E}{R}e^{-t/RC}} \tag{10.13}$$

The factor $e^{-t/RC}$ is an exponential function of the form e^{-x}, where $x = -t/RC$ and $e = 2.71828 \ldots$. A plot of e^{-x} for $x \geq 0$ appears in Fig. 10.29. Exponentials are mathematical functions that all students of electrical, electronic, or computer systems must become very familiar with. They will appear throughout the analysis to follow in this course, and in succeeding courses.

Our current interest in the function e^{-x} is limited to values of x greater than zero, as noted by the curve of Fig. 10.25. All modern-day scientific calculators have the function e^x. To obtain e^{-x}, the sign of x must be changed using the sign key before the exponential function is keyed in. The magnitude of e^{-x} has been listed in Table 10.3 for a range of values of x. Note the rapidly decreasing magnitude of e^{-x} with increasing value of x.

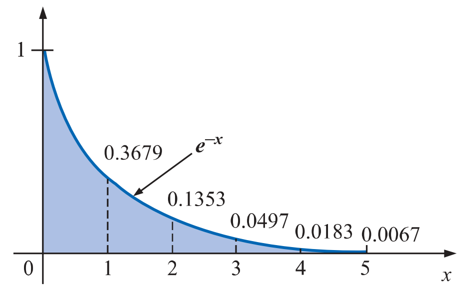


FIG. 10.29
The e^{-x} function ($x \geq 0$).

TABLE 10.3
Selected values of e^{-x}.

$x = 0$	$e^{-x} = e^{-0} = \dfrac{1}{e^0} = \dfrac{1}{1} = 1$
$x = 1$	$e^{-1} = \dfrac{1}{e} = \dfrac{1}{2.71828\ldots} = 0.3679$
$x = 2$	$e^{-2} = \dfrac{1}{e^2} = 0.1353$
$x = 5$	$e^{-5} = \dfrac{1}{e^5} = 0.00674$
$x = 10$	$e^{-10} = \dfrac{1}{e^{10}} = 0.0000454$
$x = 100$	$e^{-100} = \dfrac{1}{e^{100}} = 3.72 \times 10^{-44}$

The factor RC in Eq. (10.13) is called the *time constant* of the system and has the units of time as follows:

$$RC = \left(\frac{V}{I}\right)\left(\frac{Q}{V}\right) = \left(\frac{\cancel{V}}{\cancel{Q}/t}\right)\left(\frac{\cancel{Q}}{\cancel{V}}\right) = t$$

Its symbol is the Greek letter τ (tau), and its unit of measure is the second. Thus,

$$\boxed{\tau = RC} \qquad \text{(seconds, s)} \qquad \textbf{(10.14)}$$

If we substitute $\tau = RC$ into the exponential function $e^{-t/RC}$, we obtain $e^{-t/\tau}$. In one time constant, $e^{-t/\tau} = e^{-\tau/\tau} = e^{-1} = 0.3679$, or the function equals 36.79% of its maximum value of 1. At $t = 2\tau$, $e^{-t/\tau} = e^{-2\tau/\tau} = e^{-2} = 0.1353$, and the function has decayed to only 13.53% of its maximum value.

The magnitude of $e^{-t/\tau}$ and the percentage change between time constants have been tabulated in Tables 10.4 and 10.5, respectively. Note that the current has dropped 63.2% (100% − 36.8%) in the first time constant but only 0.4% between the fifth and sixth time constants. The rate of change of i_C is therefore quite sensitive to the time constant determined by the network parameters R and C. For this reason, the universal time constant chart of Fig. 10.30 is provided to permit a more accurate estimate of the value of the function e^{-x} for specific time intervals related to the time constant. The term *universal* is used because the axes are not scaled to specific values.

TABLE 10.4
i_C versus τ (charging phase).

t	Magnitude	
0	100%	
1τ	36.8%	
2τ	13.5%	
3τ	5.0%	
4τ	1.8%	
5τ	**0.67%**	← Less than 1% of maximum
6τ	0.24%	

TABLE 10.5
Change in i_C between time constants.

$(0 \rightarrow 1)\tau$	63.2%	
$(1 \rightarrow 2)\tau$	23.3%	
$(2 \rightarrow 3)\tau$	8.6%	
$(3 \rightarrow 4)\tau$	3.0%	
$(4 \rightarrow 5)\tau$	**1.2%**	
$(5 \rightarrow 6)\tau$	0.4%	← Less than 1%

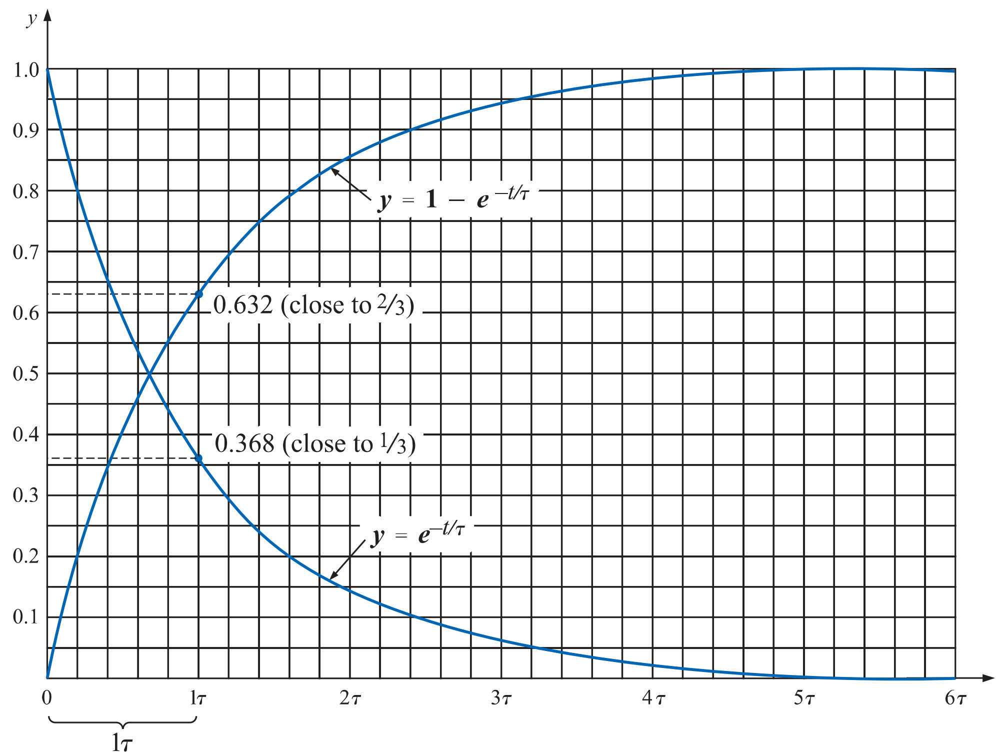


FIG. 10.30
Universal time constant chart.

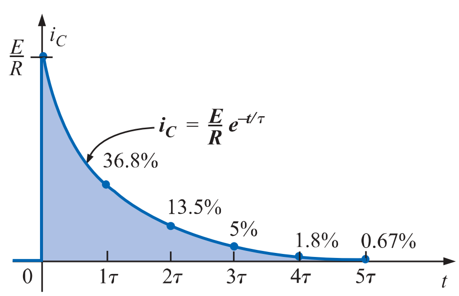


FIG. 10.31
i_C versus t during the charging phase.

Returning to Eq. (10.13), we find that the multiplying factor E/R is the maximum value that the current i_C can attain, as shown in Fig. 10.25. Substituting $t = 0$ s into Eq. (10.13) yields

$$i_C = \frac{E}{R}e^{-t/RC} = \frac{E}{R}e^{-0} = \frac{E}{R}$$

verifying our earlier conclusion.

For increasing values of t, the magnitude of $e^{-t/\tau}$, and therefore the value of i_C, will decrease, as shown in Fig. 10.31. Since the magnitude of i_C is less than 1% of its maximum after five time constants, we will assume the following for future analysis:

The current i_C of a capacitive network is essentially zero after five time constants of the charging phase have passed in a dc network.

Since C is usually found in microfarads or picofarads, the time constant $\tau = RC$ will never be greater than a few seconds unless R is very large.

Let us now turn our attention to the charging voltage across the capacitor. Through further mathematical analysis, the following equation for the voltage across the capacitor can be determined:

$$\boxed{v_C = E(1 - e^{-t/RC})} \qquad \textbf{(10.15)}$$

Note the presence of the same factor $e^{-t/RC}$ and the function $(1 - e^{-t/RC})$ appearing in Fig. 10.30. Since $e^{-t/\tau}$ is a decaying function, the factor $(1 - e^{-t/\tau})$ will grow toward a maximum value of 1 with time, as shown in Fig. 10.30. In addition, since E is the multiplying factor, we can conclude that, for all practical purposes, the voltage v_C is E volts after five time constants of the charging phase. A plot of v_C versus t is provided in Fig. 10.32.

If we keep R constant and reduce C, the product RC will decrease, and the rise time of five time constants will decrease. The change in transient behavior of the voltage v_C is plotted in Fig. 10.33 for various values of C. The product RC will always have some numerical value, even though it may be very small in some cases. For this reason:

The voltage across a capacitor cannot change instantaneously.

In fact, the capacitance of a network is also a measure of how much it will oppose a change in voltage across the network. The larger the capacitance, the larger the time constant, and the longer it takes to charge up to its final value (curve of C_3 in Fig. 10.33). A lesser capacitance would permit the voltage to build up more quickly since the time constant is less (curve of C_1 in Fig. 10.33).

The rate at which charge is deposited on the plates during the charging phase can be found by substituting the following for v_C in Eq. (10.15):

$$v_C = \frac{q}{C}$$

and $$\boxed{q = Cv_C = CE(1 - e^{-t/\tau})} \quad \textit{charging} \qquad \textbf{(10.16)}$$

indicating that the charging rate is very high during the first few time constants and less than 1% after five time constants.

The voltage across the resistor is determined by Ohm's law:

$$v_R = i_R R = Ri_C = R\frac{E}{R}e^{-t/\tau}$$

or $$\boxed{v_R = Ee^{-t/\tau}} \qquad \textbf{(10.17)}$$

A plot of v_R appears in Fig. 10.34.

Applying Kirchhoff's voltage law to the circuit of Fig. 10.24 will result in

$$v_C = E - v_R$$

Substituting Eq. (10.17):

$$v_C = E - Ee^{-t/\tau}$$

Factoring gives $v_C = E(1 - e^{-t/\tau})$, as obtained earlier.

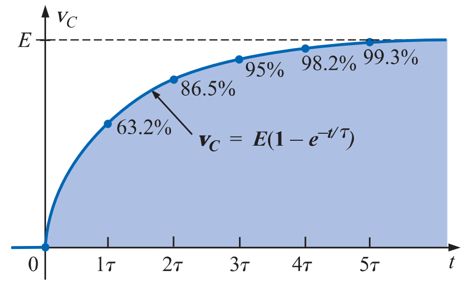


FIG. 10.32
v_C versus t during the charging phase.

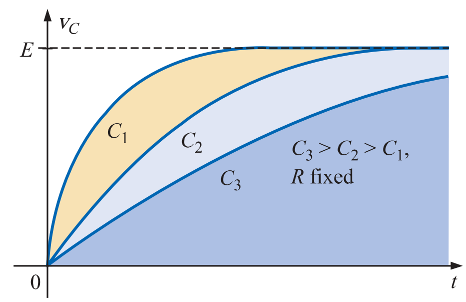


FIG. 10.33
Effect of C on the charging phase.

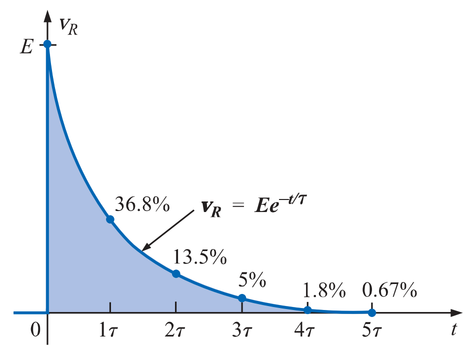


FIG. 10.34
v_R versus t during the charging phase.

EXAMPLE 10.5

a. Find the mathematical expressions for the transient behavior of v_C, i_C, and v_R for the circuit of Fig. 10.35 when the switch is moved to position 1. Plot the curves of v_C, i_C, and v_R.
b. How much time must pass before it can be assumed, for all practical purposes, that $i_C \cong 0$ A and $v_C \cong E$ volts?

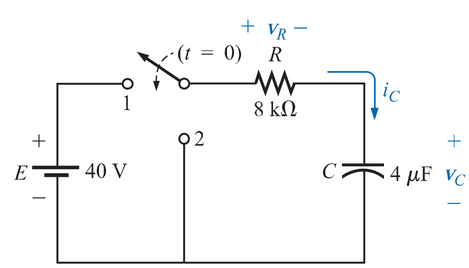


FIG. 10.35
Example 10.5.

Solutions:

a. $\tau = RC = (8 \times 10^3\ \Omega)(4 \times 10^{-6}\ \text{F}) = 32 \times 10^{-3}\ \text{s} = \mathbf{32\ ms}$

By Eq. (10.15),

$$v_C = E(1 - e^{-t/\tau}) = \mathbf{40(1 - e^{-t/(32\times10^{-3})})}$$

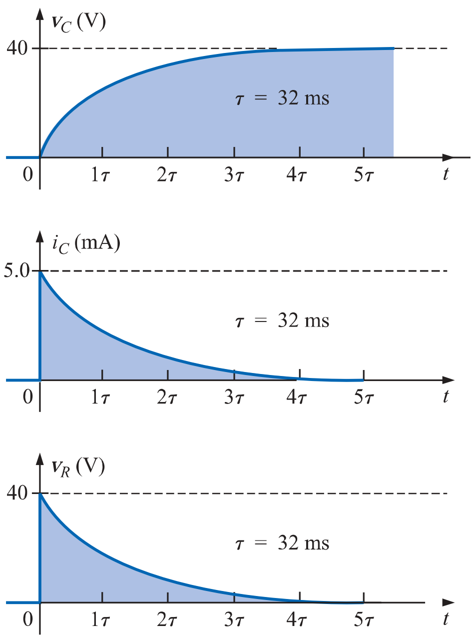


FIG. 10.36
Waveforms for the network of Fig. 10.35.

By Eq. (10.13),

$$i_C = \frac{E}{R} e^{-t/\tau} = \frac{40\text{ V}}{8\text{ k}\Omega} e^{-t/(32\times10^{-3})}$$
$$= \mathbf{(5 \times 10^{-3})e^{-t/(32\times10^{-3})}}$$

By Eq. (10.17),

$$v_R = Ee^{-t/\tau} = \mathbf{40e^{-t/(32\times10^{-3})}}$$

The curves appear in Fig. 10.36.

b. $5\tau = 5(32\text{ ms}) = \mathbf{160\text{ ms}}$

Once the voltage across the capacitor has reached the input voltage *E,* the capacitor is fully charged and will remain in this state if no further changes are made in the circuit.

If the switch of Fig. 10.24 is opened, as shown in Fig. 10.37(a), the capacitor will retain its charge for a period of time determined by its leakage current. For capacitors such as the mica and ceramic, the leakage current ($i_{\text{leakage}} = v_C/R_{\text{leakage}}$) is very small, enabling the capacitor to retain its charge, and hence the potential difference across its plates, for a long time. For electrolytic capacitors, which have very high leakage currents, the capacitor will discharge more rapidly, as shown in Fig. 10.37(b). In any event, to ensure that they are completely discharged, capacitors should be shorted by a lead or a screwdriver before they are handled.

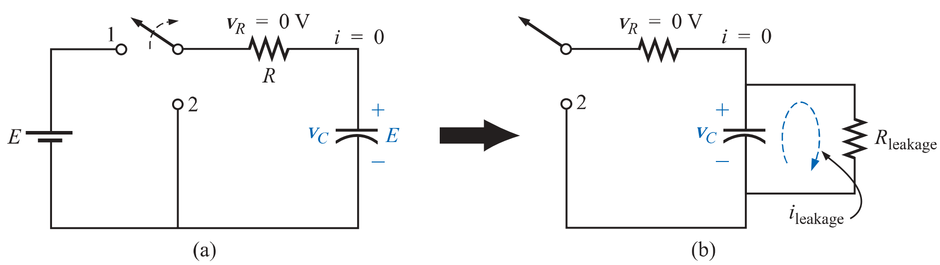


FIG. 10.37
Effect of the leakage current on the steady-state behavior of a capacitor.

10.8 DISCHARGE PHASE

The network of Fig. 10.24 is designed to both charge and discharge the capacitor. When the switch is placed in position 1, the capacitor will charge toward the supply voltage, as described in the last section. At any point in the charging process, if the switch is moved to position 2, the capacitor will begin to discharge at a rate sensitive to the same time constant $\tau = RC$. The established voltage across the capacitor will create a flow of charge in the closed path that will eventually discharge the capacitor completely. In essence, the capacitor functions like a battery with a decreasing terminal voltage. Note in particular that the current i_C has reversed direction, changing the polarity of the voltage across *R.*

If the capacitor had charged to the full battery voltage as indicated in Fig. 10.38, the equation for the decaying voltage across the capacitor would be the following:

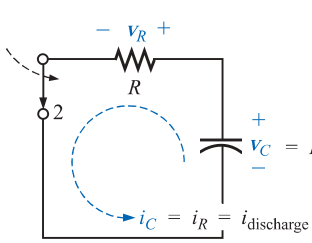


FIG. 10.38
Demonstrating the discharge behavior of a capacitive network.

$$\boxed{v_C = Ee^{-t/RC}} \quad \textit{discharging} \qquad \textbf{(10.18)}$$

which employs the function e^{-x} and the same time constant used above. The resulting curve will have the same shape as the curve for i_C and v_R in the last section. During the discharge phase, the current i_C will also decrease with time, as defined by the following equation:

$$\boxed{i_C = \frac{E}{R}e^{-t/RC}} \quad \textit{discharging} \tag{10.19}$$

The voltage $v_R = v_C$, and

$$\boxed{v_R = Ee^{-t/RC}} \quad \textit{discharging} \tag{10.20}$$

The complete discharge will occur, for all practical purposes, in five time constants. If the switch is moved between terminals 1 and 2 every five time constants, the wave shapes of Fig. 10.39 will result for v_C, i_C, and v_R. For each curve, the current direction and voltage polarities were defined by Fig. 10.24. Since the polarity of v_C is the same for both the charging and the discharging phases, the entire curve lies above the axis. The current i_C reverses direction during the charging and discharging phases, producing a negative pulse for both the current and the voltage v_R. Note that the voltage v_C never changes magnitude instantaneously but that the current i_C has the ability to change instantaneously, as demonstrated by its vertical rises and drops to maximum values.

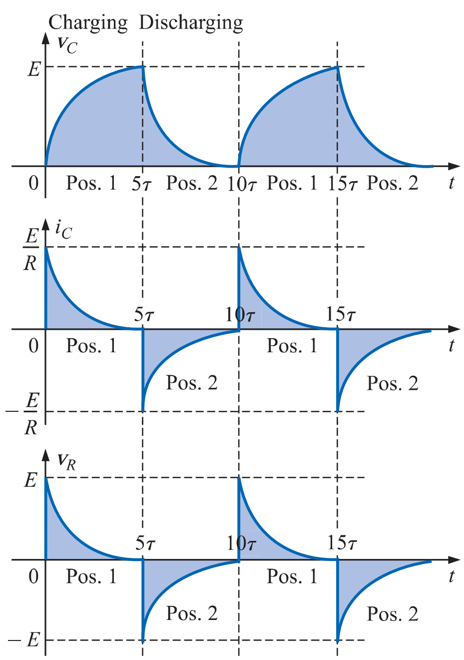


FIG. 10.39
The charging and discharging cycles for the network of Fig. 10.24.

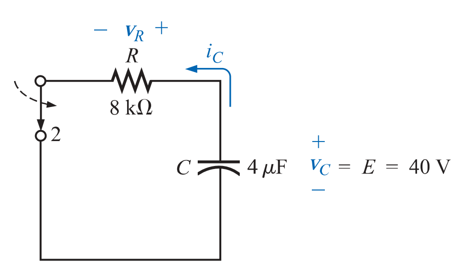


FIG. 10.40
Example 10.6.

EXAMPLE 10.6 After v_C in Example 10.5 has reached its final value of 40 V, the switch is thrown into position 2, as shown in Fig. 10.40. Find the mathematical expressions for the transient behavior of v_C, i_C,

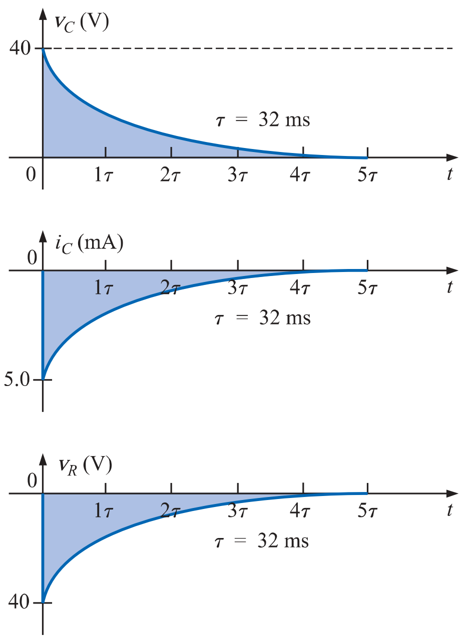


FIG. 10.41
The waveforms for the network of Fig. 10.40.

and v_R after the closing of the switch. Plot the curves for v_C, i_C, and v_R using the defined directions and polarities of Fig. 10.35. Assume that $t = 0$ when the switch is moved to position 2.

Solution:

$$\tau = 32 \text{ ms}$$

By Eq. (10.18),

$$v_C = Ee^{-t/\tau} = \mathbf{40e^{-t/(32\times10^{-3})}}$$

By Eq. (10.19),

$$i_C = -\frac{E}{R}e^{-t/\tau} = \mathbf{-(5 \times 10^{-3})e^{-t/(32\times10^{-3})}}$$

By Eq. (10.20),

$$v_R = -Ee^{-t/\tau} = \mathbf{-40e^{-t/(32\times10^{-3})}}$$

The curves appear in Fig. 10.41.

The preceding discussion and examples apply to situations in which the capacitor charges to the battery voltage. If the charging phase is disrupted before reaching the supply voltage, the capacitive voltage will be less, and the equation for the discharging voltage v_C will take on the form

$$\boxed{v_C = V_i e^{-t/RC}} \qquad \textbf{(10.21)}$$

where V_i is the starting or initial voltage for the discharge phase. The equation for the decaying current is also modified by simply substituting V_i for E; that is,

$$\boxed{i_C = \frac{V_i}{R}e^{-t/\tau} = I_i e^{-t/\tau}} \qquad \textbf{(10.22)}$$

Use of the above equations will be demonstrated in Examples 10.7 and 10.8.

EXAMPLE 10.7

a. Find the mathematical expression for the transient behavior of the voltage across the capacitor of Fig. 10.42 if the switch is thrown into position 1 at $t = 0$ s.
b. Repeat part (a) for i_C.
c. Find the mathematical expressions for the response of v_C and i_C if the switch is thrown into position 2 at 30 ms (assuming that the leakage resistance of the capacitor is infinite ohms).
d. Find the mathematical expressions for the voltage v_C and current i_C if the switch is thrown into position 3 at $t = 48$ ms.
e. Plot the waveforms obtained in parts (a) through (d) on the same time axis for the voltage v_C and the current i_C using the defined polarity and current direction of Fig. 10.42.

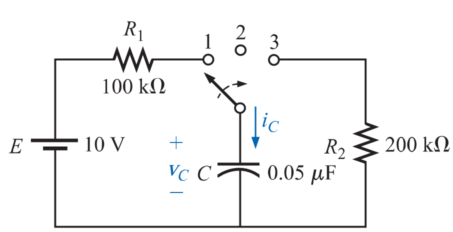


FIG. 10.42
Example 10.7.

Solutions:

a. *Charging phase:*

$$v_C = E(1 - e^{-t/\tau})$$
$$\tau = R_1C = (100 \times 10^3\ \Omega)(0.05 \times 10^{-6}\ \text{F}) = 5 \times 10^{-3}\ \text{s} = 5\ \text{ms}$$
$$\mathbf{v_C = 10(1 - e^{-t/(5\times10^{-3})})}$$

b. $$i_C = \frac{E}{R_1}e^{-t/\tau} = \frac{10\ \text{V}}{100 \times 10^3\ \Omega}e^{-t/(5\times10^{-3})}$$
$$\mathbf{i_C = (0.1 \times 10^{-3})e^{-t/(5\times10^{-3})}}$$

c. *Storage phase:*

$$v_C = E = \mathbf{10\ V}$$
$$i_C = \mathbf{0\ A}$$

d. *Discharge phase* (starting at 48 ms with $t = 0$ s for the following equations):

$$v_C = Ee^{-t/\tau'}$$
$$\tau' = R_2C = (200 \times 10^3\ \Omega)(0.05 \times 10^{-6}\ \text{F}) = 10 \times 10^{-3}\ \text{s} = 10\ \text{ms}$$
$$\mathbf{v_C = 10e^{-t/(10\times10^{-3})}}$$
$$i_C = -\frac{E}{R_2}e^{-t/\tau'} = -\frac{10\ \text{V}}{200 \times 10^3\ \Omega}e^{-t/(10\times10^{-3})}$$
$$\mathbf{i_C = -(0.05 \times 10^{-3})e^{-t/(10\times10^{-3})}}$$

e. See Fig. 10.43.

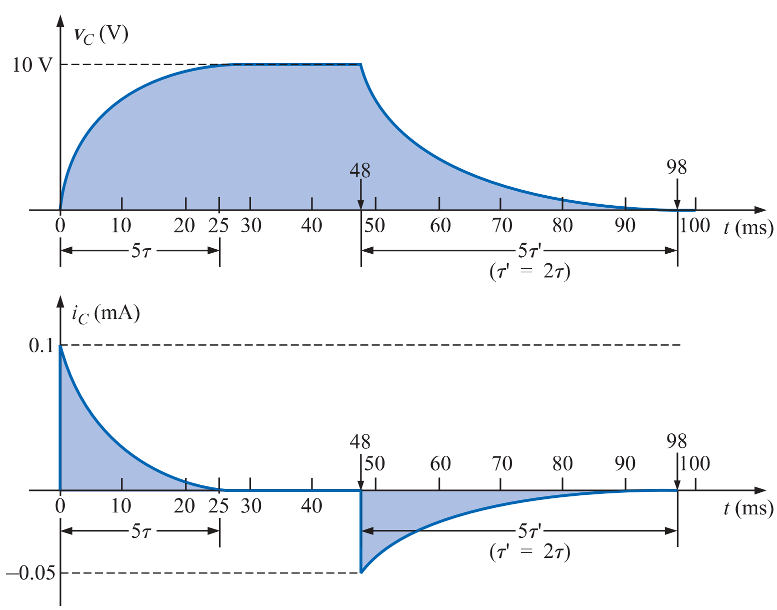


FIG. 10.43
The waveforms for the network of Fig. 10.42.

EXAMPLE 10.8

a. Find the mathematical expression for the transient behavior of the voltage across the capacitor of Fig. 10.44 if the switch is thrown into position 1 at $t = 0$ s.

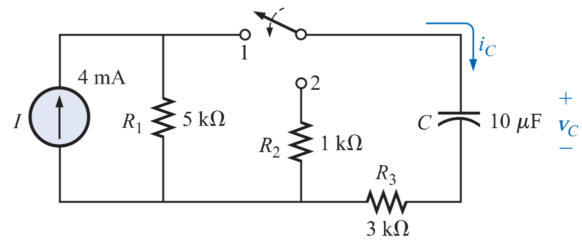


FIG. 10.44
Example 10.8.

b. Repeat part (a) for i_C.
c. Find the mathematical expression for the response of v_C and i_C if the switch is thrown into position 2 at $t = 1\tau$ of the charging phase.
d. Plot the waveforms obtained in parts (a) through (c) on the same time axis for the voltage v_C and the current i_C using the defined polarity and current direction of Fig. 10.44.

Solutions:

a. *Charging phase:* Converting the current source to a voltage source will result in the network of Fig. 10.45.

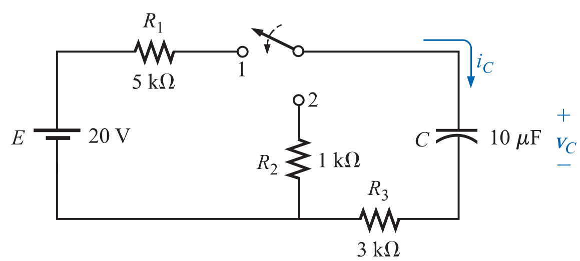


FIG. 10.45
The charging phase for the network of Fig. 10.44.

$$v_C = E(1 - e^{-t/\tau_1})$$
$$\tau_1 = (R_1 + R_3)C = (5\text{ k}\Omega + 3\text{ k}\Omega)(10 \times 10^{-6}\text{ F})$$
$$= 80\text{ ms}$$
$$v_C = \mathbf{20(1 - e^{-t/(80\times10^{-3})})}$$

b. $$i_C = \frac{E}{R_1 + R_3}e^{-t/\tau_1}$$
$$= \frac{20\text{ V}}{8\text{ k}\Omega}e^{-t/(80\times10^{-3})}$$
$$i_C = \mathbf{(2.5 \times 10^{-3})e^{-t/(80\times10^{-3})}}$$

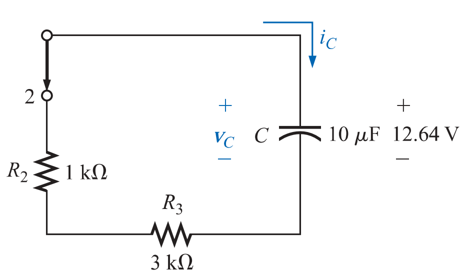


FIG. 10.46
Network of Fig. 10.45 when the switch is moved to position 2 at $t = 1\tau_1$.

c. At $t = 1\tau_1$, $v_C = 0.632E = 0.632(20\text{ V}) = 12.64\text{ V}$, resulting in the network of Fig. 10.46. Then $v_C = V_i e^{-t/\tau_2}$ with

$$\tau_2 = (R_2 + R_3)C = (1\text{ k}\Omega + 3\text{ k}\Omega)(10 \times 10^{-6}\text{ F})$$
$$= 40\text{ ms}$$

and $$v_C = \mathbf{12.64}\boldsymbol{e^{-t/(40\times10^{-3})}}$$

At $t = 1\tau_1$, i_C drops to (0.368)(2.5 mA) = 0.92 mA. Then it switches to

$$i_C = -I_i e^{-t/\tau_2}$$
$$= -\frac{V_i}{R_2 + R_3}e^{-t/\tau_2} = -\frac{12.64\text{ V}}{1\text{ k}\Omega + 3\text{ k}\Omega}e^{-t/(40\times10^{-3})}$$
$$i_C = \mathbf{-3.16 \times 10^{-3}}\boldsymbol{e^{-t/(40\times10^{-3})}}$$

d. See Fig. 10.47.

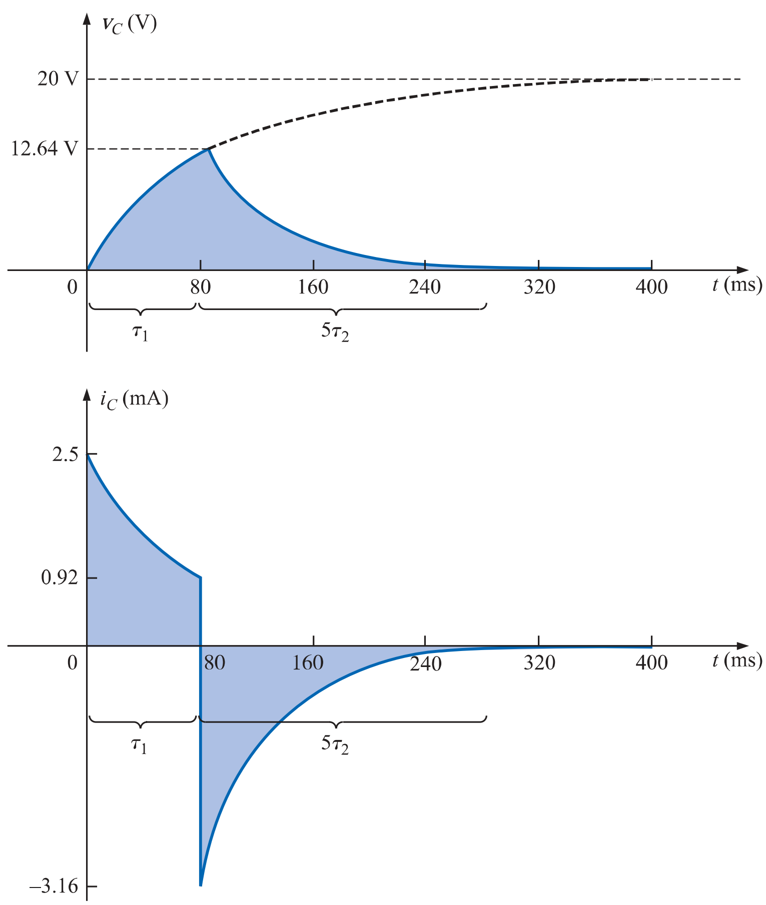


FIG. 10.47
The waveforms for the network of Fig. 10.44.

10.9 INITIAL VALUES

In all the examples examined in the previous sections, the capacitor was uncharged before the switch was thrown. We will now examine the effect of a charge, and therefore a voltage ($V = Q/C$), on the plates at the instant the switching action takes place. The voltage across the capacitor at this instant is called the *initial* value, as shown for the general waveform of Fig. 10.48. Once the switch is thrown, the transient

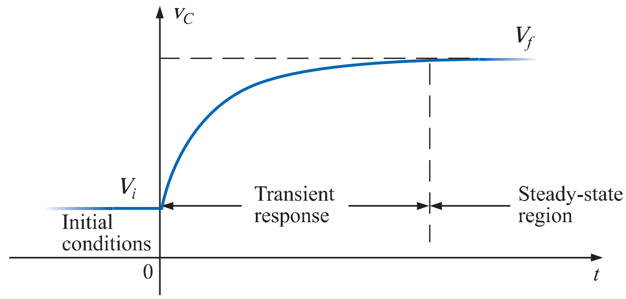


FIG. 10.48
Defining the regions associated with a transient response.

phase will commence until a leveling off occurs after five time constants. This region of relatively fixed value that follows the transient response is called the *steady-state* region, and the resulting value is called the *steady-state* or *final* value. The steady-state value is found by simply substituting the open-circuit equivalent for the capacitor and finding the voltage across the plates. Using the transient equation developed in the previous section, an equation for the voltage v_C can be written for the entire time interval of Fig. 10.48; that is,

$$v_C = V_i + (V_f - V_i)(1 - e^{-t/\tau})$$

However, by multiplying through and rearranging terms:

$$\begin{aligned} v_C &= V_i + V_f - V_f e^{-t/\tau} - V_i + V_i e^{-t/\tau} \\ &= V_f - V_f e^{-t/\tau} + V_i e^{-t/\tau} \end{aligned}$$

we find

$$\boxed{v_C = V_f + (V_i - V_f)e^{-t/\tau}} \qquad \textbf{(10.23)}$$

If you are required to draw the waveform for the voltage v_C from the initial value to the final value, start by drawing a line at the initial and steady-state levels, and then add the transient response (sensitive to the time constant) between the two levels. The example to follow will clarify the procedure.

EXAMPLE 10.9 The capacitor of Fig. 10.49 has an initial voltage of 4 V.

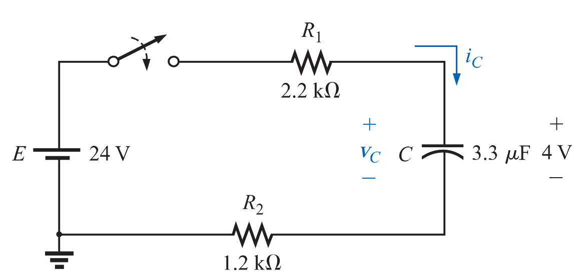


FIG. 10.49
Example 10.9.

a. Find the mathematical expression for the voltage across the capacitor once the switch is closed.
b. Find the mathematical expression for the current during the transient period.
c. Sketch the waveform for each from initial value to final value.

Solutions:

a. Substituting the open-circuit equivalent for the capacitor will result in a final or steady-state voltage v_C of 24 V.

The time constant is determined by

$$\begin{aligned}\tau &= (R_1 + R_2)C \\ &= (2.2\ \text{k}\Omega + 1.2\ \text{k}\Omega)(3.3\ \mu\text{F}) \\ &= 11.22\ \text{ms}\end{aligned}$$

with $$5\tau = 56.1\ \text{ms}$$

Applying Eq. (10.23):

$$\begin{aligned}v_C &= V_f + (V_i - V_f)e^{-t/\tau} \\ &= 24\ \text{V} + (4\ \text{V} - 24\ \text{V})e^{-t/11.22\text{ms}}\end{aligned}$$

and $$\mathbf{v_C = 24\ V - 20\ Ve^{-t/11.22ms}}$$

b. Since the voltage across the capacitor is constant at 4 V prior to the closing of the switch, the current (whose level is sensitive only to changes in voltage across the capacitor) must have an initial value of 0 mA. At the instant the switch is closed, the voltage across the capacitor cannot change instantaneously, so the voltage across the resistive elements at this instant is the applied voltage less the initial voltage across the capacitor. The resulting peak current is

$$I_m = \frac{E - V_C}{R_1 + R_2} = \frac{24\ \text{V} - 4\ \text{V}}{2.2\ \text{k}\Omega + 1.2\ \text{k}\Omega} = \frac{20\ \text{V}}{3.4\ \text{k}\Omega} = 5.88\ \text{mA}$$

The current will then decay (with the same time constant as the voltage v_C) to zero because the capacitor is approaching its open-circuit equivalence.

The equation for i_C is therefore:

$$i_C = \mathbf{5.88\ mA}e^{-t/11.22\text{ms}}$$

c. See Fig. 10.50.

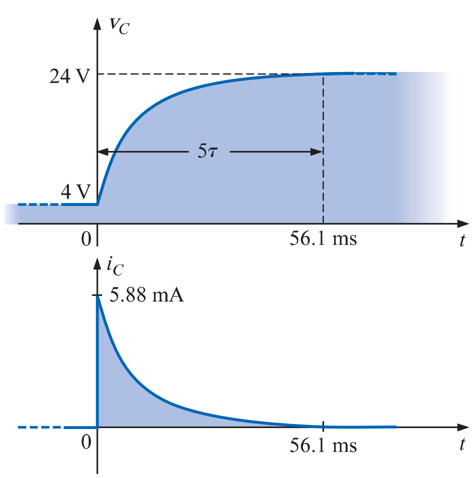


FIG. 10.50
v_C and i_C for the network of Fig. 10.49.

The initial and final values of the voltage were drawn first, and then the transient response was included between these levels. For the current, the waveform begins and ends at zero, with the peak value having a sign sensitive to the defined direction of i_C in Fig. 10.49.

Let us now test the validity of the equation for v_C by substituting $t = 0$ s to reflect the instant the switch is closed.

$$e^{-t/\tau} = e^{-0} = 1$$

and $\quad v_C = 24\text{ V} - 20\text{ V}e^{-t/\tau} = 24\text{ V} - 20\text{ V} = 4\text{ V}$

When $t > 5\tau$,

$$e^{-t/\tau} \cong 0$$

and $\quad v_C = 24\text{ V} - 20\text{ V}e^{-t/\tau} = 24\text{ V} - 0\text{ V} = 24\text{ V}$

10.10 INSTANTANEOUS VALUES

On occasion it will be necessary to determine the voltage or current at a particular instant of time that is not an integral multiple of τ, as in the previous sections. For example, if

$$v_C = 20(1 - e^{-t/(2\times10^{-3})})$$

the voltage v_C may be required at $t = 5$ ms, which does not correspond to a particular value of τ. Figure 10.30 reveals that $(1 - e^{-t/\tau})$ is approximately 0.93 at $t = 5\text{ ms} = 2.5\tau$, resulting in $v_C = 20(0.93) = 18.6$ V. Additional accuracy can be obtained simply by substituting $t = 5$ ms into the equation and solving for v_C using a calculator or table to determine $e^{-2.5}$. Thus,

$$\begin{aligned} v_C &= 20(1 - e^{-5\text{ms}/2\text{ms}}) \\ &= 20(1 - e^{-2.5}) \\ &= 20(1 - 0.082) \\ &= 20(0.918) \\ &= \mathbf{18.36\ V} \end{aligned}$$

The results are close, but accuracy beyond the tenths place is suspect using Fig. 10.30. The above procedure can also be applied to any other equation introduced in this chapter for currents or other voltages.

There are also occasions when the time to reach a particular voltage or current is required. The procedure is complicated somewhat by the use of natural logs ($\log_e$, or ln), but today's calculators are equipped to handle the operation with ease. There are two forms that require some development. First, consider the following sequence:

$$v_C = E(1 - e^{-t/\tau})$$

$$\frac{v_C}{E} = 1 - e^{-t/\tau}$$

$$1 - \frac{v_C}{E} = e^{-t/\tau}$$

$$\log_e\left(1 - \frac{v_C}{E}\right) = \log_e e^{-t/\tau}$$

$$\log_e\left(1 - \frac{v_C}{E}\right) = -\frac{t}{\tau}$$

12

Inductors

12.1 INTRODUCTION

We have examined the resistor and the capacitor in detail. In this chapter we shall consider a third element, the **inductor,** which has a number of response characteristics similar in many respects to those of the capacitor. In fact, some sections of this chapter will proceed parallel to those for the capacitor to emphasize the similarity that exists between the two elements.

12.2 FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

If a conductor is moved through a magnetic field so that it cuts magnetic lines of flux, a voltage will be induced across the conductor, as shown in Fig. 12.1. The greater the number of flux lines cut per unit time (by increasing the speed with which the conductor passes through the field), or the stronger the magnetic field strength (for the same tra-

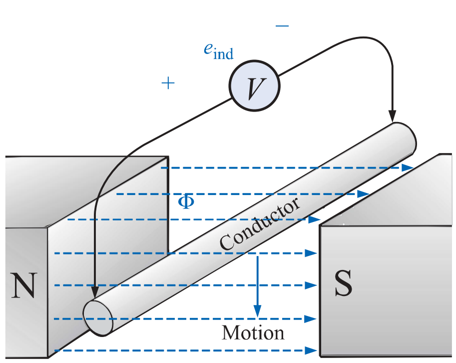


FIG. 12.1
Generating an induced voltage by moving a conductor through a magnetic field.

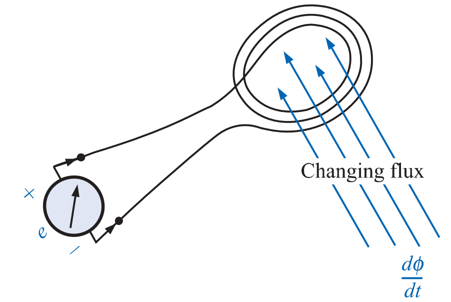


FIG. 12.2
Demonstrating Faraday's law.

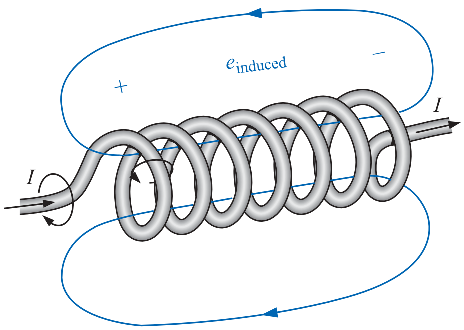


FIG. 12.3
Demonstrating the effect of Lenz's law.

American (Albany, NY; Princeton, NJ)
(1797–1878)
Physicist and Mathematician
Professor of Natural Philosophy, Princeton University

Courtesy of the Smithsonian Institution Photo No. 59,054

In the early 1800s the title Professor of Natural Philosophy was applied to educators in the sciences. As a student and teacher at the Albany Academy, Henry performed extensive research in the area of electromagnetism. He improved the design of *electromagnets* by insulating the coil wire to permit a tighter wrap on the core. One of his earlier designs was capable of lifting 3600 pounds. In 1832 he discovered and delivered a paper on *self-induction*. This was followed by the construction of an effective *electric telegraph transmitter and receiver* and extensive research on the oscillatory nature of lightning and discharges from a *Leyden jar*. In 1845 he was appointed the first Secretary of the Smithsonian.

FIG. 12.4
Joseph Henry.

versing speed), the greater will be the induced voltage across the conductor. If the conductor is held fixed and the magnetic field is moved so that its flux lines cut the conductor, the same effect will be produced.

If a coil of N turns is placed in the region of a changing flux, as in Fig. 12.2, a voltage will be induced across the coil as determined by **Faraday's law:**

$$\boxed{e = N\frac{d\phi}{dt}} \qquad \text{(volts, V)} \tag{12.1}$$

where N represents the number of turns of the coil and $d\phi/dt$ is the instantaneous change in flux (in webers) linking the coil. The term *linking* refers to the flux within the turns of wire. The term *changing* simply indicates that either the strength of the field linking the coil changes in magnitude or the coil is moved through the field in such a way that the number of flux lines through the coil changes with time.

If the flux linking the coil ceases to change, such as when the coil simply sits still in a magnetic field of fixed strength, $d\phi/dt = 0$, and the induced voltage $e = N(d\phi/dt) = N(0) = 0$.

12.3 LENZ'S LAW

In Section 11.2 it was shown that the magnetic flux linking a coil of N turns with a current I has the distribution of Fig. 12.3.

If the current increases in magnitude, the flux linking the coil also increases. It was shown in Section 12.2, however, that a changing flux linking a coil induces a voltage across the coil. For this coil, therefore, an induced voltage is developed *across* the coil due to the change in current *through* the coil. The polarity of this induced voltage tends to establish a current in the coil that produces a flux that will oppose any change in the original flux. In other words, the induced effect (e_{ind}) is a result of the increasing current through the coil. However, the resulting induced voltage will tend to establish a current that will oppose the increasing change in current through the coil. Keep in mind that this is all occurring simultaneously. The instant the current begins to increase in magnitude, there will be an opposing effect trying to limit the change. It is "choking" the change in current through the coil. Hence, the term **choke** is often applied to the inductor or coil. In fact, we will find shortly that the current through a coil cannot change instantaneously. A period of time determined by the coil and the resistance of the circuit is required before the inductor discontinues its opposition to a momentary change in current. Recall a similar situation for the voltage across a capacitor in Chapter 10. The reaction above is true for increasing or decreasing levels of current through the coil. This effect is an example of a general principle known as **Lenz's law,** which states that

an induced effect is always such as to oppose the cause that produced it.

12.4 SELF-INDUCTANCE

The ability of a coil to oppose any change in current is a measure of the **self-inductance** L of the coil. For brevity, the prefix *self* is usually dropped. Inductance is measured in henries (H), after the American physicist Joseph Henry (Fig. 12.4).

Inductors are coils of various dimensions designed to introduce specified amounts of inductance into a circuit. The inductance of a coil varies directly with the magnetic properties of the coil. Ferromagnetic materials, therefore, are frequently employed to increase the inductance by increasing the flux linking the coil.

A close approximation, in terms of physical dimensions, for the inductance of the coils of Fig. 12.5 can be found using the following equation:

$$\boxed{L = \frac{N^2 \mu A}{l}} \qquad \text{(henries, H)} \tag{12.2}$$

where N represents the number of turns; μ, the permeability of the core (as introduced in Section 11.4; recall that μ is not a constant but depends on the level of B and H since $\mu = B/H$); A, the area of the core in square meters; and l, the mean length of the core in meters.

Substituting $\mu = \mu_r \mu_o$ into Eq. (12.2) yields

$$L = \frac{N^2 \mu_r \mu_o A}{l} = \mu_r \frac{N^2 \mu_o A}{l}$$

and

$$\boxed{L = \mu_r L_o} \tag{12.3}$$

where L_o is the inductance of the coil with an air core. In other words, the inductance of a coil with a ferromagnetic core is the relative permeability of the core times the inductance achieved with an air core.

Equations for the inductance of coils different from those shown above can be found in reference handbooks. Most of the equations are more complex than those just described.

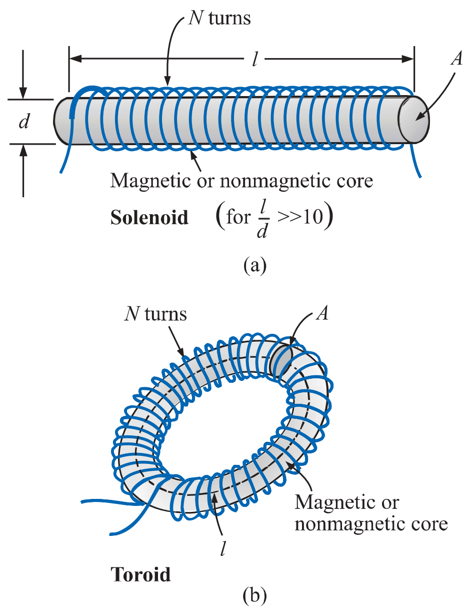


FIG. 12.5
Inductor configurations for which Equation (12.2) is appropriate.

EXAMPLE 12.1 Find the inductance of the air-core coil of Fig. 12.6.

Solution:

$$\mu = \mu_r \mu_o = (1)(\mu_o) = \mu_o$$

$$A = \frac{\pi d^2}{4} = \frac{(\pi)(4 \times 10^{-3}\text{ m})^2}{4} = 12.57 \times 10^{-6}\text{ m}^2$$

$$L_o = \frac{N^2 \mu_o A}{l} = \frac{(100\text{ t})^2(4\pi \times 10^{-7}\text{ Wb/A·m})(12.57 \times 10^{-6}\text{ m}^2)}{0.1\text{ m}}$$

$$= \mathbf{1.58\ \mu H}$$

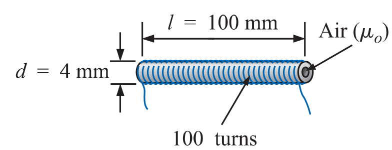


FIG. 12.6
Example 12.1.

EXAMPLE 12.2 Repeat Example 12.1, but with an iron core and conditions such that $\mu_r = 2000$.

Solution: By Eq. (12.3),

$$L = \mu_r L_o = (2000)(1.58 \times 10^{-6}\text{ H}) = \mathbf{3.16\ mH}$$

12.5 TYPES OF INDUCTORS

Practical Equivalent

Inductors, like capacitors, are not ideal. Associated with every inductor are a resistance equal to the resistance of the turns and a stray capaci-

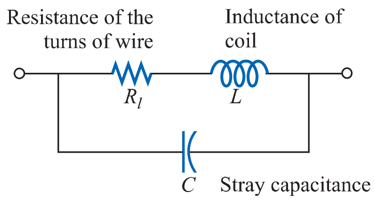


FIG. 12.7
Complete equivalent model for an inductor.

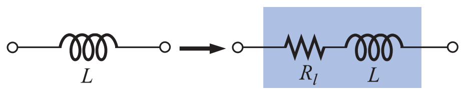


FIG. 12.8
Practical equivalent model for an inductor.

tance due to the capacitance between the turns of the coil. To include these effects, the equivalent circuit for the inductor is as shown in Fig. 12.7. However, for most applications considered in this text, the stray capacitance appearing in Fig. 12.7 can be ignored, resulting in the equivalent model of Fig. 12.8. The resistance R_l can play an important role in the analysis of networks with inductive elements. For most applications, we have been able to treat the capacitor as an ideal element and maintain a high degree of accuracy. For the inductor, however, R_l must often be included in the analysis and can have a pronounced effect on the response of a system (see Chapter 20, "Resonance"). The level of R_l can extend from a few ohms to a few hundred ohms. Keep in mind that the longer or thinner the wire used in the construction of the inductor, the greater will be the dc resistance as determined by $R = \rho l / A$. Our initial analysis will treat the inductor as an ideal element. Once a general feeling for the response of the element is established, the effects of R_l will be included.

Symbols

The primary function of the inductor, however, is to introduce inductance—not resistance or capacitance—into the network. For this reason, the symbols employed for inductance are as shown in Fig. 12.9.

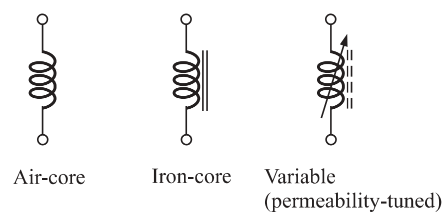


FIG. 12.9
Inductor symbols.

Appearance

All inductors, like capacitors, can be listed under two general headings: *fixed* and *variable.* The fixed air-core and iron-core inductors were described in the last section. The permeability-tuned variable coil has a ferromagnetic shaft that can be moved within the coil to vary the flux linkages of the coil and thereby its inductance. Several fixed and variable inductors appear in Fig. 12.10.

Testing

The primary reasons for inductor failure are shorts that develop between the windings and open circuits in the windings due to factors such as excessive currents, overheating, and age. The open-circuit condition can be checked easily with an ohmmeter (∞ ohms indication), but the short-circuit condition is harder to check because the resistance of many good inductors is relatively small and the shorting of a few windings will not adversely affect the total resistance. Of course, if one is aware of the typical resistance of the coil, it can be compared to the

the voltage across the coil is not determined solely by the magnitude of the change in current through the coil (Δi), but also by the rate of change of current through the coil ($\Delta i/\Delta t$).

A similar statement was made for the current of a capacitor due to a change in voltage across the capacitor.

A careful examination of Fig. 12.13 will also reveal that the area under the positive pulse from 2 ms to 4 ms equals the area under the negative pulse from 4 ms to 9 ms. In Section 12.13, we will find that the area under the curves represents the energy stored or released by the inductor. From 2 ms to 4 ms, the inductor is storing energy, whereas from 4 ms to 9 ms, the inductor is releasing the energy stored. For the full period zero to 10 ms, energy has simply been stored and released; there has been no dissipation as experienced for the resistive elements. Over a full cycle, both the ideal capacitor and inductor do not consume energy but simply store and release it in their respective forms.

12.7 *R-L* TRANSIENTS: STORAGE CYCLE

The changing voltages and current that result during the storing of energy in the form of a magnetic field by an inductor in a dc circuit can best be described using the circuit of Fig. 12.14. At the instant the switch is closed, the inductance of the coil will prevent an instantaneous change in current through the coil. The potential drop across the coil, v_L, will equal the impressed voltage E as determined by Kirchhoff's voltage law since $v_R = iR = (0)R = 0$ V. The current i_L will then build up from zero, establishing a voltage drop across the resistor and a corresponding drop in v_L. The current will continue to increase until the voltage across the inductor drops to zero volts and the full impressed voltage appears across the resistor. Initially, the current i_L increases quite rapidly, followed by a continually decreasing rate until it reaches its maximum value of E/R.

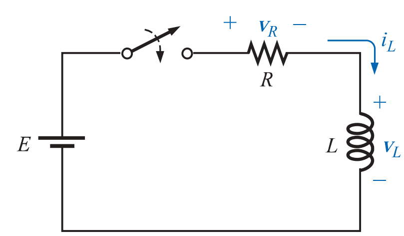


FIG. 12.14
Basic R-L transient network.

You will recall from the discussion of capacitors that a capacitor has a short-circuit equivalent when the switch is first closed and an open-circuit equivalent when steady-state conditions are established. The inductor assumes the opposite equivalents for each stage. The instant the switch of Fig. 12.14 is closed, the equivalent network will appear as shown in Fig. 12.15. Note the correspondence with the earlier comments regarding the levels of voltage and current. The inductor obviously meets all the requirements for an open-circuit equivalent: $v_L = E$ volts, and $i_L = 0$ A.

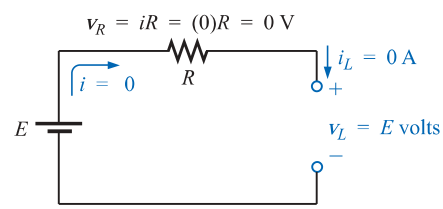


FIG. 12.15
Circuit of Fig. 12.14 the instant the switch is closed.

When steady-state conditions have been established and the storage phase is complete, the "equivalent" network will appear as shown in Fig. 12.16. The network clearly reveals the following:

An ideal inductor ($R_l = 0\ \Omega$) assumes a short-circuit equivalent in a dc network once steady-state conditions have been established.

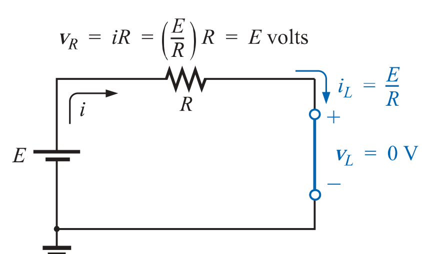


FIG. 12.16
Circuit of Fig. 12.14 under steady-state conditions.

Fortunately, the mathematical equations for the voltages and current for the storage phase are similar in many respects to those encountered for the *R-C* network. The experience gained with these equations in Chapter 10 will undoubtedly make the analysis of *R-L* networks somewhat easier to understand.

The equation for the current i_L during the storage phase is the following:

$$\boxed{i_L = I_m(1 - e^{-t/\tau}) = \frac{E}{R}(1 - e^{-t/(L/R)})} \qquad \textbf{(12.8)}$$

Note the factor $(1 - e^{-t/\tau})$, which also appeared for the voltage v_C of a capacitor during the charging phase. A plot of the equation is given in Fig. 12.17, clearly indicating that the maximum steady-state value of i_L is E/R, and that the rate of change in current decreases as time passes. The abscissa is scaled in time constants, with τ for inductive circuits defined by the following:

$$\boxed{\tau = \frac{L}{R}} \qquad \text{(seconds, s)} \qquad \textbf{(12.9)}$$

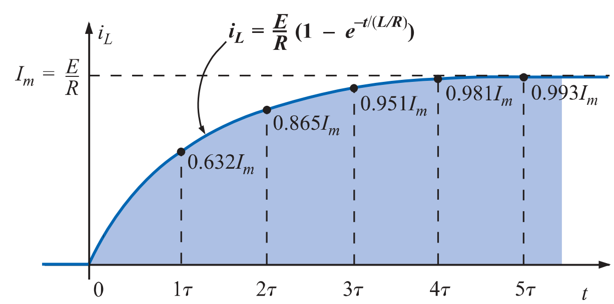


FIG. 12.17
Plotting the waveform for i_L during the storage cycle.

The fact that τ has the units of time can be verified by taking the equation for the induced voltage

$$v_L = L\frac{di}{dt}$$

and solving for L: $$L = \frac{v_L}{di/dt}$$

which leads to the ratio

$$\tau = \frac{L}{R} = \frac{\dfrac{v_L}{di/dt}}{R} = \frac{v_L}{\dfrac{di}{dt}R} \longrightarrow \frac{V}{\dfrac{IR}{t}} = \frac{\cancel{V}}{\dfrac{\cancel{V}}{t}} = t \qquad \text{(s)}$$

Our experience with the factor $(1 - e^{-t/\tau})$ verifies the level of 63.2% after one time constant, 86.5% after two time constants, and so on. For convenience, Figure 10.29 is repeated as Fig. 12.18 to evaluate the functions $(1 - e^{-t/\tau})$ and $e^{-t/\tau}$ at various values of τ.

If we keep R constant and increase L, the ratio L/R increases and the rise time increases. The change in transient behavior for the current i_L is plotted in Fig. 12.19 for various values of L. Note again the duality between these curves and those obtained for the R-C network in Fig. 10.32.

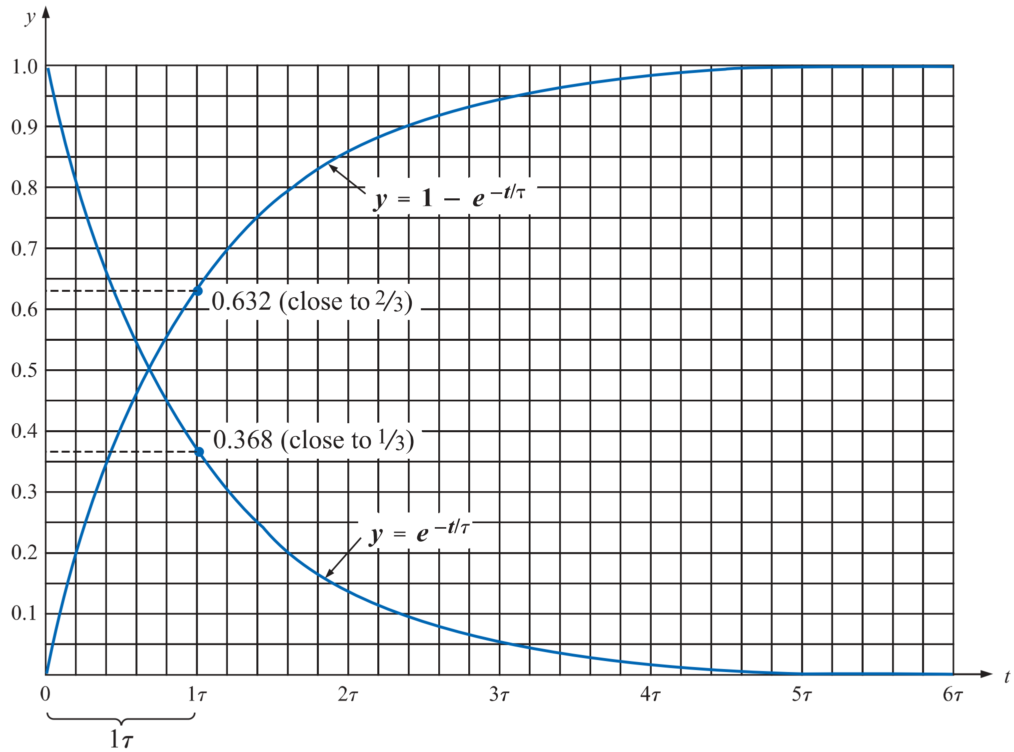


FIG. 12.18
Plotting the functions $y = 1 - e^{-t/\tau}$ and $y = e^{-t/\tau}$.

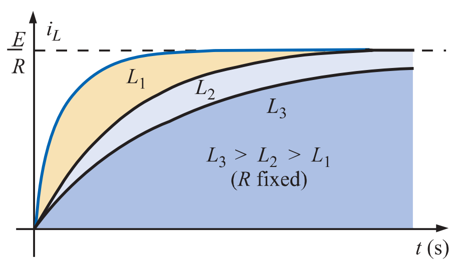


FIG. 12.19
Effect of L on the shape of the i_L storage waveform.

For most practical applications, we will assume that

the storage phase has passed and steady-state conditions have been established once a period of time equal to five time constants has occurred.

In addition, since L/R will always have some numerical value, even though it may be very small, the period 5τ will always be greater than zero, confirming the fact that

the current cannot change instantaneously in an inductive network.

In fact, the larger the inductance, the more the circuit will oppose a rapid buildup in current level.

Figures 12.15 and 12.16 clearly reveal that the voltage across the coil jumps to E volts when the switch is closed and decays to zero volts with time. The decay occurs in an exponential manner, and v_L during the storage phase can be described mathematically by the following equation:

$$\boxed{v_L = Ee^{-t/\tau}} \tag{12.10}$$

A plot of v_L appears in Fig. 12.20 with the time axis again divided into equal increments of τ. Obviously, the voltage v_L will decrease to zero volts at the same rate the current presses toward its maximum value.

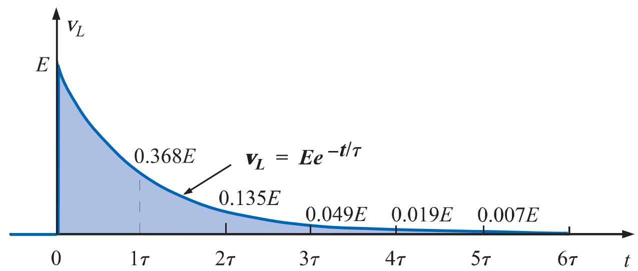


FIG. 12.20
Plotting the voltage v_R versus time for the network of Fig. 12.14.

In five time constants, $i_L = E/R$, $v_L = 0$ V, and the inductor can be replaced by its short-circuit equivalent.

Since $$v_R = i_R R = i_L R$$

then $$v_R = \left[\frac{E}{R}(1 - e^{-t/\tau})\right]R$$

and $$\boxed{v_R = E(1 - e^{-t/\tau})} \tag{12.11}$$

and the curve for v_R will have the same shape as obtained for i_L.

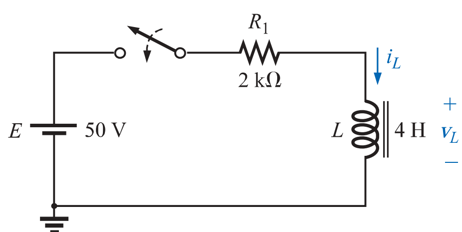


FIG. 12.21
Example 12.4.

EXAMPLE 12.4 Find the mathematical expressions for the transient behavior of i_L and v_L for the circuit of Fig. 12.21 after the closing of the switch. Sketch the resulting curves.

Solution:

$$\tau = \frac{L}{R_1} = \frac{4\text{ H}}{2\text{ k}\Omega} = 2\text{ ms}$$

By Eq. (12.8),

$$I_m = \frac{E}{R_1} = \frac{50}{2\text{ k}\Omega} = 25 \times 10^{-3}\text{ A} = 25\text{ mA}$$

and $$i_L = \mathbf{(25 \times 10^{-3})(1 - e^{-t/(2\times10^{-3})})}$$

By Eq. (12.10),

$$v_L = \mathbf{50e^{-t/(2\times10^{-3})}}$$

Both waveforms appear in Fig. 12.22.

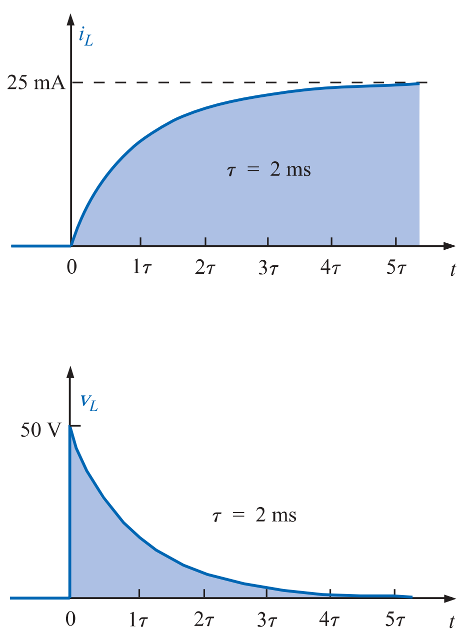


FIG. 12.22
i_L and v_L for the network of Fig. 12.21.

12.8 INITIAL VALUES

This section will parallel Section 10.9 (Initial Values—Capacitors) on the effect of *initial values* on the transient phase. Since the current through a coil cannot change instantaneously, the current through a coil will begin the *transient phase* at the *initial value* established by the network (note Fig. 12.23) before the switch was closed. It will then pass through the transient phase until it reaches the *steady-state* (or *final*) level after about five time constants. The steady-state level of the inductor current can be found by simply substituting its short-circuit equivalent (or R_l for the practical equivalent) and finding the resulting current through the element.

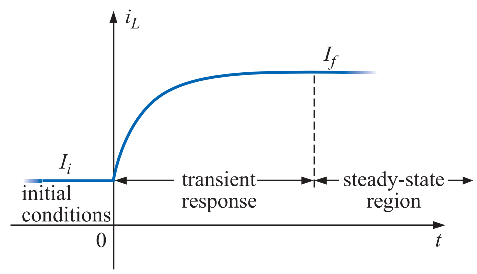


FIG. 12.23
Defining the three phases of a transient waveform.

Using the transient equation developed in the previous section, an equation for the current i_L can be written for the entire time interval of Fig. 12.23; that is,

$$i_L = I_i + (I_f - I_i)(1 - e^{-t/\tau})$$

with $(I_f - I_i)$ representing the total change during the transient phase. However, by multiplying through and rearranging terms:

$$\begin{aligned} i_L &= I_i + I_f - I_f e^{-t/\tau} - I_i + I_i e^{-t/\tau} \\ &= I_f - I_f e^{-t/\tau} + I_i e^{-t/\tau} \end{aligned}$$

we find

$$\boxed{i_L = I_f + (I_i - I_f)e^{-t/\tau}} \qquad \textbf{(12.12)}$$

If you are required to draw the waveform for the current i_L from initial value to final value, start by drawing a line at the initial value and steady-state levels, and then add the transient response (sensitive to the time constant) between the two levels. The following example will clarify the procedure.

EXAMPLE 12.5 The inductor of Fig. 12.24 has an initial current level of 4 mA in the direction shown. (Specific methods to establish the initial current will be presented in the sections and problems to follow.)

a. Find the mathematical expression for the current through the coil once the switch is closed.
b. Find the mathematical expression for the voltage across the coil during the same transient period.
c. Sketch the waveform for each from initial value to final value.

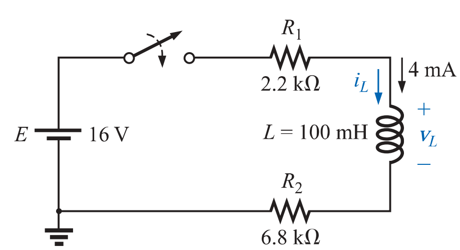


FIG. 12.24
Example 12.5.

Solutions:

a. Substituting the short-circuit equivalent for the inductor will result in a final or steady-state current determined by Ohm's law:

$$I_f = \frac{E}{R_1 + R_2} = \frac{16\text{ V}}{2.2\text{ k}\Omega + 6.8\text{ k}\Omega} = \frac{16\text{ V}}{9\text{ k}\Omega} = 1.78\text{ mA}$$

The time constant is determined by

$$\tau = \frac{L}{R_T} = \frac{100\text{ mH}}{2.2\text{ k}\Omega + 6.8\text{ k}\Omega} = \frac{100\text{ mH}}{9\text{ k}\Omega} = 11.11\ \mu\text{s}$$

Applying Eq. (12.12):

$$\begin{aligned} i_L &= I_f + (I_i - I_f)e^{-t/\tau} \\ &= 1.78\text{ mA} + (4\text{ mA} - 1.78\text{ mA})e^{-t/11.11\,\mu\text{s}} \\ &= \mathbf{1.78\ mA + 2.22\ mA}\boldsymbol{e^{-t/11.11\,\mu s}} \end{aligned}$$

b. Since the current through the inductor is constant at 4 mA prior to the closing of the switch, the voltage (whose level is sensitive only to changes in current through the coil) must have an initial value of 0 V. At the instant the switch is closed, the current through the coil cannot change instantaneously, so the current through the resistive elements will be 4 mA. The resulting peak voltage at $t = 0$ s can then be found using Kirchhoff's voltage law as follows:

$$\begin{aligned} V_m &= E - V_{R_1} - V_{R_2} \\ &= 16\text{ V} - (4\text{ mA})(2.2\text{ k}\Omega) - (4\text{ mA})(6.8\text{ k}\Omega) \\ &= 16\text{ V} - 8.8\text{ V} - 27.2\text{ V} = 16\text{ V} - 36\text{ V} \\ &= -20\text{ V} \end{aligned}$$

Note the minus sign to indicate that the polarity of the voltage v_L is opposite to the defined polarity of Fig. 12.24.

The voltage will then decay (with the same time constant as the current i_L) to zero because the inductor is approaching its short-circuit equivalence.

The equation for v_L is therefore:

$$\boldsymbol{v_L = -20e^{-t/11.11\,\mu s}}$$

c. See Fig. 12.25. The initial and final values of the current were drawn first, and then the transient response was included between these levels. For the voltage, the waveform begins and ends at zero, with the peak value having a sign sensitive to the defined polarity of v_L in Fig. 12.24.

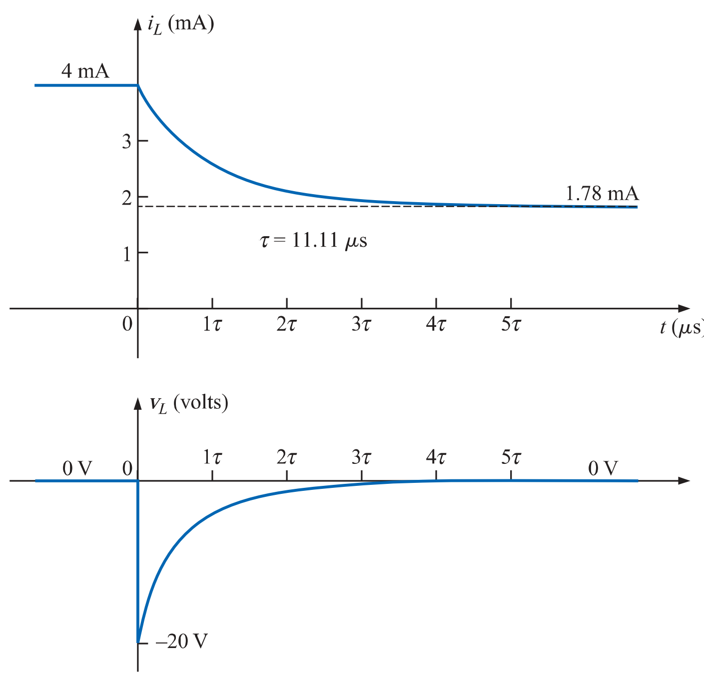


FIG. 12.25
i_L and v_L for the network of Fig. 12.24.

Let us now test the validity of the equation for i_L by substituting $t = 0$ s to reflect the instant the switch is closed.

$$e^{-t/\tau} = e^{-0} = 1$$

and $\quad i_L = 1.78 \text{ mA} + 2.22 \text{ mA}e^{-t/\tau} = 1.78 \text{ mA} + 2.22 \text{ mA}$
$\quad = 4 \text{ mA}$

When $t > 5\tau$,

$$e^{-t/\tau} \cong 0$$

and $\quad i_L = 1.78 \text{ mA} + 2.22 \text{ mA}e^{-t/\tau} = 1.78 \text{ mA}$

12.9 *R-L* TRANSIENTS: DECAY PHASE

In the analysis of *R-C* circuits, we found that the capacitor could hold its charge and store energy in the form of an electric field for a period of time determined by the leakage factors. In *R-L* circuits, the energy is stored in the form of a magnetic field established by the current through the coil. Unlike the capacitor, however, an isolated inductor cannot continue to store energy since the absence of a closed path would cause the current to drop to zero, releasing the energy stored in the form of a magnetic field. If the series *R-L* circuit of Fig. 12.26 had reached steady-state conditions and the switch were quickly opened, a spark would probably occur across the contacts due to the rapid change in current from a maximum of E/R to zero amperes. The change in current di/dt of the equation $v_L = L(di/dt)$ would establish a high voltage v_L across the coil that in conjunction with the applied voltage E appears across the points of the switch. This is the same mechanism as applied in the ignition system of a car to ignite the fuel in the cylinder. Some 25,000 V are generated by the rapid decrease in ignition coil current that occurs when the switch in the system is opened. (In older systems, the "points" in the distributor served as the switch.) This inductive reaction is significant when you consider that the only independent source in a car is a 12-V battery.

If opening the switch to move it to another position will cause such a rapid discharge in stored energy, how can the decay phase of an *R-L* circuit be analyzed in much the same manner as for the *R-C* circuit? The solution is to use a network such as that appearing in Fig. 12.27(a). When the switch is closed, the voltage across the resistor R_2 is E volts, and the *R-L* branch will respond in the same manner as described above, with the same waveforms and levels. A Thévenin network of E in parallel with R_2 would simply result in the source as

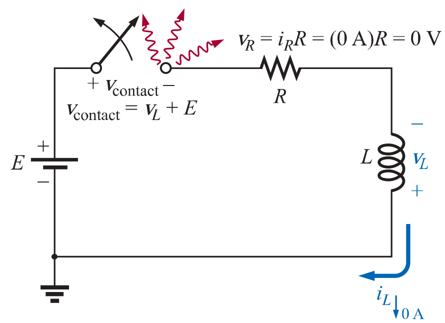


FIG. 12.26
Demonstrating the effect of opening a switch in series with an inductor with a steady-state current.

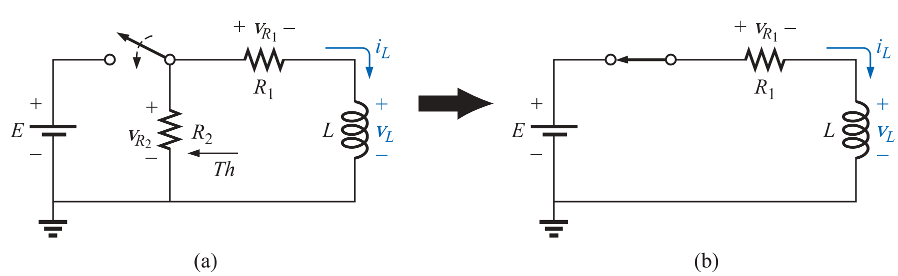


FIG. 12.27
Initiating the storage phase for the inductor L by closing the switch.

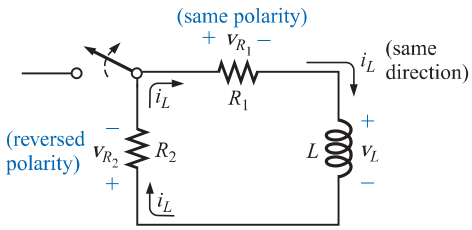


FIG. 12.28
Network of Fig. 12.27 the instant the switch is opened.

shown in Fig. 12.27(b) since R_2 would be shorted out by the short-circuit replacement of the voltage source E when the Thévenin resistance is determined.

After the storage phase has passed and steady-state conditions are established, the switch can be opened without the sparking effect or rapid discharge due to the resistor R_2, which provides a complete path for the current i_L. In fact, for clarity the discharge path is isolated in Fig. 12.28. The voltage v_L across the inductor will reverse polarity and have a magnitude determined by

$$\boxed{v_L = -(v_{R_1} + v_{R_2})} \tag{12.13}$$

Recall that the voltage across an inductor can change instantaneously but the current cannot. The result is that the current i_L must maintain the same direction and magnitude as shown in Fig. 12.28. Therefore, the instant after the switch is opened, i_L is still $I_m = E/R_1$, and

$$\begin{aligned} v_L &= -(v_{R_1} + v_{R_2}) = -(i_1R_1 + i_2R_2) \\ &= -i_L(R_1 + R_2) = -\frac{E}{R_1}(R_1 + R_2) = -\left(\frac{R_1}{R_1} + \frac{R_2}{R_1}\right)E \end{aligned}$$

and

$$\boxed{v_L = -\left(1 + \frac{R_2}{R_1}\right)E} \tag{12.14}$$

which is bigger than E volts by the ratio R_2/R_1. In other words, when the switch is opened, the voltage across the inductor will reverse polarity and drop instantaneously from E to $-[1 + (R_2/R_1)]E$ volts.

As an inductor releases its stored energy, the voltage across the coil will decay to zero in the following manner:

$$\boxed{v_L = -V_i e^{-t/\tau'}} \tag{12.15}$$

with
$$V_i = \left(1 + \frac{R_2}{R_1}\right)E$$

and
$$\tau' = \frac{L}{R_T} = \frac{L}{R_1 + R_2}$$

The current will decay from a maximum of $I_m = E/R_1$ to zero. Using Eq. (12.20), $I_i = E/R_1$ and $I_f = 0$ A so that

$$\begin{aligned} i_L &= I_f + (I_i - I_f)e^{-t/\tau'} \\ &= 0\text{ A} + \left(\frac{E}{R_1} - 0\text{ A}\right)e^{-t/\tau'} \end{aligned}$$

and

$$\boxed{i_L = \frac{E}{R_1}e^{-t/\tau'}} \tag{12.16}$$

with
$$\tau' = \frac{L}{R_1 + R_2}$$

The mathematical expression for the voltage across either resistor can then be determined using Ohm's law:

$$v_{R_1} = i_{R_1}R_1 = i_L R_1$$
$$= \frac{E}{R_1}R_1 e^{-t/\tau'}$$

and $$\boxed{v_{R_1} = Ee^{-t/\tau'}} \tag{12.17}$$

The voltage v_{R_1} has the same polarity as during the storage phase since the current i_L has the same direction. The voltage v_{R_2} is expressed as follows using the defined polarity of Fig. 12.27:

$$v_{R_2} = -i_{R_2}R_2 = -i_L R_2$$
$$= -\frac{E}{R_1}R_2 e^{-t/\tau'}$$

and $$\boxed{v_{R_2} = -\frac{R_2}{R_1}Ee^{-t/\tau'}} \tag{12.18}$$

EXAMPLE 12.6 The resistor R_2 was added to the network of Fig. 12.21, as shown in Fig. 12.29.

a. Find the mathematical expressions for i_L, v_L, v_{R_1}, and v_{R_2} for five time constants of the storage phase.
b. Find the mathematical expressions for i_L, v_L, v_{R_1}, and v_{R_2} if the switch is opened after five time constants of the storage phase.
c. Sketch the waveforms for each voltage and current for both phases covered by this example and Example 12.4 if five time constants pass between phases. Use the defined polarities of Fig. 12.27.

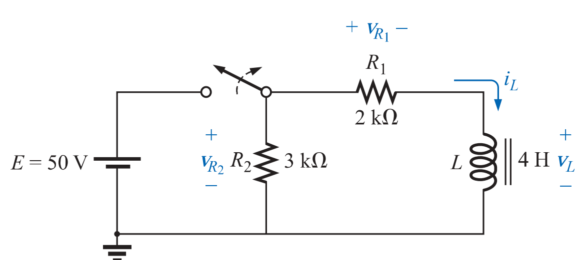


FIG. 12.29
Defined polarities for v_{R_1}, v_{R_2}, v_L, and current direction for i_L for Example 12.6.

Solutions:

a. $\tau = \dfrac{L}{R} = \dfrac{4\text{ H}}{2\text{ k}\Omega} = 2\text{ ms}$

Eq. (12.10): $v_L = Ee^{-t/\tau}$

$$v_L = \mathbf{50e^{-t/2\times10^{-3}}}$$

Eq. (12.8): $i_L = I_m(1 - e^{-t/\tau})$

$$I_m = \frac{E}{R_1} = \frac{50\text{ V}}{2\text{ k}\Omega} = 25\text{ mA}$$

$$i_L = \mathbf{25 \times 10^{-3}(1 - e^{-t/2\times10^{-3}})}$$

Eq. (12.11): $v_{R_1} = E(1 - e^{-t/\tau})$

$$v_{R_1} = \mathbf{50(1 - e^{-t/2\times10^{-3}})}$$

$$v_{R_2} = \mathbf{50\text{ V}}$$

b. $\tau' = \dfrac{L}{R_1 + R_2} = \dfrac{4\text{ H}}{2\text{ k}\Omega + 3\text{ k}\Omega} = \dfrac{4\text{ H}}{5 \times 10^3\ \Omega} = 0.8 \times 10^{-3}\text{ s}$
$= 0.8\text{ ms}$

By Eq. (12.15),

$$V_i = \left(1 + \frac{R_2}{R_1}\right)E = \left(1 + \frac{3\text{ k}\Omega}{2\text{ k}\Omega}\right)(50\text{ V}) = 125\text{ V}$$

and $\qquad v_L = -V_i e^{-t/\tau'} = \mathbf{-125e^{-t/(0.8\times10^{-3})}}$

By Eq. (12.16),

$$I_i = I_m = \frac{E}{R_1} = \frac{50\text{ V}}{2\text{ k}\Omega} = 25\text{ mA}$$

and $\qquad i_L = \mathbf{(25 \times 10^{-3})e^{-t/(0.8\times10^{-3})}}$

By Eq. (12.17),

$$v_{R_1} = Ee^{-t/\tau'} = \mathbf{50e^{-t/(0.8\times10^{-3})}}$$

By Eq. (12.18),

$$v_{R_2} = -\frac{R_2}{R_1}Ee^{-t/\tau'} = -\frac{3\text{ k}\Omega}{2\text{ k}\Omega}(50\text{ V})e^{-t/\tau'} = \mathbf{-75e^{-t/(0.8\times10^{-3})}}$$

c. See Fig. 12.30 (opposite page).

In the preceding analysis, it was assumed that steady-state conditions were established during the charging phase and $I_m = E/R_1$, with $v_L = 0$ V. However, if the switch of Fig. 12.28 is opened before i_L reaches its maximum value, the equation for the decaying current of Fig. 12.28 must change to

$$\boxed{i_L = I_i e^{-t/\tau'}} \qquad \textbf{(12.19)}$$

where I_i is the starting or initial current. Equation (12.15) would be modified as follows:

$$\boxed{v_L = -V_i e^{-t/\tau'}} \qquad \textbf{(12.20)}$$

with $\qquad V_i = I_i(R_1 + R_2)$

12.10 INSTANTANEOUS VALUES

The development presented in Section 10.10 for capacitive networks can also be applied to *R-L* networks to determine instantaneous voltages, currents, and time. The instantaneous values of any voltage or current can be determined by simply inserting *t* into the equation and using

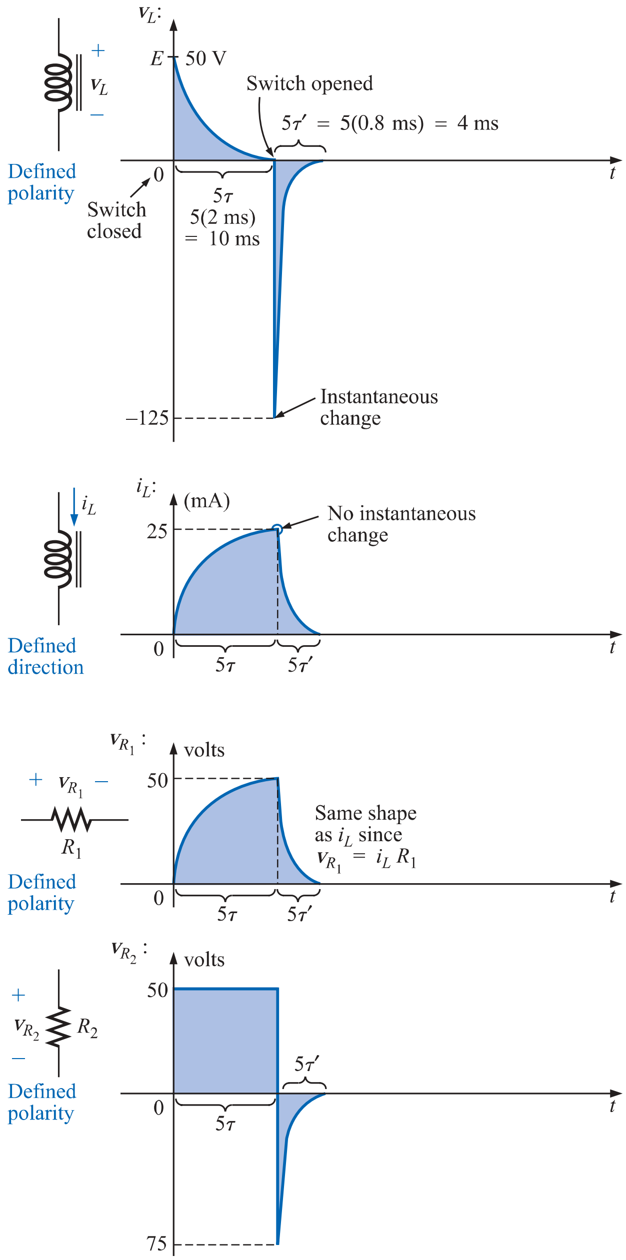


FIG. 12.30
The various voltages and the current for the network of Fig. 12.29.

a calculator or table to determine the magnitude of the exponential term.

The similarity between the equations $v_C = E(1 - e^{-t/\tau})$ and $i_L = I_m(1 - e^{-t/\tau})$ results in a derivation of the following for t that is identical to that used to obtain Eq. (10.24):

$$\boxed{t = \tau \log_e\left(\frac{I_m}{I_m - i_L}\right)} \qquad \textbf{(12.21)}$$