Lecture: Waves and Oscillations

Ref book: Physics for Engineers - Giasuddin Ahmad (Part-1)

University Physics - Sears, Zemansky, Young & Freedman

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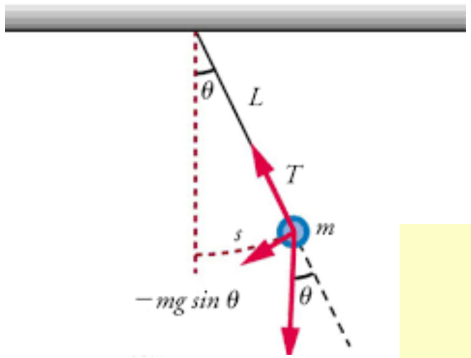
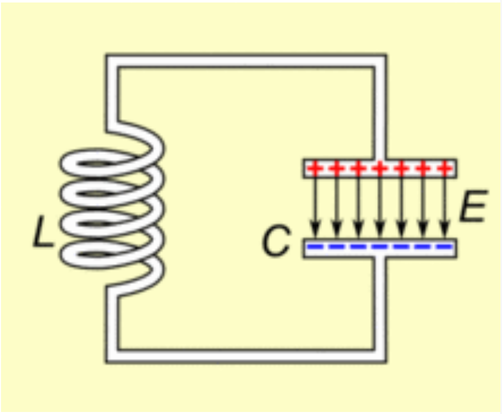
Web ref provided on slides

Harmonic Motion





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Simple Harmonic Motion

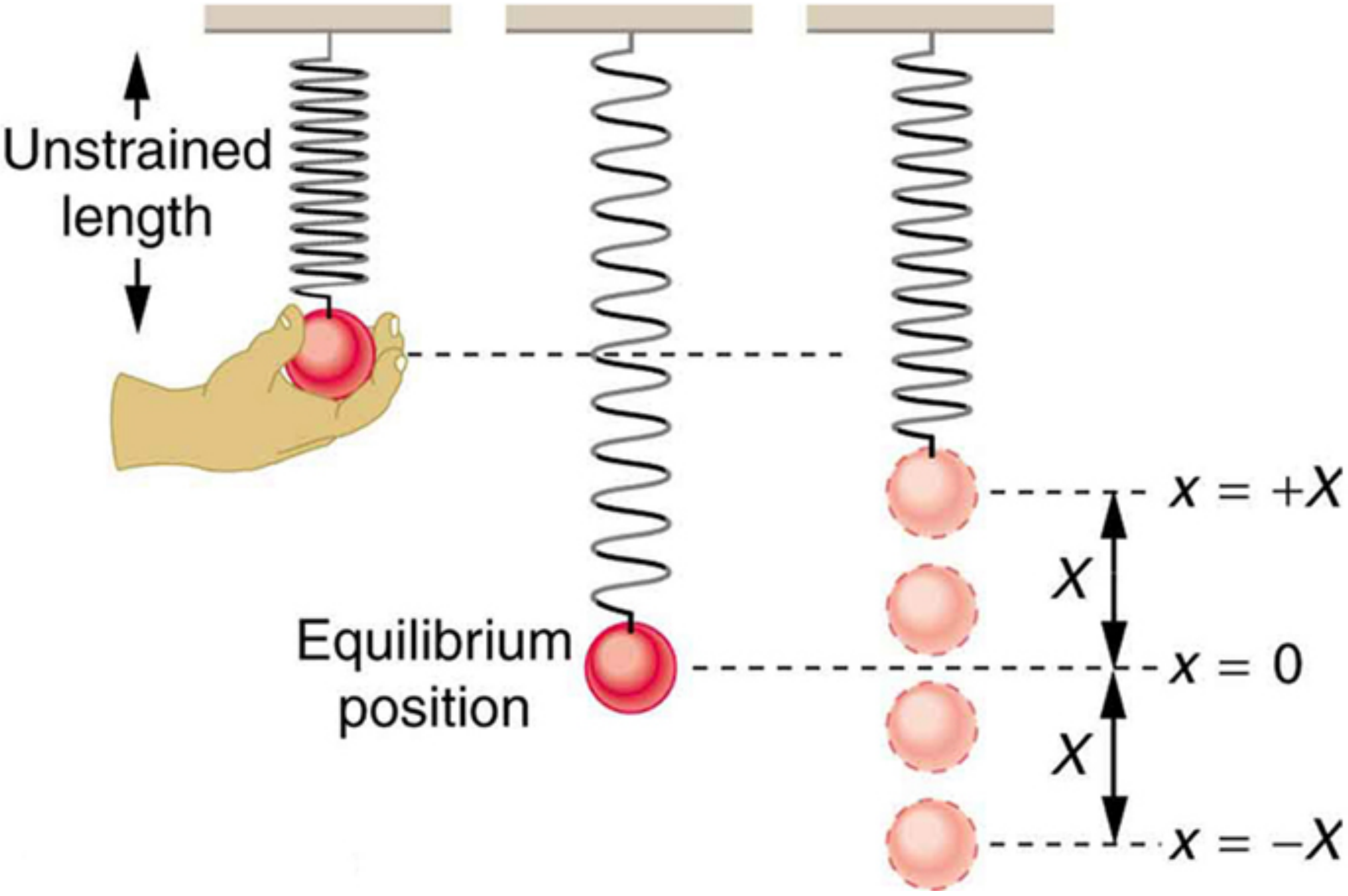
Periodic Motion: A motion which repeats itself in equal intervals of time is periodic motion. For example, the motion of the hands of a clock, the motion of the wheels of a car and the motion of a merry-go-round.

Oscillatory Motion: An oscillatory motion is a periodic motion in which an object moves to and fro about its equilibrium position. The object performs the same set of movements again and again after a fixed time. One such set of movements is an Oscillation. The motion of a simple pendulum, the motion of leaves vibrating in a breeze and the motion of a cradle are all examples of oscillatory motion.

SHM: To-and-fro motion under the action of a restoring force. Simple harmonic motion is the simplest example of oscillatory motion.

Ref: toppr.com

Simple Harmonic Motion: Graphs



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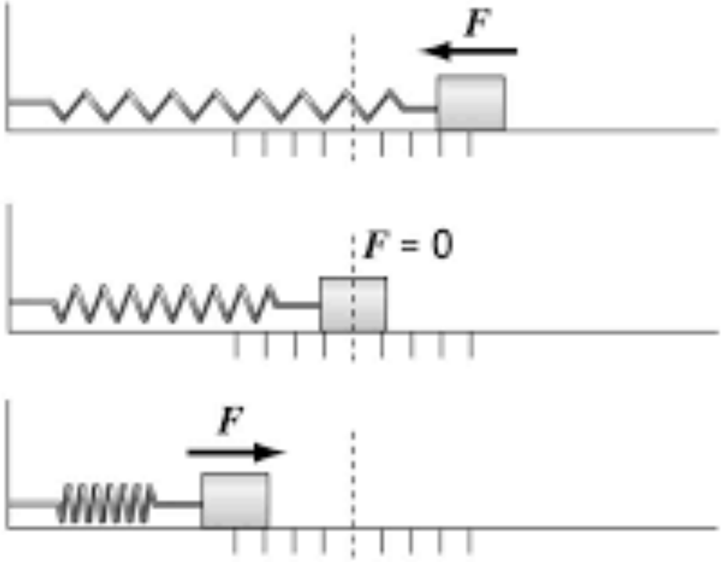
Simple Harmonic Motion: Equation

Hooke's Law: The extension of an elastic object is directly proportional to the force applied to it. Or,

The restoring force applied to an elastic object (such as a spring) is proportional to the displacement (or, extension) and in the opposite direction of that displacement.



Hooke's Law:

Restoring force,

$$\vec{\mathbf{F}}_{restore} = -k\Delta\vec{\mathbf{x}}$$

where $\Delta\vec{\mathbf{x}} = \vec{\mathbf{x}} - \vec{\mathbf{x}}_0$

and k is the "spring constant" [N m^{-1}]

tart with the
ıomentum principle: $\dfrac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{F}}_{net}$

For horizontal forces on the mass: $\dfrac{dp_x}{dt} = -kx$

$$\therefore \frac{d(mv_x)}{dt} = -kx \quad \text{or} \quad \frac{d}{dt}\left(m\frac{dx}{dt}\right) = -kx$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Ref: google

Simple Harmonic Motion: Equation

We can combine the constants *k* and *m* by making the substitution:
$\frac{k}{m} = \omega_0^2$, which results

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0.$$

Some solutions of this equation are

$x = A \sin(\omega_0 t+\varphi)$
$x = A \cos(\omega_0 t+\varphi)$

This solutions can be proved to be the solutions of the above differential equation (see lecture).

$$x(t) = A\cos(\omega_0 t + \phi)$$

$$v(t) = \frac{dx(t)}{dt} = -A\omega_0 \sin(\omega_0 t + \phi)$$

$$a(t) = \frac{d^2x(t)}{dt^2} = \frac{dv(t)}{dt} = -A\omega_0^2 \cos(\omega_0 t + \phi)$$

… acceleration = – (constant) . (displacement)

$$= -A\omega_0^2 \cos(\omega_0 t + \phi)$$

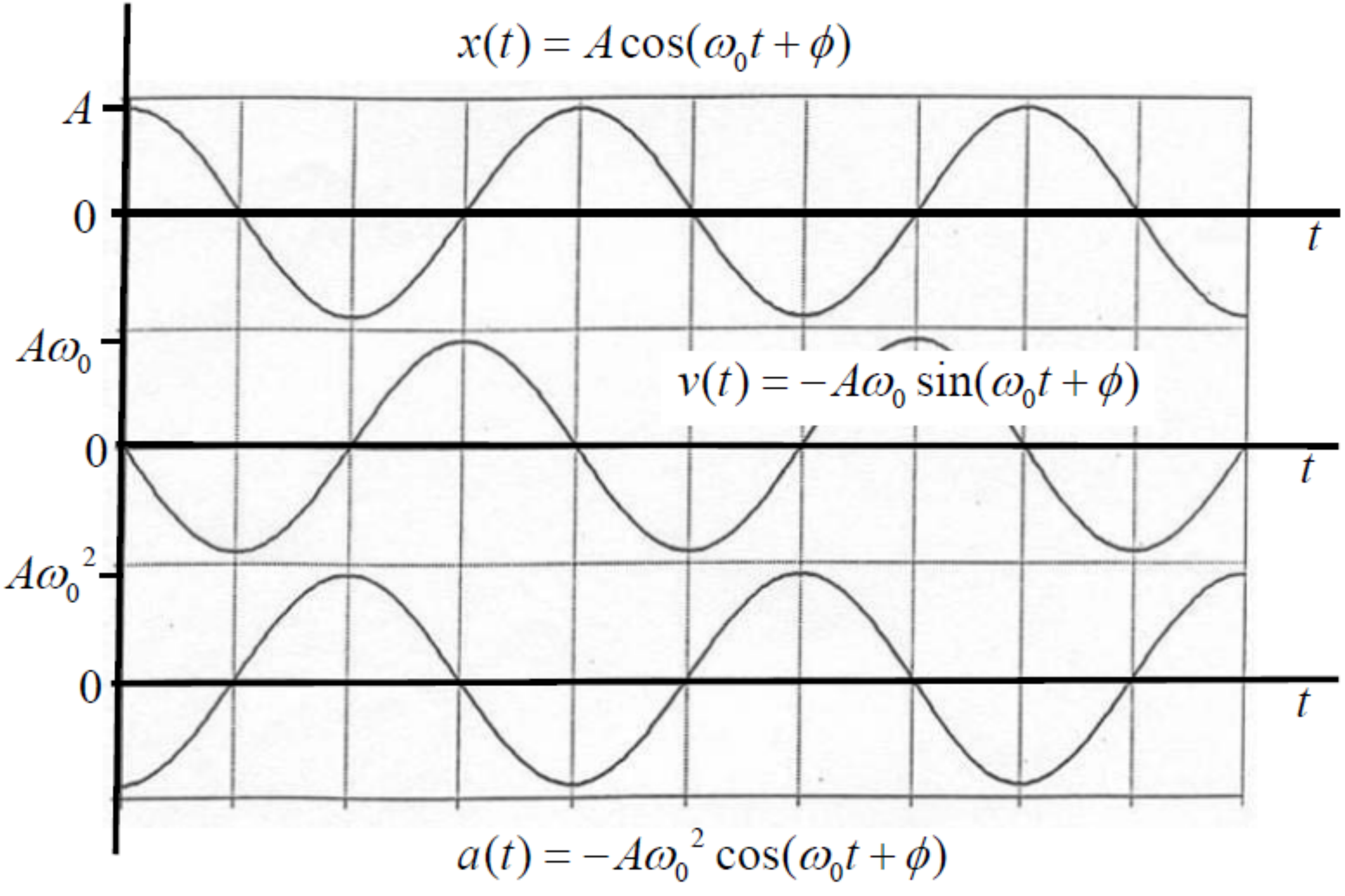
$$= A\omega_0^2 \cos(\omega_0 t + \phi + \pi)$$

Phase difference between acceleration and displacement is π

Phase difference between v and x (and v & a) is $\frac{\pi}{2}$

Ref: google image

Simple Harmonic Motion: Graphs



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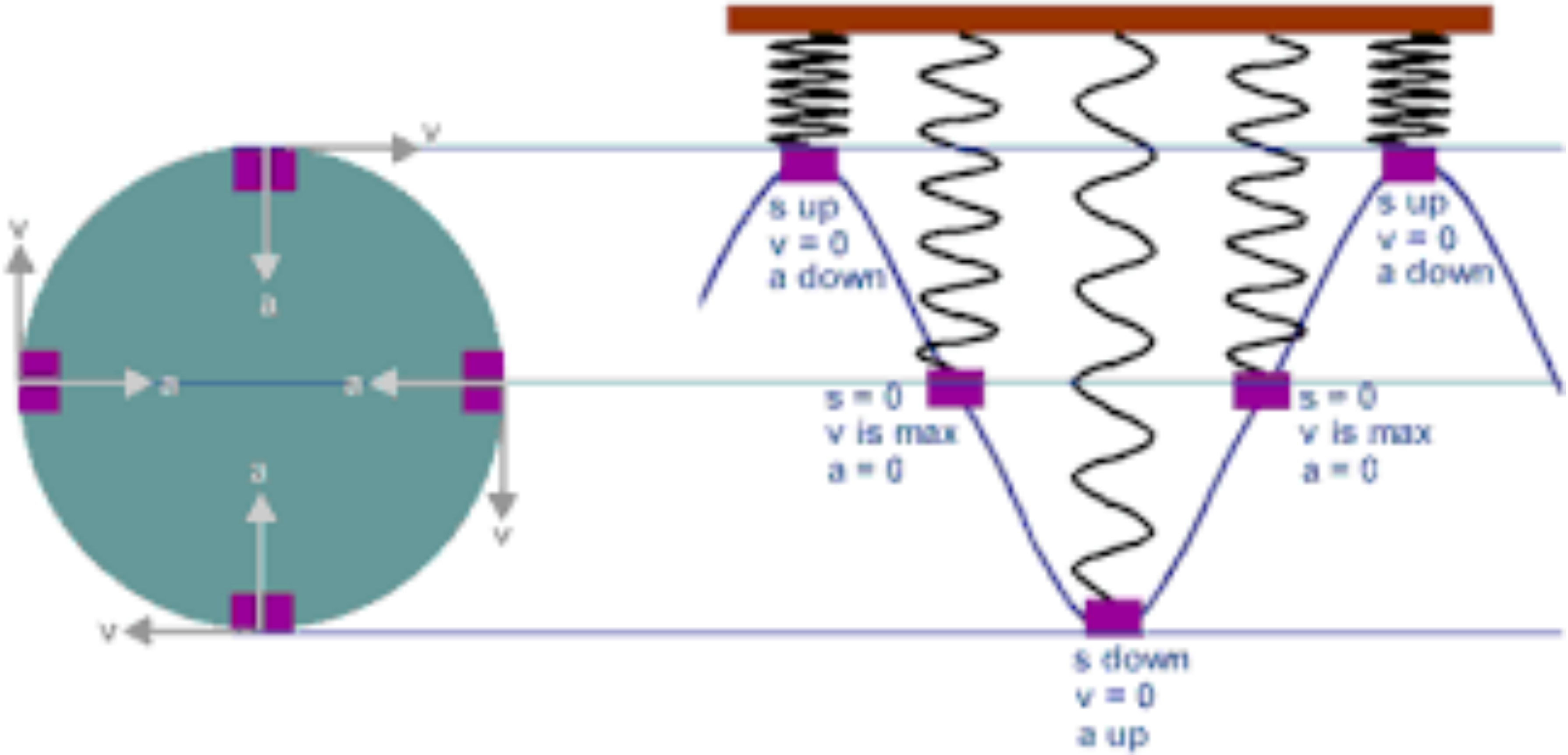
Simple Harmonic Motion: Energy



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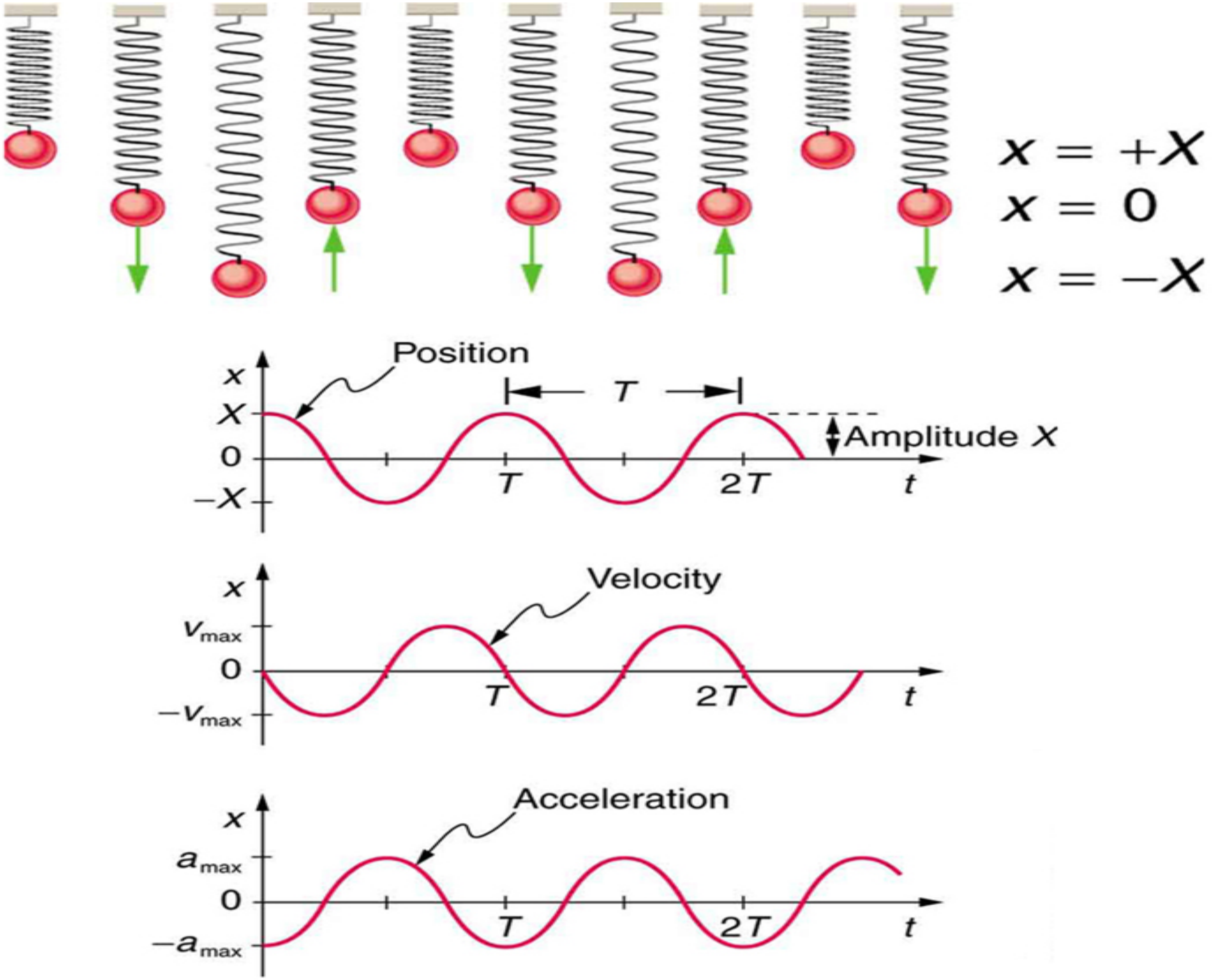
Simple Harmonic Motion: Graphs



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Simple Harmonic Motion: Equation

Another Method:

F ∝ -x

or, F= -kx,

where *x* is the displacement from equilibrium and *k* is called the spring constant, which is characteristic of a **spring** which is defined as the ratio of the **force** affecting the **spring** to the displacement caused by it.

Since the acceleration:

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2},$

Newton's second law becomes:

$-kx = m\,\frac{d^2x}{dt^2},$

which is called a second-order differential equation because it contains a second derivative.

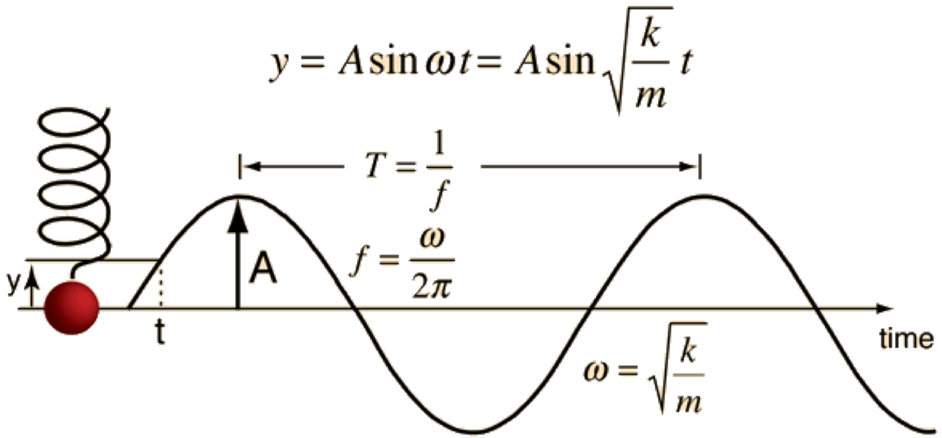
For vertical motion:

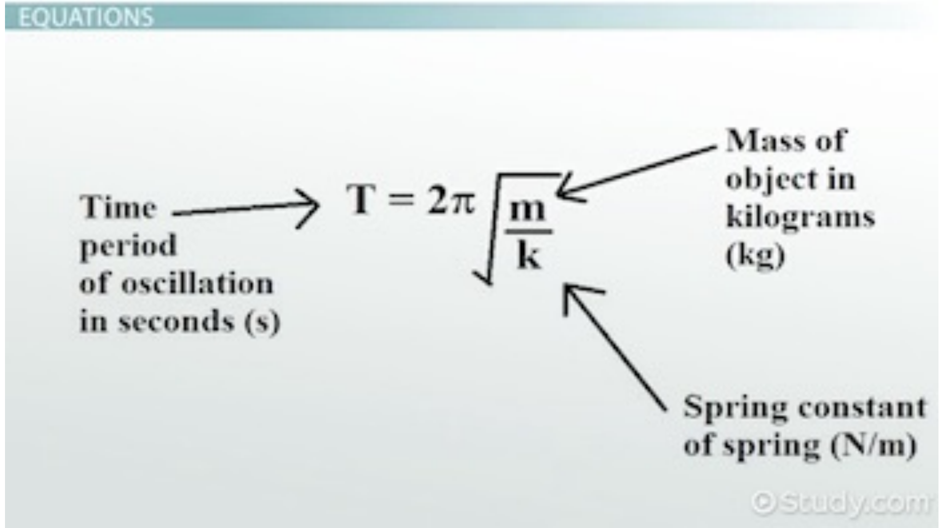
$m\,\frac{d^2y}{dt^2} + \omega^2 y = 0.$

Some solutions of this equation are:

$y = A\sin(\omega t+\varphi)$

$y = A\cos(\omega t+\varphi)$





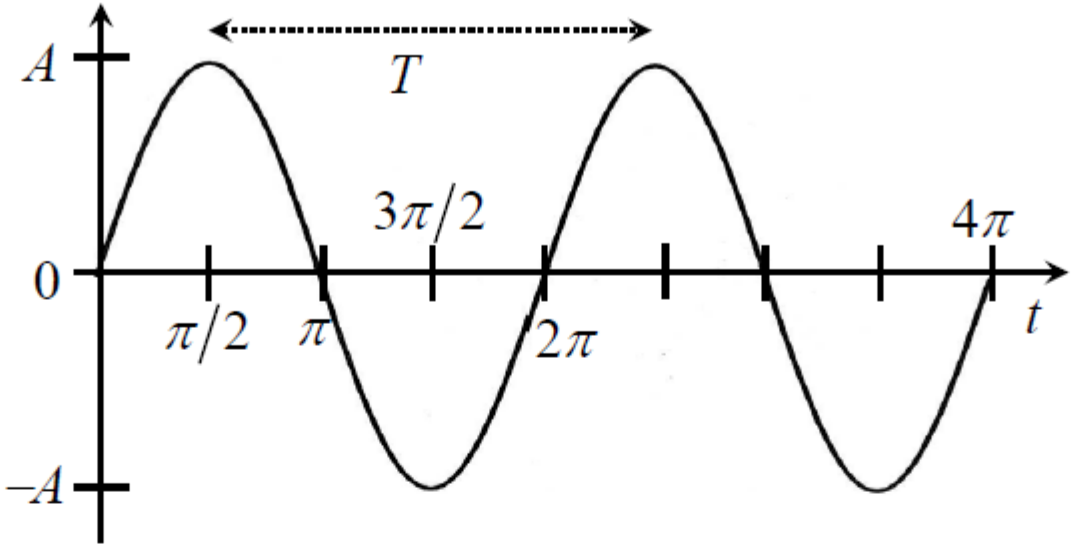
Simple Harmonic Motion: Definition

Definitions of some related quantities for $y = A\sin(\omega t+\varphi)$

Amplitude: The amplitude of the motion, denoted by A, is the maximum magnitude of displacement from the equilibrium position. It is always positive

Period: The period T, is the time required for one oscillation.

Frequency: The frequency, f, is the number of cycles in a unit tine.



$\omega T = 2\pi$ where T : period (s)

ω : angular frequency (rad s^{-1})

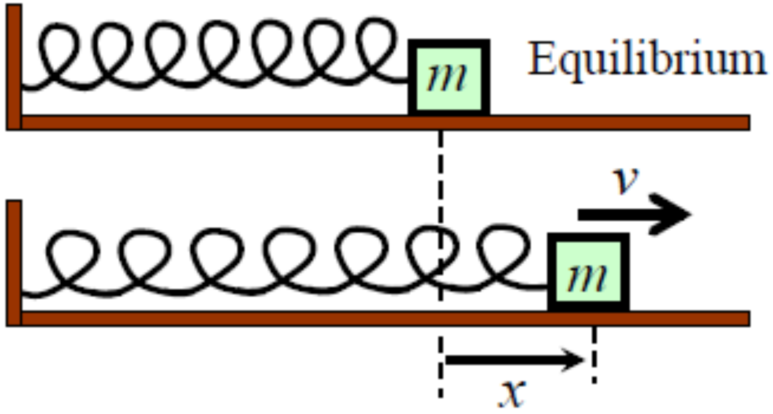
some books use υ $\rightarrow$ $f = \dfrac{1}{T}$ where f : frequency (Hz)

A : Amplitude

ϕ : phase angle, initial phase or phase constant

Ref: google

Simple Harmonic Motion: Energy



Suppose that the mass has a speed v when it has displacement x

Kinetic energy of mass = $\frac{1}{2}mv^2$

Potential energy of spring = $\int_0^x F dx' = \int_0^x kx' dx' = \frac{1}{2}kx^2$

There are no dissipative mechanisms in our model (no friction).
… the total energy of the mass-spring system is conserved.

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$$

Ref: google

Simple Harmonic Motion: Energy

For our mass-spring system: $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$

$$\therefore \frac{d}{dt}\left(\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2\right) = 0$$

$$\therefore mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0$$

$$\therefore mv\frac{dv}{dt} + kxv = 0$$

$$\therefore m\frac{dv}{dt} + kx = 0$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

… as before

Simple Harmonic Motion: Energy

For the mass-spring system: $x = A\cos(\omega_0 t + \phi)$

Potential energy = $\frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$

k.e. = $\frac{1}{2}mv^2 = \frac{1}{2}m[-A\omega_0\sin(\omega_0 t + \phi)]^2 = \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi)$

Total energy = p.e. + k.e

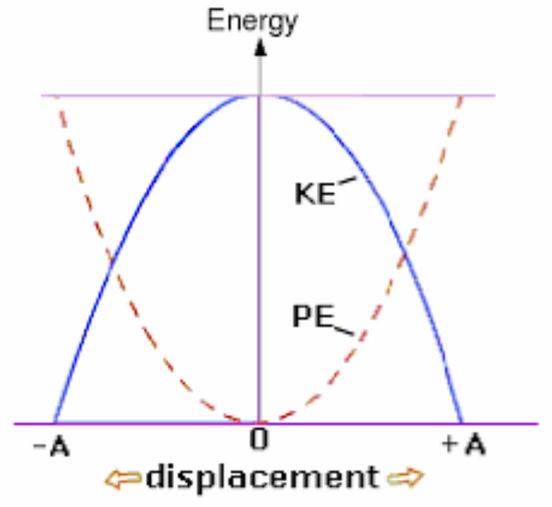
$$= \tfrac{1}{2}kA^2\cos^2(\omega_0 t + \phi) + \tfrac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi)$$

$$= \tfrac{1}{2}kA^2 \quad (= \tfrac{1}{2}m\omega_0^{\,2}A^2) \qquad (\therefore E \propto A^2)$$

We can now write: $\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$

$$\therefore v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} \quad \text{or} \quad v(x) = \pm\omega_0\sqrt{A^2 - x^2}$$

$$E = KE + PE = \frac{1}{2}kA^2$$



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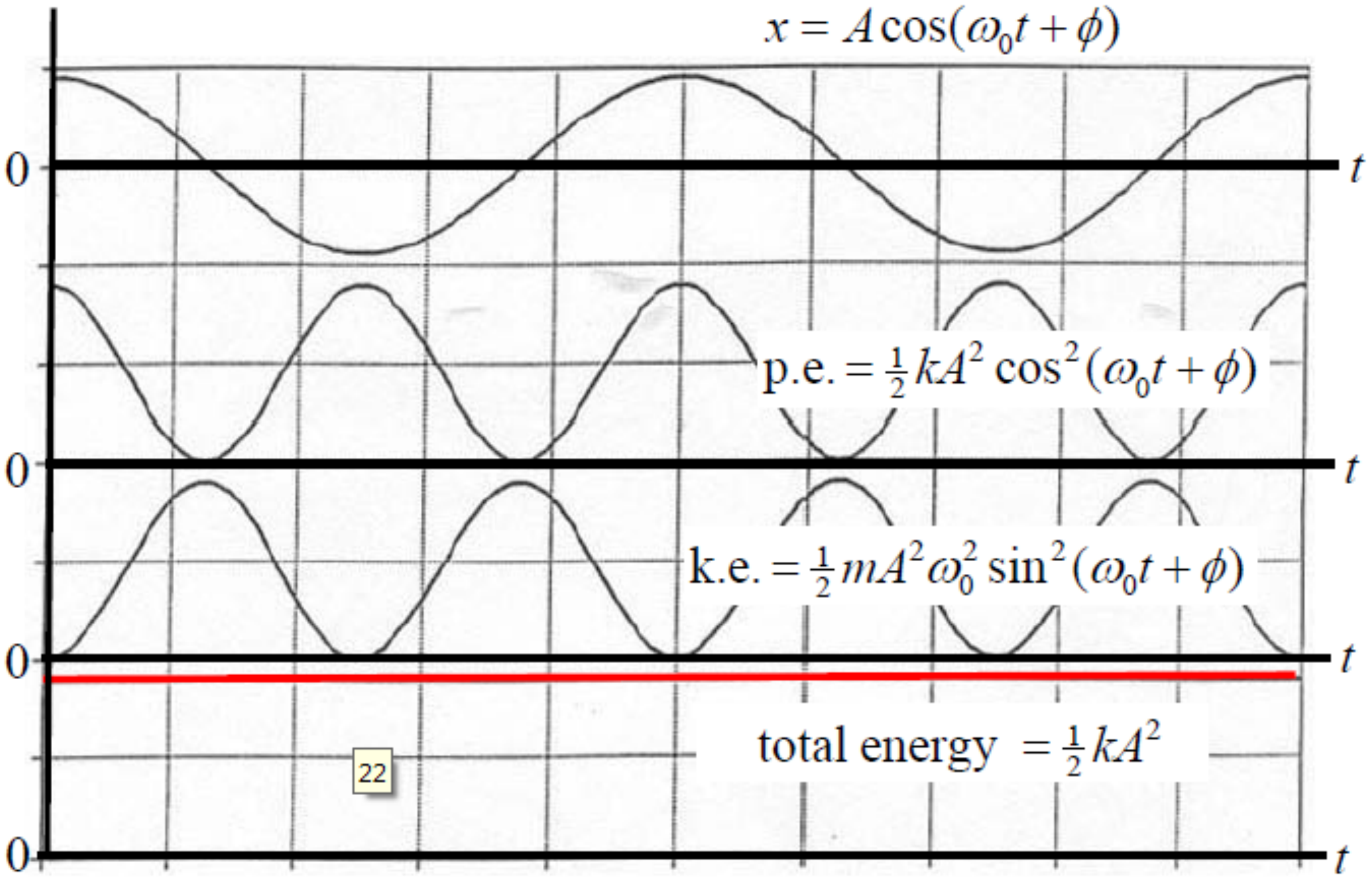
Simple Harmonic Motion: Energy



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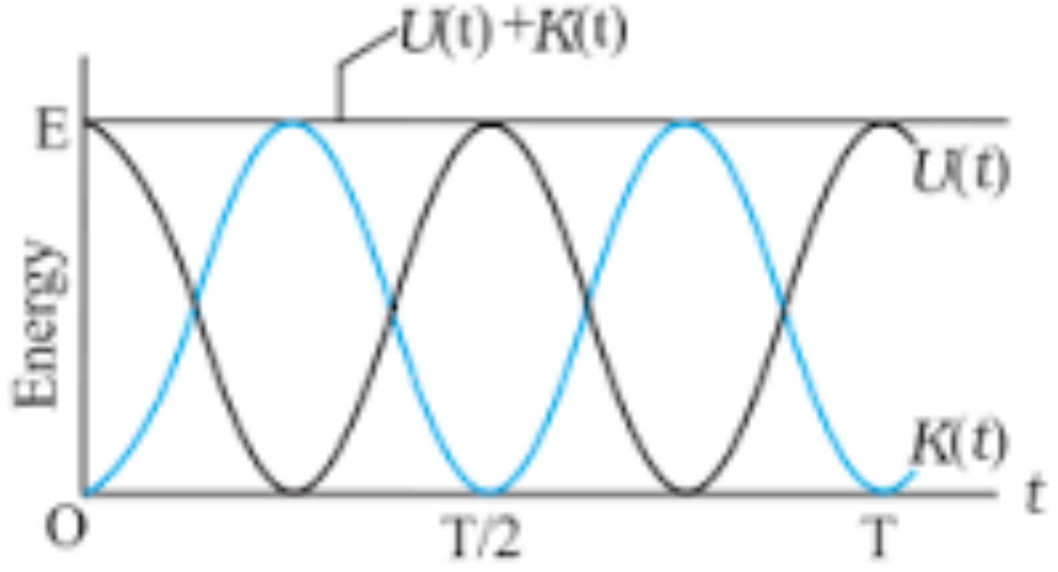
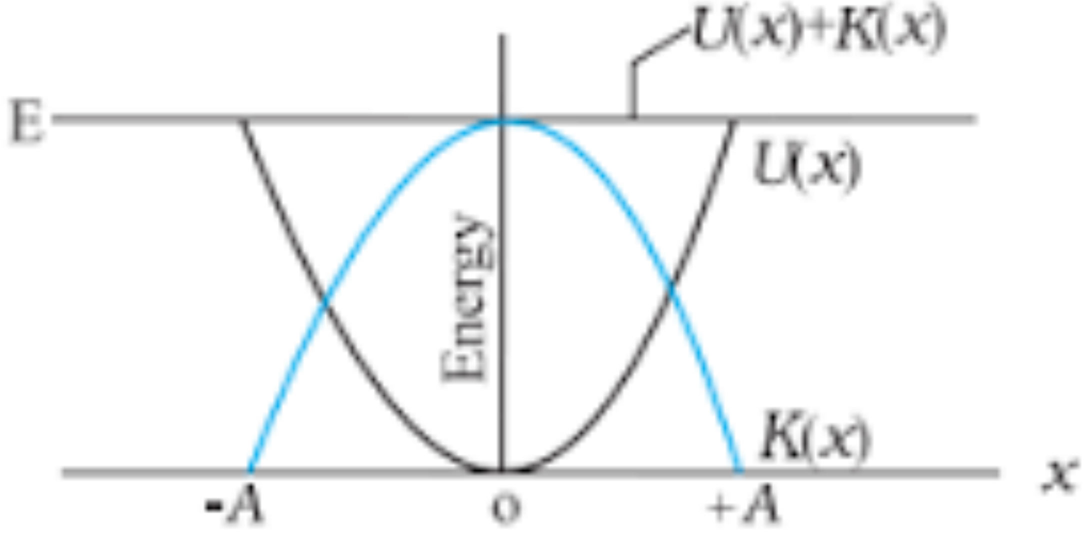
Simple Harmonic Motion: Energy





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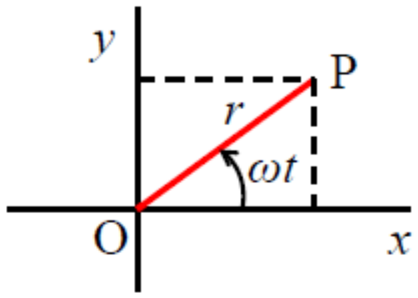
Simple Harmonic Motion: Complex Plane

Complex numbers

Consider a vector $\overrightarrow{OP}$ of length r which rotates with angular velocity ω



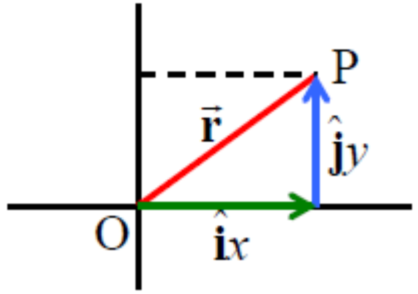
The point P has coordinates

$$x = r\cos\omega t \qquad y = r\sin\omega t$$

We see that the x coordinate of P, or the projection of $\overrightarrow{OP}$ onto the x-axis, executes SHM

Can also introduce the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$

and write $\vec{\mathbf{r}} = \hat{\mathbf{i}}x + \hat{\mathbf{j}}y$

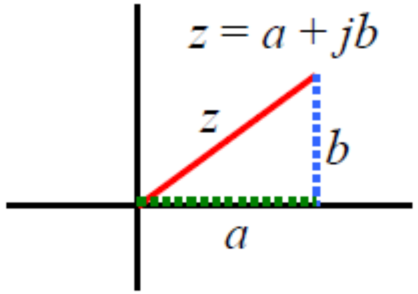


Simple Harmonic Motion: Complex Plane

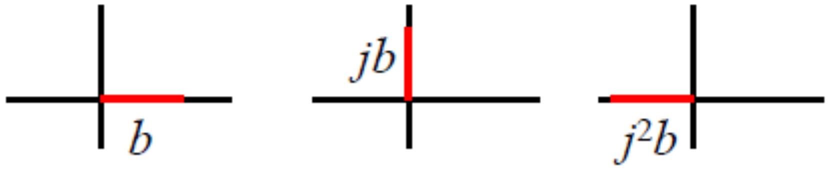
Complex numbers …2

Modify our notation to $z = x + jy$

… where x means a displacement in the x-direction and jy means a displacement in the y-direction

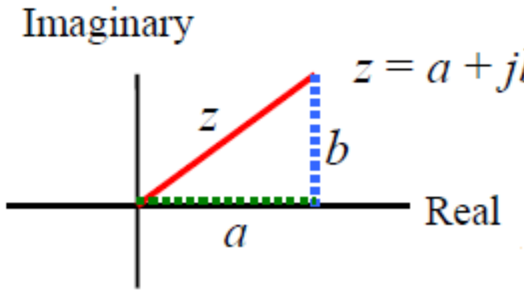


We can also think of j as a rotation through $\pi/2$ anticlockwise



Hence $j^2 = -1$

… really talking about vectors in the complex number plane:



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Simple Harmonic Motion: Complex Plane

Complex numbers …3

From Taylor's theorem: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

therefore $e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots$

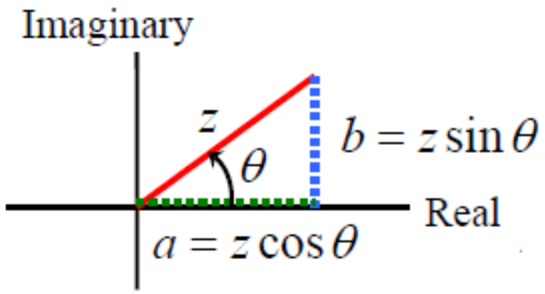
and $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots$ and $j\sin\theta = j\theta - \frac{j\theta^3}{3!} + \ldots$

Hence $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$ **Euler relation**

Then $z = a + jb \;= |z| e^{j\theta}$

where $|z| = \sqrt{a^2 + b^2}$

$\tan\theta = b/a$



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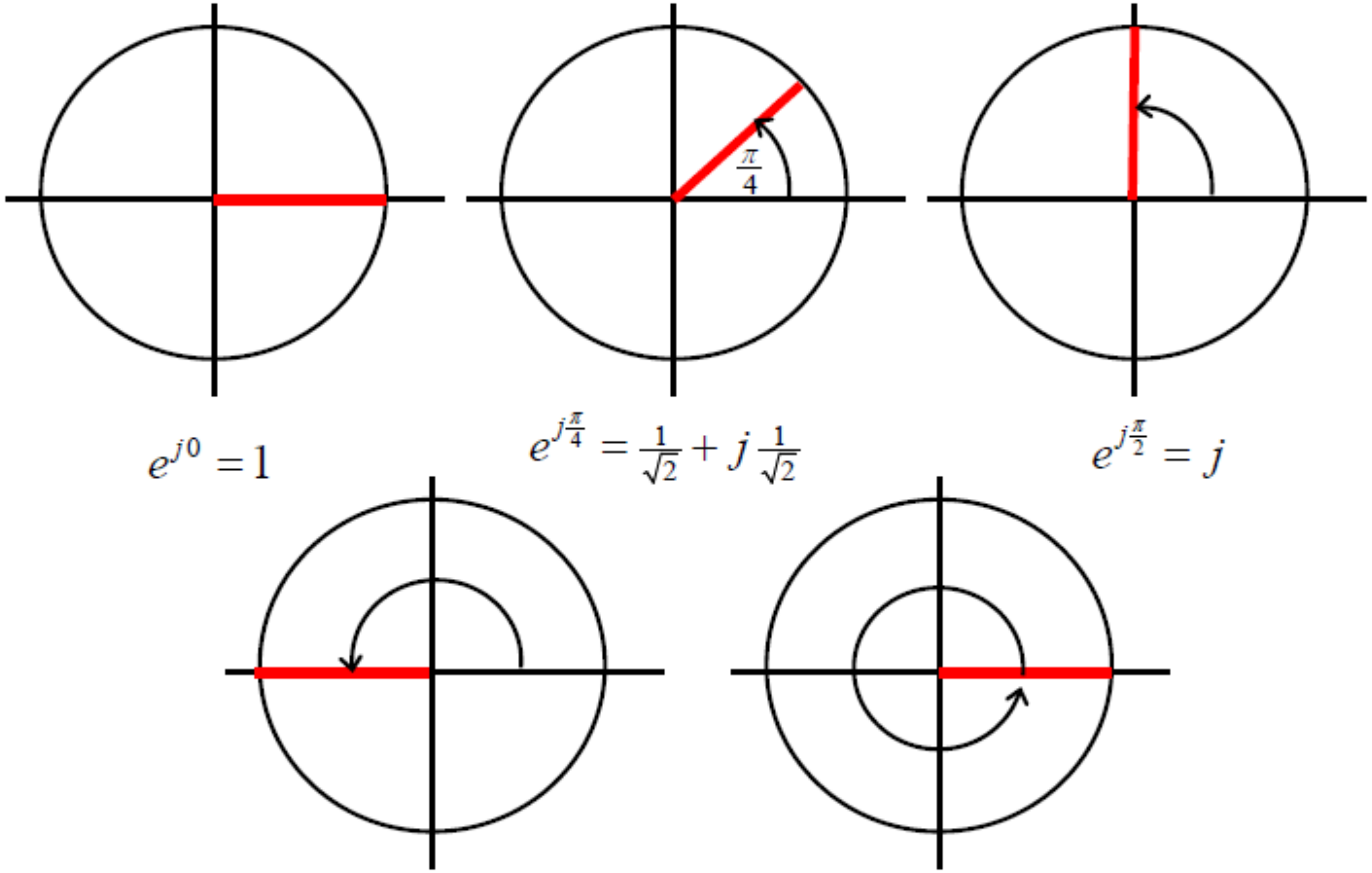
Simple Harmonic Motion: Complex Plane



$e^{j0} = 1$

$e^{j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$e^{j\frac{\pi}{2}} = j$

$e^{j\pi} = -1$

$e^{j2\pi} = 1$

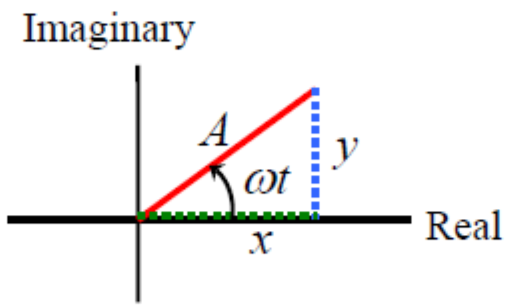
Ref: google image

Simple Harmonic Motion: Complex Plane

Complex numbers …4

For our rotating vectors:

$$z = x + jy$$
$$= A\cos\omega t + jA\sin\omega t$$
$$= A\left(\cos\omega t + j\sin\omega t\right)$$
$$= Ae^{j\omega t}$$



Now write: $Ae^{j(\omega t+\phi)} = A\cos(\omega t+\phi) + jA\sin(\omega t+\phi)$

... and remember that the physical quantity x (e.g. a displacement) is the real part of z :

i.e. $x = \mathrm{Re}[z]$

Ref: google image

Simple Harmonic Motion: Complex Plane

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

Using $z = x + jy$ becomes $\frac{d^2z}{dt^2} + \omega_0^2 z = 0$

Try $z = Ae^{j(\omega t+\phi)}$

$$\therefore A(j\omega)^2 e^{j(\omega t+\phi)} + \omega^2 Ae^{j(\omega t+\phi)} = 0$$

Therefore $z = Ae^{j(\omega t+\phi)}$ is the most general solution

A and ϕ are determined from the initial conditions.

Take real part of z:

$$x = \mathrm{Re}[z] = A\cos(\omega_0 t + \phi)$$

3

Ref: google image

Simple Harmonic Motion: Complex Plane

$$x = A\cos(\omega_0 t + \phi) \qquad\qquad z = Ae^{j(\omega t+\phi)}$$

$$x = \mathrm{Re}[z]$$

$$\frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \phi) \qquad\qquad \frac{dz}{dt} = j\omega Ae^{j(\omega t+\phi)} = j\omega z$$

$$\frac{dx}{dt} = \mathrm{Re}\left[\frac{dz}{dt}\right]$$

$$\frac{d^2x}{dt^2} = -A\omega_0{}^2 \cos(\omega_0 t + \phi) \qquad\qquad \frac{d^2z}{dt^2} = (j\omega)^2\, Ae^{j(\omega t+\phi)} = -\omega^2 z$$

$$\frac{d^2x}{dt^2} = \mathrm{Re}\left[\frac{d^2z}{dt^2}\right]$$

Ref: google image

Simple Harmonic Motion: Complex Plane

Solving $\frac{d^2x(t)}{dt^2} + \omega_0^2 x(t) = 0$

Let $x = Be^{pt}$

Then $\frac{dx}{dt} = Bpe^{pt}$ and $\frac{d^2x}{dt^2} = Bp^2e^{pt}$

Substituting into DE: $Bp^2e^{pt} + B\omega_0^2e^{pt} = 0$

This holds true for all t if and only if $p^2 = -\omega_0^2$ or $p = \pm j\omega_0$

$$\therefore x = B_1e^{j\omega_0 t} + B_2e^{-j\omega_0 t}$$

How to get B_1 and B_2 ? … need to know the initial conditions

… but consider $v = \frac{dx}{dt} = j\omega_0B_1e^{j\omega_0 t} - j\omega_0B_2e^{-j\omega_0 t}$

Ref: google image

Simple Harmonic Motion: Complex Plane

Solving $\dfrac{d^2x(t)}{dt^2}+\omega_0^2 x(t)=0$ **continued**

$$v=\frac{dx}{dt}=j\omega_0 B_1 e^{j\omega_0 t}-j\omega_0 B_2 e^{-j\omega_0 t}$$

At $t=0$, $v(0)=j\omega B_1-j\omega_0 B_2$

Choose $v(0)=0$

Since v must be real, then $B_1=B_2=B$

i.e. $$\begin{aligned}x&=Be^{j\omega_0 t}+Be^{-j\omega_0 t}\\&=2B\cos\omega_0 t\end{aligned}$$

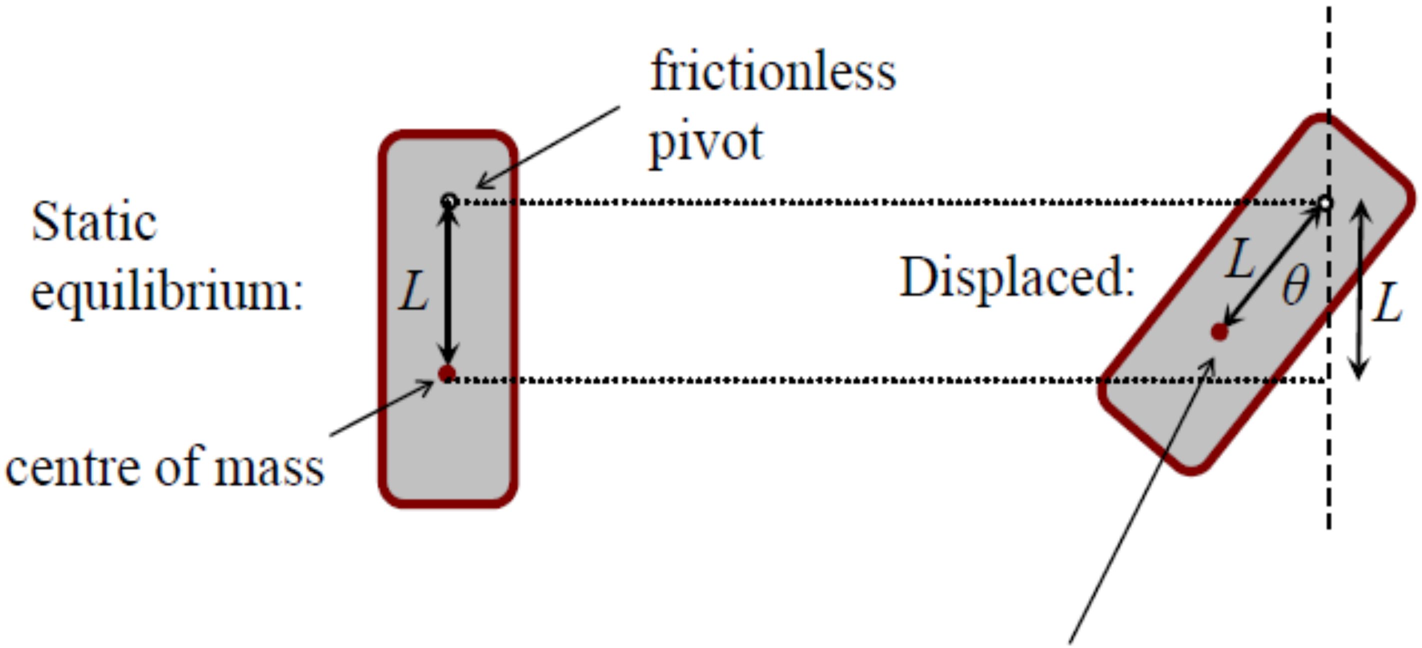
$$\therefore x=A\cos\omega_0 t$$

[… with a little more effort we could have got
the more general solution $x=A\cos\left(\omega_0 t+\phi\right)$]

Ref: google image

Simple Harmonic Motion: Pendulum

The pendulum: general case



In displaced position, centre of mass is $L - L\cos\theta$ above the equilibrium position.

Recall $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$ For small angles, $\cos\theta \approx 1 - \frac{\theta^2}{2}$

Gravitational potential energy = $mgL(1-\cos\theta) = mgL\frac{\theta^2}{2}$

Simple Harmonic Motion: Pendulum

Gravitational potential energy = $\frac{1}{2}mgL\theta^2$

Kinetic energy = $\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2$

Total energy = $\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}mgL\theta^2$ = constant

$$\therefore I\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgL\theta\frac{d\theta}{dt} = 0$$ … true for all $\frac{d\theta}{dt}$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{mgL}{I}\theta = -\omega_0^2\theta$$ where $\omega_0 = \sqrt{\frac{mgL}{I}}$

Equation of SHM

Ref: google

Simple Harmonic Motion: Pendulum

The moment of inertia of the pendulum about an passing through the point of suspension is

$$= mK^2 + mL^2$$

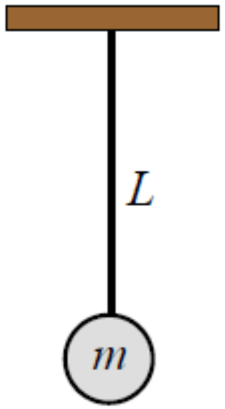
Therefore, $\omega_0 = \sqrt{\dfrac{gL}{K^2 + L^2}}$

Time Period

$$T = 2\pi\sqrt{\frac{K^2 + L^2}{Lg}}$$

Ref: google image

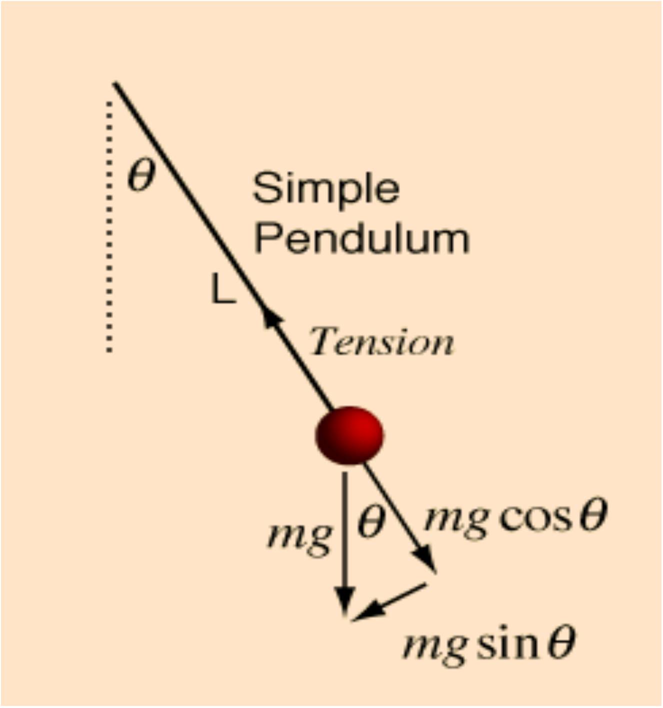
Simple Harmonic Motion: Simple Pendulum



The simple pendulum

$$I = mL^2$$

$$\omega_0 = \sqrt{\frac{mgL}{mL^2}} = \sqrt{\frac{g}{L}} \qquad T = 2\pi\sqrt{\frac{L}{g}}$$



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Simple Harmonic Motion: Simple Pendulum

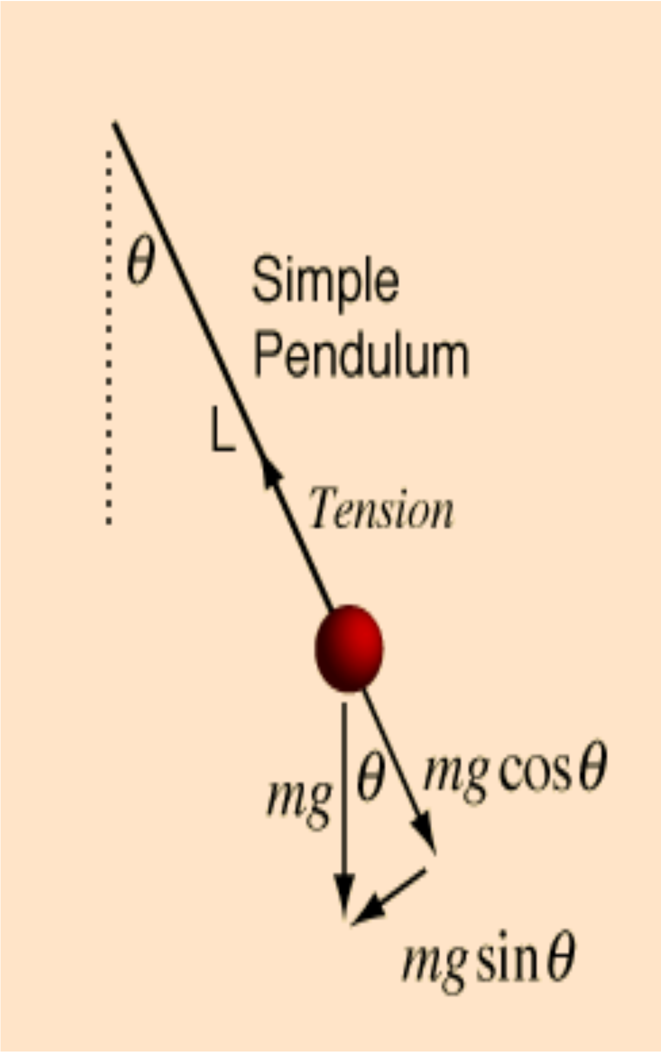
Restoring force

$$F = -mg\sin\theta$$

If the angle θ is very small sinθ is very nearly equal to θ. The displacement along the arc is

$$x = L\theta$$

Therefore, $$F = -mg\theta$$



Simple Harmonic Motion: Simple Pendulum

Acceleration $$\frac{d^2x}{dt^2} = L\frac{d^2\theta}{dt^2}$$

$$mL\frac{d^2\theta}{dt^2} = -mg\theta$$

$$Force = mL\frac{d^2\theta}{dt^2}$$

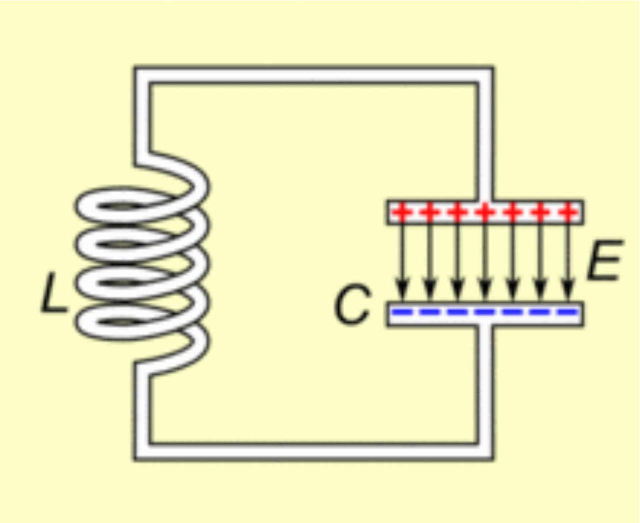
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

$$\omega^2 = \frac{g}{L}$$

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Simple Harmonic Motion: LC Circuit



An **LC circuit**, also called a **resonant circuit**, **tank circuit**, or **tuned circuit**, consists of an inductor, represented by the letter L, and a capacitor, represented by the letter C. When connected together, they can act as an electrical resonator.

Voltage across capacitor at any instant $V_C = \dfrac{Q}{C}$

Q is the charge on the capacitor and C is capacitance of capacitor.

Voltage across inductor at the same instant $V_L = L\dfrac{di}{dt}$

i is the current flowing and *L* is inductance of inductor.

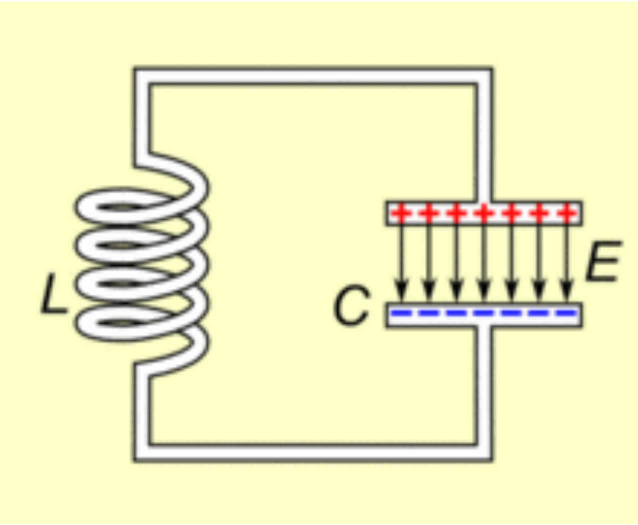
Ref: google

Simple Harmonic Motion: LC Circuit

Kirchhoff's voltage law:

$$\frac{Q}{C} + L\frac{di}{dt} = 0$$

$$\frac{d^2 i}{dt^2} + \frac{1}{LC} i = 0$$



Similar to differential equation of SHM:

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0, \quad \text{with } \omega_0^{\ 2} = \frac{1}{LC}$$

Ref: google

Simple Harmonic Motion: LC Circuit

Solution of the differential equation is

$$Q(t) = Q_0 \cos(\omega_0 t + \phi)$$

Current in the circuit

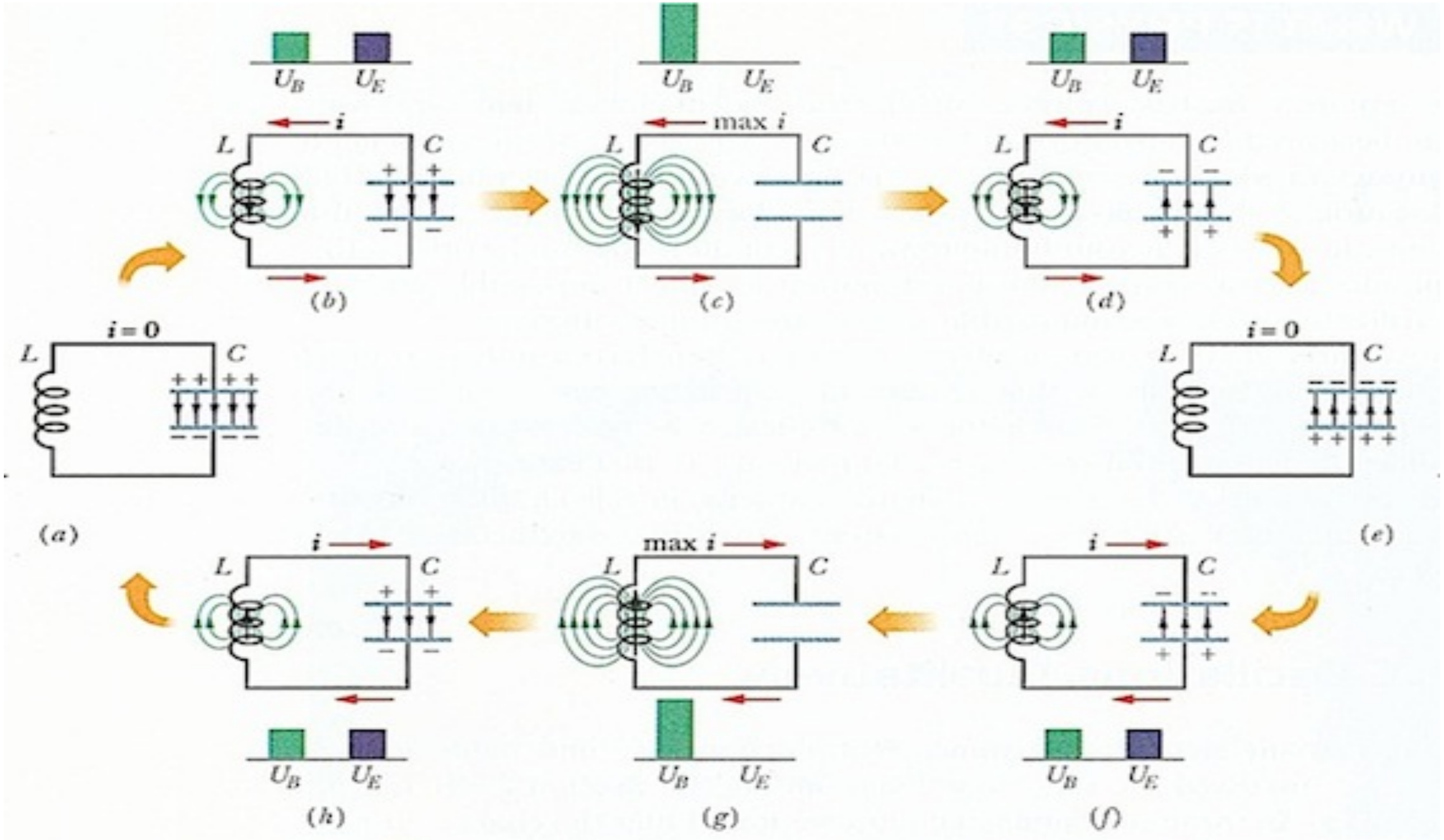
$$i(t) = -i_0 \sin(\omega_0 t + \phi)$$

Time Period $\quad T = 2\pi\sqrt{LC}$

Frequency $\quad f = \dfrac{1}{2\pi\sqrt{LC}}$

Ref: google

Simple Harmonic Motion: LC Circuit



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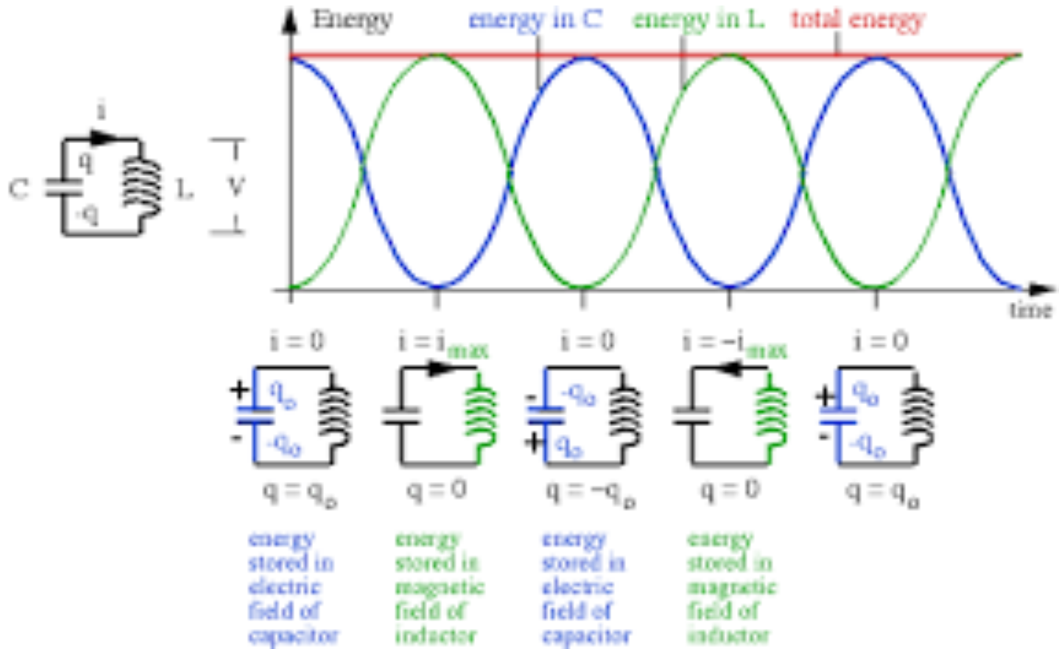
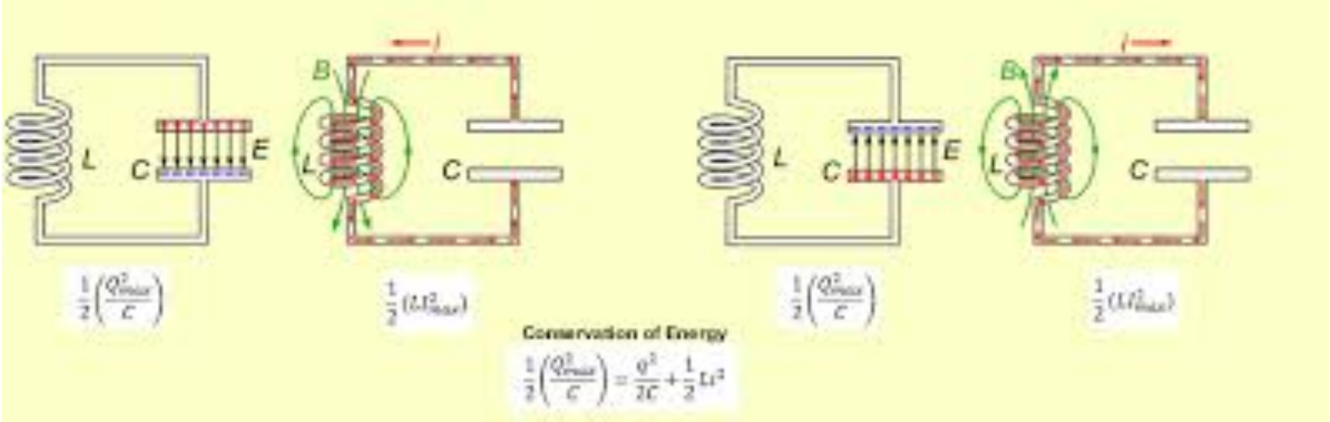
Simple Harmonic Motion: LC Circuit





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Simple Harmonic Motion: LC Circuit

Mechanical	Electrical
displacement x	charge Q
velocity v	current I
mass m	inductance L
spring constant k	$\frac{1}{\text{capacitance}}$ $\frac{1}{C}$
$\omega_0 = \sqrt{\frac{k}{m}}$	$\omega_0 = \sqrt{\frac{1}{LC}}$
potential energy: $\frac{1}{2}kx^2$	Electric energy stored in capacitor: $\frac{1}{2}\frac{Q^2}{C}$
kinetic energy: $\frac{1}{2}mv^2$	Magnetic energy stored in inductor: $\frac{1}{2}LI^2$

Ref: google image

Simple Harmonic Motion: Sample Problems

A 0.42-kg block is attached to the end of a horizontal ideal spring and rests on a frictionless surface. The block is pulled so that the spring stretches by 2.1 cm relative to its unstrained length. When the block is released, it moves with an acceleration of 9.0 m/s^2. What is the spring constant of the spring?

180 N/m

Ref: google image

Simple Harmonic Motion: Sample Problems

Energy calculations.

For the simple harmonic oscillation where k = 19.6 N/m, A = 0.100 m, x = -(0.100 m) cos 8.08t, and v = (0.808 m/s) sin 8.08t, determine (a) the total energy, (b) the kinetic and potential energies as a function of time, (c) the velocity when the mass is 0.050 m from equilibrium, (d) the kinetic and potential energies at half amplitude ($x = \pm A/2$).

Ref: google image

Simple Harmonic Motion: Sample Problems

a. $E = \frac{1}{2}kA^2 = \frac{1}{2}\cdot 19.6\text{N/m}\cdot(0.100\text{m})^2 = 9.80\times 10^{-2}\,\text{J}.$

b. $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2\omega t = (9.80\times 10^{-2}\,\text{J})\cos^2 8.08t,$

$K = E - U = (9.80\times 10^{-2}\,\text{J})\sin^2 8.08t.$

c. $K = E - U, \quad \frac{1}{2}mv^2 = \frac{1}{2}kA^2 - \frac{1}{2}kx^2,$

$$v = \sqrt{\frac{k}{m}(A^2 - x^2)} = \omega\sqrt{A^2 - x^2}$$

$$= 8.08\text{Hz}\cdot\sqrt{(0.100\text{m})^2 - (0.050\text{m})^2} = 0.70\text{m/s}.$$

d. $U = \frac{1}{2}kx^2 = \frac{1}{2}k\left(\frac{A}{2}\right)^2 = \frac{1}{4}E = 2.5\times 10^{-2}\,\text{J},$

$E = K - U = 7.3\times 10^{-2}\,\text{J}.$

Ref: google image

Simple Harmonic Motion: Sample Problems

A 500 g block on a spring is pulled a distance of 20 cm and released. The subsequent oscillations are measured to have a period of 0.80 s. At what position (or positions) is the speed of the block 1.0 m/s?

$$T = 0.80\text{ s} \quad \text{so} \quad \omega = \frac{2\pi}{T} = \frac{2\pi}{(0.80\text{ s})} = 7.85\text{ rad/s}$$

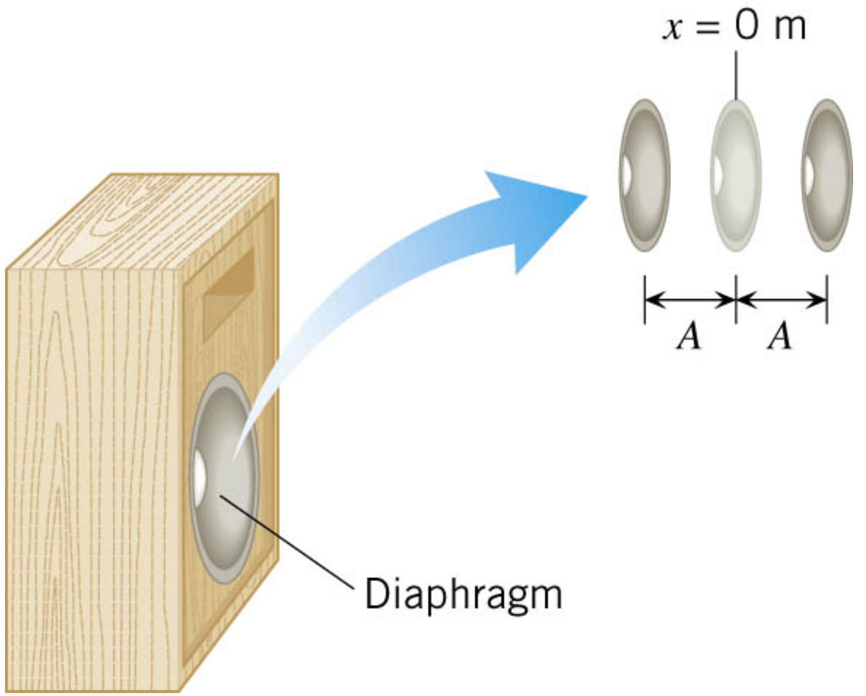
$$v = \sqrt{\frac{k}{m}\left(A^2 - x^2\right)} = \omega\sqrt{A^2 - x^2}$$

$$x = \pm\sqrt{A^2 - \left(\frac{v}{\omega}\right)^2} = \pm\sqrt{(0.20\text{ m})^2 - \left(\frac{(1.0\text{ m/s})}{(7.85\text{ rad/s})}\right)^2} = \pm 0.154\text{ m} = \pm 15.4\text{ cm}$$

Ref: google image

Simple Harmonic Motion: Sample Problems

The diaphragm of a loudspeaker moves back and forth in simple harmonic motion to create sound. The frequency of the motion is $f = 1.0$ kHz and the amplitude is $A = 0.20$ mm.



(a) What is the maximum speed of the diaphragm?

(b) Where in the motion does this maximum speed occur?

Ref: google image

Simple Harmonic Motion: Sample Problems

(a)

$$v_{\max} = A\omega = A(2\pi f) = (0.20 \times 10^{-3}\,\mathrm{m})(2\pi)(1.0 \times 10^{3}\,\mathrm{Hz}) = \boxed{1.3\,\mathrm{m/s}}$$

(b) The speed of the diaphragm is zero when the diaphragm momentarily comes to rest at either end of its motion: $x = +A$ and $x = -A$. Its maximum speed occurs midway between these two positions, or at $x = 0$ m.

Ref: google image

Simple Harmonic Motion: Sample Problems

What is the oscillation period of an FM radio station that broadcasts at 100 MHz?

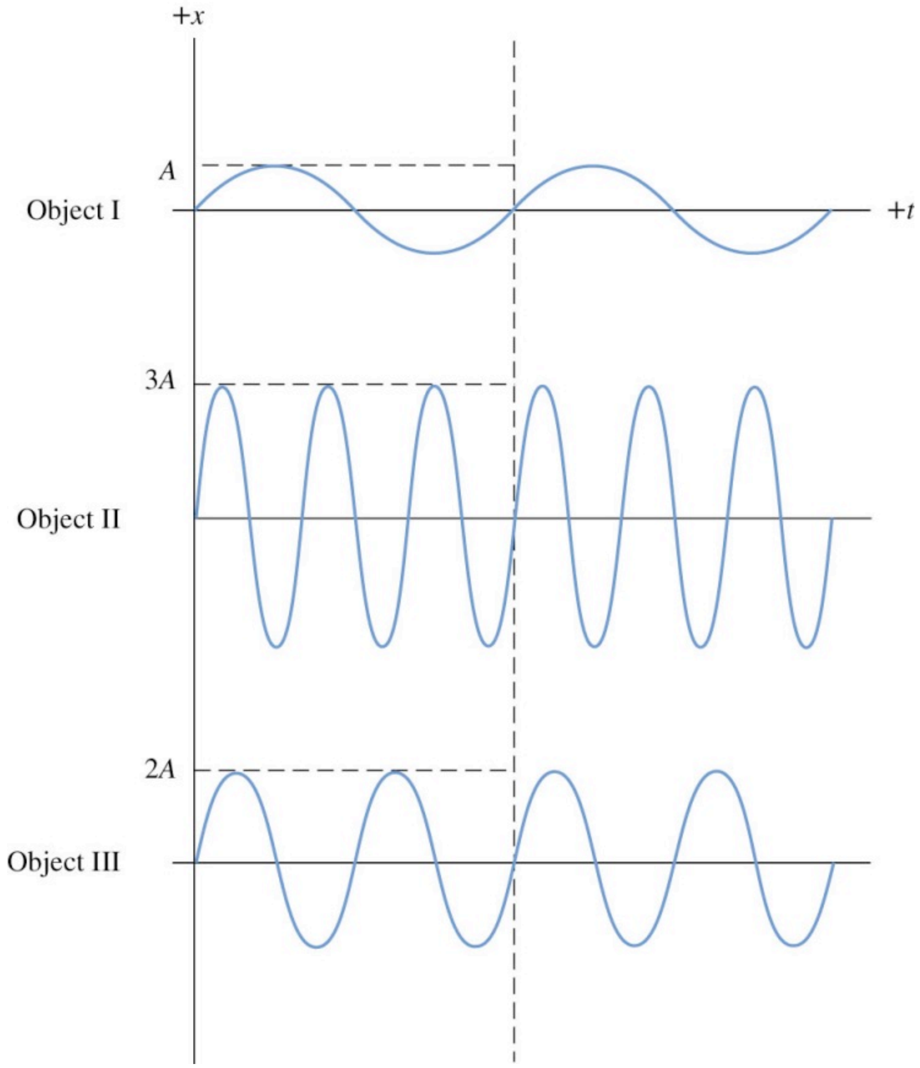
$$f = 100 \text{ MHz} = 1.0 \times 10^{8} \text{ Hz}$$

$$T = 1/f = \frac{1}{1.0 \times 10^{8} \text{ Hz}} = 1.0 \times 10^{-8} \text{ s} = 10 \text{ ns}$$

Note that 1/Hz = s

Ref: google image

Simple Harmonic Motion: Sample Problems



The drawing shows plots of the displacement *x* versus the time *t* for three objects undergoing simple harmonic motion. Which object, I, II, or III, has the greatest maximum velocity?

II

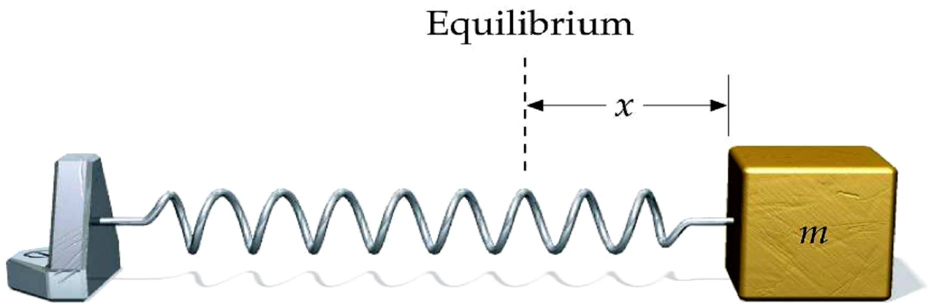
Ref: google image

Simple Harmonic Motion: Sample Problems

A 2.00 kg block is attached to a spring as shown. The force constant of the spring is $k =$ 196 N/m. The block is held a distance of 5.00 cm from equilibrium and released at $t = 0$.

(a) Find the angular frequency ω, the frequency f, and the period T.

(b) Write an equation for x vs. time.



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Simple Harmonic Motion: Sample Problems

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{(196\ \text{N/m})}{(2.00\ \text{kg})}} = 9.90\ \text{rad/s}$$

$$f = \frac{\omega}{2\pi} = \frac{(9.90\ \text{rad/s})}{2\pi} = 1.58\ \text{Hz}$$

$$T = 1/f = 0.635\ \text{s}$$

$$A = 5.00\ \text{cm} \quad \text{and} \quad \delta = 0$$

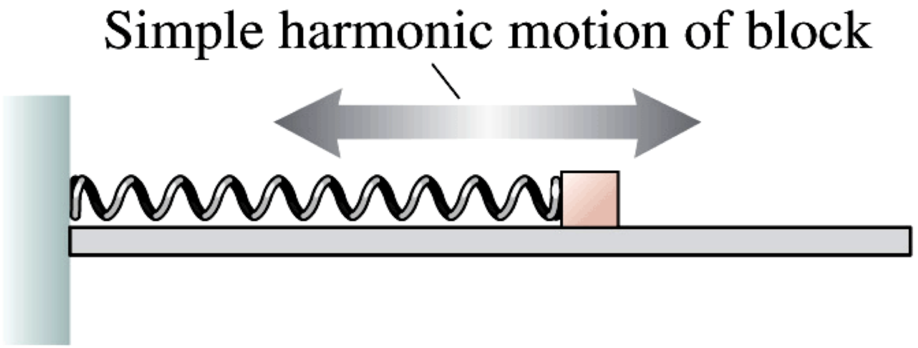
$$\boxed{x = (5.00\ \text{cm})\cos\left[(9.90\ \text{rad/s})t\right]}$$

Ref: google image

Simple Harmonic Motion: Sample Problems

An air-track glider is attached to a spring, pulled 20 cm to the right, and released at t-=0. It makes 15 complete oscillations in 10 s.

a. What is the period of oscillation?
b. What is the object's maximum speed?
c. What is its position and velocity at t=0.80 s?



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Simple Harmonic Motion: Sample Problems

$$f = \frac{15 \text{ oscillations}}{10 \text{ s}}$$

$$= 1.5 \text{ oscillations/s} = 1.5 \text{ Hz}$$

$$T = 1/f = 0.667 \text{ s}$$

$$v_{\text{max}} = \frac{2\pi A}{T} = \frac{2\pi(0.20 \text{ m})}{(0.667 \text{ s})} = 1.88 \text{ m/s}$$

$$x = A\cos\frac{2\pi t}{T} = (0.20 \text{ m})\cos\frac{2\pi(0.80 \text{ s})}{(0.667 \text{ s})} = 0.062 \text{ m} = 6.2 \text{ cm}$$

$$v = -v_{\text{max}}\sin\frac{2\pi t}{T} = -(1.88 \text{ m/s})\sin\frac{2\pi(0.80 \text{ s})}{(0.667 \text{ s})} = -1.79 \text{ m/s}$$

Ref: google image

Simple Harmonic Motion: Sample Problems

A mass, oscillating in simple harmonic motion, starts at $x = A$ and has period T.

At what time, as a fraction of T, does the mass first pass through $x = \frac{1}{2}A$?

$$x = \tfrac{1}{2}A = A\cos\frac{2\pi t}{T}$$

$$t = \frac{T}{2\pi}\cos^{-1}\left(\tfrac{1}{2}\right) = \frac{T}{2\pi}\frac{\pi}{3} = \tfrac{1}{6}T$$

Ref: google image

Simple Harmonic Motion: Sample Problems

A particle execute s simple harmonic motion given by the equation

$$y = 12\sin\left(\frac{2\pi t}{10} + \frac{\pi}{4}\right)$$

Calculate (i) amplitude, (ii) frequency, (iii) displacement at t= 1.25s, (iv) velocity at t= 2.5s (v) acceleration at t= 5s.

Simple Harmonic Motion: Sample Problems

A particle execute s simple harmonic motion given by the equation

$$y = 10\sin(10t - \frac{\pi}{6})$$

Calculate (i) frequency, (ii) time period (iii) the maximum displacement (iv)the maximum velocity (v) the maximum acceleration acceleration.

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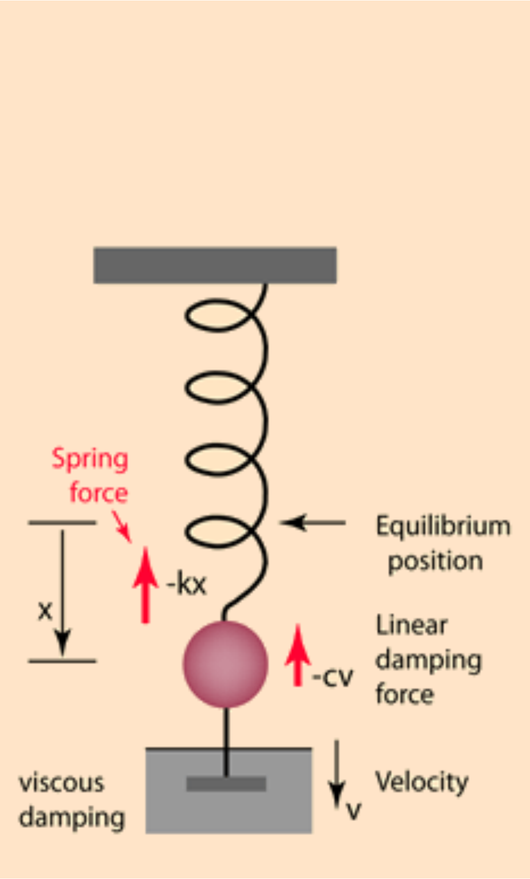
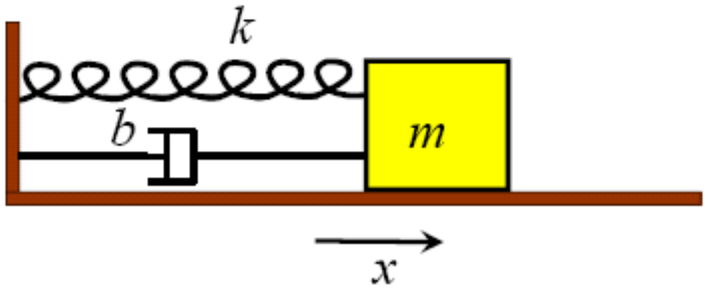
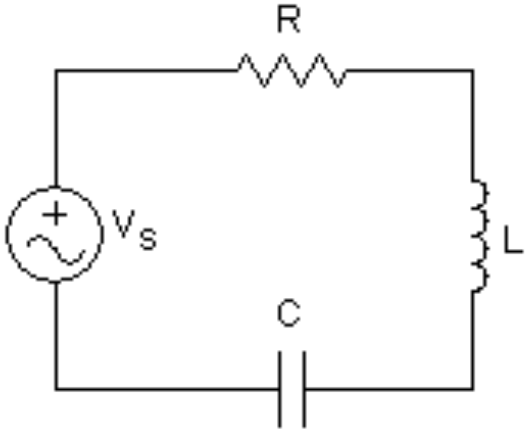
Damped Oscillations







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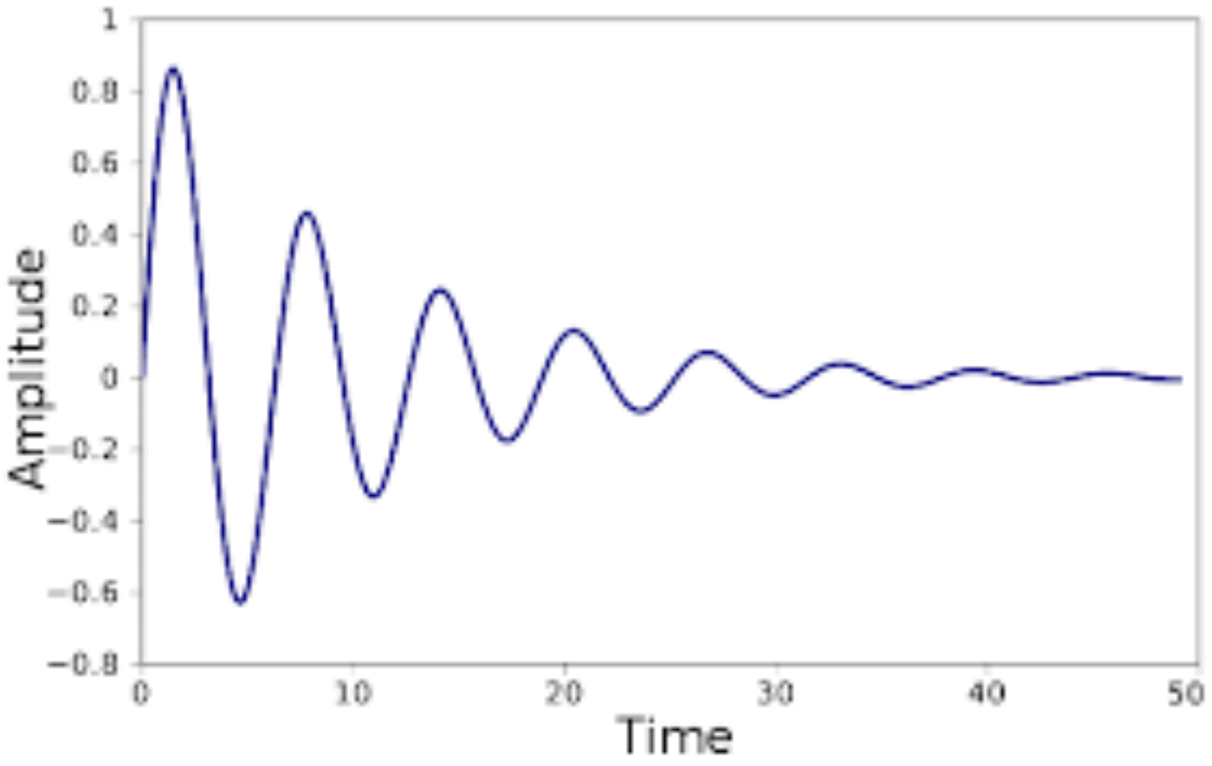
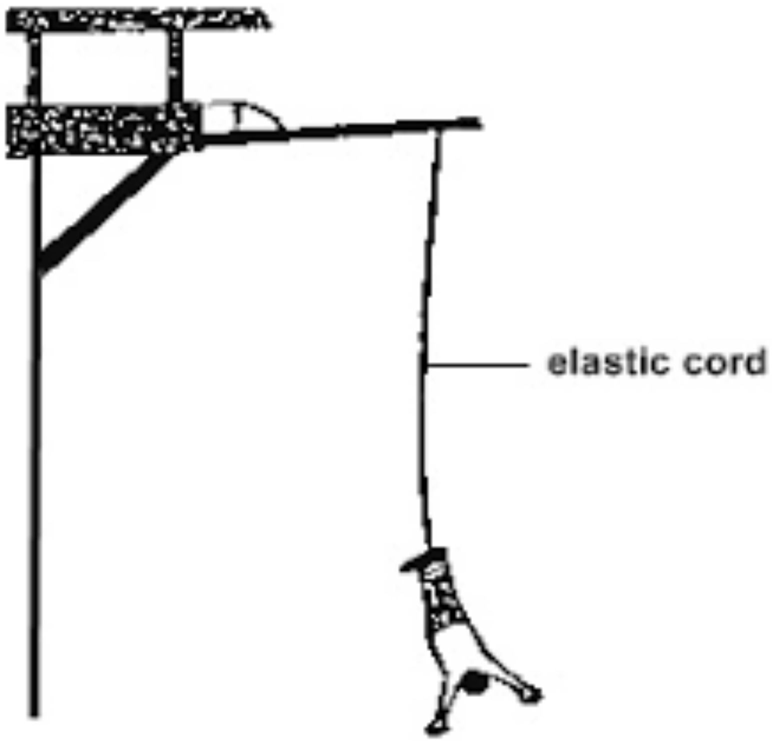
Damped Oscillations: Definition

Damped Harmonic motion: When oscillating bodies do not move back and forth between Precisely fixed limits because frictional force dissipate the energy and amplitude of oscillation Decreases with time and finally die out. Such harmonic motion is called Damped Harmonic Motion.

Ref: google

Damped Oscillations: Example





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Damped Oscillations: Equation

In theses systems the damping force $F' = -bv$

For horizontal forces on the mass: $ma = -kx - bv$

or $m\dfrac{d^2x}{dt^2} = -kx - b\dfrac{dx}{dt}$

or $\dfrac{d^2x}{dt^2} + \gamma\dfrac{dx}{dt} + \omega_0{}^2 x = 0$ where $\begin{cases} \omega_0 = \sqrt{\dfrac{k}{m}} \\ \gamma = \dfrac{b}{m} \end{cases}$

γ : "damping constant" unit: s^{-1}

- "life time" $= \dfrac{1}{\gamma}$

Ref: google

Damped Oscillations: Equation

$$p = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^{\ 2}}$$

We can distinguish three cases:

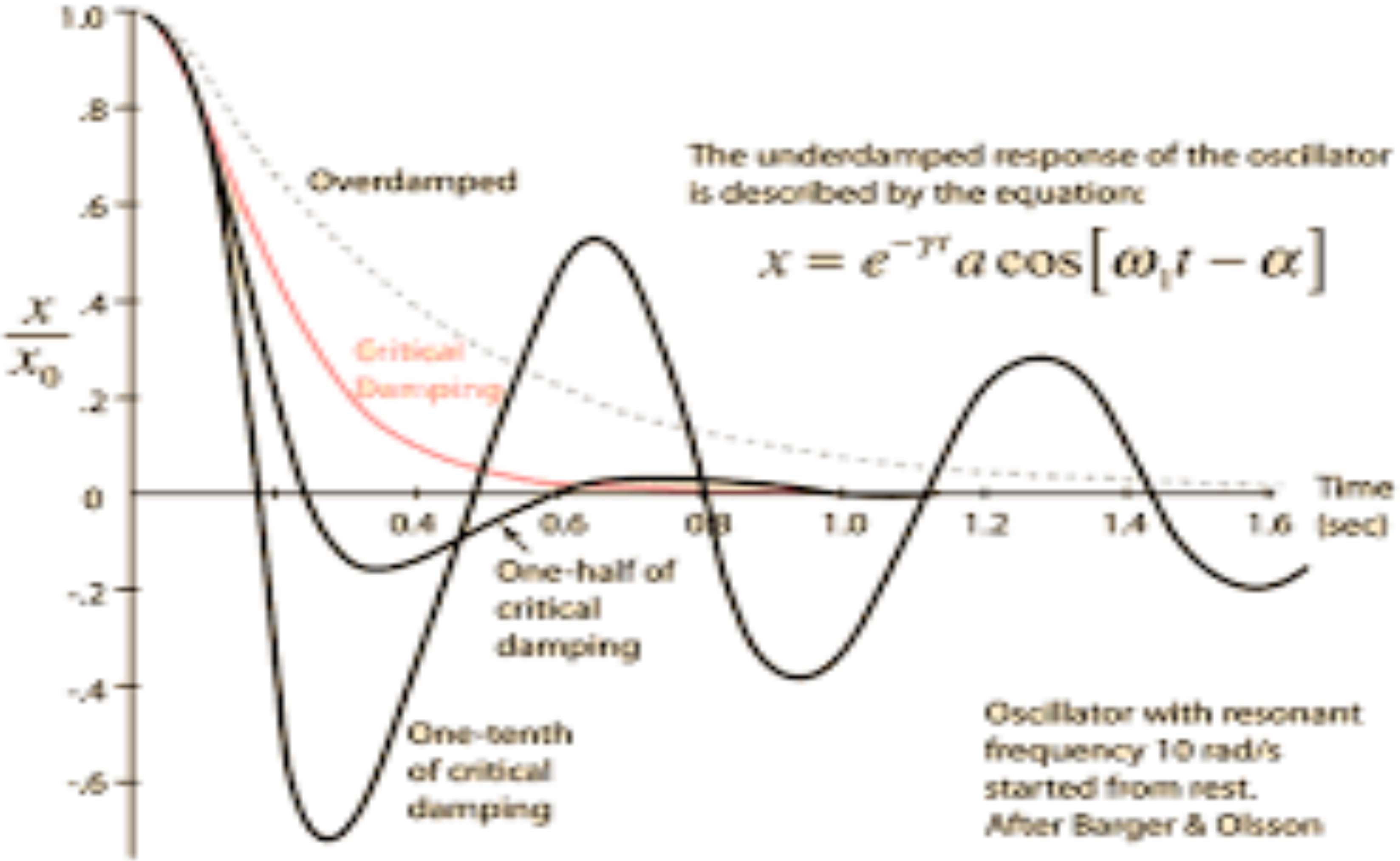
(i) $\omega_0^{\ 2} > \frac{\gamma^2}{4}$ **Oscillatory behaviour**

(ii) $\omega_0^{\ 2} = \frac{\gamma^2}{4}$ **Critical damping**

(iii) $\omega_0^{\ 2} < \frac{\gamma^2}{4}$ **Overdamping**

Ref: google

Damped Oscillations: Equation



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Damped Oscillations: Equation

$$\text{Case (i):} \qquad \omega_0^2 > \frac{\gamma^2}{4}$$

$$\therefore \sqrt{\gamma^2/4 - \omega_0^2} = \sqrt{-(\omega_0^2 - \gamma^2/4)}$$

Put $\quad \omega_1^2 = \omega_0^2 - \gamma^2/4$

$$\therefore p = -\frac{\gamma}{2} \pm \sqrt{-\omega_1^2} = -\frac{\gamma}{2} \pm j\omega_1$$

The solution will be:

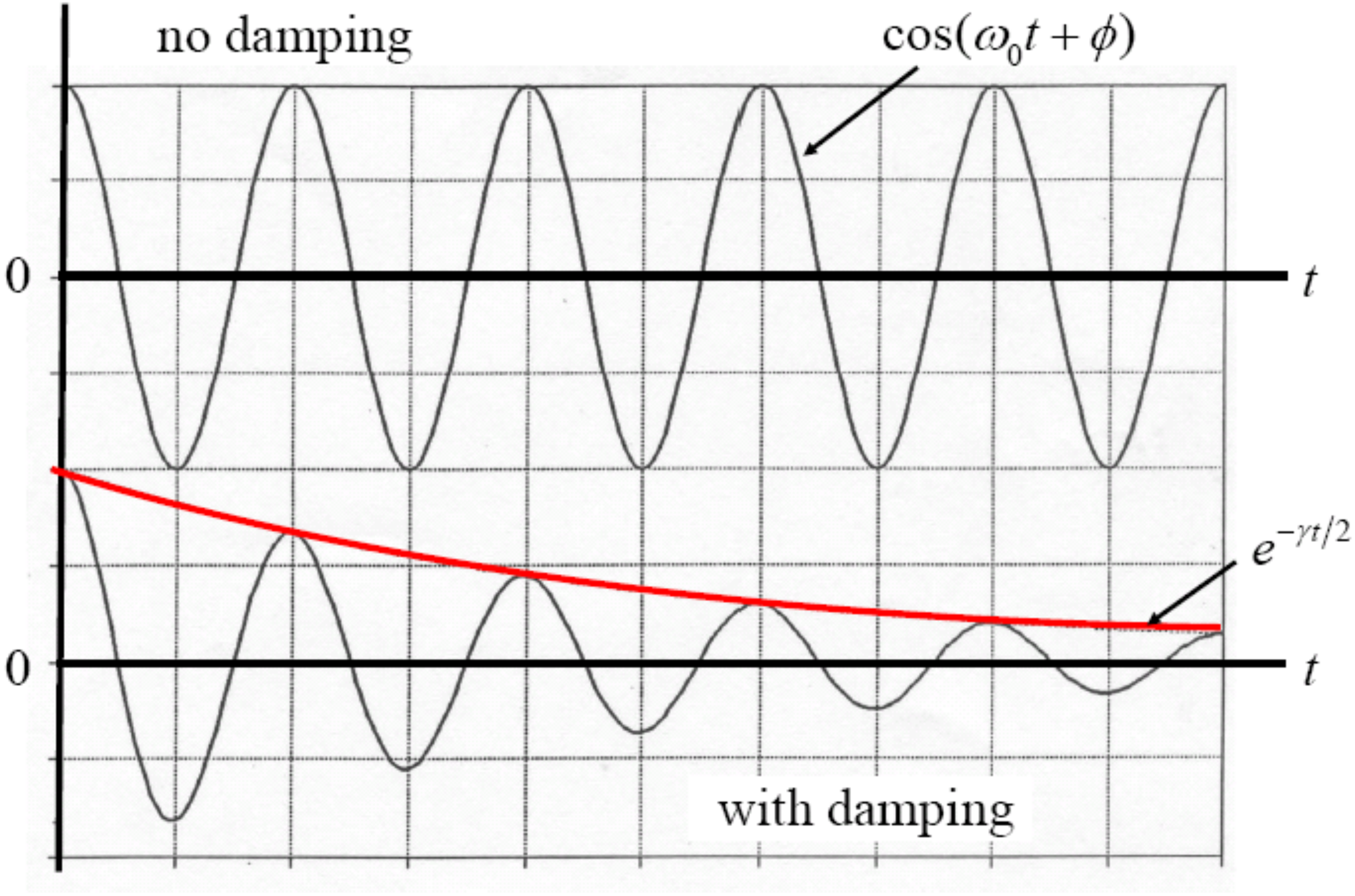
$$x = B_1 e^{\left(-\frac{\gamma}{2}+j\omega_1\right)t} + B_2 e^{\left(-\frac{\gamma}{2}-j\omega_1\right)t} = e^{-\frac{\gamma}{2}t}\left\{B_1 e^{j\omega_1 t} + B_2 e^{-j\omega_1 t}\right\}$$

… leading to $\quad x(t) = Ae^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \phi)$

This is an **oscillatory solution** $\; A\cos(\omega_1 t + \phi) \;$ multiplied by a damping factor $\; e^{-\gamma t/2}$.

As $\; \gamma \to 0 \;$ we approach our undamped oscillator.

Damped Oscillations: Graph



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Damped Oscillations: Equation

Case (ii): $\omega_0^{\ 2} = \dfrac{\gamma^2}{4}$

The two roots coincide: $p = -\dfrac{\gamma}{2}$

The solution will be $x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$

The condition $\omega_0^{\ 2} = \gamma^2/4$ is referred to as the "**critical damping**" condition.

If $\omega_0^{\ 2} < \gamma^2/4$ a system released from rest will oscillate.

As γ is increased the oscillations decay more rapidly, until at $\omega_0^{\ 2} = \gamma^2/4$ oscillation no longer occurs.

[… many practical applications …]

Ref: google

Damped Oscillations: Equation

Case (iii): $\omega_0{}^2 < \frac{\gamma^2}{4}$

$$p = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0{}^2}$$

$$= -\frac{\gamma}{2} \pm \lambda \qquad \text{say}$$

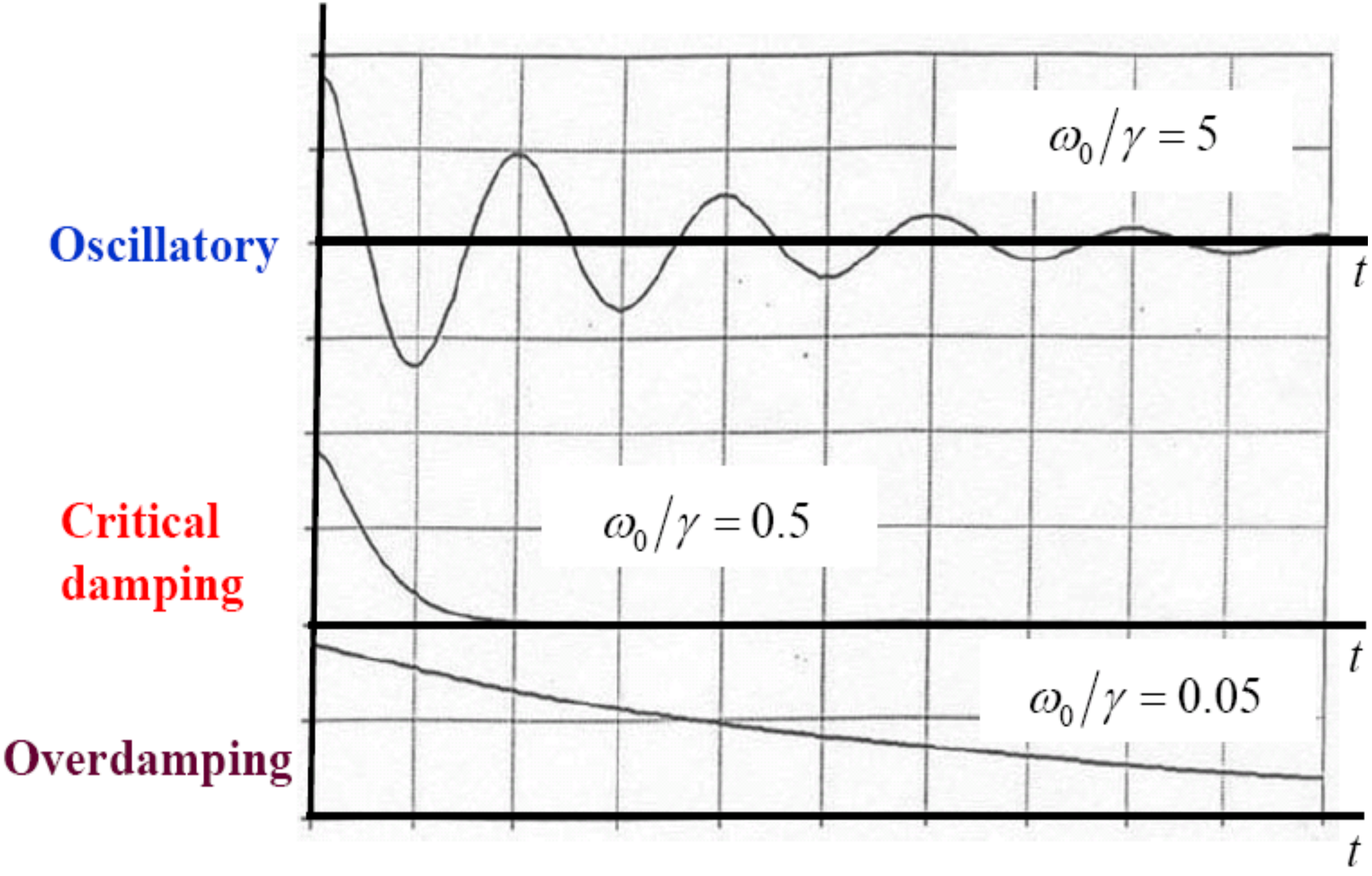
The solution will be $x(t) = B_1 e^{\left(-\frac{\gamma}{2}+\lambda\right)t} + B_2 e^{\left(-\frac{\gamma}{2}-\lambda\right)t}$

The condition $\omega_0{}^2 < \frac{\gamma^2}{4}$ is referred to as **overdamping**

… a slower approach to the rest position is observed.

Ref: google

Damped Oscillations: Graph



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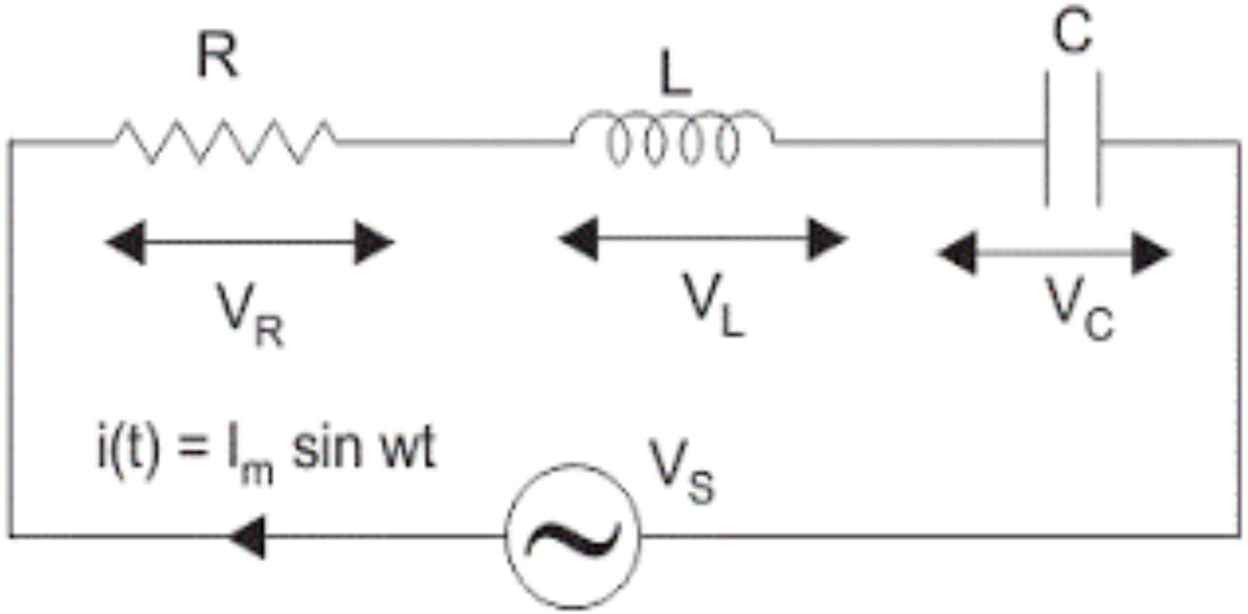
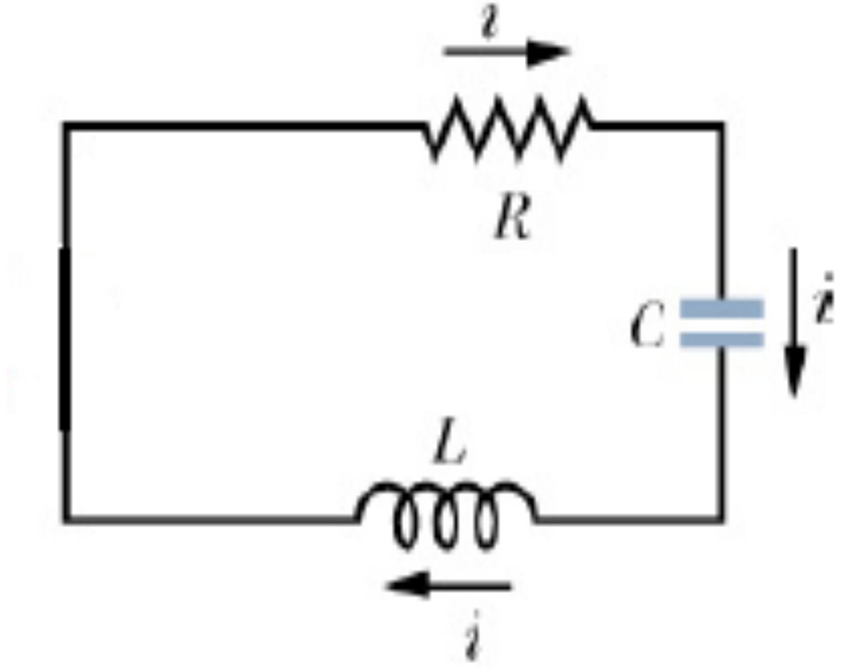
Damped Oscillations: LRC Circuit





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Damped Oscillations: LRC Circuit

- Voltage across resistor R $$V_R = iR$$
- Voltage across capacitor C $$V_C = \frac{Q}{C}$$
- Voltage across inductor L $$V_L = L\frac{di}{dt}$$
- According to

Kirchhoff's voltage law

$$iR + \frac{Q}{C} + L\frac{di}{dt} = 0$$

Ref: google

Damped Oscillations: LRC Circuit

Rewrite the equation

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

Comparing with the equation

$$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0{}^2 x = 0$$

Where

$$\gamma = \frac{R}{L} \qquad \omega_0 = \sqrt{\frac{1}{LC}}$$

Ref: google

Damped Oscillations: LRC Circuit

Three distinguish cases are

i) $$\frac{1}{LC} > \frac{R^2}{4L^2}$$ Oscillatory behavior

ii) $$\frac{1}{LC} = \frac{R^2}{4L^2}$$ Critical damping

iii) $$\frac{1}{LC} < \frac{R^2}{4L^2}$$ Over damping

Ref: google

Damped Oscillations: LRC Circuit

Case i) $\frac{1}{LC} > \frac{R^2}{4L^2}$

Solution of the differential equation

$$Q(t) = Ae^{-\frac{R}{2L}t} \cos(\omega_1 t + \phi)$$

Where $\omega_1 = \sqrt{(\frac{1}{LC} - \frac{R^2}{4L^2})}$

Frequency of oscillation $f = \frac{1}{2\pi}\sqrt{(\frac{1}{LC} - \frac{R^2}{4L^2})}$

Ref: google

Harmonic Motion: Combination at Right Angle

$$x = A_1 \cos(\omega_1 t + \phi_1)$$

$$y = A_2 \cos(\omega_2 t + \phi_2)$$

Consider case where frequencies are equal and let initial phase difference be ϕ

Write $\quad x = A_1 \cos(\omega_0 t) \quad$ and $\quad y = A_2 \cos(\omega_0 t + \phi)$

Case 1 : $\phi = 0 \quad \left.\begin{array}{l} x = A_1 \cos(\omega_0 t) \\ y = A_2 \cos(\omega_0 t) \end{array}\right\} \; y = \dfrac{A_2}{A_1} x$ $\quad$ Rectilinear motion

Case 2 : $\phi = \pi/2 \quad x = A_1 \cos(\omega_0 t)$

$$y = A_2 \cos(\omega_0 t + \pi/2) = -A_2 \sin(\omega_0 t)$$

$$\therefore \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$$

Elliptical path in clockwise direction

Ref: google image

Harmonic Motion: Combination at Right Angle

$$\text{Case 3}: \ \phi = \pi \quad \left.\begin{array}{l} x = A_1 \cos(\omega_0 t) \\ y = A_2 \cos(\omega_0 t + \pi) = -A_2 \cos(\omega_0 t) \end{array}\right\} \ y = -\frac{A_2}{A_1} x$$

$$\text{Case 4}: \ \phi = 3\pi/2 \quad x = A_1 \cos(\omega_0 t)$$

$$y = A_2 \cos(\omega_0 t + 3\pi/2) = +A_2 \sin(\omega_0 t)$$

$$\therefore \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = -1$$

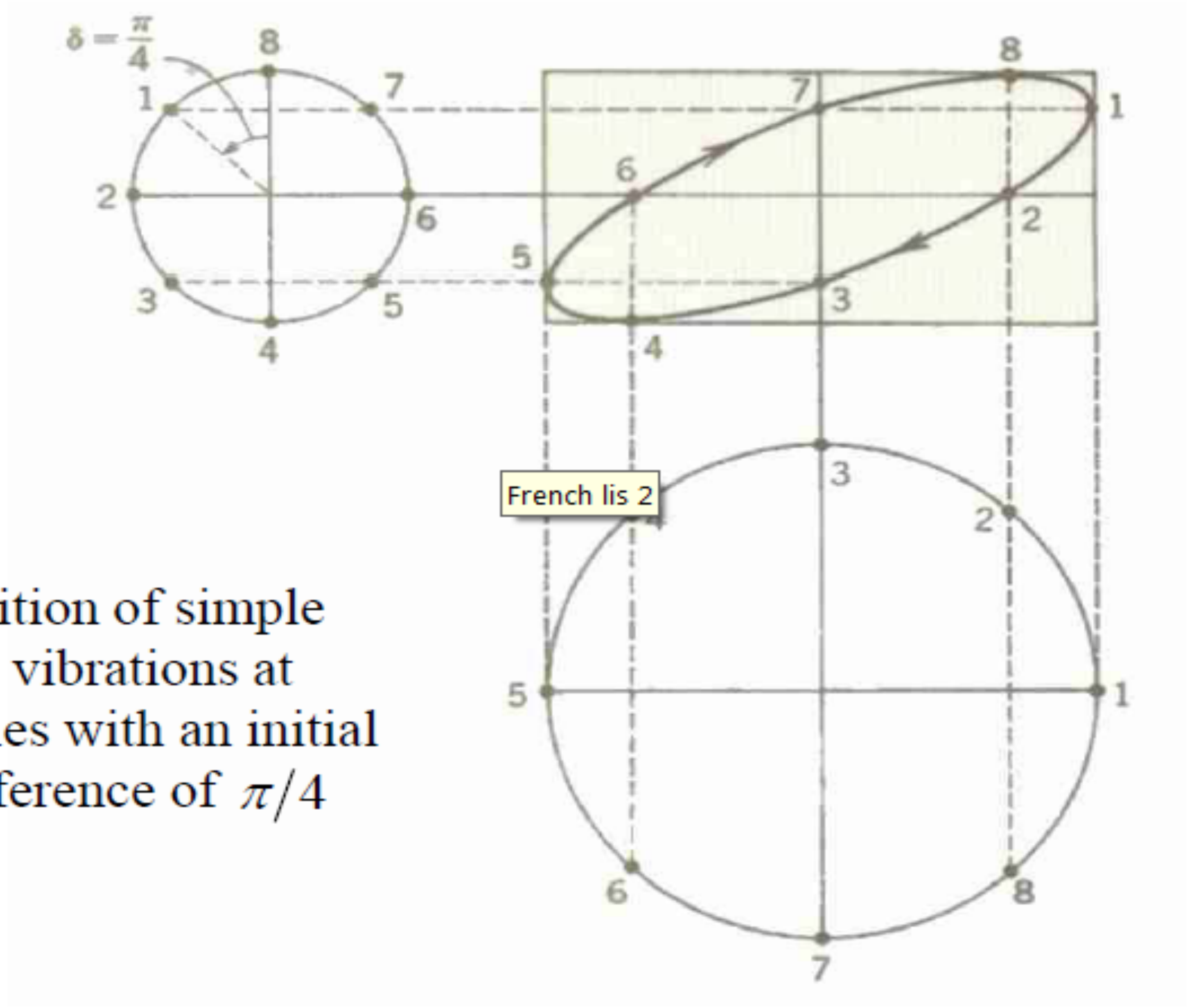
Elliptical path in anticlockwise direction

$$\text{Case 5}: \ \phi = \pi/4 \quad \begin{array}{l} x = A_1 \cos(\omega_0 t) \\ y = A_2 \cos(\omega_0 t + \pi/4) \end{array}$$

Harder to see … use a graphical approach …

Ref: google image

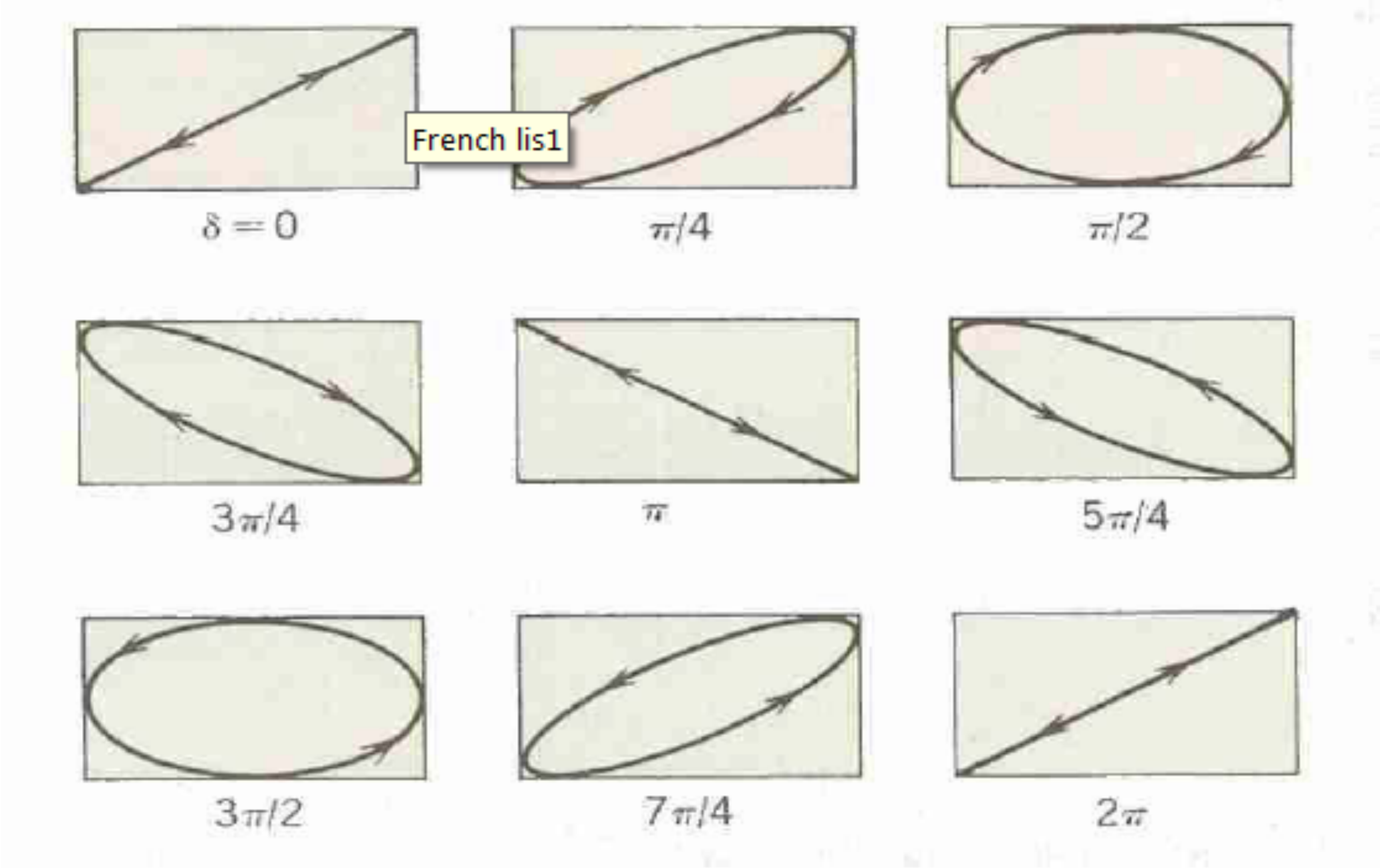
Harmonic Motion: Combination at Right Angle



Superposition of simple harmonic vibrations at right angles with an initial phase difference of $\pi/4$

Ref: google image

Harmonic Motion: Combination at Right Angle



Superposition of two perpendicular simple harmonic motions of the same frequency for various initial phase differences.

Harmonic Motion: Combination at Right Angle

Lssajous' Figures: When particle is influenced simultaneously by two simple harmonic motion at right angles to each other , the resultant motion of the particle traces a curve. This curves are called Lissajous' figures. The shape of the curves Depend on the time period, phase difference and amplitude of the constituent vibrations.

Ref: google image

Damped Harmonic Motion: Sample Problems

A capacitor 1.0μF, an inductor 0.2h and a resistance 800Ω are joined in series. Is the circuit oscillatory? Find the frequency of oscillation.

Find whether the discharge of capacitor through the following inductive circuit is oscillatory.
C = 0.1μF, L = 10mh, R 200 Ω
If Oscillatory, find the frequency of oscillation.

Ref: google image