CHAPTER II

COMPOSITION OF SIMPLE HARMONIC MOTIONS

Introduction - Composition of two simple harmonic vibrations of same frequency but different phase and amplitude - Composition of two simple harmonic vibrations at right angles to each other having equal frequencies but differing in phases and amplitudes - Lissajous' figures - Composition of two simple harmonic motion at right angles to each other and having time periods in the ratio 1:2 - Experimental determination of Lissajous' figures - Uses of Lissajous' figures - Solved problems - Exercises.

2.1 Introduction

Very often problems are encountered in physics where a particle is simultaneously acted upon by more than one simple harmonic vibrations acting either along the same straight line or at right angles to each other. The resultant displacement, velocity, acceleration, etc., of the particle is given by the vector (algebraic) sum of the corresponding quantities due to the individual waves. There are two general methods for solving such problems based on (i) a graphical treatment and (ii) analytical treatment of the dynamics of the particle. The analytical treatment is based on finding the vector sum of the individual motions either with the help of trigonometric functions or writing them as complex quantities. This is usually the easliest to handle and is at the same time most informative. This method of treatment will be followed in the ensuing articles.

2.2 Composition of two simple harmonic vibrations of same frequency but different phase and amplitude

Let a particle in a medium be simultaneously acted upon by two simple harmonic vibrations of same frequency but different phase and amplitude given by the following equations.

$$y_1 = a_1 \sin(wt + \alpha_1)$$

$$y_2 = a_2 \sin(wt + \alpha_2) \qquad (2.1)$$

(2.1)

where y_1 and y_2 are the displacements of the particles due to the individual vibrations of amplitudes a_1 and a_2 and angles of epochs α_1 and α_2 respectively. The two vibrations have the same angular frequency w. The resultant displacement y of the particle will be given by the vector sum of the individual displacements so that

$$
\begin{aligned}
y &= y_1 + y_2 \\
&= a_1 \sin (wt + \alpha_1) + a_2 \sin (wt + \alpha_2) \\
&= a_1 (\sin wt \cos \alpha_1 + \cos wt \sin \alpha_1) \\
&\quad + a_2 (\sin wt \cos \alpha_2 + \cos wt \sin \alpha_2) \\
&= (a_1 \cos \alpha_1 + a_2 \cos \alpha_2) \sin wt \\
&\quad + (a_1 \sin \alpha_1 + a_2 \sin \alpha_2) \cos wt \qquad (2.3)
\end{aligned}
$$

The amplitudes a_1 and a_2 and the angles of epoch α_1 and α_2 of the two vibrations are constant.

Hence putting

$$a_1 \cos \alpha_1 + a_2 \cos \alpha_2 = A \cos \phi$$

and $a_1 \sin \alpha_1 + a_2 \sin \alpha_2 = A \sin \phi$

the resultant amplitude can be written as

$$
\begin{aligned}
y &= A \cos \phi \sin wt + A \sin \phi \cos wt \\
&= A \sin (wt + \phi) \qquad (2.4)
\end{aligned}
$$

Thus the equation of the resultant vibration as given by eqn. (2.4) is simple harmonic and is very much similar to either eqn. (2.1) or (2.2). The amplitude of the resultant vibration is A while the epoch angle is ϕ, the time period of the resultant vibration remaining same as the original vibrations. The values of A and ϕ in eqn. (2.4) can be determined as follows :

$$
\begin{aligned}
A^2 \sin^2 \phi + A^2 \cos^2 \phi &= a_1^2 \sin^2 \alpha_1 + a_2^2 \sin^2 \alpha_2 + 2a_1 a_2 \sin \alpha_1 \sin \alpha_2 \\
&\quad + a_1^2 \cos^2 \alpha_1 + a_2^2 \cos^2 \alpha_2 + 2a_1a_2 \cos \alpha_1 \cos \alpha_2 \\
&= a_2^2 (\sin^2 \alpha_1 + \cos^2 \alpha_1) + a_2^2 (\sin^2 \alpha_2 + \cos^2 \alpha_2) + 2a_1a_2 (\sin \alpha_1 \\
&\quad \sin \alpha_2 + \cos \alpha_1 \cos \alpha_2)
\end{aligned}
$$

or, $A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos (\alpha_1 - \alpha_2)$ (2.5)

and

$$\tan\phi = \frac{A\sin\phi}{A\cos\phi}$$

$$= \frac{a_1 \sin\alpha_1 + a_2 \sin\alpha_2}{a_1 \cos\alpha_1 + a_2 \cos\alpha_2}$$

special cases :

(i) *same phase* : if the two simple harmonic vibrations are in the same phase, then $\alpha_1 = \alpha_2 = \alpha$ (say). Thus if the two vibrations acting on the particle are in the same phase or if the phase difference

$(\alpha_1 - \alpha_2) = 0, 2\pi, 4\pi, \ldots\ldots = 2n\pi$ where $n = 0, 1, 2, \ldots\ldots$, then we get from eqn. (2.5)

$\cos(\alpha_1 - \alpha_2) = 1$ and

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2$$

$$= (a_1 + a_2)^2$$

or, $A = a_1 + a_2$

and $\tan\phi = \dfrac{(a_1 + a_2)\sin\alpha}{(a_1 + a_2)\sin\alpha} = \tan\alpha.$

In that case eqn. (2.4) can be rewritten as

$$y = (a_1 + a_2)\sin(wt + \alpha).$$

(ii) *opposite phase* : If the two vibrations acting on the particle are in opposite phase *i.e.*, if the phase difference $(\alpha_1 - \alpha_2) = \pi, 3\pi, 5\pi, \ldots\ldots = (2n + 1)\pi$ where $n = 0, 1, 2, \ldots\ldots$, we have $\cos(\alpha_1 - \alpha_2) = -1$ and

$$A^2 = a_1^2 + a_2^2 - 2a_1a_2$$

$$= (a_1 - a_2)^2$$

or, $A = a_1 - a_2.$

If, in addition, the amplitudes of the individual vibrations are equal, *i.e.*,

$a_1 = a_2 = a$ (say), then

for the *same phase* condition

$$A = 2a, \text{ and } A^2 = 4a^2$$

while in case of *opposite phase*

$$A = 0 \ (i.e., \text{ the resultant vibration is zero}).$$

Example 2.1. *Two simple harmonic motions acting simultaneously on a particle are given by the equations*

$$y_1 = 2 \sin (wt + \pi/6)$$

$$y_2 = 3 \sin (wt + \pi/3)$$

Calculate (i) amplitude, (ii) phase constant and (iii) time period of the resultant vibration.

What is the equation of the resultant vibration ?

Soln :

$$y_1 = 2 \sin (wt + \pi/6) \quad \text{(i)}$$

$$y_2 = 3 \sin (wt + \pi/3) \quad \text{(ii)}$$

The equations are similar to the equations

$$y_1 = a_1 \sin (wt + \alpha_1) \quad \text{(iii)}$$

$$y_2 = a_2 \sin (wt + \alpha_2) \quad \text{(iv)}$$

The equation of the resultant vibration is given by

$$y = A \sin (wt + \phi) \quad \text{(v)}$$

where $A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos (\alpha_1 - \alpha_2)$ (vi)

$$\text{and } \phi = \tan^{-1}\left(\frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2}\right) \quad \text{(vii)}$$

Here $a_1 = 2, \quad a_2 = 3, \quad \alpha_1 = \pi/6, \quad \alpha_2 = \pi/3$

Hence,

(i) the resultant amplitude

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos (\alpha_2 - \alpha_2)}$$

$$= \sqrt{4+9+(2)(2)(3)\cos\left(-\frac{\pi}{6}\right)}$$

$$= 4.939$$

(ii) $$\phi = \tan^{-1}\left[\frac{(2)(0.5)+(3)(0.866)}{(2)(0.866)+(3)(0.5)}\right]$$

$$= \tan^{-1}[1.114]$$

$$= 48.1° \simeq \frac{4\pi}{15}$$

$\therefore$ phase constant $= (wt + \phi)$

$$= (wt + \frac{4\pi}{15})$$

(iii) the resultant time period is the same as the time period of the individual vibrations. The equation of the resultant vibration is

$$y = 4.939 \sin (wt + \frac{4\pi}{15}).$$

2.3 *Composition of two simple harmonic vibrations at right angles to each other having equal frequencies but differing in phases and amplitudes*

Let us consider two simple harmonic motions of the same frequency (*i.e.*, time period) but of amplitude a and b and having their vibrations mutually perpendicular to one another (*i.e.*, if one vibrates along the X-axis, the other vibrates along the Y-axis). If ϕ is the phase difference between the two vibrations, then their equations can be written as

$$x = a \sin (wt + \phi) \qquad (2.7)$$

and $$y = b \sin wt \qquad (2.8)$$

The vibrations given by eqn. (2.7) leads the vibration given by eqn. (2.8).

From eqn. (2.7) we get

$$\frac{x}{a} = \sin(wt + \phi)$$

$$= \sin wt \cos\phi + \cos wt \sin\phi$$

$$= \sin wt \cos\phi + \sqrt{1-\sin^2 wt}\ \sin\phi \qquad (2.9)$$

Again from eqn. (2.8) we have

$$\sin wt = \frac{y}{b} \qquad (2.10)$$

Substituting this value of sin wt in eqn. (2.9), we get

$$\frac{x}{a} = \frac{y}{b}\cos\varphi + \sqrt{1-\frac{y^2}{b^2}}\,.\sin\phi$$

$$\text{or, } \left(\frac{x}{a} - \frac{y}{b}\cos\varphi\right) = \sqrt{1-\frac{y^2}{b^2}}\,.\sin\phi$$

Squaring both sides, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\cos\varphi - 2.\frac{x}{a}.\frac{y}{b}\cos\varphi = \left(1-\frac{y^2}{b^2}\right)\sin^2\phi$$

$$\text{or, } \frac{x^2}{a^2} + \frac{y^2}{b^2}\cos^2\varphi + \frac{y^2}{b^2}\sin^2\varphi - 2.\frac{xy}{ab}\cos\varphi = \sin^2\varphi$$

$$\text{or, } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2\,xy}{ab}\cos\phi = \sin^2\phi \qquad (2.11)$$

Eqn. (2.11) gives the general equation of the resultant vibration of the two vibrations given by eqns. (2.7) and (2.8). This is the general equation of a conic whose shape will depend upon the value of the phase difference ϕ between the two vibrations.

case I :

$\phi = 0, 2\pi, 4\pi, \ldots = 2n\pi$

where $n = 0, 1, 2, \ldots$

Since there is no phase difference between the two vibrations, $\phi = 0$ and hence $\sin\phi = 0$ and $\cos\phi = 1$. Putting these values in eqn. (2.11), we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2\,xy}{ab}=0$$

$$\text{or,}\quad \left(\frac{x}{a}-\frac{y}{b}\right)^2=0$$

$$\text{or,}\quad \pm\left(\frac{x}{a}-\frac{y}{b}\right)=0$$

$$\text{or,}\quad y=\frac{b}{a}\,x \tag{2.12}$$

Eqn. (2.12) is the equation of a straight line passing through the origin and inclined to the direction of first motion i.e., the X-axis, at an angle $\tan^{-1}\frac{b}{a}$ (Fig. 2.1 (i).). The resultant amplitude is $\sqrt{a^2+b^2}$ If in addition a = b, then the line will be inclined at an angle of 45°.

case II :

$$\phi=\frac{\pi}{4}\ \text{radian}$$

when $\phi=\frac{\pi}{4}$ rad., $\qquad \cos\phi=\sin\phi=\frac{1}{\sqrt{2}}$

Putting these values in eqn. (2.11), we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{\sqrt{2}\,xy}{ab}=\frac{1}{2} \tag{2.13}$$

Eqn. (2.13) represents the equation of an oblique ellipse inscribed in a rectangle whose length parallel to the X-axis is 2a and breadth 2b. The ellipse touches the rectangle at points $\left(\pm a,\ \pm\frac{b}{\sqrt{2}}\right)$ and $\left(\pm\frac{a}{\sqrt{2}},\ \pm b,\right)$ [Fig. 2.1 (ii)]

case III :

$$\phi=\frac{\pi}{2}\ \text{radian.}$$

when $\phi = \frac{\pi}{2}$ rad., $\sin \phi = 1$ and $\cos \phi = 0$.

Hence eqn. (2.11) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (2.14)$$

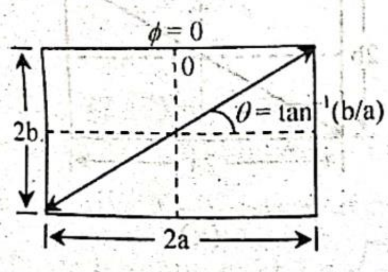
Eqn. (2.14) represents a symmetrical ellipse whose centre coincides with the origin. The semi-major and semi-minor axes of length 2a and 2b respectively coincide with the co-ordinate axes (Fig. 2.11 (iii)]

If in addition a = b, *i.e.*, the amplitudes of the two vibrations are equal; then eqn. (2.14) reduces to

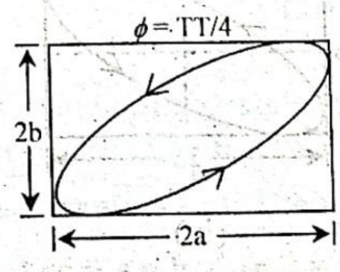
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\text{or,} \quad x^2 + y^2 = a^2 \qquad (2.15),$$

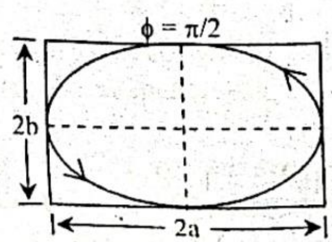
which is the equation of a circle of radius a. The locus of the particle becomes a circle [Fig. 2.1 (iv)].



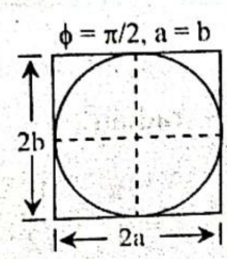
(i)



(ii)



(iii)



(iv)

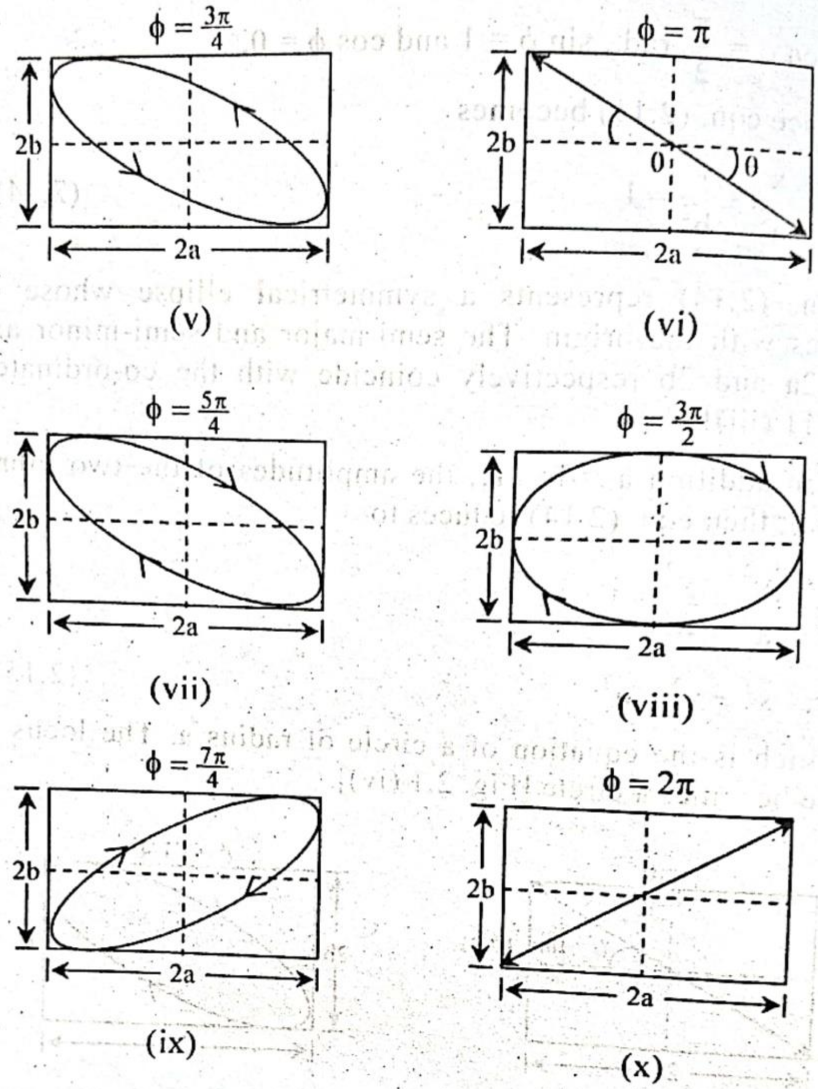


Fig. 2.1

Case IV :

$$\phi = \frac{3\pi}{4} \text{ radian}$$

In that case $\sin\dfrac{3\pi}{4} = \dfrac{1}{\sqrt{2}}$ and $\cos\dfrac{3\pi}{4} = -\dfrac{1}{\sqrt{2}}$

Hence eqn. (2.11) become

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{\sqrt{2}\,xy}{ab}=\frac{1}{2} \qquad (2.16)$$

Eqn. (2.16) represents again the equation of an oblique ellipse with its axes rotated by $\frac{\pi}{2}$ with respect to that in case II. [Fig. 2.1(v)].

case V :

$$\phi = \pi \text{ radian}$$

when $\phi = \pi$ radian, $\sin\phi = 0$ and $\cos\phi = -1$. Hence eqn. (2.11) becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2\,xy}{ab}=0$$

$$\text{or,} \quad \left(\frac{x}{a}+\frac{y}{b}\right)^2=0$$

$$\text{or,} \quad \pm\left(\frac{x}{a}+\frac{y}{b}\right)=0$$

$$\text{or,} \quad y=-\frac{b}{a}x \qquad (2.17)$$

Eqn. (2.17) represents again the equation of a straight line with a negative slope as shown in Fig. 2.1 (vi). The line is now inclined with negative X-direction at an angle of $\tan^{-1}(-\frac{b}{a})$. The resultant amplitude is again given by $\sqrt{a^2+b^2}$. And if a = b, then $\theta = 45°$.

When $\phi = \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$ and 2π respectively the shapes of the curve given by eqn. (2.11) will be as shown in (vii), (viii), (ix) and (x) of Fig. 2.1 respectively.

2.4 *Composition of two simple harmonic motion at right angles to each other and having time periods in the ratio 1 :2.*

Let the equations of the two simple harmonic motion be

$$x = a\sin(2wt+\phi) \qquad (2.18)$$

and $y = b \sin wt$ (2.19)

where a is the amplitude for the motion along the X-axis and b is the amplitude for the motion along the Y-axis. The phase difference between the two vibrations is ϕ.

From eqn. (2.18)

$$\frac{x}{a} = \sin(2wt + \phi)$$

$$= \sin 2wt \cos\phi + \cos 2wt \sin\phi$$

$$= 2 \sin wt \cos wt \cos\phi + (1 - 2\sin^2 wt)\sin\phi \qquad (2.20)$$

And from eqn. (2.19) we have

$$\frac{x}{a} = \sin wt$$

$$\therefore \quad \cos wt = \sqrt{1 - \sin^2 wt}$$

$$= \sqrt{1 - \frac{y^2}{b^2}}$$

Substituting these value of *sin* wt and *cos* wt in eqn. (2.20) we get,

$$\frac{x}{a} = 2.\frac{y}{b}.\sqrt{1 - \frac{y^2}{b^2}}.\cos\phi + \left(1 - 2.\frac{y^2}{b^2}\right)\sin\phi$$

$$\text{or, } \left[\frac{x}{a} - \left(1 - \frac{2y^2}{b^2}\right)\sin\phi\right] = \frac{2y}{b}\cos\phi\sqrt{1 - \frac{y^2}{b^2}}$$

$$\text{or, } \left[\left(\frac{x}{a} - \sin\phi\right) + \frac{2y^2}{b^2}\sin\phi\right] = \frac{2y\cos\phi}{b^2}\sqrt{1 - \frac{y^2}{b^2}}$$

Squaring both sides,

$$\therefore \left(\frac{x}{a} - \sin\varphi\right)^2 + \frac{4y^4}{b^4}\sin^2\varphi + 2\left(\frac{x}{a} - \sin\varphi\right)\frac{2y^2}{b^2}\sin\varphi$$

$$= \frac{4y^2\cos^2\varphi}{b^2}\left(1 - \frac{y^2}{b^2}\right)$$

$$\text{or, } \left(\frac{x}{a} - \sin\varphi\right)^2 + \frac{4y^4}{b^4}\left(\sin^2\varphi + \cos^2\varphi\right)$$

$$- \frac{4y^2}{b^2}\left(\sin^2\varphi + \cos^2\varphi\right) + \frac{4y^2}{b^2} \cdot \frac{x}{a} \sin\varphi = 0$$

$$\text{or, } \left(\frac{x}{a} - \sin\varphi\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a}\sin\varphi - 1\right) = 0 \qquad (2.21)$$

Eqn. (2.21) represents the general equation of a curve having two loops, for any difference in phase and amplitude; the actual shape of the curve will of course depend upon the phase difference ϕ between the two vibrations. The resulting curves for different values of ϕ are shown in Fig. 2.2. Some of these cases are discussed below :

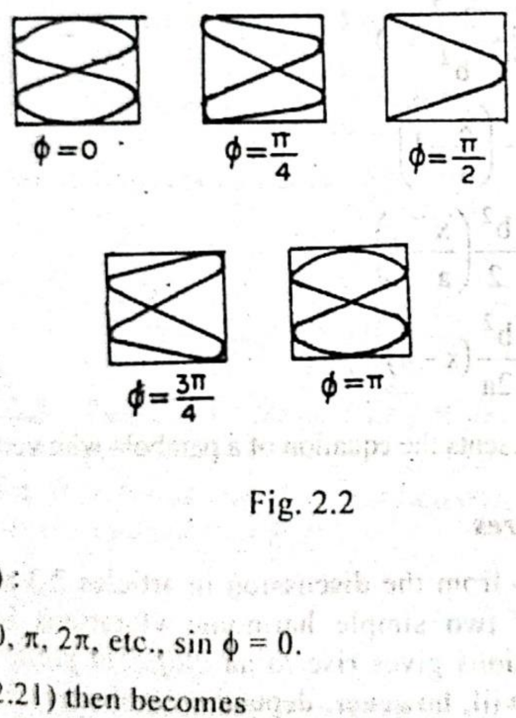


Fig. 2.2

case (1) :

If $\phi = 0, \pi, 2\pi$, etc., $\sin\phi = 0$.

Eqn. (2.21) then becomes

$$\frac{x^2}{a^2} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1\right) = 0 \qquad (2.22)$$

The above equation represents the figure of eight and has two loops (Fig. 2.2).

case (II) :

If $\phi = \frac{\pi}{2}$, $\sin\phi = +1$

Then eqn. (2.21) becomes

$$\left(\frac{x}{a}-1\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2}+\frac{x}{a}-1\right) = 0$$

$$\text{or, } \left(\frac{x}{a}-1\right)^2 + \frac{4y^2}{b^2}\left(\frac{x}{a}-1\right) + \frac{4y^4}{b^4} = 0$$

$$\text{or, } \left[\left(\frac{x}{a}-1\right)^2 + \frac{2y^2}{b^2}\right]^2 = 0$$

$$\text{or, } \left(\frac{x}{a}-1\right) + \frac{2y^2}{b^2} = 0$$

$$\text{or, } \frac{2y^2}{b^2} = -\left(\frac{x}{a}-1\right)$$

$$\text{or, } y^2 = -\frac{b^2}{2}\left(\frac{x}{a}-1\right)$$

$$\text{or, } y^2 = -\frac{b^2}{2a}(x-a) \tag{2.23}$$

Eqn. (2.23) represents the equation of a parabola with vertex at (a, 0).

2.5 *Lissajous' figures*

As can be seen from the discussion in articles 2.3 and 2.4 that the composition of two simple harmonic vibrations in mutually perpendicular directions gives rise to an *elliptical path*. The actual shape of the curve will, however, depend upon the phase difference ϕ between the two vibrations – and also on the ratio of the frequencies of the component vibrations. These figures or curves were first produced optically by Lissajous by reflecting a beam of light from two mirrors, in turn attached to two forks vibrating at right angles to one another. These figures are now known as *Lissajous' figures*.

When two rectangular simple harmonic vibrations whose periods are nearly but not exactly equal act simultaneously on a particle, then the pattern of Lissajous' figures migrates slowly through the sequence shown in Figs. 2.1 and 2.2. The number of such complete sequences gone through per second is equal to the difference of frequencies of the component simple harmonic vibrations.

Example 2.2. *In an experiment to obtain Lissajous' figures, one tuning fork is of frequency 256 Hz and a circular figure occurs after every ten seconds. What deductions may be made about the frequency of the other tuning fork ?*

Soln.

Frequency of A = 256 Hz.

Time for one complete cycle = 10 seconds

$\therefore$ difference in frequencies $= \frac{1}{10} = 0.1$ Hz.

So, the possible frequency of B is

either $256 + 0.1 = 256.1$ Hz

or $256 - 0.1 = 255.9$ Hz.

Example 2.3. *Two tuning forks A and B are used to produce Lissajous' figures. The frequency of A is slightly greater than that of B and is 200 Hz. It is found that the figure completes its cycle in 5 seconds. What is the frequency of B ?*

Soln.

Frequency of A = 200 Hz.

Time for one complete cycle = 5 seconds

$\therefore$ the difference in frequencies $= \frac{1}{5} = 0.2$ Hz.

Since the frequency of A is larger than that of B, the frequency of B

$= 200 - 0.2 = 199.8$ Hz.

Example 2.4. *Two tuning forks A and B are of nearly equal frequencies. Frequency of A is 256 Hz. When the two tuning forks are used to obtain Lissajous' figures, it is found that the complete cycle of changes takes place in 20 seconds. When the tuning fork B is loaded with a little wax, the time taken for one complete cycle of change is 10 seconds. Calculate the original frequency of B.*

Soln.

Frequency of A = 256 Hz.

Time for one complete cycle = 20 secs.

$\therefore$ Difference in frequencies = $\frac{1}{20}$ = **0.05 Hz.**

So, possible frequency of B is

either 256 + 0.05 = 256.05 Hz.

or, 256 − 0.05 = 255.95 Hz.

After loading, time for a complete change of cycle is 10 seconds, i.e., the time decreases. Suppose the frequency of B is 256.05 Hz. After loading, the frequency of B will be lowered and its difference with the frequency of A becomes less. Therefore, the time taken for a complete change of cycle will be more than 20 seconds. Hence the frequency of B cannot be 256.05 Hz.

Suppose the frequency of B = 255.95 Hz. After loading its difference with the frequency of A is increased. Therefore, the cycle of change will take place in less time.

Hence the original frequency of B = 255.95 Hz.

2.6 Experimental determination of Lissajous' figures

A and B are two turning forks with frequencies in the ratio of 1:2. The prongs of A vibrate in a horizontal plane while the prong of B vibrate in a vertical plane. A strong beam of light from an electric arc S is made to converge on a mirror M_1 attached to one of the prongs of the fork A with the help of a convergent lens. The arrangement is such that after reflection from the mirror M_1, the light is again reflected from the mirror M_2 attached to one of the prongs of B. It is further adjusted so that after reflection from M_1 and M_2, the spot of light is obtained at O, the centre