United International University
PHY 210
Waves and Oscillation, Electricity and Magnetism

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Part 1

Simple harmonic oscillators

Damped oscillators

Driven oscillators

Resonance

The Wave Equation

Part II

Electricity and Magnetism

1. Concept of Charge and Coulomb's Law
2. Electric Fields
3. Gauss' Law
4. Electric potential
5. Capacitance
6. Current and Resistance
7. Circuit
8. Magnetic Fields

What to do in this course:

1. Read the relevant sections in the textbook
   … the course notes will guide you.

2. Do all homework and problem sets.

3. Get help early … from Course Teacher.

4. there are no shortcuts … put effort in to understand things …

Problem-solving and homework

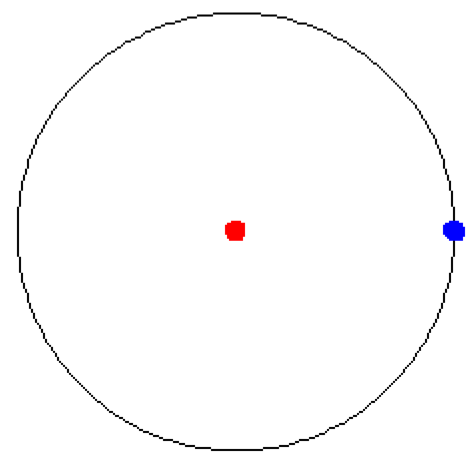
Each Chapter you will be given a take-home problem set to complete and hand in for marks ... In addition to this, you need to work through the problems in *Book, in you own time, at home.*

You will not be asked to hand these in for marks. Get help from you friends, the course Teacher ...

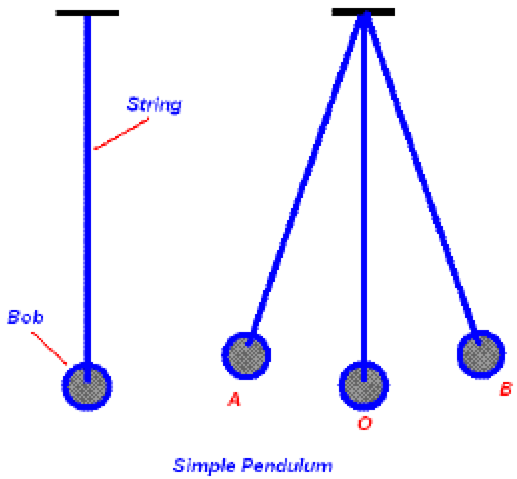
Do not take shortcuts. Mastering these problems is a fundamental aspect of this course.

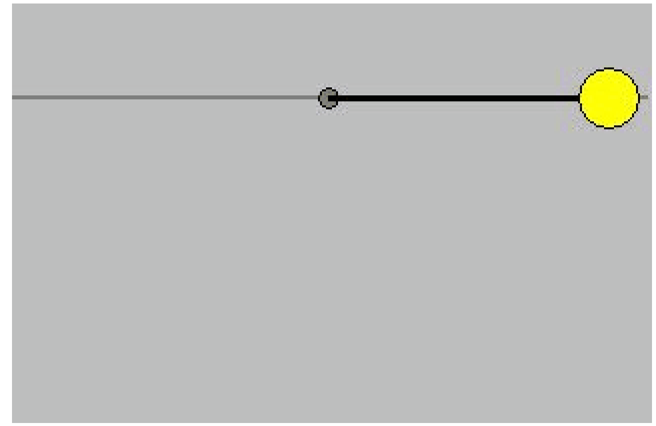
Different Kind of Motions

Linear Motion



Uniform Circular Motion





Oscillatory Motion (Simple Pendulum)

Oscillatory Motion (Spring Mass)

Oscillatory Phenomena

… observed in many physical systems …
from the very small…(e.g. dipole resonance in nuclei)… to the very large (earthquake waves, stars,…)

Mechanical systems …… to lasers
…… from violin strings …… to electrical systems

Oscillatory Motion

Motion which is periodic in time, that is, motion that repeats itself in time.

Examples:

- Power line oscillates when the wind blows past it
- Earthquake oscillations move buildings

- Block attached to a spring
- Motion of a swing
- Motion of a pendulum
- Vibrations of a stringed musical instrument
- Motion of a cantilever
- Oscillations of houses, bridges, …
- All clocks use simple harmonic motion

Periodic Motion: Many kind of motion repeat over and over, such as, the vibrations of quartz crystal in a watch, swinging pendulum in a clock and back-and-forth motion of a piston in an engine. This kind of motions are called periodic motion.

Amplitude: The amplitude of the motion, denoted by A, is the maximum magnitude of displacement from the equilibrium position. It is always positive

Time Period: The period T, is the time required for one oscillation.

Frequency: The frequency, f, is the number of complete oscillations in a unit time.

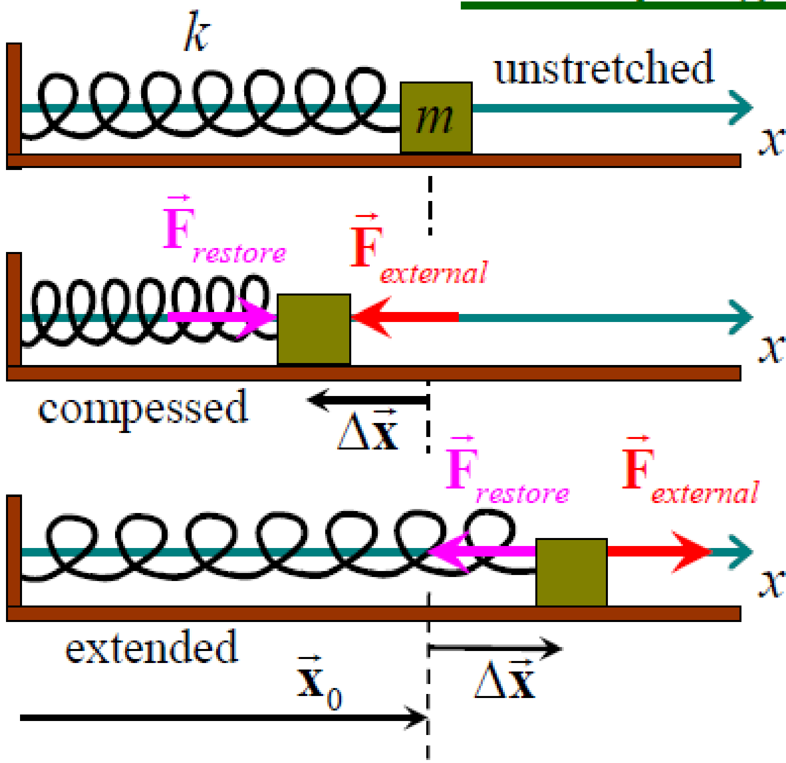
Simple Harmonic Motion: The simplest kind of oscillation occurs when the restoring force F_x is directly proportional to the displacement from the equilibrium x.

When the restoring force is directly proportional to the displacement from the equilibrium as given by equation

$$F_x = -kx$$

The oscillation is called Simple Harmonic Motion(SHM).

Mass-spring oscillator



Hooke's Law:

Restoring force,

$$\vec{F}_{restore} = -k\Delta\vec{x}$$

where $\Delta\vec{x} = \vec{x} - \vec{x}_0$

and k is the "spring constant" [N m^{-1}]

Start with the momentum principle: $\dfrac{d\vec{p}}{dt} = \vec{F}_{net}$

For horizontal forces on the mass: $\dfrac{dp_x}{dt} = -kx$

$$\therefore \frac{d(mv_x)}{dt} = -kx \quad \text{or} \quad \frac{d}{dt}\left(m\frac{dx}{dt}\right) = -kx$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Mass-spring oscillator ...2

$$\frac{d^2x(t)}{dt^2} = \frac{-k}{m}x(t)$$

... a second order differential equation
... we know that if we displace a mass-spring system from its rest position and then release it, it will perform SHM ...

Guess a trial solution: $x(t) = A\cos(\omega t + \phi)$

then $\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi)$

and substitute into our DE: $-A\omega^2\cos(\omega t + \phi) = -A\frac{k}{m}\cos(\omega t + \phi)$

... which is true provided $\omega^2 = \frac{k}{m}$

Therefore our solution is $x(t) = A\cos(\omega t + \phi)$ where $\omega = \sqrt{\frac{k}{m}}$

Mass-spring oscillator ...3

We will write a particular value of ω as ω_0 , as the **natural angular frequency** of the oscillator – the frequency that it "wants" to oscillate at.

Mass-spring system: $\omega_0 = \sqrt{\dfrac{k}{m}}$ and $T = 2\pi\sqrt{\dfrac{m}{k}}$

System parameters: $m, k \longrightarrow \omega_0$

Initial conditions: $\longrightarrow A, \phi$

(Note that ω_0 is independent of A)

What is ω?

Let us now find the physical significance of constant ω. If the time t in displacement equation is increased by 2π/ω, the function becomes

$$x = A\cos[\omega(t + 2\pi / \omega) + \phi]$$

$$= A\cos[\omega t + 2\pi + \phi]$$

$$= A\cos(\omega t + \phi]$$

The function merely repeat itself after a time 2π/ω. Therefore 2π/ω is period of the motion.

We have

$$\omega^2 = \frac{k}{m}$$

We have got

$$T = \frac{2\pi}{\omega}$$

Therefore,

$$\omega = \frac{2\pi}{T}$$

The quantity ω is called angular Frequency.

Simple harmonic oscillator

$$x(t) = A\cos(\omega_0 t + \phi)$$

$$v(t) = \frac{dx(t)}{dt} = -A\omega_0 \sin(\omega_0 t + \phi)$$

$$a(t) = \frac{d^2x(t)}{dt^2} = \frac{dv(t)}{dt} = -A\omega_0{}^2 \cos(\omega_0 t + \phi)$$

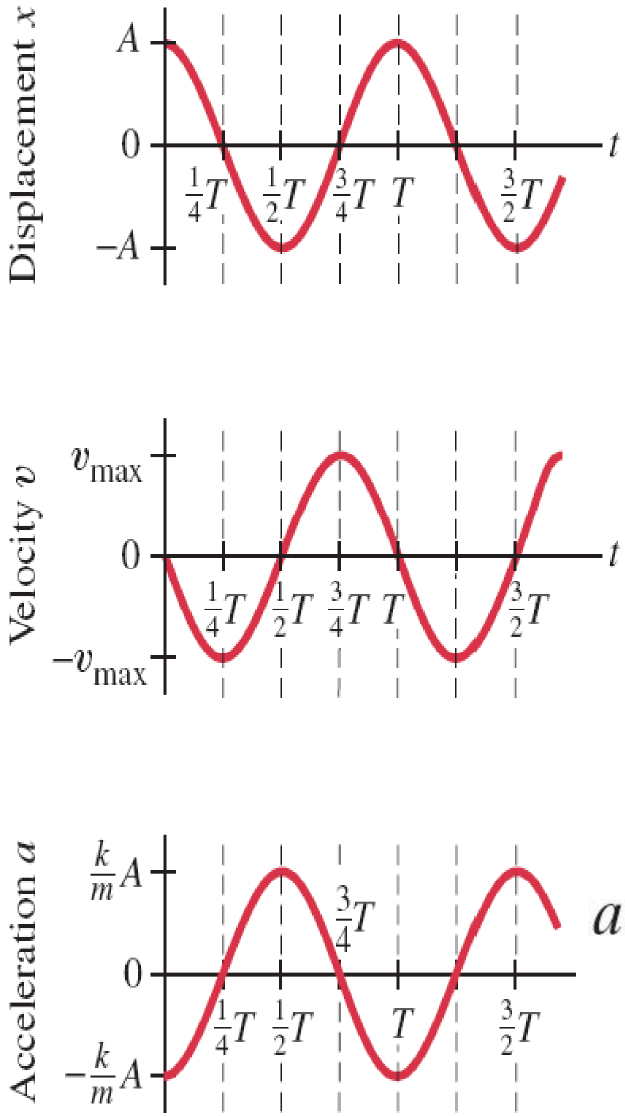
… acceleration = – (constant) . (displacement)

$$= -A\omega_0^2 \cos(\omega_0 t + \phi)$$

$$= A\omega_0^2 \cos(\omega_0 t + \phi + \pi)$$

Phase difference between acceleration and displacement is π

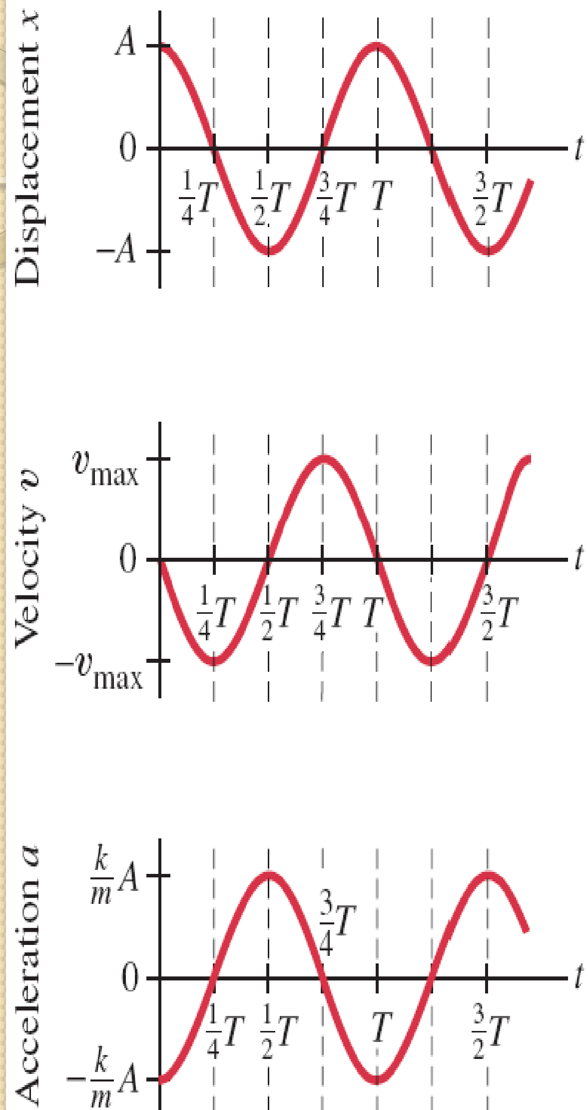
Phase difference between v and x (and v & a) is $\frac{\pi}{2}$



The velocity and acceleration for simple harmonic motion can be found by differentiating the displacement:

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi).$$



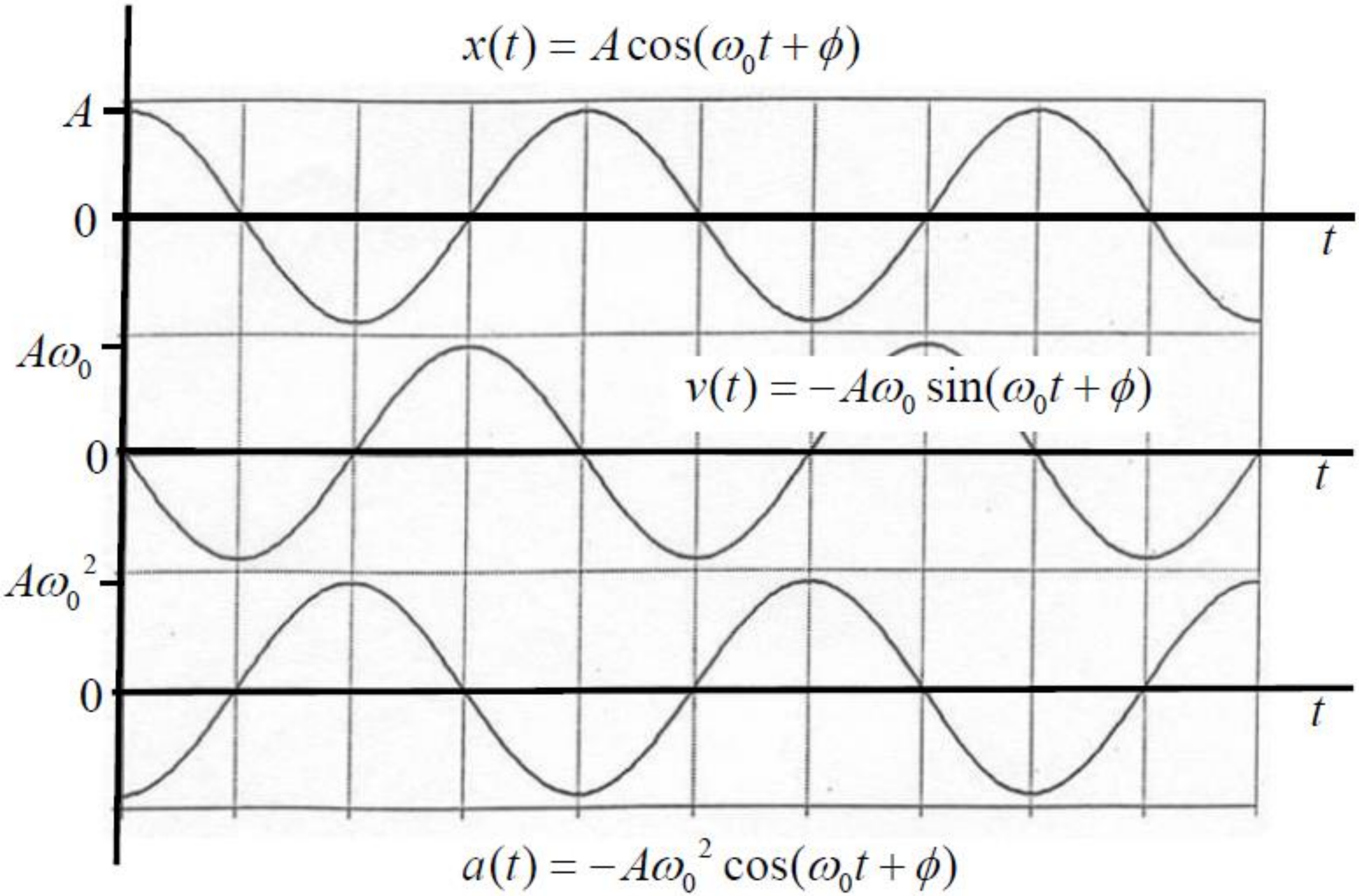
$$x = A\cos(\omega t + \phi),$$

$$v = -\omega A \sin(\omega t + \phi)$$

$$= \omega A \cos(\omega t + \phi + \frac{\pi}{2}),$$

$$a = -\omega^2 A \cos(\omega t + \phi)$$

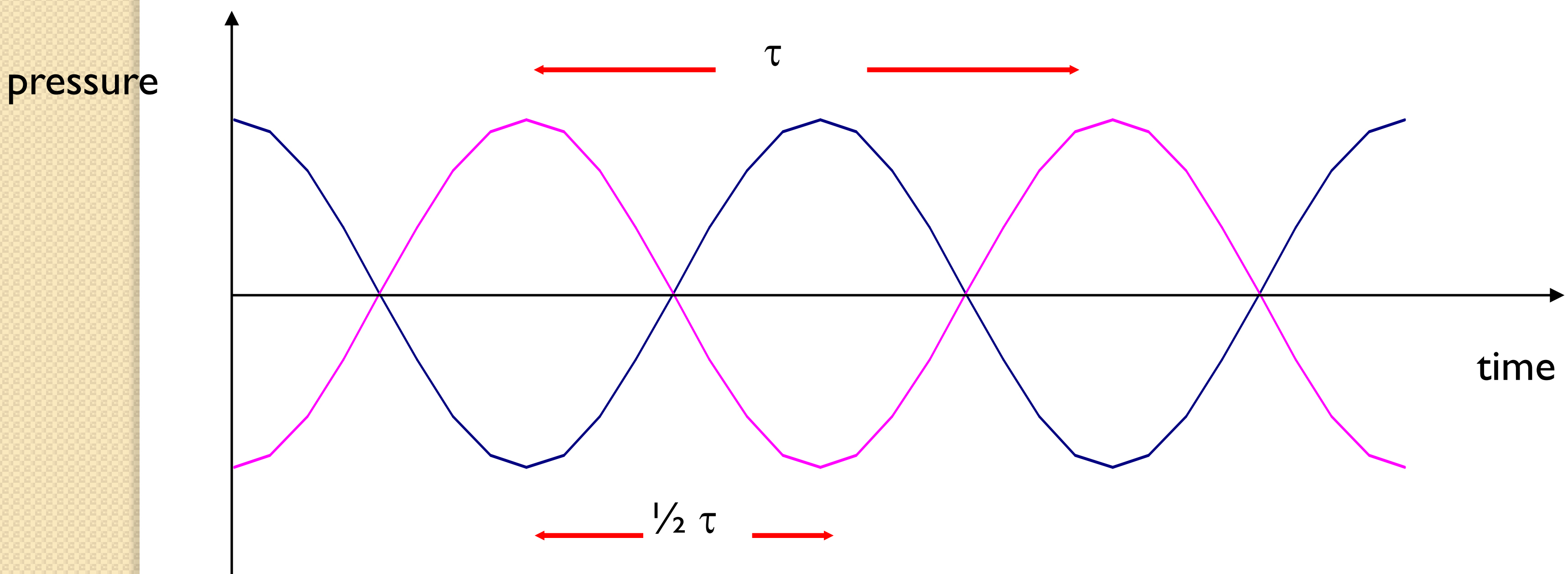
$$= -\omega^2 A \sin(\omega t + \phi + \frac{\pi}{2}).$$


$x(t) = A\cos(\omega_0 t + \phi)$
A
0
t
$A\omega_0$
$v(t) = -A\omega_0 \sin(\omega_0 t + \phi)$
0
t
$A\omega_0{}^2$
0
t
$a(t) = -A\omega_0{}^2 \cos(\omega_0 t + \phi)$

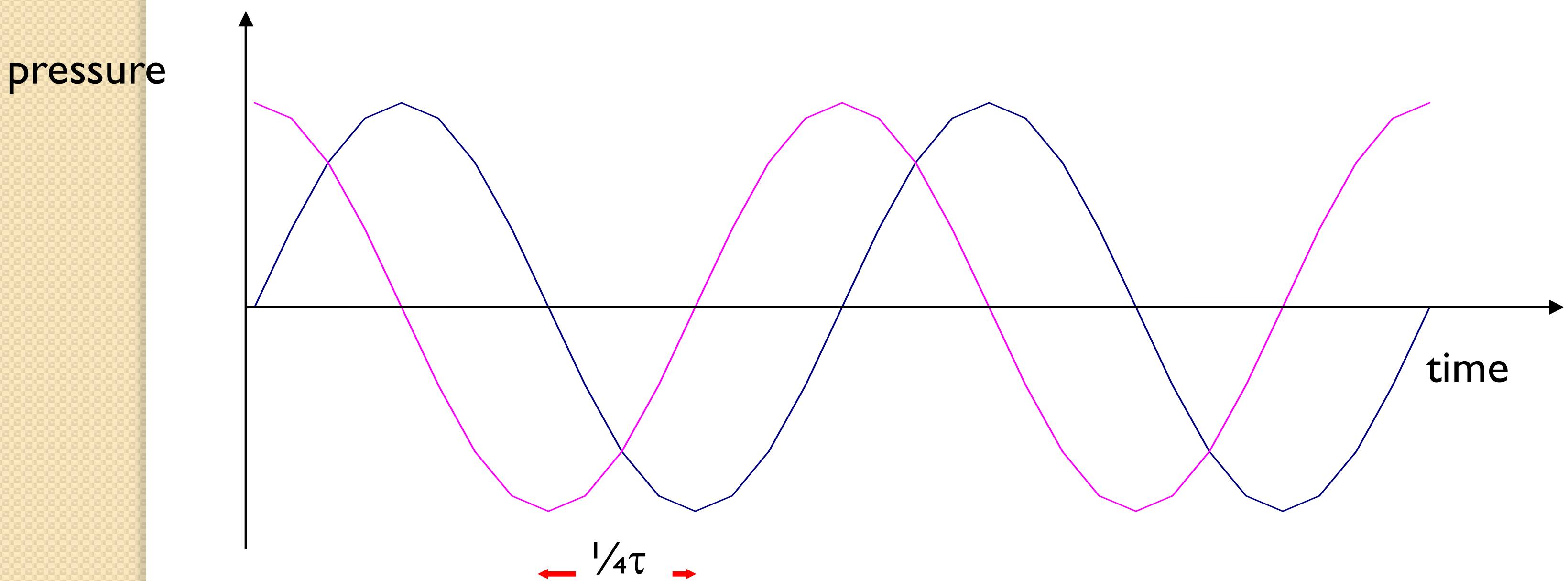
Phase Difference

- The phase of periodic wave describes where the wave is in its cycle
- Phase difference is used to describe the phase position of one wave relative to another

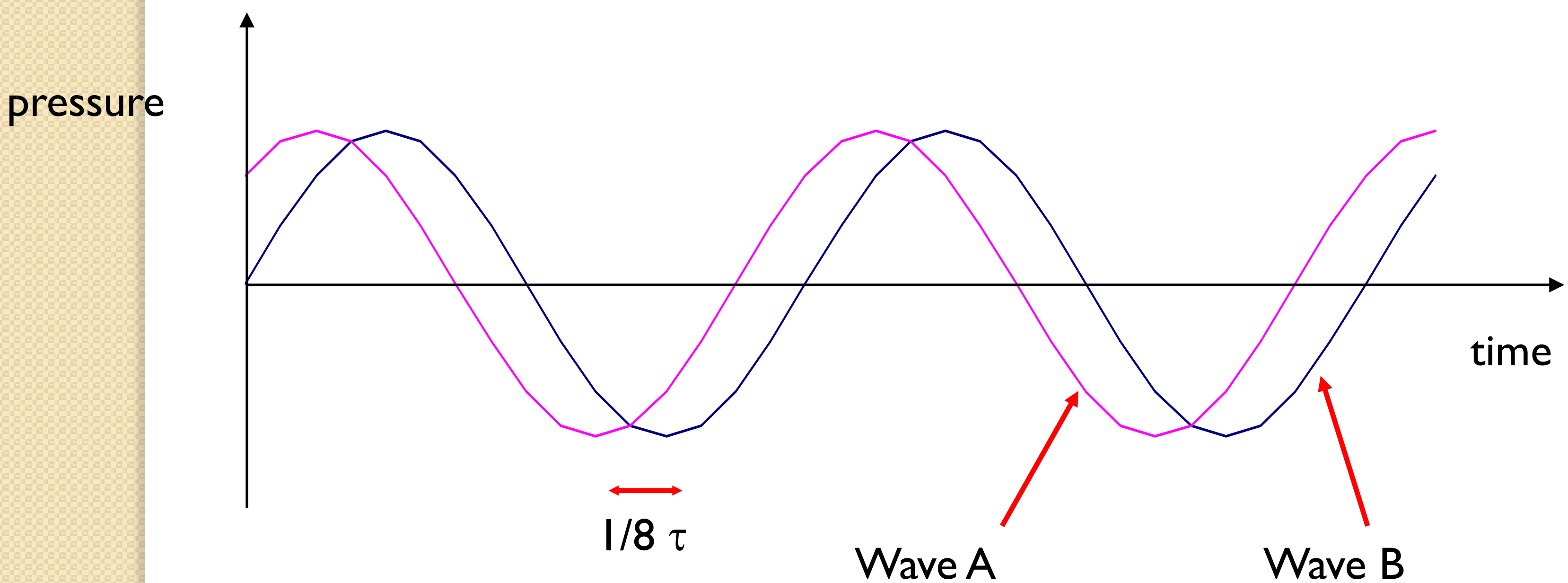
Phase Difference 180°



Phase Difference 90°



Phase Difference 45°



Simple harmonic oscillator

At $t = 0$, write $x = x_0$ and $v = v_0$.

Then at $t = 0$:

$$\left.\begin{aligned} x_0 &= A\cos(\phi) \\ v_0 &= -\omega_0 A\sin(\phi) \end{aligned}\right\} \quad \tan\phi = -\frac{v_0}{\omega_0 x_0}$$

… and $\quad x_0{}^2 + \left(\dfrac{v_0}{\omega_0}\right)^2 = A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$

$$\therefore A = \sqrt{x_0{}^2 + \left(\frac{v_0}{\omega_0}\right)^2}$$

Ex. A block of mass 680gm is fastened to a spring of spring constant 65N/m. The block is pulled a distance 11cm from its equlibrum on a frictionless table and released

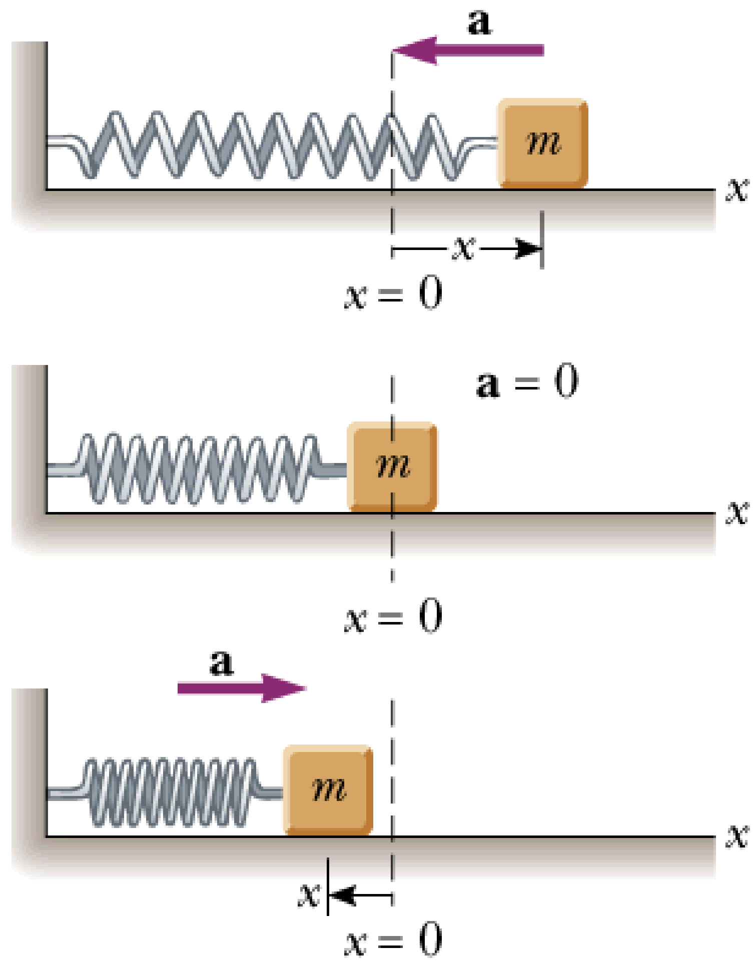
(a) What are the angular frequency, the frequency, and the period of the motion?

(b) What is amplitude of the motion?

(c) What is the maximum speed of the block?

Example

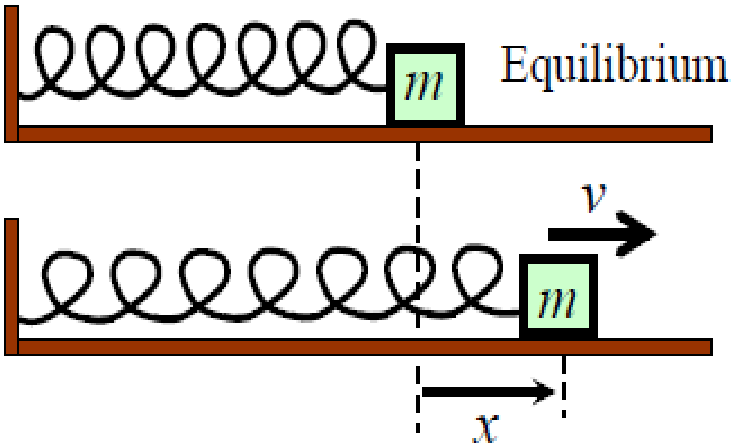
A spring stretches by 3.90 cm when a 10.0 g mass is hung from it. A 25.0 g mass attached to this spring oscillates in simple harmonic motion.



(a) Calculate the period of the motion.

(b) Calculate frequency and the angular velocity of the motion.

Mass-spring oscillator: an energy approach



Suppose that the mass has a speed v when it has displacement x

Kinetic energy of mass = $\frac{1}{2}mv^2$

Potential energy of spring = $\int_0^x F\,dx' = \int_0^x kx'\,dx' = \frac{1}{2}kx^2$

There are no dissipative mechanisms in our model (no friction).
… the total energy of the mass-spring system is conserved.

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \text{constant}$$

Mass-spring oscillator: an energy approach …2

For our mass-spring system: $\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \text{constant}$

$$\therefore \frac{d}{dt}\left(\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2\right) = 0$$

$$\therefore mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0$$

$$\therefore mv\frac{dv}{dt} + kxv = 0$$

$$\therefore m\frac{dv}{dt} + kx = 0$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

… as before

Mass-spring oscillator: an energy approach …3

For the mass-spring system: $x = A\cos(\omega_0 t + \phi)$

Potential energy = $\frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$

k.e. = $\frac{1}{2}mv^2 = \frac{1}{2}m[-A\omega_0 \sin(\omega_0 t + \phi)]^2 = \frac{1}{2}mA^2\omega_0^2 \sin^2(\omega_0 t + \phi)$

Total energy = p.e. + k.e

$$= \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi) + \frac{1}{2}mA^2\omega_0^2 \sin^2(\omega_0 t + \phi)$$

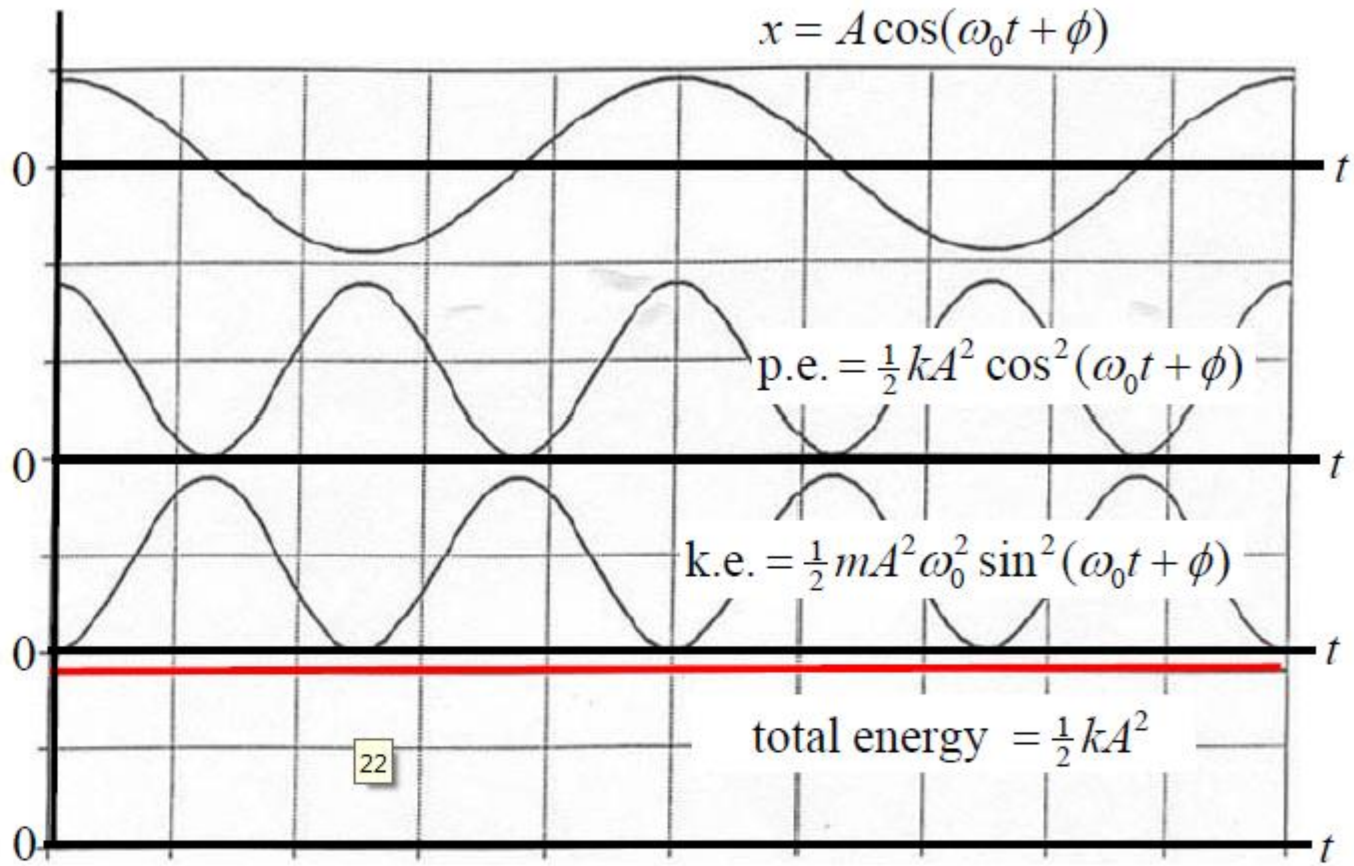
$$= \frac{1}{2}kA^2 \quad (= \frac{1}{2}m{\omega_0}^2 A^2) \qquad (\therefore E \propto A^2)$$

We can now write: $\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$

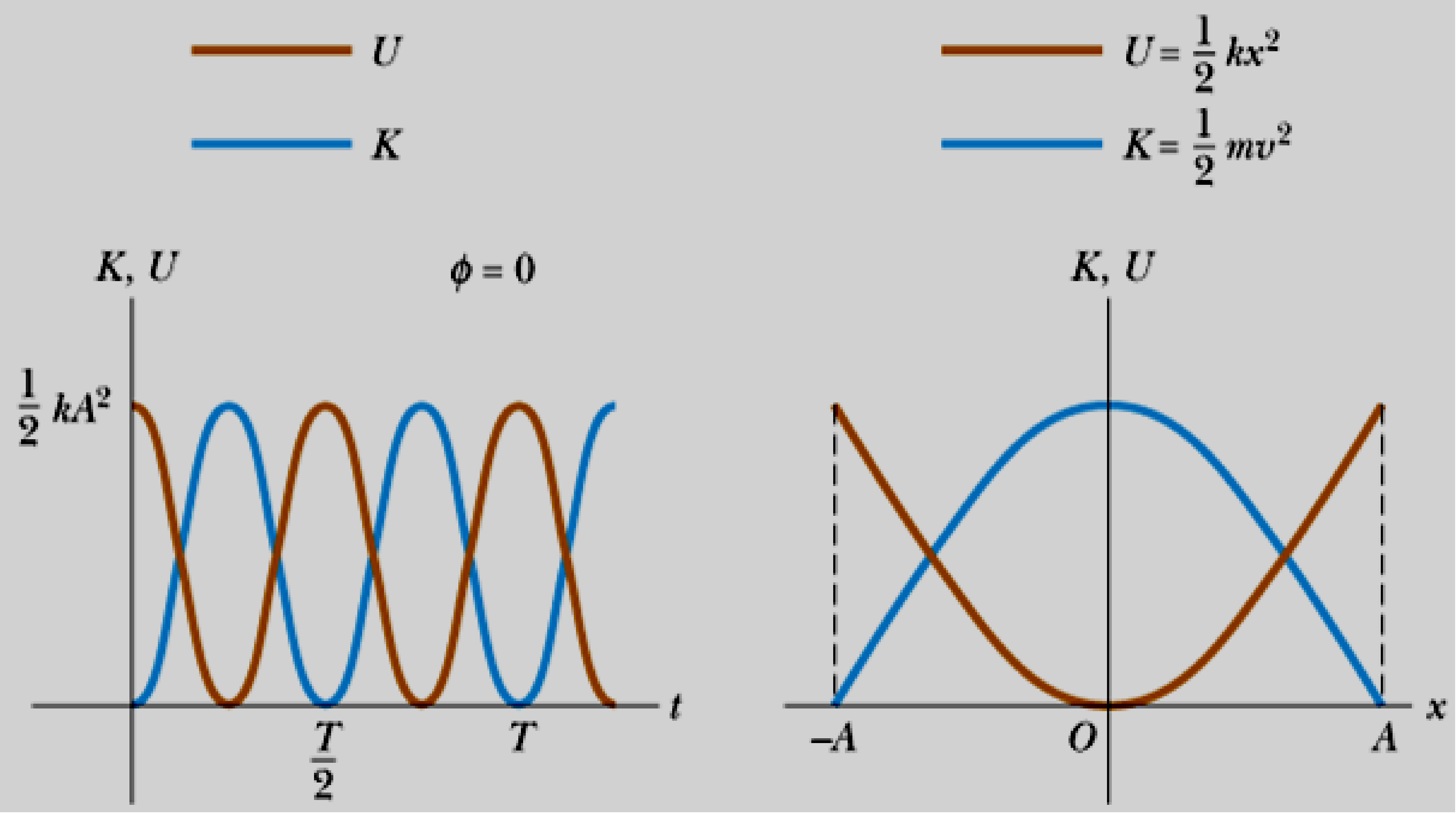
$$\therefore v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} \quad \text{or} \quad v(x) = \pm\omega_0\sqrt{A^2 - x^2}$$

(useful)

Energy of the mass-spring simple harmonic oscillator



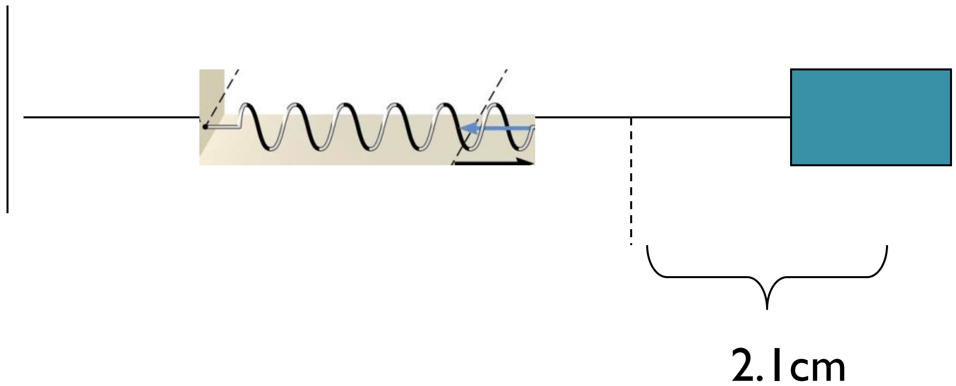
K.E and P.E of SHM



Check Your Understanding

A 0.42-kg block is attached to the end of a horizontal ideal spring and rests on a frictionless surface. The block is pulled so that the spring stretches by 2.1 cm relative to its unstrained length. When the block is released, it moves with an acceleration of 9.0 m/s^2. What is the spring constant of the spring?

180 N/m



kx = ma

$$k \times \frac{2.1}{100} = 0.42 \times 9.0 m/s^2$$

$$k = \frac{0.42 \times 9.0}{2.1} \times 100 = 180 N/m$$

Energy in the SHO

Energy calculations.

For the simple harmonic oscillation where k = 19.6 N/m, A = 0.100 m, x = -(0.100 m) cos 8.08t, and v = (0.808 m/s) sin 8.08t, determine (a) the total energy, (b) the kinetic and potential energies as a function of time, (c) the velocity when the mass is 0.050 m from equilibrium, (d) the kinetic and potential energies at half amplitude ($x = \pm A/2$).

Solution:

a. $E = \frac{1}{2}kA^2 = \frac{1}{2} \times 19.6\,\mathrm{N/m} \times (0.100\,\mathrm{m})^2 = 9.80 \times 10^{-2}\,\mathrm{J}.$

b. $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2 \omega t = (9.80 \times 10^{-2}\,\mathrm{J}) \cos^2 8.08t,$

$K = E - U = (9.80 \times 10^{-2}\,\mathrm{J}) \sin^2 8.08t.$

Solution:

c. $K = E - U,\ \frac{1}{2}mv^2 = \frac{1}{2}kA^2 - \frac{1}{2}kx^2,$

$$v = \sqrt{\frac{k}{m}(A^2 - x^2)} = \omega\sqrt{A^2 - x^2}$$

$$= 8.08\text{Hz} \cdot \sqrt{(0.100\text{m})^2 - (0.050\text{m})^2} = 0.70\text{m/s}.$$

d. $U = \frac{1}{2}kx^2 = \frac{1}{2}k\left(\frac{A}{2}\right)^2 = \frac{1}{4}E = 2.5\times 10^{-2}\,\text{J},$

$$K = E - U = 7.3\times 10^{-2}\,\text{J}.$$

Example: Using Conservation of Energy

A 500 g block on a spring is pulled a distance of 20 cm and released. The subsequent oscillations are measured to have a period of 0.80 s. At what position (or positions) is the speed of the block 1.0 m/s?

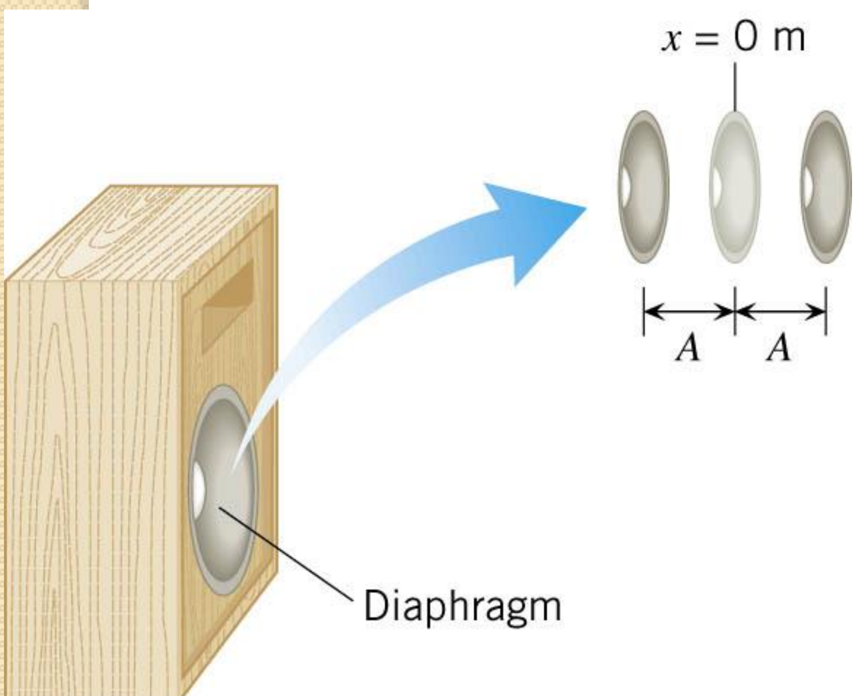
$$T = 0.80 \text{ s} \quad \text{so} \quad \omega = \frac{2\pi}{T} = \frac{2\pi}{(0.80 \text{ s})} = 7.85 \text{ rad/s}$$

$$v = \sqrt{\frac{k}{m}\left(A^2 - x^2\right)} = \omega\sqrt{A^2 - x^2}$$

$$x = \pm\sqrt{A^2 - \left(\frac{v}{\omega}\right)^2} = \pm\sqrt{(0.20 \text{ m})^2 - \left(\frac{(1.0 \text{ m/s})}{(7.85 \text{ rad/s})}\right)^2} = \pm 0.154 \text{ m} = \pm 15.4 \text{ cm}$$

Example: The Maximum Speed of a Loudspeaker Diaphragm

The diaphragm of a loudspeaker moves back and forth in simple harmonic motion to create sound. The frequency of the motion is f = 1.0 kHz and the amplitude is A = 0.20 mm.



(a) What is the maximum speed of the diaphragm?

(b) Where in the motion does this maximum speed occur?

(a)

$$v_{max} = A\omega = A(2\pi f) = (0.20 \times 10^{-3}\ \text{m})(2\pi)(1.0 \times 10^{3}\ \text{Hz}) = \boxed{1.3\ \text{m/s}}$$

(b) The speed of the diaphragm is zero when the diaphragm momentarily comes to rest at either end of its motion: *x* = +*A* and *x* = –*A*. Its maximum speed occurs midway between these two positions, or at *x* = 0 m.

Example: Radio Station Frequency and Period

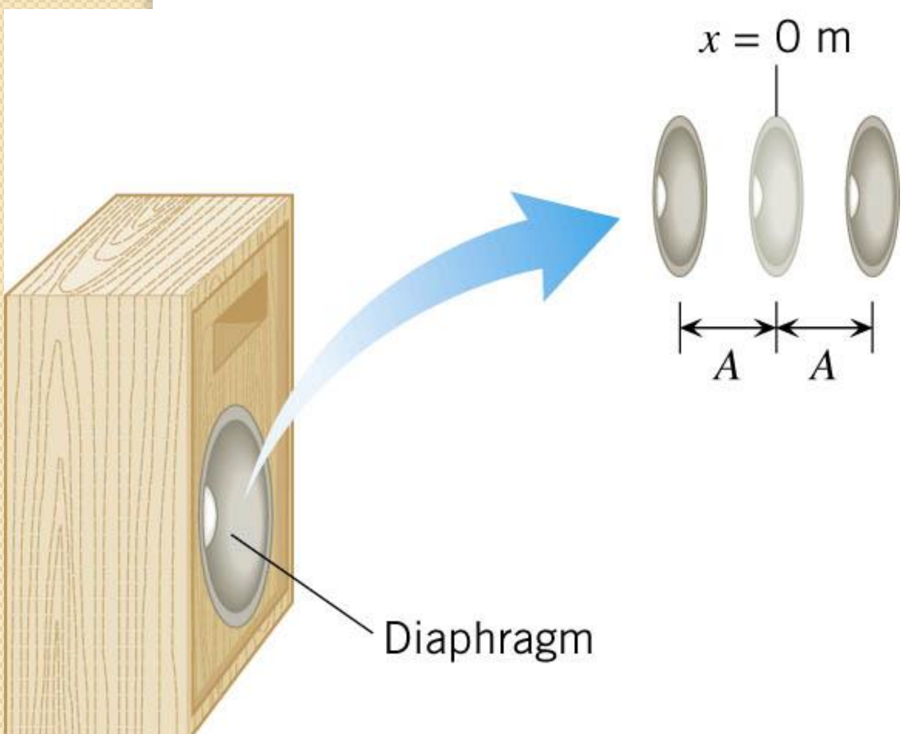
What is the oscillation period of an FM radio station that broadcasts at 100 MHz?

$$f = 100 \text{ MHz} = 1.0 \times 10^{8} \text{ Hz}$$

$$T = 1/f = \frac{1}{1.0 \times 10^{8} \text{ Hz}} = 1.0 \times 10^{-8} \text{ s} = 10 \text{ ns}$$

Note that 1/Hz = s

Example : The Loudspeaker Revisited—The Maximum Acceleration



A loudspeaker diaphragm is vibrating at a frequency of $f = 1.0$ kHz, and the amplitude of the motion is $A = 0.20$ mm.

(a) What is the maximum acceleration of the diaphragm, and

(b) where does this maximum acceleration occur?

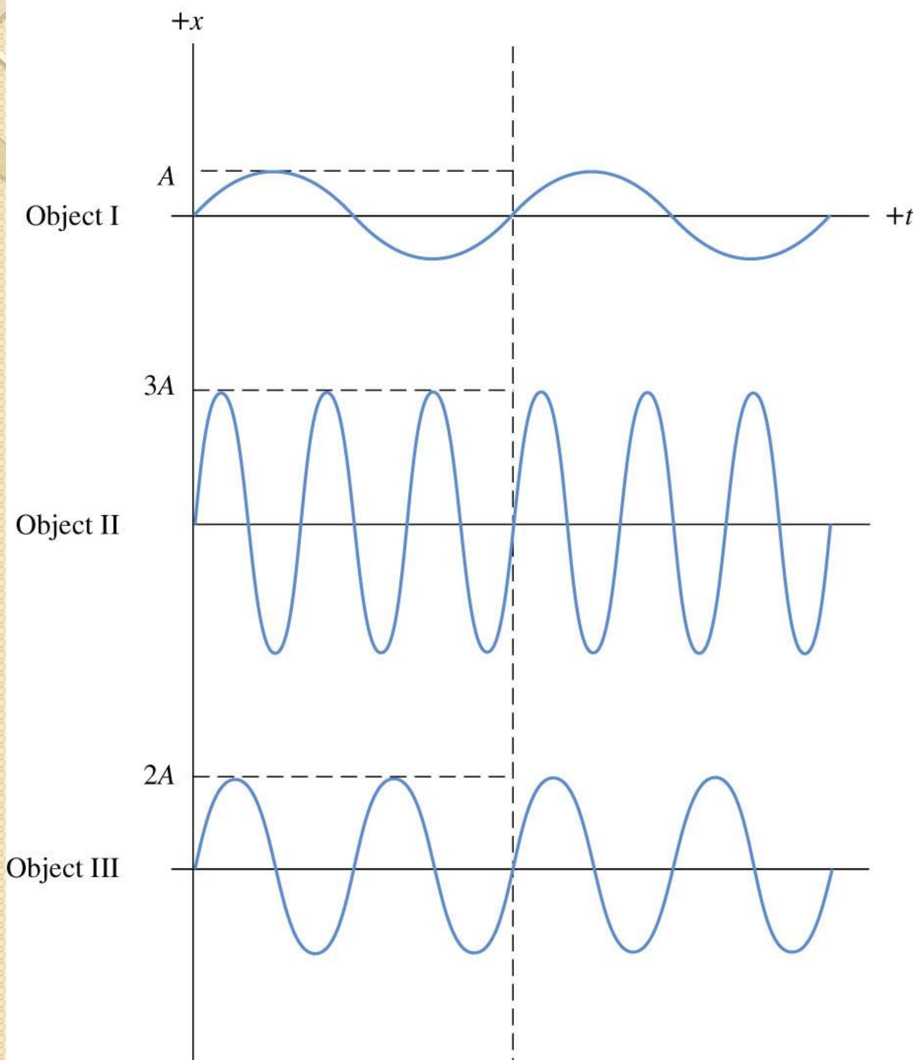
(a)

$$a_{\max} = A\omega^2 = A(2\pi f)^2 = (0.20 \times 10^{-3}\ \mathrm{m})\left[2\pi(1.0 \times 10^3\ \mathrm{Hz})\right]^2$$

$$= \boxed{7.9 \times 10^3\ \mathrm{m/s^2}}$$

(b) the maximum acceleration occurs at *x* = +*A* and *x* = –*A*

Check Your Understanding



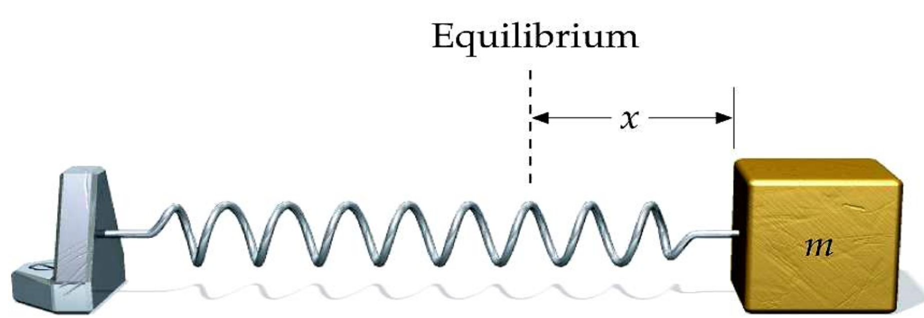
The drawing shows plots of the displacement *x* versus the time *t* for three objects undergoing simple harmonic motion. Which object, I, II, or III, has the greatest maximum velocity?

II

Example: A Block on a Spring

A 2.00 kg block is attached to a spring as shown. The force constant of the spring is $k = 196$ N/m. The block is held a distance of 5.00 cm from equilibrium and released at $t = 0$.

(a) Find the angular frequency ω, the frequency f, and the period T.
(b) Write an equation for x vs. time.



$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{(196\ \text{N/m})}{(2.00\ \text{kg})}} = 9.90\ \text{rad/s}$$

$$f = \frac{\omega}{2\pi} = \frac{(9.90\ \text{rad/s})}{2\pi} = 1.58\ \text{Hz}$$

$$T = 1/f = 0.635\ \text{s}$$

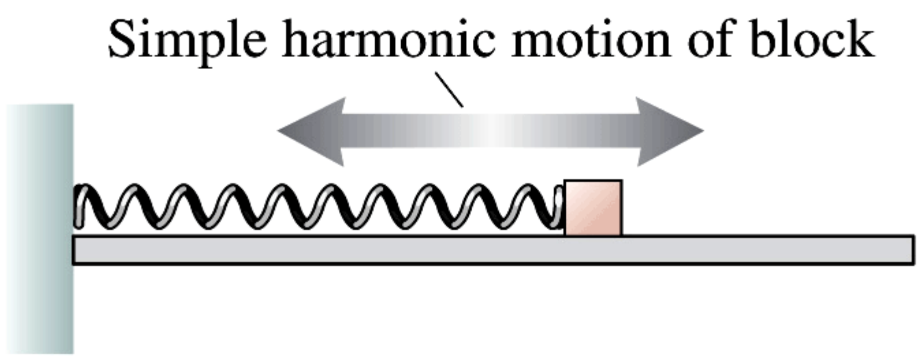
$$A = 5.00\ \text{cm} \ \text{ and } \ \delta = 0$$

$$\boxed{x = (5.00\ \text{cm})\cos\left[(9.90\ \text{rad/s})t\right]}$$

Example: A System in SHM

An air-track glider is attached to a spring, pulled 20 cm to the right, and released at t-=0. It makes 15 complete oscillations in 10 s.

a. What is the period of oscillation?
b. What is the object's maximum speed?
c. What is its position and velocity at t=0.80 s?



$$f = \frac{15 \text{ oscillations}}{10 \text{ s}}$$

$$= 1.5 \text{ oscillations/s} = 1.5 \text{ Hz}$$

$$T = 1/f = 0.667 \text{ s}$$

$$v_{\max} = \frac{2\pi A}{T} = \frac{2\pi(0.20 \text{ m})}{(0.667 \text{ s})} = 1.88 \text{ m/s}$$

$$x = A\cos\frac{2\pi t}{T} = (0.20 \text{ m})\cos\frac{2\pi(0.80 \text{ s})}{(0.667 \text{ s})} = 0.062 \text{ m} = 6.2 \text{ cm}$$

$$v = -v_{\max}\sin\frac{2\pi t}{T} = -(1.88 \text{ m/s})\sin\frac{2\pi(0.80 \text{ s})}{(0.667 \text{ s})} = -1.79 \text{ m/s}$$

Example: Finding the Time

A mass, oscillating in simple harmonic motion, starts at $x = A$ and has period T.

At what time, as a fraction of T, does the mass first pass through $x = \frac{1}{2}A$?

$$x = \tfrac{1}{2}A = A\cos\frac{2\pi t}{T}$$

$$t = \frac{T}{2\pi}\cos^{-1}\left(\tfrac{1}{2}\right) = \frac{T}{2\pi}\frac{\pi}{3} = \tfrac{1}{6}T$$

Ex. A particle execute s simple harmonic motion given by the equation

$$y = 12\sin(\frac{2\pi t}{10} + \frac{\pi}{4})$$

Calculate (i) amplitude, (ii) frequency, (iii) displacement at t= 1.25s, (iv) velocity at t= 2.5s (v) acceleration at t= 5s.

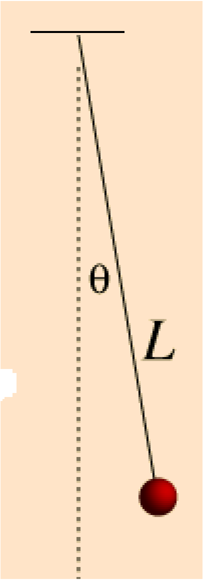
Ex. A particle execute s simple harmonic motion given by the equation

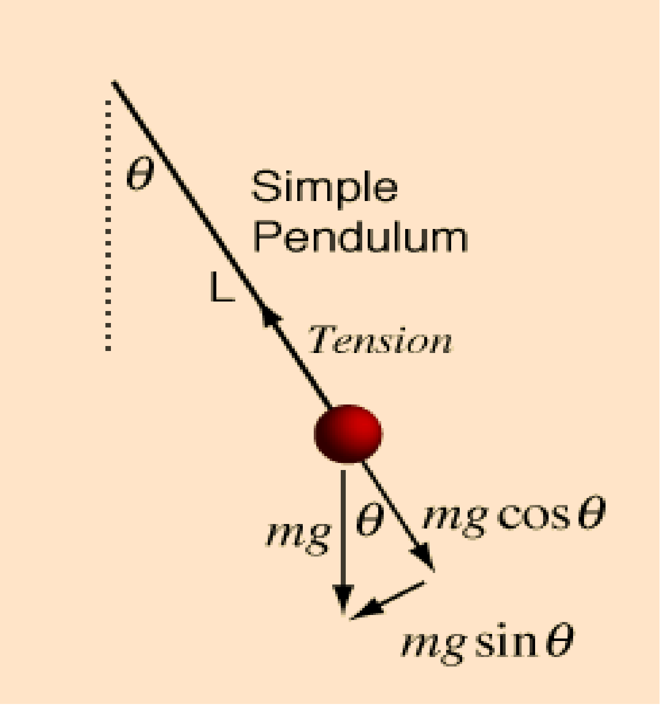
$$y = 10\sin(10t - \frac{\pi}{6})$$

Calculate (i) frequency, (ii) time period (iii) the maximum displacement (iv)the maximum velocity (v) the maximum acceleration acceleration.

Simple Pendulum

A *simple pendulum* consists of a particle of mass *m*, attached to a frictionless point by a cable of length *L* and negligible mass.





From the above figure restoring force

$$F = -mg\sin\theta$$

If the angle θ is very small sinθ is very nearly equal to θ. The displacement along the arc is

$$x = L\theta$$

Therefore, $$F = -mg\theta$$

Acceleration $\frac{d^2x}{dt^2} = L\frac{d^2\theta}{dt^2}$

$$Force = mL\frac{d^2\theta}{dt^2}$$

$$mL\frac{d^2\theta}{dt^2} = -mg\,\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Where

$$\omega^2 = \frac{g}{L}$$

And

$$T = 2\pi\sqrt{\frac{L}{g}}$$

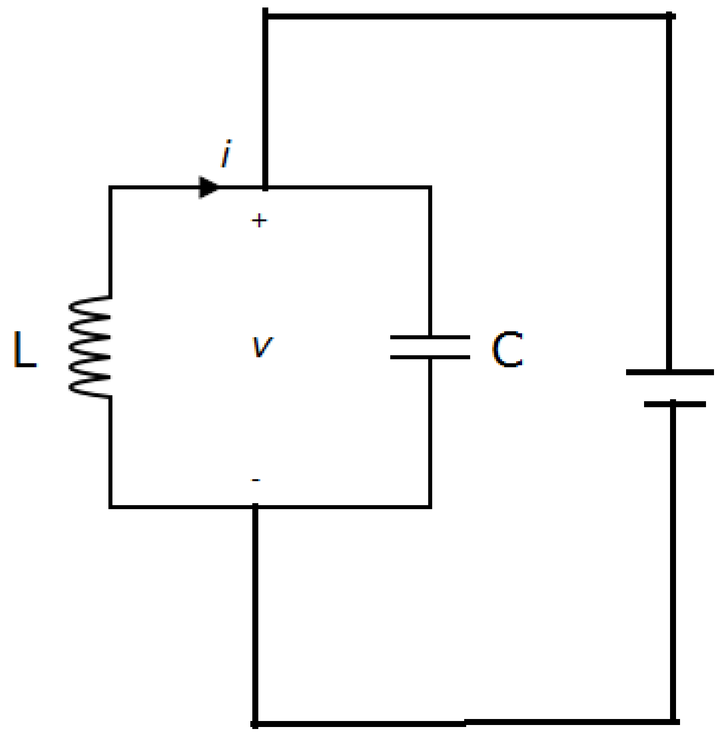
Example 10. Keeping Time

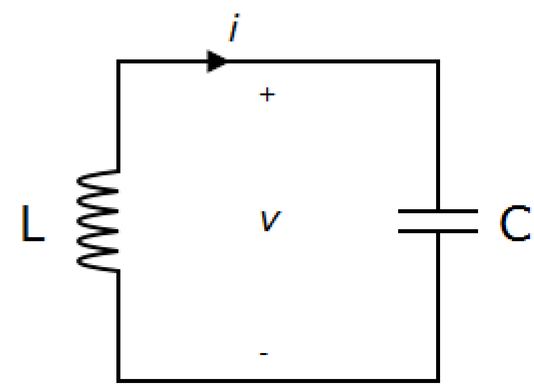
Determine the length of a simple pendulum that will swing back and forth in simple harmonic motion with a period of 1.00 s.

$$f = \frac{1}{2\pi}\sqrt{g/L}$$

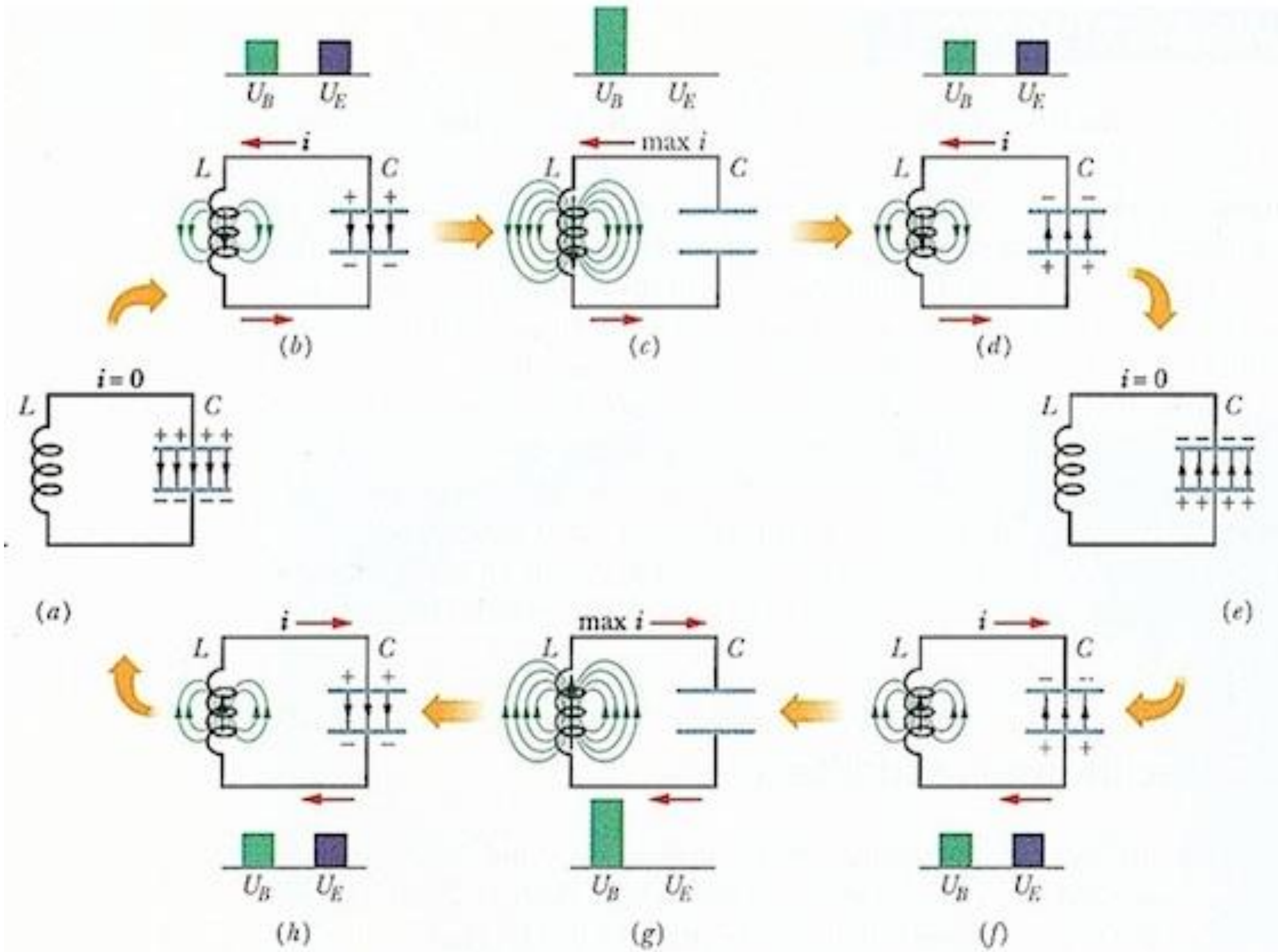
$$L = \frac{T^2 g}{4\pi^2} = \frac{(1.00\,\mathrm{s})^2(9.80\,\mathrm{m/s^2})}{4\pi^2} = \boxed{0.248\,\mathrm{m}}$$

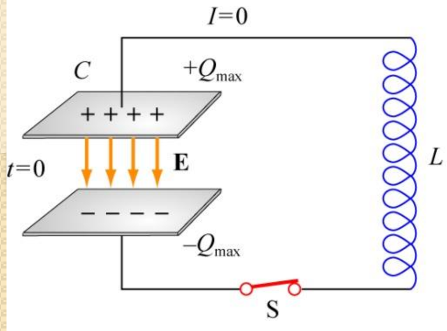
LC Circuit

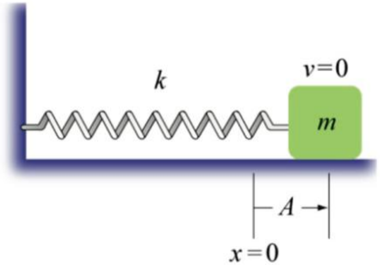




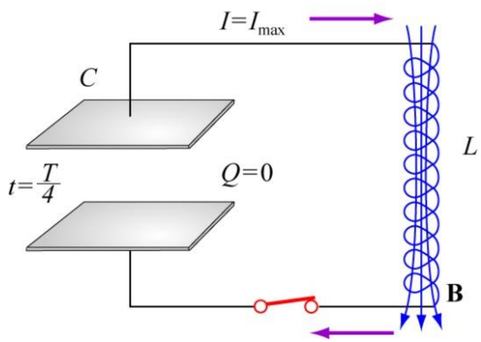
Charging discharging of an LC Circuit

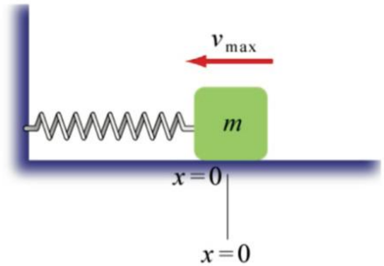






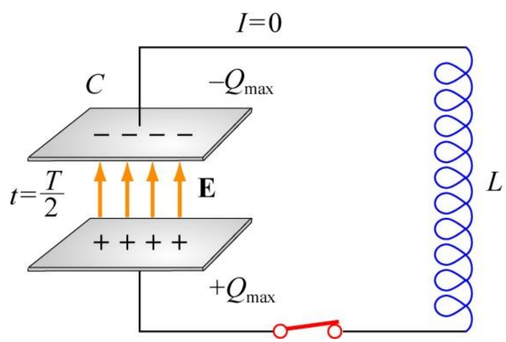
Energy stored
in electric field

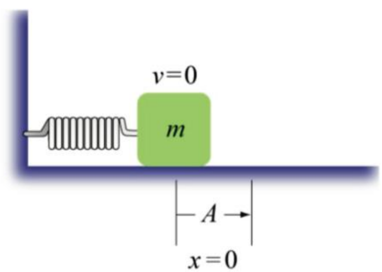




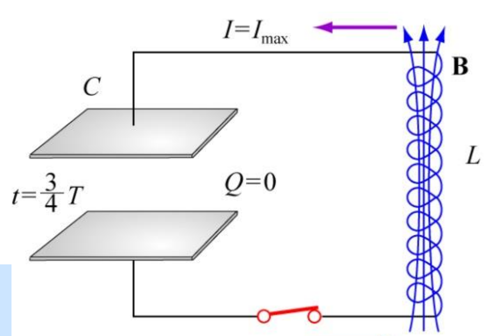
Energy stored in
magnetic field

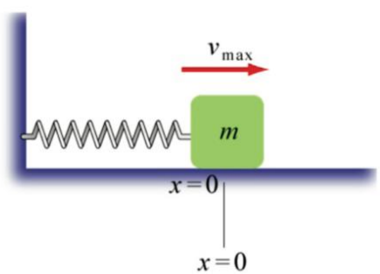
Energy stored
in electric field





Energy stored in
magnetic field





An **LC circuit**, also called a **resonant circuit**, **tank circuit**, or **tuned circuit**, consists of an inductor, represented by the letter L, and a capacitor, represented by the letter C. When connected together, they can act as an electrical resonator or oscillator.

Voltage across capacitor at any instant

$$V_C = \frac{Q}{C}$$

Q is the charge on the capacitor and C is capacitance of capacitor.

Voltage across inductor at the same instant

$$V_L = L\frac{di}{dt}$$

$$\frac{Q}{C} + L\frac{di}{dt} = 0$$ Kirchhoff's voltage law

$$\frac{d^2 i}{dt^2} + \frac{1}{LC} i = 0$$

Similar to differential equation of SHM

$$\frac{d^2 x}{dt^2} + \omega_0 x = 0, \qquad {\omega_0}^2 = \frac{1}{LC}$$

Time Period

$$T = 2\pi\sqrt{LC}$$

Frequency

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Solution of the differential equation is

$$i(t) = -i_0 \sin(\omega_0 t + \phi)$$

Current in the circuit

$$i(t) = -i_0 \sin(\omega_0 t + \phi)$$

Mechanical	Electrical
displacement x	charge Q
velocity v	current I
mass m	inductance L
spring constant k	$\frac{1}{\text{capacitance}}$ $\frac{1}{C}$
$\omega_0 = \sqrt{\frac{k}{m}}$	$\omega_0 = \sqrt{\frac{1}{LC}}$
potential energy: $\frac{1}{2}kx^2$	Electric energy stored in capacitor: $\frac{1}{2}\frac{Q^2}{C}$
kinetic energy: $\frac{1}{2}mv^2$	Magnetic energy stored in inductor: $\frac{1}{2}LI^2$

Damped oscillations

We have thus far neglected all dissipative mechanisms

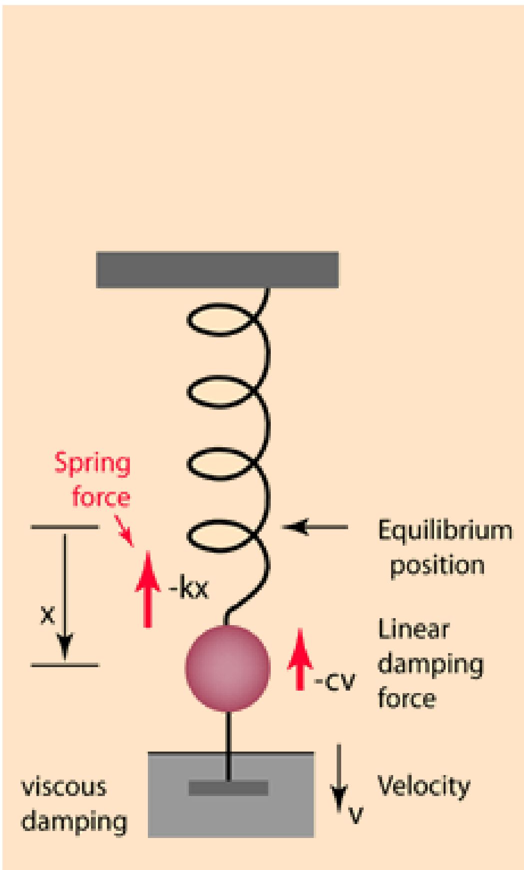
… our oscillations can continue oscillating with the same amplitude forever …

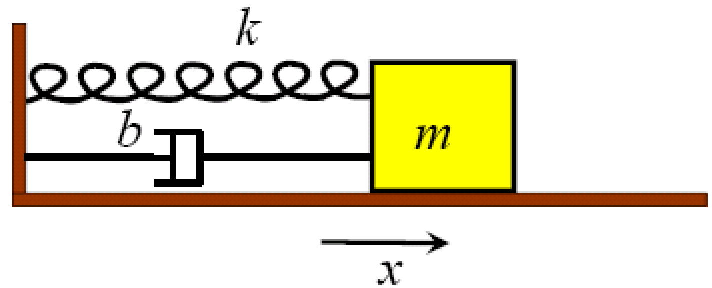
Various physical damping mechanisms will contribute towards the damping…

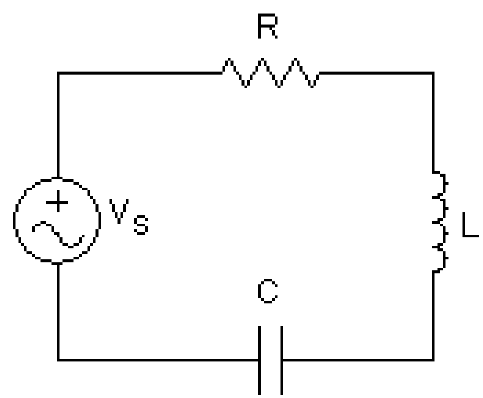
- friction between mass and table
- "air resistance"
- internal friction in spring
- …….

… model these by introducing a damping force which is proportional to the velocity of the oscillator …

Examples some damped oscillating systems

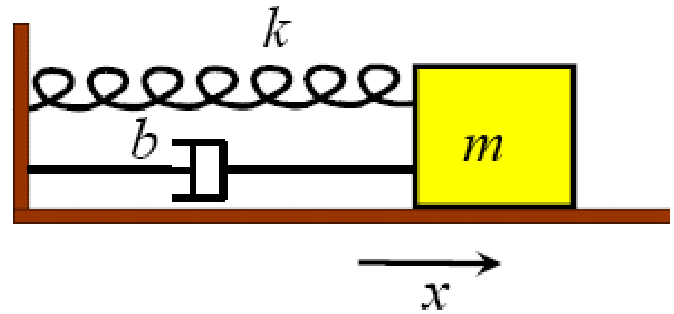






Damped Harmonic motion: When oscillating bodies do not move back and forth between Precisely fixed limits because frictional force dissipate the energy and amplitude of oscillation Decreases with time and finally die out. Such harmonic motion is called Damped Harmonic Motion.

Damped mass-spring system



In theses systems the damping force

$$F' = -bv$$

For horizontal forces on the mass: $ma = -kx - bv$

or $$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

or $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0{}^2 x = 0$$ where $$\begin{cases} \omega_0 = \sqrt{\dfrac{k}{m}} \\ \gamma = \dfrac{b}{m} \end{cases}$$

γ : "damping constant" unit: s^{-1}

- "life time" $= \dfrac{1}{\gamma}$

Damped oscillator equation

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0{}^2x = 0$$

Let $x = Be^{pt}$

Then $\dfrac{dx}{dt} = Bpe^{pt}$ and $\dfrac{d^2x}{dt^2} = Bp^2e^{pt}$

Substituting into DE: $Bp^2e^{pt} + \gamma Bpe^{pt} + \omega_0{}^2Be^{pt} = 0$

Thus $p^2 + \gamma p + \omega_0{}^2 = 0$

$$\therefore p = \tfrac{1}{2}\left\{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0{}^2}\right\}$$

or $p = -\dfrac{\gamma}{2} \pm \sqrt{\dfrac{\gamma^2}{4} - \omega_0{}^2}$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Damped oscillator equation …2

$$p = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - {\omega_0}^2}$$

We can distinguish three cases:

(i) $\quad {\omega_0}^2 > \dfrac{\gamma^2}{4}$ **Oscillatory behaviour**

(ii) $\quad {\omega_0}^2 = \dfrac{\gamma^2}{4}$ **Critical damping**

(iii) $\quad {\omega_0}^2 < \dfrac{\gamma^2}{4}$ **Overdamping**

$$\boxed{\text{Case (i):} \qquad \omega_0^{\,2} > \frac{\gamma^2}{4}}$$

$$\therefore \sqrt{\gamma^2/4 - \omega_0^{\,2}} = \sqrt{-\left(\omega_0^{\,2} - \gamma^2/4\right)}$$

Put $\quad \omega_1^{\,2} = \omega_0^{\,2} - \gamma^2/4$

$$\therefore p = -\frac{\gamma}{2} \pm \sqrt{-\omega_1^{\,2}} = -\frac{\gamma}{2} \pm j\omega_1$$

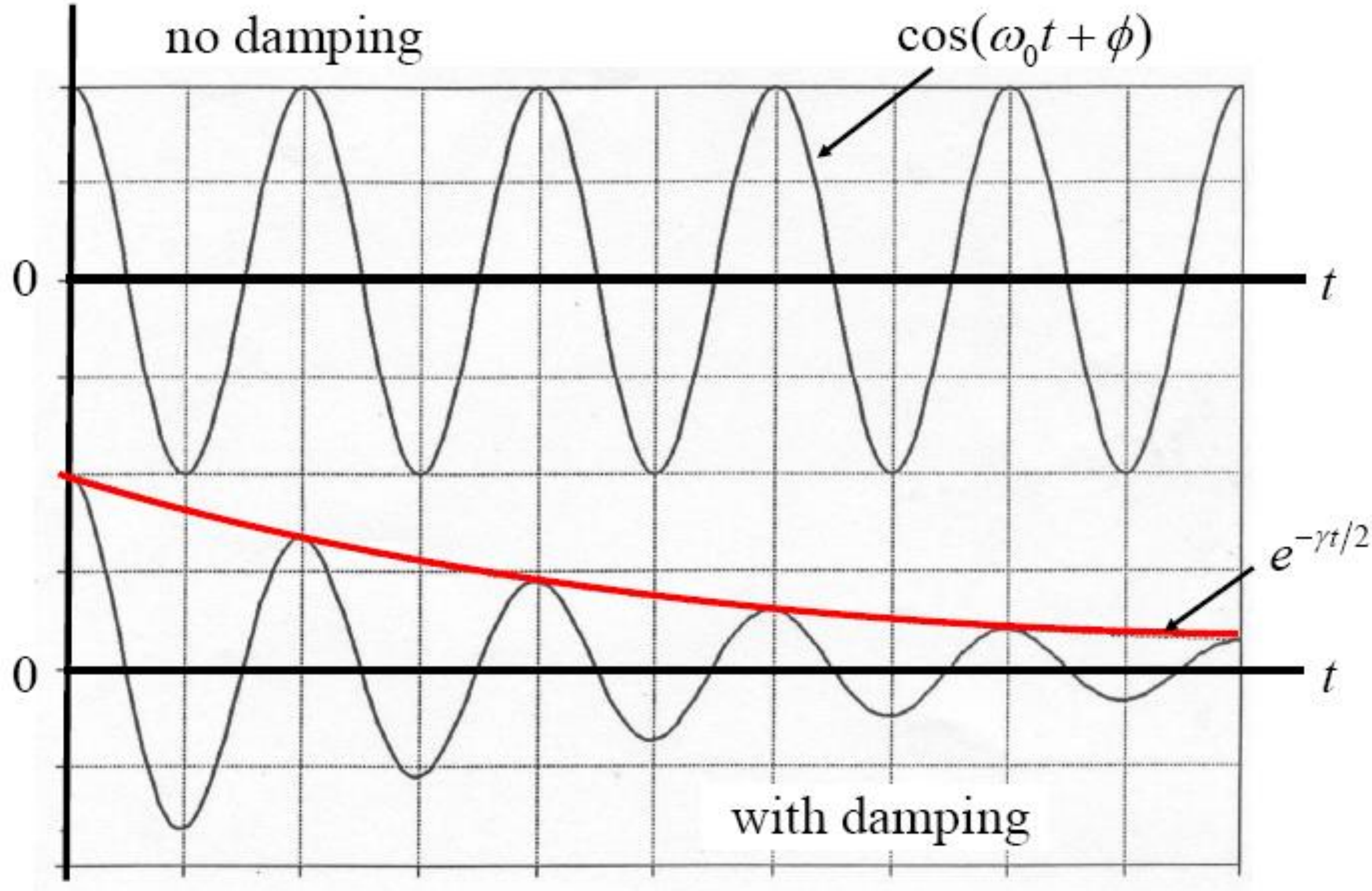
The solution will be:

$$x = B_1 e^{\left(-\frac{\gamma}{2}+j\omega_1\right)t} + B_2 e^{\left(-\frac{\gamma}{2}-j\omega_1\right)t} = e^{-\frac{\gamma}{2}t}\left\{B_1 e^{j\omega_1 t} + B_2 e^{-j\omega_1 t}\right\}$$

… leading to $\quad x(t) = Ae^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \phi)$

This is an **oscillatory solution** $\; A\cos(\omega_1 t + \phi) \;$ multiplied by a damping factor $\; e^{-\gamma t/2}$.

As $\;\gamma \to 0\;$ we approach our undamped oscillator.


no damping
$\cos(\omega_0 t + \phi)$
0
t
$e^{-\gamma t/2}$
0
t
with damping

Case (ii): $\omega_0{}^2 = \dfrac{\gamma^2}{4}$

The two roots coincide: $p = -\dfrac{\gamma}{2}$

The solution will be $x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$

The condition $\omega_0{}^2 = \gamma^2/4$ is referred to as the "**critical damping**" condition.

If $\omega_0{}^2 < \gamma^2/4$ a system released from rest will oscillate.

As γ is increased the oscillations decay more rapidly, until at $\omega_0{}^2 = \gamma^2/4$ oscillation no longer occurs.

[… many practical applications …]

$$\boxed{\text{Case (iii):} \quad \omega_0^{\ 2} < \frac{\gamma^2}{4}}$$

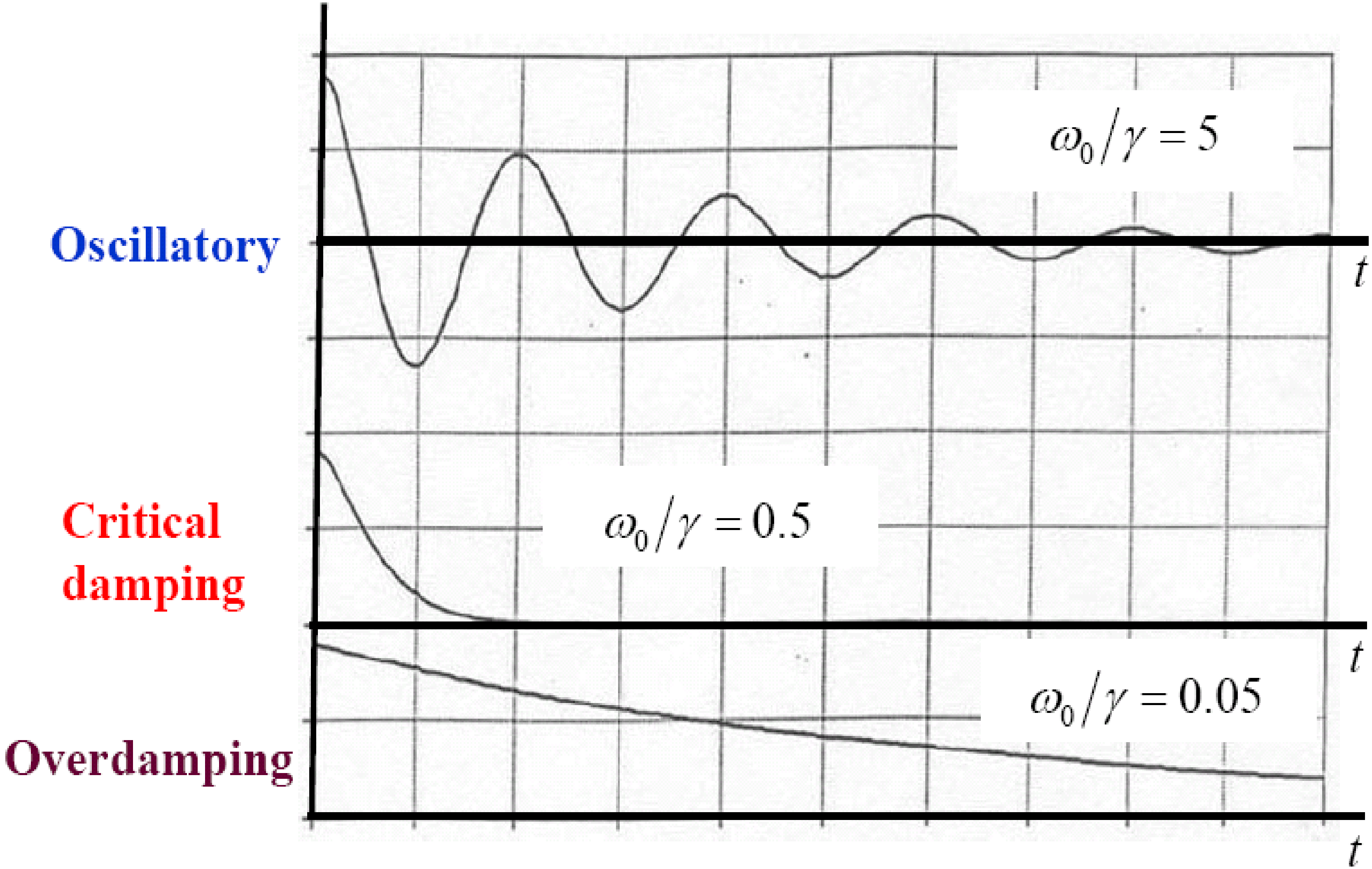
$$p = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^{\ 2}}$$

$$= -\frac{\gamma}{2} \pm \lambda \quad \text{say}$$

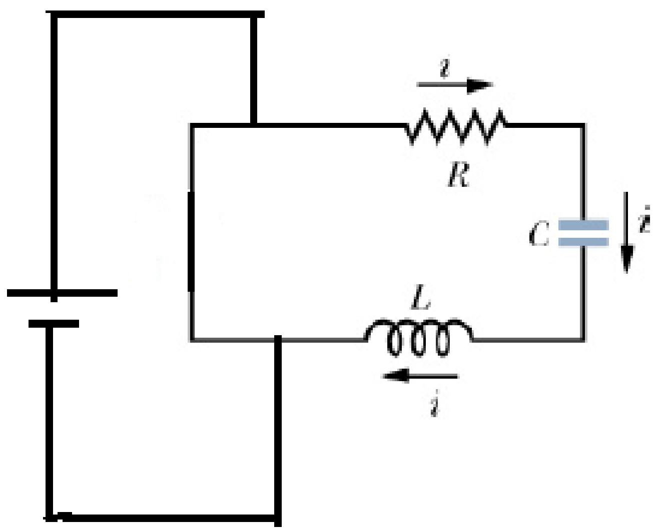
The solution will be $x(t) = B_1 e^{\left(-\frac{\gamma}{2}+\lambda\right)t} + B_2 e^{\left(-\frac{\gamma}{2}-\lambda\right)t}$

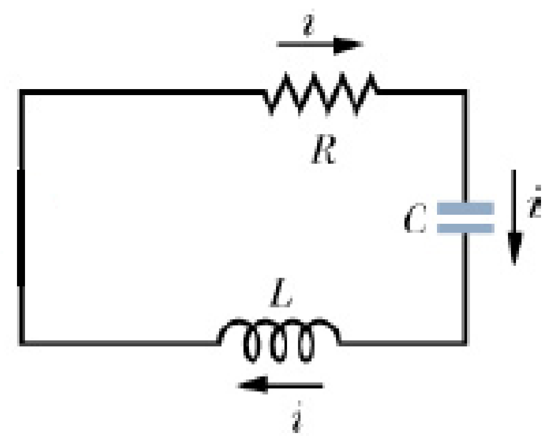
The condition $\omega_0^{\ 2} < \dfrac{\gamma^2}{4}$ is referred to as **overdamping**

… a slower approach to the rest position is observed.


Oscillatory
$\omega_0/\gamma = 5$
t
Critical damping
$\omega_0/\gamma = 0.5$
t
Overdamping
$\omega_0/\gamma = 0.05$
t

RLC circuit





- Voltage across resistor R $V_R = iR$
- Voltage across capacitor C $V_C = \frac{Q}{C}$
- Voltage across inductor L $V_L = L\frac{di}{dt}$
- According to Kirchhoff's voltage law

$$iR + \frac{Q}{C} + L\frac{di}{dt} = 0$$

Rewrite the equation

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

Comparing with the equation

$$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0{}^2 x = 0$$

Where

$$\gamma = \frac{R}{L} \qquad \omega_0 = \sqrt{\frac{1}{LC}}$$

Three distinguish cases are

i) $\frac{1}{LC} > \frac{R^2}{4L^2}$ Oscillatory behavior

ii) $\frac{1}{LC} = \frac{R^2}{4L^2}$ Critical damping

iii) $\frac{1}{LC} < \frac{R^2}{4L^2}$ Over damping

Case i) $\frac{1}{LC} > \frac{R^2}{4L^2}$

Solution of the differential equation

$$Q(t) = Ae^{-\frac{R}{2L}t}\cos(\omega_1 t + \phi)$$

Where $$\omega_1 = \sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

Frequency of oscillation

$$f = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

EX. A capacitor $1.0\mu F$, an inductor $0.2h$ and a resistance 800Ω are joined in series. Is the circuit oscillatory? Find the frequency of oscillation.

Ex. Find whether the discharge of capacitor through the following inductive circuit is oscillatory.

$C = 0.1\mu F$, $L = 10mh$, $R\ 200\ \Omega$

If Oscillatory, find the frequency of oscillation.

Combination of two vibrations at right angles

$$x = A_1 \cos(\omega_1 t + \phi_1)$$

$$y = A_2 \cos(\omega_2 t + \phi_2)$$

Consider case where frequencies are equal and let initial phase difference be ϕ

Write $x = A_1 \cos(\omega_0 t)$ and $y = A_2 \cos(\omega_0 t + \phi)$

Case 1 : $\phi = 0$ $\left.\begin{aligned} x &= A_1 \cos(\omega_0 t) \\ y &= A_2 \cos(\omega_0 t) \end{aligned}\right\} \; y = \dfrac{A_2}{A_1} x$ Rectilinear motion

Case 2 : $\phi = \pi/2$ $x = A_1 \cos(\omega_0 t)$

$$y = A_2 \cos(\omega_0 t + \pi/2) = -A_2 \sin(\omega_0 t)$$

$$\therefore \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$$

Elliptical path in clockwise direction

Combination of two vibrations at right angles …2

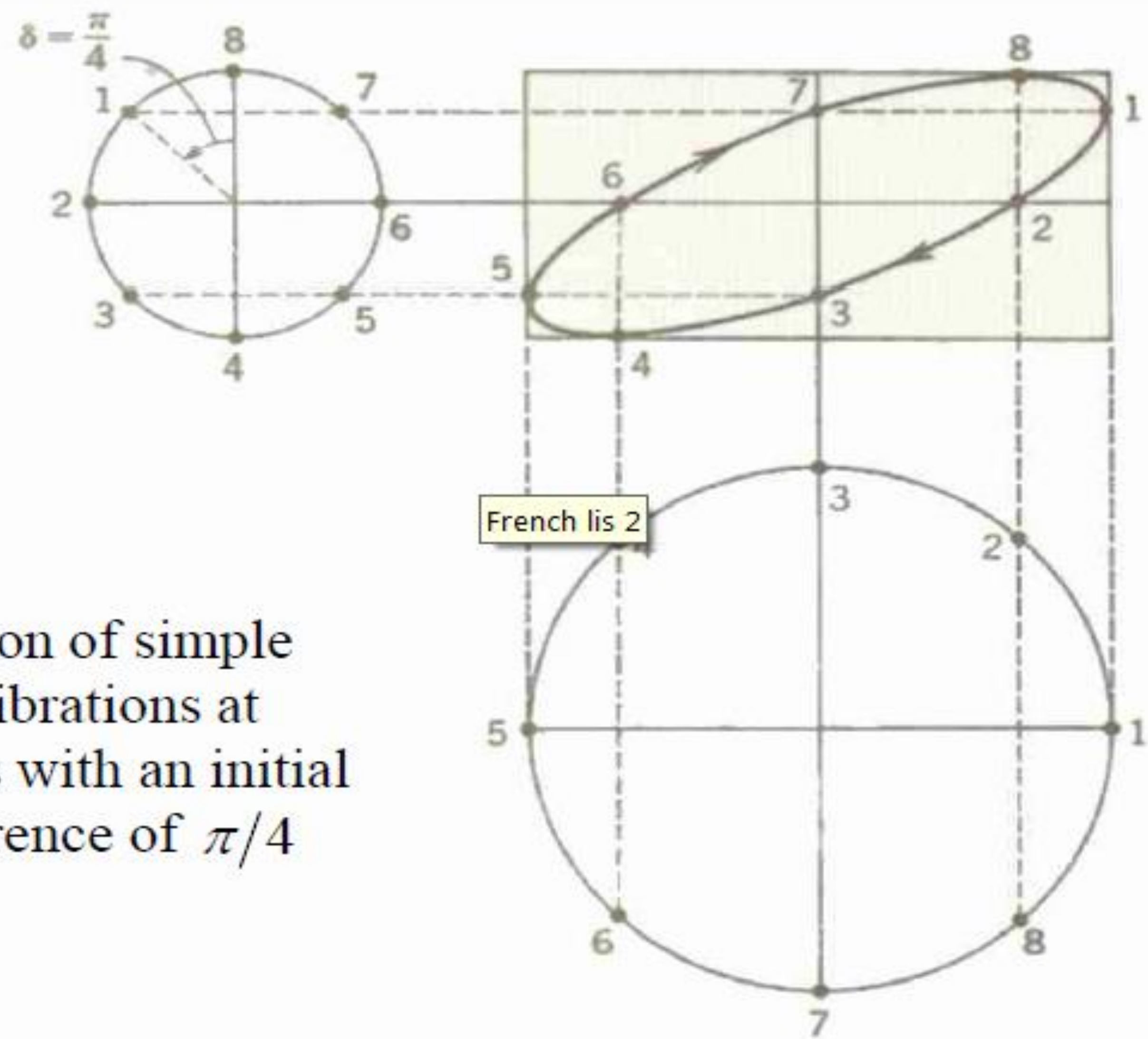
Case 3 : $\phi = \pi$
$$\left.\begin{aligned} x &= A_1\cos(\omega_0 t) \\ y &= A_2\cos(\omega_0 t + \pi) = -A_2\cos(\omega_0 t) \end{aligned}\right\} \quad y = -\frac{A_2}{A_1}x$$

Case 4 : $\phi = 3\pi/2$
$$x = A_1\cos(\omega_0 t)$$
$$y = A_2\cos(\omega_0 t + 3\pi/2) = +A_2\sin(\omega_0 t)$$
$$\therefore \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = -1$$

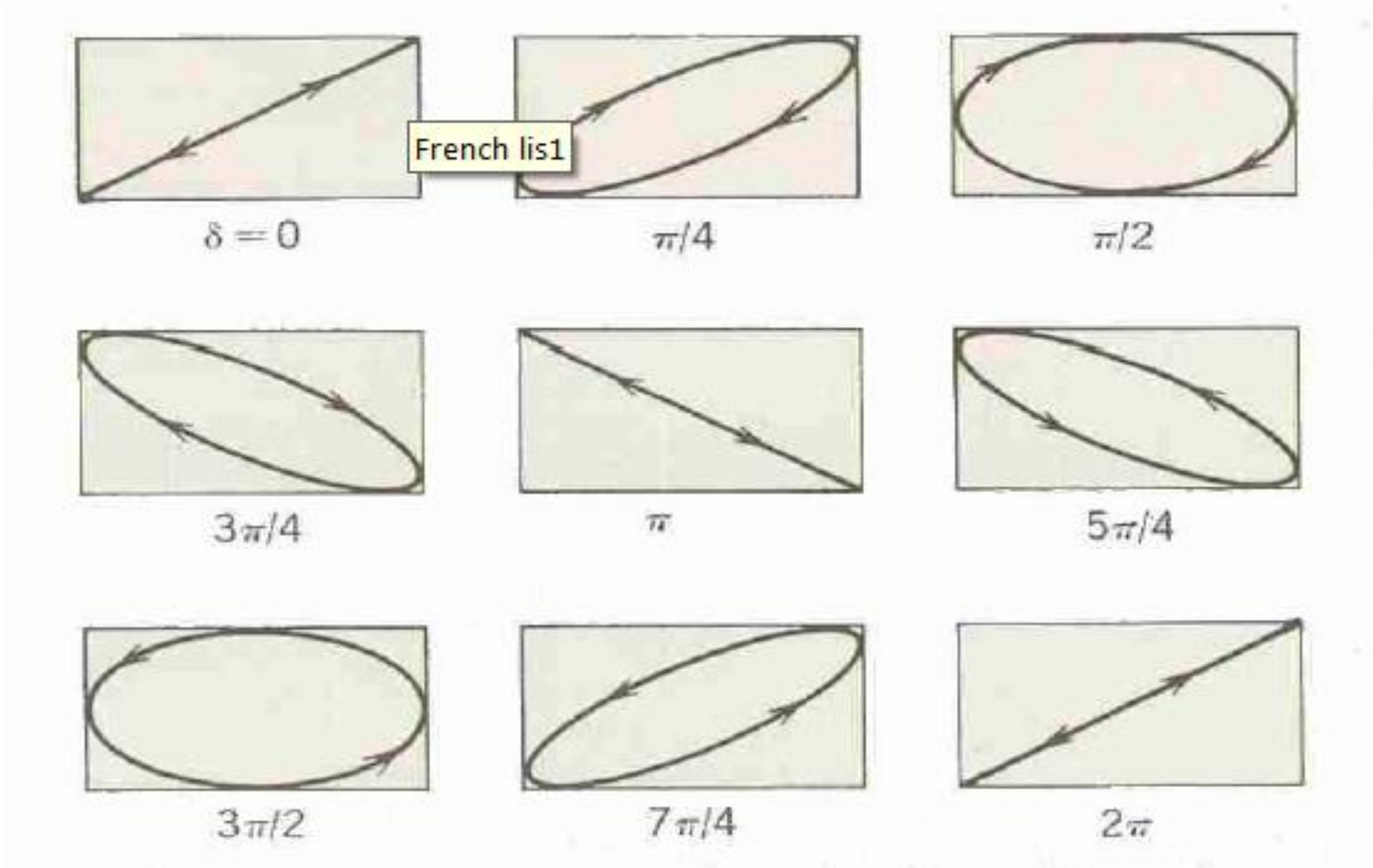
Elliptical path in anticlockwise direction

Case 5 : $\phi = \pi/4$
$$x = A_1\cos(\omega_0 t)$$
$$y = A_2\cos(\omega_0 t + \pi/4)$$

Harder to see … use a graphical approach …



Superposition of simple harmonic vibrations at right angles with an initial phase difference of $\pi/4$



Superposition of two perpendicular simple harmonic motions of the same frequency for various initial phase differences.

Lssajous' Figures: When particle is influenced simultaneously by two simple harmonic motion at right angles to each Other , the resultant motion of the particle traces a curve. This curves are called Lissajous' figures. The shape of the curves Depend on the time period, phase difference and amplitude of the constituent vibrations.

Wave Motion

- ✦ Differential Equation of Wave motion
- ✦ Progressive and Standing Waves
- ✦ Group and Phase Velocity
- ✦ Power and Intensity of Wave Motion

Review: Simple Harmonic Motion

- The position x of an object moving in simple harmonic motion as a function of time has the following form:

$$x = A \cos(\omega t + \phi)$$

i.e. the object periodically moves back and forth between the amplitudes $x=+A$ and $x=-A$.

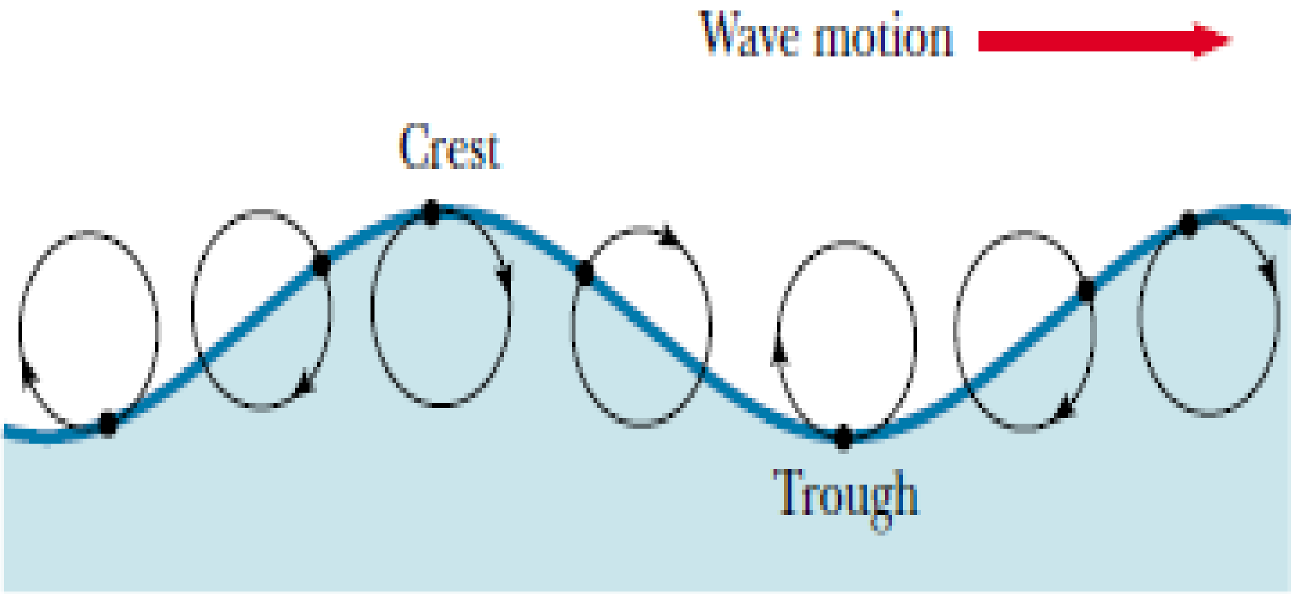
The time it takes for the object to make one full cycle is the period $T=2\pi/\omega=1/f$, where f is the frequency of the motion.

Thus, the angular speed in terms of T and f reads

$$\boldsymbol{\omega = 2\pi/T \quad \text{and} \quad \omega = 2\pi f}$$

What is a wave ?

- Nature of waves:
  - A wave is a traveling disturbance that transports energy from place to place.
  - There are two basic types of waves: transverse and longitudinal.
  - Transverse: the disturbance travels perpendicular to the direction of travel of the wave.
  - Longitudinal: the disturbance occurs parallel to the line of travel of the wave.


Wave motion
Crest
Trough

- Examples:
  - ➔ Longitudinal: Sound waves (e.g. air moves back & forth)
  - ➔ Transverse: Light waves (electromagnetic waves, i.e. electric and magnetic disturbances)

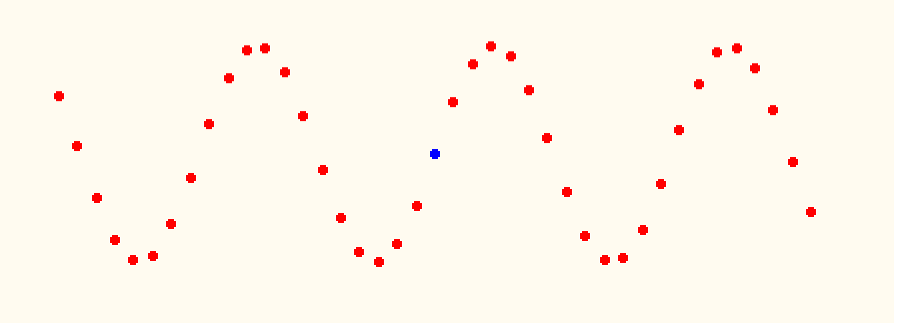
The source of the wave, i.e. the disturbance, moves continuously in

simple harmonic motion, generating an entire wave, where each part of the wave also performs a simple harmonic motion.

Types of Waves

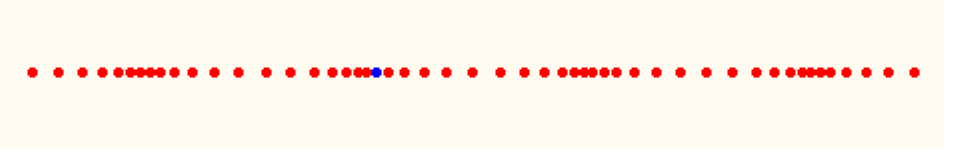
- **Transverse:** The medium oscillates perpendicular to the direction the wave is moving.

  ➔ Water waves

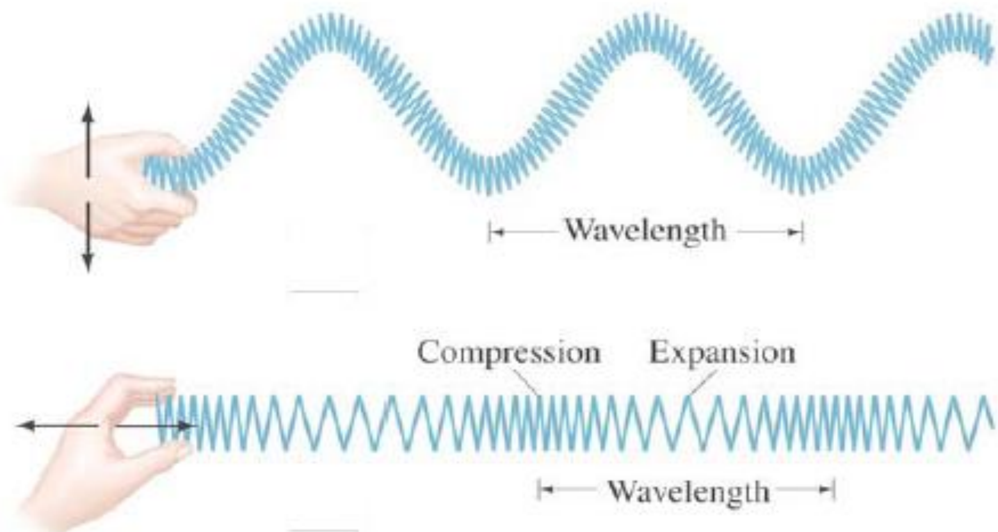


- **Longitudinal:** The medium oscillates in the same direction as the wave is moving

  ➔ Sound



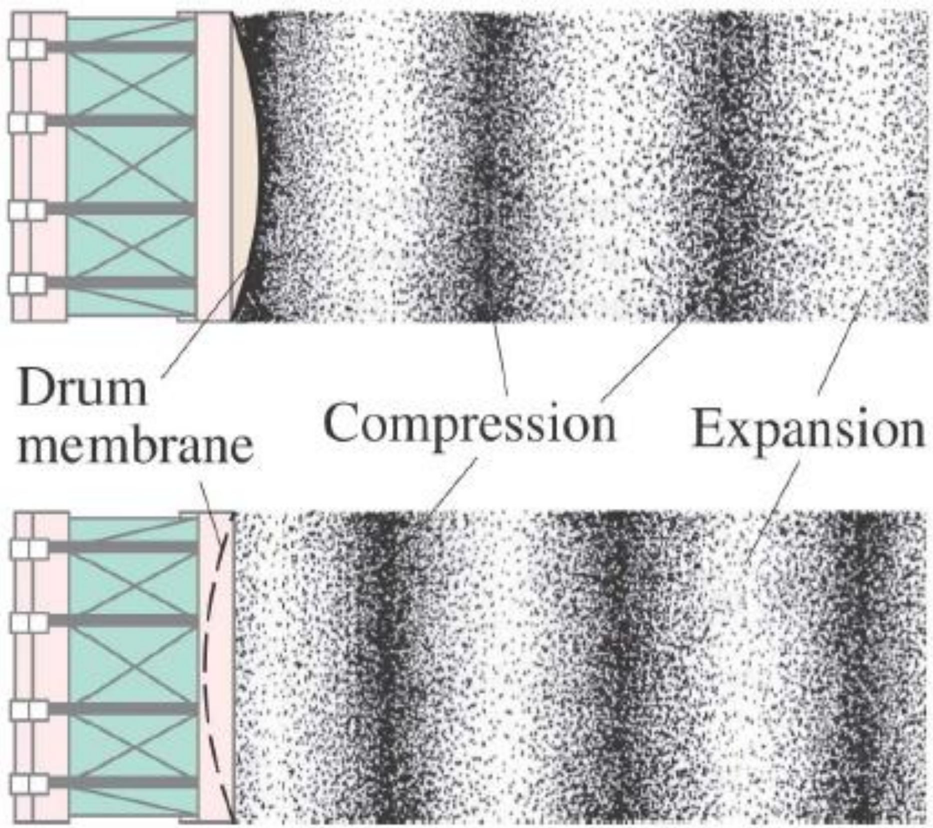
Types of Waves: Transverse and Longitudinal



The motion of particles in a wave can be either **perpendicular** to the wave direction **(transverse)** or **parallel** to it **(longitudinal)**.

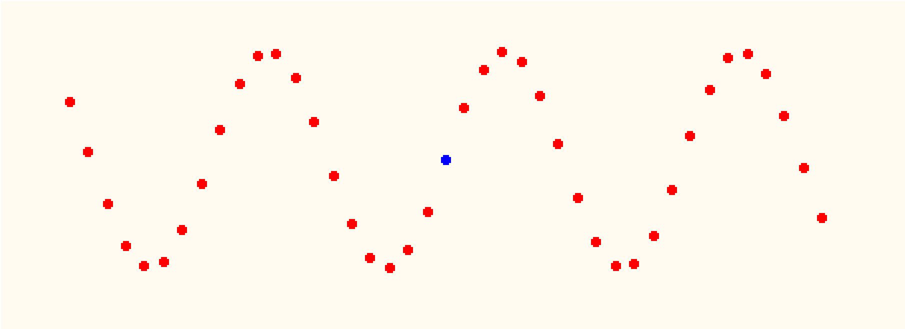
Types of Waves: Transverse and Longitudinal

Sound waves are longitudinal waves:



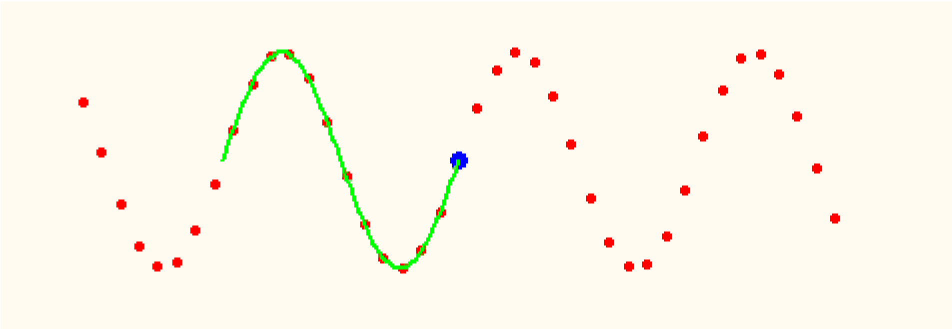
Wave Properties...

- Period: The time T for a point on the wave to undergo one complete oscillation.



- Speed: The wave moves one wavelength λ in one period T so its speed is $v = \lambda / T$.

$$\boxed{v = \frac{\lambda}{T}}$$



Wave Properties...

- The speed of a wave is a *constant* that depends only on the medium, not on the amplitude, wavelength or period:

  λ and T are related !

  $\lambda = vT$ or $\lambda = 2\pi v / \omega$ (since $T = 2\pi / \omega$)

  or $\lambda = v / f$ (since $T = 1/ f$)

- Recall f = cycles/sec or revolutions/sec

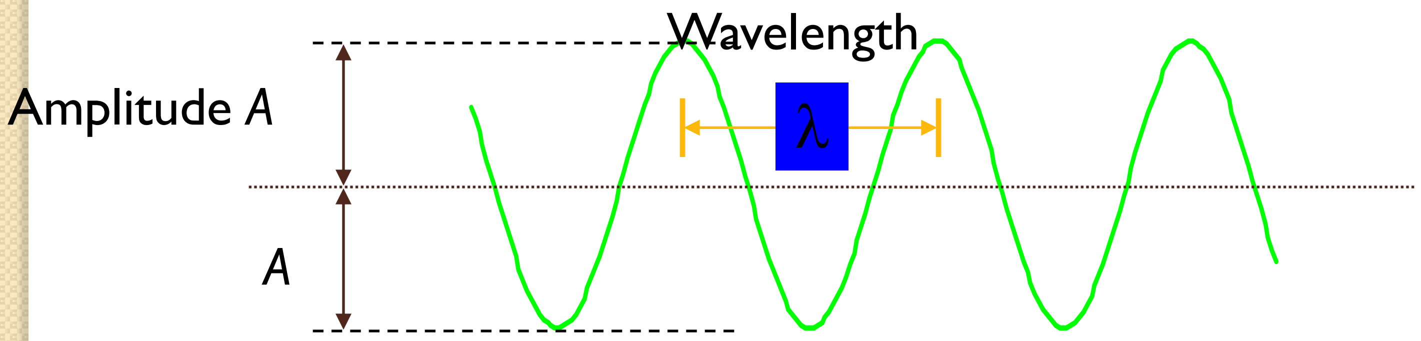
  $\omega = 2\pi f$

Is the speed of a wave particle the same as the speed of the wave ?

No. Wave particle performs simple harmonic motion: $v = A\,\omega \sin \omega t$.

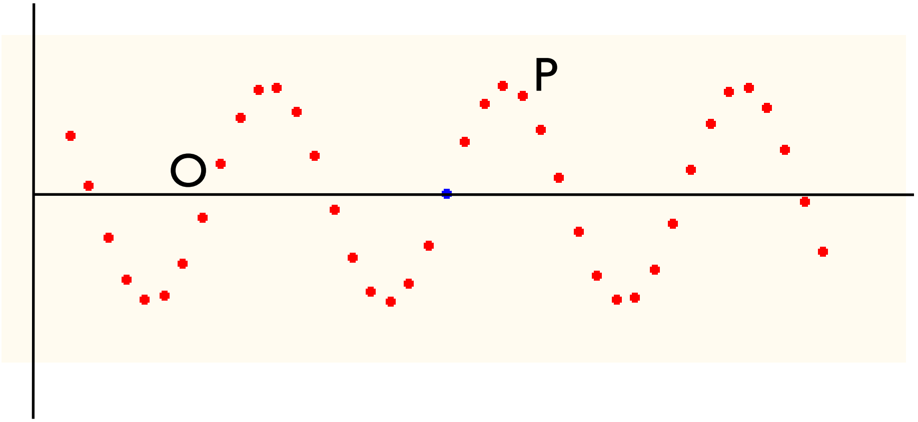
Wave Properties

- Amplitude: The maximum displacement A of a point on the
- wave.
- Wavelength: The distance λ between identical points
- on the wave.



Equation for a Progressive Wave

The simplest type of wave is the one in which the particles of the medium are set into simple harmonic vibrations as the wave passes through it. The wave is then called a simple harmonic wave.



Consider a particle O in the medium.
The displacement at any instant of time is given by

$$y = A \sin \omega t \ldots\ldots\ldots (1)$$

Where A is the amplitude, ω is the angular frequency of the wave. Consider a particle P at a distance x from the particle O on its right. Let the wave travel with a velocity v from left to right. Since it takes some time for the disturbance to reach P, its displacement can be written as

$$y = A \sin(\omega t - \phi) \ldots\ldots\ldots(2)$$

Where ϕ is the phase difference between the particles O and P.

We know that a path difference of λ corresponds to a phase difference of 2π radians. Hence a path difference of x corresponds to a phase difference of

$$\frac{2\pi}{\lambda} . x$$

$$\phi = \frac{2\pi x}{\lambda}$$

Displacement of particle P is

$$y = A\sin(\omega t - \frac{2\pi x}{\lambda})\ldots\ldots\ldots(3)$$

But $\omega = \frac{2\pi}{T}$

$$y = A\sin(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda})\ldots\ldots\ldots(5)$$

$$y = A\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

But $v = \frac{\lambda}{T}$ or $T = \frac{\lambda}{v}$

$$y = A\sin 2\pi\left(\frac{vt}{\lambda} - \frac{x}{\lambda}\right)$$

$$y = A\sin\frac{2\pi}{\lambda}(vt - x)\ldots\ldots\ldots\ldots(6)$$

Similarly, for a particle at a distance x to the left of 0, the equation for the displacement is given by

$$y = A\sin\frac{2\pi}{\lambda}(vt + x)\ldots\ldots\ldots\ldots(7)$$

Differential equation for wave motion

We have wave equation

$$y = A\sin\frac{2\pi}{\lambda}(vt - x)\ldots\ldots\ldots(1)$$

Differentiating equation with respect to x,
We get,

$$\frac{dy}{dx} = A\frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x)\ldots\ldots\ldots(2)$$

$\frac{dy}{dx}$ represents the strain or the compression. When $\frac{dy}{dx}$ is positive, a rarefaction takes place and when $\frac{dy}{dx}$ is negative, a compression takes place.

The velocity of the particle whose displacement y is represented by equation, is obtained by differentiating it with respect to t, since velocity is the rate of change of displacement with respect to time.

$$\frac{dy}{dt} = A\frac{2\pi v}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x)\ldots\ldots\ldots(3)$$

Comparing equations (2) and (3) we get,

$$\frac{dy}{dt} = v\frac{dy}{dx} \quad \text{..........}(4)$$

Particle velocity = wave velocity x slope of the displacement curve or strain.

Differentiating equation (2)

$$\frac{d^2y}{dx^2} = -A\frac{4\pi^2}{\lambda^2}\sin\frac{2\pi}{\lambda}(vt - x) \quad \text{......... .}(5)$$

Differentiating equation (3)

$$\frac{d^2y}{dt^2} = -A\frac{4\pi^2v^2}{\lambda^2}\sin\frac{2\pi}{\lambda}(vt-x)\text{......... .}(6)$$

Comparing equs. (5) and (6)

$$\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}\text{.........}(\ 7)$$

Equation (7) represents the differential equation of wave velocity

Ex. The equation of a traveling wave is

$$y = 4.0\sin\pi(0.10x - 2t)$$

Find (i) wavelength, (ii) speed and (iii) frequency of oscillating particle of the wave

Ex. When a simple harmonic wave is propagated through a medium, the displacement of a paricle in cm at any instant is

$$y = 10\sin\frac{2\pi}{100}(36000t - 20)$$

Calculate the amplitude, wave velocity, wavelength, frequency and period of the oscillating particle.

- Ex. The equation of a traveling wave is

$$y = 4.0 \sin \pi (0.10x - 2t)$$

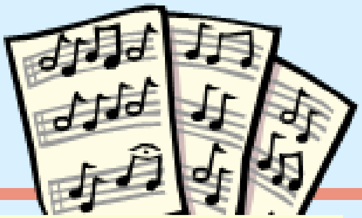
Find (i) amplitude (ii) wavelength (iii) speed (iv) frequency of wave

When a simple harmonic wave is propagated through a medium, the displacement of the particle at any instant of time is given by

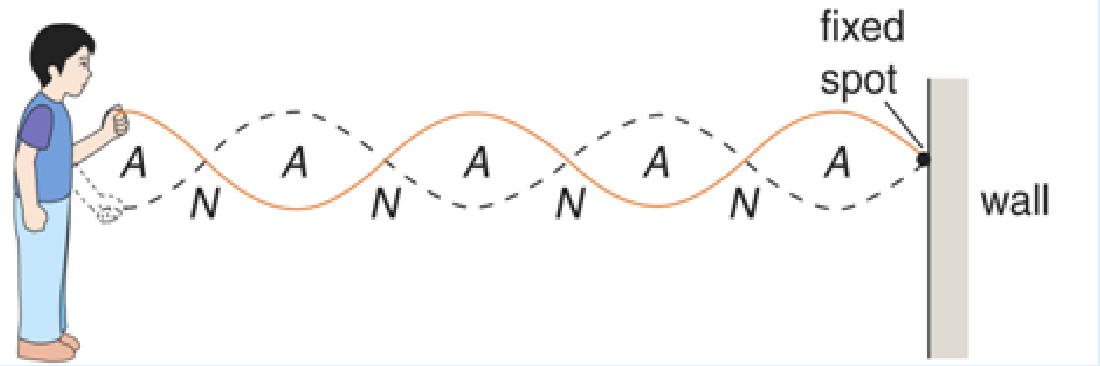
$$y = 5.0 \sin \pi (360t - 0.15x)$$

calculate
(i)the amplitude of the vibrating particle,
(ii)wave velocity,
(ii)wave length,
(iv)frequency and
(v) time period.

Ex. A simple harmonic wave of amplitude 8units travels a line of particles in the direction of positive X axis. At any instant for a particle at a distance of 10cm from the origin, the displacement is +6units and at a distance a particle from the origin is 25units, the displacement is +4units. Calculate the wavelength.

Introduction to stationary waves

Stationary waves (standing waves) produced in
- string of guitar or piano
- rope (one end tie to wall)

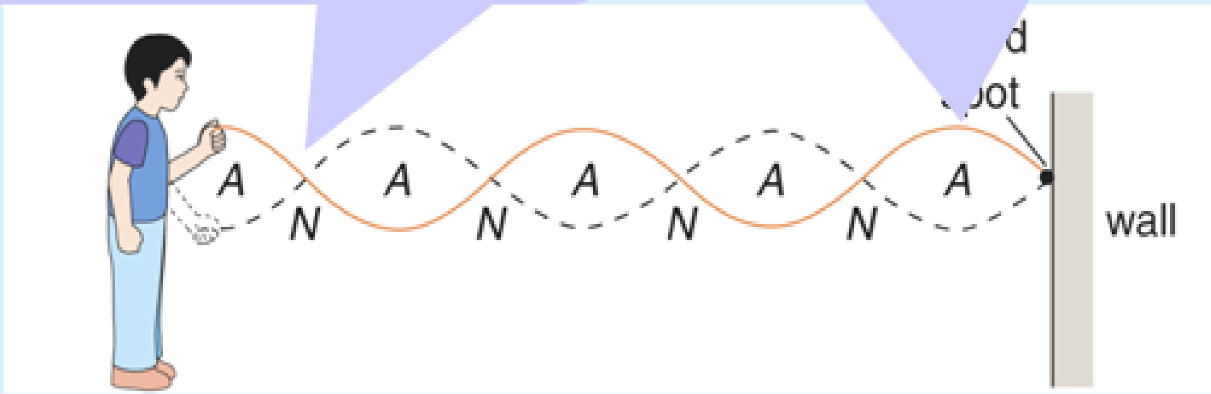
- **Incident wave** is reflected by the wall to from **reflected wave**
- **superposition** of the waves produces stationary wave



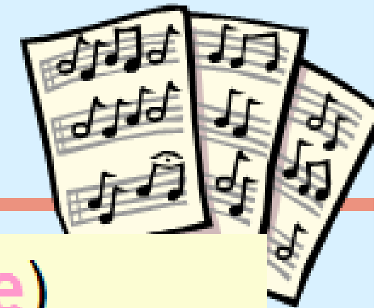
Introduction to stationary waves

N – **nodes** (remain stationary)

A – **antinodes** (largest amplitude of oscillation)



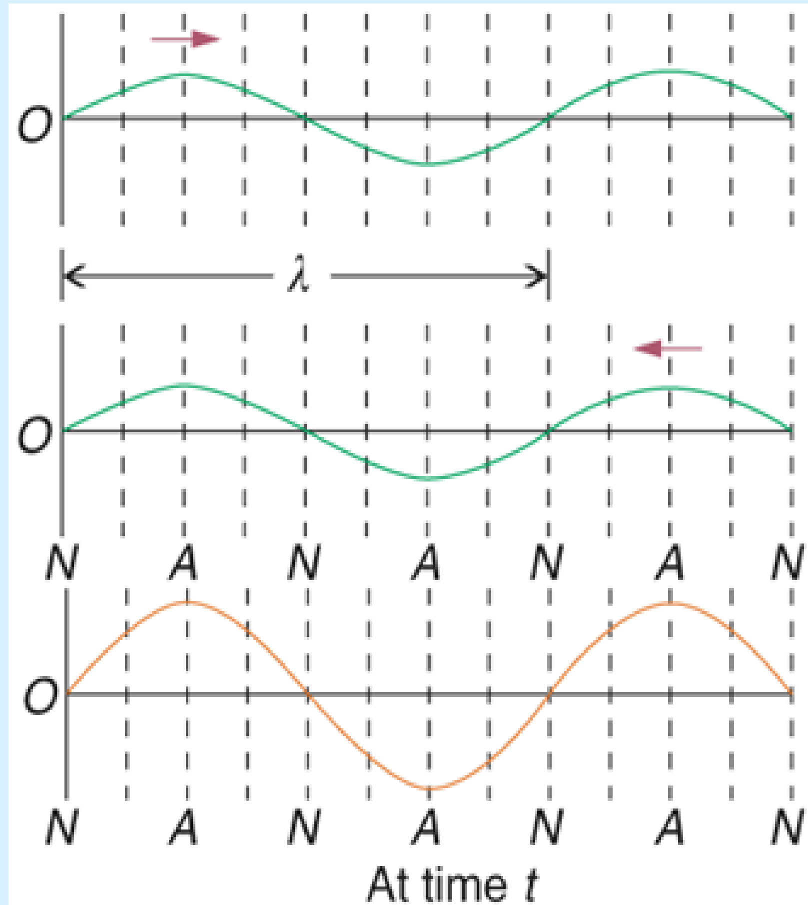
A stationary wave is produced by the superposition of two **progressive waves** of the same amplitude and frequency but travelling in opposite directions.

Introduction to stationary waves

(a) at time t (constructive interference)

(1) To right

(2) To left

(1) + (2) = (3)



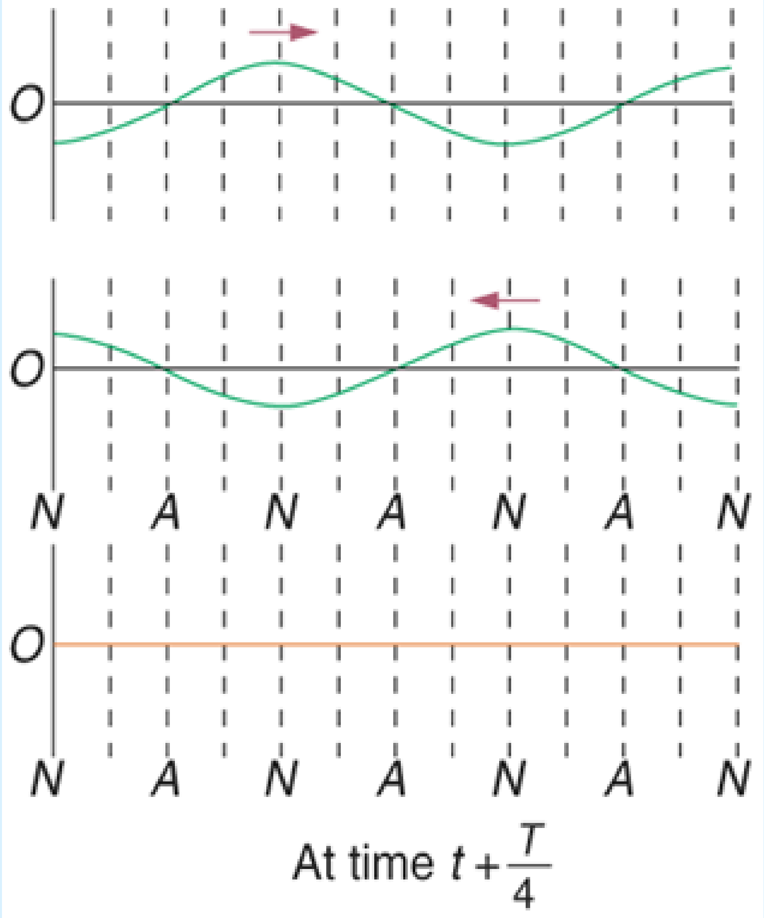
Introduction to stationary waves

(b) at time t + $T/4$ (destructive interference)

(1)

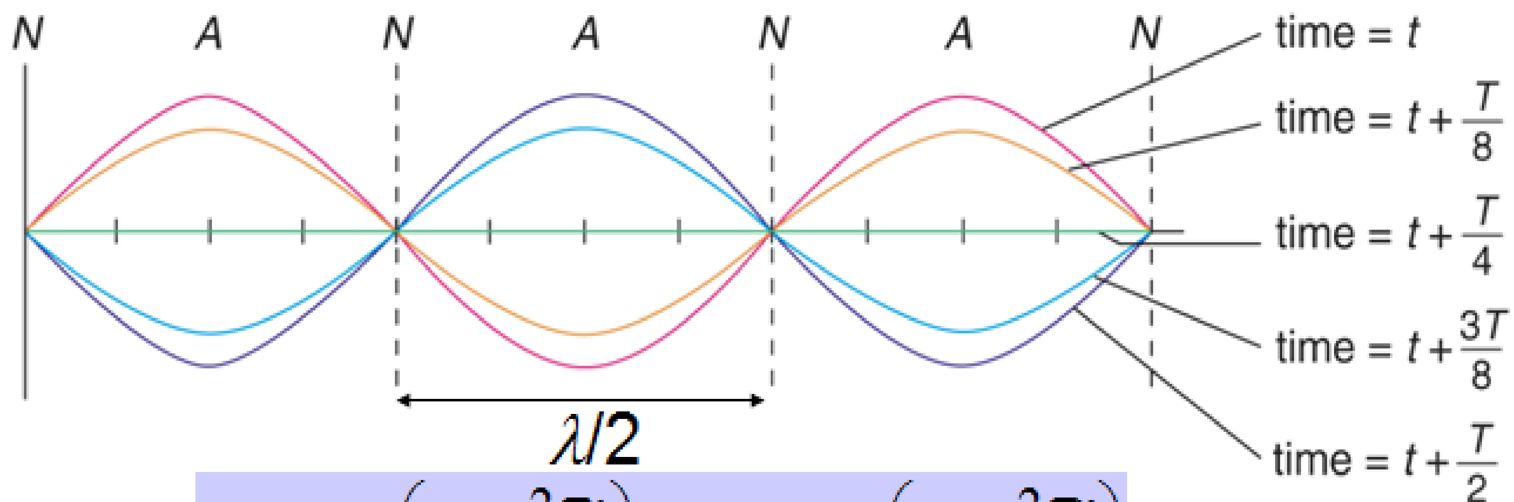
(2)

(1) + (2) = (3)



Introduction to stationary waves

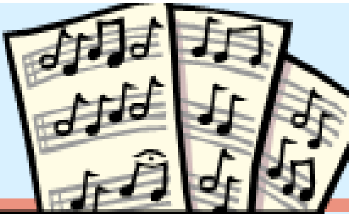
Vibration of stationary wave at various instants



$$y_1 = a\sin\left(\omega t - \frac{2\pi x}{\lambda}\right), \quad y_2 = a\sin\left(\omega t + \frac{2\pi x}{\lambda}\right)$$

$$y = y_1 + y_2 = \left(2a\cos\frac{2\pi x}{\lambda}\right)\sin\omega t$$

Amplitude (*A*) of resultant stationary wave

The positions of antinodes

Progressive wave	Stationary wave
Energy is transferred along the direction of propagation.	No energy is transferred along the direction of propagation.
The wave profile moves in the direction of propagation.	The wave profile does not move in the direction of propagation.
Every point along the direction of propagation is displaced.	There are points known as nodes where no displacement occurs.
Every point has the same amplitude.	Points between two successive nodes have different amplitudes.
Neighbouring points are not in phase.	All points between two successive nodes vibrate in phase with one other.

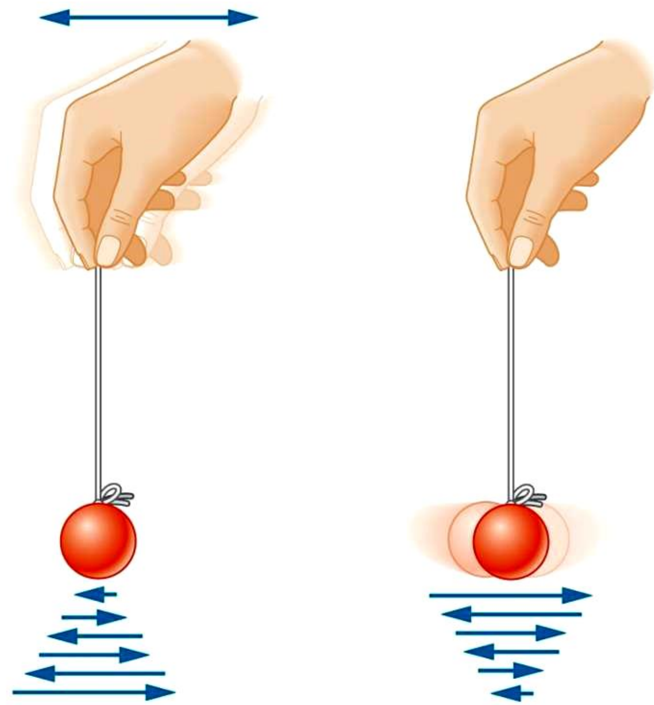
Forced Oscillations; Resonance

Forced vibrations occur when there is a periodic driving force. This force may or may not have the same period as the natural frequency of the system.

If the frequency is the same as the natural frequency, the amplitude can become quite large. This is called resonance.

Figure : Caption: (a) Large-amplitude oscillations of the Tacoma Narrows Bridge, due to gusty winds, led to its collapse (1940). (b) Collapse of a freeway in California, due to the 1989 earthquake.


m



Equation of Driven Harmonic Motion.

- Consider what happens when we apply a time-dependent force $F(t)$ to a system that normally would carry out SHM with an angular frequency ω_0.

- Assume the external force $F(\mathrm{t}) = mF_0\sin(\omega t)$. The equation of motion can now be written as

$$\frac{d^2x}{dt^2} = -\omega_0^{\ 2}x + F_0\sin(\omega t)$$

- The steady state motion of this system will be harmonic motion with an angular frequency equal to the angular frequency of the driving force.

- Consider the general solution

$$x(t) = A\cos(\omega t + \phi)$$

- The parameters in this solution must be chosen such that the equation of motion is satisfied. This requires that

$$-\omega^2 A\cos(\omega t + \phi) + \omega_0{}^2 A\cos(\omega t + \phi) - F_0 \sin(\omega t) = 0$$

- This equation can be rewritten as

$$\left(\omega_0{}^2 - \omega^2\right) A\cos(\omega t)\cos(\phi) -$$
$$\left(\omega_0{}^2 - \omega^2\right) A\sin(\omega t)\sin(\phi) - F_0 \sin(\omega t) = 0$$

- Our general solution must thus satisfy the following condition:

$$\left(\omega_0^{\,2} - \omega^2\right) A \cos(\omega t) \cos(\phi) - \left\{\left(\omega_0^{\,2} - \omega^2\right) A \sin(\phi) - F_0\right\} \sin(\omega t) = 0$$

- Since this equation must be satisfied at all time, we must require that the coefficients of cos(ωt) and sin(ωt) are 0. This requires that

$$\left(\omega_0^{\,2} - \omega^2\right) A \cos(\phi) = 0$$

and

$$\left(\omega_0^{\,2} - \omega^2\right) A \sin(\phi) - F_0 = 0$$

- The interesting solutions are solutions where $A \neq 0$ and $\omega \neq \omega_0$. In this case, our general solution can only satisfy the equation of motion if

$$\cos(\phi) = 0$$

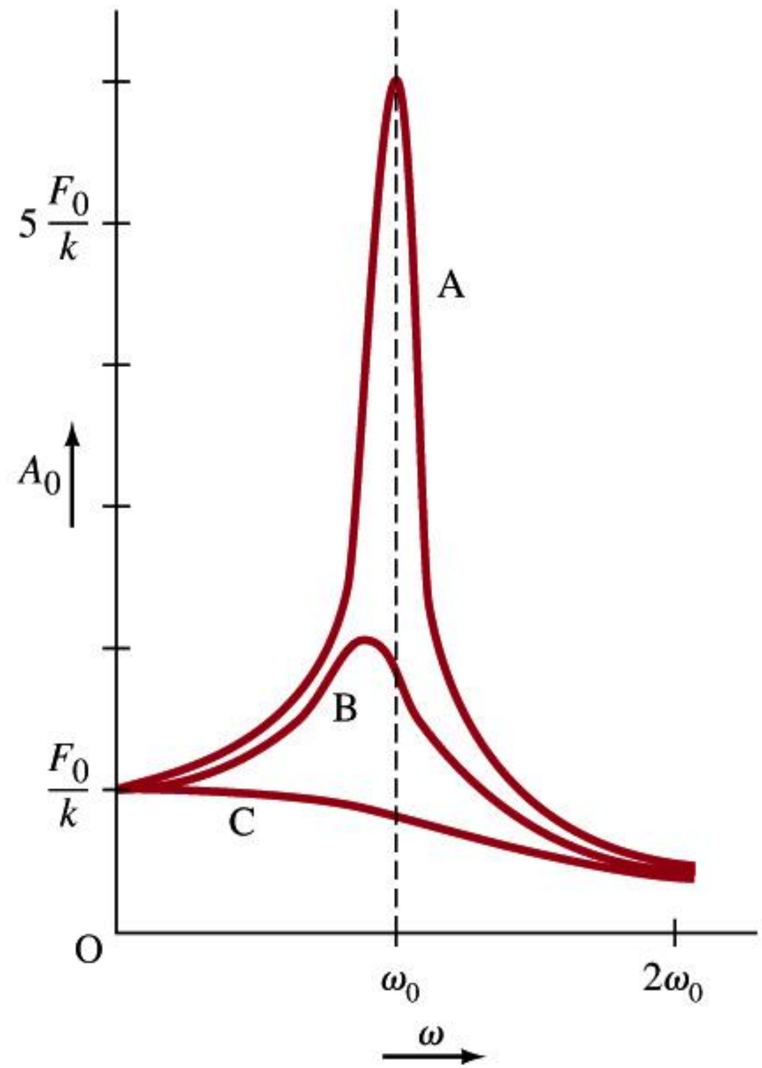
and

$$\left(\omega_0^{\,2} - \omega^2\right) A \sin(\phi) - F_0 = \left(\omega_0^{\,2} - \omega^2\right) A - F_0 = 0$$

- The amplitude of the motion is thus equal to

$$A = \frac{F_0}{\left(\omega_0^{\,2} - \omega^2\right)}$$

- If the driving force has a frequency close to the natural frequency of the system, the resulting amplitudes can be very large even for small driving amplitudes. The system is said to be in resonance.
- In realistic systems, there will also be a damping force. Whether or not resonance behavior will be observed will depend on the strength of the damping term.



$5\,\frac{F_0}{k}$
A
A_0
B
$\frac{F_0}{k}$
C
O
ω_0
$2\omega_0$
ω

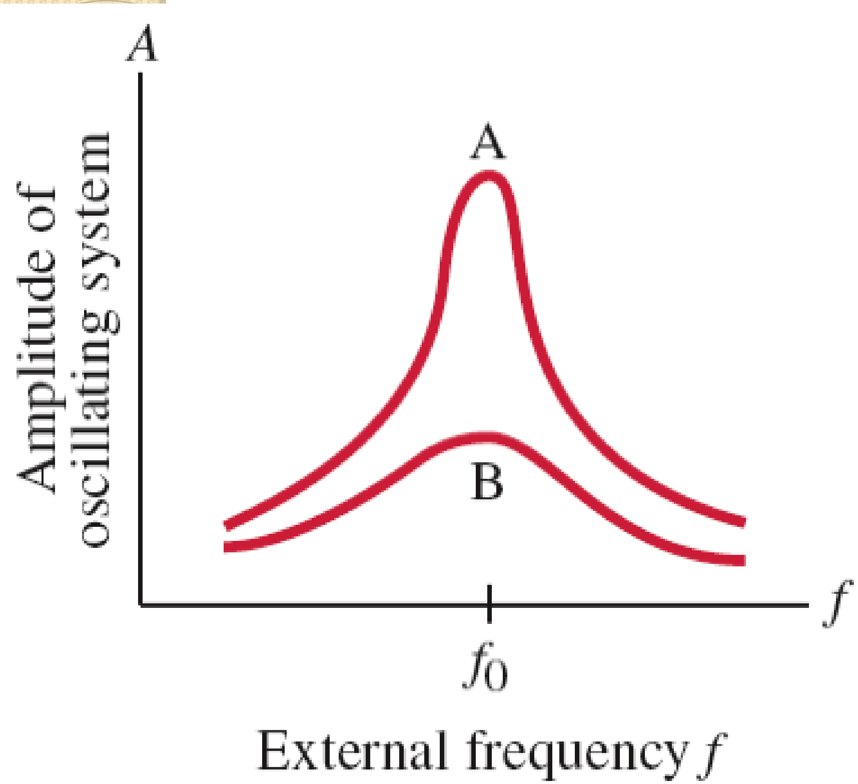
Driven Harmonic Motion.

Done for today!
Thursday: Temperature and Heat!

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

Unusually Strong Cyclone Off the Brazilian Coast: A lot of Rotational Motion!
Credit: Jacques Descloitres, MODIS Land Rapid Response Team, GSFC, NASA

Forced Oscillations; Resonance



The sharpness of the resonant peak depends on the damping. If the damping is small (A) it can be quite sharp; if the damping is larger (B) it is less sharp.

Like damping, resonance can be wanted or unwanted. Musical instruments and TV/radio receivers depend on it.

Forced Oscillations; Resonance

The equation of motion for a forced oscillator is:

$$\boxed{ma = -kx - bv + F_0 \cos \omega t.}$$

The solution is: $$x = A_0 \sin(\omega t + \phi_0),$$

where $$A_0 = \frac{F_0}{m\sqrt{(\omega^2 - \omega_0^2)^2 + b^2\omega^2/m^2}}$$

and

$$\phi_0 = \tan^{-1} \frac{\omega_0^2 - \omega^2}{\omega(b/m)}.$$

Forced Oscillations; Resonance

The width of the resonant peak can be characterized by the Q factor:

$$Q = \frac{m\omega_0}{b}.$$