Soln.

$$C = 4\pi\epsilon_0 R$$

$$= (4)(3.14)(8.85 \times 10^{-12} F/m)(6370 \times 10^3 m)$$

$$= 7.08 \times 10^{-4} F$$

$$= 708 \times 10^{-6} = 708\ \mu F.$$

Example 4.5 The space between the conductors of a long coaxial cable, used to transmit TV signals, has an inner radius a = 0.15mm and an outer radius b = 2.1 mm. What is the capacitance per unit length of this cable?

Soln.

The capacitance of a coaxial cable is given by (eqn. 4.12)

$$C = 2\pi\epsilon_0 \frac{l}{\ln(b/a)}$$ where l is the length of the cable.

Hence capacitance per unit length is

$$\frac{C}{l} = \frac{2\pi\epsilon_0}{\ln(b/a)}$$

Here, $b = 2.1$ mm $= 2.1 \times 10^{-3}$ m

$a = 0.15$ mm $= 0.15 \times 10^{-3}$ m.

$$= \frac{(2)(3.14)(8.85\times10^{-12} F/m)}{\ln\left(\frac{2.1\times10^{-3} m}{0.15\times10^{-3} m}\right)}$$

$$= \frac{55.578\times10^{-12}}{2.64} F/m$$

$$= 21 \times 10^{-12} F/m = 21 pF.$$

4.3 Capacitors in series and parallel

In analyzing electric circuits, very often it is desirable to know the *equivalent capacitance* of two or more capacitors that are connected in a certain way. By equivalent capacitance is meant the capacitance of a single capacitor that can be substituted for the combination with no change in the operation of the rest of the circuit. With such a replacement, the circuit can be simplified, so that we can solve for unknown quantities in the circuit more easily. In a circuit, capacitors can be combined either in series or parallel.

(a) Capacitors connected in series

Fig. 4.4 shows three capacitors connected *in series* to a battery B, maintaining a potential difference of V volts across the left and right terminals of the series combination. The potential differences produced across the individual capacitors are V_1, V_2, and V_3, the corresponding capacitances being C_1, C_2 and C_3 respectively. The connection is said to be series if $V_1 + V_2 + V_3 = V$.

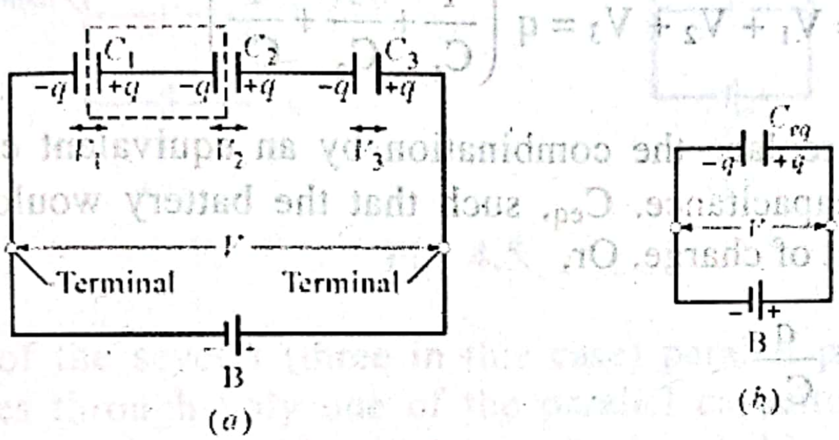


Fig. 4.4

Although the potential differences produced across the capacitors are different (depending on the individual capacitances), each capacitor has the same charge q, irrespective of its capacitance. To understand this let us consider the part of the circuit enclosed by the dashed line in the figure. Since this portion of the circuit is electrically isolated from the rest of the circuit, this may be regarded as a *floating* circuit. Let us assume that the battery puts a charge $-q$ on the left hand plate of C_1. Since a capacitor carries equal and opposite charges on its plates, a charge $+q$ appears on the right hand plate of C_1. Since the floating portion of the circuit is electrically isolated, initially it carries no net charge, and no charge can be transferred to it. Thus if a charge $+q$ appears on the right hand plate of C_1, then a charge $-q$ must appear on the left-hand plate of C_2. The presence of charge on the plates of the capacitors, not directly connected to the battery can be similarly explained. The result is that the left hand plate of *every* capacitor of the series connection carries a charge q of one sign while the right hand plate of every capacitor carries a charge of equal magnitude q but of opposite sign.

Another characteristic feature of series connection is that in going from one terminal to the other terminal, one must pass through *all* the circuit elements *in succession*.

For each individual circuit element we can write

$$V_1 = \frac{q}{C_1}, \quad V_2 = \frac{q}{C_2}, \quad \text{and} \quad V_3 = \frac{q}{C_3}$$

The potential difference of the series combination is then

$$V = V_1 + V_2 + V_3 = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right) \qquad \text{(i)}$$

Let us replace the combination by an equivalent capacitor of equivalent capacitance. C_{eq}, such that the battery would move the same amount of charge. Or,

$$V = \frac{q}{C_{eq}} \qquad \text{(ii)}$$

Combing eqns. (i) and (ii) we obtain

$$C_{eq} = \frac{q}{V} = \frac{q}{q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)}$$

$$\text{or,} \quad \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \qquad (4.13)$$

Thus the equivalent capacitance is the reciprocal of the sum of the reciprocals of the individual capacitances.

(ii) Capacitors connected in parallel

Fig. 4.5 shows three capacitors connected in *parallel* to a battery B. The terminals of the battery are wired directly to the plates of the three capacitors. The connection is said to be parallel if the same potential difference V, which is the potential difference between the terminal of the battery, is applied across each of the capacitors. In going from one terminal to the other terminal, we can

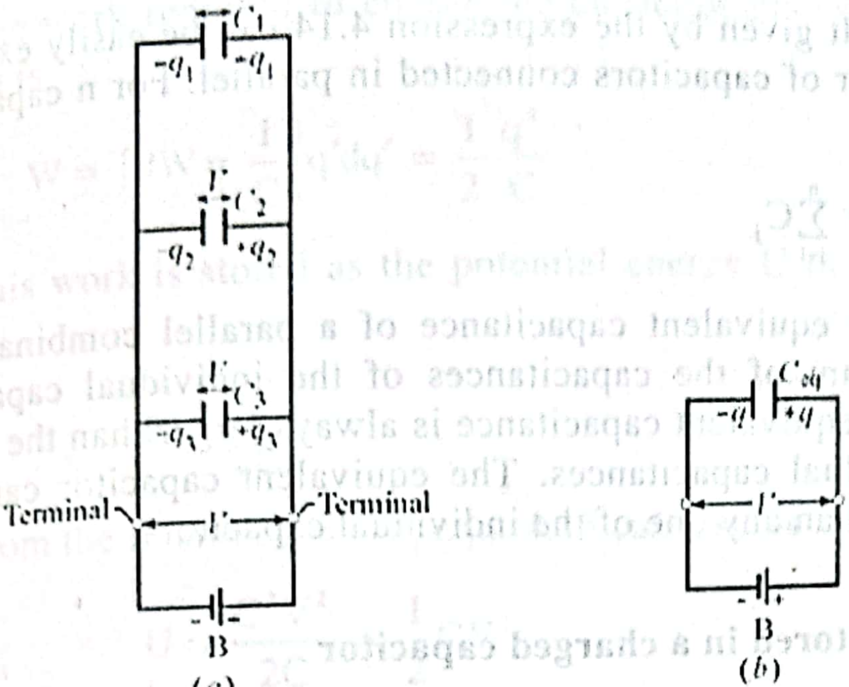


Fig. 4.5

take one of the several (three in this case) parallel paths, each of which goes through only one of the parallel capacitors. The total charge q that is delivered by the battery to the combination is shared among the capacitors. While calculating the equivalent capacitance C_{eq} of the single capacitor that can replace the combination we must remember that both the potential difference applied across the combination and the total charge q must not be changed.

For each capacitor we can write

$$q_1 = C_1V, \qquad q_2 = C_2V, \qquad \text{and} \qquad q_3 = C_3V.$$

where q_1, q_2, and q_3 are the charges stored in the first, second and the third capacitors respectively and C_1, C_2 and C_3 are the corresponding capacitances of the capacitors.

The total charge on the parallel combination is then

$$q = q_1 + q_2 + q_3 = C_1V + C_2V + C_3V$$
$$= (C_1 + C_2 + C_3)\,V$$

The equivalent capacitance, with the same total charge q and the applied potential difference V as the combination is then

$$C_{eq} = \frac{q}{V} = \frac{(C_1 + C_2 + C_3)V}{V} = C_1 + C_2 + C_3 \qquad (4.14)$$

The result given by the expression 4.14 can be easily extended to any number of capacitors connected in parallel. For n capacitors, we have

$$C_{eq} = \sum_{j=1}^{n} C_j$$

Thus the equivalent capacitance of a parallel combination is simply the sum of the capacitances of the individual capacitors. Note that the equivalent capacitance is always larger than the largest of the individual capacitances. The equivalent capacitor can store more charge than any one of the individual capacitors.

4.4 Energy stored in a charged capacitor

In chapter III, we have seen that any charge configuration has a certain electric potential energy U, equal to the work W that must be done by an external agent to assemble the charge configuration from its individual components, originally assumed to be infinitely far apart and at rest. Similarly, charging of a capacitor needs work to be done by an external agent. The process of charging an uncharged capacitor can be visualized as electrons being removed from one plate by some external agent. The electric field that builds up in the space between the plates will be in a direction that tends to oppose further transfer. Thus as charge accumulates on the capacitor plates, increasingly larger amounts of work will be required to transfer additional electrons. This work is stored in the form of electric potential energy U in the electric field between the plates. In practice this work is done by a battery at the expense of its store of chemical energy. The energy stored can be easily recovered by allowing the capacitor to discharge in a circuit.

Suppose that, at a given instant, a charge q' has been transferred from one plate to the other. The potential difference V' between the plates at that instant will be q'/C where C is the capacitance.

If an extra amount of charge dq' is to be transferred then, according to eqn. 3.2, the increase in work required will be

$$dW = V'dq' = \frac{q'}{C}dq'$$

The work required to charge the capacitor plates up to a final value q is

$$W = \int dW = \frac{1}{C}\int_0^q q'dq' = \frac{1}{2}\frac{q^2}{C}$$

This work is stored as the potential energy U in the capacitor, so that

$$U = \frac{q^2}{2C} \qquad (4.15)$$

From the relation q = CV, eqn. 4.15 can also be written as

$$U = \frac{C^2V^2}{2C} = \frac{1}{2}CV^2 \qquad (4.16)$$

4.5 Energy density of electric field

It is reasonable to suppose that the energy stored in a capacitor resides in the electric field between its plates just as the energy carried by electromagnetic waves can be regarded as residing in its electric field. As q or V in eqns. 4.15 and 4.16 increases, so does the electric field E; when q or V becomes zero, E vanishes.

In a parallel-plate capacitor, the electric field, neglecting fringing, has the same value for all points between the plates. Thus the energy density U, defined as the stored energy per unit volume, should also be same everywhere between the plates. If A is the area of the capacitor plate and d the distance of separation between the plates, then the volume between the plates is Ad. Hence

$$U = \frac{U}{Ad} = \frac{\frac{1}{2}CV^2}{Ad}$$

Now the capacitance C for a parallel plate capacitor is given by $C = \frac{\epsilon_0 A}{d}$. Eqn. 4.16 thus becomes

$$u = \frac{\frac{1}{2}\frac{\epsilon_0 A}{d}V^2}{Ad} = \frac{1}{2}\epsilon_0\left(\frac{V}{d}\right)^2$$

But V/d is the electric field strength E between the capacitor plates, so that

$$u = \frac{1}{2}\epsilon_0 E^2 \qquad (4.17)$$

Although derived for a parallel-plate capacitor, the equations hold no matter what the geometry of the capacitor is. *If an electric field E exists at any point in space (or vacuum), then the point may be regarded as the site of stored energy of* $\frac{1}{2}\epsilon_0 E^2$ *per unit volume.* In general, since E varies with location, u is also a function of coordinates. For the special case of a parallel-plate capacitor, E and U have same value anywhere between the plates.

4.6 Sharing of charges

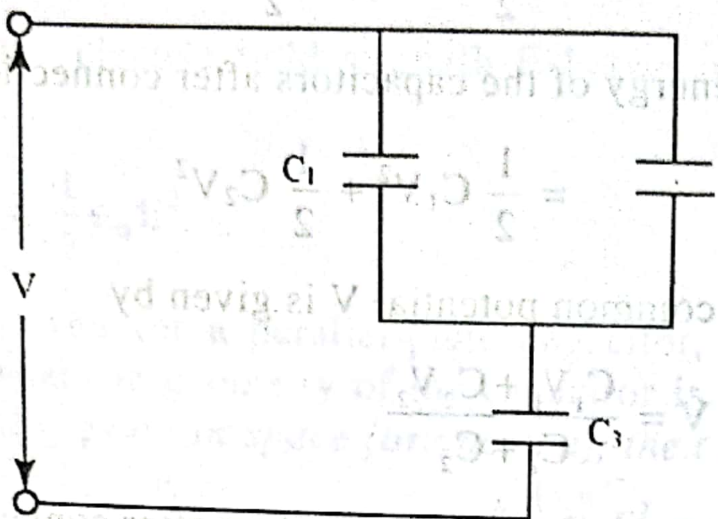
If two charged capacitors of capacitances C_1 and C_2 be joined by a conductor, some charges will flow from the conductor at higher potential to the conductor at lower potential till the potentials are equalized. Let q_1 and q_2 be the charges on the capacitors after redistribution of charges and let the common potential of the capacitors be V. Then

$$V = \frac{q_1}{C_1} = \frac{q_2}{C_2} = \frac{q_1 + q_2}{C_1 + C_2}$$

Loss of energy due to sharing of charges

When two capacitors of capacitances C_1 and C_2 and potentials V_1 and V_2 are joined by a conductor, electric charges will flow from the capacitor at higher potential to the capacitor at lower potential. Due to this flow of charge, the energy of the system will decrease. The decrease in energy is due to the fact that during the flow of electricity some energy is dissipated in the form of heat.

Example 4.6 *Find the equivalent capacitance of the combination as illustrated in the following figure. Assume $C_1 = 10\mu F$, $C_2 = 5\mu F$, $C_3 = 4\mu F$ and $V = 100$ volts.*



Soln.

C_1 and C_2 are connected in parallel. Their equivalent capacitance $C_p = C_1 + C_2 = 10 + 5 = 15\mu F$.

Now, considering C_p connected in series with C_3, we get

$$\frac{1}{C} = \frac{1}{C_p} + \frac{1}{C_3} = \frac{1}{15} + \frac{1}{4} = \frac{4+15}{60} = \frac{19}{60}$$

$$\therefore \quad C = \frac{60}{19}\,\mu F = 3.2\mu F.$$

Example 4.7 *Two capacitors have a capacity of $5\mu F$ when connected in parallel and $1.2\mu F$ when connected in series. Calculate their individual capacitances.*

Soln.

Let C_1 and C_2 be the individual capacitances when the two are connected in parallel, we have

$$C_1 + C_2 = 5 \qquad \text{(i)}$$

when they are connected in series, we have

$$\frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{1.2}; \quad \text{or,} \quad \frac{1}{1.2} = \frac{C_1 + C_2}{C_1 C_2} \qquad \text{(ii)}$$

From (i) and (ii)

$$\frac{1}{1.2} = \frac{5}{C_1C_2}, \quad \text{or,} \quad C_1C_2 = 6 \qquad \text{(iii)}$$

$$\text{Now } (C_1 - C_2)^2 = (C_1 + C_2)^2 - 4C_1C_2$$
$$= 5^2 - 4.6 = 1$$

$\therefore\ C_1 - C_2 \pm 1$

Let $C_1 - C_2 = +1$

Then, combining with eqn. (i), we get $2C_1 = 6$; or $C_1 = 3\mu F$.

Hence $C_2 = 2\mu F$.

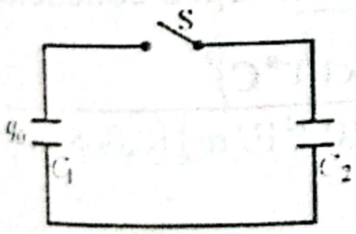
If $C_1 - C_2 = -1$, $2C_1 = 4$ or $C_1 = 2\mu F$ and $C_2 = 3\mu F$.

$\therefore\ C_1 = 2\mu F$ or $3\mu F$

$C_2 = 3\mu F$ or $2\mu F$.

Example 4.8 *A 3.55μF capacitor C_1 is charged to a potential difference $V_0 = 6.30V$, using a battery. The charging battery is then removed and the capacitor is connected to an uncharged 8.95μF capacitor C_2 in series – as in the figure. After the switch S is closed, charge flows from C_1 to C_2 until an equilibrium is established, with both capacitors at the same potential difference V. (i) what is the common potential difference V? (ii) what is the energy stored in the electric field before and after the switch S is closed?*

Soln.



(i) when the capacitor C_1 is connected to the capacitor C_2, the original charge q_0 is shared by the two capacitors. Or, $q_0 = q_1 + q_2$.

Applying the relation $q = CV$, we get

$$C_1V_0 = C_1V_1 + C_2V; \quad \text{or} \quad V = \frac{C_1}{C_1 + C_2}V_0$$

$$\text{or, } V = \left(\frac{3.55}{3.55 + 8.95}\right)6.30 = 1.79V.$$

(ii) The initial stored energy

$$U_i = \frac{1}{2}C_1 V_0^2 = \frac{1}{2}(3.55 \times 10^{-6}F)(6.30V)^2 \simeq 7.05 \times 10^{-5}\text{Joules.}$$

The final energy

$$U_f = \frac{1}{2}C_1V^2 + \frac{1}{2}C_2V^2$$

$$= \frac{1}{2}(3.55 \times 10^{-6}F)(1.79V)^2 + \frac{1}{2}(8.95 \times 10^{-6}F)(1.79V)^2$$

$$= 5.69 \times 10^{-6}J + 14.33 \times 10^{-6}J$$

$$= 20.02 \times 10^{-6}J = 2.0 \times 10^{-6}J.$$

Thus the final energy is less than initial energy by about 72% This is not a violation of conservation of energy. The 'missing' energy appears as the thermal energy in the connecting wires.

Example 4.9 *An isolated conducting sphere whose radius R is 6.85 cm carries a charge q = 1.25nC. (i) How much energy is stored in the electric field of this charged conductor? (ii) What is the energy density at the surface of the sphere?*

Soln.

(i) The capacitance of an isolated conducting sphere of radius R is $C = 4\pi\epsilon_0 R$.

The energy stored in the electricfield of this charged conductor,

$$U = \frac{q^2}{2C} = \frac{q^2}{8\pi\epsilon_0 R} = \frac{(1.25\times10^{-9}C)^2}{(8)(3.14)(8.85\times10^{-12}F/m)(0.0685m)}$$

$$= 1.03 \times 10^{-7}J = 103nJ.$$

(ii) The energy density,

$$u = \frac{1}{2}\epsilon_0 E^2$$

where $E = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{R^2}$

$$\therefore \quad u = \frac{1}{2}\epsilon_0 E^2 = \frac{q^2}{32\pi^2 \epsilon_0 R^4}$$

$$= \frac{(1.25\times10^{-9}C)^2}{(32)(3.14)(8.85\times10^{-12}C^2/N.m^2)(0.0685m)^4}$$

$$= 2.54 \times 10^{-5} J/m^3 = 25.4\mu J/m^3.$$

Example 4.10 A plane-parallel plate capacitor has circular plates of radius r = 10.0 cm, separated by a distance d = 1.00 mm. (i) How much charge is stored on each plate when their electric potential difference has the value V = 100V? (ii) Calculate the electric field, the electric field energy density, and the energy stored in the capacitor.

Soln.

$$A = \pi r^2 = (3.14)(0.1m)^2$$

$$d = 1mm = 1.0 \times 10^{-3}m$$

$$\text{(i)} \quad C = \frac{\epsilon_0 A}{d}$$

$$= \frac{(8.85\times10^{-12}C^2/N.m^2)(3.14\times10^{-2}m^2)}{1.00\times10^{-3}m}$$

$$= 2.8 \times 10^{-10}F.$$

The magnitude of charge on each plate,

$$q = CV = (2.8 \times 10^{-10}F)(100 V) = 2.8 \times 10^{-8}C = 28nC.$$

$$\text{(ii)} \quad E = \frac{V}{d} = \frac{100V}{1.0\times10^{-3}m} = 10^5 V/m = 100 kV/m$$

$$\text{energy density, } u = \frac{1}{2}\epsilon_0 E^2$$

$$= \left(\frac{1}{2}\right)(8.85\times10^{-12}C^2/N.m^2)(100000 V/m)^2$$

$$= 0.044 J/m^3.$$

$$\text{energy stored, } u = \frac{1}{2}CV^2$$

$$= \left(\frac{1}{2}\right)(2.8\times10^{-10}F)(100V)^2$$

$$= 1.4 \times 10^{-6}J = 1.4\mu J.$$

$$[U = u.\pi r^2.d = (4.4 \times 10^{-2}J/m^3)\ \pi\ (0.1m)^2\ (0.001m) = 1.4\mu J]$$

Example 4.11 Compute the energy stored in a 60 – pF capacitor (i) when charged to a potential difference of 2kV, (ii) when the charge on each plate is 30 nC.

Soln.

$$\text{(i)}\ E = \frac{1}{2}CV^2 = \frac{1}{2}\ (60 \times 10^{-12}F)\ (2000V)^2$$

$$= 1.2 \times 10^{-4}J.$$

$$\text{(ii)}\ E = \frac{1}{2}\ \frac{q^2}{C} = \left(\frac{1}{2}\right)\frac{(30\times10^{-9}C)^2}{(60\times10^{-12}F)} = 7.5 \times 10^{-6}J.$$

4.7 Capacitance with dielectric

Up to this point the capacitance was calculated on the assumption that there is no material in the space between the plates of the capacitor. What happens if the space between the plates is filled with a *dielectric*, which is an insulating material such as mineral oil, glass or plastic? The effect of the presence of a dielectric in the space between the plates of a capacitor was first investigated by Michael Faraday.

Faraday constructed two identical capacitors, filling one with dielectric and the other with air under normal conditions. Faraday's experiments showed that when both the capacitors were charged to the *same potential difference, the charge on the capacitor with dielectric was greater than that on the other.* Since q is larger for the same V with the dielectric present, it follows from relation C = q/V that the capacitance of a capacitor *increases* if a dielectric is placed between the plates. If C is the capacitance of the capacitor when a dielectric material is present and C_o the capacitance when no dielectric material is present, then the ratio C/C_o is called the *dielectric constant k* of the dielectric material. Or,

$$k = \frac{C}{C_0}$$

k is a dimensionless factor by which the capacitance of a capacitor increases when a dielectric material is inserted in between the plates, relative to its capacitance when no dielectric is present. It is assumed, unless otherwise stated, that the dielectric completely fills the space between the plates.

The dielectric constant is a fundamental property of the dielectric material and is independent of the size or shape of the conductor. The dielectric constant of some dielectric materials is given in the table below. The dielectric constant of vacuum is unity by definition and for most practical applications, air and vacuum are equivalent in their dielectric effects.

SOME PROPERTIES OF DIELECTRIC MATERIALS

Material	Dielectric constant k	Dielectric strength kV/mm
Vacuum	1 (exact)	∞
Air (1 atm)	1.00059	3
Polystyrene	2.6	24
Paper	3.5	16
Transformer oil	4.5	12
Pyrex	4.7	14
Mica	5.4	160
Porcelain	6.5	4
Silicon	12	
Water (25°C)	78.5	
Water (20°C)	80.4	
Titania ceramic	130	
Strontium titanate	310	8

The values quoted in the table were measured at room temperature.

Some insight into Faraday's experiment is provided by Fig. 4.6. A capacitor is initially charged to a charge q by the battery B which then remains connected to the capacitor to ensure that the potential difference V and the electric field E between the plates remain constant. After the dielectric slab is inserted between the plates, the

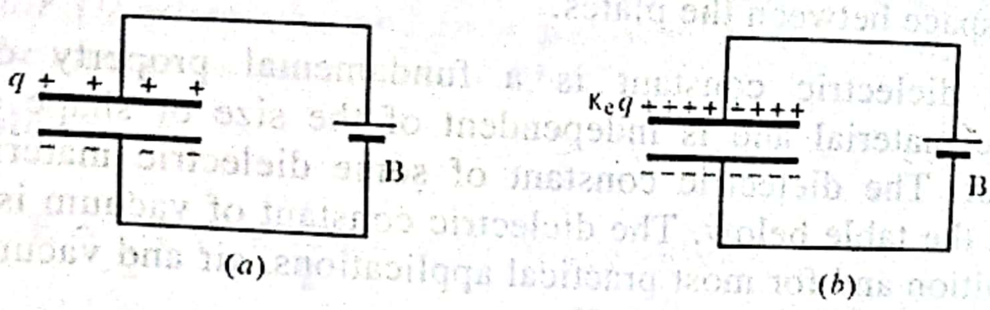


Fig. 4.6

charge increases by a factor k to a value kq. The additional charge $(k - 1)q$ is delivered to the capacitor plates by the battery as the dielectric slab is inserted.

Alternately, suppose the battery is disconnected after the capacitor is charged to a charge q as in Fig. 4.7. As the dielectric slab is now inserted, the charge remains constant as there is no path for transfer of charge, but according to relation C = q/V, the potential difference must decrease by a factor k to allow for the increase in capacitance by the same factor. Thus the potential difference decreases by a factor k from V to V/k. Similarly the electric field also decreases from E to E/k.

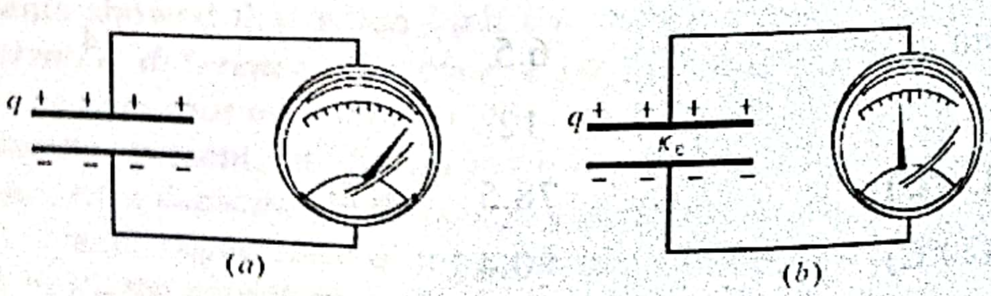


Fig. 4.7

The purpose of a capacitor is to store charge. Thus the presence of a dielectric permits a capacitor to store a factor k more charge for the same potential difference. However, the presence of a dielectric also limits the potential difference that can be maintained across the plates. If this limit is exceeded, the dielectric material breaks down and forms a conducting path between the plates. Every dielectric material has a characteristic *dielectric strength* which is defined as the maximum value of the electric field that it can tolerate without breakdown. A few such values are also listed in the Table.

For a parallel plate capacitor filled with dielectric, the capacitance is given by

$$C = \frac{k\epsilon_0 A}{d} = kC_0$$

where C_0 is the capacitance of the capacitor with air between the plates. The equation suggests that the effect of a dielectric can be summed up in more general terms as

In a region completely filled up by a dielectric, all electrostatic equations containing the permittivity constant ϵ_0 are to be modified by replacing that constant by $k\epsilon_0$.

For a point charge q imbedded in a dielectric, the electric field is given by

$$E = \frac{1}{4\pi k\epsilon_0} \qquad (4.18)$$

Eqn. 4.18 gives the total field in the dielectric. As can be seen, the field due to the Coulomb charge is still given by Coulomb's law (without the factor k) but the dielectric itself produces another electric field, which when combined with the field due to the point charge gives eqn. 4.18.

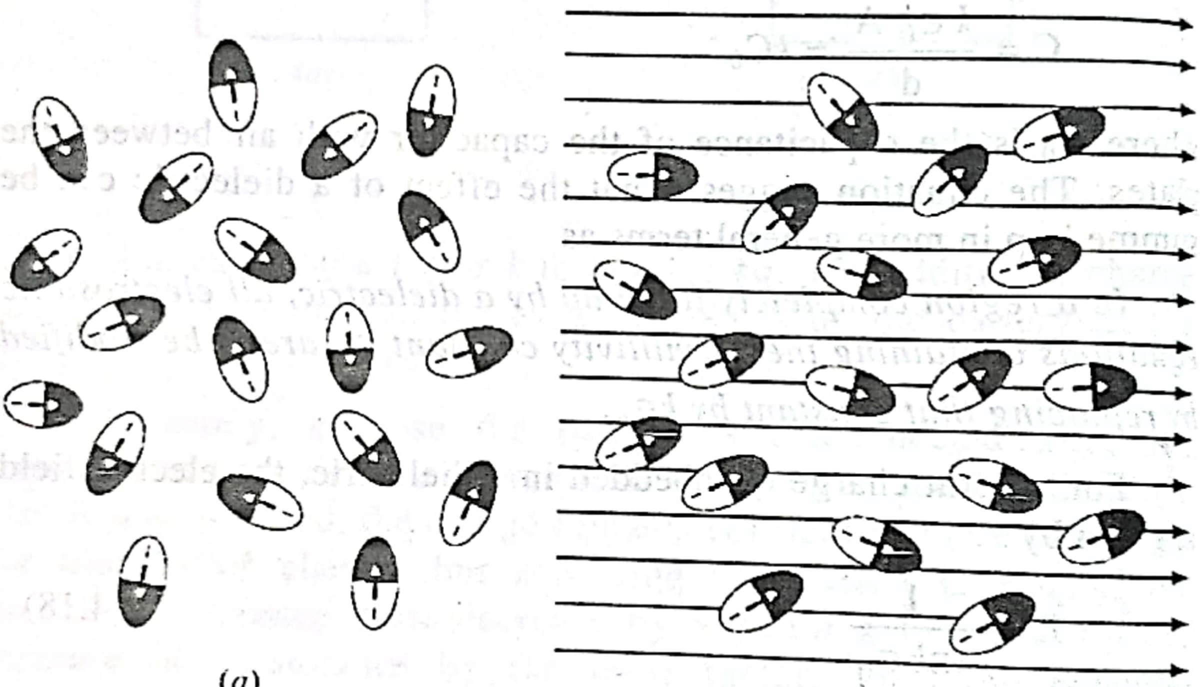
In a similar manner, the expression for the electric field just outside an isolated conductor immersed in a dielectric becomes

$$E = \frac{\sigma}{k\epsilon_0}$$

It is obvious from both of the above expressions that, for a fixed distribution of charges, the effect of a dielectric is to weaken the electric field that would otherwise be present.

4.8 Dielectric: an atomic view

Dielectric materials may be either *polar* or *non-polar.* In the case of molecules of polar dielectrics, *the centres of positive and negative charges do not coincide.* These molecules have permanent electric dipole moments. Some examples of polar molecules are N_2O, H_2O and HCl. In the absence of electric fields, the *electric dipole moments* of these polar molecules are in random orientation (Fig. 4.8) and cancel each other. So even though each molecule has a dipole moment, the average moment per unit volume is zero.



(*a*)

(*b*)

Fig. 4.8

If now an electric field is applied, the dipole moments of these molecules align themselves parallel to the direction of the electric field. Because the molecules are in constant thermal agitation, the degree of alignment is not complete [Fig. 4.8(b)] but increases as the applied electric field increases or as the temperature decreases.

In case of non-polar molecules the centres of positive charges and negative charges coincide; so these molecules do not have *any permanent dipole moment.* Oxygen (O_2), nitrogen (N_2) and hydrogen (H_2) molecules are some common examples of non-polar molecules.

When a non-polar molecule is placed in an electric field, the centres of positive and negative charges get displaced and the molecules are said to have been *polarized*. Such a molecule is then called *induced electric dipole* and its dipole moment is called *induced electric dipole moment*. The induced electric dipole moment is present only when the electric field is present. It is proportional to the electric field (for normal field strengths) and is created already lined up with the electric field.

Let us consider a slab of a dielectric material as shown in Fig. 4.9. The arrangement of negative and positive charges within the molecules of the dielectric is shown in the figure. This alignment of the dipole moments of the permanent or induced dipoles with the direction of the applied electric field is call *polarization* and the dielectric material and its molecules are said to be *polarized*.

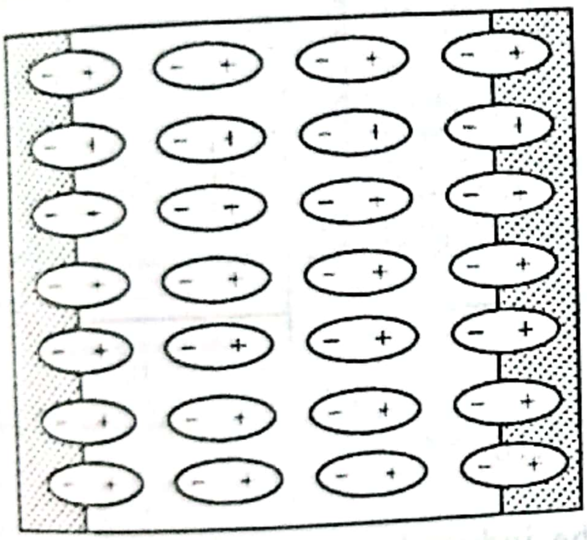


Fig. 4.9

Surface layers extremely thin are shown as shaded part of the figure. Excess positive charge is shown in one thin layer and an equal excess negative charge is shown in the other layer. *The induced charges on the surfaces of the dielectric are due to these layers*. It may be noted that these charges are not free but are bound to the molecules lying in the surface or near the surface. For this reason, these charges are referred to as *bound charges*. Also within the remaining dielectric, the *net charge per unit volume remains zero*. So though the dielectric is polarized, yet it remains *electrically neutral*. Also in polarization, the internal state of the slab is characterized not by an excess charge but by the relative displacement of the charges within it.

Let us use a parallel-plate capacitor, carrying a fixed charge q and not connected to a battery to provide a uniform electric field E_0 into which we place a dielectric slab (Fig. 4.10). The overall effect of alignment and induction is to separate the centre of positive charge of the entire slab slightly from the centre of the negative charge. Although the slab as a whole remains electrically neutral; it has become *polarized*. The net effect is a pile up of positive charge on the right face of the slab and negative charge on the left face, there being no excess charge in any given volume element within the slab. Since the slab as a whole is electrically neutral, the positive induced surface charge must be equal in magnitude to the negative induced surface charge. It should be noted that in this process electrons in the dielectric are displaced from their equilibrium positions by distances that are considerably less than an atomic diameter. There is no transfer of charge over macroscopic distance such as that occurs when a current is set up in the conductor.

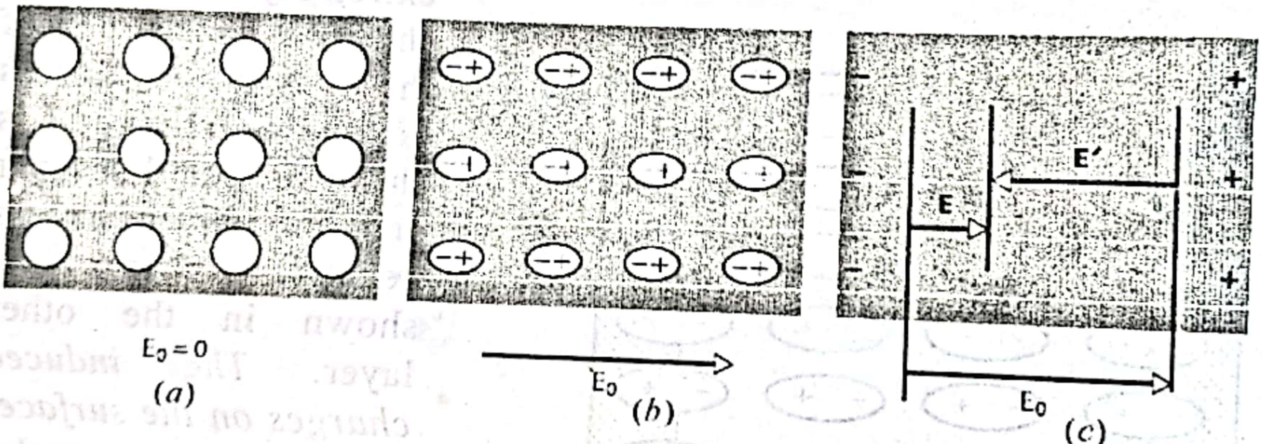


Fig. 4.10

As shown in Fig. 4.10(c), the induced surface charges always appear in such a way that the electric field E' set up by them opposes the external electric field E_0. The resultant field E in the dielectric is the vector sum of E_0 and E'. It points in the same direction as E_0 but is smaller. *If a dielectric is placed in an electric field, induced surface charges appear which tend to weaken original field within the dielectric.*

This weakening of the electric field reveals itself in Fig. 4.7 as a reduction in potential difference between the plates of a charged isolated capacitor when a dielectric is introduced between the plates.

The relation V = Ed for a parallel plate capacitor holds whether or not dielectric is present and shows that the reduction in V described in Fig. 4.7 is directly connected to the reduction in E. More specifically, if a dielectric slab is introduced into a charged parallel-plate capacitor, then

$$\frac{E_0}{E} = \frac{V_0}{V_d} = k$$

4.9 Gauss' law as applied to a dielectric

So far Gauss' law has been confined to situations in which no dielectric is present. We shall now apply the law to a parallel-plate capacitor the space between whose plates is filled with a material of dielectric constant k.

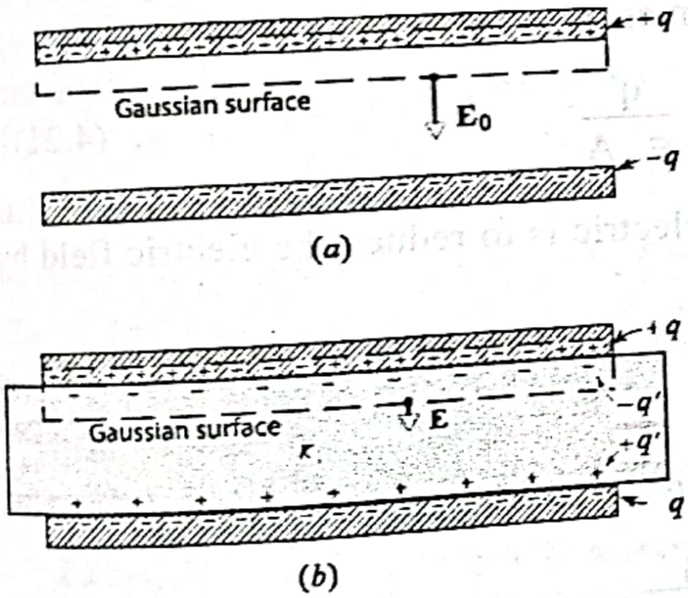


Fig. 4.11

Fig. 4.11 shows a parallel-plate capacitor both with and without the dielectric. The charge q on the plates is assumed to be same in both cases. Gaussian surfaces have been drawn as in Fig. 4.11.

In the absence of the dielectric, Gauss' law gives us

$$\in_0 \oint E.ds = \in_0 E_0 A = q$$

where A is the plate area of the capacitor and E_0 is the magnitude of the electric field in the empty space between the plates. E_0 is given by

$$E_0 = \frac{q}{\in_0 A} \qquad (4.19)$$

When the empty space between the plates is completely filled up with the dielectric, Gauss' law gives

$$\in_0 \oint \mathbf{E}.d\mathbf{s} = \in_0 EA = q - q' \tag{4.20}$$

where E is now the electric field between the plates. The charges q and q' must be distinguished from each other. q is the free charge on the capacitor plates and q' is the induced surface charge which appears on the surfaces of the dielectric under the influence of the electric field E_0, already existing between the capacitor plates at the time of introduction of the dielectric. The induced charge on the dielectric surface adjacent to the positive plate is $-q'$ while that near the negative plate being $+q'$. Both the charges $+q$ and $-q'$ lie within the Gaussian surface and are opposite in sign. Hence the net charge enclosed within the Gaussian surface is $q + (-q') = q - q'$ as shown in Fig. 4.12.

Fig. 4.20 can be rewritten as

$$E = \frac{q-q'}{\in_0 A} = \frac{q}{\in_0 A} - \frac{q'}{\in_0 A} \tag{4.21}$$

Now the effect of the dielectric is to reduce the electric field by a factor k. So

$$E = \frac{E_0}{k} = \frac{q}{k \in_0 A}$$

Inserting this value of E in eqn. 4.21 we obtain

$$\frac{q}{k \in_0 A} = \frac{q}{\in_0 A} - \frac{q'}{\in_0 A}$$

$$\text{or,} \quad q - q' = \frac{q}{k}$$

$$\text{or,} \quad q' = q\left(1 - \frac{q}{k}\right) \tag{4.22}$$

Eqn. 4.22 shows that the induced surface charge q' is always less that the free charge q and is equal to zero if no dielectric is present *i.e.*, $k = 1$ (vacuum).

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Substituting the value of q − q' in eqn. 4.20 Gauss' law for the situation when the dielectric is present can be written as

$$\epsilon_0 \oint \mathbf{E} . d\mathbf{S} = \frac{q}{k}$$

Rearranging, we obtain

$$\epsilon_0 \oint k . \mathbf{E} . d\mathbf{S} = q \qquad (4.23)$$

Eqn. 4.23 is the general form in which Gauss' law is usually written when dielectric is present. Although derived for a parallel-plate capacitor, this important relation is true generally.

The following important points may be noticed regarding eqn. 4.23.

(i) The flux integral now deals with kE, not with E.

(ii) The charge enclosed by the Gaussian surface is taken to be q, *i.e.*, the free charge only. The induced surface charge is deliberately ignored on the right side of the equation, having been taken fully into account by introducing the dielectric constant k on the left side of the equation.

(iii) Eqn. 4.23 differs from the original statement of Gauss' law (eqn. 2.5) only in that ϵ_0 in the latter equation has been replaced by $k\epsilon_0$ in full accord with the statement in Art. 4.7. Moreover, k is taken inside the integral to allow for cases in which k is not constant over the entire Gaussian surface.

***Example 4.12** A parallel-plate capacitor has plates with area A and separation d. A battery charges the plates to a potential difference V_0. The battery is then disconnected and a dielectric slab of thickness d is introduced. Calculate the stored energy both before and after the slab is introduced and account for any difference.*

Soln.

The energy U_0 before the introduction of the slab is

$$U_0 = \frac{1}{2} C_0 V_0^2$$

After the introduction of the slab, the energy is

$$U = \frac{1}{2}CV^2$$

where C is the new capacitance and V is the potential difference between the plates.

But $C = kC_0$ and $V = V_0/k$

where k is the dielectric constant of the slab.

$$\therefore\ U = \frac{1}{2}kC_0\left(\frac{V_0}{k}\right)^2 = \frac{1}{2}\frac{C_0V_0^2}{k} = \frac{1}{k}U_0.$$

Thus the energy after the introduction of the slab is less by a factor $\frac{1}{k}$.

The person who introduces the slab will be aware of this 'missing energy'. He would feel a "tug" on the slab and would have to restrain it if he wants to insert the slab without acceleration. This means that he would have to do negative work on the slab. Alternately, the (condenser + slab) system would do positive work on it.

***Example 4.13** A certain parallel-plate capacitor cosists of two plates, each with area 200 cm², separated by a 0.4 cm air gap. (i) Compute its capacitance. (ii) If the capacitor is connected across a 500-V source, what is the charge on it, the energy stored in it, and the value of E between the plates? (iii) If a liquid with k = 2.60 is poured between the plates so as to fill the air gap, how much additional charge will flow onto the capacitor from the 500-V source?*

Soln.

(i) For a parallel-plate capacitor

$$C = \frac{\epsilon_0 A}{d} = \frac{(8.85\times10^{-12})(0.02)}{0.004}$$

$$= 44 \times 10^{-12}\text{F}$$

$$= 44\ \text{pF}.$$

$A = 200\ \text{cm}^2$

$= 0.02\text{m}^2$

$d = 0.4\ \text{cm}$

$= 0.004\text{m}$

(ii) $q = CV = (4.4 \times 10^{-11}\text{F})\ (500\text{V})$

$$= 22 \times 10^{-9}C = 22\ nC.$$

$$\text{energy} = \frac{1}{2}CV^2 = \left(\frac{1}{2}\right)(44\times10^{-12}F).(500V)^2$$

$$= 550 \times 10^{-8}J$$

$$= 5.5\ mJ.$$

$$E = \frac{V}{d} = \frac{500V}{0.004m} = 125 \times 10^3 V/m = 125kV/m.$$

(iii)The capacitor will now have a capacitance 2.60 times larger than before. Therefore, the charge on the plates of the capacitor will be

$$q' = kCV = (2.6)(44 \times 10^{-12}F)(500V)$$

$$= 572 \times 10^{-10}C$$

$$= 57.2\ nC.$$

So, the additional charge that must flow onto it now

$$= (57.2 - 22)\ nC = 35.2\ nC.$$

Example 4.14 *An isolated conducting sphere whose radius R is 6.85 cm carries a charge $q = 1.25nC$. (i) How much energy is stored in the electric field of this charged conductor? (ii) What is the energy density at the surface of the sphere? (iii) What is the radius R_0 of a spherical surface such that one-half of the stored potential energy lies within it?*

Soln.

$$U = \frac{q^2}{2C}$$

Now $C = 4\pi\epsilon_0 R$

$$\therefore\ U = \frac{q^2}{8\pi\epsilon_0 R} = \frac{(1.25\times10^{-9}C)^2}{(8)(3.14)(8.85\times10^{-12}F/m)(0.0685m)}$$

$$= 1.03 \times 10^{-7}J = 103\ nJ$$

(ii) Energy density, $U = \frac{1}{2}\epsilon_0 E^2$

The electric field strength at the surface of the charged conductor is given by

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}$$

$$\therefore\ U = \frac{1}{2}\epsilon_0 E^2 = \frac{q^2}{32\pi^2 \epsilon_0 R^4}$$

$$= \frac{\left(1.25\times10^{-9}C\right)^2}{(32)\,(3.14)^2\,\left(8.85\times10^{-12}C^2/N.m^2\right)(0.0685m)^4}$$

$$= 2.54 \times 10^{-5}\ J/m^3$$

$$= 25.4\ \mu J/m^3.$$

(iii)The energy that lies in a spherical shell between radii r and r + dr is given by

$$dU = (4\pi r^2)\ (dr)\ (u)$$

where $(4\pi r^2)$ (dr) is the volume of the spherical shell and u is the energy density.

$$\therefore\ dU = \frac{q^2}{32\pi^2 \epsilon_0 r^4} 4\pi r^2 dr = \frac{q^2}{8\pi\epsilon_0}\frac{dr}{r^2}$$

The total energy U is found by integration,

$$\text{or,}\ U = \int dU = \frac{q^2}{8\pi\epsilon_0}\int_R^\infty \frac{dr}{r^2} = \frac{q^2}{8\pi\epsilon_0 R}$$

[U can also be found from the relation $U = \frac{q^2}{2C}$ where $C = 4\pi\epsilon_0 R$]

Now what will be the radius R_0 of a spherical surface such that half the stored energy lies within it?

In the equation just obtained, let us put

$$\frac{1}{2}U = \frac{q^2}{8\pi\epsilon_0}\int_R^{R_0}\frac{dr}{r^2}$$

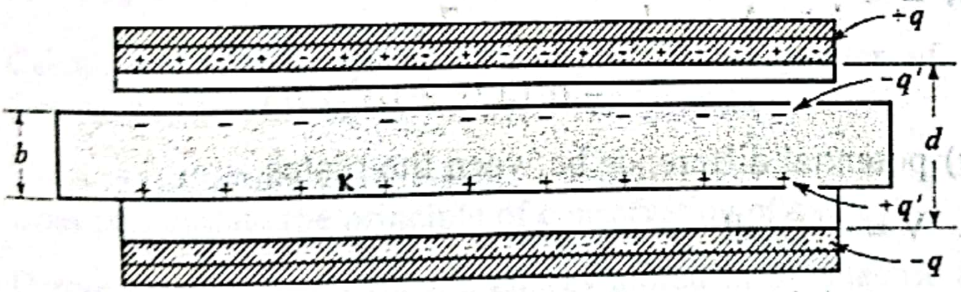
or, $$\frac{q^2}{16\pi\epsilon_0 R} = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{R}-\frac{1}{R_0}\right)$$

or, $$\frac{1}{R}-\frac{1}{R_0} = \frac{1}{2R}$$

or, $R_0 = 2R$.

Example 4.15 Fig shows a dielectric slab of thickness b and dielectric constant k placed between the plates of a parallel-plate capacitor of plate area A and separation d. A potential difference V_0 is applied with no dielectric present. The battery is then disconnected and the dielectric slab inserted. Calculate (i) the capacitance C_0 before the slab is inserted, (ii) the free charge q, (iii) the electric field strength in the air gap, (iv) the electric field strength in the dielectric, (v) the potential difference between the plates and (vi) the capacitance with the slab in place.

Assume $A = 100\text{cm}^2$, $d = 1.0$ cm, $b = 0.50$ cm, $k = 7.0$ and $V_0 =$ 100 volts.



Soln.

(i) Capacitance C_0 before insertion of the slab

$$C_0 = \frac{\epsilon_0 A}{d} = \frac{(8.9\times10^{-12}\,\text{coul}^2/\text{nt-m}^2)(10^{-2}\text{m}^2)}{10^{-2}\text{m}}$$

$$= 8.9\times10^{-12}\text{F} = 8.9\ \text{pF}.$$

(ii) The free charge q

$$q = C_0V_0 = (8.9 \times 10^{-12}F)\,(100\ V)$$
$$= 8.9 \times 10^{-10}\ \text{coul.}$$

(iii)Electric field strength E_0 in the air gap

From Gauss' law for a dielectric, we have

$$\epsilon_0 \oint kE.d\underline{S} = \epsilon_0\, kE_0A = q$$

for air $k = 1$, hence

$$\epsilon_0 E_0\, A = q$$

$$\text{or, } E_0 = \frac{q}{\epsilon_0\, A} = \frac{8.9\times10^{-10}\,\text{coul}}{\left(8.9\times10^{-12}\,\text{coul}^2/\text{nt}-\text{m}^2\right)\left(10^{-2}\,\text{m}^2\right)}$$
$$= 1.0 \times 10^4\ \text{volts/m.}$$

(iv)Electric field strength E in the dielectric

From Gauss' law

$$\epsilon_0 \oint \mathbf{E}\,.\,d\mathbf{S} = q$$

$$\text{or, } \epsilon_0 k\, E\, A = q$$

$$\text{or, } E = \frac{q}{k\epsilon_0\, A} = \frac{E_0}{k} = \frac{1.0\times10^4\,\text{V/m}}{7}$$
$$= 0.1428 \times 10^4\ \text{V/m.}$$

(v) potential difference between the plates

$$V = -\int \mathbf{E}\,.\,d\mathbf{l}$$

For a straight perpendicular path from the lower plate (L) to th upper plate (U),

$$V = -\int_L^U E\cos 180^\circ dl = \int_L^U E\, dl$$
$$= E_0\,(d - b) + Eb$$

$$\text{or, } V = (1.0 \times 10^4\text{V/m})\,(5 \times 10^{-3}\text{m})$$
$$+ (0.1428 \times 10^4\text{V/m})\,(5 \times 10^{-3}\text{m})$$

$= 50V + 7.140V = 57.14V.$

(vi) Capacitance C with the slab in place.

$$C = \frac{q}{V} = \frac{8.9\times10^{-10}\text{coul}}{57.14V}$$

$$= 15.57 \times 10^{-12}F$$

$$= 15.57pF.$$

EXERCISES

1. What is a capacitor? Explain the term capacitance. What do you mean by the charge of a capacitor?

2. What guidelines would you follow to calculate the capacitance of a capacitor? Hence calculate the capacitance of a parallel-plate capacitor.

3. Calculate the capacitance of a spherical capacitor of radii a and b. Hence show that the capacitance of an isolated conductor of radius R is $4\pi\epsilon_0 R$.

4. Calculate the capacitance of a cylindrical capacitor of length L formed by two coaxial cylinders of radii a and b.

5. Show that there is always a loss of energy due to sharing of charge. Does this violate the principle of conservation of energy?

6. Derive an expression for the energy stored in the electric field of a charged capacitor. What is energy density? Obtain an expression for it.

7. What is a dielectric material? For a given potential difference does a capacitor store more or less charge with a dielectric than it does without dielectric (vacuum)?

8. A capacitor is charged by using a battery, which is then disconnected. A dielectric slab is then introduced between the plates. Describe qualitatively what happens to the charge, the capacitance, the potential difference, the electric field strength, and the stored energy.