CHAPTER III

DAMPED AND FORCED HARMONIC OSCILLATION

Motion in a resisting medium-Logarithmic decrement of damped oscillatory system-Power dissipation in damped harmonic oscillation : Quality factor-Example of damped harmonic oscillation : the LCR circuit-Forced vibration : Quality factor-Sharpness of resonance-Phase of the driven oscillator-Power absorption-Driven LCR circuit-Solved problems-Exercises.

3.1 Motion in a resisting medium

When the bob of a simple pendulum is displaced from its mean position and then released, it will execute simple harmonic motion. The lone force acting on the system is the restoring force trying perpetually to bring it back to its mean position. The time period of oscillation depends only on the length of the pendulum and the acceleration due to gravity of the place. As there is no loss of energy by friction or otherwise, the pendulum will go on oscillating with the same time period and amplitude for any length of time. These types of vibrations will be undamped and are referred to as *undamped free vibration.*

In actual practice a simple harmonic oscillator almost always vibrates in a resisting medium, like air, oil, etc. Consequently when the oscillator vibrates in such a medium, energy is dissipated in each vibration in overcoming the opposing frictional or viscous forces. The amplitude of vibration, therefore, goes on decreasing progressively with time. Such forces, which are *non-conservative* in nature, have thus a damping effect on the oscillations and are aptly referred to as *damping, resistive* or *dissipative* forces. In the absence of any such forces, the oscillations will continue indefinitely, without any change in amplitude (Fig. 3.1). But in the presence of these forces, the amplitude of vibration decreases continuously with time and finally the oscillations die out. Such vibrations are called *free damped vibrations.*

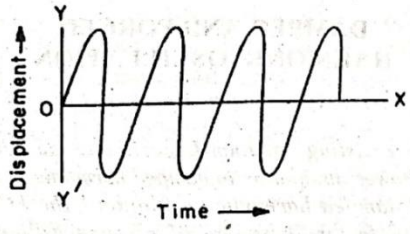

Fig. 3.1

Thus a body executing simple harmonic oscillations in a damping medium will be simultaneously subjected to the following two opposing forces :

(i) the restoring force acting on the body which is proportional to the displacement of the body and acts in a direction opposite to the displacement. Let this force be $-ay$ where a is the force constant.

(ii) a resistive or damping force. It has been shown by *Mayevski* that at ordinary velocities, the opposing, resistive or damping force is, to a first approximation, proportional to the velocity of the oscillating body.

As most cases of interest to us fall in the category of ordinary velocities, the damping or resistive force may thus be represented by

$$F = -bv = -b\frac{dy}{dt}$$

where b is a constant of proportionality. b is a positive constant called *damping coefficient* of the medium and may be looked upon as the resistive or dissipative force per unit velocity. The negative sign signifies a restraining influence on the vibration of the particle.

Thus, the differential equation of motion of a body executing damped harmonic oscillations may be written as

$$m\frac{d^2y}{dt^2} = -ay - b\,.\,\frac{dy}{dt}$$

$$\text{or, } m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ay = 0$$

$$\text{or, } \frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{a}{m}y = 0$$

$$\text{or, } \frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + w^2y = 0 \qquad (3.1)$$

$$\text{where } 2\lambda = \frac{b}{m} \text{ and } w^2 = a/m.$$

Eqn. (3.1) is referred to as the *differential equation of a damped harmonic oscillator*.

Solution of eqn. (3.1) : Eqn. (3.1) which is a *homogeneous linear type differential equation of the second order*, must have at least one solution of the form $y = Ae^{kt}$ where A and k are both arbitrary constants. Let this be used as a trial solution.

Differentiating $y = Ae^{kt}$ with respect to time, we get

$$\frac{dy}{dt} = kAe^{kt} \text{ and } \frac{d^2y}{dt^2} = k^2Ae^{kt}$$

Substituting these values in eqn. (3.1), we have

$$k^2Ae^{kt} + 2\lambda kAe^{kt} + w^2Ae^{kt} = 0$$

Dividing throughout by Ae^{kt}, we have

$$k^2 + 2\lambda k + w^2 = 0 \qquad (3.2)$$

Eqn. (3.2) is clearly a *quadratic equation* in k, the solution of which is

$$k = -\lambda \pm \sqrt{\lambda^2 - w^2}$$

The differential equation (3.1) is, therefore, satisfied by two values of *y*, *viz*,

$$y = Ae^{(-\lambda + \sqrt{\lambda^2 - w^2})t}$$

and

$$y = Ae^{(-\lambda - \sqrt{\lambda^2 - w^2})t}$$

$$\text{or, } m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ay = 0$$

$$\text{or, } \frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{a}{m}y = 0$$

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and

$$y = Ae^{(-\lambda - \sqrt{\lambda^2 - w^2})t}$$

The equation being a linear one, the linear sum of the two linearly independent solutions of the equation is also a – and, indeed, the most general solution. Thus the general solution is

$$y = A_1 e^{(-\lambda+\sqrt{\lambda^2-w^2})t} + A_2 e^{(-\lambda-\sqrt{\lambda^2-w^2})t} \qquad (3.3)$$

where A_1 and A_2 are two arbitrary constants.

The values of the arbitrary constants A_1 and A_2 may be determined as follows:

Differentiating eqn. (3.3) with respect to t, we get

$$\frac{dy}{dt} = (-\lambda+\sqrt{\lambda^2-w^2})A_1 e^{(-\lambda+\sqrt{\lambda^2-w^2})t}$$

$$+(-\lambda-\sqrt{\lambda^2-w^2})A_2 e^{(-\lambda-\sqrt{\lambda^2-w^2})t} \qquad (3.4)$$

Let the maximum value of the displacement y be $y_{max} = a_o$, say at time $t = 0$. Then we have from eqn. (3.3)

$$y_{max} = a_o = A_1 + A_2 \qquad (3.5)$$

Again, the velocity is zero at maximum displacement, *i.e.*, $\frac{dy}{dt} = 0$ at $t = 0$.

Hence from eqn. (3.4), we have

$$(-\lambda+\sqrt{\lambda^2-w^2})A_1 + (-\lambda-\sqrt{\lambda^2-w^2})A_2 = 0$$

$$\text{or, } -\lambda(A_1+A_2) + \sqrt{\lambda^2-w^2}\,(A_1-A_2) = 0$$

$$\text{or, } \sqrt{\lambda^2-w^2}\,(A_1-A_2) = \lambda(A_1+A_2) = \lambda a_O$$

$$\text{or, } A_1 - A_2 = \frac{\lambda a_O}{\sqrt{\lambda^2-w^2}} \qquad (3.6)$$

Adding eqns. (3.5) and (3.6), we have

$$2A_1 = a_0 + \frac{\lambda a_O}{\sqrt{\lambda^2-w^2}}$$

$$\text{or,}\quad A_1 = \frac{1}{2}\left(a_0 + \frac{\lambda a_O}{\sqrt{\lambda^2 - w^2}}\right) = \frac{1}{2}a_0\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - w^2}}\right)$$

$$\text{and}\quad A_2 = a_0 - \frac{1}{2}a_0\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - w^2}}\right) = \frac{1}{2}a_0\left(1 - \frac{\lambda}{\sqrt{\lambda^2 - w^2}}\right)$$

Substituting these values of A_1 and A_2 in eqn. (3.3), we have,

$$y = \frac{1}{2}a_0\, e^{-\lambda t}\left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - w^2}}\right)e^{\left(\sqrt{\lambda^2 - w^2}\right)t} + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - w^2}}\right)e^{\left(-\sqrt{\lambda^2 - w^2}\right)t}\right] \qquad (3.7)$$

Three important cases now arise :

(i) $\lambda^2 > w^2$ **:** Damping is large; $\sqrt{\lambda^2 - w^2}$ is clearly a real quantity with a *positive value, less than* λ. Thus each of the two terms on the right hand side of eqn. (3.7) has an exponential term with a negative power and hence each decreases exponentially with time. In this case the particle does not vibrate. The displacement, after attaining its maximum value falls off asymptotically to zero [curve (i) Fig. 3.2]. There is thus no oscillation and the motion is, therefore, called *overdamped* or *apeeriodic*, or *dead beat*. Examples are a dead beat galvanometer or a pendulum oscillating in a viscous fluid like oil.

(ii) $\lambda^2 = w^2$ **:** In this case, $\sqrt{\lambda^2 - w^2}$ is obviously equal to zero, so that each of the two terms on the right hand side of eqn. (3.7) becomes infinite and the solution breaks down.

Let us, however, consider that λ^2 is not quite equal to w^2, but very nearly so, so that $\sqrt{\lambda^2 - w^2} = h$, *a very small quantity but not zero*. Then from eqn. (3.3), we have

$$y = A_1 e^{(-\lambda + h)t} + A_2 e^{(-\lambda - h)t}$$

$$= e^{-\lambda t}(A_1 e^{ht} + A_2 e^{-ht})$$

$$\text{or, } y = e^{-\lambda t}\left[A_1\left(1 + ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + \ldots\right)\right.$$

$$\left. + A_2\left(1 - ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + \ldots\right)\right]$$

Neglecting terms containing second and higher powers of h, we get

$$y = e^{-\lambda t}[A_1(1 + ht) + A_2(1 - ht)]$$

$$= e^{-\lambda t}[(A_1 + A_2) + (A_1 - A_2)\,ht] \qquad (3.8)$$

Putting $(A_1 + A_2) = M$ and $(A_1 - A_2)\,h = N$, we have

$$y = e^{-\lambda t}(M + Nt)$$

Recalling that $y = y_{max} = a_0$ and $\frac{dy}{dt} = 0$ at $t = 0$, we have

$$M = y_{max} = a_0 \text{ and } N = \lambda a_0.$$

$$\left[A_1 - A_2 = \frac{\lambda a_0}{\sqrt{\lambda^2 - w^2}} = \frac{\lambda a_0}{h}; \quad \text{or, } (A_1 - A_2)\,h = N = \lambda a_0\right].$$

Eqn. (3.8), therefore, becomes

$$y = e^{-\lambda t}(a_0 + \lambda a_0 t)$$

$$= a_0 e^{-\lambda t}(1 + \lambda t) \qquad (3.9)$$

The second term in eqn. (3.9), viz, $a_0\lambda t e^{-\lambda t}$ decays less rapidly than the first term, viz, $a_0 e^{-\lambda t}$ and the displacement of the oscillator first increases but as t increases the exponential factor $e^{-\lambda t}$ becomes more important and the displacement decreases rapidly reaching the value zero for a finite value of t. The oscillator just ceases to oscillate and its motion just becomes *aperiodic* or *non-oscillatory*. This is called the case of *critical damping* [curve (ii) in Fig. 3.2] the necessary condition for which is that *λ^2 should be almost, but not quite, equal to w^2* (i.e., $\lambda^2 \rightarrow w^2$).

This principle finds application in many pointer-type instruments like galvanometers where the pointer moves at once to, and stays at, the correct position, without any annoying oscillations.

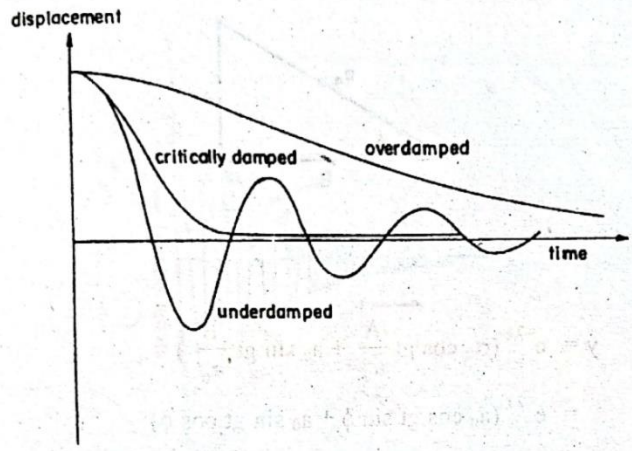

Fig. 3.2

It may be seen that although the motion is non-oscillatory both in case of critical damping and overdamping the time taken to reach near the equilibrium position from a given displacement becomes greater and greater with the increase of damping.

(iii) $\lambda^2 < w^2$: the quantity $\sqrt{\lambda^2 - w^2}$ is clearly imaginary, say, equal to ig where $i\sqrt{-1}$ and $g = \sqrt{w^2 - \lambda^2}$ a real quantity.

Eqn. (3.3) then becomes

$$y = A_1 e^{(-\lambda + ig)t} + A_2 e^{(-\lambda - ig)t}$$

or, $y = e^{-\lambda t}(A_1 e^{igt} + A_2 e^{-igt})$

$$= e^{-\lambda t}[A_1 (\cos gt + i \sin gt) + A_2 (\cos gt - i \sin gt)]$$

$$= e^{-\lambda t}[(A_1 + A_2) \cos gt + i (A_1 - A_2) \sin gt]$$

Putting $(A_1 + A_2) = A$ and

$i (A_1 - A_2) = B$, we have

$$y = e^{-\lambda t}(A \cos gt + B \sin gt)$$

if A, B and a_0 be related as shown in Fig. 3.3.

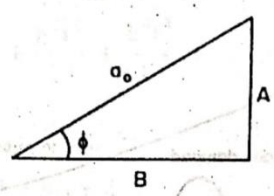

Fig. 3.3

$$y = e^{-\lambda t}\left(\alpha_0 \cos gt \frac{A}{a_0} + a_0 \sin gt \frac{B}{a_0}\right)$$

$$= e^{-\lambda t}(a_0 \cos gt \sin \phi + a_0 \sin gt \cos \phi)$$

$$= e^{-\lambda t}\, a_0 \sin (gt + \phi)$$

$$= a_0 e^{-\lambda t} \sin (gt + \phi) \qquad (3.10)$$

Eqn. (3.10) is the equation of a damped harmonic oscillator with *amplitude* $a_0 e^{-\lambda t}$ and *frequency* $g/2\pi = (\sqrt{w^2 - \lambda^2}\,) / 2\pi)$. It is so called because the sine term in the equation suggests the oscillatory character of the motion and the exponential term, the gradual damping out of the oscillations [curve (iii) Fig. 3.2].

Thus damping clearly produces two effects :

(i) The frequency of the damped harmonic oscillator, $g/2\pi$ is smaller than its natural or undamped frequency. This means that damping somewhat decreases the frequency or increases the time-period of the oscillator. In actual practice, the damping is small in majority of cases (particularly in the case of musical instruments) and its effect on the time-period or frequency of oscillation is, therefore, quite negligible.

(ii) The amplitude of the oscillation does not remain constant at a_0, which represents the amplitude in the absence of any damping, but decays exponentially with time, to zero, in accordance with the

term $e^{-\lambda t}$. The term $e^{-\lambda t}$ is called the *damping factor*. The time displacement curve of a damped harmonic oscillator is illustrated in Fig. 3.4.

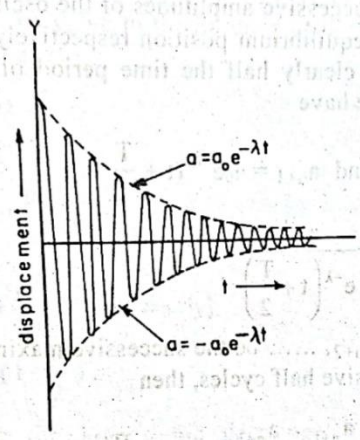

Fig. 3.4

Since the maximum values of sin (gt + ϕ) alternate between + 1 and – 1, the time displacement curve of the oscillating particle is bound by the dotted curves $a = a_0 e^{-\lambda t}$ and $a = -a_0 e^{-\lambda t}$.

Thus although the amplitude of an underdamped harmonic oscillator decreases exponentially with time, it does perform a sort of oscillatory motion. The motion does not, of course, repeat itself and is thus not periodic in the usual sense of the term. However, it still has a time period $2\pi/g = 2\pi/\sqrt{w^2 - \lambda^2}$ which is the time interval between its successive passages in the same direction past the equilibrium point. It is obviously also the time interval between successive maximum displacements on the same side of the position of equilibrium.

Note : *Even if the body vibrates in vacuum, thereby eliminating external force due to the friction of air, still the vibration is not free. Viscosity or internal friction arises within the material of the body during its vibration. This internal friction ultimately brings the vibration to a stop.*

Thus,

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy lost per period}}$$

$$= \frac{2\pi E}{P.T.}$$

where P is the rate of loss of energy = $2\lambda E$.

Now $\frac{2\pi}{T} = g$ where g is the frequency of the damped harmonic oscillator. Then

$$Q = \frac{Eg}{P} = \frac{Eg}{2\lambda E} = \frac{g}{2\lambda}$$

In the case of low damping $g \simeq w$; therefore,

$$Q = \frac{w}{2\lambda}. \qquad (3.18)$$

The quality factor, as the name suggests, measures the quality of a harmonic oscillator in so far as damping is concerned. The less the damping, the better the quality of the harmonic oscillator as an oscillator and, therefore, the higher its quality factor Q.

We also know that

$$w = \sqrt{\frac{a}{m}} \text{ and } 2\lambda = \frac{b}{m}$$

Hence,

$$Q = \frac{w}{2\lambda} = \sqrt{\frac{a}{m}} \cdot \frac{m}{b} = \frac{\sqrt{am}}{b} \qquad (3.19)$$

clearly indicating that lower the value of damping, *i.e.*, b, the higher the value of Q.

Or, as $b \rightarrow 0$, $Q \rightarrow \infty$

3.5 Example of damped harmonic oscillation – the LCR circuit

An electrical circuit which contains inductance (L), capacitance (C) and resistance (R) renders an excellent example of damped harmonic

oscillation. The resistance by itself plays the part of a resistive or dissipative force analogous to that of friction or viscosity in the case of mechanical oscillations.

Fig. 3.5 shows a circuit containing a resistance, an inductance and a capacitance connected in series. A battery can be included into the circuit through the key K. When K is pressed, the capacitor is charged. On releasing K, the battery is thrown out of the circuit and the capacitance discharges through the resistance and the inductance.

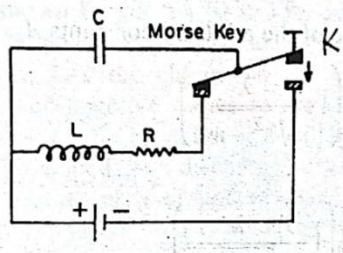

Fig. 3.5

Suppose that at a given instant t, the charge on the capacitor is Q and the current flowing through the circuit is i. Then the induced *emf.* across the inductance L is $L\frac{di}{dt}$ and the potential difference across the resistance R is Ri. Since there is no external *emf.* in the circuit now (the battery being disconnected), then according to Kirchoff's law,

$$Ri + \frac{Q}{C} = -L\frac{di}{dt}$$

$$\text{or, } L\frac{di}{dt} + Ri + \frac{Q}{C} = 0 \qquad (3.20)$$

But $i = \frac{dQ}{dt}$; therefore eqn. (3.20) becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

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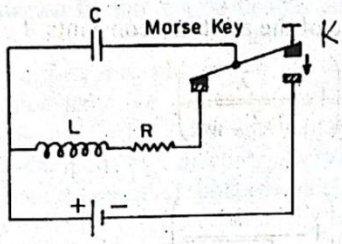

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$$\text{or, } L\frac{di}{dt} + Ri + \frac{Q}{C} = 0 \qquad (3.20)$$

But $i = \frac{dQ}{dt}$; therefore eqn. (3.20) becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$\text{or,} \quad \frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0 \qquad (3.21)$$

Eqn. (3.21) is very similar to eqn. (3.1), the differential equation of damped harmonic oscillation, with the difference that the displacement y is replaced by charge Q, 2λ by R/L and w^2 by $1/LC$.

Proceeding in the same manner as described in Art. 3.1, the solution of eqn. (3.21) is

$$Q = A_1 e^{\left(-\lambda+\sqrt{\lambda^2-w^2}\right)t} + A_1 e^{\left(-\lambda-\sqrt{\lambda^2-w^2}\right)t}$$

where the value of the arbitrary constants A_1, and A_2 are

$$A_1 = \frac{1}{2}Q_0\left(1+\frac{\lambda}{\sqrt{\lambda^2-w^2}}\right)$$

and

$$A_2 = \frac{1}{2}Q_0\left(1-\frac{\lambda}{\sqrt{\lambda^2-w^2}}\right)$$

Thus,

$$Q = \frac{1}{2}Q_0 e^{-\lambda t} \times \left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-w^2}}\right) e^{\left(\sqrt{\lambda^2-w^2}\right)t} + \left(1-\frac{\lambda}{\sqrt{\lambda^2-w^2}}\right) e^{\left(-\sqrt{\lambda^2-w^2}\right)t}\right] \qquad (3.22)$$

Again the following three cases arise :

(i) when the damping is large, *i.e.*,

$$\lambda^2 > w^2 \left(\text{or, } \frac{R^2}{4L^2} > \frac{1}{LC} \text{ or, } R^2 > \frac{4L}{C}\right)$$

both the terms on the right hand side of eqn. (3.22) have exponential terms with a negative power. The charge (Q), therefore, decays exponentially with time without changing sign *i.e.*, without

making any oscillation. Thus the discharge of the capacitor is non-oscillatory or damping [curve (i) Fig. 3.6].

(ii) when the damping is critical *i.e.*, when $\lambda^2 = w^2$ (or, $R^2 = 4L/C$). Here the discharge of the condenser just becomes aperiodic and dies down in the shortest possible time as shown by curve (ii), Fig. (3.6).

(iii) when the damping is small, i.e., $\lambda^2 < w^2$ (or, $R^2 < \frac{4L}{C}$). The equation for the discharge of charge takes the form

$$Q = Q_0 e^{-\lambda t} \sin (gt + \phi) \qquad (3.23)$$

viz. that of a damped oscillator (see Art. 3.1). The discharge curve of the capacitor is shown by curve (iii) in Fig. 3.6. Thus although the charge decreases exponentially with time, it does perform a sort of oscillatory motion, *i.e.*, the charge on the condenser alternately acquires positive and negative values before ultimately decaying to zero. The discharge of the capacitor is oscillatory in character, the oscillations being damped. The discharge curve is identical in form with the displacement curve of underdamped harmonic oscillator (curve (iii) Fig. 3.2) and is similarly bound by the curves

$$Q = + Q_0 e^{-\lambda t} \text{ and } Q = - Q_0 e^{-\lambda t}$$

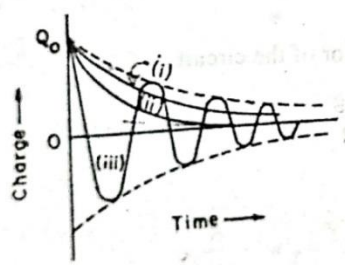

Fig. 3.6

Differentiating eqn. (3.23) with respect to time, we have

$$\frac{dQ}{dt} = \frac{dQ_0}{dt} e^{-\lambda t} \sin (gt + \phi)$$

Since $\frac{dQ}{dt} = i$ and $\frac{dQ_0}{dt} = i_0$, the maximum value or the amplitude of the current, we have

$$i = i_0 e^{-\lambda t} \sin (gt + \phi)$$

The frequency of oscillation is

$$n = \frac{g}{2\pi} = \frac{\sqrt{w^2 - \lambda^2}}{2\pi}$$

$$= \frac{1}{2\pi} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \qquad (3.24)$$

Thus the resistance R alone is responsible for the damping of the oscillations in this case (the energy dissipated appearing as heat) as can be seen from eqn (3.24). In the absence of R *i.e.*, if $R = 0$, the frequency of oscillation becomes

$$n = \frac{1}{2\pi} \sqrt{\frac{1}{LC}}$$

The same as that of an LC circuit, with no damping of the oscillation.

The quality factor of the circuit,

$$Q = \frac{g}{2\lambda} = \frac{Lg}{R}$$

where

$$g = \sqrt{w^2 - \lambda^2} = \sqrt{w^2 - \frac{R^2}{4L^2}}$$

Since R is small, $g \simeq w$, and, therefore, the

$$\text{Quality factor, } Q = \frac{Lw}{R} \qquad (3.25)$$

Some useful relations for solving problems :

From Art. 3.1, we know that for ordinary velocities, the opposing resistive force is given by

$$F = -bv = -b\frac{dy}{dt}$$

If there is no other force acting on the body, then according to Newton's second law of motion

$$F = m\frac{d^2y}{dt^2} = m\frac{dv}{dt} = -bv$$

$$\text{or,} \quad \frac{dv}{dt} + \frac{b}{m}v = 0 \qquad \text{(i)}$$

where $\frac{b}{m}$, denoted by 2λ in the earlier articles, is the resistive force per unit mass per unit velocity. λ is the damping constant of the medium. The term $\frac{m}{b}$ (reciprocal of $\frac{b}{m}$) is also denoted by a constant τ ($= \frac{1}{2\lambda}$).

Eqn. (i) may be rewritten

$$\frac{dv}{v} = -\frac{dt}{\tau}$$

which on integration yields

$$v = v_0 e^{-t/\tau} \qquad \text{(ii)}$$

where v_0 is the velocity of the body at time $t = 0$.

Eqn. (ii) clearly shows that velocity decreases exponentially with time. This is expressed by saying that velocity is damped with time constant τ.

When $t = \tau$, then

$$v = v_0 e^{-\tau/\tau} = v_0 e^{-1} = 0.368\, v_0$$

Thus the *time constant* τ may be defined as the time in which the velocity of the oscillating particle falls to $1/e^{th}$ (*i.e.*, 0.368 or

roughly, one-third) of its initial value. τ, therefore, has the dimension of time and is also referred to as the relaxation time.

Now the kinetic energy of the oscillating particle is given by $T = \frac{1}{2} mv^2$. Substituting the value of v from eqn. (ii), we have $T = \frac{1}{2} mv_0^2 e^{-2t/\tau}$. If $\frac{1}{2} mv_0^2 = T_0$, the *initial kinetic energy of the the particle, then* $T = T_0 e^{-2t/\tau}$. *This indicates that like velocity, K.E. of the oscillating particle also falls exponentially with time, but with a relaxation time half that for velocity, i.e.,* $\tau/2$, *which, of course is expected since KE.* $\propto$ *(velocity)*^{2}.

Example 3.1. *An underdamped harmonic oscillator has its amplitude reduced to 1/10th of its initial value after 100 oscillations. Its time period is 1.15 sec. (i) Calculate the damping constant and the relaxation time, (ii) If the observed value of the first amplitude of the oscillator be 2 cm, what would be its value in the absence of damping ?*

(iii) Calculate the time in which its amplitude and its energy falls to 1/eth of its undamped value. If the mass of the oscillator be 1.127 gm, calculate its average rate of loss of energy.

Soln.

(i) The first amplitude (a_1) obviously occurs at t = T/4 where T is the time-period of oscillation.

Now the amplitude of a damped harmonic oscillator is given by $a = a_0 e^{-\lambda t}$ where a_0 is its amplitude in the absence of damping and λ, the damping constant. Then for the first amplitude we have

$$a_1 = a_0 e^{-\lambda T/4}$$

successive amplitudes occur at intervals of T/2 sec each, the 201th amplitude after 100 oscillations occurs after (200 T/2 + T/4) = (100 T + T/4) secs. Therefore, we have

$$a_{201} = a_0 e^{-\lambda(100\,T + T/4)}$$

Hence $$\frac{a_{201}}{a_1} = \frac{a_0 e^{-\lambda(100T+T/4)}}{a_0 e^{-\lambda T/4}}$$

$$= e^{-100\lambda T} = \frac{1}{10}$$

$$\text{or, } e^{-100\lambda(1.15)} = 10^{-1}$$

$$\text{or, } e^{115\lambda} = 10$$

$$\text{or, } 115\lambda = \log_e 10$$

$$\text{or, } 115\lambda = 2.30$$

$$\text{or, } \lambda = 0.02.$$

Thus the damping constant = **0.02.**

The relaxation time $\tau = \frac{1}{2\lambda} = \frac{1}{2 \times 0.02}$ = **25 sec.**

(ii) The first amplitude,

$$a_1 = a_0 e^{-\lambda T/4} = a_0 e^{-\left(0.02 \times \frac{1.15}{4}\right)}$$

$$\text{or, } 2 = a_0 e^{-\frac{0.023}{4}}$$

$$\text{or, } a_0 = 2.e^{\frac{0.023}{4}}$$

$$\text{or, } \log_{10} a_0 = \log_{10} 2 + \frac{0.023}{4} \log_{10} e$$

$$\text{or, } \log_{10} a_0 = 0.3010 + \frac{0.023}{4} \times \frac{1}{2.3}$$

$$\text{or, } a_0 = \mathbf{2.011\ cm.}$$

Thus the undamped amplitude of the oscillator would be **2.011 cm.**

(iii) Let the amplitude to be reduced to $1/e^{th}$ of its undamped value after time t from the start. Then

$$a = \frac{a_0}{e} = a_0 e^{-1} = a_0 e^{-\lambda t}$$

$$\text{or, } \lambda t = 1 \text{ or, } t = \frac{1}{0.02} = 50 \text{ sec.}$$

The amplitude of the oscillator falls to $1/e^{th}$ of its undamped value in **50 sec.**

Similarly from the relation $E = E_0 e^{-2\lambda t}$, we have

$$E = \frac{E_0}{e} = E_0\, e^{-1} = E_0 e^{-2\lambda t}$$

or, $2\lambda t = 1$

$$\text{or, } t = \frac{1}{2\lambda} = \frac{1}{2 \times 0.02} = 25 \text{ sec.}$$

Thus the energy of the oscillator falls to $1/e^{th}$ of its value in the absence of damping in **25 sec.**

The average rate of loss of energy

$$= \frac{E}{\tau} = 2\lambda E.$$

Now, $E = \frac{1}{2} m\, a_0^2 w^2 e^{-2\lambda t}$

$m = 1.127$ gm

$a_0 = 2.011$ cm

$w = \frac{2\pi}{T} = \frac{2\pi}{1.15}$

$\lambda = 0.02$

$t = 25$ sec.

rate of loss of energy

$$= 2 \times 0.02 \times \frac{1}{2} \times 1.12 \times (2.011)^2 \times \left(\frac{2\pi}{1.15}\right)^2 \times e^{-2(0.02 \times 25)}$$

$$= \frac{0.02 \times 1.127 \times (2.011)^2 \times 4\pi^2}{(1.15)^2 \times e}$$

$$= \frac{0.02 \times 1.127 \times (2.011)^2 \times 4\pi^2}{(1.15)^2 \times 2.7813}$$

= 1 erg/sec.

Example 3.2. *A massless spring, suspended from a rigid support, carries a mass of 500 gm at its lower end and the system oscillates with a frequency of 5/sec. If the amplitude is reduced to half its undamped value in 20 sec, calculate (i) the force constant of the spring, (ii) the relaxation time of the system and (iii) its quality factor.*

(iv) What would be the quality factor of the system if the suspended mass be reduced to 150 gm ?

Soln.

(i) The force constant *a* of the mass-spring system is given by *w*

$$= \sqrt{\frac{a}{m}}$$

where w = angular frequency

$= 2\pi n = 2\pi . 5 = 10\pi$ rad / sec.

m = mass of the system = 500 gm.

Then, $w^2 = \frac{a}{m}$

or, $a = m.w^2 = 500 \times (10\,\pi)^2$

$= 4.934 \times 10^5$ dynes/cm.

(ii) The amplitude is reduced to half its undamped value in 20 sec.

or, from $a = a_0 e^{-\lambda t}$ we have

$$a = \frac{a_0}{2} = a_0 e^{-\lambda \times 20}$$

or, $e^{-20\lambda} = \frac{1}{2}$

or, $20\,\lambda = \log_e 2$

or, $20\,\lambda = 0.693$

or, $\lambda = \frac{0.693}{20}$

The relaxation time $\tau = \frac{1}{2\lambda}$

$$\therefore \quad \tau = \frac{20}{2 \times 0.693} = \mathbf{14.44\ sec.}$$

(iii) The quality factor of the system,

$$Q = \frac{w}{2\lambda} = \frac{10\pi \times 20}{2 \times 0.693} = \mathbf{453.7}$$

(iv) The force constant *a* remains the same irrespective of the mass suspended from the spring and so does the damping factor b.

The damping factor $b = 2\lambda m$

$$= 2 \times \frac{0.693}{20} \times 500$$

$$= 115 \times 0.3010$$

when the mass is reduced to 150 gm, the frequency $w' = \sqrt{\frac{a}{m'}}$

$$= \sqrt{\frac{5\pi^2 \times 10^4}{150}} = 18.26\pi.$$

The relaxation time now, $\tau' = \frac{1}{2\lambda} = \frac{m'}{b} = \frac{150}{115 \times 0.3010}$

So the quality factor now,

$$Q' = \frac{w'}{2\lambda} = w' \cdot \tau'$$

$$= \frac{18.26\pi \times 150}{115 \times 0.3010} = \mathbf{248.5.}$$

Example 3.3. *Deduce the frequency and quality factor for a circuit with L = 2 mH, C = 5μF and R = 0.2 ohm.*

Soln.

The circuit is equivalent to a damped (or underdamped) harmonic oscillator. The angular frequency of oscillation is given by $g = \sqrt{w^2 - \lambda^2}$ where *w* is the natural angular frequency of the oscillator and λ is the damping constant. Hence the frequency of oscillation

$$n = \frac{g}{2\pi} = \frac{\sqrt{w^2 - \lambda^2}}{2\pi}$$

$$\text{or,}\quad n = \frac{1}{2\pi} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Here L = 2mH = 0.002 henry

$C = 5\mu F = 5 \times 10^{-6}$ farad

R = 0.2 ohm.

$$\therefore\quad n = \frac{1}{2\pi}\sqrt{\frac{1}{0.002 \times 5 \times 10^{-6}} - \frac{(0.2)^2}{4 \times (0.002)^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{4 \times 10^8 - 10^4}{4}}$$

$$= \frac{10^2}{2\pi}\sqrt{10^4 - 1} = \frac{10^4}{2\pi}$$

$$= 1.592 \times 10^3 \text{ cps.}$$

Since the resistance is small, the quality factor $Q = \frac{Lw}{R} = \frac{Lg}{R}$

where $g = 2\pi n$

$$= \frac{2\pi \times 10^4}{2\pi} = 10^4 \text{ rad/sec.}$$

$\therefore$ Quality factor for the circuit,

$$Q = \frac{0.002 \times 10^4}{0.2} = 0.01 \times 10^4 = \mathbf{100.}$$

Example 3.4. A particle of mass 3 gm is subjected to an elastic force of 48 dyne – cm^{-1} and a damping force of 12 dyne – cm^{-1} sec. If the motion is oscillatory, find its period.

Soln.

$$w = \sqrt{\frac{a}{m}} = \sqrt{\frac{48 \text{ dyne} - \text{cm}^{-1}}{3 \text{ gm}}} = 4 \text{ sec}^{-1}$$

and $2\lambda = \frac{b}{m}$; or, $\lambda = \frac{b}{2m}$

$$= \frac{12}{2 \times 3} = 2 \text{ sec}^{-1}.$$

since $w > \lambda$, the motion is oscillatory.

Frequency of the oscillatory motion,

$$\frac{g}{2\pi} = \frac{\sqrt{w^2 - \lambda^2}}{2\pi}$$

Hence the period of the motion

$$= \frac{1}{g/2\pi} = \frac{2\pi}{\sqrt{w^2 - \lambda^2}}$$

$$= \frac{2\pi}{\sqrt{4^2 - 2^2}} = \frac{2\pi}{\sqrt{12}}$$

= 1.81 sec. approx.

Example 3.5. *Suppose a tuning fork in air has a frequency $g/2\pi$ = 200 cps and its oscillations die away to 1/e of its former amplitude in one second. Show that the reduction in frequency brought about by air damping is exceedingly small.*

Soln.

For damped vibrations,

$$\frac{g}{2\pi} = \frac{\sqrt{w^2 - \lambda^2}}{2\pi}$$

Let the amplitude at time t be given by

$$y = Ae^{-\lambda t}$$

Then the amplitude after one second,

$$y' = \frac{y}{e} = Ae^{-\lambda(t+1)} \qquad \text{(ii)}$$

Dividing (ii) by (i)

$$\frac{1}{e} = e^{-\lambda}; \quad \text{or, } e^{-1} = e^{-\lambda}$$

or, $\lambda = 1$

Hence, $\frac{g}{2\pi} = 200 = \frac{\sqrt{w^2 - 1}}{2\pi}$

Therefore, the value of w is

$$w = [(2\pi \times 200)^2 + 1]^{1/2}$$

Hence $\frac{w}{2\pi} = [(200)^2 + \frac{1}{4\pi^2}]^{1/2}$

Since $\frac{1}{4\pi^2}$ is negligible as compared to $(200)^2$, damping due to air has only a negligible effect on the frequency of the tuning fork.

3.6 *Forced vibration*

The time period of a body executing simple harmonic motion depends upon the dimensions of the body and its elastic properties. When the body oscillates in a medium like air, its oscillations, as we know get damped, *i.e.*, its amplitude falls exponentially with time to zero due to dissipation of energy.

If, however, an external periodic force is constantly applied to the oscillator, of a frequency not necessarily the same as the natural frequency of the oscillator, a sort of tussle ensues between the damping force and the applied force. The damping force tends to retard the motion of the body and the applied force tends to maintain it. Initially, the amplitude of the oscillations increases, then decreases with time, becomes minimum and again increases. After some initial erratic movements, the body ultimately succumbs to the applied or driving force and settles down to oscillating with the frequency of the applied or driving force and a constant amplitude and phase so long as the applied force remains operative. Such *vibrations of the body are called forced (or driven) vibrations.* The amplitude of this forced vibration of the body depends on the difference between the natural frequency and the frequency of the

applied force. The amplitude will be large if difference in frequencies is small and vice versa.

Let the periodic force which is applied on a damped harmonic oscillator be $F = F_0 \sin pt$. This is obviously a sinusoidal force of amplitude F_0 and frequency $p/2\pi$.

Now the damping ad restoring forces on the oscillator are $-b\frac{dy}{dt}$ and $-ay$ respectively where b and a have the same meaning as in Art. 3.1. Hence the equation of motion for forced vibration may be written as

$$m\frac{d^2y}{dt^2} = -b\frac{dy}{dt} - ay + F$$

$$\text{or, } m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ay = F_0 \sin pt$$

$$\text{or, } \frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + w^2 y = f_0 \sin pt \qquad (3.26)$$

$$\text{where } 2\lambda = \frac{b}{m},\ w^2 = \frac{a}{m} \text{ and } f_0 = \frac{F}{m}$$

After the initial tussle between the damping and the applied force dies down, a steady state is attained and the oscillator settles down to oscillate with the frequency $p/2\pi$ of the applied force and a constant amplitude. Let a particular solution of the equation of motion of the body as given by eqn. (3.26), after the steady state has been attained, be

$$y = A \sin (pt - \theta) \qquad (3.27)$$

where A is the amplitude and θ is the possible phase difference between the applied force and the displacement of the oscillator.

Then, we have

$$\frac{dy}{dt} = Ap \cos (pt - \theta) \qquad (3.28)$$

$$\text{and } \frac{d^2y}{dt^2} = -Ap^2 \sin (pt - \theta) = -p^2 y \qquad (3.29)$$

Substituting these values of $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in eqn. (3.26), we get

$$-Ap^2 \sin(pt-\theta) + 2\lambda Ap \cos(pt-\theta) + w^2 A \sin(pt-\theta) = f_0 \sin pt$$

$$= f_0 \sin[(pt-\theta)+\theta]$$

$$= f_0 \sin(pt-\theta)\cos\theta + f_0 \cos(pt-\theta)\sin\theta$$

or, $A(w^2-p^2)\sin(pt-\theta) + 2\lambda Ap \cos(pt-\theta)$

$$= f_0 \cos\theta \sin(pt-\theta) + f_0 \sin\theta \cos(pt-\theta) \qquad (3.30)$$

If this solution is to hold good for all values of t, the respective coefficients of sin (pt – θ) and cos (pt – θ) on either side of eqn. (3.30) must be equal. Thus we must have

$$A(w^2-p^2) = f_0 \cos\theta \quad \ldots\ldots\ldots \quad (i)$$

$$\text{and } 2\lambda Ap = f_0^2 \sin\theta \quad \ldots\ldots\ldots \quad (ii)$$

Squaring and adding eqns. (i) and (ii), we have

$$A^2(w^2-p^2)^2 + 4\lambda^2 A^2 p^2 = f_0^2 \cos^2\theta + f_0^2 \sin^2\theta$$

or, $A^2[(w^2-p^2)^2 + 4\lambda^2 p^2 = f_0^2$

$$\text{or, } A^2 = \frac{f_0^2}{(w^2-p^2)+4\lambda^2 p^2}$$

Thus the amplitude of the driven or forced oscillator is

$$A = \frac{f_0^2}{(w^2-p^2)+4\lambda^2 p^2} \qquad (3.31)$$

Only the positive value of the square root has been considered in eqn. (3.31). Its negative value will mean opposite phase but then θ too will change by π.

Thus there will be no effect on the value of A.

The value of θ is given by the expression

$$\tan\theta = \frac{f_o \sin\theta}{f_o \cos\theta} = \frac{2\lambda Ap}{A\left(w^2 - p^2\right)} = \frac{2\lambda p}{\left(w^2 - p^2\right)}$$

Thus the phase difference between the driven or forced oscillator and the applied force is

$$\theta = \tan^{-1} \frac{2\lambda p}{\left(w^2 - p^2\right)} \qquad (3.32)$$

Since sin θ is positive, it follows that θ must lie between 0 to π.

Substituting these value in eqn. (3.27) we get

$$y = \frac{f_o}{\sqrt{\left(w^2 - p^2\right)^2 + 4\lambda^2 p^2}} \times \sin\left[\, pt - \tan^{-1} \frac{2\lambda p}{\left(w^2 - p^2\right)} \,\right] \qquad (3.33)$$

Eqn. (3.33) represents a S.H.M. of frequency $p/2\pi$ *i.e.*, the same as that of the driving force, but lagging behind it in phase by $\theta = \tan^{-1} [2\lambda p/(w^2 - p^2)]$ where θ lies between 0 and π.

Now, because of the presence of the term $f_0 \sin pt$ which contains neither the variable (y) or its derivative; eqn. (3.26) is an inhomogeneous differential equation and hence $y = A \sin (pt - \theta)$ is not really its complete solution. The solution will, therefore, be complete only if we add to it a complementary function which is a solution of the related homogeneous equation $\frac{d^2y}{dt^2} + 2\lambda \frac{dy}{dt} + w^2 y = 0$

One such solution, as we know, is

$y = a_0 e^{-\lambda t} \sin (gt + \phi)$ which represents a damped harmonic oscillation of frequency $\frac{g}{2\pi} = \frac{\sqrt{w^2 - \lambda^2}}{2\pi}$. The addition of this term to the solution $y = A \sin (pt - \theta)$ does not, in any way, impair the validity of the latter, since the term, taken by itself, reduces the left hand side of eqn. (3.26) to zero.

Thus the complete solution of eqn. (3.26) is

$$y = a_0 e^{-\lambda t} \sin(gt + \phi) + A \sin(pt - \theta)$$

where, the first term on the right hand side represents an initial damped oscillation of frequency $g/2\pi$ with its amplitude decaying exponentially to zero while the second term represents a forced vibration of frequency $p/2\pi$ and a constant amplitude A. The former (damped) oscillation dies out quickly and the latter alone then remains effective, so that one is left with $y = A \sin(pt - \theta)$ as the equation of motion of the forced (or driven) oscillation, with its amplitude A and phase angle θ as given by eqns. (3.31) and (3.32) respectively.

Maximum displacement of a driven oscillator — Resonance.

Although the period of the forced vibration will eventually be same as that of the applied force, its amplitude, irrespective of the magnitude of the damping forces present, will depend on the ratio between its natural frequency and the frequency of the applied force. However, the amplitude will have a definite value for a particular amount of damping. This means that if the damping (resisting) forces acting on the body are changed without changing the ratio between its natural frequency and the frequency of the applied force, the amplitude will change due only to change in damping.

When the damping *i.e.*, λ has a finite value greater than zero, the value of the amplitude, as given by eqn. (3.31) will obviously be maximum when the denominator in the equation has its minimum value, *i.e.*, when

$$\frac{d}{dp}[(w^2 - p^2)^2 + 4\lambda^2 p^2] = 0$$

$$\text{or, } -2(w^2 - p^2) \cdot 2p + 4\lambda^2 2p = 0$$

$$\text{or, } (w^2 - p^2) = 2\lambda^2$$

$$\text{or, } p^2 = w^2 - 2\lambda^2$$

$$\text{or, } p = \sqrt{w^2 - 2\lambda^2}$$

Thus, the amplitude will be maximum when the driving frequency $p/2\pi = \frac{\sqrt{w^2 - 2\lambda^2}}{2\pi}$. Let us denote this particular value of the frequency by $p_r/2\pi$. This state of vibration when the amplitude of the driven oscillator becomes maximum is called *resonance* (or *amplitude resonance*). The particular forced frequency $(p_r/2\pi)$ for which resonance takes place is called *resonant frequency*.

In the absence of damping, resonance takes place when the natural frequency of the oscillator is equal to the frequency of the driving force. But when damping is present, the resonant frequency $p_r/2\pi = \left(\sqrt{w^2 - 2\lambda^2}\right)/2\pi$ is smaller than the natural, undamped frequency $w/2\pi$ or the natural, damped frequency $g/2\pi = \left(\sqrt{w^2 - \lambda^2}\right)/2\pi$ of the oscillator.

Substituting $p_r = \sqrt{w^2 - \lambda^2}$ for p in the expression for A as given by eqn. (3.31), we have

maximum amplitude,

$$A_{max} = \frac{f_o}{\sqrt{(w^2 - p^2) + 4\lambda^2 p^2}}$$

$$= \frac{f_o}{\sqrt{\left[w^2 - (w^2 - 2\lambda^2)\right]^2 + 4\lambda^2 (w^2 - 2\lambda^2)}}$$

$$= \frac{f_o}{\sqrt{4\lambda^4 + 4\lambda^2 w^2 - 8\lambda^4}}$$

$$= \frac{f_o}{\sqrt{4\lambda^2 w^2 - 4\lambda^4}}$$

$$= \frac{f_o}{\sqrt{4\lambda^4 (w^2 - \lambda^2)}}$$

$$= \frac{f_o}{2\lambda\sqrt{w^2 - \lambda^2}} = \frac{f_o}{2\lambda (w^2 - \lambda^2)^{1/2}} \qquad \text{(i)}$$

If there is no damping, *i.e.*, if $\lambda = 0$, then the amplitude A_{max} tends to infinity. This, however, never happens since damping is never actually zero. When it is very small λ^2 will be negligibly small compared to w^2 and the maximum amplitude becomes $A_{max} = f_0/2\lambda w$. Hence smaller the value of λ larger the value of the maximum amplitude. Fig. 3.7 illustrates this point.

Further, it is clear from the expression for A itself (eqn. 3.31) that the magnitude of A is controlled by the factor $(w^2 - p^2)$ *i.e., A depends upon the relative values of p and w*. Thus the value of A diminishes both when $p > w$ and when $p < w$. Fig. 3.7 shows a number of curves plotted with amplitude A along the Y-axis and p along the X-axis. Each curve corresponds to a given value of damping (λ). It will be noted that

(i) The peak value of amplitude $A_{max} = f_0/2\lambda w$ occurs when $(w^2 - p^2) = 0$ *i.e.*, when $p = w$, which represents the condition for amplitude resonance. If, in addition there is no damping *i.e.*, $\lambda = 0$, A_{max} becomes infinity for this value of p, the curve being asymptotic to the Y- or the amplitude axis. Since damping is never zero, we do not obtain such a curve in actual practice but only those of the type shown for light and moderate damping.

(ii) The maximum value of A is different for different degree of damping, becoming less and less as damping increases. But, provided the damping is not too large, the maximum value of A always occurs at, or very nearly at $(w^2 - p^2) = 0$. This means that the resonant frequency p_r in all these cases is equal to or very nearly equal to w.

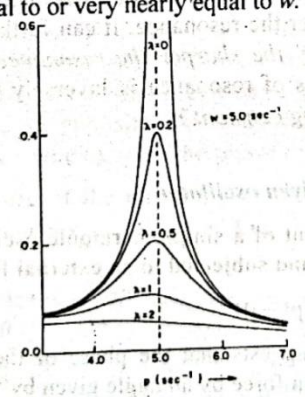

Fig. 3.7

It is clear from relation (3.38) above, that the average power absorbed per cycle will have its maximum value, when p = *w*, so that,

$$P_{max} = \frac{1}{4} \cdot \frac{mf_0^2}{\lambda} \qquad (3.41)$$

Thus the resonant frequency for the average power absorbed *i.e.*, the frequency at which the value of the average power absorbed becomes maximum, turns out to be the same as the resonant frequency for velocity. Or, $p = p_r = w$.

It should be noted that the power resonance takes place at $p_r = w$ whereas the amplitude resonance takes place at $p_r = \sqrt{w^2 - 2\lambda^2}$

3.11 *Driven LCR circuit*

An *LCR* circuit, *i.e.*, a circuit containing an inductance *L*, a capacitance *C* and a resistance *R* connected in series acts as a damped oscillator (Art. 3.5). If an external source of alternating electro-motive force (*emf*) is included in the circuit, also in series, the circuit now acts as a driven oscillator. The source of *emf* supplies the necessary energy to maintain the oscillation.

Let *i* be the current in the circuit at a given instant and *Q* the charge on the capacitor at that instant. Then the potential drop across the resistance is *Ri*, that across the inductance is $-L\frac{di}{dt}$ and that across the capacitance is $\frac{Q}{C}$.

If $E = E_0 \sin pt$ be the external *emf*, impressed on the circuit, then we have

$$E - L\frac{di}{dt} = \frac{Q}{C} + Ri$$

$$\text{or, } L\frac{di}{dt} + Ri + \frac{Q}{C} = E = E_0 \sin pt \qquad (3.41)$$

$$\text{or, } \frac{di}{dt} + \frac{R}{L}i + \frac{Q}{LC} = \frac{E_0}{L} \sin pt$$

Since $i = \frac{dQ}{dt}$, the above relation may be written as

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = \frac{E_0}{L}\sin pt \qquad (3.42)$$

Eqn. (3.41) is identical in form with eqn. (3.26), with Q replacing y and $\frac{R}{L}, \frac{Q}{LC}$ and $\frac{E_0}{L}$ replacing 2λ, w^2 and f_0 respectively.

Similarly, the steady state solution of eqn. (3.42) is

$$Q = \frac{E_0/L}{\sqrt{\left(\frac{1}{LC} - p^2\right) + \left(\frac{pR}{L}\right)^2}} \sin(pt - \theta)$$

where $\theta = \tan^{-1}\frac{pR/L}{1/LC - p^2}$

The value of the current flowing through the circuit can be easily obtained by differentiating this expression for Q with respect to t. Alternately, however, the value of the current can be obtained directly in the manner described below.

Let $i = i_0 \sin(pt - \theta)$ be a trial solution of eqn. (3.41).

Then

$$\frac{di}{dt} = i_0\, p \cos(pt - \theta)$$

and $Q = \int i\,dt = \int i_0 \sin(pt - \theta).\, dt = -\frac{i_0}{p}\cos(pt - \theta)$

Substituting these values in eqn. (3.40), we get

$$L\,p\,i_0 \cos(pt - \theta) + Ri_0 \sin(pt - \theta) - \frac{i_0}{Cp}\cos(pt - \theta) = E_0 \sin pt$$

or, $i_0\,[R \sin(pt - \theta) + (Lp - 1/Cp)\cos(pt - \theta)] = E_0 \sin pt \qquad (3.43)$

Putting $R = a\cos\theta$ and $(Lp - 1/Cp) = a\sin\theta$, we have

$$a = \sqrt{R^2 + \left(Lp - \frac{1}{Cp}\right)^2} \text{ and } \tan\theta = \frac{Lp - 1/Cp}{R}$$

Substituting these values of R and (Lp – 1/Cp) in relation (3.43) above, we have

$$i_0 a\,[\cos\theta \sin(pt - \theta) + \sin\theta \cos(pt - \theta)] = E_0 \sin pt$$

$$\text{or, } i_0 a \sin[(pt - \theta) + \theta] = i_0 a \sin pt = E_0 \sin pt$$

$$\text{or, } i_0 \cdot \sqrt{R^2 + (Lp - 1/Cp)^2} \cdot \sin pt = E_0 \sin pt$$

From the above relation we find that

$$i_0 \sqrt{R^2 + (Lp - 1/Cp)^2} = E_0$$

$$\text{or, } i_0 = \frac{E_0}{\sqrt{R^2 + (Lp - 1/Cp)^2}} \quad (3.44)$$

$$\therefore \quad i = i_0 \sin(pt - \theta)$$

$$= \frac{E_0}{\sqrt{R^2 + (Lp - 1/Cp)^2}} \sin(pt - \theta) \quad (3.45)$$

$$\text{where } \theta = \tan^{-1} \frac{Lp - \frac{1}{Cp}}{R}$$

It can be readily seen from eqn. (3.44) that the denominator $\sqrt{R^2 + (Lp - 1/Cp)^2}$ functions as the effective resistance in the circuit. It is called the *impedance* of the circuit and denoted by the letter *Z* and measured in *ohms*.

The impedance itself is made up to two parts, the ohmic resistance *R* and the quantity *(Lp – 1/Lp)*, called the *reactance*. The reactance usually denoted by *X*, is also measured in ohms although *L* and *C* are individually measured in henry and farad respectively. The relation between resistance, reactance, impedance and the phase angle *θ* can best be illustrated as in Fig. 3.9.

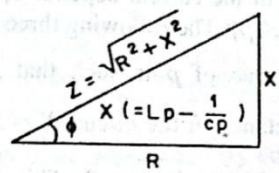

Fig. 3.9

The reactance too is made up of two parts : *Lp*, the reactance due to inductance (*inductive reactance*) and 1/Cp, the reactance due to capacitance (*capacitive reactance*). Inductive and capacitive reactances are usually denoted by X_L and X_C respectively.

Hence $X = X_L + X_C$

Thus, we have

$$Z = \sqrt{R^2 + X^2}$$

or, *Impedance*

$$= \sqrt{\text{Resistance}^2 + \text{Reactance}^2}$$

The maximum (or peak) value of the current is given by

$$i_0 = \frac{E_0}{Z} = \frac{E_0}{\sqrt{R^2 + X^2}}$$

Eqn. (3.45) can, therefore, be written as

$$i = \frac{E_0}{Z} \sin(pt - \theta)$$

$$= i_0 \sin(pt - \theta)$$

$$\text{and } \theta = \tan^{-1} \frac{Lp - 1/Cp}{R} = \tan^{-1} \frac{X_L - X_C}{R}$$

is the difference in phase between the current and the *emf* in the circuit. The ohmic resistance is quite independent of the frequency and, as such, has no effect on the phase angle. Hence the phase angle is determined by the reactance $(X_L - X_C)$. *In an inductive circuit the emf leads the current whereas in a capacitive circuit the current leads the emf.* When both inductance and capacitance are present in a circuit, the

lead or *lag* of the *emf*, or the current depends upon the relative values of X_L (or Lp) and X_C (or $1/Cp$). The following three cases then arise :

(i) When the value of p is such that $X_L = X_C$. In this case, $Lp = \frac{1}{Cp}$; or the reactance in the circuit $X = X_L - X_C = 0$, so that the impedance in the circuit = resistance R. This is the least value of the impedance and hence the value of the current in the circuit will be maximum.

$$\text{or,} \quad i_0 = \frac{E_0}{R}$$

When current in the circuit becomes maximum, we have the case of resonance. From the condition of resonance we have

$$Lp = \frac{1}{C_p}; \text{ or, } p^2 = \frac{1}{LC}$$

$$\text{or,} \quad p = \frac{1}{\sqrt{LC}} = w, \text{ the natural frequency of the circuit.}$$

Hence, the resonance frequency

$$= \frac{p}{2\pi} = \frac{w}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

The circuit under this condition is said to be *resonance series circuit*.

Further. Since $X_L = X_C$,

$$\tan\theta = \frac{X_L - X_C}{R} = 0$$

or, $\theta = 0$. Thus when $X_L = X_C$, the current in the circuit is in phase with the emf.

(ii) when p has a value such that $X_L > X_C$:

since $Lp > \frac{1}{Cp}$, the net reactance in the circuit is inductive, $\tan\theta$ is now a positive quantity and therefore, $\theta = \frac{\pi}{2}$. The *applied emf thus leads the current* by $\frac{\pi}{2}$ and the value of p is greater than w.

(iii) when the value of p is such that $X_L < X_C$:

since Lp is now less that $\frac{1}{Cp}$, the net reactance in the circuit is capacitive ; tan θ is now a negative quantity and therefore, $\theta = -\frac{\pi}{2}$.

The *current thus leads the applied emf* by $\frac{\pi}{2}$ and the value of p is less than *w*.

The changes in the maximum value of the current i_0 and the value of the phase angle θ with *p* are shown in Fig. 3.10 respectively, for different values of *R*. It can be seen that lower the value of *R*, the sharper the resonance, and the *current lags or leads the emf* according as *p* is greater than or less than *w*. R corresponds to λ in the case of mechanical oscillators.

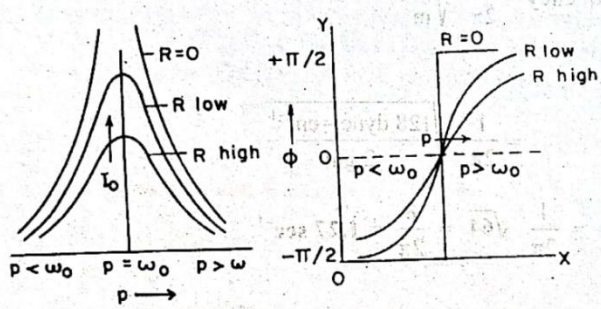

Fig. 3.10

Example 3.6. *A particle of mass 2 gm is free to vibrate under the action of an elastic force of 128 dyne–cm* ^{1} *and a damping force of 8 dyne–cm* ^{1} *sec. A periodically varying outside force of maximum value 256 dynes is applied to the particle. Find the frequency for displacement resonance and velocity resonance.*

Soln.

$$\text{frequency} = \frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{w^2 - 2\lambda^2}$$

Now $w^2 = \frac{a}{m} = \frac{128 \text{ dyne} - \text{cm}^{-1}}{2 \text{ gm}}$ $\sqrt{w^2 - 2\lambda^2}$ $= 64 \text{ sec}^{-2}$

and $\lambda = \frac{b}{2m} = \frac{8 \text{ dyne} - \text{cm}^{-1} \text{ sec}}{2 \times 2 \text{ gm}} = 2 \text{ sec}^{-2}$

$$\therefore \quad \text{frequency} = \frac{1}{2\pi} \sqrt{64 - 4 \times 2}$$

$$= \frac{1}{2\pi} \sqrt{64 - 8}$$

$$= \mathbf{1.19 \text{ sec}^{-1}}.$$

For velocity resonance,

$$\text{frequency} = \frac{1}{2\pi} \sqrt{\frac{a}{m}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{128 \text{ dyne} - \text{cm}^{-1}}{2 \text{ gm}}}$$

$$= \frac{1}{2\pi} \sqrt{64} = \frac{8}{2\pi} = \mathbf{1.27 \text{ sec}^{-1}}.$$

Example 3.7. *In an experiment on forced oscillations, the frequency of a sinusoidal driving force is changed while its amplitude is kept constant. It is found that the amplitude of vibrations is 0.01 mm at very low frequency of the driving force and goes upto a maximum of 5.0 mm at driving frequency 200 cps. Calculate the quality factor and the relaxation time of the system.*

Soln.

The quality factor of the driven oscillator,

$$Q = \frac{A_{max}}{A} = \frac{f_o / 2\lambda w}{f_o / w^2}$$

$$= \frac{w}{2\lambda} = w\tau \left(\tau = \frac{1}{2\lambda}\right)$$

A_{max} = 5.0 mm and A = 0.01 mm.

$$\therefore \quad Q = w\tau = \frac{5}{0.01} = \mathbf{500}$$

$$\text{or,} \quad \tau = \frac{Q}{w}$$

where $w = 2\pi n = 2\pi \times 200 = 400\,\pi$.

$$\therefore \quad \tau = \frac{500}{400\pi} \simeq \mathbf{0.40\ sec.}$$

Example 3.8. *A harmonic oscillator of quality factor 10 is subjected to a sinusoidal applied force of frequency one and a half times the natural frequency of the oscillator. If the damping be small, obtain (i) the amplitude of the forced oscillation in terms of its maximum amplitude and (ii) the angle θ by which it will be out of phase with the driving force.*

Soln.

The amplitude of a driven oscillator is given by

$$A = \frac{f_0}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}}$$

$$= \frac{f_0}{\sqrt{w^4\left(1 - \frac{p^2}{w^2}\right)^2 + w^4 \cdot \frac{4\lambda^2 p^2}{w^4}}}$$

$$= \frac{f_0}{w^2\sqrt{\left(1 - \frac{p^2}{w^2}\right)^2 + 4\,\frac{\lambda^2 p^2}{w^4}}}$$

where w is the natural frequency of the oscillator and p, the angular frequency of the driving force.

Now, the quality factor,

$$Q = \frac{w}{2\lambda} = 10\ ; \quad \text{or,}\ \frac{2\lambda}{w} = \frac{1}{10}$$

and $\frac{p}{w} = \frac{3}{2}$

$$\therefore \quad A = \frac{f_0}{w^2\sqrt{\left(1-\frac{9}{4}\right)^2 + \left(\frac{2\lambda}{w}\right)^2 \frac{p^2}{w^2}}}$$

$$= \frac{f_0}{w^2\sqrt{\frac{25}{16} + \frac{1}{100} \times \frac{9}{4}}}$$

$$= \frac{f_0}{w^2\left(\frac{634}{400}\right)^{1/2}} = \frac{20 f_0}{w^2 . \sqrt{634}}$$

And, for low damping, amplitude is maximum when $p = w$

or, $\frac{p}{w} = 1$. Then

$$A_{max} = \frac{f_0}{w^2\sqrt{(1-1)^2 + \frac{1}{100} \times 1}}$$

$$= \frac{f_0}{w^2\left(\frac{1}{100}\right)^{1/2}} = \frac{10 f_0}{w^2}$$

$$\therefore \quad \frac{A}{A_{max}} = \frac{20 f_0}{w^2\sqrt{634}} \times \frac{w^2}{10 f_0}$$

$$= \frac{2}{\sqrt{634}} \approx 0.08$$

Or, $A = 0.08\ A_{max}$.

Thus, the amplitude of the forced oscillation will be 0.08 times the maximum amplitude.

$$\text{Now, } \tan\theta = \frac{2\lambda p}{w^2 - p^2}$$

$$= \frac{p.w/10}{w^2 - p^2} = \frac{\frac{3w}{2} \cdot \frac{w}{10}}{w^2 - \frac{9w^2}{4}}$$

$$= \frac{3w^2}{20} \times \frac{4}{5w^2} = \frac{3}{5} = 0.12.$$

or, $\theta = \tan^{1} 0.12 = 6°51'$.

Therefore, the forced oscillation is $(180° - 6°51') = \mathbf{173°9'}$ out of phase with the driving force.

Example 3.9. *A harmonic oscillator consisting of a 50 gm mass attached to a massless spring has a quality factor 200. If it oscillates with an amplitude of 2 cm in resonance with a periodic force of frequency 20 c.p.s., calculate (i) the average energy stored in it and (ii) the rate of dissipation of energy.*

Soln.

In case of resonant vibration,

The average energy stored in the oscillator = maximum potential energy.

Now, the maximum potential energy is given by $\frac{1}{2}$ a y^2 where a is the force constant and y is the displacement of the oscillator. Since $a = mw^2$, the maximum P.E. = $\frac{1}{2}$ mw^2y^2.

Here m = 50 gm, $w = 2\pi n = 2\pi \times 20$

$= 40\pi$ and y = 2 cm.

$\therefore$ the average energy absorbed or stored up in the oscillator

$$= \frac{1}{2} \times 50 \times (40\pi)^2 \times 22$$

$$= 16 \times 10^4 \times \pi^2$$

$$= \mathbf{1.58 \times 10^6 \text{ ergs.}}$$

By definition, the quality factor

$$Q = 2\pi . \frac{\text{average energy stored}}{\text{energy dissipated per cycle}}$$

$\therefore$ energy dissipated per cycle

$$= 2\pi . \frac{\text{average energy stored}}{Q}$$

$$= 2\pi . \frac{1.58 \times 10^6}{200}$$

$$= \pi \times 1.58 \times 10^4 \text{ ergs.}$$

Since there are 20 cycles to the second, energy dissipated per second or the rate of dissipation of energy

$$= 20\pi \times 1.58 \times 10^4$$

$$= 9.926 \times 10^5 \simeq 10^6 \text{ erg/sec.}$$

Example 3.10. *A resistance of 10 ohms is joined in series with an inductance of 0.5 henry. What capacitance should be put in series with the combination to obtain maximum current ? What will be the potential difference across the resistance, inductance and capacitance ? The current is being supplied by a 200 volts and 50 cycles A.C. mains.*

Soln.

The impedance in a series LCR circuit,

$$Z = R^2 + \left(Lp - \frac{1}{Lp}\right)^2$$

For resonance or maximum current,

$$Z = R \text{ ; or, } Lp - \frac{1}{Cp} = 0$$

$$\text{or, } Lp = \frac{1}{Cp}$$

Therefore, $C = \frac{1}{Lp^2}$

Now L = 0.5 henry, p = 2πn

= 2π x 50 = 100π.

∴ the required capacitance,

$$C = \frac{1}{0.5 \times (100\pi)^2}$$

= 20.27 x 10^{-6} farad

= **20.27 μF.**

The maximum current will obviously be

$$i_0 = \frac{E_0}{Z} = \frac{E_0}{R} = \frac{200}{10} = 20 \text{ amp}$$

(∵ Z = R at resonance).

Potential difference across the resistance,

i_0R = 20 x 10 = 200 volts.

Also at resonance,

$X_L = L_P$ due to inductance = $X_C = \frac{1}{Cp}$ due to capacitance.

Hence potential difference across X_L

= potential difference across X_C

or, $i_0 Lp = i_0/Cp$

= 220 x 0.5 x 100 π = **3142 volts.**