CHAPTER I

SIMPLE HARMONIC MOTION

Periodic motion - Simple Harmonic Motion - Phase and epoch of a particle executing simple harmonic motion - Time period and frequency of a body executing SHM - Energy of a body executing SHM - Average value of kinetic and potential energies of a harmonic oscillator - Some examples of simple harmonic motion - Relation between simple harmonic motion and uniform circular motion - Solved problems - Exercises.

1.1 Periodic motion

A motion which repeats itself over and over again after a regular interval of time is referred to as a *periodic motion*. The time required for each repetition is called *time-period*. The motion of moon about the earth, the oscillation of a pendulum, the motion of a mass suspended from a coil spring are examples of periodic motion.

When the particle, undergoing periodic motion, covers the same path back and forth about a mean position, it is said to be executing an *oscillatory or vibratory* motion. The oscillatory motion is, therefore, a *to and fro* (forward and backward) motion. One complete to and fro motion is called an *oscillation or vibration or a cycle*. Further, the oscillatory motion is not only periodic but also *bounded*, *i.e.*, the displacement of the particle on either side of its mean position remains confined within a well-defined limit. The number of complete oscillations or cycles in unit time is called the *frequency* of vibration.

Of all the trigonometrical ratios, the sines and cosines alone are periodic as well as bounded. Thus the displacement of a particle executing an oscillatory motion is usually expressed in terms of sines or cosines or combination of both. This, coupled with the fact that this type of motion is generally associated with musical instruments, oscillatory motion is also referred to as *harmonic* motion.

Simple Harmonic Motion

Let a particle oscillate along a straight line within some fixed limits.

In case the displacement is not the same on either side of the mean position, the motion is still harmonic but not simple harmonic. However, it can be shown that even in this case the oscillation can still be regarded as simple harmonic provided the displacement is small.

1.2 Differential Equation of Simple Harmonic Motion

If F be the force acting on a particle executing simple harmonic motion and y its displacement from its mean or equilibrium position, then $F = -ky$. Again, according to Newton's laws of motion, $F = ma'$ where m is the mass of the particle and a' its acceleration.

Substituting $-ky$ for F and $\frac{d^2y}{dt^2}$ for a', we can write (for $F = ma'$)

$$-ky = m\frac{d^2y}{dt^2}$$

$$\text{or,} \quad \frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \qquad (1.2)$$

Equation (1.2) is called the differential equation of motion of a body executing simple harmonic motion, because the solution of eqn. (1.2) gives us the nature of variation of displacement with time. Thus the correct nature of the motion of the particle can be known,

Rearranging eqn. (1.2), we can write,

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y = -w^2y \qquad (1.3)$$

$$= -\mu y \qquad (1.4)$$

where $w = \sqrt{\frac{k}{m}}$ is the angular velocity of the particle and μ is a constant equal to w^2. Since $\frac{d^2y}{dt^2} = -\mu$ when $y = 1$, μ may be defined as the *acceleration per unit displacement of the particle*.

To obtain a general solution of the differential equation of simple harmonic motion, let us multiply both sides of eqn. (1.3) by $2\frac{dy}{dt}$ when we get

$$2.\frac{dy}{dt}.\frac{d^2y}{dt^2} = -w^2y.2.\frac{dy}{dt}$$

$$\text{or,} \quad 2\frac{dy}{dt}.\frac{d^2y}{dt^2} = -w^2.2.\frac{dy}{dt}.y$$

Integrating with respect to time, we have

$$\left(\frac{dy}{dt}\right)^2 = w^2y^2 + C \qquad (1.5)$$

where C is a constant of integration. C can be evaluated by applying boundary conditions. To evaluate C we recall that velocity and acceleration of a simple harmonic motion are oppositely directed. We further recall that the velocity is zero (or K.E. is zero) at maximum displacement (or amplitude).

or, $\frac{dy}{dt} = 0$ when y = a (amplitude).

Substituting these values in eqn. (1.5)

$$0 = -w^2a^2 + C \text{ ; or, } C = w^2a^2$$

Substituting this value of C in eqn. (1.5), we have

$$\left(\frac{dy}{dt}\right)^2 = w^2y^2 + w^2a^2 = w^2(a^2 + y^2) \qquad (1.6)$$

$$\text{or,} \quad \frac{dy}{dt} = \pm w\sqrt{a^2 - y^2}$$

$$= \pm\sqrt{\frac{k}{m}}\sqrt{a^2 - y^2} \qquad (1.7)$$

Eqn. (1.7) can be rearranged as

$$\frac{dy}{\sqrt{a^2 - b^2}} = w.dt$$

Integrating again with respect to time, we have

$$\sin^{-1}\frac{y}{a} = wt + \phi$$

or, $y = a \sin (wt + \phi)$ (1.8)

Equation (1.8) gives the displacement of the particle at an instant t in terms of its amplitude a and its total phase $(wt + \phi)$ and is the general solution of the differential equation of simple harmonic motion.

Expanding eqn. (1.8), we have

$$y = a \sin wt \cos \phi + a \cos wt \sin \phi$$

$$= A \sin wt + B \cos wt$$

where $A = a \cos \phi$ and $B = a \sin\phi$

In special cases where either A or B may be zero, the displacement may be written as either

$$y = A \sin wt$$

or, $y = B \cos wt$

Hence the most general form of the differential equation is

$$y = A \sin wt + B \cos wt \qquad (1.9)$$

which is a combination of both the sine and the cosine terms.

Velocity and acceleration of a body executing SHM

The displacement of a particle executing simple harmonic motion is given by

$$y = a \sin (wt + \phi)$$

(i) Hence, the velocity of the particle at any instant of time t is

$$\frac{dy}{dt} = wa \cos (wt + \phi)$$

$$= \pm\, wa \sqrt{1 - \sin^2 (wt + \phi)}$$

Now $\sin (wt + \phi) = \frac{y}{a}$.

$$\therefore \quad \frac{dy}{dt} = \pm\, wa \sqrt{1 - \frac{y^2}{a^2}}$$

$$= \pm wa\sqrt{\frac{a^2 - y^2}{a^2}}$$

$$= \pm wa\sqrt{a^2 - y^2}$$

$$= \pm\sqrt{\frac{k}{m}}\sqrt{a^2 - y^2} \qquad (1.10)$$

(ii) The acceleration of the particle at any time t is given by

$$\frac{d^2y}{dt^2} = -w^2 a \; sin\,(wt + \phi)$$

$$= w^2.a.\frac{y}{a} = -w^2.y$$

$$= -\frac{k}{m}.y. \qquad (1.11)$$

From eqns. (1.10) and (1.11) it can be seen that *maximum value of velocity, i.e.,* $v_{max} = \pm w.a$ or $\pm a\sqrt{\frac{k}{m}}$ and occurs when $y = 0$ *i.e.*, when the particle is passing through its mean position.

Similarly, the *maximum value of acceleration* occurs when y is maximum *i.e.*, the particle is at the position of one of its extreme displacements.

1.3 Phase and epoch of a particle executing simple harmonic motion

Going back to eqn. (1 8), we see that the total phase of a particle executing simple harmonic motion is made up of the *phase angle wt* and ϕ which is called the *initial phase* or *phase constant* or the *epoch* of the particle, usually denoted by the letter e. This initial phase or epoch is a direct consequence of the fact that we start to count time, not from the instant when the particle is in some standard position, like its mean position or one of its extreme position, but from the instant when it is anywhere else in between. The following cases will then arise :

(i) if we start counting time when the particle is in its mean position, *i.e.*, when $y = 0$ at $t = 0$, we have $\phi = 0$. Eqn. (1.8), therefore, reduces to

$y = a \sin wt$

(ii) *If the counting of time starts when the particle is in one of its extreme positions, i.e.*, when $y = a$ at $t = 0$, we have from eqn (1.8),

$$a = a \sin(0 + \phi)$$

$$\text{or,} \quad \sin\phi = \frac{a}{a} = 1; \quad \text{or,} \ \phi = \frac{\pi}{2}.$$

Eqn. (1.9), therefore, becomes

$$y = a \sin\left(wt + \frac{\pi}{2}\right) = a \cos wt.$$

As can be seen, both $y = a \sin wt$ and $y = a \cos wt$ represent the equations of simple harmonic motion with the difference lying only in the initial position of the vibrating particle. Either of these equations can, therefore, be taken as the general equation of simple harmonic motion. In fact, the general solution of the differential equation of simple harmonic motion is of the form

$$y = A \sin wt + B \cos wt$$

which is a combination of both the sine and cosine terms.

(iii) If, on the other hand, we start counting time from an instant t', before the particle has passed through its mean position, we have $y = 0$ at $t = t'$.

Then we have

$$0 = a \sin(wt' + \phi)$$

or, $(wt' + \phi) = 0$ whence $\phi = -wt' = -e$, say. Therefore, the expression for simple harmonic motion becomes

$$y = a \sin(wt - e)$$

(iv) Similarly, if the counting of time is started from an instant t', after the particle has passed through its mean position, we have

$$y = a \sin(wt + e)$$

1.4 Time period, frequency and angular frequency of a body executing SHM

If the time t in the relation $y = a \sin(wt + \phi)$ is increased by $2\pi/w$, then we have

$$y = a \sin\left[w\left(t + \frac{2\pi}{w}\right) + \phi\right]$$

$$\text{or,} \quad y = a \sin(wt + 2\pi + \phi)$$

$$= a \sin(wt + \phi)$$

the same as before. This indicates that the particle executing simple harmonic motion *repeats its motion after every $2\pi/w$ seconds*. In other words, the *time period of the particle* is given by

$$T = \frac{2\pi}{w} = 2\pi.\sqrt{\frac{1}{w^2}} = 2\pi\sqrt{\frac{1}{\mu}}$$

$$\text{or,} \quad T = 2\pi\sqrt{\frac{1}{\text{acceleration per unit displacement}}}$$

$$= 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

Since $w^2 = \dfrac{k}{m}$, the time period may also be expressed as

$$T = 2\pi.\sqrt{\frac{m}{k}} \qquad (1.12)$$

The number of oscillations (or vibrations) made by the particle per second (unit time) is called its *frequency of oscillation* or, simply, its *frequency*, usually denoted by the letter n.

Thus, frequency is the reciprocal of the time period.

$$\text{or,} \quad n = \frac{1}{T} = \frac{w}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

From the above relation we also have $w = 2\pi n = \dfrac{2\pi}{T}$. w is also referred to as the angular frequency of the particle. It is the angle described per second and has the unit radians per second, the same as angular velocity, also designated w. Angular frequency is very closely related to the angular velocity of a circular motion which is associated with simple harmonic motion. The use of w instead of n to describe the frequency of simple harmonic motion simplifies these expressions by avoiding factors of 2π, which usually accompany n in such expressions.

It may be noted that the angular frequency and thus the time period $T = \frac{2\pi}{w}$ are determined only by the force constant and the mass. They do not depend either on the amplitude a or the initial phase ϕ. This means that the oscillations of a simple harmonic oscillator are isochronous, i.e., they take the same time irrespective of the values of a and ϕ. This is an important property of simple harmonic motion where the frequency and time period are independent of the amplitude and epoch.

Example 1.1. *A spring, hung vertically, is found to be stretched by 0.02 m from its equilibrium position when a force of 4N acts on it. Then a 2 kg body is attached to the end of the spring and is pulled 0.04 m from its equilibrium position along the vertical line. The body is then released and it executes simple harmonic motion.*

(i) what is the force constant of the spring ?

A force of 4N on the spring produces a displacement of 0.02 m. Hence. from $F = ky$ we have $k = \frac{F}{y} = \frac{4N}{0.02m}$

= 200 N per m.

(ii) what is the force executed by the spring on the 2 kg body just before it is released ?

The spring is stretched 0.04 m. Hence the force executed by the spring is

$$F = -ky = -(200\ N.m^{-1})\ (0.04\ m)$$

$$= -8\ N.$$

The minus sign indicates that the force is directed opposite to the displacement.

(iii) what is the period and frequency of oscillation after release ?

$$T = 2\pi.\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2\text{ kg}}{200\text{ N.m}^{-1}}}$$

$$= \frac{\pi}{5}\text{ sec} = \mathbf{0.628\ sec.}$$

$$n = \frac{1}{T} = \frac{1}{0.628} = \mathbf{159}\ \boldsymbol{Hz.}$$

$$w = 2\pi n = (2)\ (3.14)\ (1.59)$$

$$= \mathbf{10}\ \boldsymbol{sec^{-1}}.$$

Also,

$$w = 2\pi n = 2\pi.\frac{1}{T} = \frac{1}{2\pi} = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{k}{m}} = \sqrt{\frac{200\text{ N.m}^{-1}}{2\text{ kg}}} = 10\ sec.^{-1}$$

(iv) what is the amplitude of motion ?

Amplitude is the initial displacement to the body, *i.e,* **0.04 m.**

(v) what is the maximum velocity of the oscillating body ?

The maximum velocity occurs when the body passes through the position of equilibrium, *i.e.*, y = 0.

Hence, from

$$v = \pm\ w\sqrt{a^2 - y^2}\text{, we get}$$

$$v_{max} = \pm\ w.a$$

$$\therefore\ v_{max} = \left(10\ sec^{-1}\right)\ (0.04\text{ m})$$

$$= \pm\ 0.4\text{ m } sec^{-1}.$$

(vii) what is the mechanical (total) energy of the oscillating system ?

Total energy = P.E + K.E

$$= \frac{1}{2}ka^2 = \frac{1}{2}\ (200)\ (0.04)^2$$

$$= \mathbf{0.16\ joules.}$$

Also, total energy

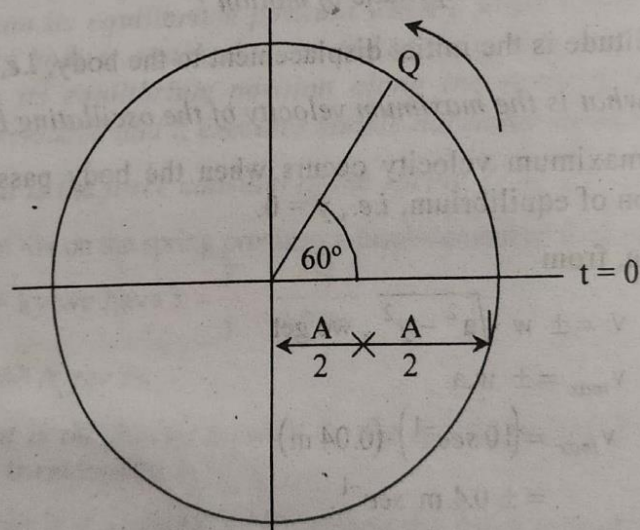
$$\frac{1}{2}ka^2 = 2\pi 2.m.a^2.n^2 \text{ (Art. 1.5)}$$

$$= 2.(3.14)^2.2.(0.04)^2.(1.59)^2$$

$$= \textbf{0.16 joules.}$$

(ix) how much time is required for the body to move half-way in to the centre from its initial position ?

The motion is neither one of constant velocity nor one of constant acceleration. The simplest method is to make use of the reference circle. While the body moves half-way in, the reference point revolves through an angle of 60°. Since the reference point moves with constant angular speed and in this example makes one complete revolution in $\frac{\pi}{5}$ sec, the time to rotate through 60° is



$$\frac{1}{6}.\frac{\pi}{5}\text{sec} = \frac{\pi}{30}\text{sec} = 0.105 \text{ sec.}$$

The time can also be computed directly from the relation,

$$y = a \sin wt$$

$$\frac{a}{2} = a \sin\ (10\ \text{sec}^{-1})\ .\ t$$

$$\sin\ (10\ \text{sec}^{-1})\ .\ t = \frac{1}{2}$$

$$10\ \text{sec}^{-1}\ .\ t = \sin^{-1}\frac{1}{2} = \frac{\pi}{3}$$

$$\therefore\ t = \frac{\pi}{3 \times 10}\ \textbf{sec}$$

(x) what is the displacement of the body as a function of time ?

The general equation for displacement of a body executing simple harmonic motion is

$$y = a \sin (wt + \phi)$$

The value of w, as already obtained, is

$$w = \frac{2\pi}{T} = 10\ \text{radian}/sec.$$

$$\therefore\ y = a\ \sin\ (10t + \phi)$$

At t = 0, y = a = 0.04 m, so that at that instant

$$y = 0.04 \sin \phi = 0.04$$

$$\therefore\ \sin \phi = 1$$

$$\text{or,}\ \phi = \sin^{-1} 1 = \frac{\pi}{2}\ \text{radian.}$$

Therefore, with a = 0.04 m, w = 10 rad./sec and $\phi = \frac{\pi}{2}$ radian, we get

$$y = 0.04 \sin (10\,t + \frac{\pi}{2})$$

$$= \mathbf{0.04 \sin 10t.}$$

Example 1.2. *A body is vibrating with simple harmonic motion of amplitude 15 cm and frequency 4 Hz. Compute (a) the maximum values of the acceleration and velocity and (b) the acceleration and velocity when the displacement is 9 cm.*

Soln. (a) $v_{max} = w.a$

$a = 15$ cm

$n = 4$ Hz

$\therefore \quad w = 2\pi n = 2 \times 3.14 \times 4$

$= 25.12$ rad/sec.

$\therefore \quad v_{max} = 25.12 \times 15 =$ **376.8 cm/sec.**

$(accln)_{max} = -w^2.a$

$= -(25.12)^2 \times 15$

$=$ **$-$ 9470 cm/sec^2.**

(b) when $y = 9$ cm

$v = w.\sqrt{a^2 - y^2} = 25.12 \quad \sqrt{(15)^2 - 9^2}$

$= 300$ cm/sec

accln. $= -w^2.y = -(25.12)^2 \times 9$

$= -5680$ cm/sec^2.

Example 1.3. *A particle executes linear harmonic motion about the point $x = 0$. At $t = 0$, it has displacement $y = 0.37$ cm and zero velocity. If the frequency of the motion is 0.25/sec, determine (a) the period, (b) the amplitude, (c) the maximum speed and (d) the maximum acceleration.*

Soln.

(a) $T = \frac{1}{n} = \frac{1}{0.25} =$ **4 sec.**

(b) $y = a \sin (wt + \delta)$

at $t = 0$, $y = 0.37$ cm.

$\therefore a \sin \delta = 0.37$ ...(i)

Again $v = \frac{dy}{dt} = wa \cos (wt + \delta)$

at $t = 0$, $v = 0$

$\therefore wa \cos \delta = 0$

or, $a \cos \delta = 0$ (ii)

from (i) and (ii),

$a^2 = (0.37)^2$

or, **$a = \pm 0.37$ cm.**

(c) $v = w.\sqrt{a^2 - y^2}$

v is maximum when $y = 0$.

$\therefore \quad v_{max} = w.a = a \times 2\pi n$

$= 0.37 \times 2 \times 3.14 \times 0.25$

= 0.5809 cm/sec.

(d) $(accln)_{max} = - w^2.a$

$= -(2\pi n)^2 . a$

$= -(2 \times 3.14 \times 0.25)^2 \times 0.37$

$= -0.912013$ cm/sec^2.

Example 1.4. *The displacement of an oscillating particle at an instant t is given by*

$$y = a \cos wt + b \sin wt.$$

Show that it is executing a simple harmonic motion.

If a = 5 cm, b = 12 cm and w = 4 radian/sec, calculate (i) the amplitude, (ii) the time period, (iii) the maximum velocity and (iv) the maximum acceleration of the particle.

Soln.

$$y = a \cos wt + b \sin wt$$

or, $\dfrac{dy}{dt} = - aw \sin wt + bw \cos wt$

or, $\dfrac{d^2y}{dt^2} = - w^2.a \cos wt - w^2b \sin wt$

$= - w^2 (a \cos wt + b \sin wt)$

$= - w^2y$

or, $\frac{d^2y}{dt^2} + w^2y = 0$

Hence the motion is simple harmonic.

(i) Let $a = A \sin \alpha$ and $b = A \cos \alpha$.

Then

$$y = A \sin\alpha \cos wt + A \cos \alpha \sin wt$$

$$= A \sin (wt + \alpha)$$

This represents a simple harmonic motion with amplitude A.

$\therefore \quad A^2 \sin^2 \alpha + A^2 \cos^2 \alpha = a^2 + b^2$

or, $A = \sqrt{a^2+b^2}$

$a = 5$ cm

$b = 12$ cm

$\therefore \quad A = \sqrt{5^2+12^2}$

$= 13$ cm

(ii) $T = \frac{2\pi}{w} = \frac{2\pi}{4} = \frac{\pi}{2}$ sec.

(iii) $v_{max} = w.A = 13 \times 4$

$= 52$ cm/sec

(iv) $(\text{accln})_{max} = -w^2.A = -(4)^2 \times 13$

$= \mathbf{-208\ cm/sec^2}$.

Example 1.5. *The positions of a particle executing simple harmonic motion along the x-axis are $x = A$ and $x = B$ at time t an 2t respectively. Show that its period of oscillation is given by*

$$T = (2\pi t) / \cos^{-1} (B/2A)$$

Soln.

$$A = a \sin wt$$

$$B = a \sin (w.\ 2t)$$

$$= a \sin 2wt$$

$$= a\ 2 \sin wt \cos wt$$

$$\therefore \quad \frac{A}{B} = \frac{a \sin wt}{a\ 2 \sin wt \cos wt}$$

$$= \frac{1}{2 \cos wt}$$

$$\text{or,} \quad \cos wt = \frac{B}{2A}$$

$$\text{or,} \quad wt = \cos^{1}\left(\frac{B}{2A}\right)$$

$$\text{or,} \quad w = \frac{1}{t} \cos^{1}\left(\frac{B}{2A}\right)$$

$$\text{or,} \quad \frac{2\pi}{T} = \frac{\cos^{-1}\left(\frac{B}{2A}\right)}{t}$$

$$\therefore \quad T = \frac{2\pi t}{\cos^{-1}\left(\frac{B}{2A}\right)}$$

***Example 1.6.** For a particle executing simple harmonic motion the displacement is 8 cm at the instant the velocity is 6 cm/sec and the displacement is 6 cm at the instant the velocity is 8 cm/sec. Calculate (i) amplitude, (ii) frequency and (iii) time period.*

Soln.

Velocity of a particle executing simple harmonic motion,

$$v = \frac{dy}{dt} = w\sqrt{a^2 - y^2}$$

Now v = 6 cm/sec when y = 8 cm.

$\therefore \quad 6 = w\sqrt{a^2 - 64}$ (i)

Again v = 8 cm/sec when y = 6 cm

$\therefore \quad 8 = w\sqrt{a^2 - 64}$ (ii)

Dividing (ii) by (i) and squaring

$$\frac{64}{36} = \frac{a^2 - 36}{a^2 - 64}$$

or, a = **10 cm.**

Substituting a = 10 cm in eqn. (i)

$\therefore \quad 6 = w\sqrt{100 - 64}$

or, w = **1 rad/sec.**

Hence frequency,

$$n = \frac{w}{2\pi} = \frac{1}{2\pi} \text{ Hz.}$$

time period

$$T = \frac{1}{n} = \mathbf{2\pi \text{ seconds.}}$$

Example 1.7. *A simple harmonic motion is represented by*

$$y = 10 \sin (10t - \frac{\pi}{6})$$

where y is measured in metres, t in seconds and the phase angle in radians. Calculate (i) the frequency, (ii) the time period, (iii) the maximum displacement, (iv) the maximum, velocity and (v) the maximum acceleration and (vi) displacement, velocity and acceleration at time t = 0 and t = 1 second.

Soln.

Here $y = 10 \sin (10t - \frac{\pi}{6})$ (1)

Comparing with the displacement equation

$y = a \sin (wt + \delta)$ (2)

we get, (i) $w = 2\pi n = 10$

or, $n = \frac{10}{2\pi}$ **= 1.6 Hz.**

(ii) time period, $T = \frac{1}{n} = \frac{2\pi}{10}$ **= 0.63 sec.**

(iii) maximum displacement (amplitude)

$a =$ **10 m.**

(iv) maximum velocity,

$v_{max} = w.a = 10 \times 10 =$ **100 m/sec.**

(v) $(\text{accln.})_{max} = -w^2a = -(10)2 \times 10$

$=$ **− 1000 m/sec².**

minus sign shows that the acceleration is directed towards the mean position.

(vi) From eqn. (1)

(a) at $t = 0$

$y = 10 \sin(-\frac{\pi}{6}) = -5$ *m.*

velocity, $\frac{dy}{dt} = a\,w \cos\delta$

$= 10 \times 10 \cos(-\frac{\pi}{6})$

$= 100 \times 0.866 =$ **86.6 m/sec.**

Acceleration, $\frac{d^2y}{dt^2} = -aw^2 \sin\delta$

$= -10 \times 10^2 \times \sin(-\frac{\pi}{6})$

$= -10 \times 100 \times 0.5$

$=$ **− 500 m/sec².**

(b) From eqn. (1), at $t = 1$,

displacement

$y = 10 \sin(10 - \frac{\pi}{6})$

$= 10 \sin \left(\frac{60 - 3.142}{6}\right)$

$= 10 \sin \left(\frac{56.858}{6}\right)$

$= 10 \sin (3\pi)$ approximately

$= 10 \sin \pi$

$= 0$

velocity, $\frac{dy}{dt} = aw \cos \left(10 - \frac{\pi}{6}\right)$

$= aw \cos (\pi)$ approximately

$= 10 \times 10 \times (-1)$

$= -100$ m/sec.

Accln., $\frac{d^2y}{dt^2} = -a w^2 \sin \left(10 - \frac{\pi}{6}\right)$

$= -aw^2 \sin (\pi)$ approximately.

$= 0.$

Example 1.8. *A particle performs simple harmonic motion given by the equation*

$y = 20 \sin (wt + \alpha)$

If the time period is 30 seconds and the particle has a displacement of 10 cm at $t = 0$, find (i) epoch, (ii) the phase angle at $t = 5$ seconds and (iii) the phase difference between two positions of the particle 15 seconds apart.

Soln.

Here

$y = 20 \sin (wt + \alpha)$

$T = 30$ secs.

$\therefore \quad w = \frac{2\pi}{T} = \frac{2\pi}{30} = \frac{\pi}{15}$ rad/sec.

$a = 0$ and $\quad v = v_{max}$

$$\therefore \quad v_{max} = wa = \left(\frac{2\pi}{3}\right)(0.075) = \mathbf{0.157\ m/sec.}$$

(iii) at $y = 0.06$ m (6 cm)

$$v = w.\sqrt{a^2 - y^2} = \frac{2\pi}{3}\sqrt{(0.075)^2 - (0.06)^2}$$

$$= \frac{2\pi}{3}(0.045) = \mathbf{0.0942\ m/sec.}$$

$$a = w^2 y = \left(\frac{4\pi^2}{9}\right)(0.06) = \mathbf{0.263\ m/sec^2}.$$

1.5 Energy of a body executing S.H.M.

The acceleration of a particle and hence the force acting on the particle executing simple harmonic motion is, as we know, directed towards its mean or equilibrium position, *i.e.*, opposite to the direction in which the displacement y increases. Work is, therefore, done during the displacement of the particle. Hence the particle possesses *potential energy (U)*. As the particle also possesses velocity, it possesses *kinetic energy (K)* too. Thus the mechanical energy E of a particle executing simple harmonic motion is partly kinetic and partly potential. *If no non-conservative forces, such as the force of friction act on the particle, the sum of its kinetic energy and potential energy remains constant*. or,

$$E = K + U = \text{constant}.$$

As the displacement increase, the potential energy increases and the kinetic energy decreases and *vice versa*. But the total energy $E\ (= K + U)$ is conserved.

Let the displacement of a particle executing simple harmonic motion at any instant be y. If the mass of the particle be m and its velocity at that instant be v, then its kinetic energy is $\frac{1}{2}mv^2$. The potential energy of the particle at the same instant is the amount of work that must be done in overcoming the force through a displacement y and is given by the relation $\int_0^y F.\,dy$ where F is the force required to maintain the displacement and dy is a small displacement.

Now the displacement is given by the relation $y = a \sin (wt + \phi)$.

Hence the acceleration $\frac{d^2y}{dt^2}$ is given by

$$\frac{d^2y}{dt^2} = -aw^2 \sin (wt + \phi) = -w^2.y.$$

Then, force F = mass x acceleration

$$= m.(-w^2y) = -mw^2.y.$$

Then the potential energy of the particle is

$$P.E. = \int_0^y F.\,dy = \int_0^y mw^2.\,y\,.\,dy$$

$$= \frac{1}{2}\,m\,.\,w^2\,.\,y^2$$

$$= \frac{1}{2}\,m\,.\,w^2\,.\,a^2 \sin^2 (wt + \phi)$$

$$= \frac{1}{2}\,k\,.\,a^2 \sin^2 (wt + \phi)\ (\because\ w^2 = \frac{k}{m}) \qquad (1.14)$$

ignoring the minus sign in the expression for F, which simply shows that the direction of the force and displacement are opposite to each other.

Now the kinetic energy of the particle is given by

$$K.E. = \frac{1}{2}\,mv^2$$

$$= \frac{1}{2}\,m\,(\frac{dy}{dt})^2$$

$$= \frac{1}{2}\,m\,.\,[\,wa \cos (wt + \phi)]^2$$

$$= \frac{1}{2}\,m\,.\,a^2\,.\,w^2 \cos^2 (wt + \phi)$$

$$= \frac{1}{2} k \cdot a^2 \cos^2 (wt + \phi) \qquad (1.15)$$

It can be seen from eqns. (1.14) and (1.15) that both the potential and kinetic energies have a maximum value of $\frac{1}{2}$ ka^2 or $\frac{1}{2}$ $m (wa)^2$. During the motion the potential energy as well as the kinetic energy vary between zero to this maximum value.

The total energy which is just the sum of the kinetic and potential energy is

$$E = K + U = \frac{1}{2} ka^2 \sin^2 (wt + \phi) + \frac{1}{2} ka^2 \cos^2 (wt + \phi)$$

$$= \frac{1}{2} ka^2. \qquad (1.16)$$

We see that the total energy, as expected, is constant and has the value $\frac{1}{2}$ ka^2. Thus, *the total energy of the system is the same as the maximum value of any one of the two forms of energy*. At the maximum displacement the kinetic energy is zero, but the potential energy has the value $\frac{1}{2}ka^2$. At the position of equilibrium the potential energy is zero but the kinetic energy has the value $\frac{1}{2}$ ka^2. At any other position the kinetic and potential energies each contribute energy whose sum is always $\frac{1}{2}$ ka^2.

Fig. 1.3 shows the kinetic, potential and total energies as a function of time while in Fig. 1.4, the same energies are plotted as a function of displacement from the equilibrium position. *The total energy of a particle executing simple harmonic motion is proportional to the square of the amplitude of the motion.*

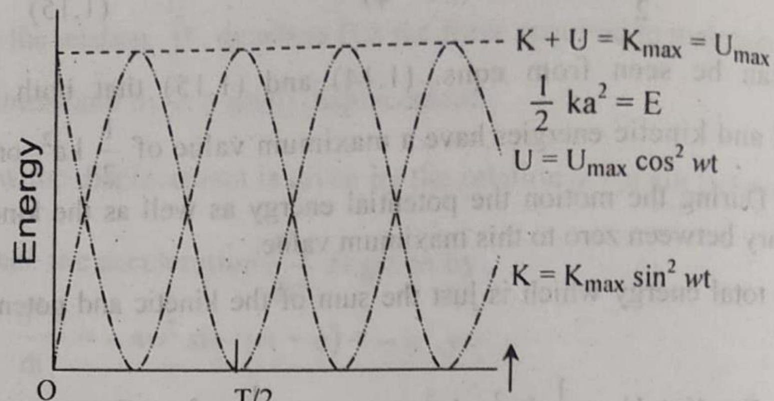


Fig. 1.3

Now

$$\frac{1}{2}\ ka^2 = \frac{1}{2}\ m\ w^2 a^2$$

$$= \frac{1}{2} m \cdot \left(\frac{2\pi}{T}\right)^2 a^2 = \frac{2\pi^2 ma^2}{T^2}$$

But, since $\frac{1}{T}$ = n, the frequency of oscillation, the total energy of the system can also be expressed as

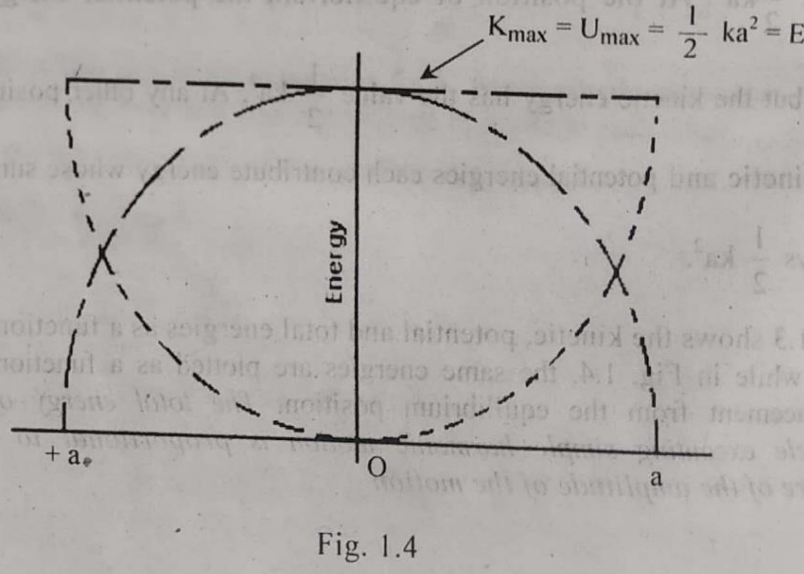


Fig. 1.4

$$E = \frac{1}{2} ka^2 = 2\pi^2 m \cdot a^2 \cdot n^2 \quad (1.17)$$

Thus, *total energy*

= maximum value of potential energy

= maximum value of kinetic energy

$$= \frac{1}{2} ka^2 = \frac{1}{2} mw^2a^2 = 2\pi^2n^2a^2m.$$

The total energy in Fig. 1.3 is represented by the upper horizontal line parallel to the time axis and touching the two curves at points representing the maximum values of kinetic and potential energies respectively. The upper horizontal line in Fig. 1.4 which passes through the two points of maximum potential energy corresponding to the two points of maximum displacement +a and -a on either side of the mean position represents the total energy. This line is also parallel to the displacement axis. Since the line representing total energy in either figure is a horizontal line, being parallel to either the time axis or the displacement axis, it follows that *total energy of the particle executing simple harmonic motion remains constant throughout and is independent of both time and displacement.*

1.6 Average value of kinetic and potential energies of a harmonic oscillator

The potential energy (P.E.) of the particle at a displacement y is given by

$$= \frac{1}{2} m\, w^2y^2$$

$$= \frac{1}{2} m \cdot w^2a^2 \sin^2 (wt + \phi)$$

So the average P.E. of the particle over a complete cycle or a whole time period T

$$= \frac{1}{T}\int_O^T \frac{1}{2} m\, w^2a^2 \sin^2 (wt + \phi)\, dt$$

$$= \frac{1}{T} \cdot \frac{m w^2 a^2}{4} \int_0^T 2 \sin^2 (wt + \phi)\, dt$$

$$= \frac{m w^2 a^2}{4T} \int_0^T [1 - \cos 2 (wt + \phi)]\, dt$$

$$= \frac{m w^2 a^2}{4T} [\int_0^T dt - \int_0^T \cos 2 (wt + \phi)]\, dt$$

The average value of both a sine and a cosine function for a complete cycle or a whole time period T is zero. We, therefore, have

average P.E. of the particle

$$= \frac{1}{4T} m w^2 a^2 \left[t \right]_0^T - 0$$

$$= \frac{1}{4T} m w^2 a^2 T$$

$$= \frac{1}{4} m w^2 a^2$$

$$= \frac{1}{4} k a^2 \qquad [\because m w^2 = K] \qquad (1.18)$$

The kinetic energy (K.E.) of the particle at displacement y is given by

$$= \frac{1}{2} m \left(\frac{dy}{dt} \right)^2$$

$$= \frac{1}{2} m \left[\frac{d}{dt} a \sin (wt + \phi) \right]^2$$

$$= \frac{1}{2} m w^2 a^2 \cos^2 (wt + \phi)$$

The average K.E. of the particle over a complete cycle or a whole time period T, as in the case of P.E., is given by

$$\frac{1}{T} \int_0^T \frac{1}{2} m w^2 a^2 \cos^2 (wt + \phi)\, dt$$

$$= \frac{m w^2 a^2}{4T} \int_0^T 2 \cos^2 (wt + \phi)\, dt$$

$$= \frac{m w^2 a^2}{4T} \int_0^T [1 + \cos 2 (wt + \phi)]\, dt$$

$$= \frac{m w^2 a^2}{4T} \left[\int_0^T dt + \int_0^T \cos 2 (wt + \phi)\, dt \right]$$

Again, the average value of a sine or cosine function over a complete cycle or a whole time period is zero. Hence

average K.E. of the particle

$$= \frac{m w^2 a^2}{4T} [t]_0^T$$

$$= \frac{m w^2 a^2}{4T} . T$$

$$= \frac{1}{4} m w^2 a^2 = \frac{1}{4} k a^2. \qquad (1.19)$$

Thus,

average value of P.E. of the particle

= average value of K.E. of the particle

$= \frac{1}{4} m w^2 a^2 = \frac{1}{4} k a^2$

= half the total energy.

Example 1.11. *(i) What is the mechanical (total) energy of the oscillating system of Example 1.1 ?*

Soln.

Total energy = P.E. + K.E. $\frac{1}{4} k a^2$

Now k = 200N/m and a = 0.04m [Ex. 1.1 (i) and (iv)]

$\therefore$ Total energy $= \frac{1}{2}$ (200) $(0.04)^2$ = **0.16 joules**

Also total energy,

$$\frac{1}{4}\ ka^2 = 2\pi^2 m\ a^2 n^2$$

$$= 2\ (3.14)^2\ (2)\ (0.04)^2\ (159)$$

$$= \mathbf{0.16\ joules}$$

From Ex. 1.1
$a = 0.04$ m
$m = 2$ Kg
$n = 159$ Hz

(ii) Compute the velocity, the acceleration and the kinetic and potential energies of the body when it has moved in half-way from its initial position toward the centre of motion.

$$v = \pm w \sqrt{a^2 - y^2}$$

At half-way,

$$y = \frac{a}{2} = \frac{0.04}{2}\ m = 0.02m$$

$$w = 10\ sec^{-1}$$

$$= \pm\ (10\ sec^{-1})\sqrt{(0.04m)^2 - (0.02m)^2}$$

$$= \pm\ \frac{2\sqrt{3}}{10}\ m/sec^{-1}$$

$$= \pm\ \mathbf{0.346\ m\ sec^{-1}}$$

$$\text{acceleration} = -\ w^2 y = -\ (10\ sec^{-1})\ .\ (0.02\ m)$$

$$= \mathbf{-\ 2.0\ m\ sec^{-2}}$$

$$K.\ E. = \frac{1}{2}\ mv^2 = \frac{1}{2}\ (2)\ (0.346)^2 \simeq \mathbf{0.12\ joules}$$

$$P.E. = \frac{1}{2}\ ky^2 = \frac{1}{2}\ (200)\ (0.02)^2 \simeq \mathbf{0.04\ joules}$$

$$\therefore \text{ Total energy} = P.E. + K.E. = (0.04 + 0.12) \text{ joules}$$

$$= \mathbf{0.16\ joules.}$$

Note that total energy is constant.

Example 1.12. *A block whose mass is 680 gm is at rest on a table top and is fastened to an anchored horizontal spring. The*