CHAPTER IV

WAVE MOTION

Wave motion : types of waves-transverse and longitudinal wave motion-some definitions connected with wave motion-Expression for a plane progressive wave-Differential equation of wave motion-Particle velocity-Phase or wave velocity-Energy density of a plane progressive wave-Energy current-intensity of a wave.

4.1 Wave motion-types of waves

Wave motion can be thought of as the transport of energy and momentum from one point in space to another without the transport of matter. For example, water waves, sound waves, light waves and radio waves are all known to carry energy of one form or the other from one place to another. In the case of water and sound wave, although a medium is necessary, there is no bulk motion of the intervening medium. In the case of light and radio waves, no intervening medium is at all necessary. Thus wave motion may be divided into two broad categories.

(i) Mechanical wave motion : This sort of wave motion is possible only in media (solid, liquid or gas) which possess intertia as well as elasticity. Water waves and sound waves are examples of this type of wave motion and are, therefore, referred to as mechanical waves.

(ii) Electromagnetic (or non-mechanical) wave motion : No material medium is necessary for the propagation of this sort of wave motion. Light and radio waves which can travel through empty space, belong to this category and are, therefore, referred to as non-mechanical or electromagnetic waves.

As sound waves are mechanical waves, only mechanical type of wave motion – to be referred simply as wave motion from now on, will be discussed in this chapter.

Production and propagation of wave motion

The production and propagation of wave motion through a medium which possesses elasticity and inertia will now be investigated. No particle of an elastic medium can be disturbed without affecting its immediate neighbour and, tending to recover its original position; it first stores up potential energy and then converts it back into kinetic energy. The neighbouring particle which has thus been disturbed then performs a similar motion, so that each successive particle repeats, in turn, the movements of its predecessor a little later than it and then hands down the same on to its successor. This results in a transference of energy from particle to particle all along the line. One complete oscillation of a particle of the medium obviously produces one single wave or a pulse and its repeated periodic motion, a succession of waves or a wavetrain.

A wave motion may thus be defined as a disturbance or a condition that travels onwards through a medium due to the repeated periodic motion of its particles about their mean or equilibrium positions, each particle repeating the movements of its predecessor a little later than it and handing it on to its successor, so that there is a regular phase difference between one particle and the next.

It must be clearly understood that what is propagated in a wave motion is only a state of motion of the matter – not the matter itself. The wave motion is a form of dynamic condition, arising out of the vibration of the particles of the medium about their mean positions, that is propagated from one point to the other point in the medium. According to the laws of physics, any dynamic condition is related to momentum and energy. Thus, in conclusion, *it may be said that in wave motion momentum and energy are transferred or propagated*. It is not a case of propagation of matter as a whole.

The simplest type of periodic motion performed by a particle is, of course, the simple harmonic motion. The corresponding wave motion is, therefore, called a simple harmonic or sinusoidal wave motion. This is the most general type of wave motion and will be dealt with in the following discussion.

It may be emphasized again that, *but for the properties of elasticity and inertia, it would not have been possible for a wave motion to be produced in or propagated through a medium*. As will be seen later, these two properties in fact determine the velocity of propagation of the wave motion through the medium. A wave may travel through a medium over fairly large distances. In order that the wave may do so without any attenuation (*i.e.*, without any decrease in its amplitude), a third property is also necessary, *viz.*, that *the medium should offer the least frictional resistance* so as not to unduly damp the periodic motion of the particles.

A wave motion which progresses onwards through a medium, with energy transferred across every section of it, is called a *travelling or a progressive wave motion* to distinguish it from what is called a *standing or stationary wave motion* in which there is no onward movement of the wave motion through the medium and hence no transference of energy across any section of it.

4.2 Transverse and longitudinal wave motion

There are two distinct types of wave motion :

(i) transverse and (ii) longitudinal.

(i) *Transverse wave motion :* In transverse wave motion, the particles of the medium oscillate up and down about their mean or equilibrium position in a direction at right angles to the direction of propagation of the wave motion itself. This form of wave motion therefore travels in the form of crests and troughs with one crest and an adjoining trough making up one wave (Fig. 4.1). A succession of waves make up a wavetrain.

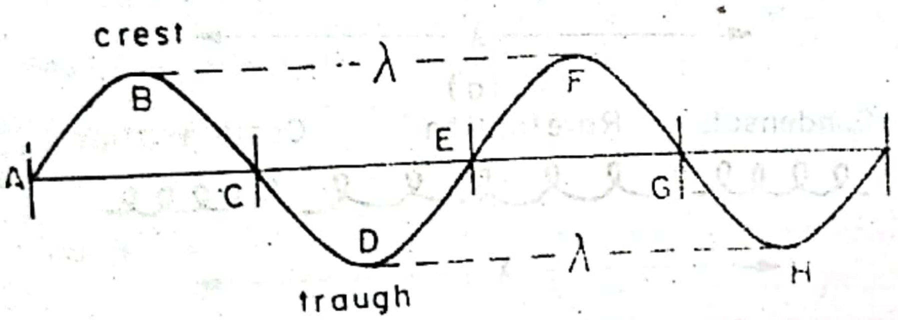

Fig. 4.1

Transverse wave motion is possible in media which possess *elasticity of shape or rigidity i.e.*, solids. Waves travelling along a stretched string is an example of transverse wave motion. Although water and liquids do not possess the property of rigidity, transverse wave motion is still possible in them because they possess another equally effective property of resisting any vertical displacement of their particles (or keeping their level). Gases, however, possess neither rigidity nor do they resist any vertical displacement of their particles (or keep their levels). *A transverse wave motion is, therefore, not possible in a gaseous medium.*

(ii) Longitudinal wave motion : In this type of wave motion, the particles of the medium oscillate to and fro about their mean or equlibrium position, along the direction of propagation of the wave motion itself. The wave motion, therefore, travels in the form of *compressions* (or *condensations*) and *rarefactions* *i.e.*, in the particles of the medium getting closer together and further apart alternately. This type of wave motion is possible in media possessing elasticity of volume, *i.e.*, in solids, liquids as well as gases. Waves produced in a spring or helix when one end of it is suddenly compressed or pulled out and then released or sound waves in air are examples of this type of wave motion (Fig. 4.2). As in the case of transverse wave motion, one compression and the adjoining rarefaction constitute a wave or pulse and a succession of them, a wavetrain

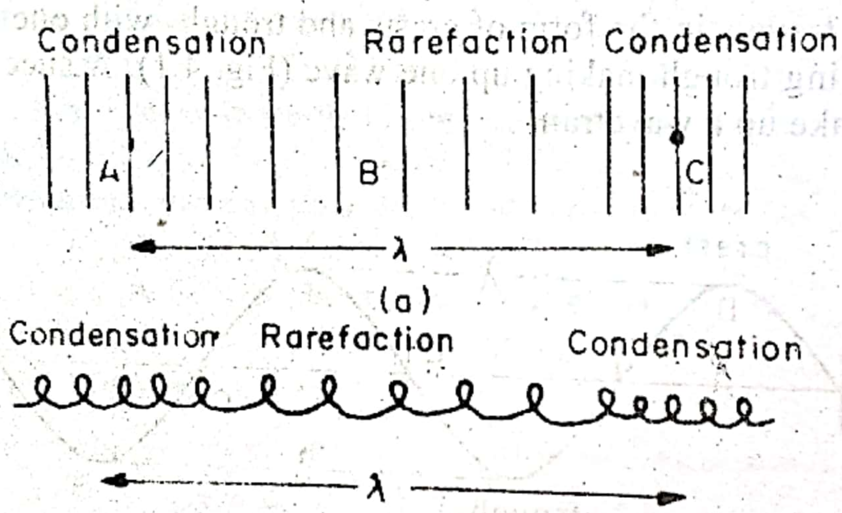

Fig. 4.2

Characteristics of wave motion

Before proceeding further, the important characteristics of wave motion, whether transverse or longitudinal, may be summarized below :

(i) Wave motion is a disturbance produced in a medium by the repeated periodic motion of the particles of the medium. It is only this disturbance which travels forward through the medium as the wave while the particles of the medium vibrate about their mean positions – they are not propagated through the medium.

(ii) The velocity with which the wave propagates through the medium and the velocity with which the particles of the medium vibrate about their mean positions are different. *While the velocity of the wave is constant, the velocity of the particles is different at different positions.* The velocity of the particle is maximum at the mean position and zero at the extreme position of the particle.

(iii) There is a phase difference between the particles of the medium. The particle ahead starts vibrating a little later than a particle just preceeding it.

4.3 Some definitions connected with wave motion

(i) ***Wavelength :*** Since a wave or a pulse is produced in the time taken by a particle of the medium to complete one full oscillation about its mean position, *wavelength may be defined as the distance travelled by the wave in the time in which the particle completes one vibration.* It may also be defined as *the distance between any two nearest particles of the medium which are in the same phase.*

(ii) Amplitude : *It is the maximum displacement of the particle from its mean position of rest.*

(iii) Time period (T) : *It is the time taken by a particle to complete one vibration.*

(iv) Frequency (n) : *The number of complete oscillations made by a particle of the medium i.e., the number of waves produced, in one second is called the frequency of the wave (or vibration).*

Suppose frequency = n

Time taken to complete n vibrations = 1 second

Time taken to complete 1 vibration $= \frac{1}{n}$ second

By definition time taken to complete one vibration is the time period (T).

$$\therefore \quad T = \frac{1}{n} \; ; \quad \text{or } nT = 1$$

Frequency x Time period = 1

(v) Angular frequency (ω) : *The rate of change of phase with time is called the angular frequency and is designated by ω.* Since in one complete cycle, a phase change of 2π occurs in a time T (the time period of the cycle), angular frequency

$$\omega = \frac{2\pi}{T} = 2\pi.n$$

and has the unit of *radian per second*, the same as that of angular velocity, also designated ω (also see Art. 1.3).

(vi) Angular wave number (k) : *the angular wave number, k, is defined as the rate of change of phase with distance.* Since in one complete cycle, a phase change of 2π takes place within a distance λ (the wavelength), $k = 2\pi/\lambda$.

(vii) Wave number (k) : *The wave number is the number of waves in a unit length of the wave pattern and is given by*

$$k = \frac{1}{\lambda} = \frac{k}{2\pi}$$

(viii) Velocity (v) : *Velocity of the wave is the distance traveled by the wave in one time period (T).*

$$\therefore \quad \text{Velocity (v)} = \frac{\text{wavelength}}{\text{time period}} = \frac{\lambda}{T}$$

or $\lambda = vT$

But, frequency x time period = 1

or, $n \times T = 1$

$$\therefore \quad T = \frac{1}{n}$$

$$\therefore \quad v = \frac{\lambda}{T} = \frac{\lambda}{1/n}$$

or $v = n\lambda$

(ix) ***Phase :*** The phase of a vibrating particle is defined as *the ratio of the displacement of the vibrating particle at any instant to the amplitude of the vibrating particle (y/a). It is also defined as the fraction of the time interval that has elapsed since the particle crossed the mean position of rest in the positive direction.* The phase is equal to the *angle* swept by the radius vector since the vibrating particle last crossed its mean position of rest.

(x) ***Wave front :*** According to origin of wave motion a vibrating particle placed at a point in a homogeneous medium, extending in all directions, communicates its motion to all its neighbouring particles. The neighbouring particles which have thus been disturbed then perform, in turn, the motion of the vibrating particle. Due to this periodic vibration of the particles a wave motion is produced which travel in every direction with equal velocity. The wave motion, therefore, reach all particles which are at equal distances from the point simultaneously. The position of all these particles can be represented by the surface of a sphere drawn with the position of the vibrating particle as the centre. With time the wave advances into spheres of gradually increasing radius. Such a sphere is known as a wavefront. A wavefront at any instant of time may, therefore, be defined as *the loci of all the neighbouring particles in the medium which are just being disturbed at that instant of time and are consequently in the same state of vibration.*

In a homogeneous medium, the wavefronts are always actually spherical. But if a wavefront is considered at a considerable distance from the source, then any small portion of the wavefront can be considered *plane*.

4.4 Expression for a plane progressive wave

A plane progressive wave is one which travels onward through the medium in a given direction without attenuation i.e., with its amplitude constant.

A progressive wave may be either transverse or longitudinal. In either case, there exists a regular phase difference between any two successive particles of the medium. A typical waveform is shown in Fig. 4.3. Let a wave originating at O, travel to the right along the x-axis. If we start counting the time at the moment when the particle at

O just passes through its mean position in the positive direction (*i.e.*, upwards in the case of transverse wave and forward in the case of longitudinal wave), the equation of motion of this particle at O is obviously

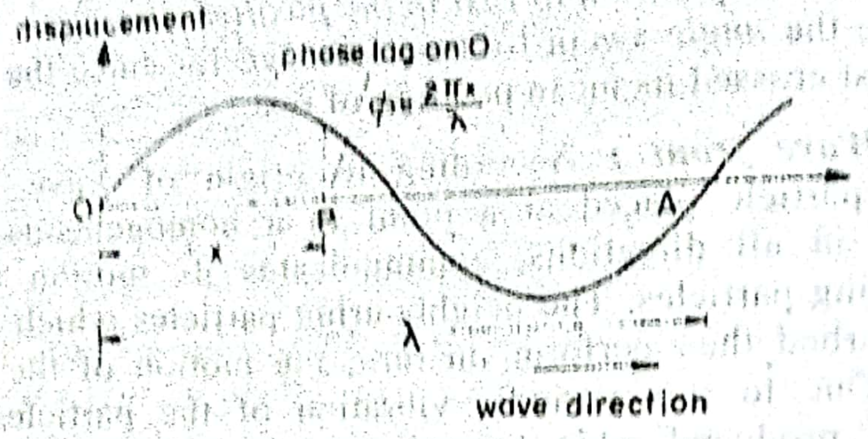

Fig. 4.3

$$y = a \sin \omega t$$

where y is the displacement of the particle at time t, a its amplitude and ω its angular velocity. Since the motion of O is received and repeated by the successive particles to the right of O, the phase lag goes on increasing as we proceed away from O towards the right. Thus for a particle at P which is at a distance x away from O, let this phase difference be ϕ. Hence the equation of motion of the particle at P is

$$y = a \sin (\omega t - \phi).$$

For a difference in path of λ, *i.e.*, one wavelength, the corresponding difference in phase is 2π. Hence for a distance x, the corresponding phase difference is $\frac{2\pi}{\lambda} \cdot x$ *i.e.*, $\phi = \frac{2\pi}{\lambda} x$. Substituting this value of ϕ in the above expression for y, we get

$$y = a \sin \left(\omega t - \frac{2\pi}{\lambda} x\right)$$

$$\text{or,} \quad y = a \sin (\omega t - kx) \tag{4.1}$$

where $k = \frac{2\pi}{\lambda}$ is referred to as the propagation constant or the angular wave number.

Now $\omega = \frac{2\pi}{T} = 2\pi n$. Again $n = \frac{v}{\lambda}$ (from $v = n\lambda$) where n is the frequency of the particle or the wave. Hence $\omega = \frac{2\pi v}{\lambda}$. Eqn. (4.1) then becomes

$$y = a \sin \frac{2\pi v}{\lambda} t - \frac{2\pi}{\lambda} x$$

$$\text{or,} \quad y = a \sin \frac{2\pi}{\lambda}(vt - x) \tag{4.2}$$

$$\text{or,} \quad y = a \sin k(vt - x) \tag{4.3}$$

Any one of the expressions written above or any one of their variations, such as those given below [eqns. 4.4 – 4.7], is referred to as the equation of a plane progressive wave motion in the positive direction of x.

Eqn. (4.2) can be written as

$$y = a \sin \frac{2\pi v}{\lambda}\left(t - \frac{x}{v}\right) \tag{4.4}$$

$$y = a \sin 2\pi n\left(t - \frac{x}{v}\right) \tag{4.5}$$

[since $n\lambda = v$; or, $v/\lambda = n$]

$$y = a \sin \frac{2\pi v}{\lambda}\left(t - \frac{x}{v}\right) \tag{4.6}$$

$(\because n = 1/T)$

$$y = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \tag{4.7}$$

(since $v = n\lambda = \lambda/T$)

The expression most commonly used is given by equation (4.2) Similarly, if the wave travels towards the left, x becomes negative and we have

$$y = a \sin \frac{2\pi}{\lambda}(vt + x) \tag{4.8}$$

or, $y = a \sin(\omega t + kx)$ (4.9)

The phase or the wave velocity is also given by

$$v = \lambda/T = \lambda \Big/ \frac{2\pi}{\omega} \qquad \left(\text{from } \omega = \frac{2\pi}{T}\right)$$

$$= \frac{\lambda\omega}{2\pi} = \omega \Big/ \frac{2\pi}{\lambda} = \omega/k \tag{4.10}$$

$$\left(\text{since } k = \frac{2\pi}{\lambda}\right)$$

It should be emphasized that in deducing the above expressions, it has been assumed that at t = 0, the particle O just passes through its mean position in the positive direction, *i.e.*, at t = 0, y = 0. If this is not the case and the particle is said to have an initial phase θ, say, then the equation of the wave becomes

$$y = a \sin \frac{2\pi}{\lambda}(vt - x + \theta) \tag{4.11}$$

and $y = a \sin(\omega t - kt + \theta)$ (4.12)

where θ is referred to as the initial phase or phase constant.

If θ = 90°, we have y = a, at x = 0 and t = 0.

The following points emerge from discussion of the wave motion :

(i) For a constant value of x (*i.e.*, at a given point x) the displacement y varies simple harmonically with time, completing one full cycle in time λ/v. which, therefore, gives the periodic time T of the wave. This also gives the frequency n of the wave since frequency n = 1/T = v/λ.

(ii) For a constant value of t (*i.e.*, at a given instant t), the displacement y varies simple harmonically with distance from the origin. For x = λ, the displacement (y) is restored to its original value, so that the wavelength = λ.

(iii) The phase lag for a distance x is given by $\frac{2\pi}{\lambda}$. x. Hence or a distance λ, the phase lag is $\frac{2\pi}{\lambda}.\lambda = 2\pi$ which, in effect, is the same thing as zero. Thus particles separated by a distance λ or an integral multiple of λ are in the same phase.

(iv) If the time t is increased by δt and the distance x by $v\delta t$, the value of y remains the same. This shows that a disturbance at one point is repeated after a time δt at a point $v.\delta t$ further away. This means, in other words, that the disturbance or the wave travels with a velocity v without any attenuation.

4.5 Phase (or wave) velocity

The compressions and rarefactions of longitudinal wave or crest and trough of transverse wave advances through a medium with a constant velocity. In other words, advance of phase through a medium takes place with same velocity. This velocity of advance is know as *phase velocity*.

The equation of a plane progressive wave, travelling in a medium along the direction of positive x-axis, is given by

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \qquad \text{(i)}$$

where v is referred to as the wave velocity and is equal to λ/T.

Rearranging, eqn. (i) can be written in the form

$$y = a \sin \frac{2\pi vt}{\lambda} - \frac{2\pi}{\lambda} x$$

$$= a \sin (2\pi nt - kx) \qquad \text{(ii)}$$

since $v/\lambda = 1/T = n$ is the frequency of the wave and $2\pi/\lambda = k$ is the propagation constant of the wave.

Further, $2\pi n = \omega$ is the angular frequency of the wave. Hence eqn. (ii) reduces to

$$y = a \sin (\omega t - kx) \qquad \text{(iii)}$$

Now (wt – kx) in eqn. (iii) is the constant phase of the wave which travels along the positive direction of the x-axis and its velocity, *i.e.*, the phase velocity of the wave should given by dx/dt.

Since $(\omega t - kx)$ is a constant quantity, we have $\frac{d}{dt}(\omega t - kx) = 0$

$$\text{or,} \quad \omega - k\frac{dx}{dt} = 0$$

$$\text{or,} \quad \omega = k\cdot\frac{dx}{dt}$$

Thus the phase velocity of the wave,

$$\frac{dx}{dt} = \frac{\omega}{k} \tag{4.13}$$

But $\omega/k = \frac{2\pi n}{2\pi/\lambda} = n\lambda = \lambda/T$ = wave velocity v. Thus, for a single wave, in any given medium, wave velocity = phase velocity = v = $\lambda/T = \omega/k$.

***Exaple 4.1** The displacement (in metres) of a particle executing simple harmonic motion at any instant of time is given by*

$$y = 0.1 \sin 2\pi (340t - 0.15).$$

Calculate (i) the amplitude of the vibrating particle (ii) wave velocity (iii) wave length (iv) frequency and (v) time period.

Soln.

The general equation of a simple harmonic wave is given by

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \tag{i}$$

$$\text{Here } y = 0.1 \sin 2\pi (340t - x) \tag{ii}$$

Comparing eqns. (i) and (ii)

a = **0.1 metre**

λ = **1 metre**

wave velocity **v = 340 m/sec.**

$$\text{Frequency } n = \frac{v}{\lambda} = \frac{340}{1} = \textbf{340 Hz.}$$

$$\text{Time period, } T = \frac{1}{n} = \frac{1}{340} \textbf{ second.}$$

***Example 4.2** A simple harmonic wave of amplitude 8 units traverses a line of particles in the direction of positive x-axis. At any given instant of time, the displacement of a particle at a distance of 10 cm from the origin is +6 units while that for a particle at a distance of 25 cm from the origin is +4 units. Calculate the wavelength.*

Soln.

The equation of a simple harmonic wave can be written as

$$y = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

$$\text{or,} \quad y = \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

In the first case,

$y = +6$, $a = 8$ and $x = 10$ cms.

$$\therefore \quad \frac{6}{8} = \sin 2\pi \left(\frac{t}{T} - \frac{25}{\lambda}\right) \qquad \text{(i)}$$

In the second case,

$y = 4$, $a = 8$ and $x = 25$ cms.

$$\therefore \quad \frac{4}{8} = \sin 2\pi \left(\frac{t}{T} - \frac{25}{\lambda}\right) \qquad \text{(ii)}$$

From eqn. (i)

$$0.75 = \sin 2\pi \left(\frac{t}{T} - \frac{10}{\lambda}\right)$$

$$\text{But } \sin\left(\frac{48.6\pi}{180}\right) = 0.75.$$

$$\therefore \quad 2\pi \left(\frac{t}{T} - \frac{10}{\lambda}\right) = \frac{48.6}{180}$$

And from (ii)

$$0.5 = \sin 2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right)$$

But $\sin\frac{\pi}{6} = 0.5$

$$\therefore\quad 2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right) = \frac{\pi}{6} \qquad \text{(v)}$$

$$\text{or,}\quad \frac{t}{T} - \frac{25}{\lambda} = \frac{1}{12} \qquad \text{(vi)}$$

Subtracting (vi) from (iv)

$$\frac{25}{\lambda} - \frac{10}{\lambda} = \frac{48.6}{360} - \frac{1}{12}$$

or, λ = **290.8 cm.**

Example 4.3 *The velocity of a simple harmonic wave is 30 cm/s. At a time $t = 0$ the displacement of a particle is given by*

$$y = 4\sin 2\pi\left(\frac{x}{100}\right)$$

Find the equation for the displacement at a time $t = 2$ sec.

Soln.

The general equation of a simple harmonic wave is

$$y = a\sin\frac{2\pi}{\lambda}(vt - x)$$

$$= a\sin\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right)$$

when $t = 0$,

$$y = a\sin\left(-\frac{2\pi x}{\lambda}\right)$$

$$= -a\sin\frac{2\pi x}{\lambda} \qquad \text{(i)}$$

At $t = 0$, the given equation is

$$y = 4 \sin 2\pi \left(\frac{x}{100}\right) \qquad \text{(ii)}$$

Comparing eqns. (i) and (ii), we get

$$a = -4 \text{ and } \lambda = 100 \text{ cm.}$$

At t = 2 sec.

$$y = a \sin \frac{2\pi}{\lambda} (vt - x).$$

here a = – 4 cm. λ = 100 cm and t = 2 sec.

$$\therefore \quad y = -4 \sin \frac{2\pi}{100} (30 \times 2 - x)$$

$$= -4 \sin \left[\frac{6\pi}{5} - 2\pi \left(\frac{x}{100}\right)\right]$$

$$= 4 \sin \left[2\pi \left(\frac{x}{100}\right) - \frac{6\pi}{5} \right].$$

***Example 4.4** A plane progressive wave travelling along the + x – direction has the following characteristics : a = 0.2 cm. v = 360 cm/sec and λ = 60 cm.*

(a) Write down the equation for it (i) when displacement is zero at x = 0 and t = 0 and (ii) when displacement is maximum at x = 0 and t = 0

(b) Obtain the displacement in either case at x = 120 cm and t = 2 sec.

Soln.

(a) The general equation of a plane progressive wave

$$y = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda} + \theta\right)$$

$$= a \sin 2\pi \left(nt - \frac{x}{\lambda} + \theta\right)$$

where θ is the initial phase.

$$= 0.2 \sin 2\pi \left(6t - \frac{x}{60} + \theta\right)$$

where $n = \frac{v}{\lambda} = \frac{360}{60} = 6$

(i) Now $y = 0$ at $x = 0$ and $t = 0$

$\therefore \quad 0 = 0.2 \sin 2\pi (0 - 0 + \theta)$

or, $\theta = 0$.

$\therefore \quad \mathbf{y = 0.2 \sin 2\pi \left(6t - \frac{x}{60}\right)}$

(ii) again $y = 0.2$ at $x = 0$ and $t = 0$

$\therefore \quad 0.2 = 0.2 \sin 2\pi (0 - 0 + \theta)$

or, $\theta = \frac{\pi}{2}$

$\therefore \quad y = 0.2 \sin 2\pi \left(6t - \frac{x}{\lambda} + \frac{\pi}{2}\right)$

$\mathbf{= 0.2 \cos 2\pi \left(6t - \frac{x}{\lambda}\right)}$

(b) when $y = 0$ at $x = 0$ and $t = 0$, the equation of a progressive wave is

$$y = 0.2 \sin 2\pi \left(6t - \frac{x}{\lambda}\right)$$

Here $t = 2$ sec, $x = 120$ cm, $\lambda = 60$ cm

$\therefore \quad y = 0.2 \sin 2\pi \left(6 \times 2 - \frac{120}{60}\right)$

$= 0.2 \sin 2\pi \times 10$

$= \mathbf{0}$

Again when $y = 0.2$ at $x = 0$ and $t = 0$, the equation is

$$y = 0.2 \cos 2\pi \left(6t - \frac{x}{60}\right)$$

$$= 0.2 \cos 2\pi \left(6 \times 2 - \frac{120}{60}\right)$$

$$= 0.2 \cos 2\pi \times 10$$

$$= \mathbf{0.2\ cm.}$$

***Example 4.5.** A plane progressive wave train of frequency 400 cycles per second has a phase velocity of 480 m/sec. (i) How far apart are two points 30° out of phase? (ii) What is the phase difference between two displacements at a given point at times 10^{-3} sec apart?*

Soln.

The equation of a plane progressive wave is given by

$$y = a \sin \frac{2\pi}{\lambda}(vt - x)$$

where $\frac{2\pi}{\lambda}(vt - x)$ is the phase angle of a point at a distance x from the origin at time t.

$\therefore$ phase angle of a point at a distance x_1 from the origin at time $t = \frac{2\pi}{\lambda}(vt - x_1)$

and phase angle of point at a distance x_2 from the origin at time $t = \frac{2\pi}{\lambda}(vt - x_2)$.

Hence phase difference between the two points

$$= \frac{2\pi}{\lambda}(vt - x_1) - \frac{2\pi}{\lambda}(vt - x_2)$$

$$= \frac{2\pi}{\lambda}(x_2 - x_1) = \frac{2\pi v}{\lambda} \cdot \frac{(x_2 - x_1)}{v}$$

$$= 2\pi n \frac{(x_2 - x_1)}{v}$$

since $\frac{v}{\lambda} = n$, the frequency of the wave. The phase difference between the two points

$$= 30^\circ = \frac{30 \times \pi}{180} \text{ rad} = \frac{\pi}{6} \text{ rad.}$$

$$\therefore\ 2\pi n \frac{(x_2 - x_1)}{v} = \frac{\pi}{6}$$

$$\text{or, } 2n \frac{(x_2 - x_1)}{v} = \frac{6}{1}$$

Here $n = 400$, $v = 480$ m/s $= 480 \times 10^2$ cm/sec.

$$\therefore\ \frac{2 \times 400 \times (x_2 - x_1)}{480 \times 10^2} = \frac{1}{6}$$

$$\text{or, } x_2 - x_1 = \frac{480 \times 10^2}{2 \times 400 \times 6}$$

$$= 10 \text{ cm} = \mathbf{0.1 \ m.}$$

(ii) Again, the phase angle at a point x from the origin at time $t_1 = \frac{2\pi}{\lambda}(vt_1 - x)$

and the phase at the same point at time t_2

$$= \frac{2\pi}{\lambda}(vt_2 - x)$$

$\therefore$ phase difference at the point at times $(t_2 - t_1)$ sec apart

$$= \frac{2\pi}{\lambda}(vt_2 - x) - \frac{2\pi}{\lambda}(vt_1 - x)$$

$$= \frac{2\pi v}{\lambda}(t_2 - t_1) = 2\pi n\,(t_2 - t_1)$$

Here $t_2 - t_1 = 10^{-3}$ sec.

$\therefore$ phase difference $= 2 \times \pi \times 400 \times 10^{-3}$

$= 0.8\pi$ rad. $=$ **144°.**

Example 4.6. *A sinusoidal wave traveling along a string is described by*

$y(x, t) = 0.00327 \sin(72.1\,x - 2.72\,t)$

in which the numerical constants are in SI units (0.00327 m, 72.1 rad/m, and 2.72 rad/s). What are (i) the amplitude, (ii) wavelength, (iii) period, (iv) wave number, (v) frequency, and (vi) speed of the wave?

Soln : The expression for a sinusoidal wave can also be written as

$$y(x, t) = y_m \sin(kx - wt)$$

where y_m is the amplitude of the wave.

Comparison with the given equation gives

(i) amplitude = 0.00327 m = **3.27 mm.**

(ii) k = 72.1 rad/m and w = 2.72 rad/s.

$$k = \frac{2\pi}{\lambda}; \quad \text{or } \lambda = \frac{2\pi}{k} = \frac{2\pi \text{ rad}}{72.1 \text{ rad/m}}$$

$$= 0.0871 \text{ m} = \mathbf{8.71\ cm.}$$

(iii) $T = \frac{2\pi}{\omega} = \frac{2\pi \text{ rad}}{2.72 \text{ rad/s}} = \mathbf{2.31\ s.}$

(iv) wave number, $k = \frac{1}{\lambda} = \frac{1}{0.0871 \text{ m}} = \mathbf{11.5\ m^{-1}}$.

(v) frequency, $n = \frac{1}{T} = \frac{1}{2.31 \text{ s}} = \mathbf{0.433\ Hz.}$

(vi) velocity, $v = \frac{\omega}{k} = \frac{2.72 \text{ rad/s}}{72.1 \text{ rad/m}} = \mathbf{0.377\ m/s}$

$$= \mathbf{3.77\ cm/s.}$$

4.6 Differential equation of wave motion

Differentiating the most general form of a simple harmonic wave

$$y = a \sin \frac{2\pi}{\lambda}(vt - x)$$

with respect to time, we get

$$\frac{dy}{dt} = \frac{2\pi av}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x)$$

Differentiating the above expression again with respect to time, we get

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2v^2}{\lambda^2}\ a\ \sin\ \frac{2\pi}{\lambda}(vt - x)$$

$$= -\frac{4\pi^2v^2}{\lambda^2}.y \qquad (4.13)$$

Similarly, differentiating eqn. (4.2) with respect to x, we get the slope of the displacement curve (also referred to as strain or compression)

$$\frac{dy}{dx} = -\frac{2\pi}{\lambda}\ a\ \cos\ \frac{2\pi}{\lambda}(vt - x)$$

Differentiating the above expression again with respect to x, we get the *rate of change of compression with distance*

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2}{\lambda^2}\ a\ \sin\ \frac{2\pi}{\lambda}(vt - x)$$

$$= -\frac{4\pi^2}{\lambda^2}.y \qquad (4.14)$$

From relations (4.13) and (4.14), we have

$$\frac{d^2y}{dt^2} = v^2\left(-\frac{4\pi^2}{\lambda^2}.y\right) = v^2\frac{d^2y}{dx^2} \qquad (4.15)$$

Eqn. (4.15) is referred to as the *differential equation of a plane or one-dimensional progressive wave.* The general differential equation of wave motion can be written as

$$\frac{d^2y}{dt^2} = K\frac{d^2y}{dx^2}$$

where $K = v^2$; or $v = \sqrt{K}$

Any equation of this form can unhesitatingly be declared to represent a plane, progressive harmonic wave, the velocity of which is given by the square root of the co-efficient of d^2y/dx^2.

Now d^2y/dx^2 gives the rate of change of compression with distance *i.e.*, the *curvature of the displacement curve.* Hence, the differential equation as given by eqn. (4.15) may be interpreted to mean that

Particle acceleration at a point $[\frac{d^2y}{dt^2}]$.

= (wave velocity)^2 $[v^2]$ x curvature of the displacement curve at the point $[\frac{d^2y}{dx^2}]$.

4.7 Particle velocity and wave velocity

The equation of a harmonic plane progressive wave is given by

$$y = a \sin \frac{2\pi}{\lambda}(vt - x)$$

where

y = displacement of a particle of the medium at a distance x from the origin and at an instant of time t.

a = amplitude

v = wave (or phase) velocity.

Differentiating the above equation with respect to time, we have particle velocity,

$$U = \frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \qquad (4.16)$$

The maximum value of the particle velocity is $U_{max} = \frac{2\pi a}{\lambda} \cdot v$.

or, *maximum particle velocity* = $\frac{2\pi a}{\lambda}$ x (*wave velocity*)

The acceleration of the particle is given by

$$\frac{d^2y}{dt^2} = \frac{4\pi^2 av^2}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x)$$

$$= -\frac{4\pi^2 v^2}{\lambda^2}\left[a \sin \frac{2\pi}{\lambda}(vt - x)\right]$$

$$= -\left(\frac{4\pi^2 v^2}{\lambda^2}\right) . y$$

The acceleration is maximum when

$y = a$

Hence the maximum acceleration

$$= -\frac{4\pi^2 v^2}{\lambda^2} . a \qquad (4.17)$$

The minus sign indicates that the acceleration is directed towards its mean position.

Now differentiating eqn. (4.2) with respect to x, we get the *slope of the displacement curve* (also referred to as strain *or compression*).

$$\frac{dy}{dt} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \qquad (4.18)$$

From eqns. (4.16) and (4.18), we get

$$U = \frac{dy}{dt} = -v . \frac{dy}{dx} \qquad (4.19)$$

Thus,

particle velocity at a point = – (wave velocity) x (slope of the displacement curve at that point

---

Example 4.7 *A train of simple harmonic waves is travelling in a gas along the positive direction of the x-axis, with an amplitude equal to 2 cm, velocity 300 metres/sec and frequency 400. Calculate the displacement, particle velocity and particle acceleration at a distance of 4 cm from the origin after an interval of 5 seconds.*

Soln.

(i) displacement (y)

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

Here a = 2 cm, v = 300 m/s = 3×10^4 cm/s, $\lambda = v/n = 3 \times 10^4/400 =$ 75 cm, x = 4 cm, t = 5 sec.

$$\therefore \quad y = 2\sin\frac{2\pi}{75}(3 \times 10^4 \times 5 - 4)$$

$$= 2\sin\left(\frac{2\pi}{75} \times 149996\right)$$

$$= 2\sin(2\pi \times 1999.9)$$

$$= 2\sin(1999 \times 2\pi + 0.9 \times 2\pi)$$

$$= 2\sin(1.8\pi)$$

$$= 2\sin(\pi + 0.8)$$

$$= -2\sin(0.8\pi)$$

$$= -2\sin\left(0.8\pi \times \frac{180}{\pi}\right)^\circ = -2\sin 144^\circ$$

$$= -2\sin(180 - 144)^\circ$$

$$= -2\sin 36^\circ$$

$$= -2 \times 0.5878 = -1.1756 \text{ cm.}$$

Thus, the displacement of the particle at a distance of 4 cm from the origin, after an interval of 5 seconds is **– 1.1756 cm.**

(ii) particle velocity (U)

$$U = \frac{dy}{dt} = \frac{2\pi v}{\lambda} a \cos \frac{dy}{dt}(vt - x)$$

As we have seen in (i)

$$\sin\frac{2\pi}{\lambda}(vt - x) = \sin 36^\circ.$$

$$\text{Hence } \frac{2\pi}{\lambda}(vt - x) = 36^\circ$$

$$\therefore \quad U = \frac{2\pi \times 3 \times 10^4}{75} \times 2 \times 0.809$$

$$= \frac{2\pi \times 3 \times 10^4}{75} \times 2 \times 0.809$$

= 4068 cm/sec = **40.68 m/sec.**

(iii) particle acceleration $\left(\frac{d^2y}{dt^2}\right)$

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2v^2}{\lambda^2}\, a \sin\frac{2\pi}{\lambda}(vt - x)$$

$$= -\frac{4\pi^2v^2}{\lambda^2} \cdot y$$

$$= -\frac{4\pi^2(3\times 10^4)^2}{(75)^2} \times (-1.1756)$$

$(\because y = 1.1756)$

$= \mathbf{7.429 \times 10^6\ cm/sec^2}$.

Example 4.8. *Which of the following are solutions of the one-dimensional wave equation? (i) $y = x^2 + v^2t^2$, (ii) $y = x^2 - v^2t^2$, (iii) $y = (x - vt)^2$, (iv) $y = 7x - 10t$, (v) $y = 2\sin x\cos vt$ and (vi) $y = \sin 2x \cos vt$*

(i) $y = x^2 + v^2t^2$

Differentiating with respect to t, we get

$$\frac{dy}{dt} = 2v^2t$$

Differentiating again, $\frac{d^2y}{dt^2} = 2v^2$

Now differentiating with respect to x,

$$\frac{dy}{dx} = 2x.$$

Differentiating again, $\frac{d^2y}{dx^2} = 2.$

Clearly, $2v^2 = v^2 . (2)$

or, $\frac{d^2y}{dt^2} = v^2 . \frac{d^2y}{dx^2}$

which is the differential equation of a one-dimensional wave.