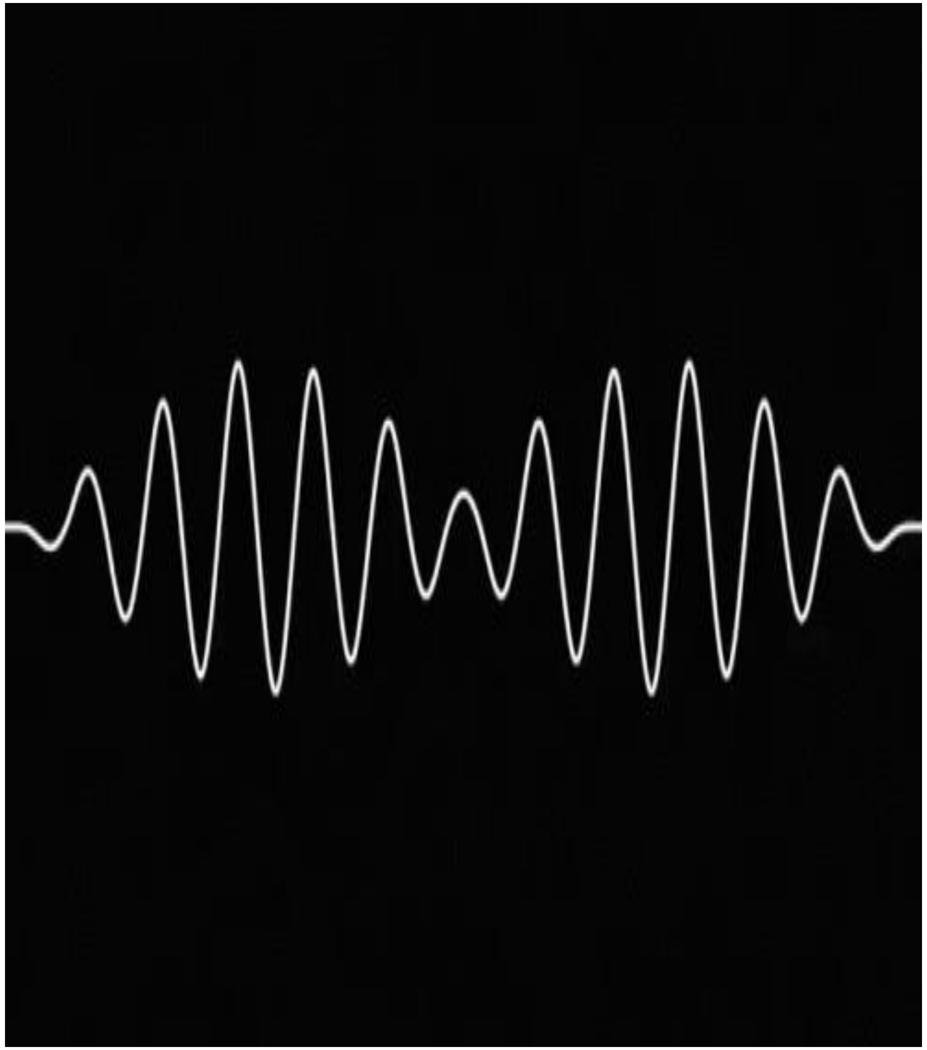
Wave

Phase Velocity and Group Velocity of Wave



Principle of Superposition of wave

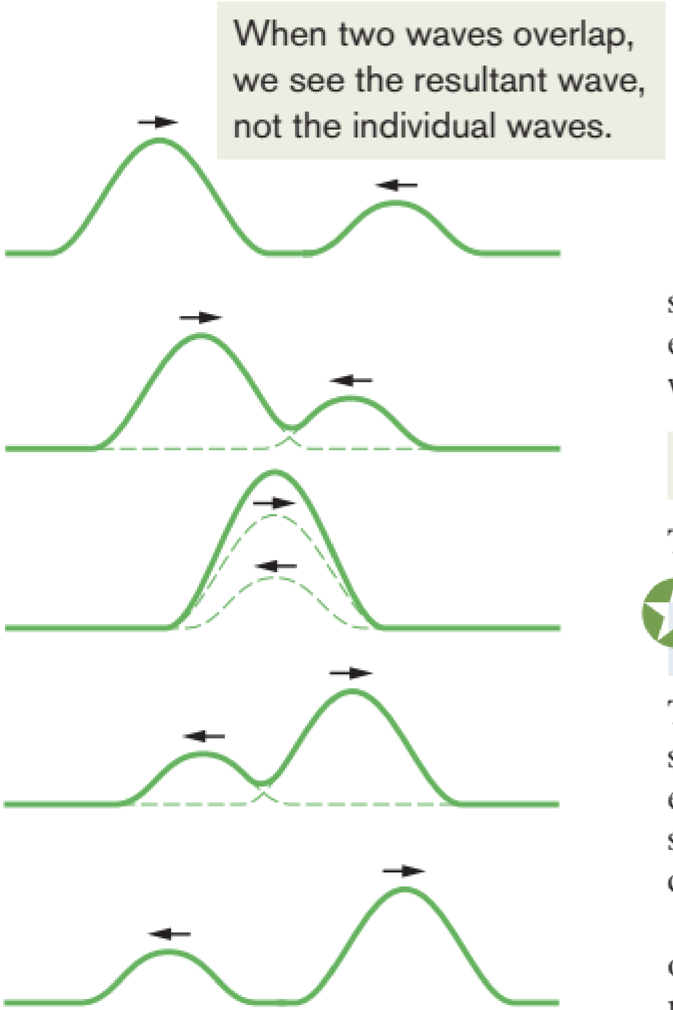


Figure 16-12 A series of snapshots that show two pulses traveling in opposite directions along a stretched string. The superposition principle applies as the pulses move through each other.

Suppose that two waves travel simultaneously along the same stretched string. Let $y_1(x, t)$ and $y_2(x, t)$ be the displacements that the string would experience if each wave traveled alone. The displacement of the string when the waves overlap is then the algebraic sum

$$y'(x, t) = y_1(x, t) + y_2(x, t). \tag{16-46}$$

This summation of displacements along the string means that

Overlapping waves algebraically add to produce a **resultant wave** (or **net wave**).

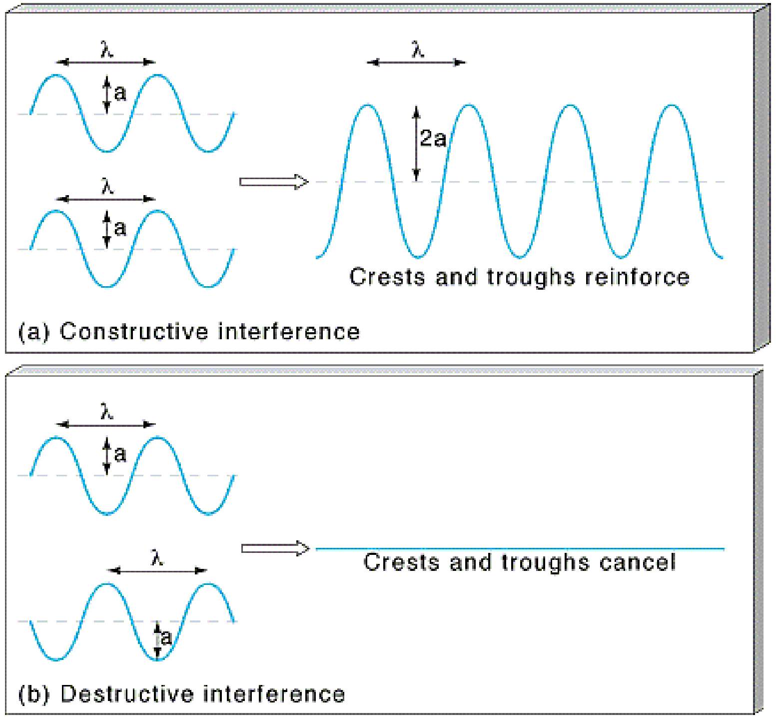
This is another example of the **principle of superposition,** which says that when several effects occur simultaneously, their net effect is the sum of the individual effects. (We should be thankful that only a simple sum is needed. If two effects somehow amplified each other, the resulting nonlinear world would be very difficult to manage and understand.)

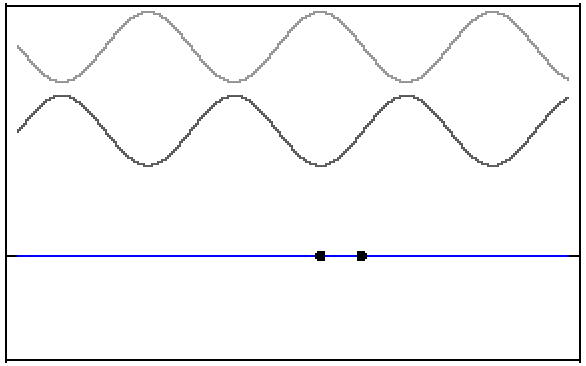
Figure 16-12 shows a sequence of snapshots of two pulses traveling in opposite directions on the same stretched string. When the pulses overlap, the resultant pulse is their sum. Moreover,

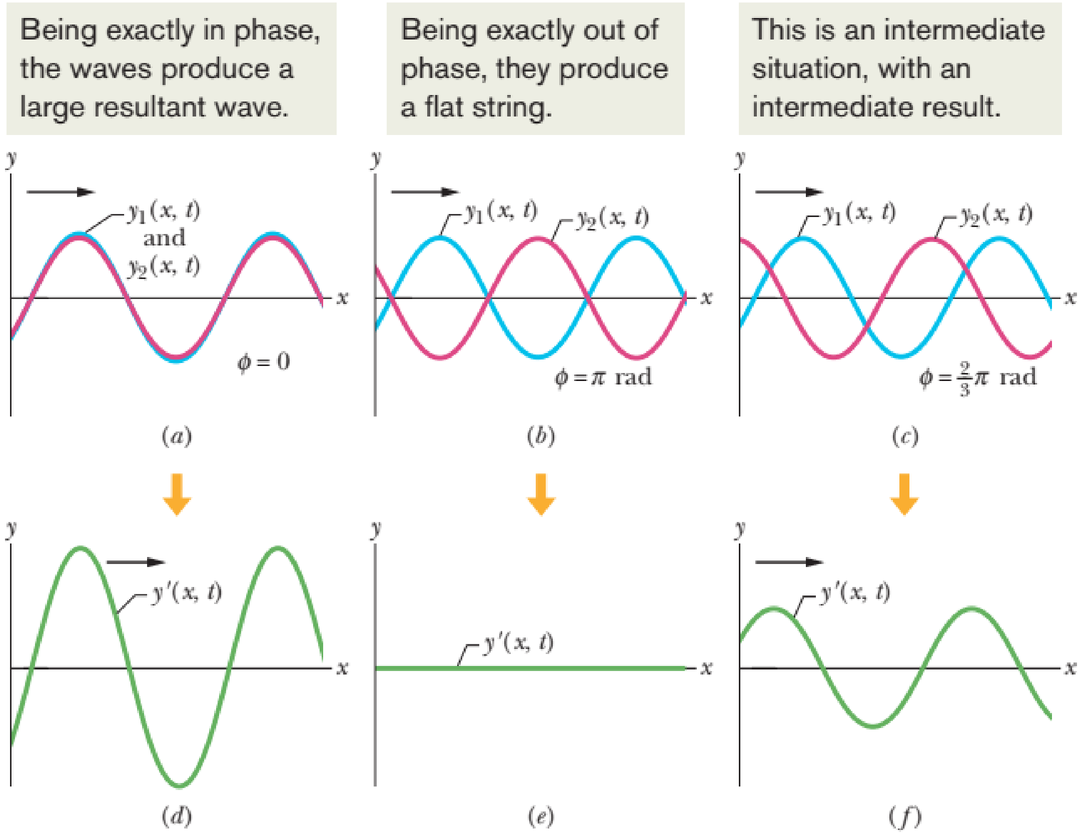
Overlapping waves do not in any way alter the travel of each other.

Interference

waves can interfere (add or cancel)







Interference

The resultant wave depends on the extent to which the waves are *in phase* (in step) with respect to each other—that is, how much one wave form is shifted from the other wave form. If the waves are exactly in phase (so that the peaks and valleys of one are exactly aligned with those of the other), they combine to double the displacement of either wave acting alone. If they are exactly out of phase (the peaks of one are exactly aligned with the valleys of the other), they combine to cancel everywhere, and the string remains straight. We call this phenomenon of combining waves **interference,** and the waves are said to **interfere.** (These terms refer only to the wave displacements; the travel of the waves is unaffected.)

Let one wave traveling along a stretched string be given by

$$y_1(x, t) = y_m \sin(kx - \omega t) \tag{16-47}$$

and another, shifted from the first, by

$$y_2(x, t) = y_m \sin(kx - \omega t + \phi). \tag{16-48}$$

Interference

From the principle of superposition (Eq. 16-46), the resultant wave is the algebraic sum of the two interfering waves and has displacement

$$y'(x,t) = y_1(x,t) + y_2(x,t)$$
$$= y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi). \qquad (16\text{-}49)$$

In Appendix E we see that we can write the sum of the sines of two angles α and β as

$$\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta). \qquad (16\text{-}50)$$

Applying this relation to Eq. 16-49 leads to

$$y'(x,t) = [2y_m \cos\tfrac{1}{2}\phi]\sin(kx - \omega t + \tfrac{1}{2}\phi). \qquad (16\text{-}51)$$

As Fig. 16-13 shows, the resultant wave is also a sinusoidal wave traveling in the direction of increasing x. It is the only wave you would actually see on the string (you would *not* see the two interfering waves of Eqs. 16-47 and 16-48).

Condition for Constructive Interference

The resultant wave differs from the interfering waves in two respects: (1) its phase constant is $\frac{1}{2}\phi$, and (2) its amplitude y'_m is the magnitude of the quantity in the brackets in Eq. 16-51:

$$y'_m = |2y_m \cos \tfrac{1}{2}\phi| \qquad \text{(amplitude).} \tag{16-52}$$

If $\phi = 0$ rad (or 0°), the two interfering waves are exactly in phase and Eq. 16-51 reduces to

$$y'(x, t) = 2y_m \sin(kx - \omega t) \qquad (\phi = 0). \tag{16-53}$$

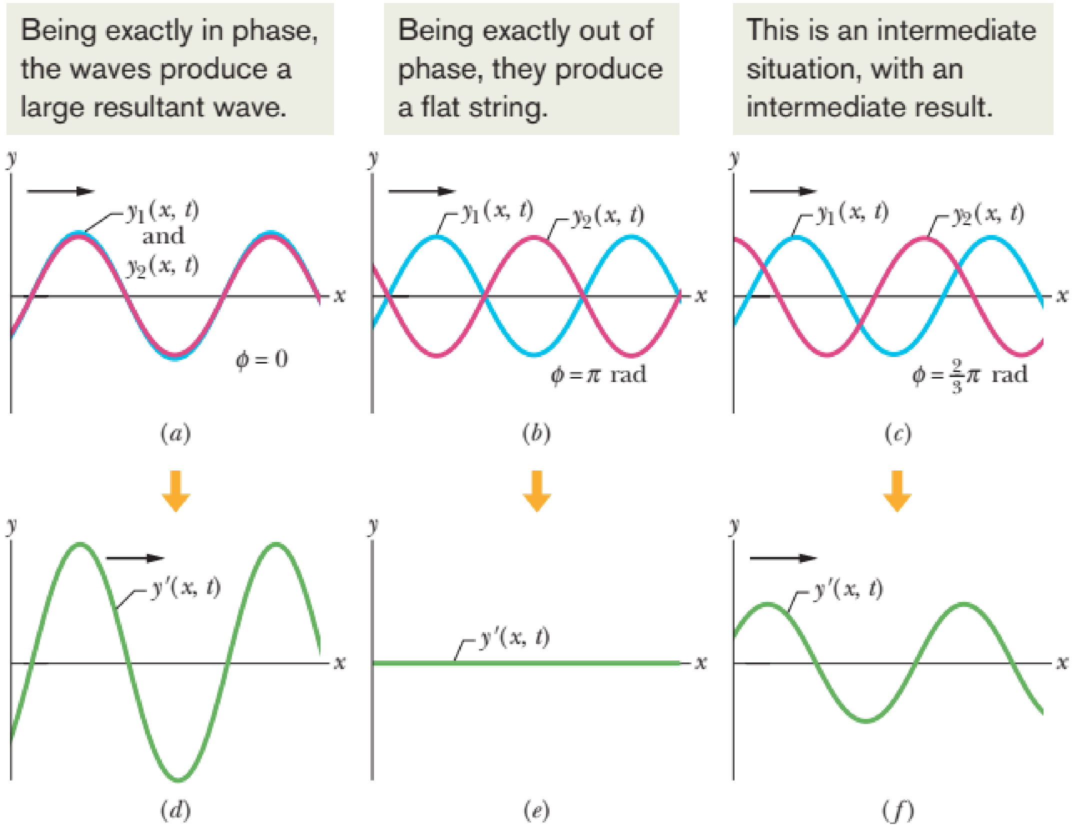
Condition for Destructive Interference

If $\phi = \pi$ rad (or 180°), the interfering waves are exactly out of phase as in Fig. 16-14*b*. Then $\cos \frac{1}{2}\phi$ becomes $\cos \pi/2 = 0$, and the amplitude of the resultant wave as given by Eq. 16-52 is zero. We then have, for all values of x and t,

$$y'(x, t) = 0 \qquad (\phi = \pi \text{ rad}). \tag{16-54}$$

The resultant wave is plotted in Fig. 16-14*e*. Although we sent two waves along the string, we see no motion of the string. This type of interference is called *fully destructive interference*.

Constructive and Destructive Interference



Ex-23:

Mathematical Problems

Two identical sinusoidal waves, moving in the same direction along a stretched string, interfere with each other. The amplitude y_m of each wave is 9.8 mm, and the phase difference ϕ between them is 100°.

(a) What is the amplitude y'_m of the resultant wave due to the interference, and what is the type of this interference?

Calculations: Because they are identical, the waves have the *same amplitude*. Thus, the amplitude y'_m of the resultant wave is given by Eq. 16-52:

$$\begin{aligned} y'_m &= |2y_m \cos \tfrac{1}{2}\phi| = |(2)(9.8\text{ mm}) \cos(100°/2)| \\ &= 13\text{ mm}. \qquad \text{(Answer)} \end{aligned}$$

We can tell that the interference is *intermediate* in two ways. The phase difference is between 0 and 180°, and, correspondingly, the amplitude y'_m is between 0 and $2y_m$ (= 19.6 mm).

(b) What phase difference, in radians and wavelengths, will give the resultant wave an amplitude of 4.9 mm?

Calculations: Now we are given y'_m and seek ϕ. From Eq. 16-52,

$$y'_m = |2y_m \cos \tfrac{1}{2}\phi|,$$

we now have

$$4.9 \text{ mm} = (2)(9.8 \text{ mm}) \cos \tfrac{1}{2}\phi,$$

which gives us (with a calculator in the radian mode)

$$\phi = 2 \cos^{-1} \frac{4.9 \text{ mm}}{(2)(9.8 \text{ mm})}$$

$$= \pm 2.636 \text{ rad} \approx \pm 2.6 \text{ rad}. \qquad \text{(Answer)}$$

There are two solutions because we can obtain the same resultant wave by letting the first wave *lead* (travel ahead of) or *lag* (travel behind) the second wave by 2.6 rad. In wavelengths, the phase difference is

$$\frac{\phi}{2\pi \text{ rad/wavelength}} = \frac{\pm 2.636 \text{ rad}}{2\pi \text{ rad/wavelength}}$$

$$= \pm 0.42 \text{ wavelength}. \qquad \text{(Answer)}$$

SUPERPOSED WAVE OF DIFFERENT FREQUENCIES (VERY SMALL DIFFERENCE) AND BEATS

We will now discuss the superposition of two waves that have same vibration direction, same amplitude **A**, but different frequency and wave number(**ω_1, k_1)** and (**ω_2, k_2**). However, the frequency difference is very small. This will generate the very interesting "beat" phenomenon.

Since the phase difference between the vibrations is continually changing, the specification of some initial nonzero phase difference is in general not of major significance in this case.

$$y_1 = A\cos(k_1 x - \omega_1 t)$$

$$y_2 = A\cos(k_2 x - \omega_2 t)$$

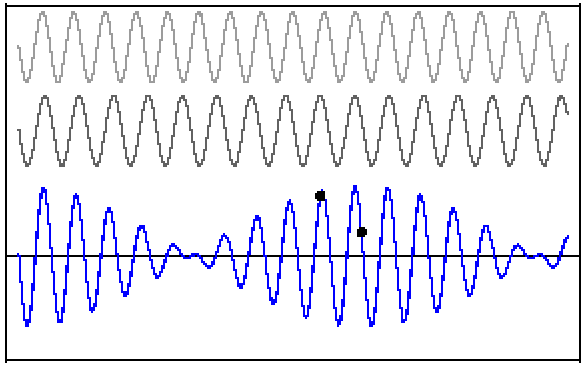
Interefering waves, generally…

Phase Velocity and Group Velocity

$$y = y_1 + y_2 = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t)$$

$$\Downarrow$$

$$y = 2A\cos\frac{1}{2}\{(k_2 - k_1)x - (\omega_2 - \omega_1)t\} \bullet \cos\frac{1}{2}\{(k_1 + k_2)x - (\omega_1 + \omega_2)t\}$$



"Beats" occur when you add two waves of slightly different frequency. They will interfere constructively in some areas and destructively in others.

Can be interpreted as a sinusoidal envelope: $$2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

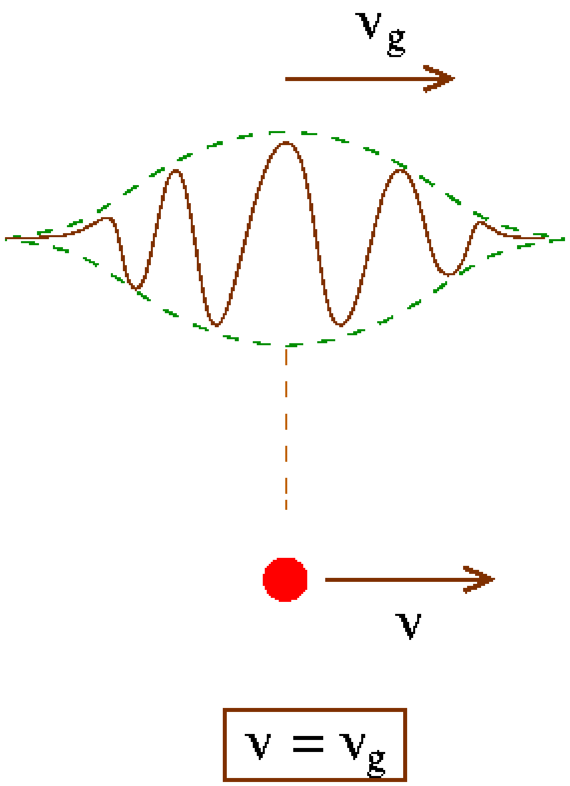
Modulating a high frequency wave within the envelope: $$\cos\left[\frac{1}{2}(k_1 + k_2)x - \frac{1}{2}(\omega_1 + \omega_2)t\right]$$

the group velocity $$\upsilon_g = \frac{(\omega_2 - \omega_1)/2}{(k_2 - k_1)/2} = \frac{\Delta\omega}{\Delta k}$$

the phase velocity $$\upsilon_p = \frac{(\omega_2 + \omega_1)/2}{(k_2 + k_1)/2} \approx \frac{\omega_1}{k_1} = \upsilon_1$$

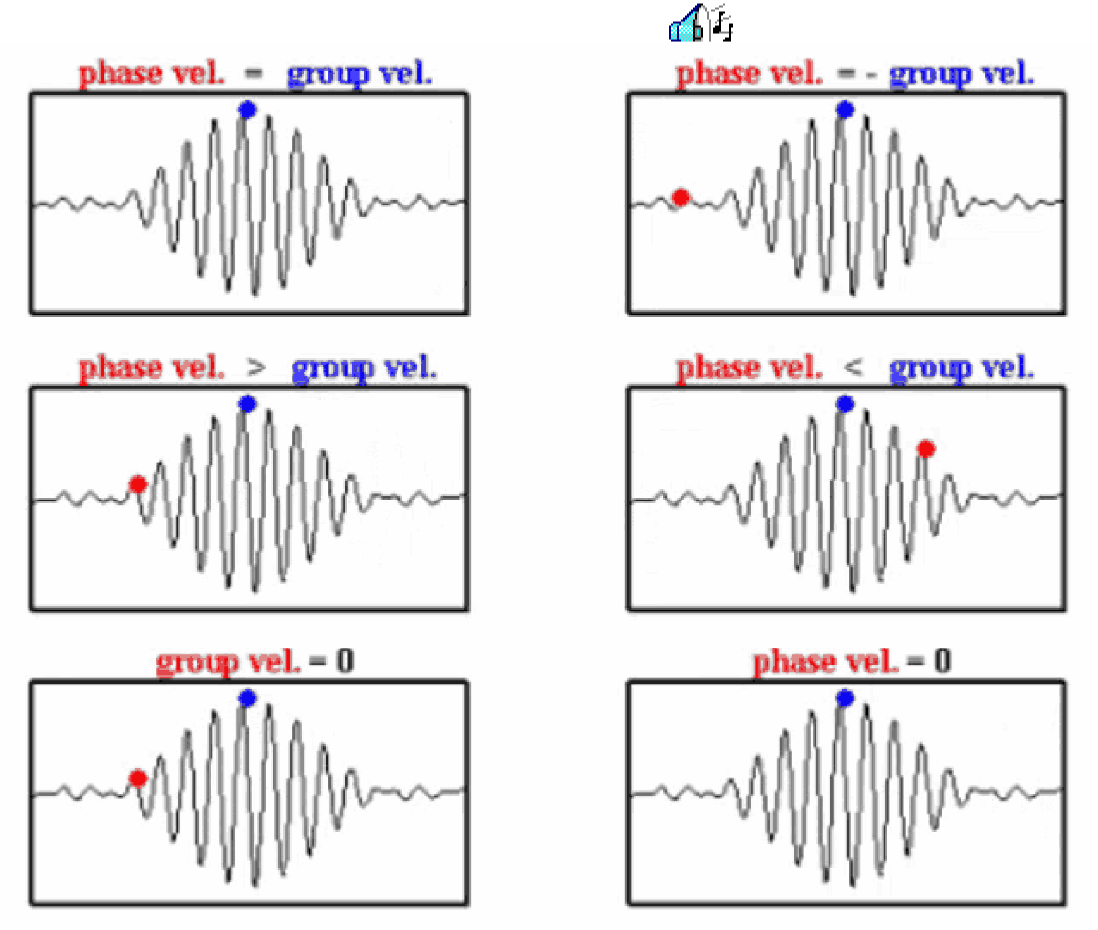
if $\upsilon_1 \approx \upsilon_2$

Phase Velocity and Group Velocity



the group velocity $\qquad \upsilon_g = \dfrac{(\omega_2 - \omega_1)/2}{(k_2 - k_1)/2} = \dfrac{\Delta\omega}{\Delta k}$

Listen to the beats!



Phase Velocity and Group Velocity

The first term is the wave and the second term is the envelope

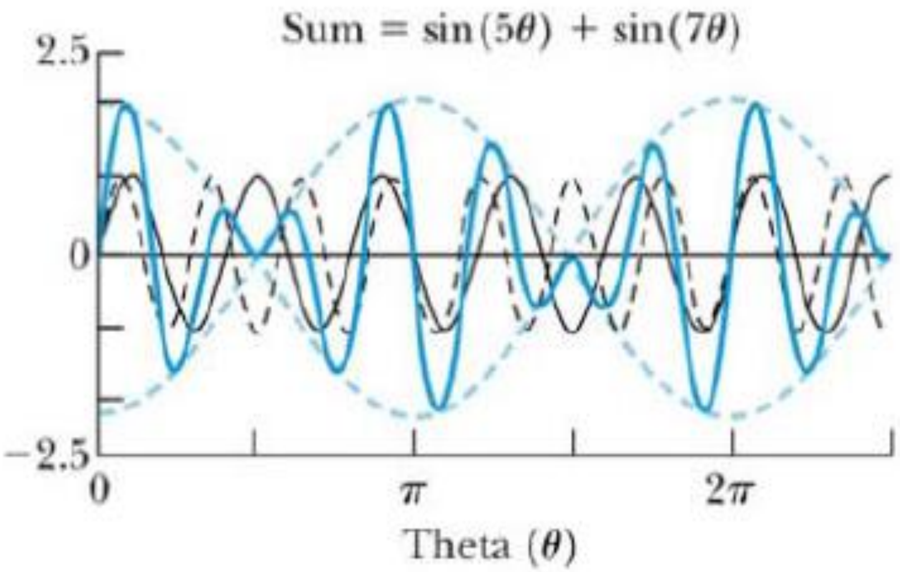
$$y = 2A\cos(k_{av}x - \omega_{av}t)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

In the previous equation,

$$y = 2A\cos(k_{av}x - \omega_{av}t)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

where $k_{av} = \dfrac{k_1 + k_2}{2}$ and $\Delta k = k_1 - k_2$

and similarly for ω



Phase Velocity and Group Velocity

- The combined wave has phase velocity

$$v_{phase} = \frac{\omega_{av}}{k_{av}}$$

- The group (envelope) has group velocity

$$v_{group} = \frac{\Delta\omega}{\Delta k}$$

- More generally the group velocity is

$$v_{group} = \frac{d\omega(k)}{dk}$$

Phase Velocity and Group Velocity

- Recall our problem with the wave velocity being slower than the particle velocity
- Using the group velocity we find

$$v_{group} = \frac{d\omega}{dk} = \frac{d\hbar\omega}{d\hbar k} = \frac{dE}{dp}$$

$$E = \frac{p^2}{2m}$$

$$v_{group} = \frac{p}{m} = V$$

- Problem solved

Phase Velocity and Group Velocity: Application

- The relation between ω and k is called the dispersion relation
- Examples
  - Photon in vacuum

$$f = \frac{c}{\lambda}$$

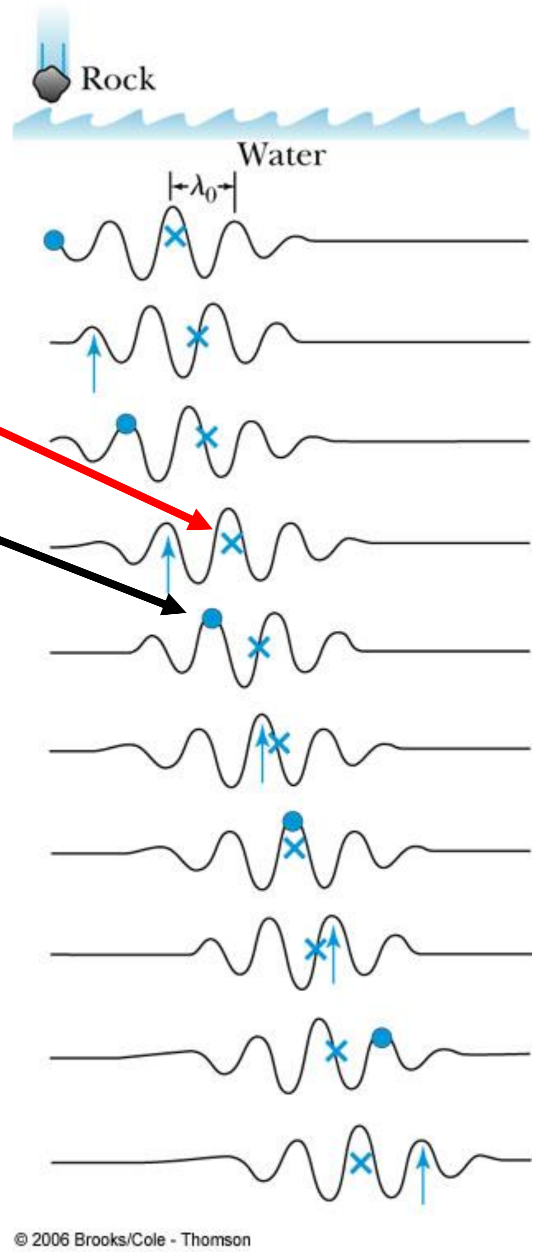
$$2\pi f = \frac{2\pi c}{\lambda}$$

$$\omega = kc$$

$$\frac{d\omega}{dk} = c$$

Phase Velocity and Group Velocity: Application

- Phase and group velocity



Phase Velocity and Group Velocity: Application

- Examples
  - Photon in a medium

$$f = \frac{c}{n(\lambda)\lambda}$$

$$2\pi f = \frac{2\pi c}{n(\lambda)\lambda}$$

$$\omega = \frac{kc}{n(k)}$$

$$\frac{d\omega}{dk} = \frac{c}{n(k)} + kc\frac{d}{dk}\frac{1}{n(k)}$$

Phase Velocity and Group Velocity: Application

- Examples
  - de Broglie waves

$$E = hf = \hbar\omega$$

$$E = \frac{p^2}{2m} = \frac{(\hbar k)^2}{2m}$$

$$\omega = \frac{\hbar k^2}{2m}$$

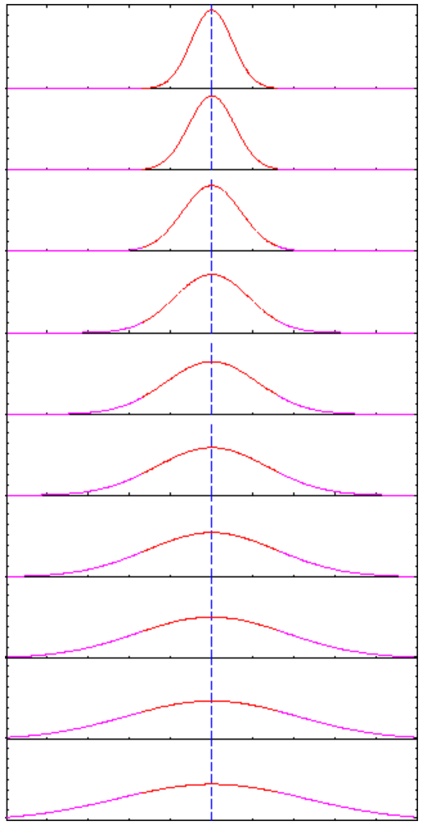
$$\frac{d\omega}{dk} = \frac{\hbar k}{m}$$

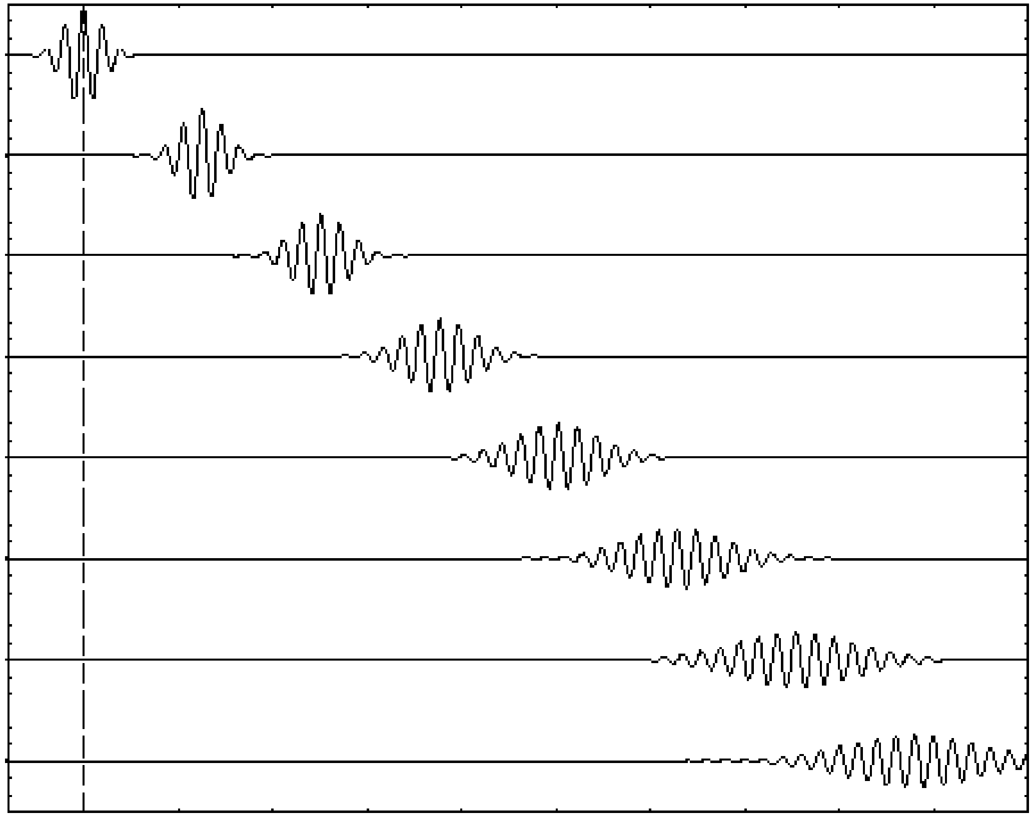
Phase Velocity and Group Velocity: Application

- Because the group velocity = $d\omega/dk$ depends on k, the wave packet will disperse with time
  - This is because each of the waves is moving at a slightly different velocity
  - Just as white light will be dispersed as it travels through a prism
- That’s why $\omega(k)$ is called a dispersion relation

Phase Velocity and Group Velocity: Application

- Spreading of a wave packet with time





- Does this mean matter waves disperse with time?

Mathematical Example

Ex-24:

Two cosine waves have phase velocity V_1= 2 cm/s, V_2= 3 cm/s, and corresponding wavelength λ_1= 4 cm, λ_2= 3 cm. Find out the angular frequency and group velocity.

Hint:

$V_p = \frac{\omega}{k} = C_p$

$\omega = V_p\, k = C_p k$

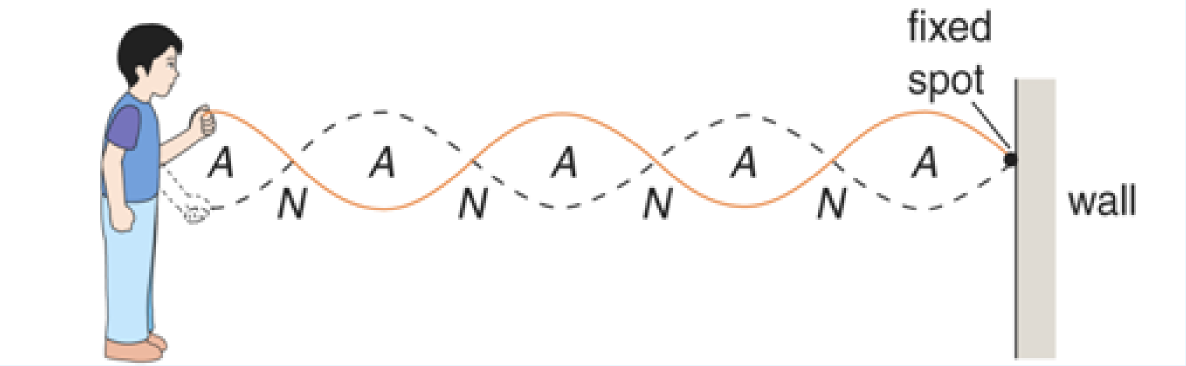
$\omega_1 = \pi$ and $\omega_2 = 2\pi$

$V_g = \frac{\Delta\omega}{\Delta k}$

Introduction to stationary waves

Stationary waves (standing waves) produced in
- string of guitar or piano
- rope (one end tie to wall)

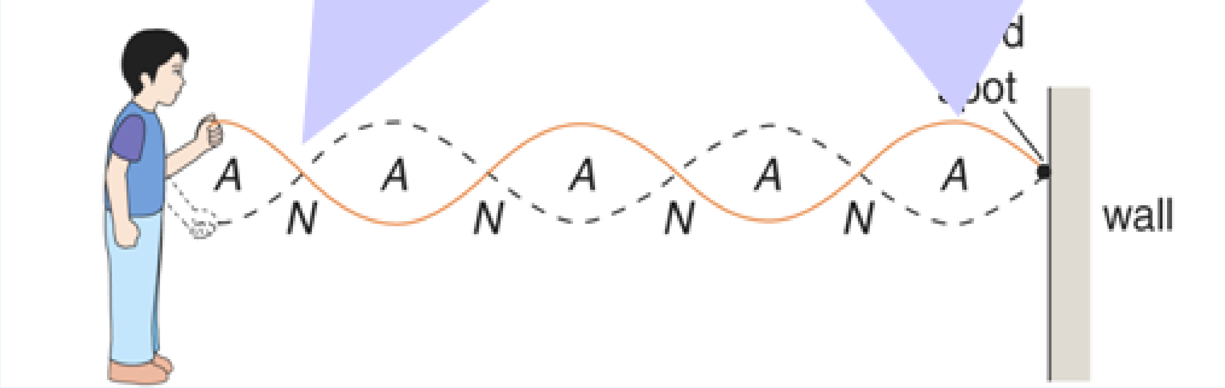
- **Incident wave** is reflected by the wall to from **reflected wave**
- **superposition** of the waves produces stationary wave



Introduction to stationary waves

N – **nodes** (remain stationary)

A – **antinodes** (largest amplitude of oscillation)



A stationary wave is produced by the superposition of two **progressive waves** of the same amplitude and frequency but travelling in opposite directions.

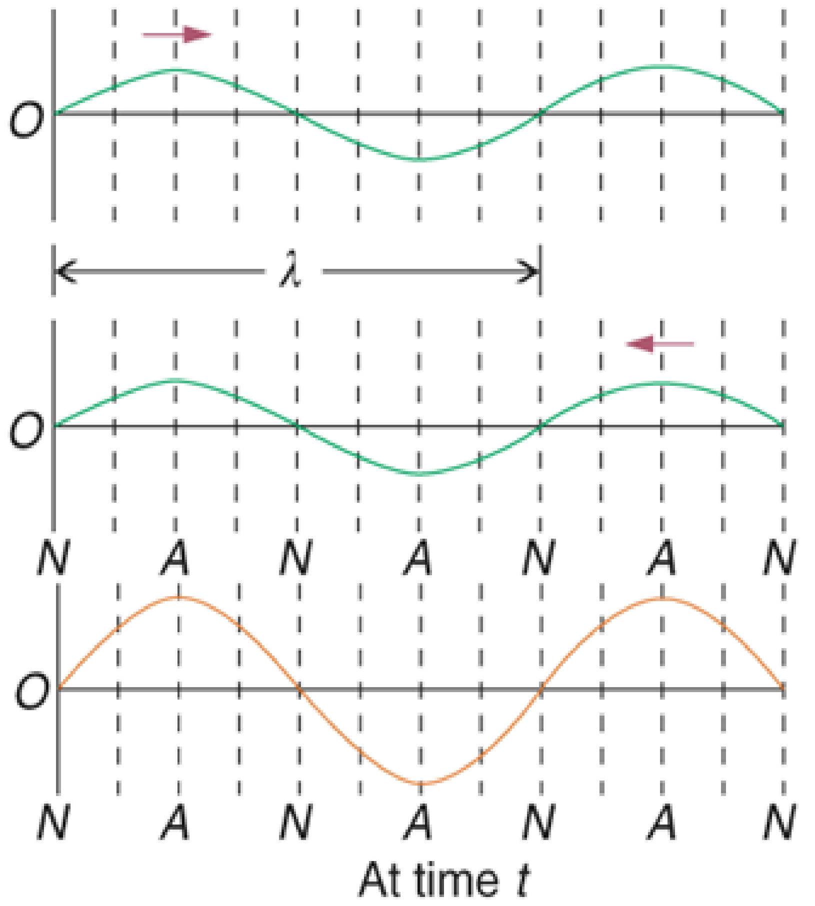
Introduction to stationary waves

(a) at time *t* (constructive interference)

(1) To right

(2) To left

(1) + (2) = (3)



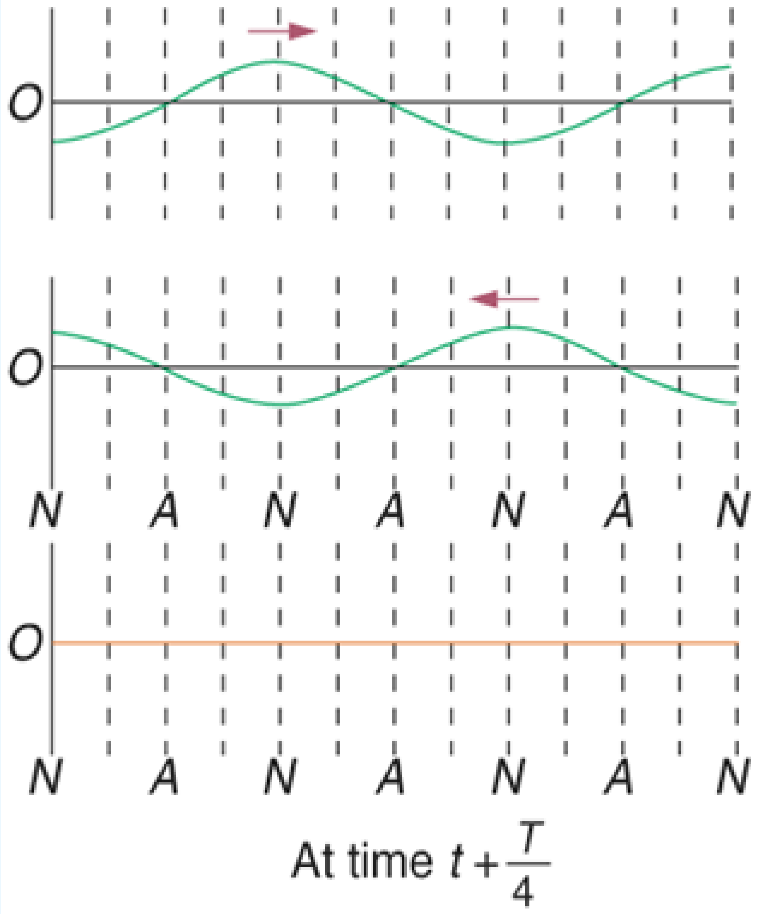
Introduction to stationary waves

(b) at time $t + T/4$ (destructive interference)

(1)

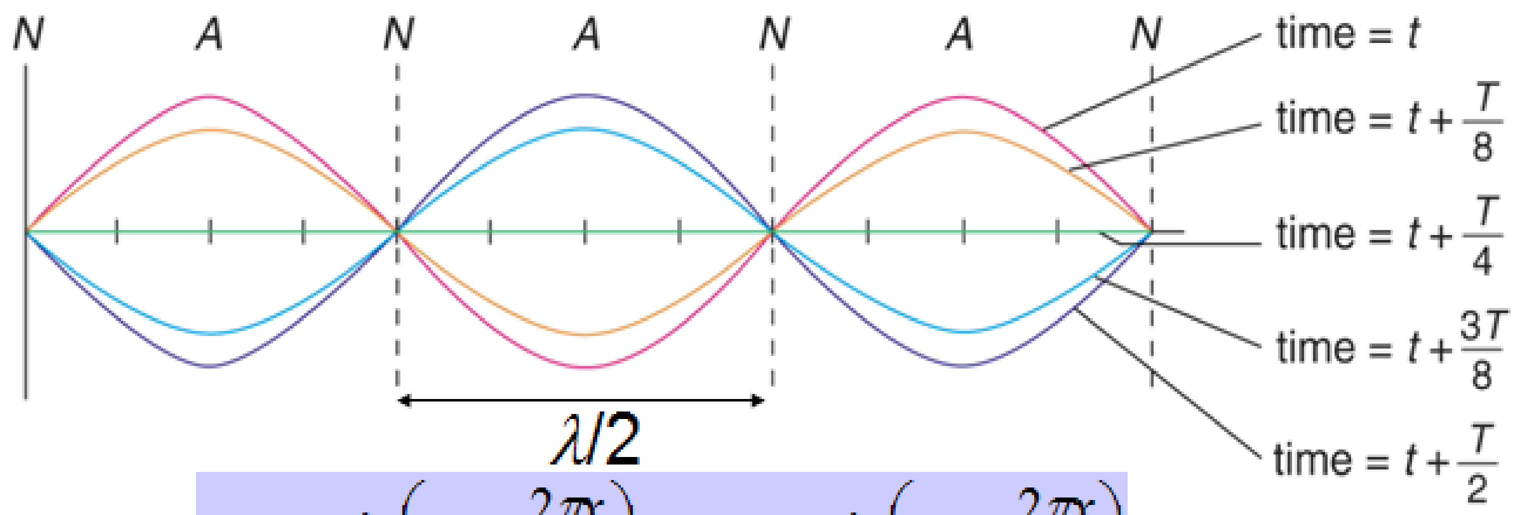
(2)

(1) + (2) = (3)



Introduction to stationary waves

Vibration of stationary wave at various instants



$$y_1 = a\sin\left(\omega t - \frac{2\pi x}{\lambda}\right),\quad y_2 = a\sin\left(\omega t + \frac{2\pi x}{\lambda}\right)$$

$$y = y_1 + y_2 = \left(2a\cos\frac{2\pi x}{\lambda}\right)\sin\omega t$$

Amplitude (A) of resultant stationary wave

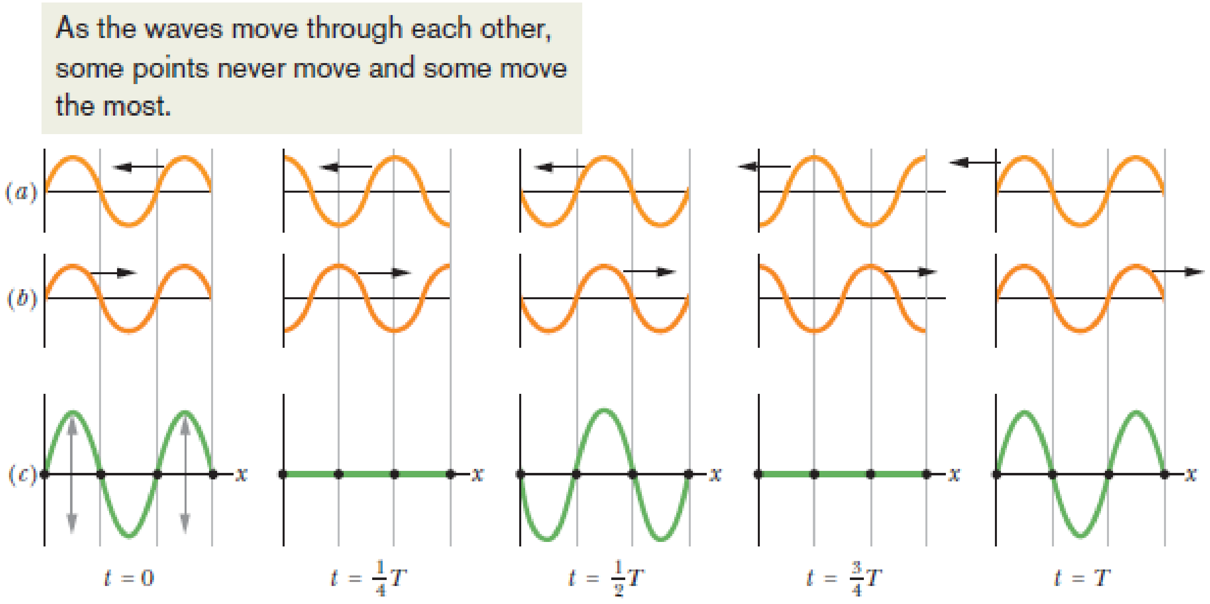
Standing Wave: Node and Antinode Points

Standing Waves

In Module 16-5, we discussed two sinusoidal waves of the same wavelength and amplitude traveling *in the same direction* along a stretched string. What if they travel in opposite directions? We can again find the resultant wave by applying the superposition principle.

Figure 16-17 suggests the situation graphically. It shows the two combining waves, one traveling to the left in Fig. 16-17*a*, the other to the right in Fig. 16-17*b*. Figure 16-17*c* shows their sum, obtained by applying the superposition

Figure 16-17 (*a*) Five snapshots of a wave traveling to the left, at the times t indicated below part (*c*) (T is the period of oscillation). (*b*) Five snapshots of a wave identical to that in (*a*) but traveling to the right, at the same times t. (*c*) Corresponding snapshots for the superposition of the two waves on the same string. At $t = 0, \frac{1}{2}T$, and T, fully constructive interference occurs because of the alignment of peaks with peaks and valleys with valleys. At $t = \frac{1}{4}T$ and $\frac{3}{4}T$, fully destructive interference occurs because of the alignment of peaks with valleys. Some points (the nodes, marked with dots) never oscillate; some points (the antinodes) oscillate the most.



Standing Wave: Node and Antinode Points

Finding the node and antinode points of different signals

If two sinusoidal waves of the same amplitude and wavelength travel in *opposite* directions along a stretched string, their interference with each other produces a standing wave.

Node and antinode from the following signals:

$$y_1(x, t) = y_m \sin(kx - \omega t)$$

$$y_2(x, t) = y_m \sin(kx + \omega t).$$

To analyze a standing wave, we represent the two waves with the equations

$$y_1(x, t) = y_m \sin(kx - \omega t) \qquad (16\text{-}58)$$

and

$$y_2(x, t) = y_m \sin(kx + \omega t). \qquad (16\text{-}59)$$

The principle of superposition gives, for the combined wave,

$$y'(x, t) = y_1(x, t) + y_2(x, t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t).$$

Applying the trigonometric relation of Eq. 16-50 leads to Fig. 16-18 and

$$y'(x, t) = [2y_m \sin kx] \cos \omega t. \qquad (16\text{-}60)$$

Standing Wave: Node and Antinode Points

This equation does not describe a traveling wave because it is not of the form of Eq. 16-17. Instead, it describes a standing wave.

The quantity $2y_m \sin kx$ in the brackets of Eq. 16-60 can be viewed as the amplitude of oscillation of the string element that is located at position x. However, since an amplitude is always positive and $\sin kx$ can be negative, we take the absolute value of the quantity $2y_m \sin kx$ to be the amplitude at x.

In a traveling sinusoidal wave, the amplitude of the wave is the same for all string elements. That is not true for a standing wave, in which the amplitude *varies with position*. In the standing wave of Eq. 16-60, for example, the amplitude is zero for values of kx that give $\sin kx = 0$. Those values are

$$kx = n\pi, \quad \text{for } n = 0, 1, 2, \ldots. \tag{16-61}$$

Substituting $k = 2\pi/\lambda$ in this equation and rearranging, we get

$$x = n\frac{\lambda}{2}, \quad \text{for } n = 0, 1, 2, \ldots \quad \text{(nodes)}, \tag{16-62}$$

as the positions of zero amplitude—the nodes—for the standing wave of Eq. 16-60. Note that adjacent nodes are separated by $\lambda/2$, half a wavelength.

The amplitude of the standing wave of Eq. 16-60 has a maximum value of $2y_m$, which occurs for values of kx that give $|\sin kx| = 1$. Those values are

$$\begin{aligned} kx &= \tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \tfrac{5}{2}\pi, \ldots \\ &= (n + \tfrac{1}{2})\pi, \quad \text{for } n = 0, 1, 2, \ldots. \end{aligned} \tag{16-63}$$

Standing Wave: Node and Antinode Points

Substituting $k = 2\pi/\lambda$ in Eq. 16-63 and rearranging, we get

$$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \quad \text{for } n = 0, 1, 2, \ldots \quad \text{(antinodes)}, \tag{16-64}$$

as the positions of maximum amplitude—the antinodes—of the standing wave of Eq. 16-60. Antinodes are separated by $\lambda/2$ and are halfway between nodes.