CHAPTER V

INTERFERENCE OF SOUND WAVES

Interference of sound waves-Energy distribution due to interference of sound waves- Interference of two waves of nearly same frequency and amplitude : Beats-analytical treatment of formation of beats, Superposition of two waves of nearly equal frequency and wavelength : group velocity-Interference by reflection : stationary or standing waves-formation of stationary waves : analytical treatment-Energy of a stationary wave-Solved problems-Exercises.

5.1 Interference of sound waves

When two or more waves combine at a particular point, they are said to interfere. According to principle of superposition, the displacement of the particles at the points of the medium where the waves meet is given by the algebraic (vector) sum of the individual displacements of the interfering waves at those points. If the two interfering waves meet at the same phase they combine to give a large displacement which is equal to the sum of their amplitudes. But when they meet in opposite phase, the resultant displacement is given by the difference of their amplitudes. In addition, if the amplitudes of the interfering waves are equal, they cancel each other producing no disturbance at all. This phenomenon of reinforcement and cancellation is called *interference*; it applies to waves of all kinds.

Let two simple harmonic waves of the same frequency be travelling along the same, or very nearly the same path and in the same direction. If a and b be the amplitudes of the two waves and if there is a phase difference ϕ between the waves, then the displacement of a particle at any instant due to each wave is given by

$$y_1 = a \sin \frac{2\pi}{\lambda} (vt - x) \qquad \text{(i)}$$

$$y_2 = b \sin \frac{2\pi}{\lambda} (vt - x + \phi) \qquad \text{(ii)}$$

where v is the velocity of propagation of each wave and λ is the wavelength.

The resultant displacement of the particle due to superposition of the two waves is

$$y = y_1 + y_2$$

$$= a \sin \frac{2\pi}{\lambda} (vt - x) + b [\sin \frac{2\pi}{\lambda} (vt - x) + \phi]$$

$$= a \sin \frac{2\pi}{\lambda} (vt - x) + b [\sin \frac{2\pi}{\lambda} (vt - x) + \cos\phi +$$

$$+ \cos \frac{2\pi}{\lambda} (vt - x) \sin \phi$$

$$= [\sin \frac{2\pi}{\lambda} (vt - x)] (a + b \cos \phi)$$

$$+ [\cos \frac{2\pi}{\lambda} (vt - x)] (b \sin \phi).$$

Let $a + b \cos \phi = A \cos \theta$

$$b \sin \phi = A \sin \theta$$

$$\therefore \quad A = \sqrt{a^2 + b^2 + 2ab \cos\phi} \qquad \text{(iii)}$$

$$\text{and } \tan \theta = \frac{b \sin \phi}{a + b \cos\phi}$$

$$\text{or, } \theta = \tan^{-1} \frac{b \sin \phi}{a + b \cos\phi} \qquad \text{(iv)}$$

$$\therefore \quad y = [\sin \frac{2\pi}{\lambda} (vt - x)] [A \cos \theta]$$

$$+ [\cos \frac{2\pi}{\lambda} (vt - x) [A \sin \theta]$$

$$= A \sin [\frac{2\pi}{\lambda} (vt - x) + \theta] \qquad (5.1)$$

Thus, the resultant vibration is also simple harmonic and has the same time period (*i.e.*, frequency and wavelength). But the amplitude and phase difference are different. What is more, the amplitude is no longer constant and has different values for different values of ϕ, the phase difference between the interfering waves.

(i) when $\phi = 0, 2\pi, 4\pi, \ldots = 2n\pi$

where n = 0, 1, 2, . . .

$$A = \sqrt{a^2 + b^2 + 2ab} = (a + b)$$

The amplitude becomes maximum when $\phi = n\pi$. Moreover, if $a = b$, $A = 2a$.

(ii) when $\phi = \pi, 3\pi, 5\pi, \ldots = (2n + 1)\pi$

where n = 0, 1, 2, . . .

$$A = \sqrt{a^2 + b^2 - 2ab} = (a - b)$$

The amplitude becomes minimum when $\phi = (2n + 1)\pi$ and is zero ($A = 0$) when $a = b$.

Thus when the phase difference is zero or a multiple of 2π (*i.e.*, the path difference is zero or a multiple of λ), the two waves reinforce each other and an intense sound is heard. The interference is known as *constructive*. When the phase difference is odd multiple of π (*i.e.*, the path difference is $\frac{\lambda}{2}$ or its odd multiple), the two waves destroy each other. The interference is known as destructive. When the amplitudes are different, a very feeble sound will be heard. If the amplitudes are equal, no sound is heard. They are graphically represented in (Fig. 5.1).

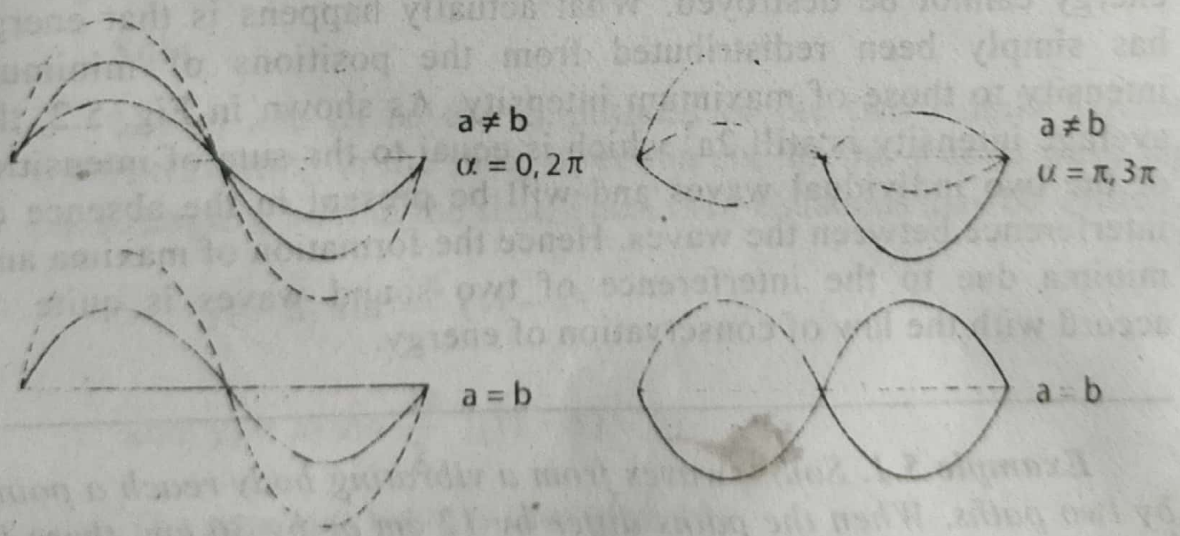

Fig. 5.1

5.2 Energy distribution due to interference of sound waves

The intensity of a wave is proportional to square of its amplitude. Hence the intensity of the resultant vibration due to superposition of two waves is given by [eqn. (iii), Art. 5.1]

$$I = A^2 = \left(\sqrt{a^2 + b^2 + 2ab\cos\phi}\right)^2$$

If a = b *i.e.*, the two waves are of the same amplitude, then

$$I = A^2 = \left(\sqrt{a^2 + a^2 + 2a^2\cos\phi}\right)^2$$

$$= \left[\sqrt{2a^2(1+\cos\phi)}\right]^2$$

$$= 2a^2(1+\cos\phi)$$

$$= 2a^2 . 2\cos^2\frac{\phi}{2}$$

$$= 4a^2\cos^2\frac{\phi}{2}. \qquad (5.2)$$

From eqn. (5.2), the intensity is maximum and is equal to $4a^2$ (four times the intensity of a single wave) at the points of maximum vibration *i.e.*, constructive interference. The intensity at the points of minima *i.e.*, the points of destructive interference according to eqn. (5.2) is zero. According to law of conservation of energy, the energy cannot be destroyed. What actually happens is that energy has simply been redistributed from the positions of minimum intensity to those of maximum intensity. As shown in Fig. 5.2, the average intensity is still $2a^2$ which is equal to the sum of intensities of the two individual waves and will be present in the absence of interference between the waves. Hence the formation of maxima and minima due to the interference of two sound waves is quite in accord with the law of conservation of energy.

Example 5.1. *Sound waves from a vibrating body reach a point by two paths. When the paths differ by 12 cm or by 36 cm, there is silence at the point. Calculate the frequency of the vibrating body if the velocity of sound in air is 330 m/sec.*

Soln.

When silence is produced, it is clearly a case of destructive interference. For, when the path difference is 12 cm, the waves arrive at the point in opposite phase and annul each other producing silence. This path difference must be equal to odd number of half wavelengths *i.e.*, equal to $(2k + 1)\frac{\lambda}{2}$ where $k = 0, 1, 2, \ldots$ or equal to $\lambda/2$, $3\lambda/2$, $5\lambda/2$, . . . so that the path difference between two successive points of silence must be equal to λ. Thus, when silence is again produced at a point when the path difference is 36 cm, this means that the difference between these two successive values of path difference must be λ.

Or, $\lambda = (36 - 12) = 24$ cm

If n be the frequency of the wave, we have from $v = n\lambda$

$$n = \frac{v}{\lambda} = \frac{33000}{24} = \textbf{1375 cycles per second.}$$

Example 5.2. *Find the resultant of two plane simple harmonic waves of the same period travelling in the same direction but differing in phase and amplitude. What is the amplitude of the resultant wave if those of the component waves be 3.0 and 4.0 respectively and their phase difference $\pi/2$ radian?*

Soln.

Let a_1 and a_2 be the amplitudes of the two simple harmonic waves travelling in the same direction and let the second wave lag a phase angle ϕ behind the first. Then, their equations may be written as

$$y_1 = a_1 \sin \frac{2\pi}{\lambda}(vt - x)$$

$$\text{and } y_2 = a_2 \sin \frac{2\pi}{\lambda}[(vt - x) - \phi]$$

The resultant wave is then given by

$$y = y_1 + y_2 = a_1 \sin \frac{2\pi}{\lambda}(vt - x) + a_2 \sin \frac{2\pi}{\lambda}[(vt - x) - \phi]$$

$$= a_1 \sin \frac{2\pi}{\lambda} (vt - x) + a_2 \sin \frac{2\pi}{\lambda} (vt - x) \cos \phi$$

$$- a_2 \cos \frac{2\pi}{\lambda} (vt - x) \sin \phi$$

$$= \sin \frac{2\pi}{\lambda} (vt - x) (a_1 + a_2 \cos \phi)$$

$$- \cos \frac{2\pi}{\lambda} (vt - x) (a_2 \sin \phi)$$

Let $a_1 + a_2 \cos \phi = a \cos \theta$

and $a_2 \sin \phi = a \sin \theta$.

Then we have,

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \cos\theta - a \cos \frac{2\pi}{\lambda} \sin \theta$$

$$= a \sin [\frac{2\pi}{\lambda} (vt - x) - \theta]$$

Thus the resultant wave is also simple harmonic having the same frequency and wavelength of the component waves. But its amplitude a is different than those of the component waves and lags in phase angle θ behind the first wave.

$$a = \sqrt{(a \cos \theta)^2 + (a \sin \theta)^2}$$

$$= \sqrt{(a_1 + a_2 \cos \phi)^2 + (a_2 \sin \phi)^2}$$

$$= \sqrt{a_1^2 + a_2^2 \cos \phi + 2a_1a_2 \cos \phi + a_2^2 \sin^2 \phi}$$

$$= \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos \phi}$$

$$\text{And } \tan \theta = \frac{a \sin \theta}{a \cos \theta} = \frac{a_2 \sin \phi}{a_1 + a_2 \cos \phi}$$

$$\text{or, } \theta = \tan^{-1} \frac{a_2 \sin \phi}{a_1 + a_2 \cos \phi}$$

Obviously, when $\phi = 0$, *i.e.*, the two component waves arrive in phase at a point, $\cos \phi = 1$ and $a = a_1 + a_2$; and when $\phi = \pi$, *i.e.*, the two component waves arrive at a point out of phase, $\cos\phi = -1$ and $a = a_1 - a_2$. At all other points the amplitude lies between these two extremes.

Therefore, when $a_1 = 3.0$ cm, $a_2 = 4.0$ cm, and $\phi = \frac{\pi}{2}$ rad;

we have

$$a = \sqrt{3^2 + 4^2 + 2.3.4 \cos\frac{\pi}{2}}$$

$$= \sqrt{9 + 16 + 0}$$

$$= \sqrt{25} = \mathbf{5.0\ cm.}$$

5.3 Interference of two waves of nearly same frequency and amplitude : formation of beats

If two waves, travelling along the same path and in the same direction, have slightly different frequencies or wavelengths and also different amplitudes, the resulting wave motion possesses certain peculiarities. The two waves arrive in phase at some points and out of phase at others, thus producing maximum displacement (and hence maximum intensity) at the former, and minimum (but not zero) displacements (and hence minimum intensity) at the later points as they proceed along. The sound heard alternates between loud and soft. This *throbbing effect* (periodic rise and fall in intensity) in the intensity of sound is called *beats*. One maximum intensity and one succeeding minimum intensity or *vice versa* constitutes one beat. The frequency of the beats is the number of intense sound heard per second.

Let us consider a layer of air some distance away from two pure notes of nearly equal frequency, say two tuning forks of frequencies 48 and 56 Hz respectively which are being sounded together. Fig. 5.3 (i) shows the variation of the displacement y_1 of the layer due to one of the notes while Fig. 5.3 (ii) shows the variation of the displacement, y_2 due to the second note. Fig. 5.3 (iii) shows the result displacement, y, of the layer due to the superposition of the waves when both of them are sounded together. To understand the variation of y, suppose that the displacements y_1 and y_2 are in phase at some instant T_1. Since the frequency of the curve in Fig. 5.3 (i) is 48 cycles per second, the variation y_1 undergoes 3 complete cycle in 1/16 seconds. During the same time the variation y_2 undergoes

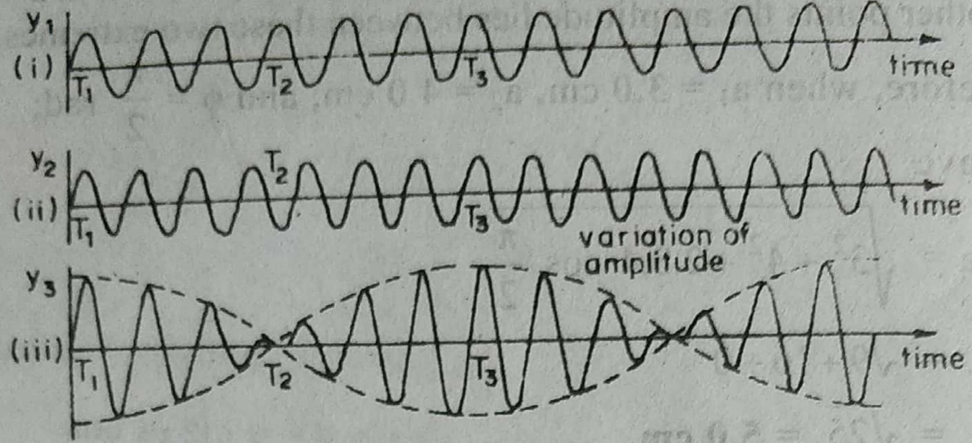

Fig. 5.3

$3\frac{1}{2}$ cycles since its frequency is 56 cycles per second. Thus y_1 and y_2 are 180° out of phase with each other at this instant and their resultant y is minimum at instant T_2. Thus T_1 T_2 represents $1/16^{th}$ of a second in Fig. 5.3 (iii). In $1/8^{th}$ of a second from T_1, y_1 has undergone 6 complete cycles while y_2 has undergone 7 complete cycles. Thus the two waves are in phase again at T_3 and their resultant is again maximum at this instant. T_1 T_3 represents $1/8^{th}$ of a second. In this way it can be seen that a loud sound is heard after every $1/8^{th}$ of a second and thus the *beat frequency* is 8 cycles per second. It will be shown below that the beat frequency is always equal to the difference of the two nearly equal frequencies.

5.4 Analytical treatment of formation of beats

Let two wavetrains of frequencies n_1 and n_2 and of amplitudes a and b respectively start with the same phase. Then the individual displacements at a point at some instant t due to the two wavetrains are

$$y_1 = a \sin w_1 t \qquad \text{where } w_1 = 2\pi n_1$$

and $$y_2 = b \sin w_2 t \qquad \text{where } w_2 = 2\pi n_2$$

Then, according to the principle of superposition, the resultant displacement is given by

$$y = y_1 + y_2$$

$$= a \sin w_1 t + b \sin w_2 t$$

$$= a \sin w_1 t + b \sin [w_1 - (w_1 - w_2)]t$$

$$= a \sin w_1 t + b[\sin w_1 t \cos (w_1 - w_2) t$$

$$- \cos w_1 t \sin (w_1 - w_2) t]$$

$$= a \sin w_1 t + b \sin w_1 t \cos (w_1 - w_2) t$$

$$- b \cos w_1 t \sin (w_1 - w_2) t$$

$$= \sin w_1 t [a + b \cos (w_1 - w_2) t]$$

$$- \cos w_1 t [b \sin (w_1 - w_2) t]$$

Let $a + b \cos (w_1 - w_2) t = A \cos \theta$ (i)

and $b \sin (w_1 - w_2) t = A \sin \theta$ (ii)

$\therefore \quad y = \sin w_1 t \, A \cos \theta - \cos w_1 t \, A \sin \theta$
$= A \sin (w_1 t - \theta)$ (iii)

Eqn. (iii) gives the resultant displacement due to the superposition of the two waves. The values of the resultant amplitude A and the phase angle θ can be obtained as follows :

From (i) and (ii)

$$A^2\cos^2\theta = a^2 + b^2 \cos^2 (w_1 - w_2) t + 2ab \cos (w_1 - w_2) t$$

$$A^2\sin^2\theta = b^2\sin^2 (w_1 - w_2) t$$

Adding we get

$$A^2 = a^2 + b^2 + 2ab \cos (w_1 - w_2) t$$

$$\text{or, } A = \sqrt{a^2 + b^2 + 2ab \cos (w_1 - w_2) t} \qquad (5.3)$$

$$\text{and } \tan \theta = \frac{A \sin \theta}{A \cos \theta} = \frac{b \sin (w_1 - w_2) t}{a + b \cos (w_1 - w_2) t}$$

$$\text{or, } \theta = \tan^{-1} \frac{b \sin (w_1 - w_2) t}{a + b \cos (w_1 - w_2) t} \qquad (5.4)$$

From eqn. (5.4) it is evident that phase angle θ changes with respect to time. Similarly eqn. (5.3) shows that the amplitude of the resultant vibration varies with time.

(a) *the amplitude is maximum* when

$$(w_1 - w_2)\,t = 2\pi\,(n_1 - n_2)\,t = 2k\pi$$

where k = 0, 1, 2, ...

The resultant amplitude is then

$$A = \sqrt{a^2 + b^2 + 2ab} = \sqrt{(a+b)^2} = (a+b)$$

Thus the resultant amplitude is maximum (= a + b)

$$\text{when } t = \frac{2k\pi}{2\pi\,(n_1 - n_2)} = \frac{k}{n_1 - n_2}$$

i.e., at time instants 0, $\frac{1}{n_1 - n_2}$, $\frac{2}{n_1 - n_2}$, ...

As the intensity of sound is directly proportional to the square of the amplitude, the maximum intensity of sound will be heard at these instants.

(b) *the amplitude is minimum* when

$$(w_1 - w_2)\,t = 2\pi\,(n_1 - n_2)\,t = (2k+1)\,\pi$$

where k = 0, 1, 2, 3, ...

The resultant amplitude is then

$$A = \sqrt{a^2 + b^2 - 2ab} = \sqrt{(a-b)^2} = (a-b)$$

Thus the resultant amplitude is minimum (= a − b)

$$\text{when } t = \frac{(2k+1)\,\pi}{2\pi\,(n_1 - n_2)} = \frac{2k+1}{2\,(n_1 - n_2)}$$

i.e., at time instants

$$\frac{1}{2(n_1 - n_2)},\ \frac{3}{2(n_1 - n_2)},\ \ldots\ldots$$

Hence the minimum intensity of sound will be heard at these instants.

Thus the time interval between successive maxima or minima is $\frac{1}{n_1 - n_2}$ sec. One minimum amplitude is present between two successive maxima and *vice versa*.

Hence, the number of beats produced per second $= \frac{1}{1/(n_1 - n_2)}$

$$= n_1 - n_2$$

Thus *the number of beats produced per second is equal to the difference in frequency of the two notes.*

The phenomenon of beats is often used to compare an unknown frequency with a known frequency, as in the case of tuning a piano with a tuning fork. The ear can detect beats upto about 10 per second. Above this, the fluctuations in loudness are too rapid to be heard. The phenomenon of beats can be used to detect the velocity of a moving object like a car. When a beam from, say, a radder source is reflected from a moving car small frequency changes take place in the reflected beam due to Doppler effect (Art. 6.1). Due to this change in frequency in the reflected radar beam, beats will be produced when the reflected beam and the original beam from the radar source superpose. By counting the number of beats and invoking the Doppler principle, the velocity of the moving car can be determined.

Example 5.3. *A tuning fork A of frequency 384 Hz gives 6 beats per second when sounded with another tuning fork B. On loading B with a little wax, the number of beats per second becomes 4. What is the frequency of B?*

Soln.

Frequency of A = 384 Hz.

Beats per second = 6

Hence frequency of B before loading is either 384 + 6 = 390 Hz

or, 384 – 6 = 378 Hz

After B is loaded, beats produced per second = 4. Then frequency of B after loading is

either 384 + 4 = 388 Hz

or, $384 - 4 = 380$ Hz.

Since frequency decreases after loading, the frequency of B before loading is **390 Hz.**

Example 5.4. *Two tuning forks A and B when sounded together give 5 beats per second. The frequency of B is 256 Hz. When A is filed, 5 beats per second is again heard. Find the frequency of A before and after filing.*

Soln.

Frequency of B = 256

Beats per second = 5

∴ Frequency of A before filing is

either $256 + 5 = 261$

or, $256 - 5 = 251$

Since after filing number of beats per second is again 5, frequency of A after filing is

Either $256 + 5 = 261$

or, $256 - 5 = 251$

Let the frequency of A before filing be 261. After filing the frequency of A increases. Therefore after filing, the frequency of A cannot be either 261 or 251. Therefore, frequency of A cannot be 261.

But if the frequency of A before filing is 251 then its frequency after filing can be 261.

Hence frequency of A

before filing = 251

after filing = 261

Example 5.5. *A note produces 6 beats per second with a tuning fork of frequency 312 and 8 beats per second with a fork of frequency 314. Find the frequency of the note.*

Soln.

In the first case,

frequency of the tuning fork = 312

baets produced per second = 6

$\therefore$ possible frequency of the note is

either, 312 + 6 = 318

or, 312 – 6 = 306

In the second case,

frequency of the tuning fork = 314

beats produced per second = 8

$\therefore$ possible frequency of the note is

either 314 + 8 = 322

or, 314 – 8 = 306

Hence the frequency of the note is **306.**

Example 5.6. *Determine (i) the velocity of sound in a gas in which two waves of lengths 50 cm and 50.5 cm produce 6 beats per second and (ii) the velocity of sound in water in which waves of lengths 500 cm and 512 cm produce 6 beats per second.*

(i) Let v be the velocity of sound in the gas. Then the frequency n_1 of the first wave

$$= \frac{v}{\lambda_1} = \frac{v}{50} \qquad \because \lambda_1 = 50 \text{ cm}$$

and the frequency n_2 of the second wave

$$= \frac{v}{\lambda_2} = \frac{v}{50.5} \qquad \because \lambda_2 = 50.5 \text{ cm}$$

But $n_1 - n_2 = 6$

$$\therefore \quad \frac{v}{\lambda_1} - \frac{v}{\lambda_2} = 6$$

$$\text{or,} \quad \frac{v}{50} - \frac{v}{50.5} = 6$$

or, $\frac{0.5\text{ v}}{2525} = 6$

or, v = 30300 cm/sec

= 303 m/sec.

(ii) Let v' be the velocity of sound in water.

Then $n_1 = \frac{v'}{\lambda_1} = \frac{v'}{500}$

and $n_2 = \frac{v'}{\lambda_2} = \frac{v'}{512}$

But $n_1 - n_2 = 6$

$\therefore \quad \frac{v'}{500} - \frac{v'}{512} = 6$

or, $\frac{12v'}{500 \times 512} = 6$

or, $v' = \frac{6 \times 256000}{12}$

= 128000 cm/sec

= 1280 m/sec.

5.5 Superposition of two waves of nearly equal frequency and wavelength : group velocity

The equation of a plane progressive wave is given by

$$y = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

$$= a \sin (wt - kx)$$

where $w = \frac{2\pi}{T} = 2\pi n$

and $k = \frac{2\pi}{\lambda}$

w and k are referred to as the angular frequency and angular wave number of the wave respectively.

Also, the term

$$\frac{w}{k} = \frac{2\pi n}{2\pi/\lambda} = n\lambda = v$$

where v is the wave velocity or phase velocity and is the velocity with which a plane progressive wavefront travels forward (see Art 4.5).

Now consider two waves of slightly different frequencies and wavelengths but having equal amplitudes moving in the same direction with slightly different velocities. Let the waves be in phase at t = 0 and x = 0. Then the waves may be represented by

$$y_1 = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

$$= a \sin (w_1 t - k_1 x) \qquad \text{(i)}$$

$$y_2 = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

$$= a \sin (w_2 t - k_2 x) \qquad \text{(ii)}$$

Fig. 5.4 shows the two waves at time t = 0, when the waves are in phase and add constructively at x = 0. Because of the difference in wavelength, they are out of phase at other places. At some distance x_1, the waves are 180° out of phase and add to zero. At an equal distance beyond this point, at x_2, the waves are in phase again. The greater the difference in wavelength, the shorter the distance x_1 in which they become out of phase. Fig. 5.4 can also represent the time dependence of the waves at x = 0. Initially they are in phase, but because of the difference in frequency, they are 180° out of phase and cancel each other at some time t_1 later. Subsequently they will again be in phase and interfere constructively. The greater the difference in frequency, the sooner they will be out of phase and cancel each other.

The waves given by eqns. (i) and (ii) can be added algebraically and the resultant wave will be given by

$$y = y_1 + y_2$$

$$= a \sin (w_1 t - k_1 x) + a \sin (w_2 t - k_2 x) \qquad \text{(iii)}$$

Applying the trigonometric relation

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$$

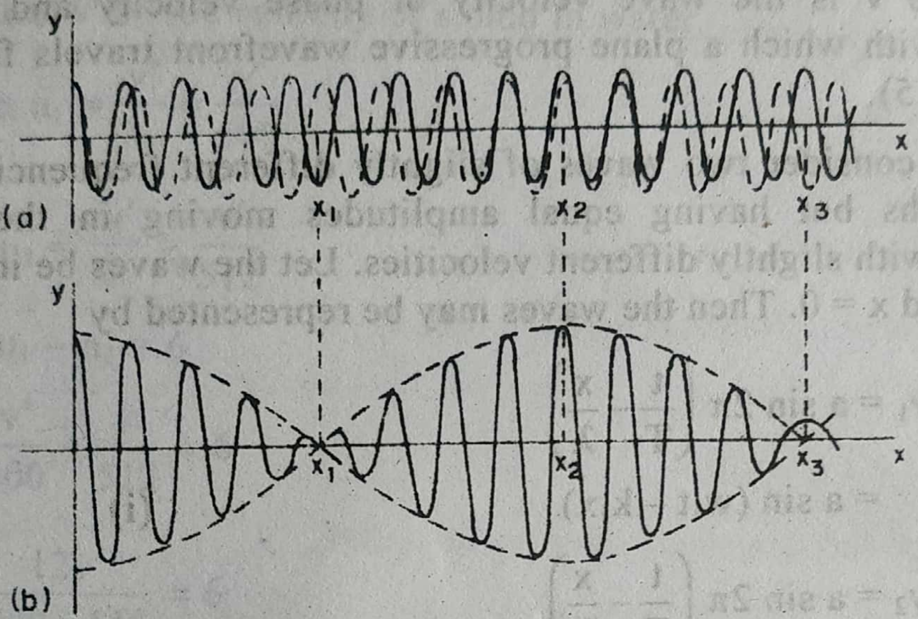

Fig. 5.4

eqn. (iii) can be written as

$$y = 2a \cos \left(\frac{w_1 - w_2}{2} \cdot t - \frac{k_1 - k_2}{2} \cdot x \right)$$

$$\sin \left(\frac{w_1 + w_2}{2} \cdot t - \frac{k_1 + k_2}{2} \cdot x \right) \qquad (5.5)$$

The above equation can be simplified if we use the notation Δk and Δw for the differences in wave number and frequency and k and w for their averages.

$\Delta k = k_1 - k_2, \qquad \Delta w = w_1 - w_2,$

$k = \frac{1}{2}(k_1 + k_2), \qquad w = \frac{1}{2}(w_1 + w_2),$

Eqn. (iv) therefore, becomes

$$y = 2a \cos \left(\frac{1}{2} \cdot \Delta w. t - \frac{1}{2} \Delta k . x \right) \sin (wt - k . x) \qquad (5.6)$$

The resultant wave is sketched in Fig. 5.4 (b). Since frequencies and wavelengths are nearly equal, Δw and Δk are small and w and k are nearly equal to the frequency and wave number of either wave. The result is a wave of about the same frequency and wavelength as the original waves but with the amplitude modulated by the factor cos $(\frac{1}{2}\Delta w \cdot t - \frac{1}{2}\Delta k \cdot x)$. The velocity of the resultant wave $v = w/k$ is nearly the same as that the individual waves and is called the phase velocity. The envelope (dashed curve) gives the cosine wave and is called the *envelope or beat wave.* The formation of this wave is similar to the formation of beats. It consists of a group of waves, each group consisting of a number of waves, with the displacement of the wave at the centre of the group the maximum, with those of others trailing off gradually to zero on either side as shown in the figure. The envelope (or the beat wave) travels as a wave of wave number $\frac{1}{2}\Delta k$ and angular frequency $\frac{1}{2}\Delta w$. The velocity with which this envelope (or the group of waves) travels can be obtained by considering the modulating factor

$$\cos\left(\frac{1}{2}\Delta w \cdot t - \frac{1}{2}\Delta k \cdot x\right)$$

$$= \cos\frac{1}{2}\Delta k\left(x - \frac{\Delta w}{\Delta k} \cdot t\right)$$

$$= \cos\frac{1}{2}\Delta k\,(x - ut)$$

where $u = \frac{\Delta w}{\Delta k}$ is the velocity of the envelope and is called the *group velocity.*

$$\therefore \quad u = \frac{\Delta w}{\Delta k}$$

But $w = v\,k$ where v is the phase velocity. Hence the relation between group velocity u and phase or wave velocity v is given by

$$u = \frac{\Delta w}{\Delta k} = \frac{\Delta}{\Delta k}(vk)$$

$$= v + k\frac{\Delta v}{\Delta k}$$

$$= v + k\frac{\Delta v}{\Delta \lambda} \cdot \frac{\Delta \lambda}{\Delta k} \qquad \text{(vi)}$$

Since $k = \dfrac{2\pi}{\lambda}$ and hence $\dfrac{\Delta \lambda}{\Delta k} = -\,2\pi/k^2$.

eqn. (vi) then becomes

$$u = v - \lambda \cdot \frac{\Delta v}{\Delta \lambda} \qquad (5.7)$$

The relation between the group velocity and the phase velocity depends on the medium through which the wave is transmitted. In a medium in which the phase velocity does not depend on the frequency of the wave, the group and phase velocities are equal. Such a medium is called a *dispersionless (or non-dispersive)* medium. Examples are waves on a perfectly flexible string, sound waves in air, and light waves in vacuum. On the other hand, if the phase velocity does depend on the frequency or wavelength, the group velocity and phase velocity are not equal. If v decreases with increasing wavelength the group velocity exceeds the wave velocity and is smaller than the wave velocity if the later increases with increase in wavelength. Examples are water waves, light waves in glass or water, and waves on a string which is not perfectly flexible. A medium for which the phase velocity depends on the frequency (or wavelength) is called a *dispersive* medium.

5.6 Interference by reflection–stationary or standing waves

A medium in which a wave is constrained to travel along a fixed linear path is known as a linear medium. As plane waves are one-dimensional, they also travel along a fixed linear path; as such they too may be regarded as waves in a linear medium. Transverse waves along a thin rod are examples of this type of wave.

If the medium be of unlimited or infinite length, the waves just continue to travel through it for an infinite time. However, if the length of the medium be limited or finite, it is referred to as a *linear bounded medium.* If a wave travelling through the medium suffer normal reflection at the boundary, we have *two identical waves - of the same* *wavelength, frequency and amplitude, travelling along the same linear path but in opposite directions*. When two such waves are superposed on each other the result is a very special case of interference in which the positions of maximum and minimum displacement remain fixed throughout. The resulting waves appear to remain stationary in space, with no onward or progressive movement. For this reason, the resulting waves are referred to as the *stationary* or *standing* waves.

Now the nature of the boundary of the linear medium where the reflection takes place may be of two types : the boundary of the medium is fixed or *rigid* or the boundary is *free* (or *yielding*). Suppose a wave pulse travels down a stretched string which is fixed at one end (Fig. 5.5). Upon arrival at the end, the wave pulse exerts an upward force on the support. However the support is rigid and as such does not move. But according to Newton's third law, the support exerts an equal but oppositely directed force on the string. This force of reaction results in the generation, at the support, of a wave pulse which travels back along the string in a direction opposite to that of the incident pulse. This is expressed by saying that the incident wave pulse has been reflected at the fixed end point of the string. It may be noticed that the reflected wave pulse returns with its transverse displacement reversed. If a wavetrain is incident on the fixed end point, a reflected wavetrain is generated at that point in the same way. The displacement at any point along the string is the sum of the displacements caused by the incident and reflected waves. Since the end point is fixed, the wavetrain is not free to move and hence the displacement of the particle there will be zero at all times. Thus the two waves must always interfere destructively at that point so as to give zero displacement there. Hence the reflected wave is always 180° out of phase with the incident wave at a fixed boundary. This is expressed by saying that on reflection from a fixed end (boundary) a wave undergoes a phase change of 180, but the wave is reflected without change of type, *i.e.*, a crest (or a condensatin) is reflected as a crest (or a condensation) and a trough (or a rarefaction) is reflected as a trough (or rarefaction), but with change of sign, *i.e.*, with the direction of the displacement of the particles reversed.

Let us now consider the reflection of a pulse at a free end of a stretched string *i.e.*, at an end that is free to move transversely – the so-called *yielding wall*. This can be achieved by attaching the end to a very light ring free to slide without friction along a transverse rod, or to a long and very much lighter string. Upon arrival at the free end,

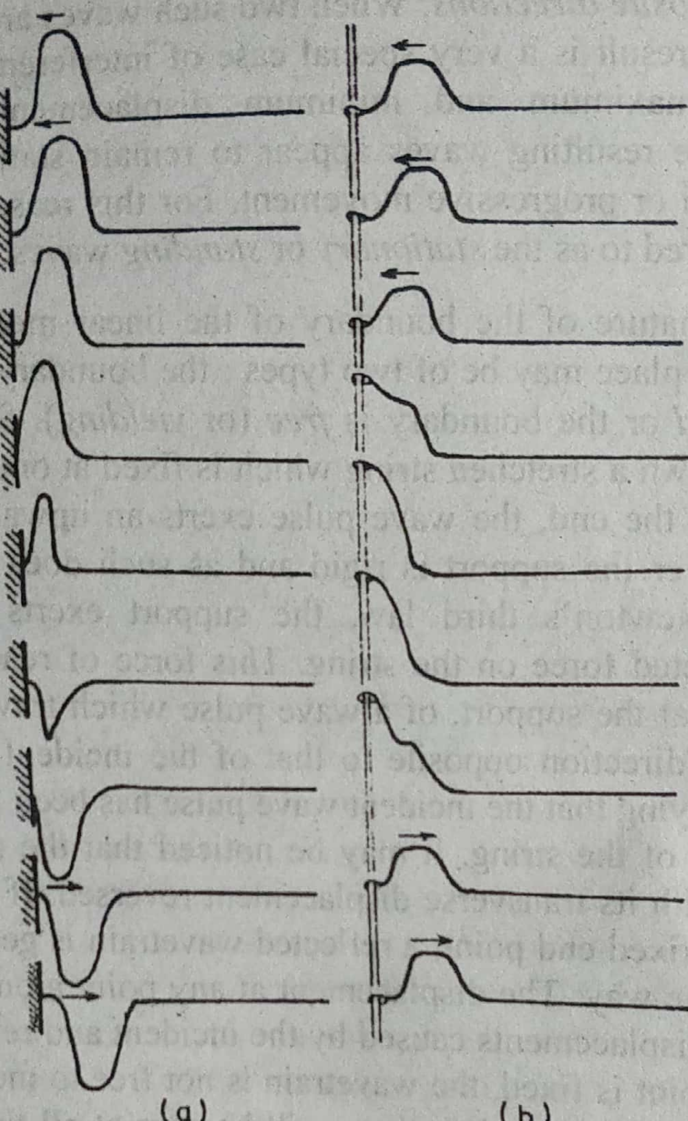

Fig. 5.5

the wave pulse exerts a force on the element of string there. This element is accelerated and its inertia carries it past the equilibrium point; it overshoots and exerts a reaction force on the string. This reaction force generates a pulse which travels back along the string in a direction opposite to that of the incident pulse, Once again we get reflection, but now at a free end. The free end or the yielding wall will obviously suffer the maximum displacement of the particles on the string. Thus the incident and the reflected wavetrains must interfere constructively at that point if we are to have a maximum there. Hence the reflected wave is always in phase with the incident wave at that point (Fig. 5.5). This is expressed by saying that at a free end a wave is reflected without change of phase.

This means in other words, that the particles of the medium continue to have their displacements in the same direction as before, with no phase change so that a crest (or condensation) is reflected back as a trough (or rarefaction) and *vice versa*.

Hence, when we have a standing wave in a string, where will be zero displacement at a fixed end and maximum displacement at a free end.

5.6 Formation of stationary waves–analytical treatment

The formation of stationary or standing waves will be analytically discussed in this section. The case when the reflection occurs at a fixed or rigid boundary and the case when the reflection occurs at a free boundary (yielding wall) will be separately discussed.

Case I : reflection occurs at a fixed or rigid boundary

Let the equation of a simple harmonic wave travelling in the positive direction of the x-axis be

$$y_1 = y_m \sin \frac{2\pi}{\lambda} (vt - x)$$

where

y_m = maximum displacement (amplitude)

λ = wavelength of the wave

v = velocity of the wave

If this wave is incident normally on, and reflected from, a fixed or rigid boundary, then the equation of the reflected wave will be

$$y_2 = -y_m \sin \frac{2\pi}{\lambda} (vt + x)$$

since both the direction of displacement of the particles and the direction of propagation of the wave itself, get reversed.

Both the incident and the reflected waves travel along the same linear path; hence they will be superposed on each other and the equation of the resultant stationary or standing wave is, therefore,

$$y = y_1 + y_2$$

$$= \left[y_m \sin \frac{2\pi}{\lambda} (vt - x)\right] + \left[-y_m \sin \frac{2\pi}{\lambda} (vt + x)\right]$$

$$= + y_m [\sin \frac{2\pi}{\lambda} (vt - x) - \sin \frac{2\pi}{\lambda} (vt + x)]$$

$$= y_m [2 \cos \frac{1}{2} (\frac{2\pi}{\lambda} vt - \frac{2\pi}{\lambda} x + \frac{2\pi}{\lambda} vt + \frac{2\pi}{\lambda} x)$$

$$\sin \frac{1}{2} (\frac{2\pi}{\lambda} vt - \frac{2\pi}{\lambda} x - \frac{2\pi}{\lambda} vt - \frac{2\pi}{\lambda} x)]$$

$$= + y_m [2 \cos \frac{2\pi}{\lambda} vt \sin (- \frac{2\pi}{\lambda} x)]$$

$$= - y_m 2 \cos 2\pi \frac{v}{\lambda} t \sin \frac{2\pi}{\lambda} x$$

$$= - y_m 2 \cos wt \sin \frac{2\pi}{\lambda} x$$

But $- y_m$ is the maximum displacement of the particles on the negative side. By definition, the maximum displacement on either side of the wave is the amplitude (a) of the wave. Hence the above equation can be written as

$$y = 2a \sin \frac{2\pi}{\lambda} x \cos wt \qquad (5.8)$$

Thus the resulting wave is also a simple harmonic wave of the same time-period and wavelength as the two constituent waves. But its amplitude has changed and is given by

$$A = 2a \sin \frac{2\pi}{\lambda} x \qquad (5.9)$$

As can be seen from eqn. 5.8 that a particle at any particular point x executes simple harmonic motion as time goes on, and that all particles vibrate with the same frequency. In a travelling wave each particle vibrates with the same amplitude.

However, in the case of standing wave the amplitude is not the same for all particles but varies with the location x of the particle. The amplitude is now a function of x (eqn. 5.9).

Differentiating eqn. 5.8 with respect to time, we obtain an expression for the velocity of the particle.

$$U = \frac{dy}{dt} = \frac{4\pi av}{\lambda} \sin\frac{2\pi x}{\lambda} \sin\frac{2\pi vt}{\lambda} \qquad (5.10)$$

The acceleration of the particle at any given instant of time is

$$\frac{d^2y}{dt^2} = \frac{8\pi^2 av^2}{\lambda^2} \sin\frac{2\pi x}{\lambda} \sin\frac{2\pi vt}{\lambda} \qquad (5.11)$$

The strain or compression at any point of the resultant vibration is given by dy/dx.

Differentiating eqn. 5.8 with respect to x

$$\frac{dy}{dx} = -\frac{4\pi a}{\lambda} \cos\frac{2\pi x}{\lambda} \cos\frac{2\pi vt}{\lambda} \qquad (5.12)$$

Eqns. 5.9 to 5.12 show that the amplitude, velocity, acceleration and strain or compression vary with position and time.

Changes with respect to position

(i) Consider the points where

$$\sin\frac{2\pi x}{\lambda} = 0 \quad \text{and} \quad \cos\frac{2\pi x}{\lambda} = \pm 1$$

From eqns. 5.8 to 5.12,

displacement $y = 0$

amplitude $A = 0$

velocity $\frac{dy}{dt} = 0$

acceleration $\frac{d^2y}{dt^2} = 0$

strain $\frac{dy}{dx} = \mp \frac{4\pi a}{\lambda} \cos\frac{4\pi vt}{\lambda}$

Thus at these positions, displacement, amplitude, velocity and acceleration are zero but the strain is maximum. These points of permanent zero displacement are called *nodes* or *nodal points*.

Now sin $\frac{2\pi x}{\lambda}$ will be zero (or cos $\frac{2\pi x}{\lambda}$ will be $\pm$ 1) when

$$\frac{2\pi x}{\lambda} = m\pi,$$

where m is an integer; 0, 1, 2, 3, ... etc.

$$\therefore \frac{2\pi x}{\lambda} = m\pi \quad \text{or} \quad x = \frac{m\lambda}{2}$$

$$\text{or, } x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2} \ldots\ldots \text{etc.}$$

Thus the nodes or nodal points are obviously equidistant and separated by $\frac{\lambda}{2}$. At x = 0, the position of interface is a node.

(ii) Consider the positions, where

$$\sin\frac{2\pi x}{\lambda} = \pm 1 \quad \text{and} \quad \cos\frac{2\pi x}{\lambda} = 0$$

From eqns. 5.8 to 5.12,

displacement $\quad y = \mp\, 2a \cos\frac{2\pi vt}{\lambda}$

amplitude $\quad A = \pm\, 2a$

velocity $\quad \frac{dy}{dt} = \pm\frac{2\pi av}{\lambda}\sin\frac{2\pi vt}{\lambda}$

acceleration $\quad \frac{d^2y}{dt^2} = \pm\frac{8\pi^2av^2}{\lambda^2}\cos\frac{2\pi vt}{\lambda}$

strain $\quad \frac{dy}{dx} = 0$

It is, therefore, clear that at these points, the displacement will always be maximum.

Now sin $\frac{2\pi x}{\lambda}$ will be maximum (or cos $\frac{2\pi x}{\lambda}$ will be zero) when

$$\frac{2\pi x}{\lambda} = (2m + 1)\frac{\pi}{2}$$

where m = 0, 1, 2, 3, ... etc.

or, $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots\ldots\ldots\ldots$ etc.

These points of maximum displacements (positive or negative) are called *antinodes* or *antinodal* points. Obviously, the antinodal points are also equidistant and separated by $\frac{\lambda}{2}$.

It may be noted that the displacement at the antinodal points also varies simple harmonically with time. But the displacement at these points is always the maximum at any given instant relative to the displacement at all other points, where the displacement lie between the two extremes, decreasing from the maximum at an antinodal point to zero at the preceding or succeeding nodal points.

It can further be seen that no two nodes can exist without an antinode in between and *vice versa*, so that the distance between a node and a succeeding or preceeding antinode is $\frac{\lambda}{4}$.

It is clear that energy is not transported along the string to the right or to the left, for energy cannot flow past the nodal points in the string which are permanently at rest. Hence, energy remains *standing* in the string although it alternates between vibrational kinetic energy and elastic potential energy. The motion is called a wave motion because it can be thought of as a superposition of waves travelling in opposite direction.

Changes with respect to time

Consider the instants of time, when

$$\sin\frac{2\pi vt}{\lambda} = 0 \qquad \text{and} \qquad \cos\frac{2\pi vt}{\lambda} = \pm 1$$

From eqns. 5.8 to 5.12

Displacement, $y = \mp 2a \sin\frac{2\pi x}{\lambda}$

the displacement is maximum (positive or negative) amplitude, $A = 2a \sin \frac{2\pi x}{\lambda}$ (independent of time)

velocity, $\frac{dy}{dt} = 0$

acceleration, $\frac{d^2y}{dt^2} = \pm \frac{8\pi^2 av^2}{\lambda^2} \sin \frac{2\pi x}{\lambda}$

strain, $\frac{dy}{dx} = \pm \frac{4\pi a}{\lambda} \cos \frac{2\pi x}{\lambda}$

These equations show that at these instants, the displacement, acceleration and strain are maximum and the velocity of the particles is zero.

Now $\cos \frac{2\pi vt}{\lambda} = \pm 1$

when $\frac{2\pi vt}{\lambda} = m\pi$ where m = 0, 1, 2, . . . etc.

But $\frac{v}{\lambda} = n = \frac{1}{T}$

$\therefore \quad t = \frac{mT}{2}$

or, $t = 0, \frac{T}{2}, T, \frac{3T}{2}, \ldots\ldots\ldots\ldots$ etc.

At these instants, although the maximum displacement of the particles are all different, each particle is at its extreme position and the velocity of the particles is zero. The pattern is, therefore, stationary at that instant. This instant is called the stationary instant.

(ii) At the instant of time, when

$\sin \frac{2\pi vt}{\lambda} = \pm 1$ and $\cos \frac{2\pi vt}{\lambda} = 0,$

displacement, $y = 0$

amplitude, $A = 2a \sin \frac{2\pi x}{\lambda}$ (independent of time)

particle velocity $\frac{dy}{dt} = \pm \frac{4\pi av}{\lambda} \sin \frac{2\pi x}{\lambda}$

acceleration $\frac{d^2y}{dt^2} = 0$

strain $\frac{dy}{dx} = 0$

These equations show that at these instants the displacement, acceleration and the strain are all zero.

Now $\cos \frac{2\pi vt}{\lambda} = 0$

when $\frac{2\pi vt}{\lambda} = (2m + 1)\frac{\pi}{2}$ where m = 0, 1, 2, 3, ... etc.

or, $2\pi \left(\frac{1}{T}\right) t = (2m + 1)\frac{\pi}{2}$

or, $t = (2m + 1)\frac{T}{4}$

i.e., when $t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}$, etc., *i.e.*, half a period apart, all the particles pass through their mean positions and have their maximum velocities although these maximum velocities are different for different particles.

case II. reflection occurs at a free boundary

As before, let the equation of a simple harmonic wave, of amplitude a and wavelength λ, travelling along the positive direction of the x-axis be

$$y_1 = a \sin \frac{2\pi}{\lambda}(vt - x)$$

Then the equation of the wave reflected at the free boundary will be

$$y_1 = a \sin \frac{2\pi}{\lambda}(vt + x)$$

since only the direction of travel of the wave is reversed and not the direction of displacement of the particles of the medium.

The equation of the resulting stationary or standing wave due to superposition of these two waves is

$$y = y_1 + y_2 = a \sin \frac{2\pi}{\lambda}(vt - x)$$

$$+ a \sin \frac{2\pi}{\lambda}(vt + x)$$

$$= 2a \sin \frac{2\pi vt}{\lambda} \cos \frac{2\pi x}{\lambda}$$

$$= 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

$$= 2a \cos \frac{2\pi x}{\lambda} \sin wt \qquad (5.13)$$

where $w = \frac{2\pi v}{\lambda} = 2\pi n$

Thus the resulting wave is also simple harmonic having the same time period and wavelength as each of the two constituent waves. However, its amplitude has changed and is no longer constant as can be seen from the relation

$$A = 2a \cos \frac{2\pi x}{\lambda}. \qquad (5.14)$$

The resultant amplitude is a function of x. The particle velocity at any instant of time

$$U = \frac{dy}{dt} = \frac{4\pi av}{\lambda} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda} \qquad (5.15)$$

Acceleration of the particle at any instant of time

$$\frac{d^2y}{dt^2} = -\frac{8\pi^2 av^2}{\lambda^2} \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda} \qquad (5.16)$$

The strain or compression at any point of the resultant vibration

$$\frac{dy}{dx} = -\frac{4\pi a}{\lambda} \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda} \qquad (5.17)$$

Eqns. 5.13 to 5.17 show that the amplitude, velocity acceleration and strain or compression vary with position and time.

Changes with respect to position

(i) Consider the instants of time, then

$$\sin\frac{2\pi x}{\lambda} = 0 \quad \text{and} \quad \cos\frac{2\pi x}{\lambda} = \pm 1$$

Then, from eqns. 5.13 to 5.17

displacement, $y = \pm 2a \sin\frac{2\pi vt}{\lambda}$

amplitude, $A = \pm 2a$

velocity, $\frac{dy}{dt} = \pm \frac{4\pi av}{\lambda} \cos\frac{2\pi vt}{\lambda}$

acceleration, $\frac{d^2y}{dt^2} = \mp \frac{8\pi^2 av^2}{\lambda^2} \sin\frac{2\pi x}{\lambda}$

and strain, $\frac{dy}{dx} = 0$

As the displacement or the amplitude is maximum, these points correspond to *antinodes*.

Now $\sin\frac{2\pi x}{\lambda} = 0$ (or $\cos\frac{2\pi x}{\lambda} = \pm 1$)

when $\frac{2\pi x}{\lambda} = m\pi$ where m = 0, 1, 2, ,3, . . . etc.

$$\therefore \quad x = \frac{m\lambda}{2}$$

or, $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots\ldots\ldots$ etc.

Thus the antinodes or the antinodal points are equidistant and separated by λ/2. At x = 0, *i.e.*, the position of interface is an antinode.

(ii) Consider the positions, where

$$\sin\frac{2\pi x}{\lambda} = \pm 1 \quad \text{and} \quad \cos\frac{2\pi x}{\lambda} = 0,$$

from eqns. 5.13 to 5.17

displacement, $y = 0$

amplitude, $A = 0$

velocity, $\frac{dy}{dt} = 0$

acceleration, $\frac{d^2y}{dt^2} = 0$

strain, $\frac{dy}{dx} = \mp \frac{4\pi a}{\lambda} \sin \frac{2\pi vt}{\lambda}$

These positions are called, as we know, the *nodal points* or *nodes*.

Now $\sin \frac{2\pi x}{\lambda} = \pm 1$ (or $\cos \frac{2\pi x}{\lambda} = 0$)

when $\frac{2\pi x}{\lambda} = (2m + 1)\frac{\pi}{2}$ where m = 0, 1, 2, ,3, . . . etc.

or $x = \frac{(2m+1)\lambda}{4}$

$\therefore \; x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots\ldots\ldots$ etc.

Thus the nodes are also equidistant and the distance between two successive nodes is $\lambda/2$.

Changes with respect to time

Consider the instant of time, when

$\sin \frac{2\pi vt}{\lambda} = 0$ and $\cos \frac{2\pi vt}{\lambda} = \pm 1$

From eqns. 5.13 to 5.17

displacement, $y = 0$

amplitude, $A = 2a \cos \frac{2\pi x}{\lambda}$ (independent of time)

velocity, $\frac{dy}{dt} = \pm \frac{4\pi av}{\lambda} \cos \frac{2\pi x}{\lambda}$

acceleration, $\frac{d^2y}{dt^2} = 0$

and strain, $\frac{dy}{dx} = 0$

Now $\sin \frac{2\pi vt}{\lambda'} = 0$ (or $\cos \frac{2\pi vt}{\lambda} = \pm 1$)

when $\frac{2\pi vt}{\lambda} = m\pi$ where m = 0, 1, 2, ,3, . . . etc.

or, $t = \frac{m\lambda}{2v}$

But $\frac{v}{\lambda} = n \text{ (frequency)} = \frac{1}{T}$

$\therefore \quad t = \frac{mT}{2}$

or, $t = 0, \frac{T}{2}, T, \frac{3T}{2}, \ldots\ldots\ldots\ldots$ etc.

Thus particle displacement will be zero while its velocity will be maximum at instants of time $t = 0, \frac{T}{2}, T, \frac{3T}{2}, \ldots\ldots\ldots\ldots$ *i.e.*, twice in each time period.

(ii) Consider the instants of time, when

$\sin \frac{2\pi vt}{\lambda} = \pm 1$ and $\cos \frac{2\pi vt}{\lambda} = 0$,

form eqns. 5.13 to 5.17

displacement, $y = \pm 2a \cos \frac{2\pi x}{\lambda}$

amplitude, $A = 2a \cos \frac{2\pi x}{\lambda}$ (independent of time)

velocity, $\frac{dy}{dt} = 0$

acceleration, $\frac{d^2y}{dt^2} = \mp \frac{4\pi a}{\lambda} \sin \frac{2\pi x}{\lambda}$

strain or compression, $\frac{dy}{dx} = \mp \frac{4\pi a}{\lambda} \sin \frac{2\pi x}{\lambda}$

Now $\sin \frac{2\pi vt}{\lambda} = \pm 1$ (or $\cos \frac{2\pi vt}{\lambda} = 0$)

when $\frac{2\pi vt}{\lambda} = (2m+1)\frac{\pi}{2}$ where m = 0, 1, 2, ,3, . . . etc.

or $t = \frac{(2m+1)\lambda}{4v}$

But $\frac{v}{\lambda} = n = \frac{1}{T}$

$\therefore \quad t = \frac{(2m+1)T}{4}$

or, $t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}, \ldots\ldots\ldots$ etc.

At these instants, all particles attain their maximum displacements (although different for different particles), suffer maximum strain and have maximum acceleration at all positions. But the velocity of the particles at these instants, will be zero. At these instants, each particle is at its extreme position and the pattern becomes stationary at that instant.

5.7 Energy of a stationary wave

When a longitudinal wave propagates through a fluid, the bulk modulus (K) of the fluid is given by

$$K = -\frac{p}{dy/dx}$$

where p is the excess pressure (volume stress) and $\frac{dy}{dx}$ is the volume strain.

$$\therefore \quad p = -K \cdot \frac{dy}{dx} \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

In the case of a stationary wave formed by reflection at a free boundary, the strain is given by

$$\frac{dy}{dx} = -\frac{4\pi a}{\lambda} \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

Also $v = \sqrt{\frac{K}{\rho}}$; or, $v^2 = \sqrt{\frac{K}{\rho}}$; or, $K = v^2\rho$.

where ρ is the density of the fluid.

Substituting the values of $\frac{dy}{dx}$ and K in eqn. (1).

$$p = v^2\rho \cdot \frac{4\pi a}{\lambda} \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda} \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

when $\sin \frac{2\pi x}{\lambda} = 1$ and $\sin \frac{2\pi vt}{\lambda} = 1$,

the excess pressure is maximum.

$$p = p_0 = v^2\rho \cdot \frac{4\pi a}{\lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

From eqns. (ii) and (iii)

$$p = p_0 \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda} \quad \ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

Putting $p_0 \sin \frac{2\pi x}{\lambda} = p_X$

eqn. (iv) becomes

$$p = p_X \sin \frac{2\pi vt}{\lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}$$

The particle velocity at a point is given by

$$U = \frac{dy}{dt} = \frac{4\pi av}{\lambda} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda} \quad \ldots\ldots \text{(vi)}$$

Putting $\frac{4\pi av}{\lambda} \cos \frac{2\pi x}{\lambda} = U_X$

we have $U = U_X \cos \frac{2\pi vt}{\lambda} \quad \ldots\ldots\ldots\ldots\ldots \text{(vii)}$

Now, the energy transferred per unit area in a small interval of time dt is equal to the work done. Or

$$dI = p.U.dt$$

So, the energy transferred during the whole time period T is given by

$$I = \int_O^T p.U.dt$$

$$= \int_O^T p_x \sin \frac{2\pi vt}{\lambda} . U_X \cos \frac{2\pi vt}{\lambda} . dt$$

Then, the rate of energy transfer or the average energy transferred per second, say

$$I_{av} = \frac{\int_O^T p.U.dt}{T}$$

$$= \frac{p_x U_X}{T} \int_O^T \sin \frac{2\pi vt}{\lambda} \cos \frac{2\pi vt}{\lambda} dt$$

$$= \frac{p_x U_X}{T} \int_O^T \sin \frac{4\pi vt}{\lambda} . dt$$

$$\text{But} \int_O^T \sin \frac{4\pi vt}{\lambda} dt = 0$$

So, *the rate of energy transfer* = 0.

Thus, there is no transference of energy across any section of the medium in the case of a stationary or standing wave.

5.9 Distinction between progressive and stationary waves

Progressive wave	Stationary wave
(i) The vibration characteristics of each particle in the path of a progressive wave is the same and is handed over from particle to particle so that there is an	(i) The vibration characteristics of each particle of the medium is its own which it does not pass on to the others. Thus there is no onward propagation of the wave through the medium – it

onward propagation of the wave through the medium.	remains confined within the space where it is produced.
(ii) The maximum displacement of all particles of the medium is same which they attain one after another.	(ii) The maximum displacement of the particles are not same but they attain their respective maximum displacements simultaneously. The displacement decreases progressively from its maximum value at an antinode to zero at the adjoining node.
(iii) All particles pass through their mean positions with the same maximum velocity but one after another.	(iii) All particles pass through their mean positions simultaneously but with different maximum velocities.
(iv) No particle of the medium is permanently at rest.	(iv) Certain particles of the medium (the displacement nodes) are permanently at rest.
(v) Every region in the path of the wave becomes successively a region of compression, normal pressure and rarefaction. Thus all particles of the medium undergo the same changes of pressure but one after the other.	(v) Condensation and rarefaction do not move along as in the progressive wave. They simply appear and disappear again to be succeeded by the opposite conditions in the same place. The change of pressure is maximum at the nodal points and zero at the antinodal points but occur simultaneously at all points.
(vi) A regular transfer of energy takes place across every section of the medium.	(vi) There is no transference of energy across any section of the medium.

Example 5.6. *A plane progressive harmonic wave is travelling with a velocity of 340 m/sec in a fluid medium of density 0.0015 gm/sec. If the amplitude of the wave be 10^{-4} cm and its frequency 300 cps, obtain the values of (i) pressure amplitude, (ii) the energy density and (iii) energy current for it.*

Soln.

(i) In the case of a plane harmonic wave

$$y = a \sin \frac{2\pi}{\lambda}(vt - x)$$

and $$p = -K \frac{dy}{dx} \text{ (see Art. 5.7)}$$

$$= K \frac{2\pi}{\lambda} a \cos \frac{2\pi}{\lambda}(vt - x)$$

Obviously p will be maximum when $\cos \frac{2\pi}{\lambda}(vt - x)$ is maximum *i.e.*, is equal to 1.

So the maximum value of p or the pressure amplitude $= K\frac{2\pi}{\lambda}a$.

Now $v = \sqrt{\frac{K}{\rho}}$; or, $K = \rho v^2$ where ρ is the density of the medium.

Hence, pressure amplitude $= \rho v^2 \cdot \frac{2\pi}{\lambda} a$.

Again, from $v = n\lambda$, we have $\frac{1}{\lambda} = \frac{n}{v}$.

$\therefore$ pressure amplitude $= 2\pi\rho v^2 a \cdot \frac{n}{v}$

$$= 2\pi \rho v a n$$

Here $\rho = 0.0015$ gm/cc, $v = 340$ m/sec $= 340 \times 10^2$ cm/sec, $a = 10^{-4}$ cm and $n = 300$ cps.

$\therefore$ pressure amplitude $= 2\pi \times 0.0015 \times 340 \times 10^{-4} \times 300$

$= \mathbf{9.6084}$ **dyne/cm^2.**

(ii) Energy density, $E = 2\pi^2 n^2 a^2 \rho$ (Art. 4.9)

$= 2\pi^2 \times (300)^2 \times (10^{-4})^2 \times 0.0015$

$= \mathbf{2.665 \times 10^{-5}}$ **erg/cc.**

(iii) Energy current (or energy flux),

$I = E.v$

$= 2.665 \times 10^{-5} \times 340 \times 10^2$

$= \mathbf{0.9059}$ **erg/cm^2 – sec.**

Example 5.7 (a). A sound wave in air, having an amplitude of 0.005 cm and frequency 700 cps. travelling along the direction of positive x-axis with a velocity of 350 m/sec suffers reflection at a free boundary. Obtain the values of (i) the displacement amplitude and (ii) the pressure amplitude in the resulting stationary wave at a point x = 50 cm. (Density of air = 1.293 gm/lītre).

(b) Also obtain the distance between (i) two successive displacement nodes and antinodes, (ii) two successive pressure nodes and antinodes, (iii) a dispalcement node and an adjacent pressure node.

Soln.

The equation of a stationary wave formed due to reflection at a free boundary is given by

$$y = 2a\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi vt}{\lambda}$$

where the displacement amplitude

$$= 2a\cos\frac{2\pi x}{\lambda}$$

$a = 0.005$ cm, $x = 50$ cm and

$$\lambda = \frac{v}{n} = \frac{350 \times 100}{700} = 50 \text{ cm}$$

$\therefore$ displacement amplitude

$$= 2 \times 0.005 \cos 2\pi \times \frac{50}{50} = \mathbf{0.01\ cm.}$$

Since the cosine term is equal to 1, the displacement amplitude is maximum and the point is, therefore, a displacement *antinode*.

(ii) excess pressure,

$$p = \rho v^2 \frac{4\pi a}{\lambda}\sin\frac{2\pi x}{\lambda}\sin\frac{2\pi vt}{\lambda}$$

$\therefore$ pressure amplitude

$$= \rho v^2 \frac{4\pi a}{\lambda}\sin\frac{4\pi x}{\lambda}$$

$$= \frac{0.001293 \times (35000)^2 \times 4\pi \times 0.005}{50} \times \sin \frac{2\pi \times 50}{50} = 0$$

Thus the point is a pressure node which is also obvious since the point is a displacement antinode.

(b) Since the distance between two successive displacement nodes or antinodes is $\lambda/2$, the distance between a displacement node and an adjacent displacement antinode is $\lambda/4$ and also because a displacement node coincides with a pressure antinode and *vice versa.*

distance between two successive displacement nodes = distance between two successive displacement antinodes

$= \lambda/2 = 50/2 =$ **25.0 cm.**

Therefore, the distance between two successive pressure nodes (or antinodes) is also $= \lambda/2 = 50/2 =$ ***25.0 cm.***

The distance between a displacement node (or pressure antinode) and an adjacent pressure node (or displacement antinode) $= \lambda/4 = 50/2 =$ ***12.5 cm.***

EXERCISES

[1] What is interference of sound waves? Distinguish between constructive and destructive interference.

[2] Analytically discuss the interference of sound and obtain the condition for maximum and minimum intensity.

[3] Show that in interference of sound waves, energy is not destroyed – it is simply redistributed from the points of minimum intensity to those of maximum intensity.

[4] What are beats? How are they produced? Analytically discuss the formation of beats and show that the number of beats per second is equal to the difference in frequency of the two notes.

[5] Distinguish between phase velocity and group velocity of a train of waves and establish a relationship between the two. Show that in a non-dispersive medium they are same.

[6] Show that the group velocity is given by the relation

$$U = v - \lambda \frac{dv}{d\lambda}$$