CHAPTER IV

CAPACITANCE

4.1 Capacitor

Two conductors of arbitrary shape, completely isolated from each other and their surroundings, form a capacitor. No matter what their shape, these conductors are called plates. When the capacitor is charged by connecting the plates to the opposite terminals of a battery, equal and opposite charges (say +q and −q) appear on the

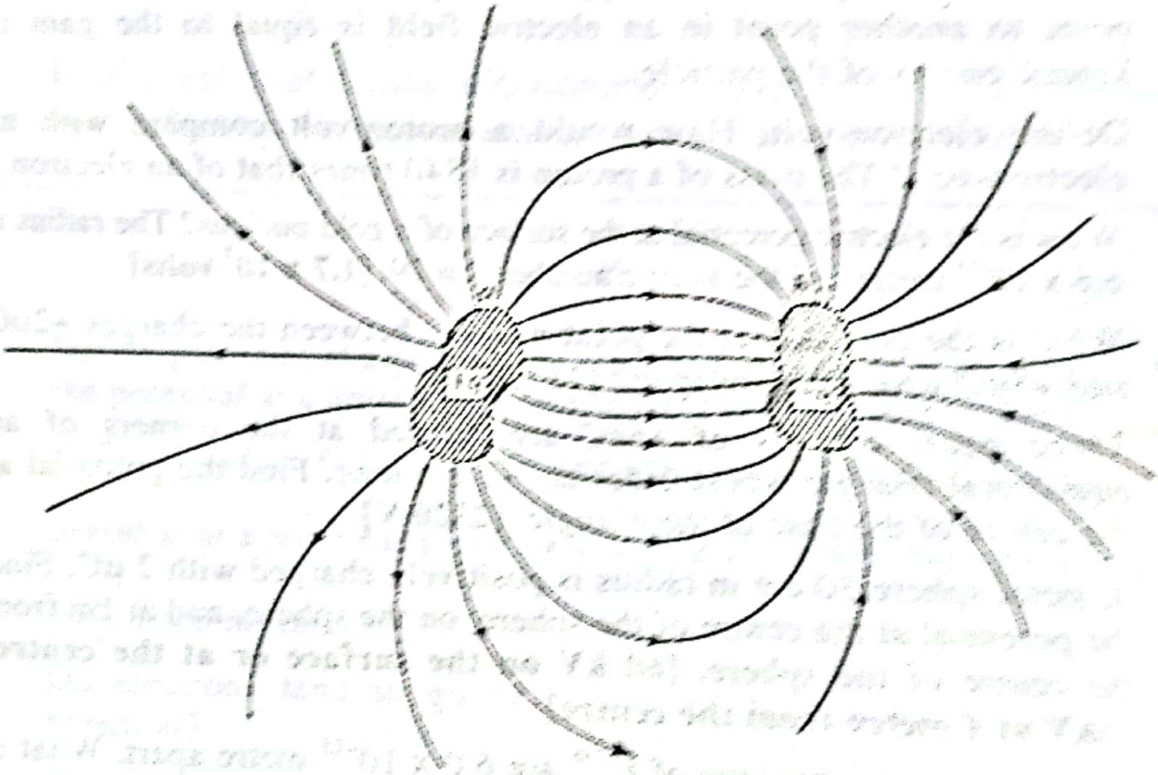

Fig. 4.1

two plates of the capacitor. By charge of a capacitor we mean the absolute value of the charge on either plate, the net charge on the capacitor being zero. The potential difference between the plates of the capacitor is the potential difference of the battery. Fig. 4.1 shows the general arrangement of a capacitor.

4.2 Capacitance

The charge q of a capacitor is found to be directly proportional to the potential difference between the plates. Or,

$$q \propto V$$

or, $q = CV$ or, $C = \frac{q}{V}$ (4.1)

The proportionality constant C is called the *capacitance* of the capacitor. Its value depends on

(i) the geometry of each plate

(ii) the spatial relationship between the plates

and (iii) the medium in which the plates are immersed.

As can be seen from eqn. (4.1), the SI unit of capacitance is *coulomb per volt*. This unit occurs so often that it is given a special name – the *farad*.

1 farad (1F) = 1 coulomb per volt (1 C/V). So *if a potential difference of 1 volt is needed to give a capacitor a charge of 1 coulomb, then the capacitance of the capacitor is said to be 1 farad.*

The farad is a large unit. Submultiples of the farad, such as *microfarad* ($1\mu F = 10^{-6}F$) and the *picofarad* ($1pF = 10^{-12}F$) are more convenient units in practice.

Calculation of capacitance

Once the geometry of a capacitor is known, its capacitance can be calculated. Since different capacitors have different plate geometries, it is wise to develop a general plan to simplify the process involved in the calculation. In brief, the plan is

(i) assume a charge q on the plates

(ii) applying Gauss' law to calculate the electric field **E** between the plates in terms of the charge on the plates.

(iii) knowing **E**, calculate the potential difference V between the plates.

(iv) calculate C from $C = \frac{q}{V}$

The calculation of the electric field and the potential difference may be simplified by making certain assumptions. These are,

(a) calculating the electric field:

The electric field is related to the charge on the plates by Gauss' law:

$$\epsilon_0 \int \mathbf{E} \cdot d\mathbf{A} = q \qquad (4.2)$$

Here q is the charge enclosed by the Gaussian surface, and the integral is carried out over that surface. Only those cases will be considered in which the Gaussian surface are such that whenever electric flux passes through it, the electric field **E** and d**A** will point in the same direction. Eqn. 4.2 then reduces to

$$q = \epsilon_0 E A$$

in which A is the area of that part of the Gaussian surface through which the flux passes. For convenience the Gaussian surface is so drawn that it completely encloses the charge on the positive plate.

(b) Calculating the potential difference:

The potential difference between the plates is related to the electric field **E** by the relation

$$V_f - V_i = -\int_i^f \mathbf{E} \cdot d\mathbf{S} \qquad (4.3)$$

the integral being evaluated along any path that starts on one plate and ends on the other. One should always choose a path that follows an electric field line from the positive plate to the negative plate as shown in Fig. 4.2, since the vectors **E** and dS point in the same direction along this path. It, therefore, follows that the quantity $V_f - V_i$ is negative. Since we are looking for V, the *absolute value* of the potential difference between the plates, we can set $V_f - V_i = -V$. Eqn. 4.3 then becomes

$$V = \int_{+}^{-} E\,dS$$

in which the + and the − signs remind us that our path of integration starts on the positive plate and ends on the negative plate.

The electric field E between the plates is the sum of the fields due to the two plates *i.e.*, $E = E_+ + E_-$ where E_+ is the field due to charges on the positive plate while E_- is that due to charges on the negative plate. By Gauss' law both E_+ and E_- are proportional to q so that E is also proportional to q. By eqn. 4.3. V is also proportional to q. This means that if q is doubled, E and V are also doubled. Because V is proportional to q, the ratio q/V is a constant and is independent of q.

(i) Capacitance of a parallel-plate capacitor

A parallel-plate capacitor formed of two parallel conducting plates of area A and separated by a distance d is shown in Fig. 4.2. If the plates are connected to the opposite terminals of a battery, then a charge +q appears on one plate and a charge −q on the other. If d is small enough compared to the plate dimensions, the electric field strength **E** between the plates will be uniform, which means that the lines of force will be parallel and evenly spaced. According to the laws of electro-magnetism, there should be some *fringing* or curving of the lines at the edges of the plates; for small enough d it can be neglected for the present purpose.

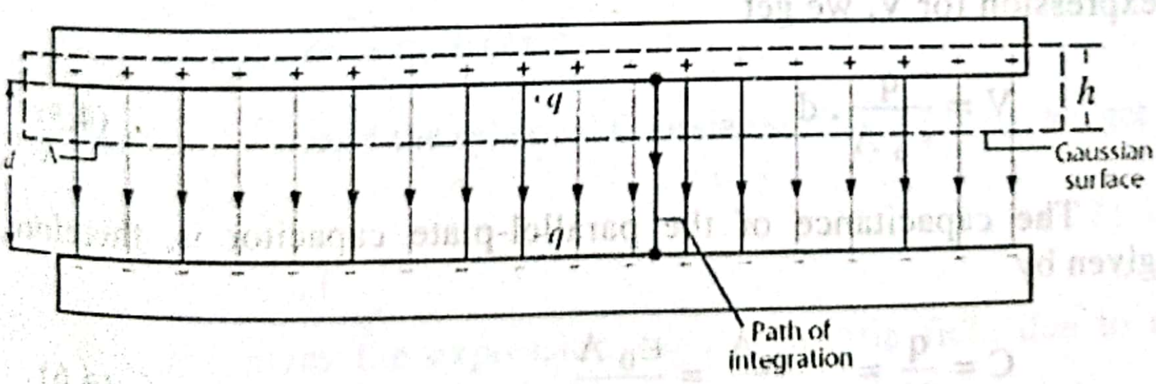

Fig. 4.2

Let us imagine a Gaussian surface of height h closed by plane caps of area A of the same shape and size of the capacitor plates. Because the electric field inside a conductor carrying a static charge is zero, the flux of **E** for the part of the Gaussian surface that lies inside the top capacitor plate is also zero. The flux **E** through the wall of the Gaussian surface is zero because, to the extent that the fringing of the lines of force can be neglected, **E** lies in the wall. Thus the only part of the Gaussian surface which contributes to the electric flux is the Gaussian surface that lies between the plates. Here **E** is constant and according to Gauss' law

$$\phi_E = \oint \mathbf{E}.d\,\mathbf{s} = E.\,A = \frac{q}{\epsilon_0}$$

$$\text{or,} \qquad \epsilon_0\, EA = q$$

$$\text{or,} \qquad E = \frac{q}{\epsilon_0\, A} \tag{4.4}$$

The potential difference V between the plates can be obtained from eqn. 4.3. Or

$$V = \int_{+}^{-} \mathbf{E}.d\,\mathbf{l} = \int_{+}^{-} Edl = E\int_{+}^{-} dl = Ed.$$

since E is constant and can be taken outside the integral and $\int dl$ is simply the plate separation d.

Substituting the value of E as given by eqn. 4.4 in the expression for V, we get

$$V = \frac{q}{\epsilon_0\, A}.\, d \tag{4.5}$$

The capacitance of the parallel-plate capacitor is, therefore given by

$$C = \frac{q}{V} = \frac{q\epsilon_0\, A}{qd} = \frac{\epsilon_0\, A}{d} \tag{4.6}$$

As can be seen from eqn. 4.6, the capacitance does indeed depend only on geometrical factors, namely, the plate area A and the plate separation d.

(ii) Capacitance of a spherical capacitor

Fig. 4.3 shows a central cross-section of a capacitor that consists of two concentric spherical shells of radii *a* and *b*. As a Gaussian surface let us draw a sphere of radius *r* concentric with the two shells. Applying Gauss' law to this surface we obtain

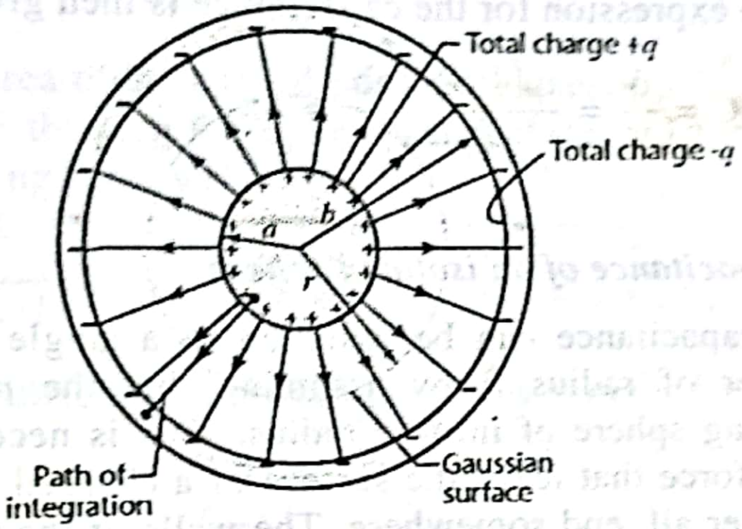

Fig. 4.3

$$q = \epsilon_0 \oint \mathbf{E} . d\,\mathbf{s} = \epsilon_0 \oint E ds$$

$$= \epsilon_0 E \oint ds = \epsilon_0 E(4\pi r^2)$$

where $4\pi r^2$ is the area of the spherical Gaussian surface. Solving, we get

$$E = \frac{q}{4\pi\epsilon_0 r^2} \tag{4.7}$$

Eqn. 4.7 gives the expression for the electric field due to a uniform spherical charge distribution.

The expression for the potential difference between the two concentric spheres is given by

$$V = \int_{+}^{-} E\,ds = \int_{a}^{b} \frac{q}{4\pi\epsilon_0 r^2} \cdot dr$$

$$= \frac{q}{4\pi\epsilon_0} \int_{a}^{b} \frac{dr}{r^2} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)$$

$$= \frac{q}{4\pi\epsilon_0} \cdot \frac{b-a}{ab} \tag{4.8}$$

In deriving eqn. 4.8 we have used the fact that here $ds = dr$.

The expression for the capacitance is then given by

$$C = \frac{q}{V} = \frac{q\,(4\pi\epsilon_0)\,ab}{q\,(b-a)} = 4\pi\epsilon_0 \frac{ab}{b-a} \tag{4.9}$$

(iii) Capacitance of an isolated sphere

A capacitance can be assigned to a single isolated spherical conductor of radius R by assuming that the *missing plate* is a conducting sphere of infinite radius. This is necessary because the lines of force that leave the surface of a charged isolated conductor must, after all, end somewhere. The walls of the room in which the conductor is housed can effectively serve as the sphere of infinite radius.

Now the capacitance of a spherical capacitor as given by eqn. 4.9, is

$$C = 4\pi\epsilon_0 \frac{ab}{b-a} = 4\pi\epsilon_0 \frac{a}{1-a/b}$$

If the second sphere is of infinite radius, then $b \to \infty$. Substituting R for a, we obtain

$$C = 4\pi\epsilon_0 R \tag{4.10}$$

(iv) Capacitance of a cylindrical capacitor

Fig. 4.3 also serves to show a cross-section of a cylindrical capacitor of length *l* formed by two co-axial cylinders of radii *a* and

b. The length of the capacitor is assumed to be much greater than its radius, i.e., $l >> b$ so that fringing of the lines of force at the ends (edge effect) can be ignored for the purpose of calculating the capacitance. Each plate contains a charge of magnitude q.

As a Gaussian surface let us construct a coaxial cylinder of radius r and length l closed by end caps. Applying Gauss' law we then obtain

$$q = \epsilon_0 \oint \mathbf{E} \cdot d\,\mathbf{s} = \epsilon_0 \oint E ds$$

$$= \epsilon_0 E \oint ds \quad = \epsilon_0 E (2\pi r l)$$

where $2\pi r l$ is the area of the curved part of the Gaussian surface, the flux being entirely through the cylindrical surface and not through the end caps. Solving for E we get

$$E = \frac{q}{2\pi \epsilon_0 r l}$$

The potential difference between the plates is given by

$$V = \int_{+}^{-} E ds = \frac{q}{2\pi \epsilon_0 l} \int_a^b \frac{dr}{r}$$

$$= \frac{q}{2\pi \epsilon_0 l} ln \cdot \frac{b}{a}$$

From the relation $C = \frac{q}{V}$, we then have

$$C = 2\pi \epsilon_0 \frac{l}{ln\left(\frac{b}{a}\right)} \qquad (4.12)$$

As can be seen from eqn. 4.12 that, like a parallel-plate capacitor; the capacitance of a cylindrical capacitor depends only on geometrical factors, in this case l, b and a.

The capacitances of various capacitors derived in this section is summarized below.

Type of capacitor	Capacitance	Equation
Paralle-plate	$\epsilon_0 \frac{A}{d}$	4.6
Spherical	$4\pi\epsilon_0 \frac{ab}{b-a}$	4.9
Isolated sphere	$4\pi\epsilon_0 R$	4.10
Cylindrical	$2\pi\epsilon_0 \frac{l}{\ln(b/a)}$	4.12

It can be seen that every expression involves the constant ϵ_0 multiplied by a quantity that has the dimension of length.

Example 4.1 A plane-parallel capacitor has circular plates of radius r = 10.0 cm, separated by a distance d = 1.00mm. How much charge is stored on each plate when their electric potential difference has the value V = 100V?

Soln.

$q = CV$ where C is the capacitance of the capacitor and is given by

$$C = \frac{\epsilon_0 A}{d}$$

Here, $A = \pi r^2 = \pi\ (0.1\text{m})^2$

$= 3.14 \times 10^{-2}\text{m}^2$

$d = 1.00 \text{ mm} = 1 \times 10^{-3}\text{m}$

$$\therefore\ C = \frac{(8.85\times10^{-12}\text{C}^2/\text{N.m}^2)(3.14\times10^{-2}\text{m}^2)}{1.00\times10^{-3}\text{m}}$$

$= 2.8 \times 10^{-10}\text{ F} = 280 \times 10^{-12}\text{F}$

$= 280\text{pF}.$

$\therefore\ q = CV = (2.8 \times 10^{-10}\text{F})\ (100\text{ V})$

$= 2.8 \times 10^{-8}$ coulomb

$= 28 \times 10^{-9}\text{ C} = 28\text{ nC}.$

Example 4.2 The area of each plate of an air-filled parallel-plate capacitor is 1.1 x 10^8 metre^2. What must be the separation between the plates, if the capacitance is to be 1.0 farad?

Soln.

$$C = \frac{\epsilon_0 A}{d}$$ Here, $A = 1.1 \times 10^8 m^2$

$C = 1$ F.

$$\text{or, } d = \frac{\epsilon_0 A}{C}$$

$$= \frac{(8.85\times10^{-12} F/m)(1.1\times10^8 m^2)}{1.0 \text{ F}}$$

$$= 9.735 \times 10^{-4} \text{ m}$$

$$= 9.735 \times 10^{-1} \text{ mm} = 0.9735 \text{ mm.}$$

Example 4.3 How much charge is stored in a capacitor consisting of two concentric spheres of radii 30 and 31 cm if the potential difference is 500V?

Soln.

$q = CV$ where

Here, $a = 30$ cm $= 0.30$ m

$b = 31$ cm $= 0.31$ m

$$C = 4\pi\epsilon_0 \frac{ab}{(b-a)}$$

$$= \frac{(4)(3.14)(8.85\times10^{-12} F/m)(0.30m)(0.31m)}{(0.31-0.30)m}$$

$$= \frac{10.34\times10^{-12}}{0.01} F = 1034 \times 10^{-12} F$$

$$= 1.034 \times 10^{-9} F$$

$$= 1.034 \text{ nF}$$

$$q = CV = (1.03 \times 10^{-9} \text{ F})(500V) = 517 \text{ nC.}$$

Example 4.4 What is the capacitance of the Earth, viewed as an isolated conducting sphere of radius 6370 km ?

Soln.

$$C = 4\pi\epsilon_0 R$$

$$= (4)(3.14)(8.85 \times 10^{-12}\text{F/m})(6370 \times 10^3\text{m})$$

$$= 7.08 \times 10^{-4}\text{F}$$

$$= 708 \times 10^{-6} = 708\ \mu\text{F}.$$

Example 4.5 *The space between the conductors of a long coaxial cable, used to transmit TV signals, has an inner radius a = 0.15mm and an outer radius b = 2.1 mm. What is the capacitance per unit length of this cable?*

Soln.

The capacitance of a coaxial cable is given by (eqn. 4.12)

$$C = 2\pi\epsilon_0 \frac{l}{\ln(b/a)}$$ where l is the length of the cable.

Hence capacitance per unit length is

$$\frac{C}{l} = \frac{2\pi\epsilon_0}{\ln(b/a)}$$

Here, $b = 2.1$ mm $= 2.1 \times 10^{-3}$m

$a = 0.15$ mm $= 0.15 \times 10^{-3}$m.

$$= \frac{(2)(3.14)(8.85\times10^{-12}\text{ F/m})}{\ln\left(\frac{2.1\times10^{-3}\text{ m}}{0.15\times10^{-3}\text{ m}}\right)}$$

$$= \frac{55.578\times10^{-12}}{2.64}\text{ F/m}$$

$$= 21 \times 10^{-12}\text{F/m} = 21\text{pF}.$$

4.3 Capacitors in series and parallel

In analyzing electric circuits, very often it is desirable to know the *equivalent capacitance* of two or more capacitors that are connected in a certain way. By equivalent capacitance is meant the capacitance of a single capacitor that can be substituted for the combination with no change in the operation of the rest of the circuit. With such a replacement, the circuit can be simplified, so