$$= \left(\frac{2\pi}{3}\right)(0.10)\cos 120°$$

$$= -0.105 \text{ m/s}.$$

$$\text{acc ln.} = \frac{d^2y}{dt^2} = -w^2 a \sin wt$$

$$= -\left(\frac{2\pi}{T}\right)^2 (a) \sin\left[\left(\frac{2\pi}{T}\right)(1)\right]$$

$$= -\left(\frac{2\pi}{3}\right)^2 (0.10) \sin 120°$$

$$= -\mathbf{0.38 \ m/s^2}.$$

(iii) E = P. E. + K. E.

$$\frac{1}{2}ka^2 = \frac{1}{2}ky^2 + \frac{1}{2}mv^2$$

$$\left(\frac{1}{2}\right)(0.88)(0.1)^2 = \left(\frac{1}{2}\right)(0.88)(0.0866)^2 + \left(\frac{1}{2}\right)(0.2)(0.105)^2$$

$$4.4 \times 10^{-3} \text{ J} = (3.3 \times 10^{-3} + 1.1 \times 10^{-3}) \text{ J}$$

$$= \mathbf{4.4 \times 10^{-3} \ J}.$$

1.7 Some examples of simple harmonic motion

Some important examples of simple harmonic motion will be examined below.

(i) Motion of a body suspended from a coil spring

Fig. 1.5 shows a coil spring whose upper end is fixed to a rigid support. A mass m is attached to its free end. Let *l* be the no-load length of the spring as shown in (a). When the load m is attached to the spring, it hangs in equilibrium with the spring extended by an amount Δl as in (b). Under this condition the upward force F exerted by the spring is equal to the weight of the body, *mg*. If the spring obey's Hooke's law (Art. 8.6), then the force on the body is given by

$$F = -k.\Delta l$$

where k is the force constant of the spring and is referred to as the *spring constant*. Again, the minus sing simply indicates that the force and displacement are oppositely directed. Since F = m g, ignoring the minus sign, we have

$$k\Delta l = mg\,; \quad \text{or, } k = \frac{mg}{\Delta l} \qquad (1.20)$$

Thus spring constant may be defined as the tension (mg) per unit displacement (Δl).

If the body is now displaced from its equilibrium position and released, it will oscillate along the vertical direction. Suppose the body is at a distance y above its equilibrium position as in (c) Then the extension of the spring is (Δl – y); the upward force it exerts on

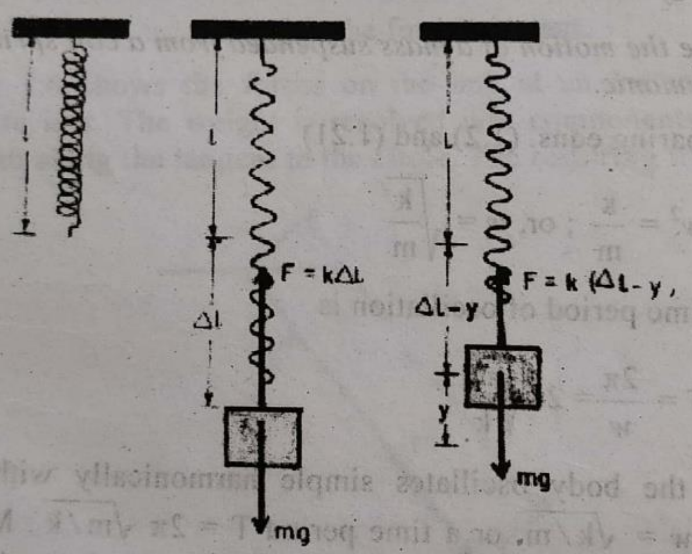

Fig. 1.5

the body is k (Δl – y) and the downward force acting on the body is mg. Hence the resultant force F on the body is

$$F = k\,(\Delta l - y) - mg = -\,ky$$

The resultant force is, therefore, proportional to the displacement of the body from its equilibrium position and is oppositely directed.

According to Newton's second law of motion,

$$F = m\frac{d^2y}{dt^2} = -ky = 0$$

$$\text{or,}\quad m\frac{d^2y}{dt^2} + ky = 0$$

$$\text{or,}\quad m\frac{d^2y}{dt^2} + \left(\frac{k}{m}\right).y = 0 \qquad (1.21)$$

The equation is similar to the differential equation of simple harmonic motion

$$\frac{d^2y}{dt^2} + w^2y = 0$$

Hence the motion of a mass suspended from a coil spring is simple harmonic.

Comparing eqns. (1.2) and (1.21)

$$w^2 = \frac{k}{m}\ ;\ \text{or,}\ w = \sqrt{\frac{k}{m}}$$

The time period of oscillation is

$$T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{m}{k}}$$

Thus the body oscillates simple harmonically with angular frequency $w = \sqrt{k/m}$ or a time period $T = 2\pi\sqrt{m/k}$. Measuring the time period, the spring constant k can be determined. The constant k depends on the shear modulus (n) of the wire, its radius (r), the radius of the coil (R) and the number of turns (N) in the coil and is given by $n = \frac{4NR^3k}{r^4}$.

Alternately, k can also be determined from the relation $k = mg/\Delta l$ where Δl is the increase in length of the spring when a small mass m is attached to its free end (eqn. 1.20). It is to be noted that $mg/\Delta l$ is constant for a given spring provided it obeys Hooke's law.

The utility of the pendulum as a time keeper is based on the fact that the period is practically independent of the amplitude. Thus, as a clock runs down and the amplitude of the swings becomes slightly smaller, the clock will keep very nearly correct time. However, for accurate time keeping, the amplitude of the swing must be kept constant despite the frictional losses that affect all mechanical systems. Even so small a change in amplitude as from 5° to 4° would cause a pendulum clock to run fast by 0.25 minute per day, an unacceptable amount even for household time-keeping. To keep the amplitude constant in a pendulum clock, energy is automatically supplied in small increments from a weight or a spring by an escapement mechanism to compensate for friction losses. The pendulum clock with escapement was invented by Christiaan Huygens.

The simple pendulum was used for early determinations of the value of acceleration due to gravity because both the period and the length are easily measured. Direct measurement by observation of free fall, in contract, is difficult because the time of fall over reasonable distance is too short for easy measurement. From eqn. (1.25), we have, in terms of l and T, $g = \dfrac{4\pi^2 l}{T}$.

(iii) The Torsional oscillator (Torsion Pendulum)

Fig. 1.8 shows a disk suspended by a wire or a shaft. One end of the wire is securely fixed to a solid support or a clamp while the other end is fixed to the centre of mass of the disk. With the disk in equilibrium, a radial line is drawn from its centre O to a point P on its rim. If the disk is rotated in a horizontal plane so that the reference line OP moves to a position OQ, the wire will be twisted. The twisted wire will exert a restoring torque on the disk tending to return the reference line to its equilibrium position. Thus if the disk is given a small twist and released, it will execute angular oscillations about its position of equilibrium. The device is then called a *torsion pendulum*, with torsion referring to the twisting.

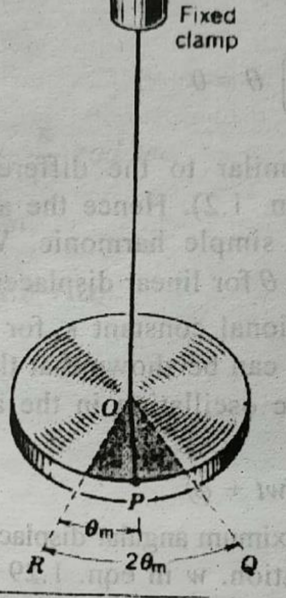

Fig. 1.8

For small twists the restoring torque is found to be proportional to the angular displacement (Hooke's law), so that

$$\tau = -k\theta \tag{1.26}$$

where k is a constant that depends on the properties of the wire and is called the *torsional constant*. The minus sign shows that the torque is directed opposite to the angular displacement θ. Eqn. 1.26 is a condition of *angular simple harmonic motion*. The equation of motion for such a system is based on the angular form of Newton's second law,

$$\tau = I\,\alpha = I\,\frac{d^2\theta}{dt^2}$$

where I is the rotational inertia of the oscillating disk.

Combining eqns. 1.26 and 1.27 we obtain

$$-K\theta = I\frac{d^2\theta}{dt^2}$$

$$\text{or,} \quad \frac{d^2\theta}{dt^2} = -\left(\frac{\kappa}{I}\right)\theta$$

$$\text{or,} \quad \frac{d^2\theta}{dt^2} + \left(\frac{\kappa}{I}\right)\theta = 0 \tag{1.28}$$

Eqn. 1.28 is similar to the differential equation of simple harmonic motion (eqn. 1.2). Hence the angular oscillations of the torsion pendulum is simple harmonic. We can simply substitute angular displacement θ for linear displacement y, rotational inertia I for mass m, and torsional constant κ for force constant k. Making these substitutions, it can be shown that the solution of eqn. 1.28 to be a simple harmonic oscillation in the angle coordinate θ and is given by

$$\theta = \theta_m \sin(wt + \varphi) \tag{1.29}$$

Here θ_m is the maximum angular displacement, that is, the amplitude of the angular oscillation. w in eqn. 1.29 means angular frequency not angular velocity. In eqn. 1.29 $w \neq d\theta / dt$.

In Fig. 1.8 the disk oscillates about the equilibrium position $\theta = 0$ the total angular range being $2\theta_m$ (from OQ to OR). By analogy with eqn. 1.12, the period of oscillation is given by

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

If κ is known and T is measured, the rotational inertia I about the axis of rotation of any oscillating rigid body can be determined. If I is known and T is measured, the torsional constant κ of any sample of wire can be determined.

Example 1.14. *A particle performs simple harmonic motion given by the equation.*

$$y = 20 \sin(wt + \alpha)$$

If the time period is 30 seconds and the particle has a displacement of 10 cm at $t = 0$, find (i) epoch, (ii) the phase angle at $t = 5$ seconds and (iii) the phase difference between two positions of the particle 15 seconds apart.

Soln.

Here

$$y = 20 \sin (wt + \alpha)$$

$$T = 30 \text{ secs.}$$

$$\therefore \quad w = \frac{2\pi}{T} = \frac{2\pi}{30} = \frac{\pi}{15} \; rad/sec.$$

(i) at $t = 0$, $y = 10$ cm.

$$\therefore \quad 10 = 20 \sin \left(\frac{\pi}{15} \times 0 + \alpha\right)$$

$$\text{or,} \quad \sin \alpha = \frac{10}{20} = 0.5$$

$$\text{or,} \quad \alpha = \frac{\pi}{6} \textbf{ radian.}$$

(ii) at $t = 5$ sec,

the phase angle $= (wt + \alpha)$

$$= \left(\frac{\pi}{15} \times 5 + \frac{\pi}{6}\right)$$

$$= \frac{\pi}{2}$$

(iii) at $t = 0$ the phase angle

$$\theta_1 = \left(\frac{\pi}{15} \times 0 + \frac{\pi}{6}\right) = \frac{\pi}{6} \text{ radian.}$$

at $t = 15$ *sec*, the phase angle

$$\theta_2 = \left(\frac{\pi}{15} \times 15 + \frac{\pi}{6}\right)$$

$$= \frac{7\pi}{6} \text{ radian.}$$

$\therefore$ the phase difference,

$$\theta_2 - \theta_1 = \frac{7\pi}{6} - \frac{\pi}{6} = \pi \textbf{ radian.}$$

Example. 1.15. *A vertically suspended spring of negligible mass and force constant R is stretched by an amount l when a body of mass m is hung on it. The body is pulled by hand an additional distance y (positive direction downward) and then released.*

(a) Show that the motion of the body is governed by $a' = -\frac{k}{m}y$, *so that the body executes harmonic motion about its equilibrium position, and*

(b) show that the period of this motion is the same as that of a simple pendulum of length l.

Soln.

When the mass m is attached to the spring it hangs in equilibrium with the spring extended by an amount l. From Hooke's law, the upward force exerted by the spring on the body is

$F = kl$ where k is the force constant of the spring.

Under the condition of equilibrium we therefore have $mg = kl$ where mg is the downward force on the spring.

When the spring is stretched by an additional length y and released, the net force acting on the spring is

$$mg - k(l + y)$$

According to Newton's second law of motion,

$$F = ma' = mg - k(l + y)$$

$$= mg - kl - ky = -ky \qquad (mg = kl)$$

$$\therefore \quad a' = -\frac{k}{m}y.$$

(b) The period of oscillation of the spring is given by

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/l}} \qquad (\text{from } mg = kl)$$

$$= 2\pi\sqrt{\frac{l}{g}}$$

The above expression is the period of oscillation of a simple pendulum of length l placed in uniform gravitational field g.

Example 1.16. *A thin rod of length L = 12.4 cm and mass m = 135 g is suspended at its midpoint from a long wire. Its period T_a of angular simple harmonic motion is measured to be 2.53 s. An irregular object X is then hung from the same wire and its period T_b of the angular SHM is found to be 4.76 s.*

(i) What is the rotational inertia of object X about its suspension axis?

The rotational inertia of the thin rod about a perpendicular axis through its midpoint is given by $\frac{1}{12}$ m L^2. Thus we have

$$I_a = \frac{1}{12} mL^2 = \left(\frac{1}{12}\right)(0.135\ \text{kg})\,(0.124\ \text{m})^2$$

$$= 1.73 \times 10^{-4}\ \text{kg.m}^2$$

Now

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}$$

The torsion constant κ, which is a property of the wire, is the same for both the objects. Only the periods and rotational inertias are different.

$$\frac{T_b}{T_a} = \sqrt{\frac{I_b}{I_a}} \quad \text{or,} \quad \frac{T_b^2}{T_a^2} = \frac{I_b}{I_a}$$

$$\text{or,}\ I_b = \frac{T_b^2}{T_a^2}.I_a = \frac{(4.76\ \text{s})^2}{(2.53\ \text{s})^2}\left(1.73 \times 10^{-4}\ \text{kg.m}^2\right)$$

$$= \mathbf{6.12 \times 10^{-4}\ kg.m^2}$$

(ii) *If both the objects are fastened together and hung from the same wire, what would be the period of oscillation ?*

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_C = 2\pi\sqrt{\frac{I_C}{\kappa}}$$

or, $= \frac{T_C}{T_a} = \sqrt{\frac{I_C}{I_a}}$

or, $T_C = T_a \sqrt{\frac{I_C}{I_a}}$

Now $I_c = I_a + I_b$

$\therefore \quad T_c = T_a\sqrt{\frac{I_c}{I_a}} = T_a\sqrt{\frac{I_a + I_b}{I_a}}$

$= Ta\sqrt{1 + \frac{I_b}{I_a}} = (2.53)\sqrt{1 + \frac{6.12 \times 10^{-4}\,Kg.m^2}{1.73 \times 10^{-4}\,Kg.m^2}}$

$= 5.37$ s.

(iv) LC circuit

Just as harmonic oscillations in mechanical system, so also we come across harmonic oscillations in electrical systems. Under suitable conditions, charge, current or voltage can execute simple harmonic motion. As an example let us consider the case of an *LC* circuit.

In an LC circuit, a capacitor of capacitance C and an inductance coil of inductance L (of negligible ohmic resistance) are connected with a battery through a Morse key K in the manner as shown in Fig. 1.9. When the key is pressed, the capacitor gets directly connected to the battery and thus gets charged. On being released, the key gets disconnected to the inductance coil. The capacitance thus discharges itself through the inductance coil.

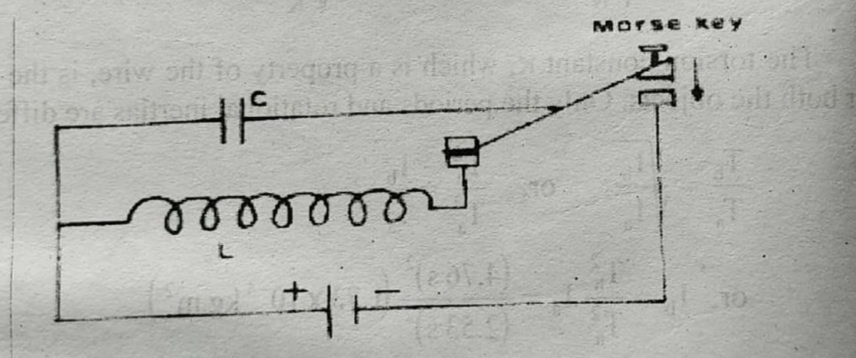

Fig. 1.9

An inductive circuit, *i.e.*, a circuit containing an inductance coil, opposes both growth and decay of current in the circuit. Thus as the capacitor discharges itself through the inductance coil and the current in the latter grows, the magnetic flux due to the current increases. This increasing magnetic flux induces an electro-motive force (*emf*) in the circuit which opposes the growth of current in it. Thus, if i be the instantaneous value of the current in the coil (or the circuit) at any given instant, then the opposing *emf* set up across the coil is $-L\frac{di}{dt}$ where $\frac{di}{dt}$ is the rate of change of current through the coil or the circuit.

If Q be the charge on the capacitor at the instant considered, then the voltage across it tending to drive the current through the circuit (or the coil) is $\frac{Q}{C}$. Since there is no external *emf* in the circuit (the battery being cut off), the net *emf* in the circuit is

$$\frac{Q}{C}+L\frac{di}{dt}=0 \qquad (1.27)$$

But current is the rate of flow of charge ;

hence $$i=\frac{dQ}{dt}$$

Eqn. (1.27) thus takes the form

$$\frac{Q}{C}+L\frac{d^2Q}{dt^2}=0$$

$$\text{or, } \frac{d^2Q}{dt^2}+\frac{Q}{LC}=0$$

$$\text{or, } \frac{d^2Q}{dt^2}+w^2Q=0 \qquad (1.28)$$

where $w^2 = \frac{1}{LC}$

or, $\frac{1}{\sqrt{LC}}$ *is a constant t.*

Eqn. (1.28) is similar to eqn. (1.2), the differential equation of simple harmonic motion with y replaced by Q, mass (m) by inductance L and the force constant by the reciprocal of the capacitance ($\frac{1}{C}$).

Thus the charge on the capacitor oscillates simple harmonically with time, *i.e.*, the discharge of the capacitor is oscillatory in nature, its time period T is given by

$$T = \frac{2\pi}{w} = 2\pi\sqrt{LC}$$

and, therefore, its frequency

$$n = \frac{1}{T} = \frac{1}{2\pi\sqrt{LC}} \qquad (1.29)$$

The solution of eqn. (1.28) is given by

$$Q = Q_o \sin(wt + \phi) \qquad (1.30)$$

where Q_o is the maximum value or amplitude of the charge and $w = \frac{1}{\sqrt{LC}}$, the angular frequency (of the variation of charge) and ϕ, the phase constant which depends, as usual, on the initial conditions.

The charge in the circuit, therefore, oscillates between $+Q_o$ and $-Q_o$ with a frequency $n = \frac{1}{2\pi\sqrt{LC}}$

Differentiating eqn. (1.29) with respect to time, we have

$$i = \frac{dQ}{dt} = Q_0\, w \cos(wt + \phi)$$

where $i = \frac{dQ}{dt}$ is the instantaneous value of the current, $Q_0 w$ is the maximum value or the amplitude of current when

$$\cos(wt + \phi) = 1.$$

Denoting this maximum value by i_0, we get

$$i = i_0 \cos(wt + \phi) \qquad (1.30)$$

Thus current in the circuit is also oscillatory in character and has the same frequency as the charge, *viz.* $n = \frac{1}{2\pi\sqrt{LC}}$.

1.8 Simple harmonic motion and uniform circular motion

Let us consider a particle P moving along the circumference of a circle with a constant angular speed w radians per seconds. Q is the perpendicular projection of P on the horizontal diameter *i.e.*, frequency but differ in phase by 90°. *When one component is at the maximum displacement, the other component is at the equilibrium point. Combining eqns. (1.33) and (1.34) we at once obtain the relation*

$$r = \sqrt{x^2 + y^2} = a$$

The point P is called the reference point and the circle on which it moves is called the reference circle. The angular frequency w of simple harmonic motion is the same as the angular speed of the reference point. The frequency of simple harmonic motion is the number of revolutions per unit time of the reference point. Hence, $n = \dfrac{w}{2\pi}$ or, $w = 2\pi n$. The time for a complete revolution of the reference point is the same as the period T of the simple harmonic motion. Hence, $T = \dfrac{2\pi}{w}$ or, $w = \dfrac{2\pi}{T}$. $(wt + \delta)$ in eqn. (1.33) or (1.34) is called the phase of the simple harmonic motion while δ is called the epoch or initial phase of the motion. The amplitude of the simple harmonic motion is the same as the radius of the reference circle.

Two-body oscillations

Many objects on the microscopic level, such as molecules, atoms, nuclei, execute oscillations that are approximately simple harmonic. One example is a diatomic molecule, in which the two atoms are bound together with a force. If displaced a small distance from its equilibrium position, the molecule will oscillate about the equilibrium position. Let us suppose that the molecule can be represented by two particles of masses m_1 and m_2 connected by a spring of force constant k, as shown in Fig. 1.11.

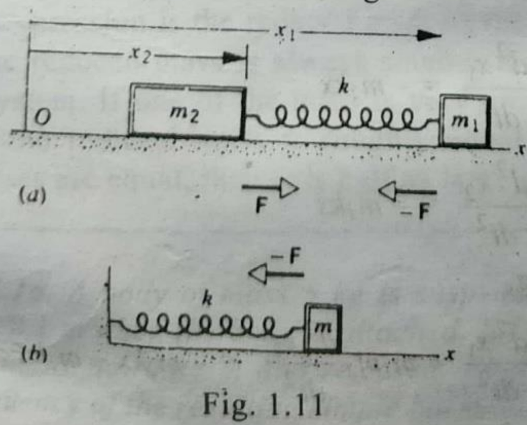

Fig. 1.11

The motion of the system can be described in terms of the separate motions of the two particles, which are located relative to the origin O by the two coordinates x_1 and x_2, as shown in Fig. 1.11 a. As we shall see, this leads to a different and often more useful description, which is given in terms of the *relative* separation and velocity of the two particles. In effect, the two coordinates x_1 and x_2 are replaced with two other coordinates: the relative separation x_1-x_2 and the location x_{cm} of the centre of mass. But, in the absence of external forces, the centre of mass moves at constant velocity and as such, its motion is of no real interest in studying the oscillation of the system. So the system can be analyzed in terms of the relative coordinate alone.

The relative separation $x_1 - x_2$ gives the length of the spring at any time. If the unstretched length of the spring is L, then the change in length of the spring is given by $x = (x_1 - x_2) - L$. The magnitude of the force that the spring exerts on *each particle* is $F = Kx$. As can be seen in Fig. 1.11 a, if the spring exerts a force $-\boldsymbol{F}$ on m_1, then it exerts a force $+\boldsymbol{F}$ on m_2.

Taking the force component along the x axis, let us apply Newton's second law separately to the two particles,

$$m_1 \frac{d^2 x_1}{dt^2} = -kx$$

$$m_2 \frac{d^2 x_2}{dt^2} = +kx$$

Multiplying the first of these equations by m_2 and the second by m_1, we get

$$m_1 m_2 \frac{d^2 x_1}{dt^2} = -m_2 kx$$

$$m_1 m_2 \frac{d^2 x_2}{dt^2} = -m_1 kx$$

Subtracting,

$$m_1 m_2 \frac{d^2 x_1}{dt^2} - m_1 m_2 \frac{d^2 x_2}{dt^2} = -m_2 kx - m_1 kx$$

$$= -kx\,(m_1 - m_2)$$

$$\text{or,}\quad \frac{m_1 m_2}{m_1 + m_2}\frac{d^2}{dt^2}(x_1 - x_2) = -kx \qquad (1.35)$$

The quantity $m_1m_2/(m_1 + m_2)$ has the dimension of mass and is known as the *reduced mass m* of the system.

The unstretched length L of the spring is a constant. Hence the derivatives of $(x_1 - x_2)$ are the same as the derivatives of x.

$$\frac{d}{dt}(x_1 - x_2) = \frac{d}{dt}(x + L) = \frac{dx}{dt}$$

Eqn. 1.35 therefore becomes

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$\text{or,}\quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad (1.36)$$

Eqn. 1.36 is identical to the differential equation of simple harmonic motion of a single oscillating mass. Hence, from the standpoint of oscillation, the system of Fig. 1.11 a can be replaced by a single particle, as represented by Fig. 1.11, whose mass is given by the reduced mass of the system. In particular, the frequency of oscillation of the system is given by

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

where m in the expression is the reduced mass of the system. It may be noted that the reduced mass is always smaller than either of the masses of the system. If one of the mass is very much smaller than the other, then the reduced mass is roughly equal to the smaller mass. If the masses are equal, then m is half as large as either mass.

Example 1.16. *A body of mass 5 kg is suspended by a spring, which stretches 0.1 m when the body is attached. It is then displaced downward an additional 0.05 m and released. Find the amplitude, period and frequency of the resulting simple harmonic motion.*

Soln.

Since the initial position is 0.05 m from equilibrium and there is no initial velocity, the amplitude.

$a = 0.05$ m.

A force of (5 kg) (9.8 m/sec^2) produces a displacement of 0.1 m. Hence from F = ky,

the force constant $k = \frac{F}{y} = \frac{mg}{y}$

$$= \frac{(5\,kg)(9.8 m/sec^2)}{0.1\,m}$$

= 490 N/m.

the time period,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{5\,kg}{490\,N/m}}$$

= 0.635 sec.

the frequency

$$n = \frac{1}{T} = \frac{1}{0.635} = \mathbf{1.57\ Hz.}$$

Example 1.17. *The scale of a spring balance reading from 0 to 10 kg is 0.25 m long. A body suspended from the balance is found to oscillate vertically with a frequency of 10/π hertz. Calculate the mass of the body attached to the spring.*

Soln.

Clearly, a mass of 10 kg suspended from the spring balance extends the spring by 0.25 m. Hence from F = ky, the force constant

$$k = \frac{F}{y} = \frac{10 \times 9.8}{0.25}\ N/m.$$

From

$T = 2\pi\sqrt{\frac{m}{k}}$ we have

$$T = \frac{1}{n} = 2\pi\sqrt{\frac{m}{k}} \qquad k = \frac{10 \times 9.8}{0.25}\ N/m.$$

$$or,\ \frac{1}{n^2} = 4\pi^2 . \frac{m}{k} \qquad n = \frac{10}{\pi}\ Hz.$$

$$or,\ m = \frac{k}{4\pi^2 . n^2} \qquad m = ?$$

$$\therefore \quad m = \frac{10 \times 9.8}{0.25 \times 4 \times (3.14)^2 \times \frac{100}{(3.14)^2}}$$

$$= \frac{98}{100}$$

$$= 0.98\, kg.$$

Example 1.18. *A body of mass 0.5 kg is suspended from a spring of negligible mass and it stretches the spring by 0.07 m. For a displacement of 0.03 m it has a downward velocity of 0.4 m/sec. Calculate (i) the time period, (ii) the frequency and (iii) the amplitude of vibration of the spring.*

Soln.

From F = ky, the force constant of the spring

$$k = \frac{F}{y} = \frac{0.5 x 9.8}{0.07}$$

(i) **the time period,**

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad m = 0.5\ kg$$

$$k = \frac{0.5 \times 9.8}{0.07}$$

$$\therefore \quad T = 2 \times 3.14 \times \sqrt{\frac{0.5 \times 0.07}{0.5 \times 9.8}} = \mathbf{0.5307\ sec.}$$

(ii) the frequency,

$$n = \frac{1}{T} = \frac{1}{0.5309} = \mathbf{1.8843\ Hz}$$

the angular frequency, $w = 2\pi n$

$$= 2 \times 3.14 \times 1.8843$$

$$= \mathbf{11.833\ rad/sec.}$$

(iii) For the amplitude, we have

$$v = w\sqrt{a^2 - y^2}$$

or, $0.4 = 11.383\sqrt{a^2 - (0.03)^2}$

or, $(0.4)^2 = (11.383)^2\left[a^2 - (0.03)^2\right]$

or, $\mathbf{a = 0.04526\,m.}$

Example 1.19. *A simple pendulum of length 100 cm has an energy equal to 2 x 10⁶ ergs when its amplitude is 4 cm. Calculate its energy when (i) its length is doubled and (ii) its amplitude is doubled.*

Soln.

The energy of a simple pendulum

= its maximum P.E.

= its maximum K.E.

$$= \frac{1}{2}\ mw^2a^2$$

Now $w = \dfrac{2\pi}{T}$ where T is the time period of the pendulum $= 2\pi.\sqrt{\dfrac{l}{g}}$

or, $w = 2\pi.\dfrac{1}{2\pi}.\sqrt{\dfrac{g}{l}} = \sqrt{\dfrac{g}{l}}$

Initially $l = 100$ cm. $\therefore\ w = \sqrt{g/100}.$

So, initially with a = 4 cm, the energy of the pendulum

$$E = \frac{1}{2}\ m.\left(\sqrt{g/100}\right)^2.\ (4)^2 = 8mg/100$$

$$= 2 \times 10^6 \text{ ergs.}$$

(i) If the length of the pendulum is doubled, $l = 200$ cm. and $w = \sqrt{g/200}$. Hence energy of the pendulum

$$E' = \frac{1}{2} . m . \left(\sqrt{g/200}\right)^2 . (4)^2$$

$$= \mathbf{8mg/200}$$

$$\therefore \quad \frac{E'}{E} = \frac{8\ mg}{200} \Big/ \frac{8\ mg}{100} = \frac{1}{2}$$

$$\text{or,} \quad E' = \frac{1}{2} . E = \frac{1}{2} \times 2 \times 10^6 \text{ ergs}$$

$$= \mathbf{10^6\ ergs.}$$

(ii) If the amplitude is doubled, a = 8 cm, its length remaining 100 cm. Hence energy of the pendulum

$$E'' = \frac{1}{2} . m . \left(\sqrt{g/100}\right)^2 . (8)^2$$

$$= 32\ mg / 100$$

$$\therefore \quad \frac{E''}{E} = \frac{32\ mg}{100} \Big/ \frac{8\ mg}{100} = 4$$

$$\text{or,} \quad E'' = 4.E = 4 \times 2 \times 10^6 \text{ ergs}$$

$$= 8 \times 10^6 \text{ ergs.}$$

Example 1.20. *A small body of mass 0.10 kg is undergoing simple harmonic motion of amplitude 1.0 metre and period 0.20 sec. (i) what is the maximum value of the force acting on it? (ii) if the oscillations are produced by a spring, what is the force constant of the spring?*

Soln.

(ii) From $T = 2\pi \sqrt{\frac{m}{k}}$, we have

$$T^2 = 4\pi^2 . \frac{m}{k}$$

T = 0.20 sec

m = 0.10 kg.

$$\therefore \quad (0.2)^2 = 4 \times (3.14)^2 \times \frac{0.10}{k}$$

$$\text{or,} \quad k = \frac{4 \times 9.86 \times 0.10}{0.04}$$

$$= \frac{3.944}{0.04} \simeq \mathbf{99\ nt/m.}$$

∴ force constant of the spring

$\simeq$ **99 nt/m.**

(i) force acting on the body is given by

F = ky.

Obviously F is maximum when y is maximum. The maximum value of y is its amplitude *i.e.*, 1.0 metre.

∴ maximum force = k.y = 99 x 1

= **99 nt.**

Example 1.21. *An oscillating mass-spring system has a mechanical (total) energy of 1.0 joule, amplitude of 0.10 metre and maximum speed of 1.0 m/sec. Find (i) the force constant of the spring, (ii) the mass, and (iii) the frequency of oscillation.*

(iii) $v_{max} = w.a$

or, $1.0 = w \times 0.1$

$$\text{or,} \quad w = \frac{1.0}{0.1} = 10$$

Now $w = 2\pi n$ where n is the frequency of oscillation

∴ $10 = 2 \times 3.14 \times n$

$$\text{or,} \quad n = \frac{10}{6.28}$$

= **1.6 cycle/sec.**

(ii) Mechanical (or total energy),

$2\pi^2 n^2 a^2 m = 1.0$ joule.

$2 \times 9.8 \times (1.6)^2 \times (0.1)^2 \times m$

$= \mathbf{1.0\ joule.}$

or, $m = \dfrac{1.0}{0.5}$ kg = **2 kg.**

(i) $T = \dfrac{1}{n} = \dfrac{1}{1.6} = 0.625$ sec.

But $T = 2\pi\sqrt{\dfrac{m}{k}}$

or, $T^2 = 4\pi^2 \dfrac{m}{k}$

or, $(0.625)^2 = \dfrac{4 \times 9.8 \times 2.0}{k}$

or, $k = \dfrac{78.4}{0.30} \simeq$ **200 nt/m.**

EXERCISES

[1] Define simple harmonic motion and discuss its characteristics.

[2] Establish the differential equation of simple harmonic motion and solve it to obtain an expression for the displacement of a particle executing simple harmonic motion.

[3] Show that for a body vibrating simple harmonically the time period is given by

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

[4] Establish the differential equation of simple harmonic motion and show that the time period is equal to 2π times the square root of displacement per unit acceleration.

[5] Define simple harmonic motion. Prove that the motion of a simple pendulum is simple harmonic. Hence obtain an expression for the time period of a simple pendulum.

[6] Solve the differential equation $\dfrac{d^2y}{dt^2} + w^2y = 0$ to obtain the expression $y = a \sin(wt + \phi)$ for the displacement of a particle executing simple harmonic motion.