United International University

PHY 101

Waves and Oscillation,

Optics and Modern Physics

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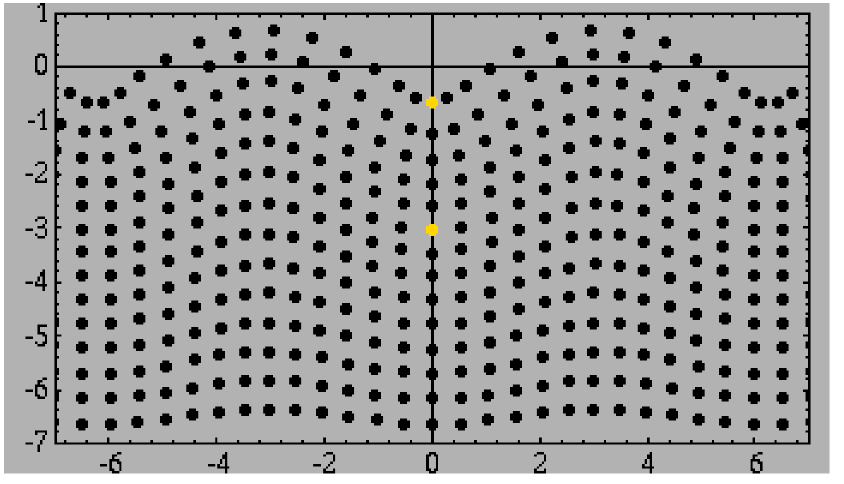
Institute of Natural Sciences



Part 1

Simple harmonic oscillators

Damped oscillators

Driven oscillators

Resonance

The Wave Equation

Part II

Theories of Light
Interference of Light
Diffraction of Light
Polarization of Light

Part III

Relativity
Particle Properties of Wave
Wave Properties of Particles
Atomic Physics
Nuclear Physics

What to do in this course:

1. Read the relevant sections in the textbook
   … the course notes will guide you.

2. Do all homework and problem sets.

3. Get help early … from Course Teacher.

4. there are no shortcuts … put effort in to understand things …

Problem-solving and homework

Each Chapter you will be given a take-home problem set to complete and hand in for marks ... In addition to this, you need to work through the problems in *Book, in you own time, at home.*

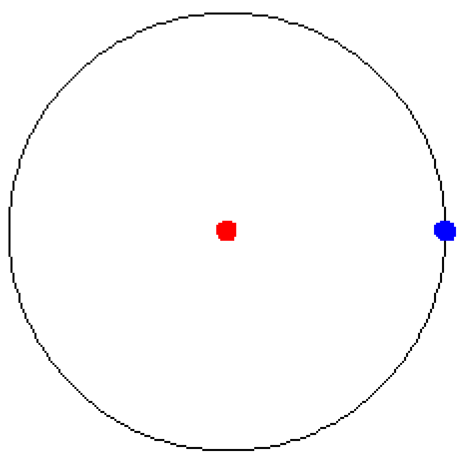
You will not be asked to hand these in for marks. Get help from you friends, the course Teacher ...

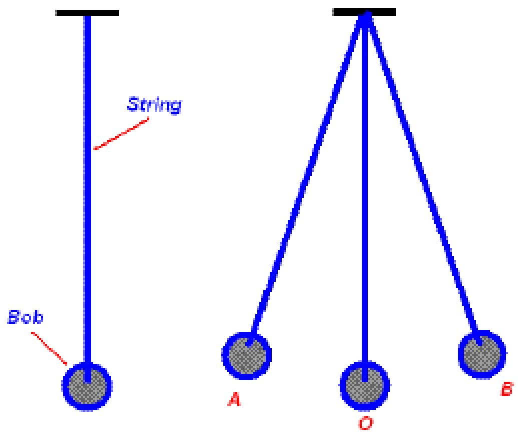
Do not take shortcuts. Mastering these problems is a fundamental aspect of this course.

Different Kind of Motions

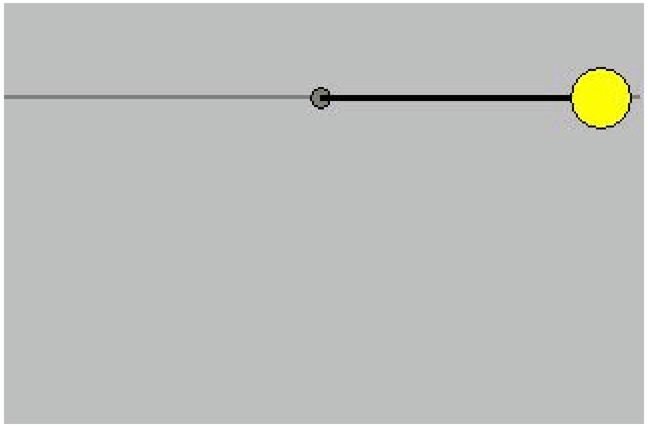
Linear Motion

Uniform Circular Motion





Simple Pendulum



Oscillatory Motion (Simple Pendulum)

Oscillatory Motion (Spring Mass)

Oscillatory Phenomena

… observed in many physical systems …
from the very small…(e.g. dipole resonance in nuclei)… to the very large (earthquake waves, stars,…)

Mechanical systems …… to lasers
…… from violin strings …… to electrical systems

Oscillatory Motion

Motion which is periodic in time, that is, motion that repeats itself in time.

Examples:

- Power line oscillates when the wind blows past it
- Earthquake oscillations move buildings

- Block attached to a spring
- Motion of a swing
- Motion of a pendulum
- Vibrations of a stringed musical instrument
- Motion of a cantilever
- Oscillations of houses, bridges, …
- All clocks use simple harmonic motion

Periodic Motion: Many kind of motion repeat over and over, such as, the vibrations of quartz crystal in a watch, swinging pendulum in a clock and back-and-forth motion of a piston in an engine. This kind of motions are called periodic motion.

Amplitude: The amplitude of the motion, denoted by A, is the maximum magnitude of displacement from the equilibrium position. It is always positive

Period: The period T, is the time required for one oscillation.

Frequency: The frequency, f, is the number of cycles in a unit time.

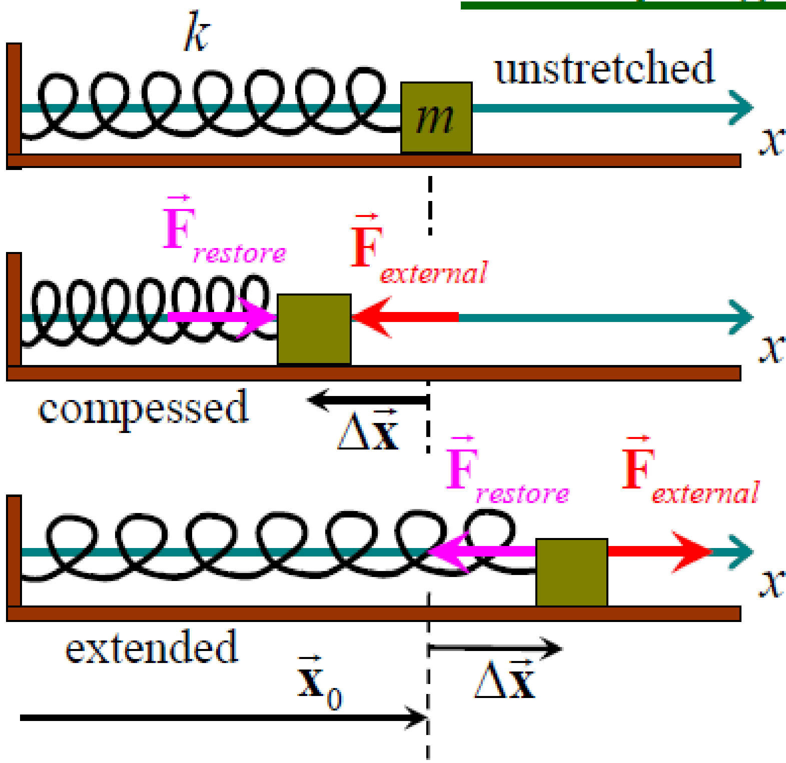
Simple Harmonic Motion: The simplest kind of oscillation occurs when the restoring force F_x is directly proportional to the displacement from the equilibrium x.

When the restoring force is directly proportional to the displacement from the equilibrium as given by equation

$$F_x = -kx$$

The oscillation is called Simple Harmonic Motion(SHM).

Mass-spring oscillator



Hooke's Law:

Restoring force,

$$\vec{\mathbf{F}}_{restore} = -k\Delta\vec{\mathbf{x}}$$

where $\Delta\vec{\mathbf{x}} = \vec{\mathbf{x}} - \vec{\mathbf{x}}_0$

and k is the "spring constant" [N m^{-1}]

Start with the momentum principle: $\dfrac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{F}}_{net}$

For horizontal forces on the mass: $\dfrac{dp_x}{dt} = -kx$

$$\therefore \frac{d(mv_x)}{dt} = -kx \quad \text{or} \quad \frac{d}{dt}\left(m\frac{dx}{dt}\right) = -kx$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Mass-spring oscillator ...2

$$\frac{d^2x(t)}{dt^2} = \frac{-k}{m}x(t)$$

... a second order differential equation
... we know that if we displace a mass-spring system from its rest position and then release it, it will perform SHM ...

Guess a trial solution: $x(t) = A\cos(\omega t + \phi)$

then $$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi)$$

and substitute into our DE: $-A\omega^2\cos(\omega t + \phi) = -A\dfrac{k}{m}\cos(\omega t + \phi)$

... which is true provided $\omega^2 = \dfrac{k}{m}$

Therefore our solution is $x(t) = A\cos(\omega t + \phi)$ where $\omega = \sqrt{\dfrac{k}{m}}$

Mass-spring oscillator ...3

We will write a particular value of ω as ω_0, as the **natural angular frequency** of the oscillator – the frequency that it "wants" to oscillate at.

Mass-spring system: $\omega_0 = \sqrt{\frac{k}{m}}$ and $T = 2\pi\sqrt{\frac{m}{k}}$

System parameters: $m, k \longrightarrow \omega_0$

Initial conditions: $\longrightarrow A, \phi$

(Note that ω_0 is independent of A)

Simple harmonic oscillator

$$x(t) = A\cos(\omega_0 t + \phi)$$

$$v(t) = \frac{dx(t)}{dt} = -A\omega_0 \sin(\omega_0 t + \phi)$$

$$a(t) = \frac{d^2x(t)}{dt^2} = \frac{dv(t)}{dt} = -A\omega_0{}^2 \cos(\omega_0 t + \phi)$$

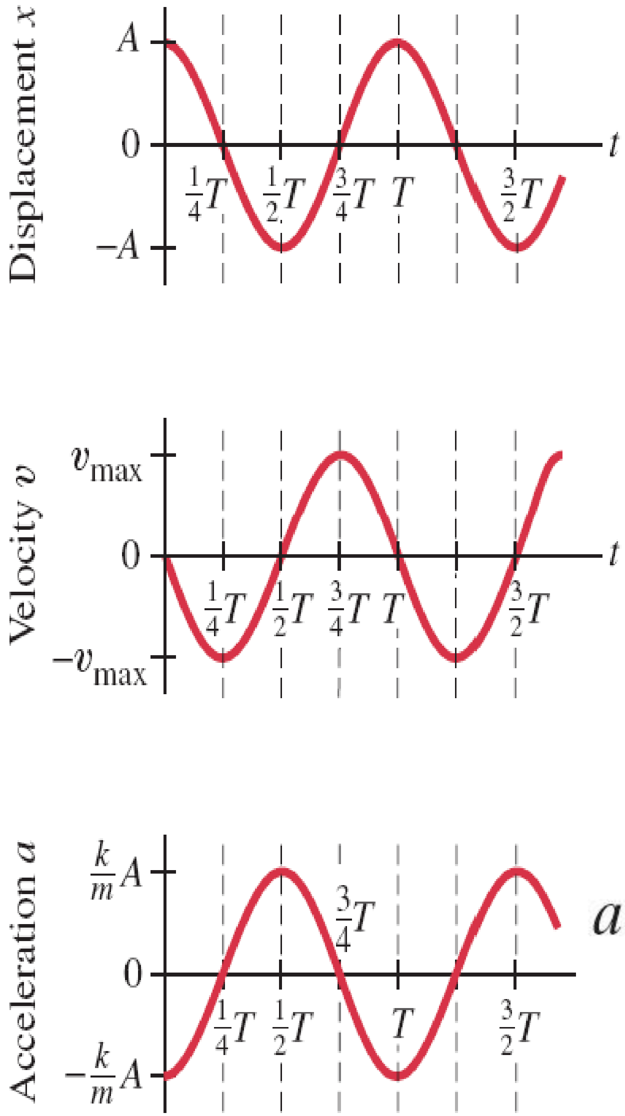
… acceleration = – (constant) . (displacement)

$$= -A\omega_0^2 \cos(\omega_0 t + \phi)$$

$$= A\omega_0^2 \cos(\omega_0 t + \phi + \pi)$$

Phase difference between acceleration and displacement is π

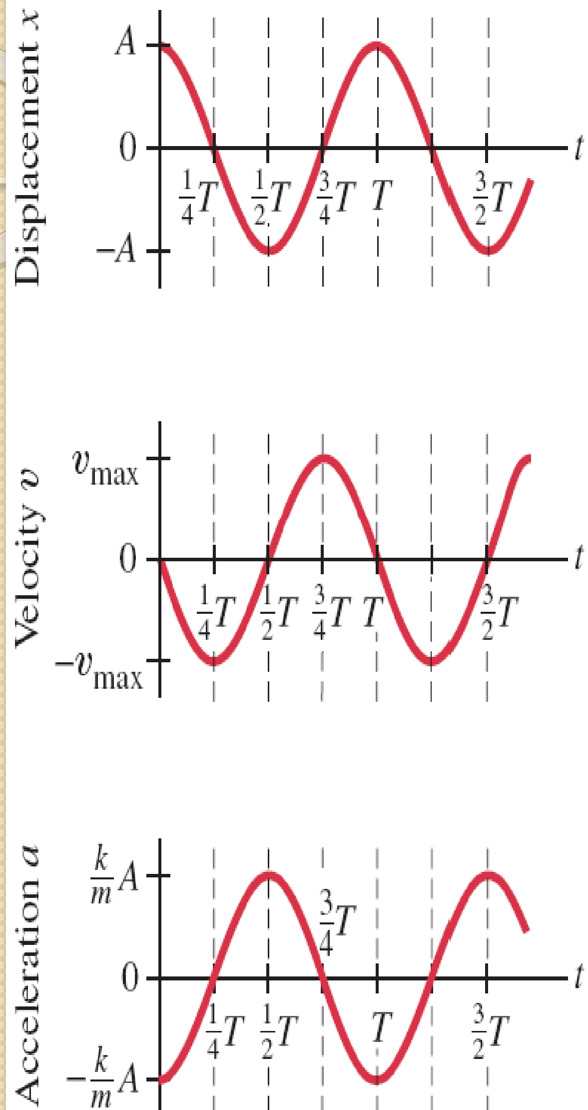
Phase difference between v and x (and v & a) is $\frac{\pi}{2}$



The velocity and acceleration for simple harmonic motion can be found by differentiating the displacement:

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi).$$



$$x = A\cos(\omega t + \phi),$$

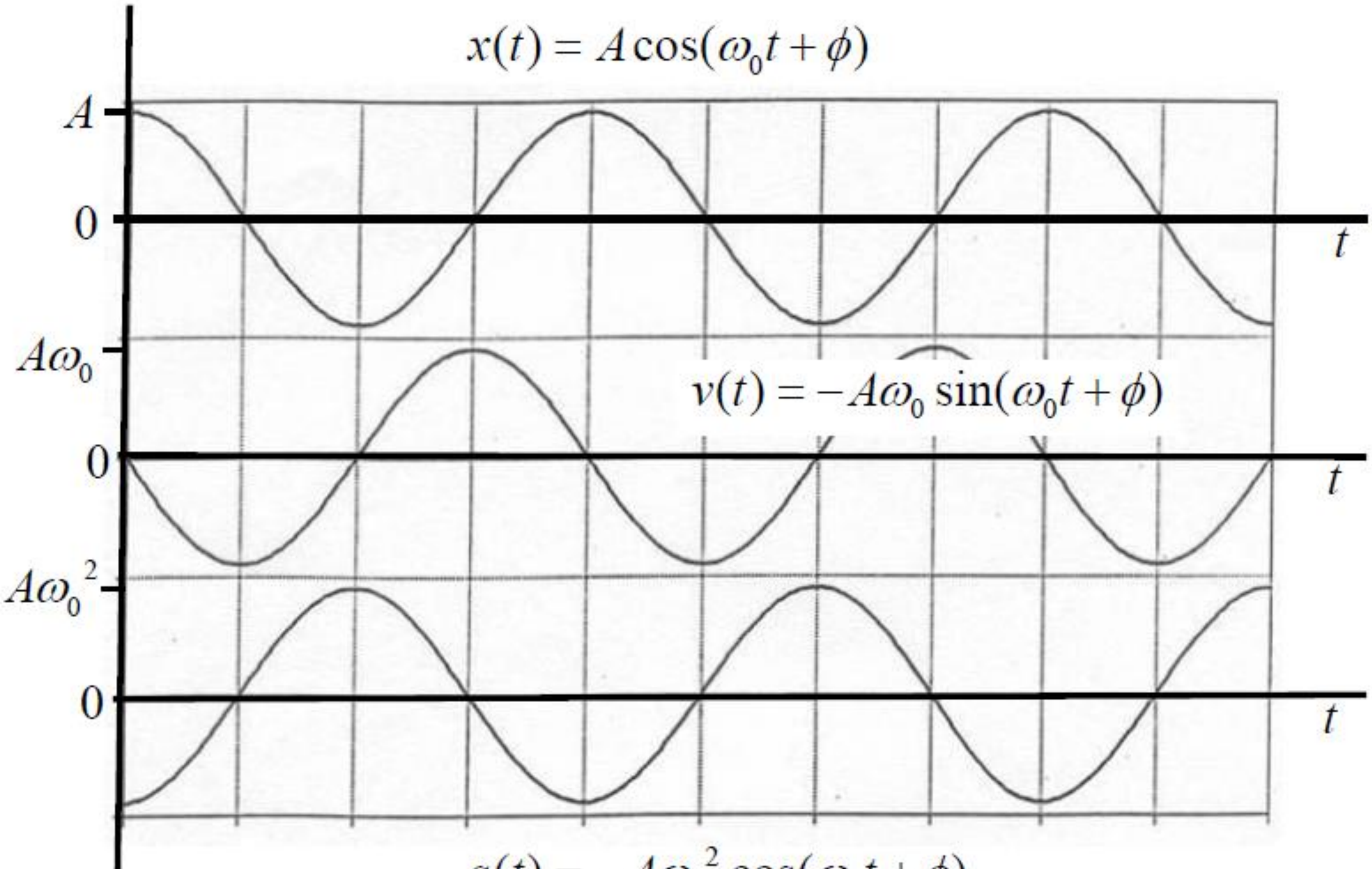
$$v = -\omega A\sin(\omega t + \phi)$$

$$= \omega A\cos(\omega t + \phi + \frac{\pi}{2}),$$

$$a = -\omega^2 A\cos(\omega t + \phi)$$

$$= -\omega^2 A\sin(\omega t + \phi + \frac{\pi}{2}).$$

$$x(t) = A\cos(\omega_0 t + \phi)$$

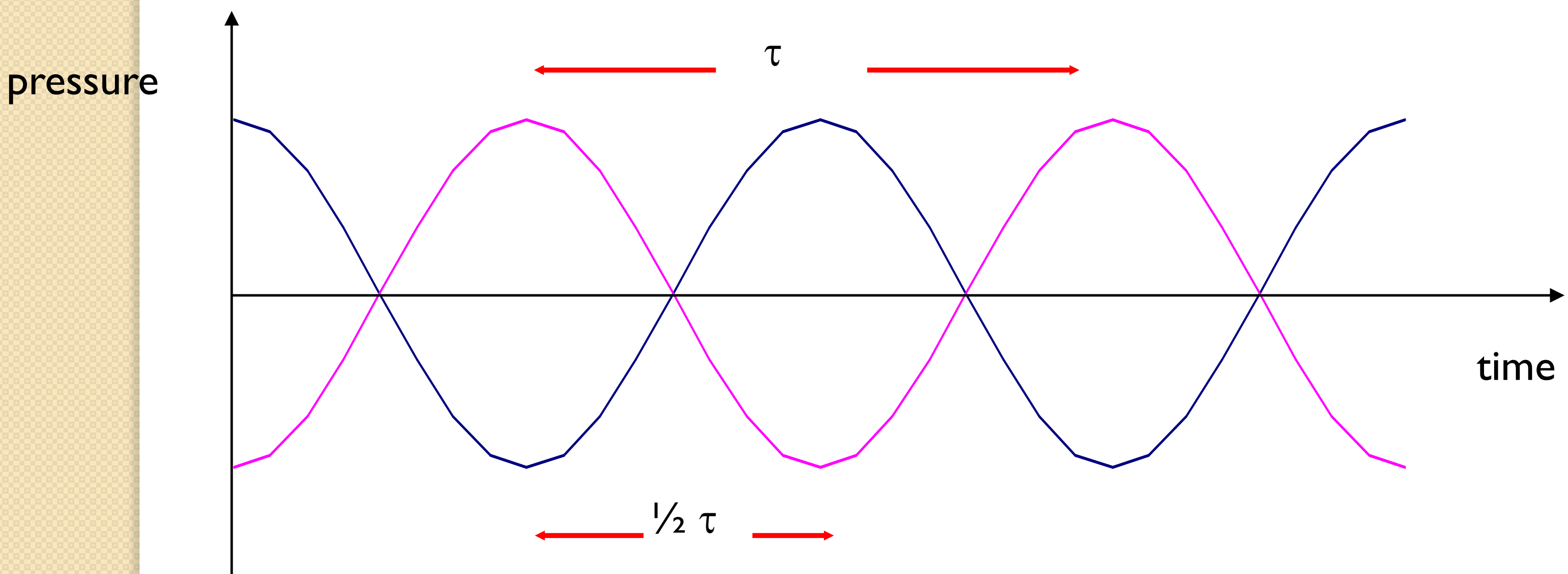


$$a(t) = -A\omega_0^2 \cos(\omega_0 t + \phi)$$

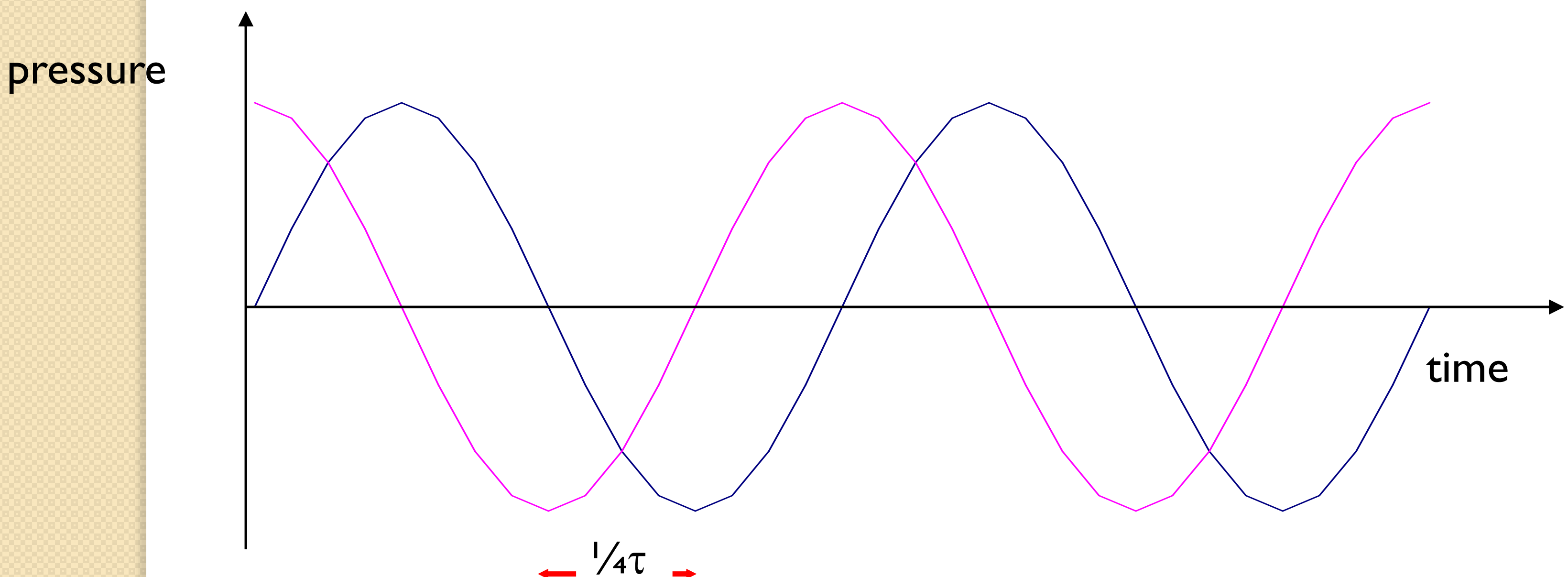
Phase Difference

- The phase of periodic wave describes where the wave is in its cycle
- Phase difference is used to describe the phase position of one wave relative to another

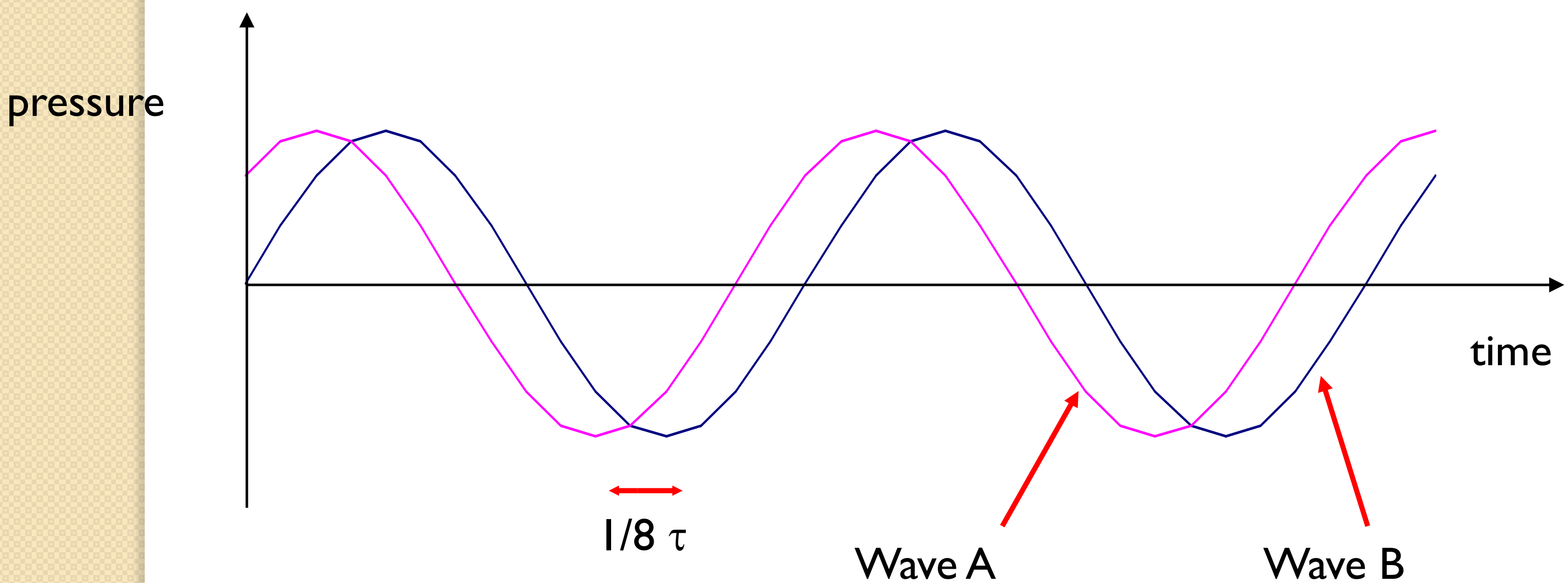
Phase Difference 180°



Phase Difference 90°



Phase Difference 45°



Simple harmonic oscillator

At $t = 0$, write $x = x_0$ and $v = v_0$.

Then at $t = 0$:

$$\left.\begin{aligned} x_0 &= A\cos(\phi) \\ v_0 &= -\omega_0 A\sin(\phi) \end{aligned}\right\} \quad \tan\phi = -\frac{v_0}{\omega_0 x_0}$$

… and $\quad x_0{}^2 + \left(\dfrac{v_0}{\omega_0}\right)^2 = A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$

$$\therefore A = \sqrt{x_0{}^2 + \left(\frac{v_0}{\omega_0}\right)^2}$$

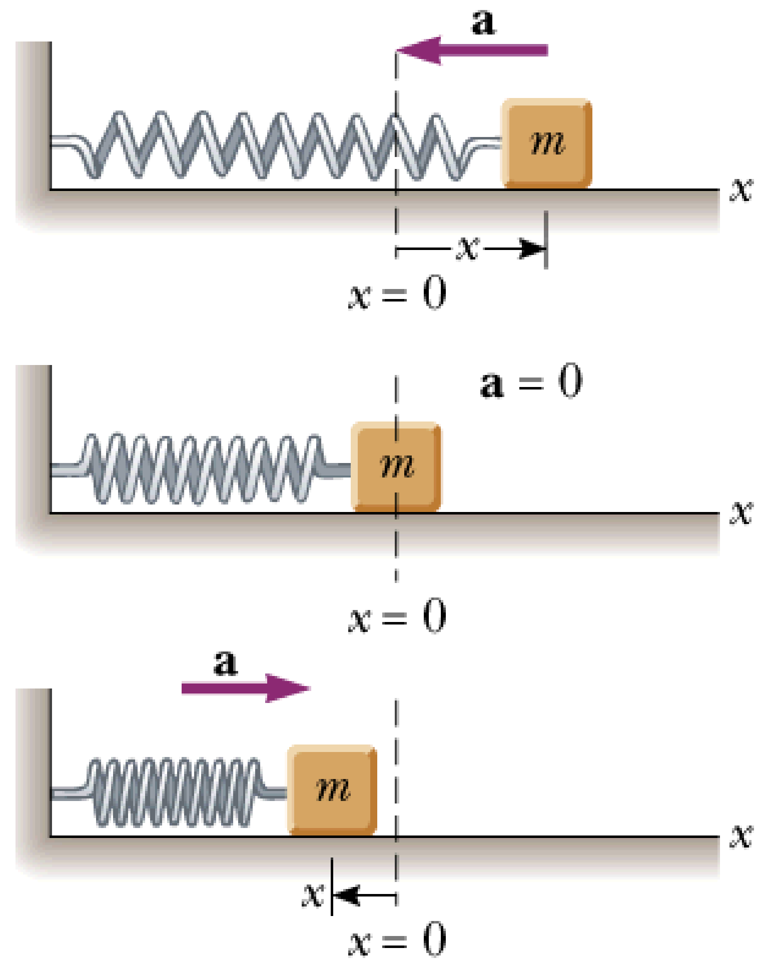
Example-1. A block of mass 680gm is fastened to a spring of spring constant 65N/m. The block is pulled a Distance 11 cm from its equlibrum on a frictionless table and released.

(a) What are the angular frequency, the frequency, and the period of the motion?

(b) What is amplitude of the motion?

(c) What is the maximum speed of the block?

Example-2

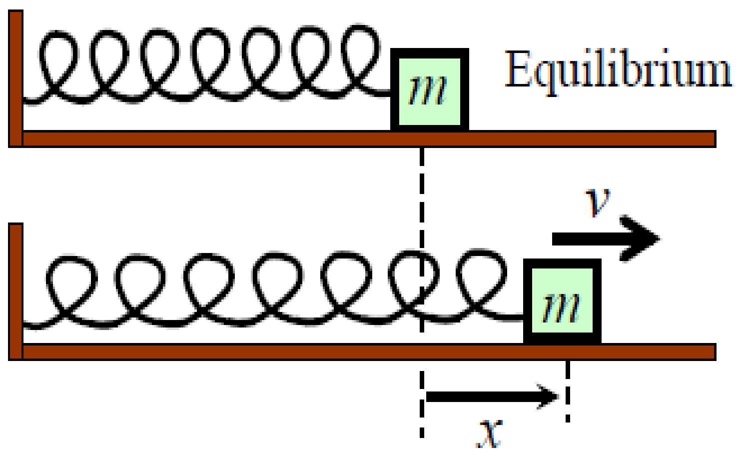


A spring stretches by 3.90 cm when a 10.0 g mass is hung from it. A 25.0 g mass attached to this spring oscillates in simple harmonic motion.

(a) Calculate the period of the motion.

(b) Calculate frequency and the angular velocity of the motion.

Mass-spring oscillator: an energy approach



Suppose that the mass has a speed v when it has displacement x

Kinetic energy of mass = $\frac{1}{2}mv^2$

Potential energy of spring = $\int_0^x F dx' = \int_0^x kx' dx' = \frac{1}{2}kx^2$

There are no dissipative mechanisms in our model (no friction).
… the total energy of the mass-spring system is conserved.

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$$

Mass-spring oscillator: an energy approach …2

For our mass-spring system: $\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \text{constant}$

$$\therefore \frac{d}{dt}\left(\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2\right) = 0$$

$$\therefore mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0$$

$$\therefore mv\frac{dv}{dt} + kxv = 0$$

$$\therefore m\frac{dv}{dt} + kx = 0$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

… as before

Mass-spring oscillator: an energy approach …3

For the mass-spring system: $x = A\cos(\omega_0 t + \phi)$

Potential energy = $\frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$

k.e. = $\frac{1}{2}mv^2 = \frac{1}{2}m[-A\omega_0\sin(\omega_0 t + \phi)]^2 = \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi)$

Total energy = p.e. + k.e

$$= \tfrac{1}{2}kA^2\cos^2(\omega_0 t + \phi) + \tfrac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi)$$

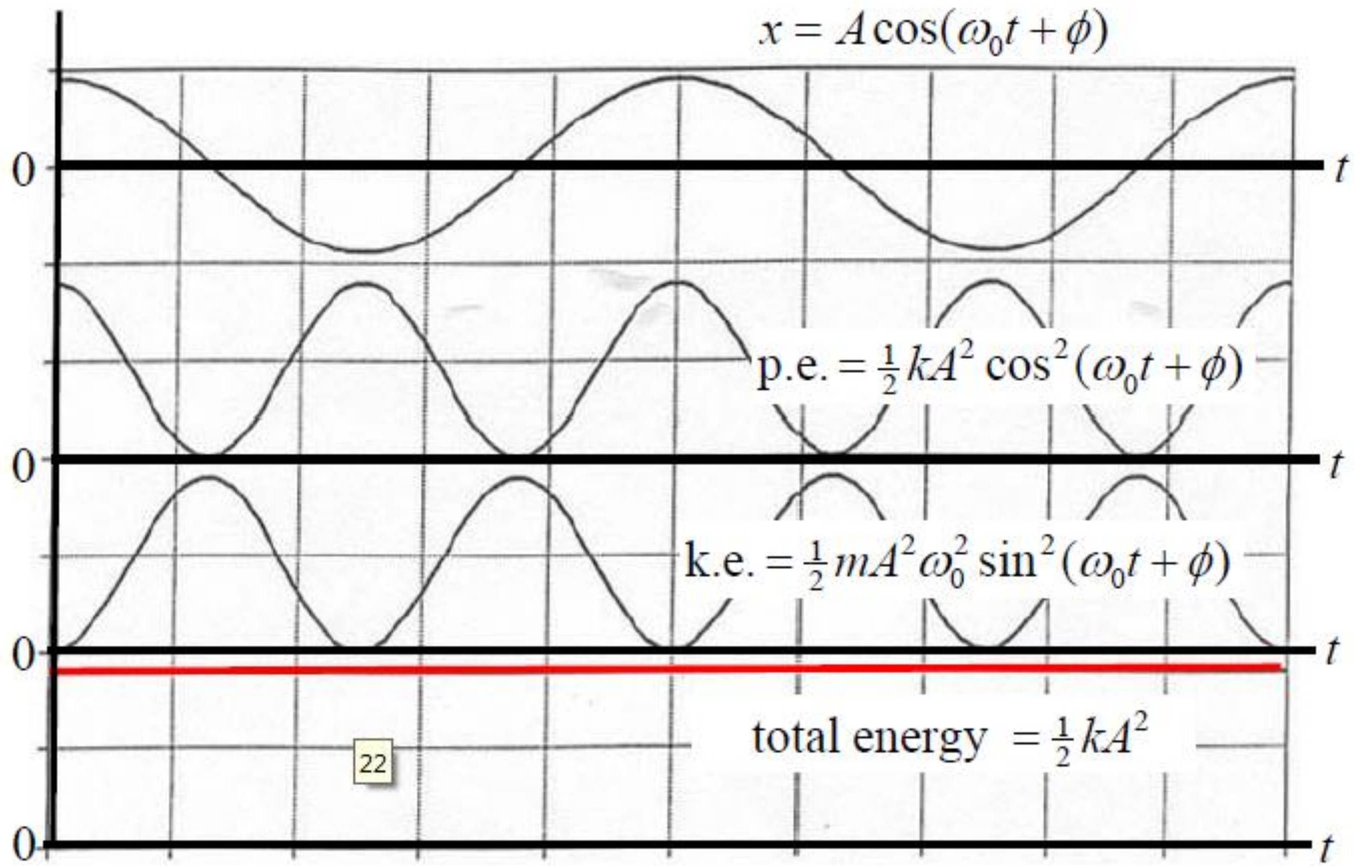
$$= \tfrac{1}{2}kA^2 \quad (= \tfrac{1}{2}m\omega_0{}^2A^2) \qquad (\therefore E \propto A^2)$$

We can now write: $\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$

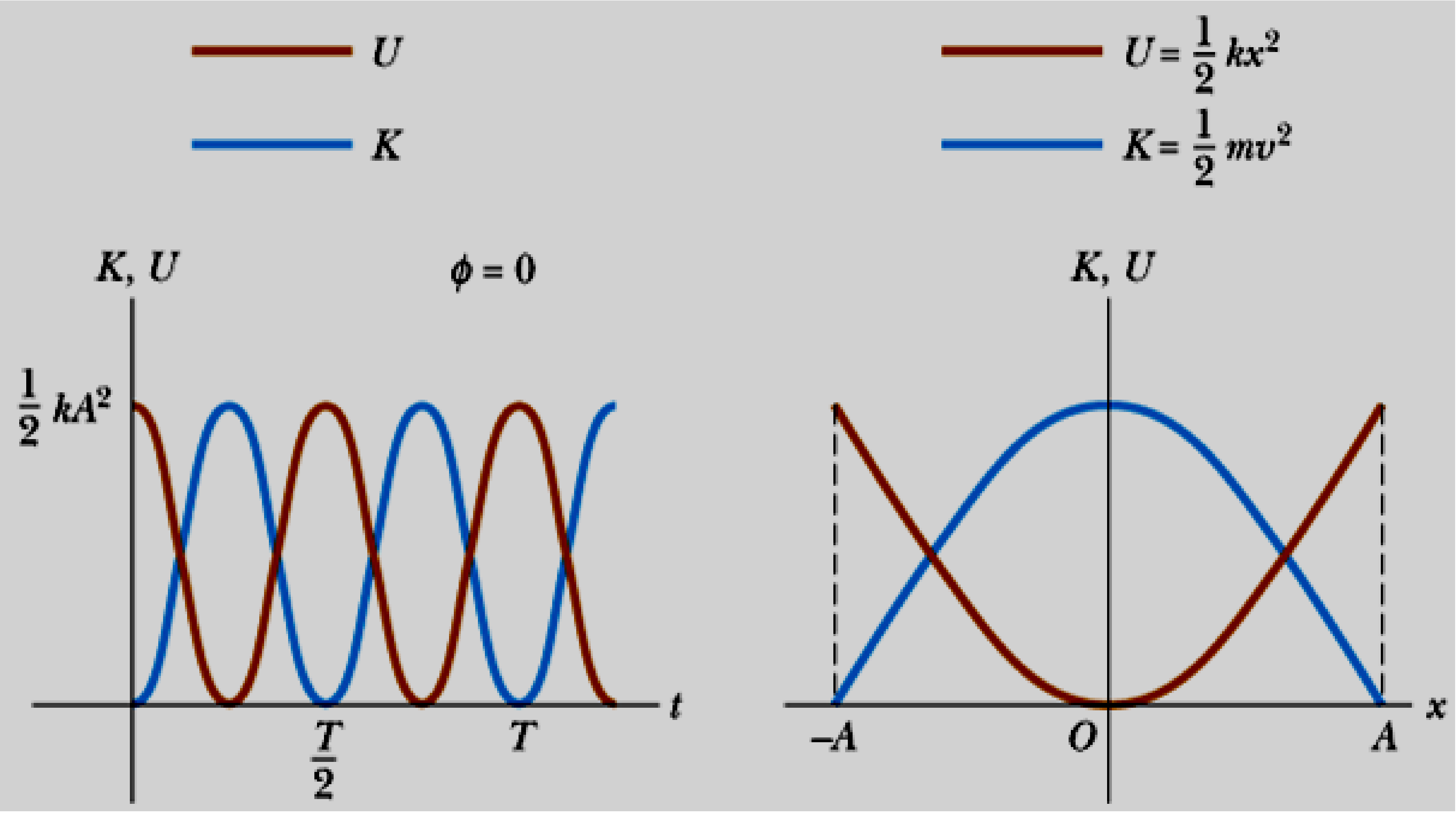
$$\therefore v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} \qquad \text{or} \qquad v(x) = \pm\omega_0\sqrt{A^2 - x^2}$$

(useful)

Energy of the mass-spring simple harmonic oscillator



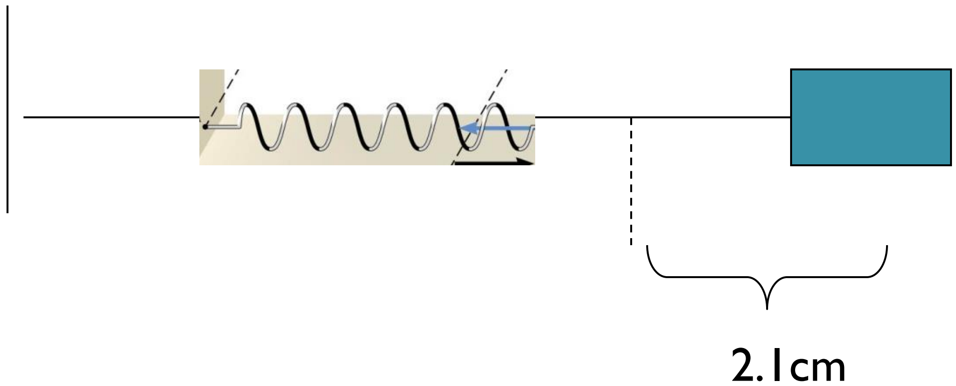
K.E and P.E of SHM



Check Your Understanding

Example-3: A 0.42-kg block is attached to the end of a horizontal ideal spring and rests on a frictionless surface. The block is pulled so that the spring stretches by 2.1 cm relative to its unstrained length. When the block is released, it moves with an acceleration of 9.0 m/s^2. What is the spring constant of the spring?

180 N/m



kx = ma

$$k \times \frac{2.1}{100} = 0.42 \times 9.0 m/s^2$$

$$k = \frac{0.42 \times 9.0}{2.1} \times 100 = 180 N/m$$

Energy in the SHO

Energy calculations.

Example-4: For the simple harmonic oscillation where k = 19.6 N/m, A = 0.100 m, x = -(0.100 m) cos 8.08t, and v = (0.808 m/s) sin 8.08t, determine (a) the total energy, (b) the kinetic and potential energies as a function of time, (c) the velocity when the mass is 0.050 m from equilibrium, (d) the kinetic and potential energies at half amplitude ($x = \pm A/2$).

Solution:

a. $E = \frac{1}{2}kA^2 = \frac{1}{2} \times 19.6\,\mathrm{N/m} \times (0.100\,\mathrm{m})^2 = 9.80 \times 10^{-2}\,\mathrm{J}.$

b. $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2\omega t = (9.80 \times 10^{-2}\,\mathrm{J})\cos^2 8.08t,$

$K = E - U = (9.80 \times 10^{-2}\,\mathrm{J})\sin^2 8.08t.$

Solution:

c. $$K = E - U, \quad \frac{1}{2}mv^2 = \frac{1}{2}kA^2 - \frac{1}{2}kx^2,$$

$$v = \sqrt{\frac{k}{m}(A^2 - x^2)} = \omega\sqrt{A^2 - x^2}$$

$$= 8.08\,\text{Hz} \times \sqrt{(0.100\,\text{m})^2 - (0.050\,\text{m})^2} = 0.70\,\text{m/s}.$$

d. $$U = \frac{1}{2}kx^2 = \frac{1}{2}k\left(\frac{A}{2}\right)^2 = \frac{1}{4}E = 2.5 \times 10^{-2}\,\text{J},$$

$$E = K - U = 7.3 \times 10^{-2}\,\text{J}.$$

Using Conservation of Energy

Example-5: A 500 g block on a spring is pulled a distance of 20 cm and released. The subsequent oscillations are measured to have a period of 0.80 s. At what position (or positions) is the speed of the block 1.0 m/s?

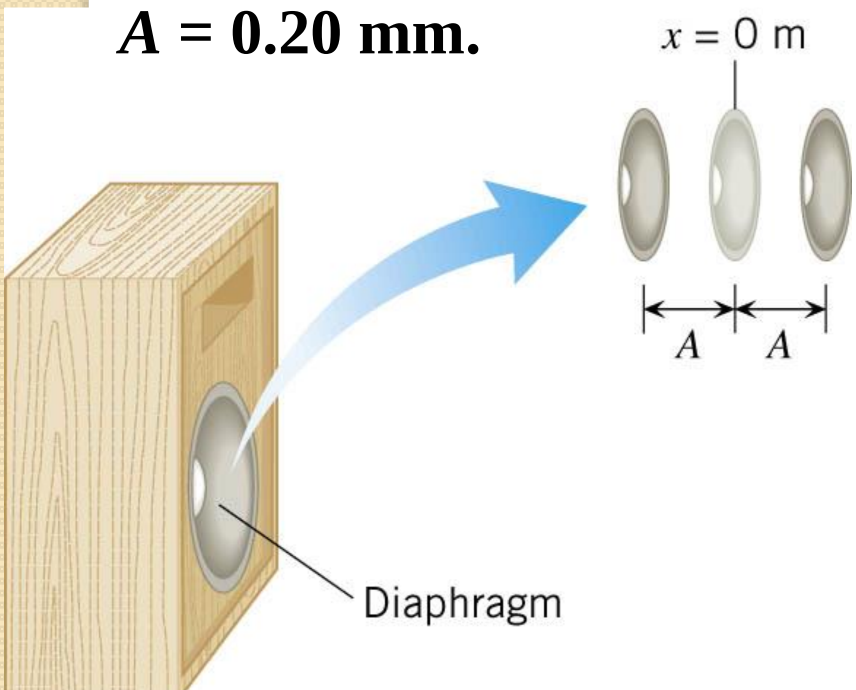
$$T = 0.80\text{ s} \quad \text{so} \quad \omega = \frac{2\pi}{T} = \frac{2\pi}{(0.80\text{ s})} = 7.85\text{ rad/s}$$

$$v = \sqrt{\frac{k}{m}\left(A^2 - x^2\right)} = \omega\sqrt{A^2 - x^2}$$

$$x = \pm\sqrt{A^2 - \left(\frac{v}{\omega}\right)^2} = \pm\sqrt{(0.20\text{ m})^2 - \left(\frac{(1.0\text{ m/s})}{(7.85\text{ rad/s})}\right)^2} = \pm 0.154\text{ m} = \pm 15.4\text{ cm}$$

Example: The Maximum Speed of a Loudspeaker Diaphragm

Example-6: The diaphragm of a loudspeaker moves back and forth in simple harmonic motion to create sound. The frequency of the motion is f = 1.0 kHz and the amplitude is A = 0.20 mm.



(a) What is the maximum speed of the diaphragm?

(b) Where in the motion does this maximum speed occur?

(a)

$$v_{max} = A\omega = A(2\pi f) = (0.20 \times 10^{-3}\ \text{m})(2\pi)(1.0 \times 10^{3}\ \text{Hz}) = \boxed{1.3\ \text{m/s}}$$

(b) The speed of the diaphragm is zero when the diaphragm momentarily comes to rest at either end of its motion: *x* = +*A* and *x* = –*A*. Its maximum speed occurs midway between these two positions, or at *x* = 0 m.

Example: Radio Station Frequency and Period

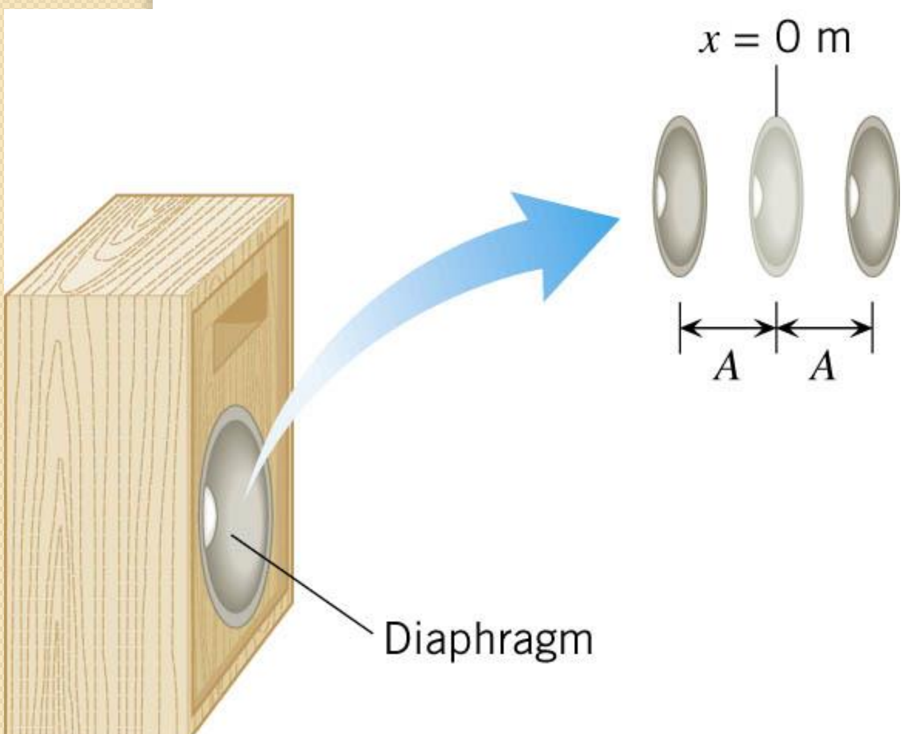
Example-7: What is the oscillation period of an FM radio station that broadcasts at 100 MHz?

$$f = 100\ \text{MHz} = 1.0 \times 10^{8}\ \text{Hz}$$

$$T = 1/f = \frac{1}{1.0 \times 10^{8}\ \text{Hz}} = 1.0 \times 10^{-8}\ \text{s} = 10\ \text{ns}$$

Note that 1/Hz = s

Example-8 : The Loudspeaker Revisited—The Maximum Acceleration



A loudspeaker diaphragm is vibrating at a frequency of f = 1.0 kHz, and the amplitude of the motion is A = 0.20 mm.

(a) What is the maximum acceleration of the diaphragm, and

(b) where does this maximum acceleration occur?

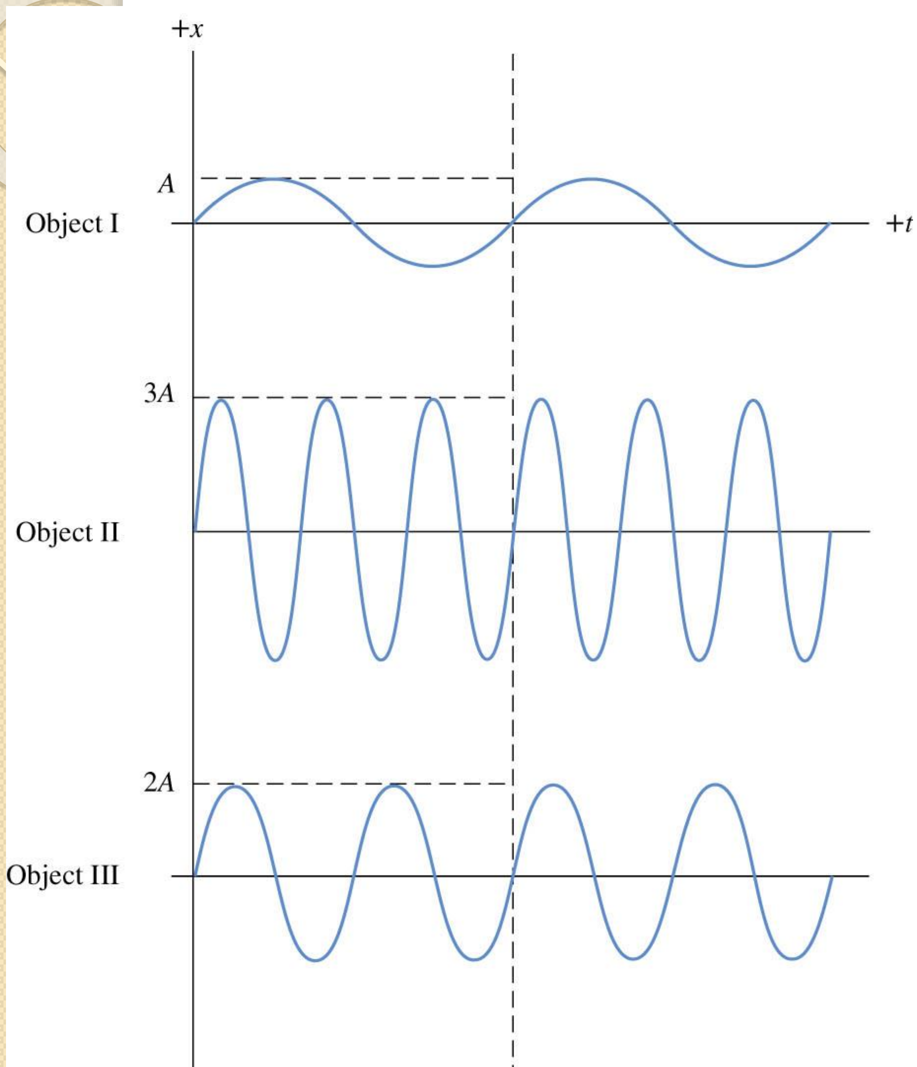
(a)

$$a_{max} = A\omega^2 = A(2\pi f)^2 = (0.20 \times 10^{-3}\text{ m})[2\pi(1.0 \times 10^3\text{ Hz})]^2$$

$$= \boxed{7.9 \times 10^3\text{ m/s}^2}$$

(b) the maximum acceleration occurs at *x* = +*A* and *x* = –*A*

Check Your Understanding



The drawing shows plots of the displacement *x* versus the time *t* for three objects undergoing simple harmonic motion. Which object, I, II, or III, has the greatest maximum velocity?

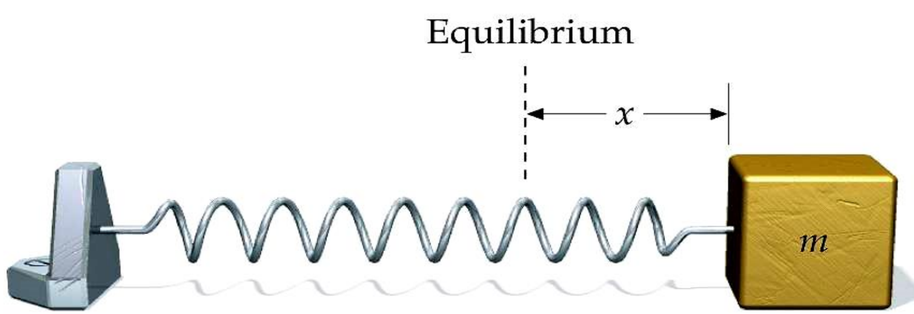
II

Example-9: A Block on a Spring

A 2.00 kg block is attached to a spring as shown. The force constant of the spring is $k = 196$ N/m. The block is held a distance of 5.00 cm from equilibrium and released at $t = 0$.

(a) Find the angular frequency ω, the frequency f, and the period T.

(b) Write an equation for x vs. time.



$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{(196 \text{ N/m})}{(2.00 \text{ kg})}} = 9.90 \text{ rad/s}$$

$$f = \frac{\omega}{2\pi} = \frac{(9.90 \text{ rad/s})}{2\pi} = 1.58 \text{ Hz}$$

$$T = 1/f = 0.635 \text{ s}$$

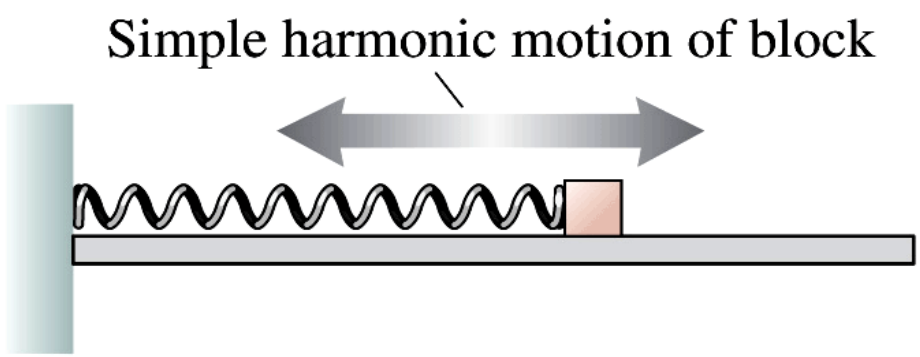
$$A = 5.00 \text{ cm} \text{ and } \delta = 0$$

$$\boxed{x = (5.00 \text{ cm}) \cos\left[(9.90 \text{ rad/s})t\right]}$$

Example-10: A System in SHM

An air-track glider is attached to a spring, pulled 20 cm to the right, and released at t-=0. It makes 15 complete oscillations in 10 s.

a. What is the period of oscillation?
b. What is the object's maximum speed?
c. What is its position and velocity at t=0.80 s?



$$f = \frac{15 \text{ oscillations}}{10 \text{ s}}$$

$$= 1.5 \text{ oscillations/s} = 1.5 \text{ Hz}$$

$$T = 1/f = 0.667 \text{ s}$$

$$v_{\max} = \frac{2\pi A}{T} = \frac{2\pi(0.20 \text{ m})}{(0.667 \text{ s})} = 1.88 \text{ m/s}$$

$$x = A\cos\frac{2\pi t}{T} = (0.20 \text{ m})\cos\frac{2\pi(0.80 \text{ s})}{(0.667 \text{ s})} = 0.062 \text{ m} = 6.2 \text{ cm}$$

$$v = -v_{\max}\sin\frac{2\pi t}{T} = -(1.88 \text{ m/s})\sin\frac{2\pi(0.80 \text{ s})}{(0.667 \text{ s})} = -1.79 \text{ m/s}$$

Example-11: Finding the Time

A mass, oscillating in simple harmonic motion, starts at $x = A$ and has period T.

At what time, as a fraction of T, does the mass first pass through $x = \frac{1}{2}A$?

$$x = \tfrac{1}{2}A = A\cos\frac{2\pi t}{T}$$

$$t = \frac{T}{2\pi}\cos^{-1}\left(\tfrac{1}{2}\right) = \frac{T}{2\pi}\frac{\pi}{3} = \tfrac{1}{6}T$$

Example-12. A particle execute s simple harmonic motion given by the equation

$$y = 12\sin\left(\frac{2\pi t}{10} + \frac{\pi}{4}\right)$$

Calculate (i) amplitude, (ii) frequency, (iii) displacement at t= 1.25s, (iv) velocity at t= 2.5s (v) acceleration at t= 5s.

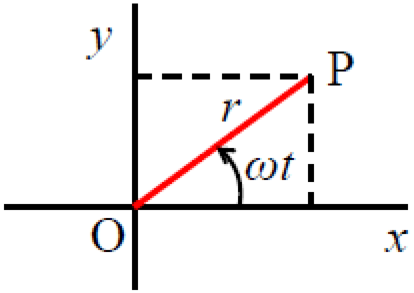
Example-13: A particle execute s simple harmonic motion given by the equation

$$y = 10\sin(10t - \frac{\pi}{6})$$

Calculate (i) frequency, (ii) time period (iii) the maximum displacement (iv)the maximum velocity (v) the maximum acceleration acceleration.

Complex numbers

Consider a vector $\overrightarrow{OP}$ of length r which rotates with angular velocity ω



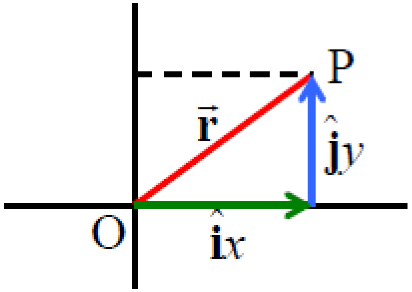
The point P has coordinates

$x = r\cos\omega t \qquad y = r\sin\omega t$

We see that the x coordinate of P, or the projection of $\overrightarrow{OP}$ onto the x-axis, executes SHM

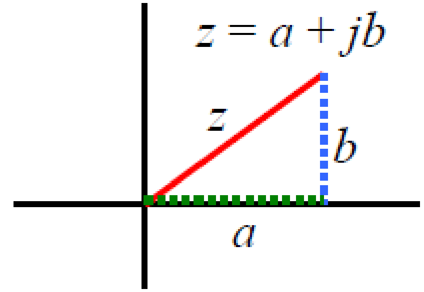
Can also introduce the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$

and write $\vec{\mathbf{r}} = \hat{\mathbf{i}}x + \hat{\mathbf{j}}y$



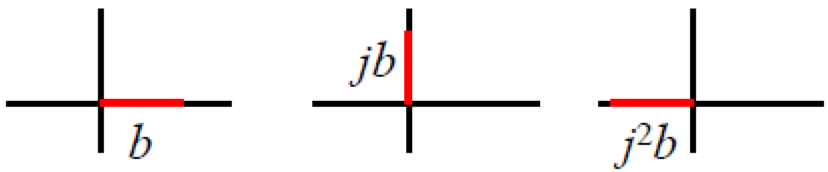
Complex numbers …2

Modify our notation to $z = x + jy$



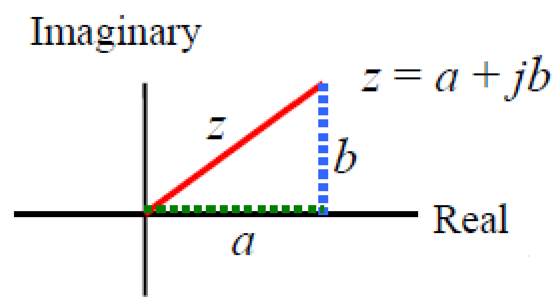
… where x means a displacement in the x-direction and jy means a displacement in the y-direction

We can also think of j as a rotation through $\pi/2$ anticlockwise



Hence $j^2 = -1$

… really talking about vectors in the complex number plane:



Complex numbers …3

From Taylor's theorem: $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

therefore $$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \dots$$

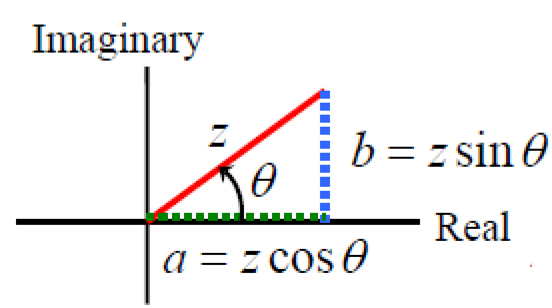
and $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots$ and $j\sin\theta = j\theta - \frac{j\theta^3}{3!} + \dots$

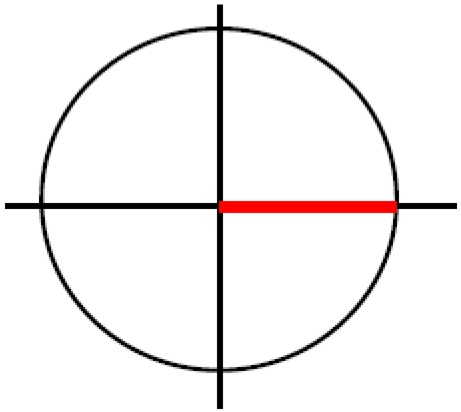
Hence $$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$ **Euler relation**

Then $z = a + jb \;= |z| e^{j\theta}$

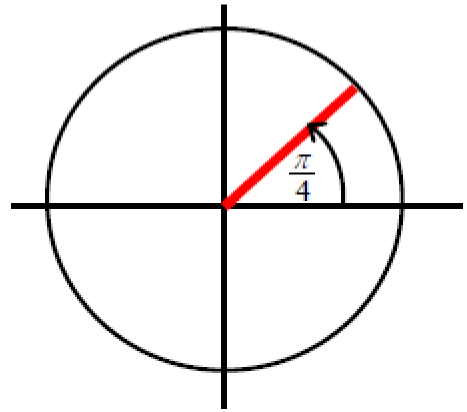
where $|z| = \sqrt{a^2 + b^2}$

$\tan\theta = b/a$

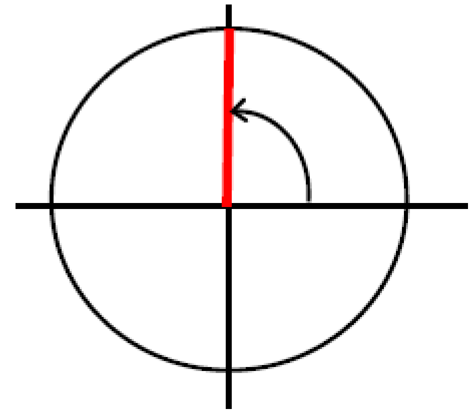




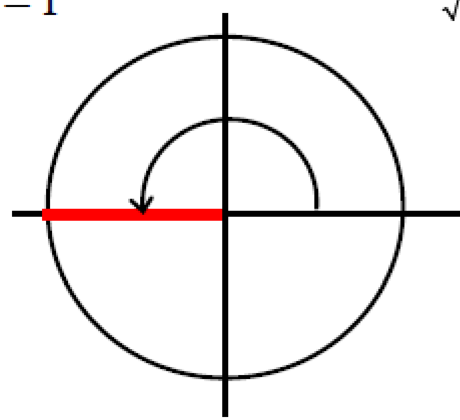
$$e^{j0} = 1$$



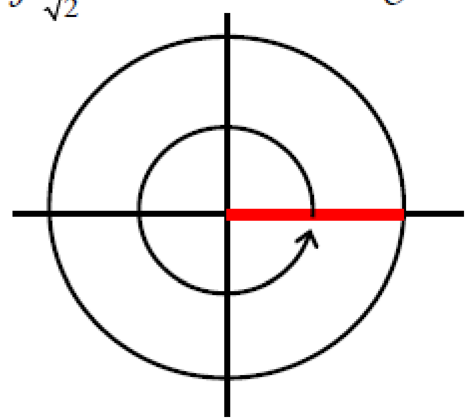
$$e^{j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$



$$e^{j\frac{\pi}{2}} = j$$



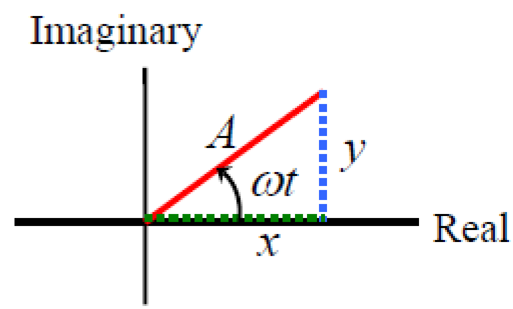
$$e^{j\pi} = -1$$



$$e^{j2\pi} = 1$$

Complex numbers …4

For our rotating vectors:



$$z = x + jy$$

$$= A\cos\omega t + jA\sin\omega t$$

$$= A\left(\cos\omega t + j\sin\omega t\right)$$

$$= Ae^{j\omega t}$$

Now write: $Ae^{j(\omega t+\phi)} = A\cos(\omega t+\phi) + jA\sin(\omega t+\phi)$

... and remember that the physical quantity x (e.g. a displacement) is the real part of z :

i.e. $x = \mathrm{Re}[z]$

SHM using complex numbers

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

Using $z = x + jy$ becomes $\dfrac{d^2z}{dt^2} + \omega_0^2 z = 0$

Try $z = Ae^{j(\omega t+\phi)}$

$$\therefore A(j\omega)^2 e^{j(\omega t+\phi)} + \omega^2 A e^{j(\omega t+\phi)} = 0$$

Therefore $z = Ae^{j(\omega t+\phi)}$ is the most general solution

A and ϕ are determined from the initial conditions.

Take real part of z:

$$x = \mathrm{Re}[z] = A\cos(\omega_0 t + \phi)$$

SHM using complex numbers

$$x = A\cos(\omega_0 t + \phi) \qquad\qquad z = Ae^{j(\omega t+\phi)}$$

$$x = \mathrm{Re}[z]$$

$$\frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \phi) \qquad\qquad \frac{dz}{dt} = j\omega A e^{j(\omega t+\phi)} = j\omega z$$

$$\frac{dx}{dt} = \mathrm{Re}\left[\frac{dz}{dt}\right]$$

$$\frac{d^2x}{dt^2} = -A\omega_0^{\,2}\cos(\omega_0 t + \phi) \qquad\qquad \frac{d^2z}{dt^2} = (j\omega)^2 A e^{j(\omega t+\phi)} = -\omega^2 z$$

$$\frac{d^2x}{dt^2} = \mathrm{Re}\left[\frac{d^2z}{dt^2}\right]$$

Solving $\dfrac{d^2x(t)}{dt^2} + \omega_0^2 x(t) = 0$

Let $x = Be^{pt}$

Then $\dfrac{dx}{dt} = Bpe^{pt}$ and $\dfrac{d^2x}{dt^2} = Bp^2e^{pt}$

Substituting into DE: $Bp^2e^{pt} + B\omega_0^2 e^{pt} = 0$

This holds true for all t if and only if $p^2 = -\omega_0^2$ or $p = \pm j\omega_0$

$$\therefore x = B_1 e^{j\omega_0 t} + B_2 e^{-j\omega_0 t}$$

How to get B_1 and B_2 ? … need to know the initial conditions

… but consider $v = \dfrac{dx}{dt} = j\omega_0 B_1 e^{j\omega_0 t} - j\omega_0 B_2 e^{-j\omega_0 t}$

Solving $\dfrac{d^2x(t)}{dt^2} + \omega_0^2 x(t) = 0$ continued

$$v = \frac{dx}{dt} = j\omega_0 B_1 e^{j\omega_0 t} - j\omega_0 B_2 e^{-j\omega_0 t}$$

At $t = 0$, $v(0) = j\omega B_1 - j\omega_0 B_2$

Choose $v(0) = 0$

Since v must be real, then $B_1 = B_2 = B$

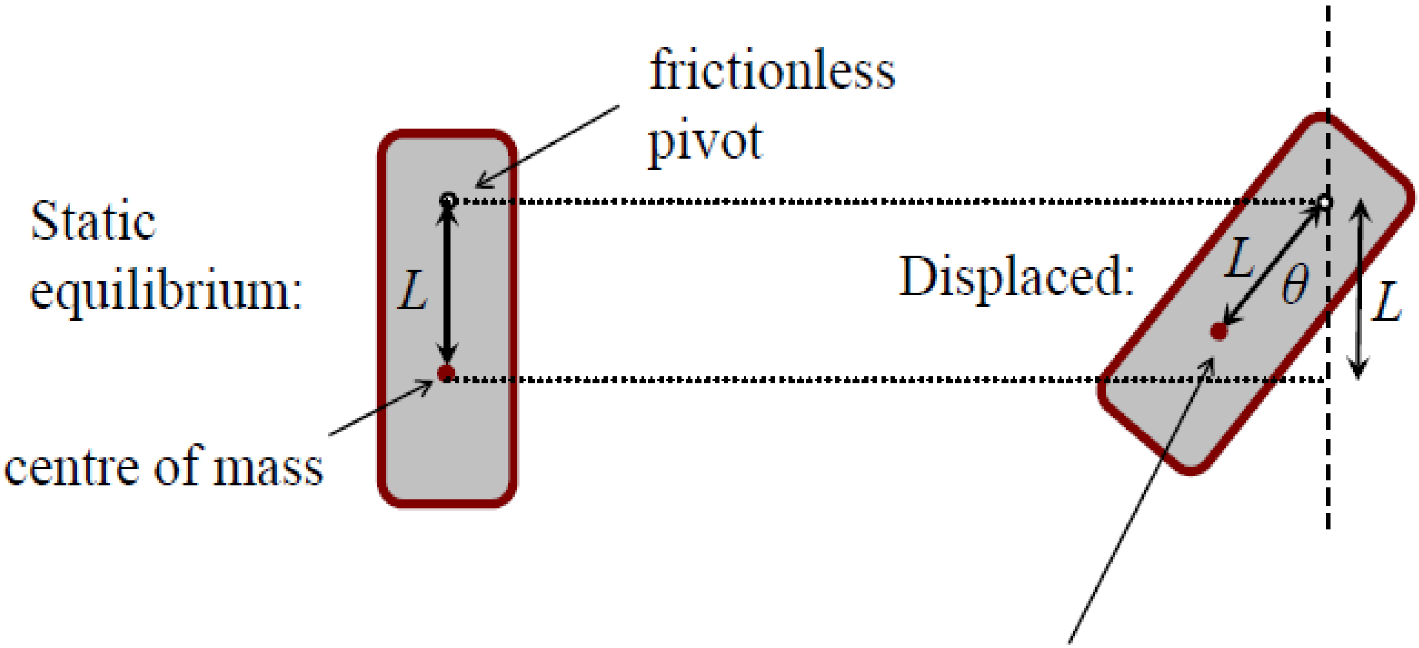
i.e. $x = Be^{j\omega_0 t} + Be^{-j\omega_0 t}$

$= 2B\cos\omega_0 t$

$\therefore x = A\cos\omega_0 t$

[… with a little more effort we could have got the more general solution $x = A\cos(\omega_0 t + \phi)$]

The pendulum: general case



In displaced position, centre of mass is $L - L\cos\theta$ above the equilibrium position.

Recall $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$ For small angles, $\cos\theta \approx 1 - \frac{\theta^2}{2}$

Gravitational potential energy = $mgL(1-\cos\theta) = mgL\frac{\theta^2}{2}$

The pendulum: general case …2

Gravitational potential energy = $\frac{1}{2}mgL\theta^2$

Kinetic energy = $\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2$ I= Moment of inertia

Total energy = $\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}mgL\theta^2$ = constant

$$\therefore I\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgL\theta\frac{d\theta}{dt} = 0$$ … true for all $\frac{d\theta}{dt}$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{mgL}{I}\theta = -\omega_0^2\theta$$ where $\omega_0 = \sqrt{\frac{mgL}{I}}$

Equation of SHM

The moment of inertia of the pendulum about an passing through the point of suspension is

$$= mK^2 + mL^2$$

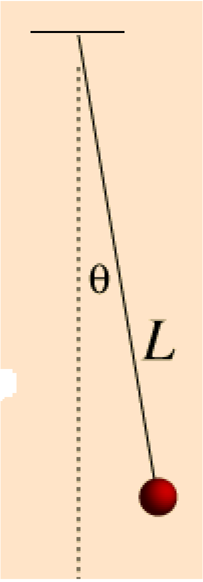
Therefore, $$\omega_0 = \sqrt{\frac{gL}{K^2 + L^2}}$$

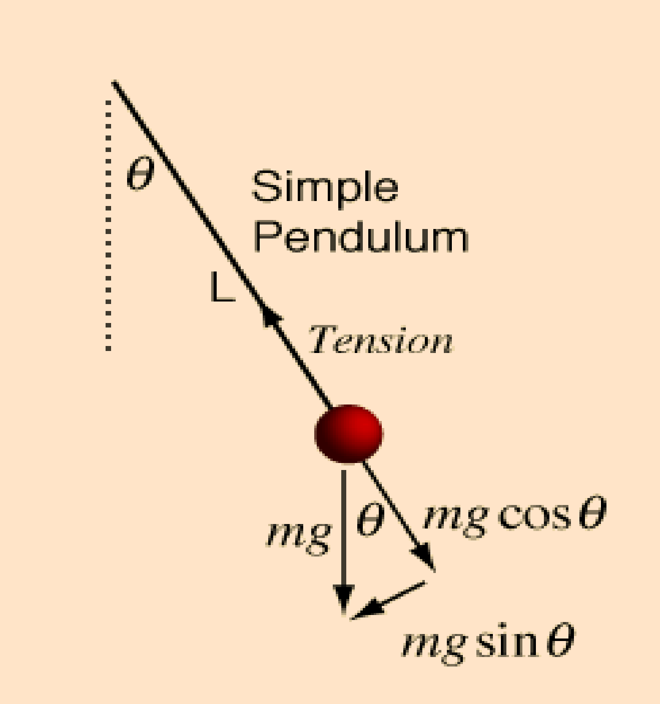
Time Period

$$T = 2\pi\sqrt{\frac{K^2 + L^2}{Lg}}$$

Simple Pendulum

A *simple pendulum* consists of a particle of mass *m*, attached to a frictionless point by a cable of length *L* and negligible mass.





From the above figure restoring force

$$F = -mg\sin\theta$$

If the angle θ is very small sinθ is very nearly equal to θ. The displacement along the arc is

$$x = L\theta$$

Therefore, $F = -mg\theta$

Acceleration $\frac{d^2x}{dt^2} = L\frac{d^2\theta}{dt^2}$

$$Force = mL\frac{d^2\theta}{dt^2}$$

$$mL\frac{d^2\theta}{dt^2} = -mg\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Where

$$\omega^2 = \frac{g}{L}$$

And

$$T = 2\pi\sqrt{\frac{L}{g}}$$

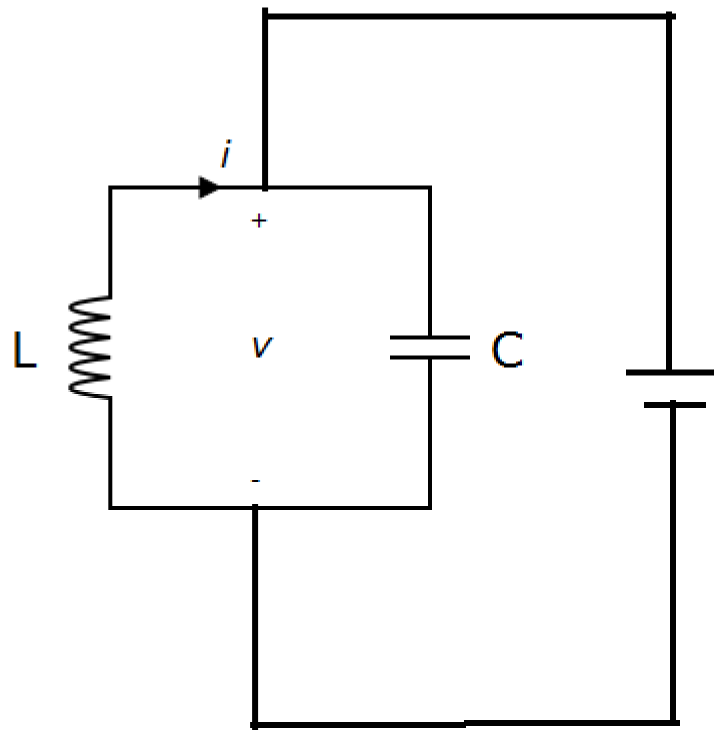
Example 14. Keeping Time

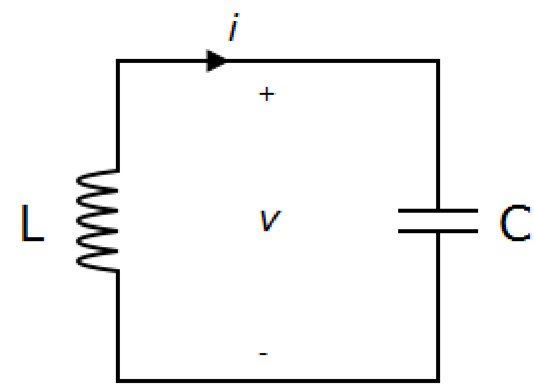
Determine the length of a simple pendulum that will swing back and forth in simple harmonic motion with a period of 1.00 s.

$$f = \frac{1}{2\pi}\sqrt{g / L}$$

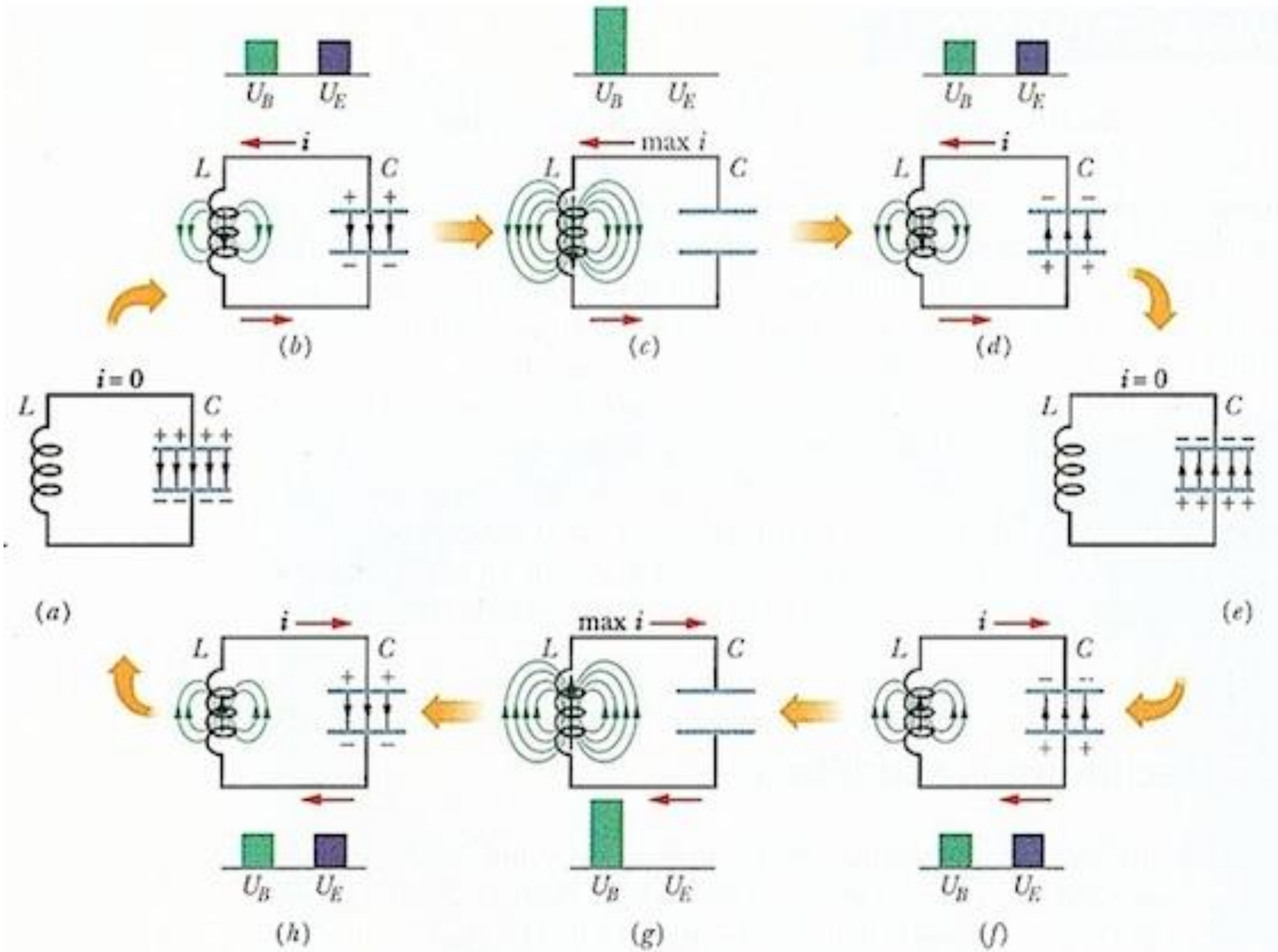
$$L = \frac{T^2 g}{4\pi^2} = \frac{(1.00\ \mathrm{s})^2 (9.80\ \mathrm{m/s^2})}{4\pi^2} = \boxed{0.248\ \mathrm{m}}$$

LC Circuit





Charging discharging of an LC Circuit



An **LC circuit**, also called a **resonant circuit**, **tank circuit**, or **tuned circuit**, consists of an inductor, represented by the letter L, and a capacitor, represented by the letter C. When connected together, they can act as an electrical resonator.

Voltage across capacitor at any instant

$$V_C = \frac{Q}{C}$$

Q is the charge on the capacitor and C is capacitance of capacitor.

Voltage across inductor at the same instant

$$V_L = L\frac{di}{dt}$$

$$\frac{Q}{C} + L\frac{di}{dt} = 0$$ Kirchhoff's voltage law

$$\frac{d^2 i}{dt^2} + \frac{1}{LC} i = 0$$

Similar to differential equation of SHM

$$\frac{d^2 x}{dt^2} + \omega_0 x = 0, \qquad \omega_0{}^2 = \frac{1}{LC}$$

Time Period

$$T = 2\pi\sqrt{LC}$$

Frequency

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Solution of the differential equation is

$$Q(t) = Q_0 \cos(\omega_0 t + \phi)$$

Current in the circuit

$$i(t) = -i_0 \sin(\omega_0 t + \phi)$$

Mechanical	Electrical
displacement x	charge Q
velocity v	current I
mass m	inductance L
spring constant k	$\frac{1}{\text{capacitance}}$ $\frac{1}{C}$
$\omega_0 = \sqrt{\frac{k}{m}}$	$\omega_0 = \sqrt{\frac{1}{LC}}$
potential energy: $\frac{1}{2}kx^2$	Electric energy stored in capacitor: $\frac{1}{2}\frac{Q^2}{C}$
kinetic energy: $\frac{1}{2}mv^2$	Magnetic energy stored in inductor: $\frac{1}{2}LI^2$

Damped oscillations

We have thus far neglected all dissipative mechanisms

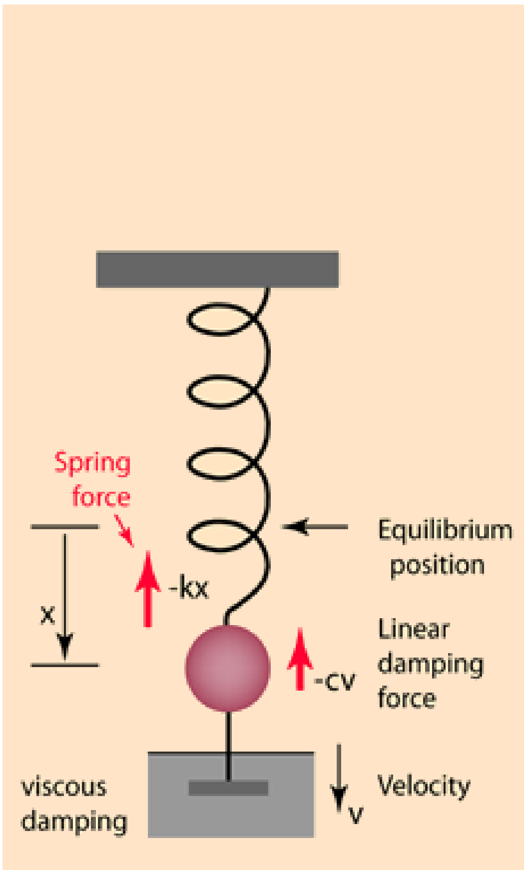
… our oscillations can continue oscillating with the same amplitude forever …

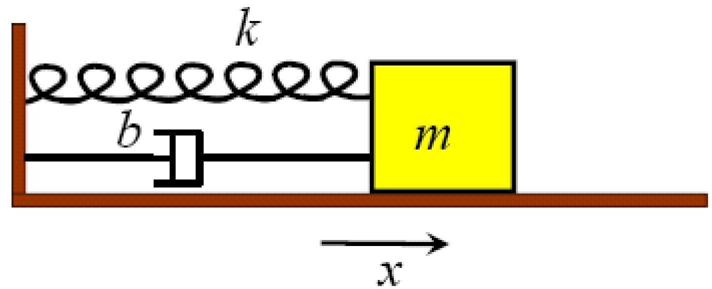
Various physical damping mechanisms will contribute towards the damping…

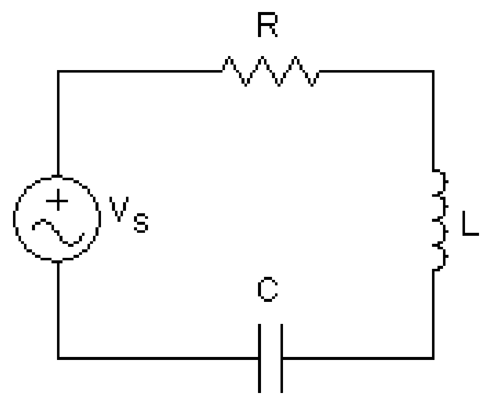
- friction between mass and table
- "air resistance"
- internal friction in spring
- …….

… model these by introducing a damping force which is proportional to the velocity of the oscillator …

Examples some damped oscillating systems

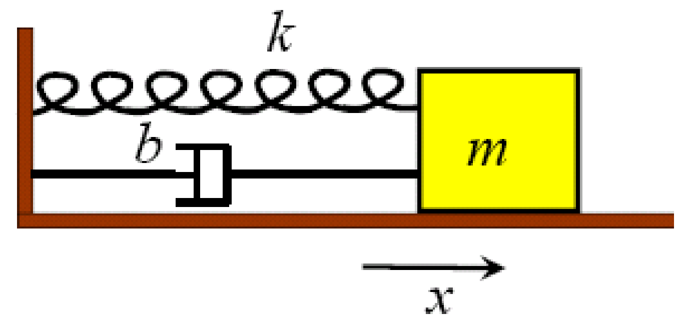






Damped Harmonic motion: When oscillating bodies do not move back and forth between Precisely fixed limits because frictional force dissipate the energy and amplitude of oscillation Decreases with time and finally die out. Such harmonic motion is called Damped Harmonic Motion.

Damped mass-spring system



In theses systems the damping force

$$F' = -bv$$

For horizontal forces on the mass: $ma = -kx - bv$

or $m\dfrac{d^2x}{dt^2} = -kx - b\dfrac{dx}{dt}$

or $\dfrac{d^2x}{dt^2} + \gamma\dfrac{dx}{dt} + \omega_0{}^2 x = 0$ where $\begin{cases} \omega_0 = \sqrt{\dfrac{k}{m}} \\ \gamma = \dfrac{b}{m} \end{cases}$

γ : "damping constant" unit: s^{-1}

- "life time" $= \dfrac{1}{\gamma}$

Damped oscillator equation

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0{}^2x=0$$

Let $x=Be^{pt}$

Then $\dfrac{dx}{dt}=Bpe^{pt}$ and $\dfrac{d^2x}{dt^2}=Bp^2e^{pt}$

Substituting into DE: $Bp^2e^{pt}+\gamma Bpe^{pt}+\omega_0{}^2Be^{pt}=0$

Thus $p^2+\gamma p+\omega_0{}^2=0$

$$\therefore p=\tfrac{1}{2}\left\{-\gamma\pm\sqrt{\gamma^2-4\omega_0{}^2}\right\}$$

or $p=-\dfrac{\gamma}{2}\pm\sqrt{\dfrac{\gamma^2}{4}-\omega_0{}^2}$

$$ax^2+bx+c=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Damped oscillator equation …2

$$p = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0{}^2}$$

We can distinguish three cases:

(i) $\omega_0{}^2 > \dfrac{\gamma^2}{4}$ **Oscillatory behaviour**

(ii) $\omega_0{}^2 = \dfrac{\gamma^2}{4}$ **Critical damping**

(iii) $\omega_0{}^2 < \dfrac{\gamma^2}{4}$ **Overdamping**

$$\boxed{\text{Case (i):} \qquad \omega_0^{\ 2} > \frac{\gamma^2}{4}}$$

$$\therefore \sqrt{\gamma^2/4 - \omega_0^{\ 2}} = \sqrt{-\left(\omega_0^{\ 2} - \gamma^2/4\right)}$$

$$\text{Put} \qquad \omega_1^{\ 2} = \omega_0^{\ 2} - \gamma^2/4$$

$$\therefore p = -\frac{\gamma}{2} \pm \sqrt{-\omega_1^{\ 2}} = -\frac{\gamma}{2} \pm j\omega_1$$

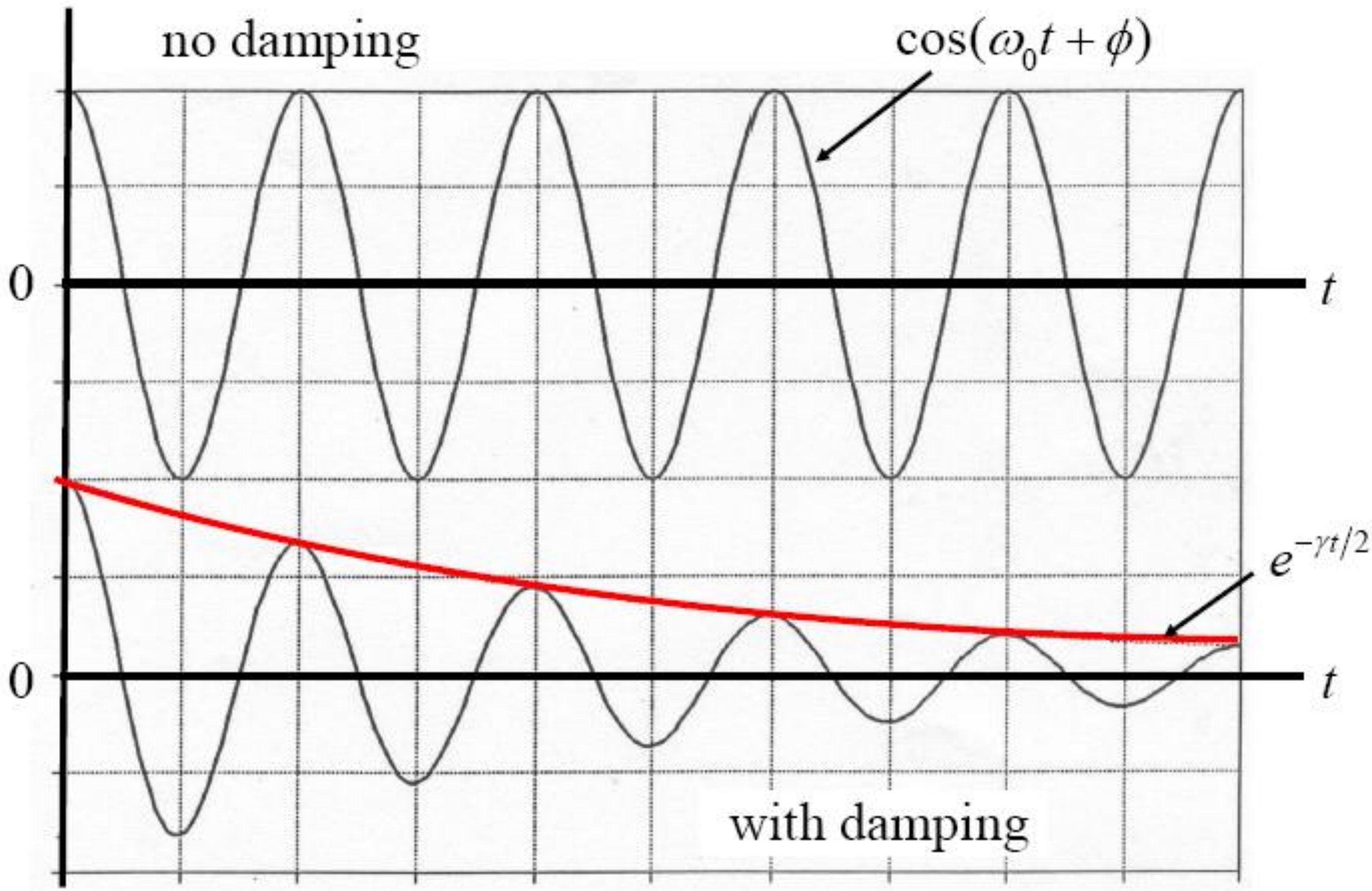
The solution will be:

$$x = B_1 e^{\left(-\frac{\gamma}{2}+j\omega_1\right)t} + B_2 e^{\left(-\frac{\gamma}{2}-j\omega_1\right)t} = e^{-\frac{\gamma}{2}t}\left\{B_1 e^{j\omega_1 t} + B_2 e^{-j\omega_1 t}\right\}$$

… leading to $x(t) = Ae^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \phi)$

This is an **oscillatory solution** $A\cos(\omega_1 t + \phi)$ multiplied by a damping factor $e^{-\gamma t/2}$.

As $\gamma \to 0$ we approach our undamped oscillator.


no damping
$\cos(\omega_0 t + \phi)$
0
t
$e^{-\gamma t/2}$
0
t
with damping

Case (ii): $\omega_0{}^2 = \dfrac{\gamma^2}{4}$

The two roots coincide: $p = -\dfrac{\gamma}{2}$

The solution will be $x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$

The condition $\omega_0{}^2 = \gamma^2/4$ is referred to as the "**critical damping**" condition.

If $\omega_0{}^2 < \gamma^2/4$ a system released from rest will oscillate.

As γ is increased the oscillations decay more rapidly, until at $\omega_0{}^2 = \gamma^2/4$ oscillation no longer occurs.

[… many practical applications …]

$$\boxed{\text{Case (iii):} \quad \omega_0{}^2 < \frac{\gamma^2}{4}}$$

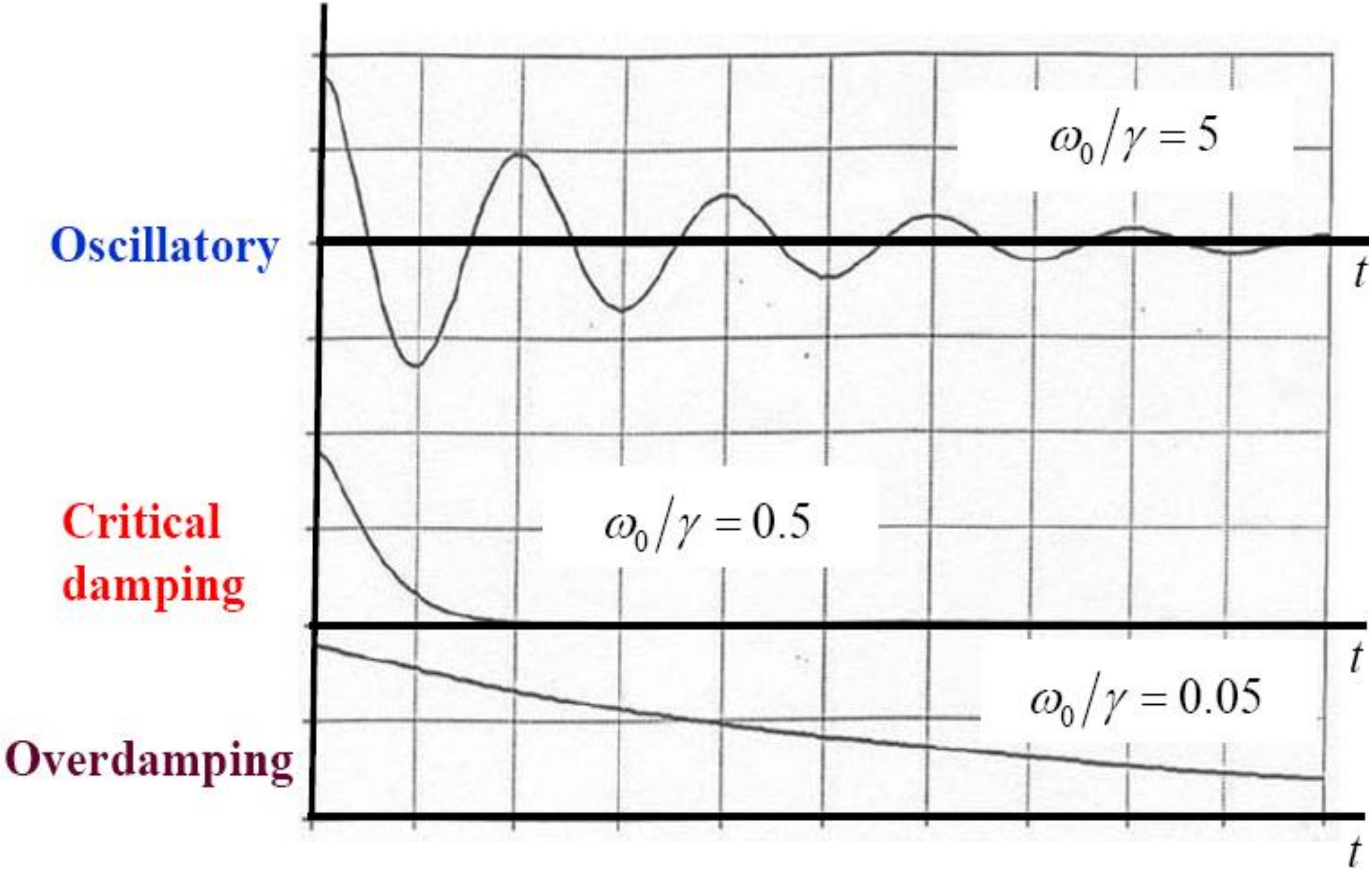
$$p = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0{}^2}$$

$$= -\frac{\gamma}{2} \pm \lambda \quad \text{say}$$

The solution will be $x(t) = B_1 e^{\left(-\frac{\gamma}{2}+\lambda\right)t} + B_2 e^{\left(-\frac{\gamma}{2}-\lambda\right)t}$

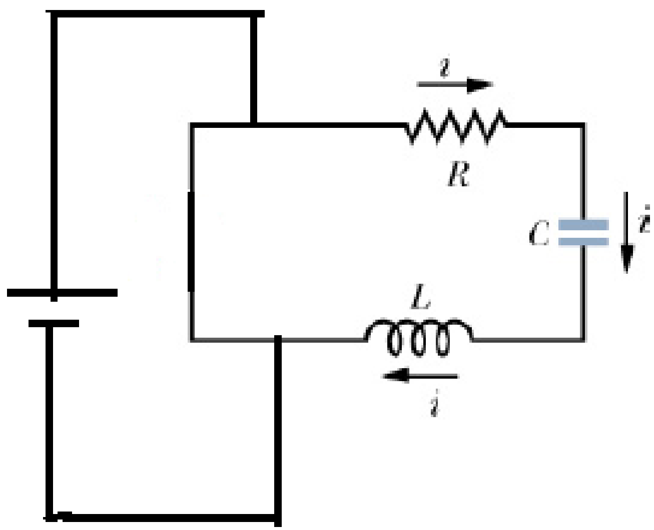
The condition $\omega_0{}^2 < \frac{\gamma^2}{4}$ is referred to as **overdamping**

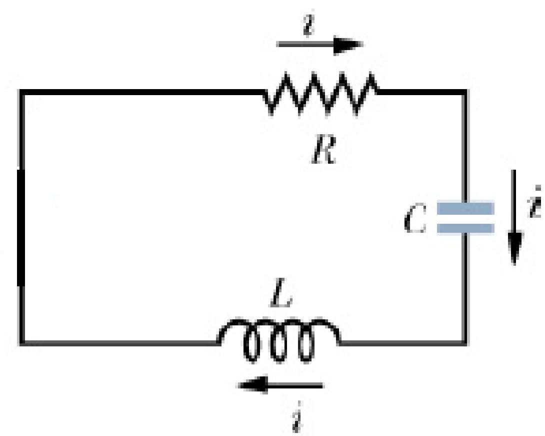
… a slower approach to the rest position is observed.



Oscillatory
$\omega_0/\gamma = 5$
t
Critical damping
$\omega_0/\gamma = 0.5$
t
Overdamping
$\omega_0/\gamma = 0.05$
t

RLC circuit





- Voltage across resistor R $V_R = iR$
- Voltage across capacitor C $V_C = \frac{Q}{C}$
- Voltage across inductor L $V_L = L\frac{di}{dt}$
- According to Kirchhoff's voltage law

$$iR + \frac{Q}{C} + L\frac{di}{dt} = 0$$

Rewrite the equation

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

Comparing with the equation

$$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0{}^2 x = 0$$

Where

$$\gamma = \frac{R}{L} \qquad \omega_0 = \sqrt{\frac{1}{LC}}$$

Three distinguish cases are

i) $\dfrac{1}{LC} > \dfrac{R^2}{4L^2}$ Oscillatory behavior

ii) $\dfrac{1}{LC} = \dfrac{R^2}{4L^2}$ Critical damping

iii) $\dfrac{1}{LC} < \dfrac{R^2}{4L^2}$ Over damping

Case i) $\frac{1}{LC} > \frac{R^2}{4L^2}$

Solution of the differential equation

$$Q(t) = Ae^{-\frac{R}{2L}t}\cos(\omega_1 t + \phi)$$

Where $\omega_1 = \sqrt{(\frac{1}{LC} - \frac{R^2}{4L^2})}$

Frequency of oscillation

$$f = \frac{1}{2\pi}\sqrt{(\frac{1}{LC} - \frac{R^2}{4L^2})}$$

EX-15: A capacitor 1.0μF, an inductor 0.2h and a resistance 800Ω are joined in series. Is the circuit oscillatory? Find the frequency of oscillation.

Ex-16: Find whether the discharge of capacitor through the following inductive circuit is oscillatory.

C = 0.1 μF, L = 10mh, R 200 Ω

If Oscillatory, find the frequency of oscillation.

Combination of two vibrations at right angles

$$x = A_1 \cos(\omega_1 t + \phi_1)$$

$$y = A_2 \cos(\omega_2 t + \phi_2)$$

Consider case where frequencies are equal and let initial phase difference be ϕ

Write $x = A_1 \cos(\omega_0 t)$ and $y = A_2 \cos(\omega_0 t + \phi)$

Case 1 : $\phi = 0$ $\left.\begin{aligned} x &= A_1 \cos(\omega_0 t) \\ y &= A_2 \cos(\omega_0 t) \end{aligned}\right\} \; y = \dfrac{A_2}{A_1} x$ Rectilinear motion

Case 2 : $\phi = \pi/2$ $x = A_1 \cos(\omega_0 t)$

$$y = A_2 \cos(\omega_0 t + \pi/2) = -A_2 \sin(\omega_0 t)$$

$$\therefore \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$$

Elliptical path in clockwise direction

Combination of two vibrations at right angles …2

$$\text{Case 3}: \ \phi = \pi \qquad \left.\begin{aligned} x &= A_1 \cos(\omega_0 t) \\ y &= A_2 \cos(\omega_0 t + \pi) = -A_2 \cos(\omega_0 t) \end{aligned}\right\} \ y = -\frac{A_2}{A_1} x$$

$$\text{Case 4}: \ \phi = 3\pi/2 \qquad x = A_1 \cos(\omega_0 t)$$

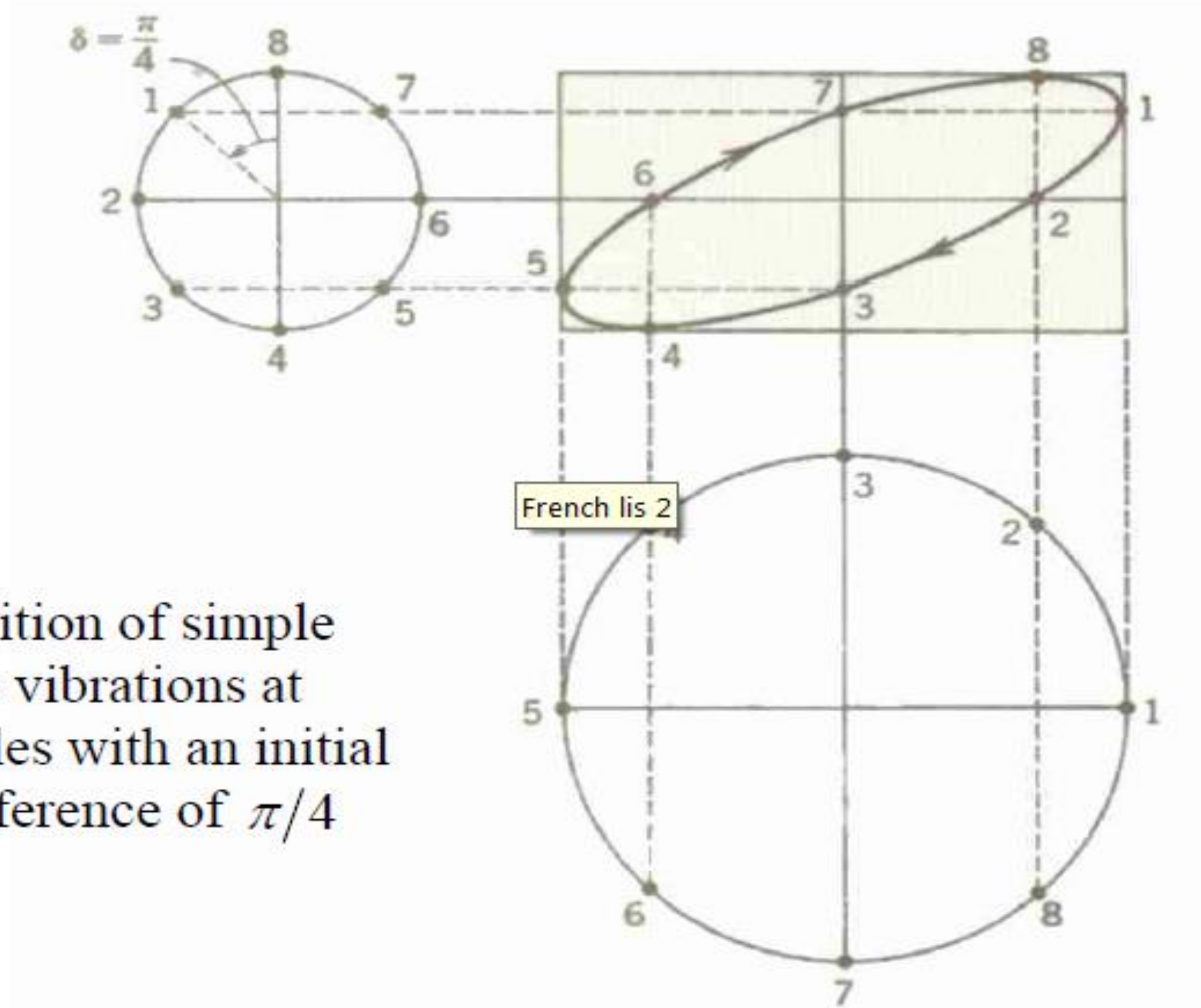
$$y = A_2 \cos(\omega_0 t + 3\pi/2) = +A_2 \sin(\omega_0 t)$$

$$\therefore \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = -1$$

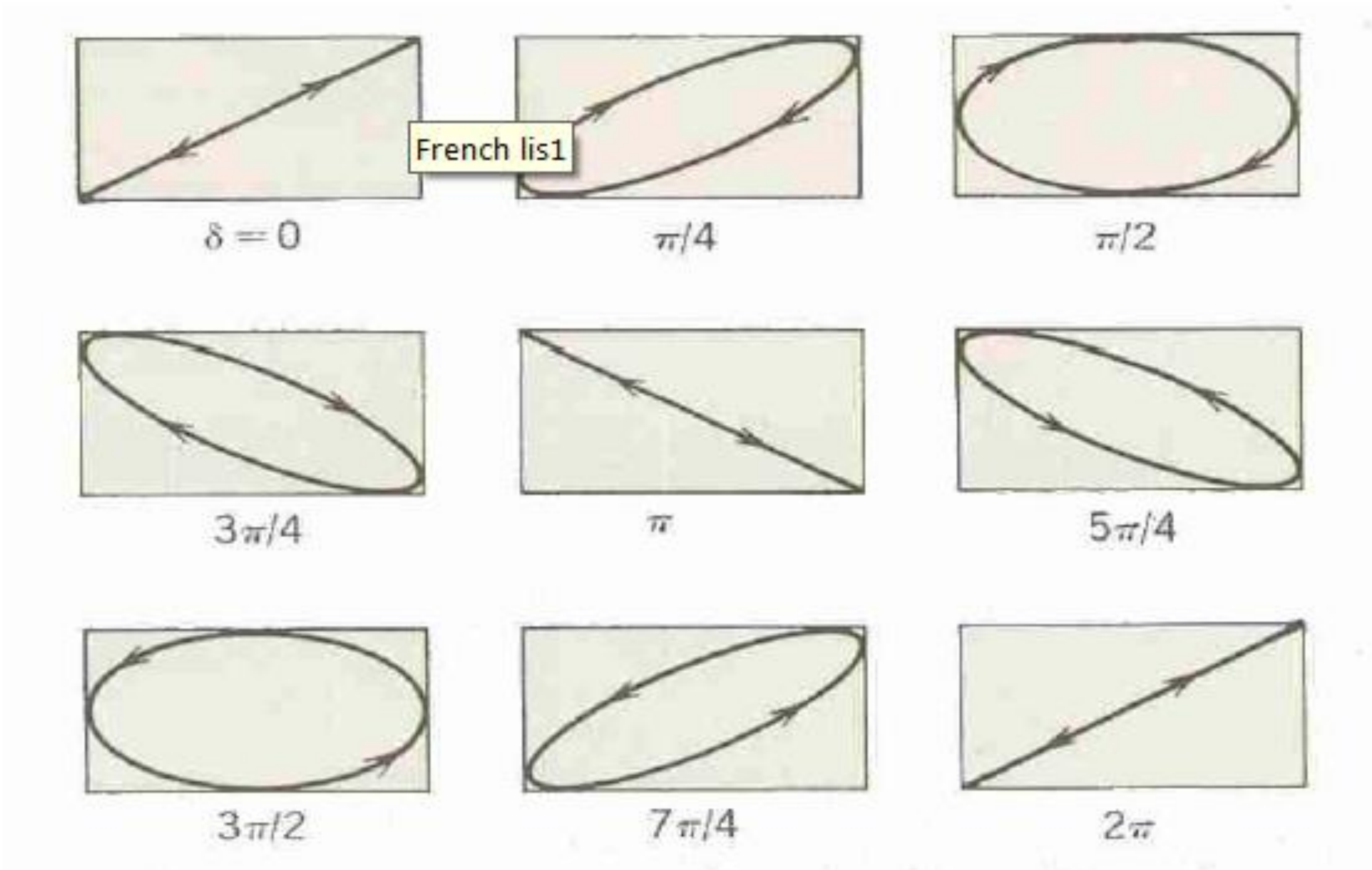
Elliptical path in anticlockwise direction

$$\text{Case 5}: \ \phi = \pi/4 \qquad \begin{aligned} x &= A_1 \cos(\omega_0 t) \\ y &= A_2 \cos(\omega_0 t + \pi/4) \end{aligned}$$

Harder to see … use a graphical approach …



Superposition of simple harmonic vibrations at right angles with an initial phase difference of $\pi/4$



Superposition of two perpendicular simple harmonic motions of the same frequency for various initial phase differences.

Lssajous' Figures: When particle is influenced simultaneously by two simple harmonic motion at right angles to each Other , the resultant motion of the particle traces a curve. This curves are called Lissajous' figures. The shape of the curves Depend on the time period, phase difference and amplitude of the constituent vibrations.

Wave Motion

- ✓ Differential Equation of Wave motion
- ✓ Progressive and Standing Waves
- ✓ Group and Phase Velocity
- ✓ Power and Intensity of Wave Motion

Review: Simple Harmonic Motion

- The position x of an object moving in simple harmonic motion as a function of time has the following form:

$$x = A\cos(\omega t+\phi)$$

i.e. the object periodically moves back and forth between the amplitudes $x=+A$ and $x=-A$.

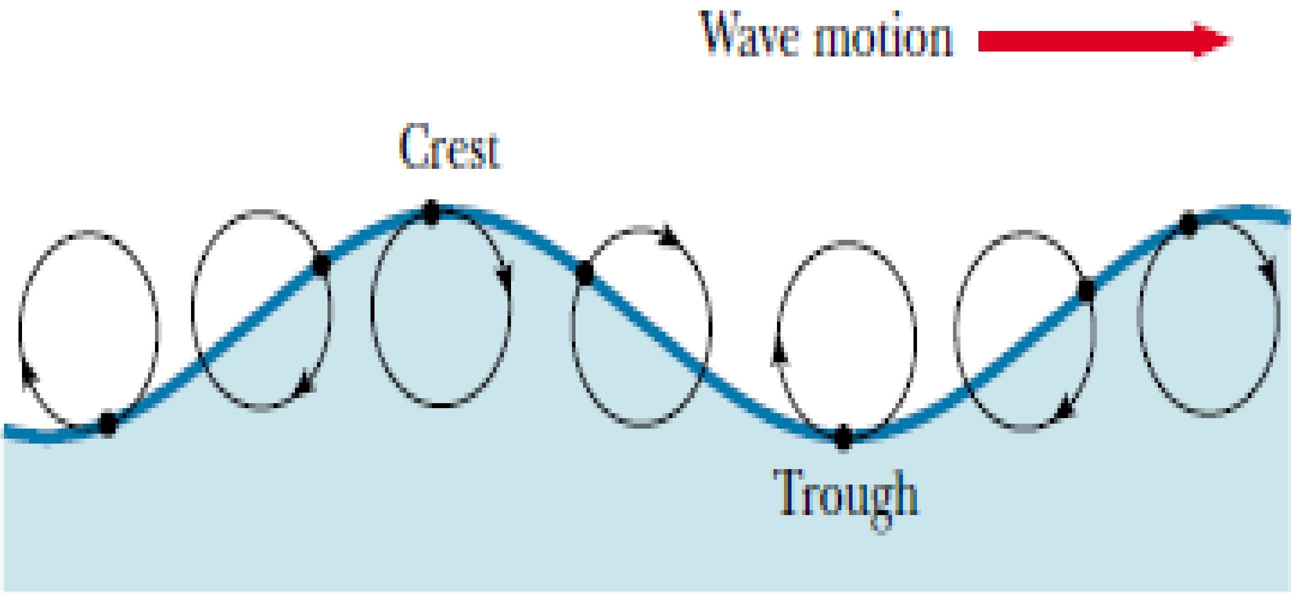
The time it takes for the object to make one full cycle is the period $T=2\pi/\omega=1/f$, where f is the frequency of the motion.

Thus, the angular speed in terms of T and f reads

$$\omega = 2\pi/T \quad \textbf{and} \quad \omega = 2\pi f$$

What is a wave ?

- Nature of waves:
  - ➔A wave is a traveling disturbance that transports energy from place to place.
  - ➔There are two basic types of waves: transverse and longitudinal.
  - ➔ Transverse: the disturbance travels perpendicular to the direction of travel of the wave.
  - ➔ Longitudinal: the disturbance occurs parallel to the line of travel of the wave.


Wave motion
Crest
Trough

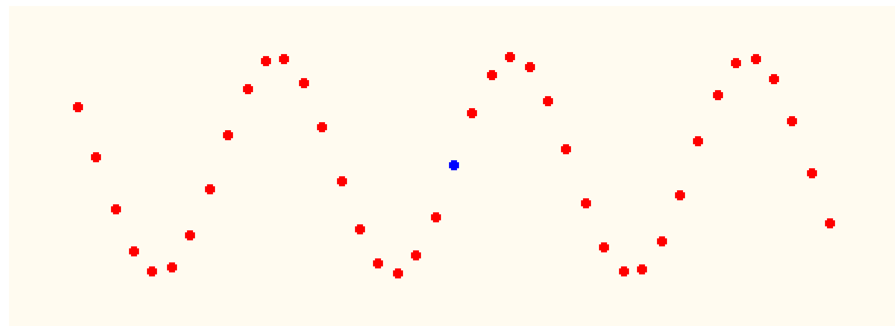
- Examples:
  - ➔ Longitudinal: Sound waves (e.g. air moves back & forth)
  - ➔ Transverse: Light waves (electromagnetic waves, i.e. electric and magnetic disturbances)

The source of the wave, i.e. the disturbance, moves continuously in

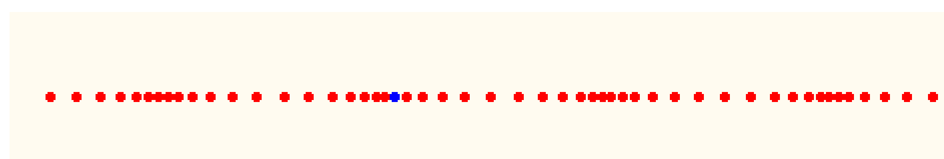
simple harmonic motion, generating an entire wave, where each part of the wave also performs a simple harmonic motion.

Types of Waves

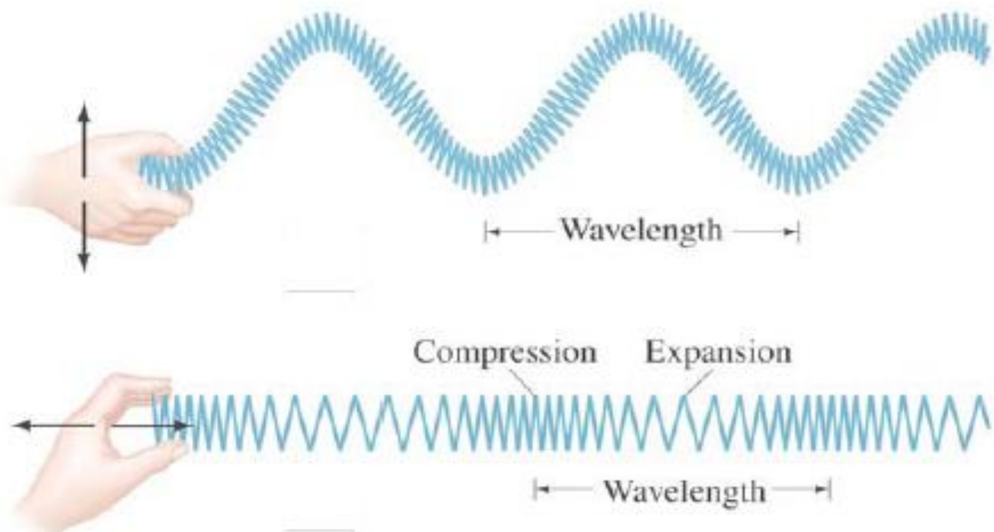
- **Transverse:** The medium oscillates perpendicular to the direction the wave is moving.
  - ➔ Water waves



- **Longitudinal:** The medium oscillates in the same direction as the wave is moving
  - ➔Sound



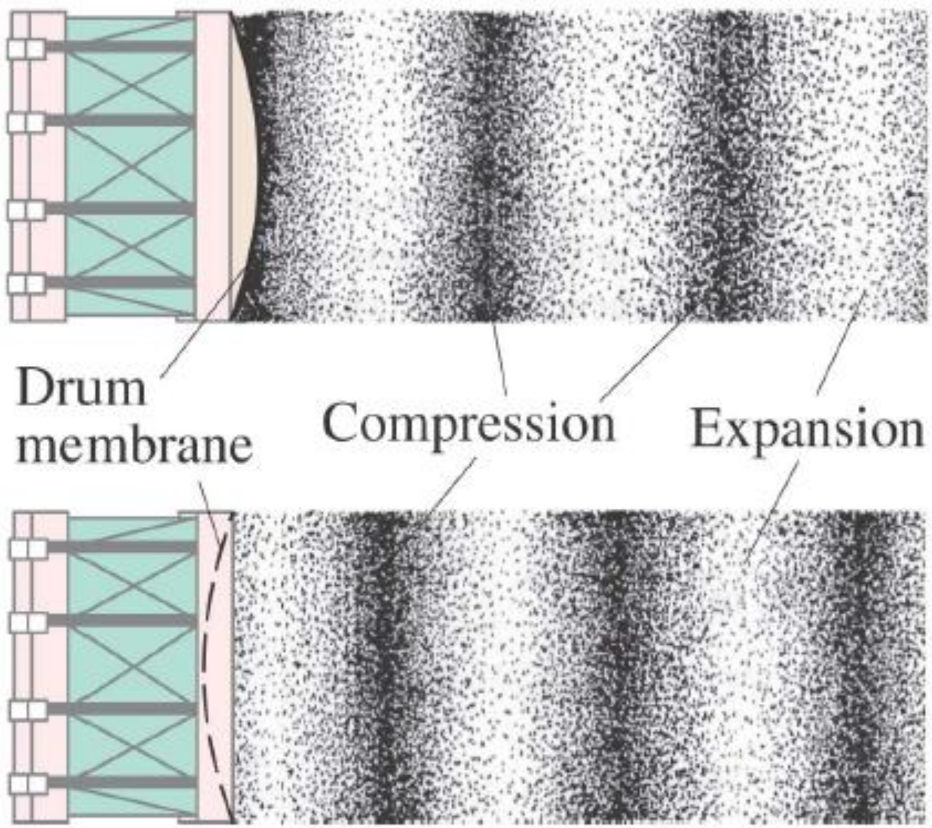
Types of Waves: Transverse and Longitudinal



The motion of particles in a wave can be either **perpendicular** to the wave direction (**transverse**) or **parallel** to it (**longitudinal**).

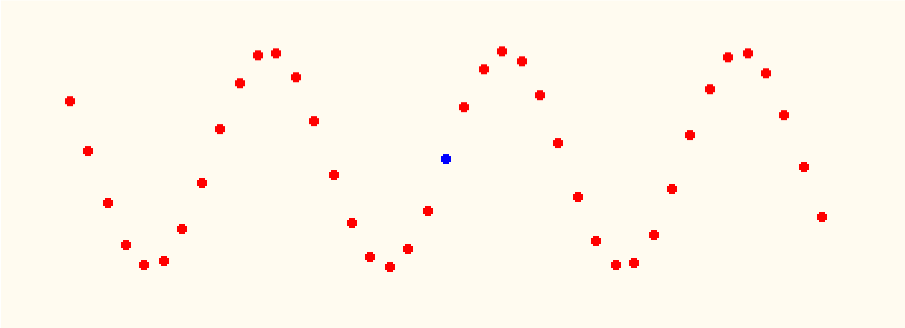
Types of Waves: Transverse and Longitudinal

Sound waves are longitudinal waves:



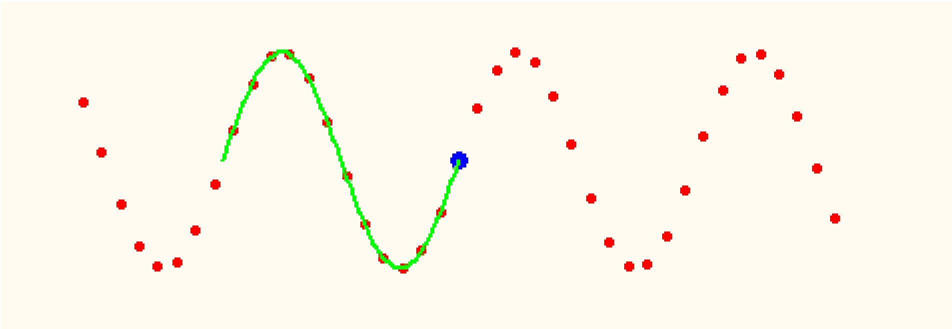
Wave Properties...

- Period: The time T for a point on the wave to undergo one complete oscillation.



- Speed: The wave moves one wavelength λ in one period T so its speed is $v = \lambda / T$.

$$\boxed{v = \frac{\lambda}{T}}$$



Wave Properties...

- The speed of a wave is a *constant* that depends only on the medium, not on the amplitude, wavelength or period:

  λ and T are related !

  $\lambda = vT$ or $\lambda = 2\pi v / \omega$ (since $T = 2\pi / \omega$)

  or $\lambda = v / f$ (since $T = 1/ f$)

- Recall f = cycles/sec or revolutions/sec

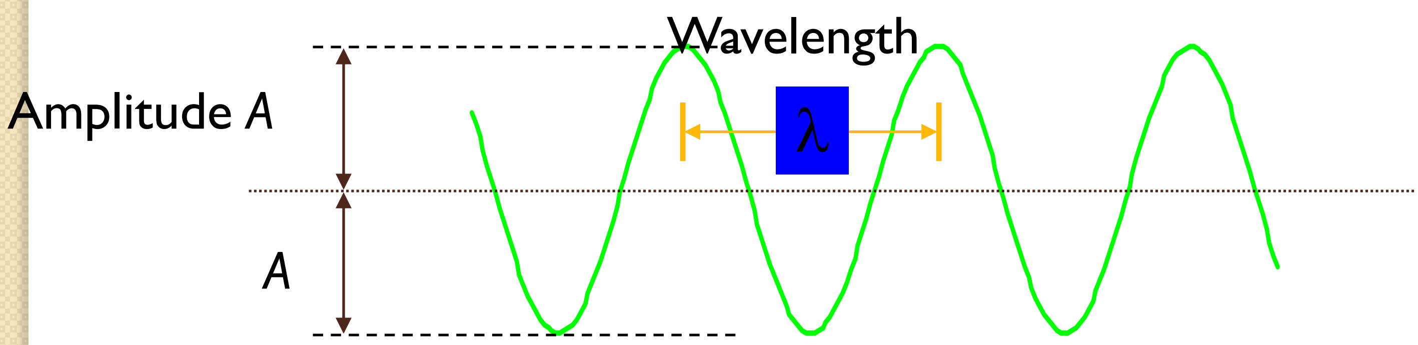
  $\omega = 2\pi f$

Is the speed of a wave particle the same as the speed of the wave ?

No. Wave particle performs simple harmonic motion: $v = A\,\omega \sin \omega t$.

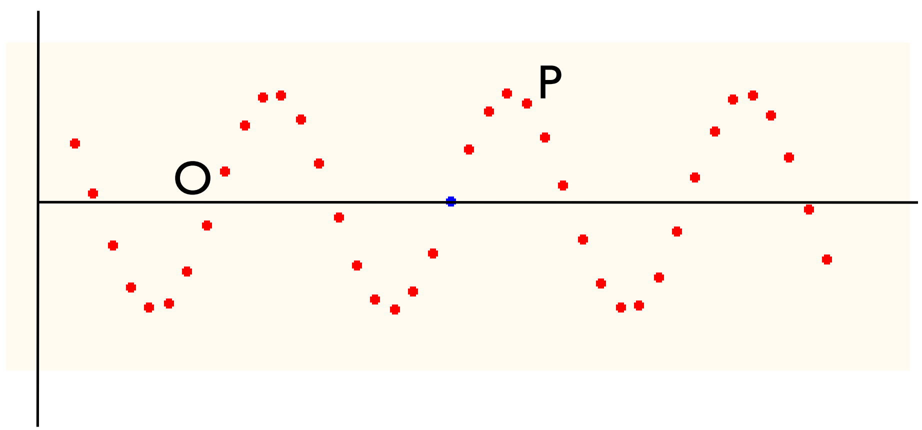
Wave Properties

- Amplitude: The maximum displacement *A* of a point on the
- wave.
- Wavelength: The distance λ between identical points
- on the wave.



Equation for a Progressive Wave

The simplest type of wave is the one in which the particles of the medium are set into simple harmonic vibrations as the wave passes through it. The wave is then called a simple harmonic wave.



Consider a particle O in the medium.
The displacement at any instant of time is given by

$$y = A \sin \omega t \ldots\ldots.(1)$$

Where A is the amplitude, ω is the angular frequency of the wave. Consider a particle P at a distance x from the particle O on its right. Let the wave travel with a velocity v from left to right. Since it takes some time for the disturbance to reach P, its displacement can be written as

$$y = A \sin(\omega t - \phi) \ldots\ldots\ldots(2)$$

Where ϕ is the phase difference between the particles O and P.

We know that a path difference of λ corresponds to a phase difference of 2π radians. Hence a path difference of x corresponds to a phase difference of

$$\frac{2\pi}{\lambda} . x$$

$$\phi = \frac{2\pi x}{\lambda}$$

Displacement of particle P is

$$y = A\sin(\omega t - \frac{2\pi x}{\lambda}) \ldots\ldots\ldots.(3)$$

$$\text{But } \omega = \frac{2\pi}{T}$$

$$y = A\sin(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}) \ldots\ldots\ldots..(5)$$

$$y = A\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

But $v = \frac{\lambda}{T}$ or $T = \frac{\lambda}{v}$

$$y = A\sin 2\pi\left(\frac{vt}{\lambda} - \frac{x}{\lambda}\right)$$

$$y = A\sin \frac{2\pi}{\lambda}(vt - x)\text{..........}(6)$$

Similarly, for a particle at a distance x to the left of 0, the equation for the displacement is given by

$$y = A\sin\frac{2\pi}{\lambda}(vt + x)\ldots\ldots\ldots\ldots\ldots(7)$$

Differential equation for wave motion

We have wave equation

$$y = A\sin\frac{2\pi}{\lambda}(vt - x)\ldots\ldots\ldots(1)$$

Differentiating equation with respect to x,
We get,

$$\frac{dy}{dx} = A\frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x)\ldots\ldots\ldots.(2)$$

$\frac{dy}{dx}$ represents the strain or the compression. When $\frac{dy}{dx}$ is positive, a rarefaction takes place and when $\frac{dy}{dx}$ is negative, a compression takes place.

The velocity of the particle whose displacement y is represented by equation, is obtained by differentiating it with respect to t, since velocity is the rate of change of displacement with respect to time.

$$\frac{dy}{dt} = A\frac{2\pi v}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x)\ldots\ldots\ldots(3)$$

Comparing equations (2) and (3) we get,

$$\frac{dy}{dt} = v\frac{dy}{dx}.........(4)$$

Particle velocity = wave velocity x slope of the displacement curve or strain.

Differentiating equation (2)

$$\frac{d^2y}{dx^2} = -A\frac{4\pi}{\lambda^2}^2 \sin\frac{2\pi}{\lambda}(vt - x).........(5)$$

Differentiating equation (3)

$$\frac{d^2y}{dt^2} = -A\frac{4\pi^2v^2}{\lambda^2}\sin\frac{2\pi}{\lambda}(vt-x)\ldots\ldots\ldots.(6)$$

Comparing equs. (5) and (6)

$$\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}\ldots\ldots\ldots(7)$$

Equation (7) represents the differential equation of wave velo0city

Ex.-17: The equation of a traveling wave is

$$y = 4.0 \sin \pi (0.10x - 2t)$$

Find (i) wavelength, (ii) speed and (iii) frequency of oscillating particle of the wave

Ex-18: When a simple harmonic wave is propagated through a medium, the displacement of a paricle in cm at any instant is

$$y = 10\sin\frac{2\pi}{100}(36000t - 20)$$

Calculate the amplitude, wave velocity, wavelength, frequency and period of the oscillating particle.

Ex-19: The equation of a traveling wave is

$$y = 3.0 \sin 2\pi(0.10t - 2x)$$

Find (i) amplitude (ii) wavelength (iii) speed (iv) frequency of wave

Ex-20: When a simple harmonic wave is propagated through a medium, the displacement of the particle at any instant of time is given by

$$y = 5.0 \sin \pi (360t - 0.15x)$$

calculate

(i)the amplitude of the vibrating particle,

(ii)wave velocity,

(ii)wave length,

(iv)frequency and

(v) time period.

Ex-21: A simple harmonic wave of amplitude 8units travels a line of particles in the direction of positive X axis. At any instant for a particle at a distance of 10cm from the origin, the displacement is +6units and at a distance a particle from the origin is 25units, the displacement is +4units. Calculate the wavelength.

Ex-22: Rank the waves represented by the following functions from the largest to the smallest according to

(i) their amplitudes, (ii) their wavelengths, (iii) their

(ii) frequencies, (iv) their periods, and (v) their speeds.

(iii) For all functions, x and y are in meters and t is in

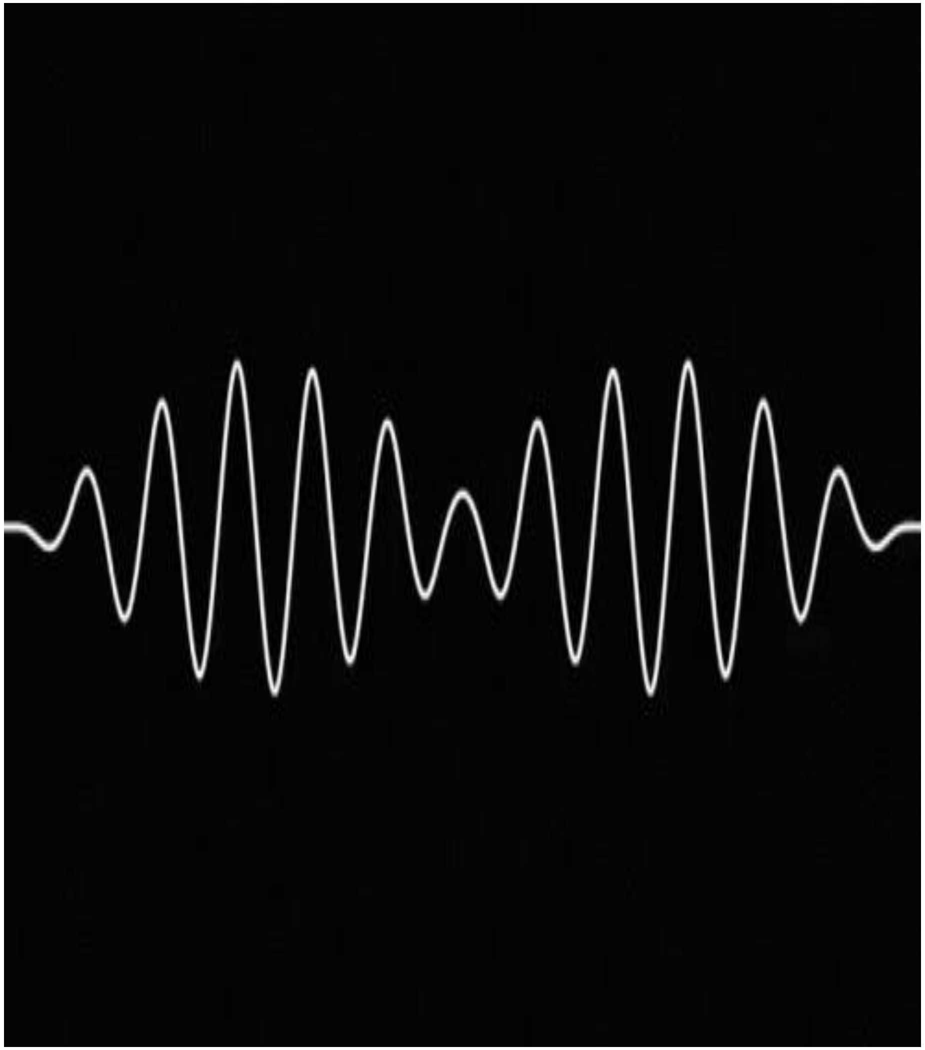
(iv) seconds. (a) $y = 4 \sin(3x - 15t)$,

(b) $y = 6 \cos(3x + 15t - 2)$, (c) $y = 8 \sin(2x + 15t)$,

(d) $y = 8 \cos(4x + 20t)$, and (e) $y = 7 \sin(6x - 24t)$

Wave

Phase Velocity and Group Velocity of Wave



Principle of Superposition of wave

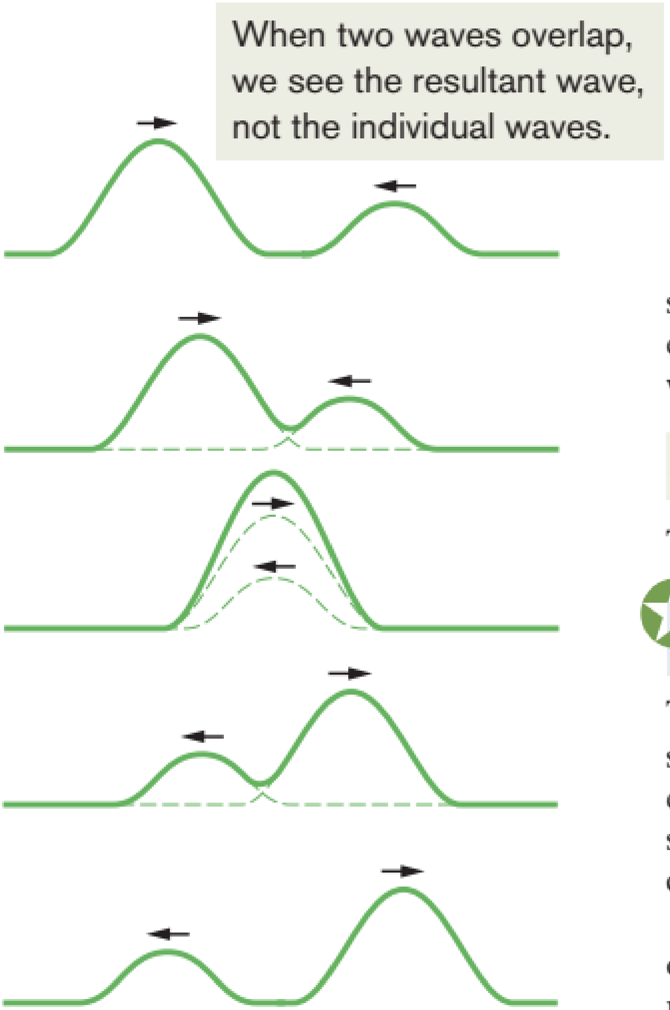


Figure 16-12 A series of snapshots that show two pulses traveling in opposite directions along a stretched string. The superposition principle applies as the pulses move through each other.

Suppose that two waves travel simultaneously along the same stretched string. Let $y_1(x, t)$ and $y_2(x, t)$ be the displacements that the string would experience if each wave traveled alone. The displacement of the string when the waves overlap is then the algebraic sum

$$y'(x,t) = y_1(x,t) + y_2(x,t). \tag{16-46}$$

This summation of displacements along the string means that

Overlapping waves algebraically add to produce a **resultant wave** (or **net wave**).

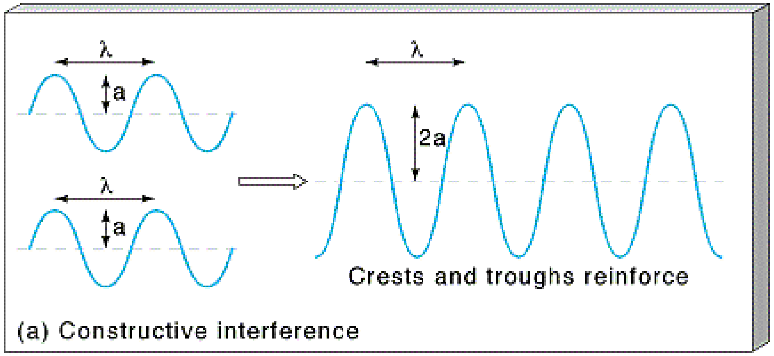
This is another example of the **principle of superposition,** which says that when several effects occur simultaneously, their net effect is the sum of the individual effects. (We should be thankful that only a simple sum is needed. If two effects somehow amplified each other, the resulting nonlinear world would be very difficult to manage and understand.)

Figure 16-12 shows a sequence of snapshots of two pulses traveling in opposite directions on the same stretched string. When the pulses overlap, the resultant pulse is their sum. Moreover,

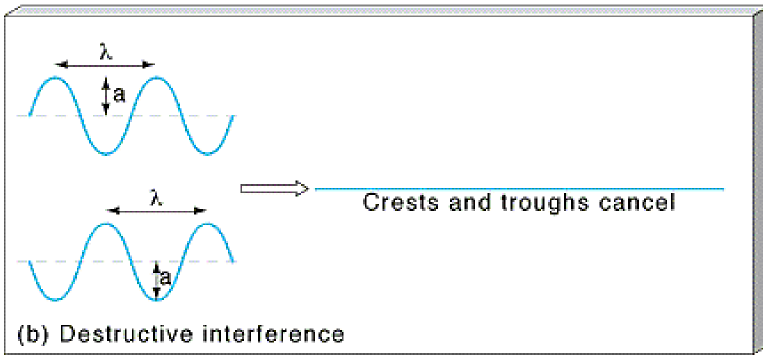
Overlapping waves do not in any way alter the travel of each other.

Interference

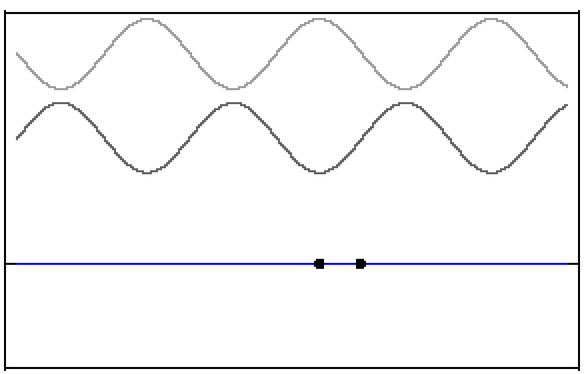
waves can interfere (add or cancel)



(a) Constructive interference



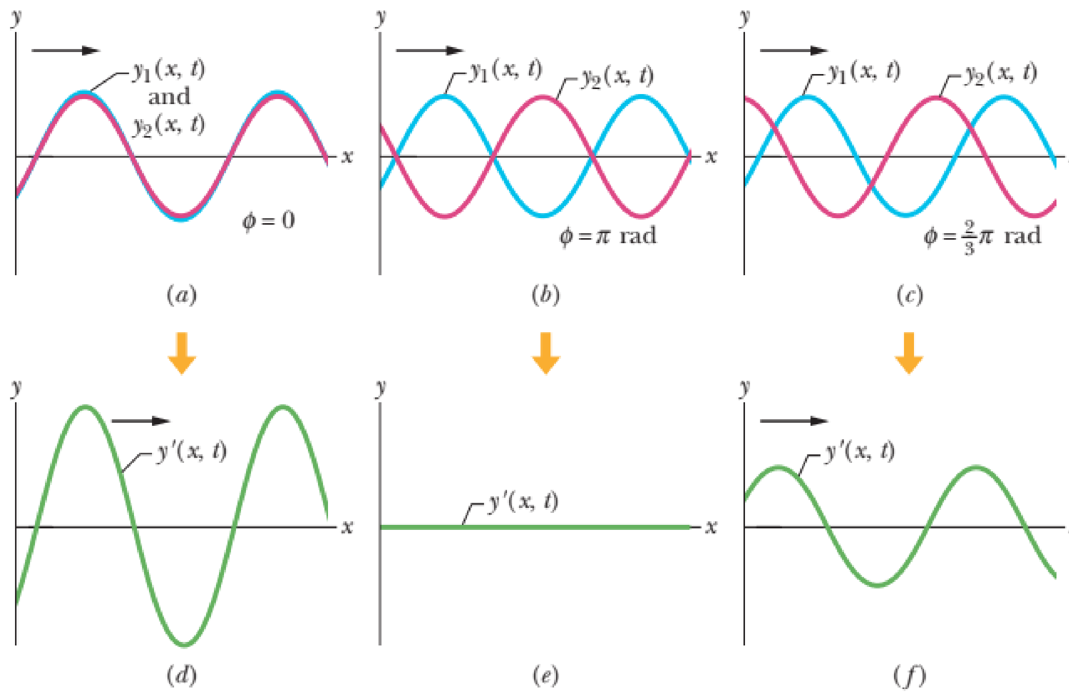
(b) Destructive interference



Being exactly in phase, the waves produce a large resultant wave.

Being exactly out of phase, they produce a flat string.

This is an intermediate situation, with an intermediate result.



Interference

The resultant wave depends on the extent to which the waves are *in phase* (in step) with respect to each other—that is, how much one wave form is shifted from the other wave form. If the waves are exactly in phase (so that the peaks and valleys of one are exactly aligned with those of the other), they combine to double the displacement of either wave acting alone. If they are exactly out of phase (the peaks of one are exactly aligned with the valleys of the other), they combine to cancel everywhere, and the string remains straight. We call this phenomenon of combining waves **interference,** and the waves are said to **interfere.** (These terms refer only to the wave displacements; the travel of the waves is unaffected.)

Let one wave traveling along a stretched string be given by

$$y_1(x, t) = y_m \sin(kx - \omega t) \tag{16-47}$$

and another, shifted from the first, by

$$y_2(x, t) = y_m \sin(kx - \omega t + \phi). \tag{16-48}$$

Interference

From the principle of superposition (Eq. 16-46), the resultant wave is the algebraic sum of the two interfering waves and has displacement

$$\begin{aligned} y'(x,t) &= y_1(x,t) + y_2(x,t) \\ &= y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi). \end{aligned} \qquad (16\text{-}49)$$

In Appendix E we see that we can write the sum of the sines of two angles α and β as

$$\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta). \qquad (16\text{-}50)$$

Applying this relation to Eq. 16-49 leads to

$$y'(x,t) = [2y_m \cos\tfrac{1}{2}\phi]\sin(kx - \omega t + \tfrac{1}{2}\phi). \qquad (16\text{-}51)$$

As Fig. 16-13 shows, the resultant wave is also a sinusoidal wave traveling in the direction of increasing x. It is the only wave you would actually see on the string (you would *not* see the two interfering waves of Eqs. 16-47 and 16-48).

Condition for Constructive Interference

The resultant wave differs from the interfering waves in two respects: (1) its phase constant is $\frac{1}{2}\phi$, and (2) its amplitude y'_m is the magnitude of the quantity in the brackets in Eq. 16-51:

$$y'_m = |2y_m \cos \tfrac{1}{2}\phi| \qquad \text{(amplitude).} \tag{16-52}$$

If $\phi = 0$ rad (or 0°), the two interfering waves are exactly in phase and Eq. 16-51 reduces to

$$y'(x, t) = 2y_m \sin(kx - \omega t) \qquad (\phi = 0). \tag{16-53}$$

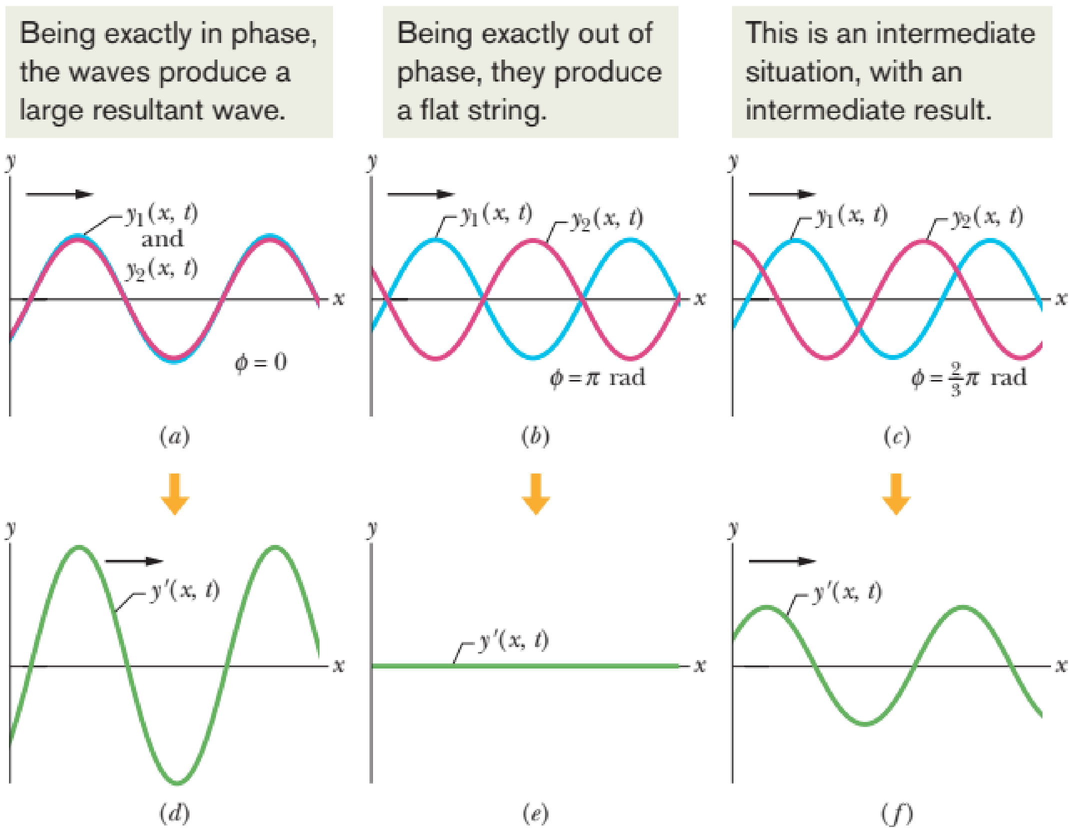
Condition for Destructive Interference

If $\phi = \pi$ rad (or 180°), the interfering waves are exactly out of phase as in Fig. 16-14*b*. Then $\cos \frac{1}{2}\phi$ becomes $\cos \pi/2 = 0$, and the amplitude of the resultant wave as given by Eq. 16-52 is zero. We then have, for all values of x and t,

$$y'(x, t) = 0 \qquad (\phi = \pi \text{ rad}). \tag{16-54}$$

The resultant wave is plotted in Fig. 16-14*e*. Although we sent two waves along the string, we see no motion of the string. This type of interference is called *fully destructive interference*.

Constructive and Destructive Interference



Ex-23:

Mathematical Problems

Two identical sinusoidal waves, moving in the same direction along a stretched string, interfere with each other. The amplitude y_m of each wave is 9.8 mm, and the phase difference ϕ between them is 100°.

(a) What is the amplitude y'_m of the resultant wave due to the interference, and what is the type of this interference?

Calculations: Because they are identical, the waves have the *same amplitude*. Thus, the amplitude y'_m of the resultant wave is given by Eq. 16-52:

$$y'_m = |2y_m \cos \tfrac{1}{2}\phi| = |(2)(9.8 \text{ mm}) \cos(100°/2)|$$
$$= 13 \text{ mm.} \qquad \text{(Answer)}$$

We can tell that the interference is *intermediate* in two ways. The phase difference is between 0 and 180°, and, correspondingly, the amplitude y'_m is between 0 and $2y_m$ (= 19.6 mm).

(b) What phase difference, in radians and wavelengths, will give the resultant wave an amplitude of 4.9 mm?

Calculations: Now we are given y'_m and seek ϕ. From Eq. 16-52,

$$y'_m = |2y_m \cos \tfrac{1}{2}\phi|,$$

we now have

$$4.9 \text{ mm} = (2)(9.8 \text{ mm}) \cos \tfrac{1}{2}\phi,$$

which gives us (with a calculator in the radian mode)

$$\begin{aligned}\phi &= 2 \cos^{-1} \frac{4.9 \text{ mm}}{(2)(9.8 \text{ mm})} \\ &= \pm 2.636 \text{ rad} \approx \pm 2.6 \text{ rad.} \qquad \text{(Answer)}\end{aligned}$$

There are two solutions because we can obtain the same resultant wave by letting the first wave *lead* (travel ahead of) or *lag* (travel behind) the second wave by 2.6 rad. In wavelengths, the phase difference is

$$\begin{aligned}\frac{\phi}{2\pi \text{ rad/wavelength}} &= \frac{\pm 2.636 \text{ rad}}{2\pi \text{ rad/wavelength}} \\ &= \pm 0.42 \text{ wavelength.} \qquad \text{(Answer)}\end{aligned}$$

SUPERPOSED WAVE OF DIFFERENT FREQUENCIES (VERY SMALL DIFFERENCE) AND BEATS

We will now discuss the superposition of two waves that have same vibration direction, same amplitude **A**, but different frequency and wave number(**ω_1**, **k_1**) and (**ω_2**, **k_2**). However, the frequency difference is very small. This will generate the very interesting "beat" phenomenon.

Since the phase difference between the vibrations is continually changing, the specification of some initial nonzero phase difference is in general not of major significance in this case.

$$y_1 = A\cos(k_1 x - \omega_1 t)$$

$$y_2 = A\cos(k_2 x - \omega_2 t)$$

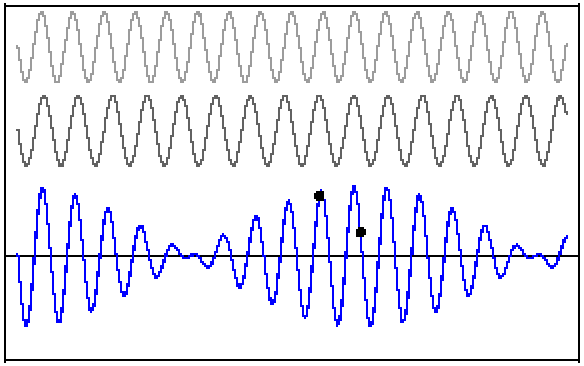
Interefering waves, generally…

Phase Velocity and Group Velocity

$$y = y_1 + y_2 = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t)$$

$$\Downarrow$$

$$y = 2A\cos\frac{1}{2}\{(k_2 - k_1)x - (\omega_2 - \omega_1)t\} \bullet \cos\frac{1}{2}\{(k_1 + k_2)x - (\omega_1 + \omega_2)t\}$$



"Beats" occur when you add two waves of slightly different frequency. They will interfere constructively in some areas and destructively in others.

Can be interpreted as a sinusoidal envelope: $$2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

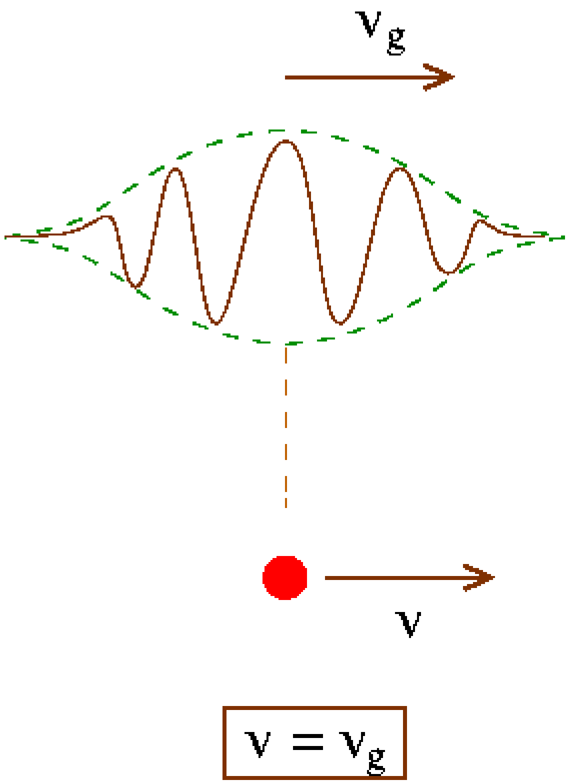
Modulating a high frequency wave within the envelope: $$\cos\left[\frac{1}{2}(k_1 + k_2)x - \frac{1}{2}(\omega_1 + \omega_2)t\right]$$

the group velocity $$\upsilon_g = \frac{(\omega_2 - \omega_1)/2}{(k_2 - k_1)/2} = \frac{\Delta\omega}{\Delta k}$$

the phase velocity $$\upsilon_p = \frac{(\omega_2 + \omega_1)/2}{(k_2 + k_1)/2} \approx \frac{\omega_1}{k_1} = \upsilon_1$$

$\boxed{\text{if } \upsilon_1 \approx \upsilon_2}$

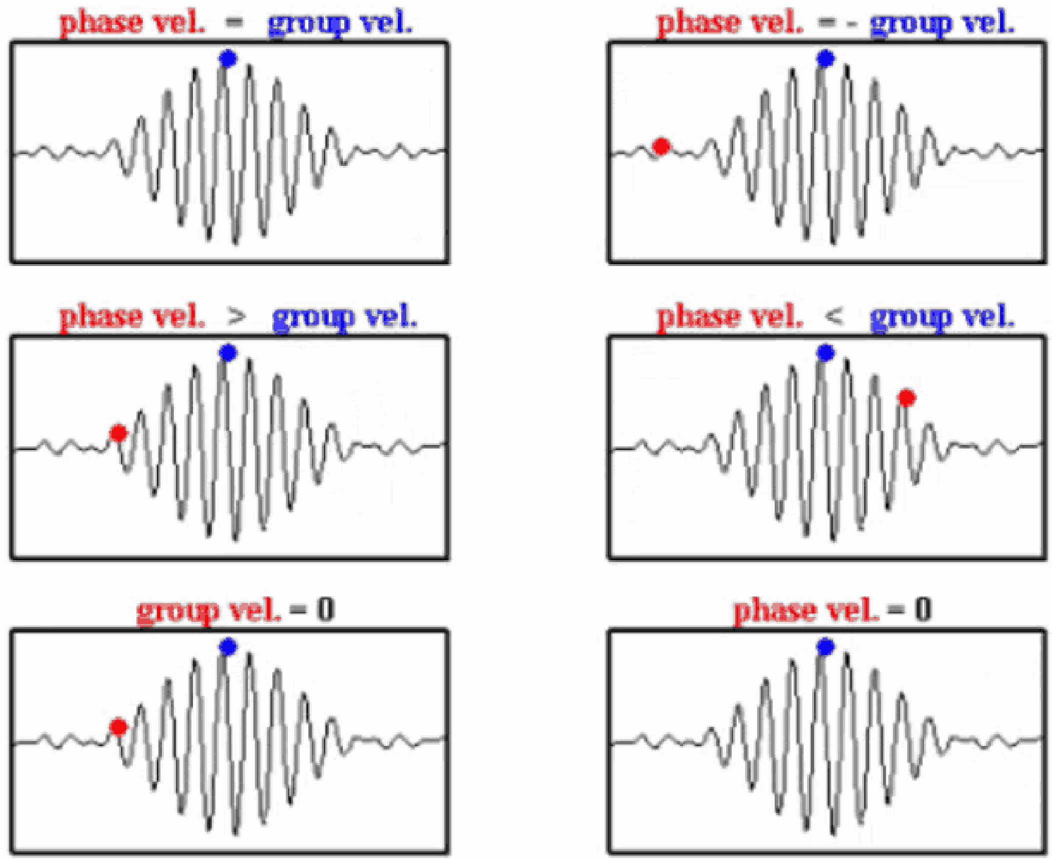
Phase Velocity and Group Velocity



the group velocity

$$\upsilon_g = \frac{(\omega_2 - \omega_1)/2}{(k_2 - k_1)/2} = \frac{\Delta\omega}{\Delta k}$$

Listen to the beats!



Phase Velocity and Group Velocity

The first term is the wave and the second term is the envelope

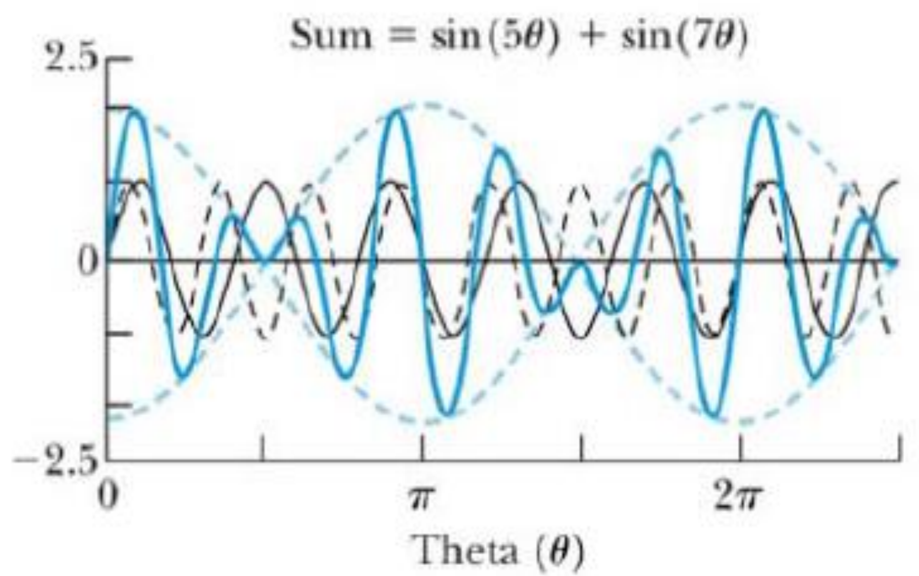
$$y = 2A\cos(k_{av}x - \omega_{av}t)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

In the previous equation,

$$y = 2A\cos(k_{av}x - \omega_{av}t)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

where $k_{av} = \dfrac{k_1 + k_2}{2}$ and $\Delta k = k_1 - k_2$

and similarly for ω



Phase Velocity and Group Velocity

- The combined wave has phase velocity

$$v_{phase} = \frac{\omega_{av}}{k_{av}}$$

- The group (envelope) has group velocity

$$v_{group} = \frac{\Delta\omega}{\Delta k}$$

- More generally the group velocity is

$$v_{group} = \frac{d\omega(k)}{dk}$$

Phase Velocity and Group Velocity

- Recall our problem with the wave velocity being slower than the particle velocity
- Using the group velocity we find

$$v_{group} = \frac{d\omega}{dk} = \frac{d\hbar\omega}{d\hbar k} = \frac{dE}{dp}$$

$$E = \frac{p^2}{2m}$$

$$v_{group} = \frac{p}{m} = V$$

- Problem solved

Phase Velocity and Group Velocity: Application

- The relation between ω and k is called the dispersion relation
- Examples
  - Photon in vacuum

$$f = \frac{c}{\lambda}$$

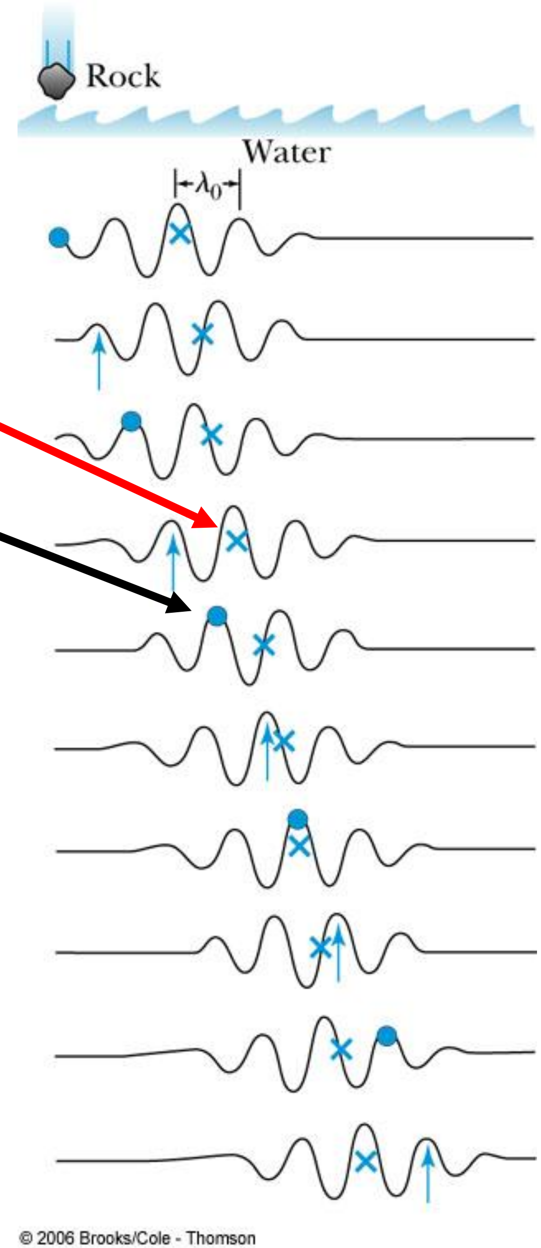
$$2\pi f = \frac{2\pi c}{\lambda}$$

$$\omega = kc$$

$$\frac{d\omega}{dk} = c$$

Phase Velocity and Group Velocity: Application

- Phase and group velocity



Phase Velocity and Group Velocity: Application

- Examples
  - Photon in a medium

$$f = \frac{c}{n(\lambda)\lambda}$$

$$2\pi f = \frac{2\pi c}{n(\lambda)\lambda}$$

$$\omega = \frac{kc}{n(k)}$$

$$\frac{d\omega}{dk} = \frac{c}{n(k)} + kc\frac{d}{dk}\frac{1}{n(k)}$$

Phase Velocity and Group Velocity: Application

- Examples
  - de Broglie waves

$$E = hf = \hbar\omega$$

$$E = \frac{p^2}{2m} = \frac{(\hbar k)^2}{2m}$$

$$\omega = \frac{\hbar k^2}{2m}$$

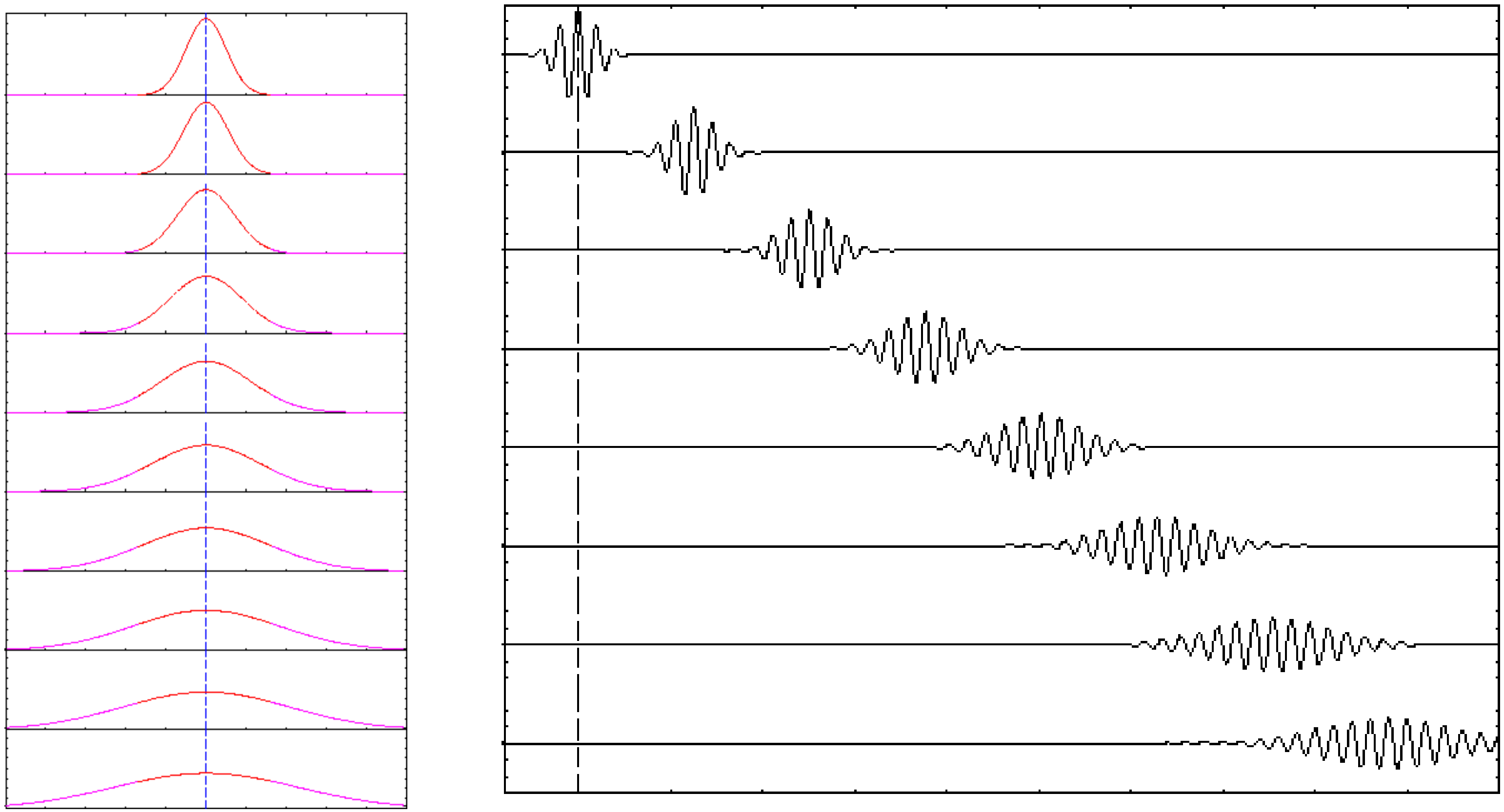
$$\frac{d\omega}{dk} = \frac{\hbar k}{m}$$

Phase Velocity and Group Velocity: Application

- Because the group velocity = $d\omega/dk$ depends on k, the wave packet will disperse with time
  - This is because each of the waves is moving at a slightly different velocity
  - Just as white light will be dispersed as it travels through a prism
- That's why $\omega(k)$ is called a dispersion relation

Phase Velocity and Group Velocity: Application

- Spreading of a wave packet with time



- Does this mean matter waves disperse with time?

Mathematical Example

Ex-24:

Two cosine waves have phase velocity V_1= 2 cm/s, V_2= 3 cm/s, and corresponding wavelength λ_1= 4 cm, λ_2= 3 cm. Find out the angular frequency and group velocity.

Hint:

$V_p = \frac{\omega}{k} = C_p$

$\omega = V_p\, k = C_p k$

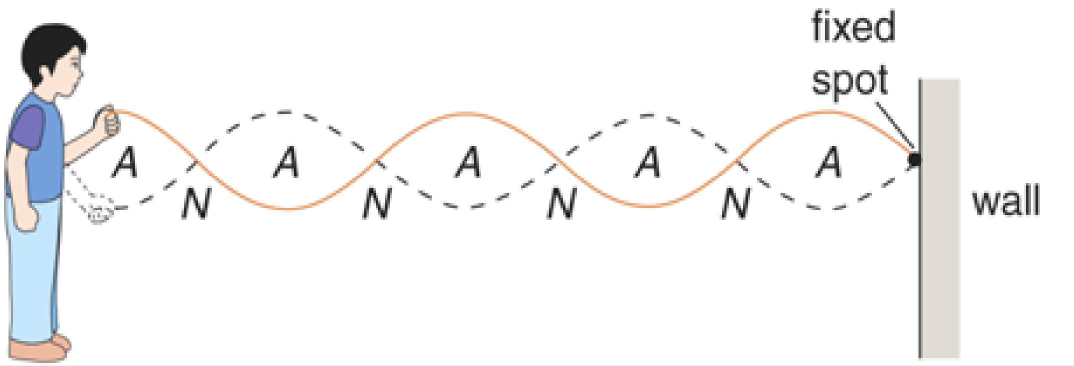
$\omega_1 = \pi$ and $\omega_2 = 2\pi$

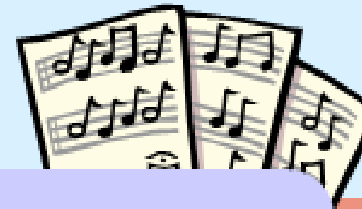
$V_g = \frac{\Delta\omega}{\Delta k}$

Introduction to stationary waves

Stationary waves (standing waves) produced in
- string of guitar or piano
- rope (one end tie to wall)

- **Incident wave** is reflected by the wall to from **reflected wave**
- **superposition** of the waves produces stationary wave

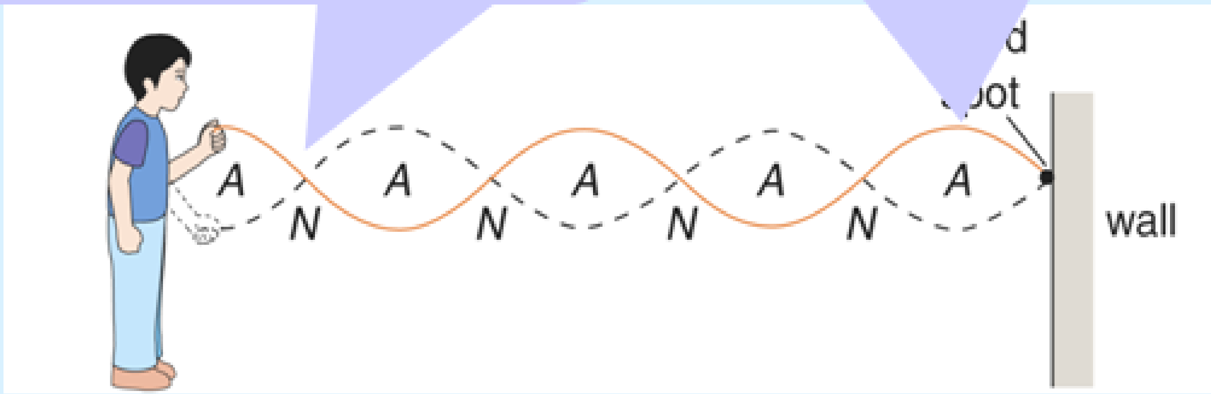




Introduction to stationary waves

N – nodes (remain stationary)

A – antinodes (largest amplitude of oscillation)



A stationary wave is produced by the superposition of two progressive waves of the same amplitude and frequency but travelling in opposite directions.

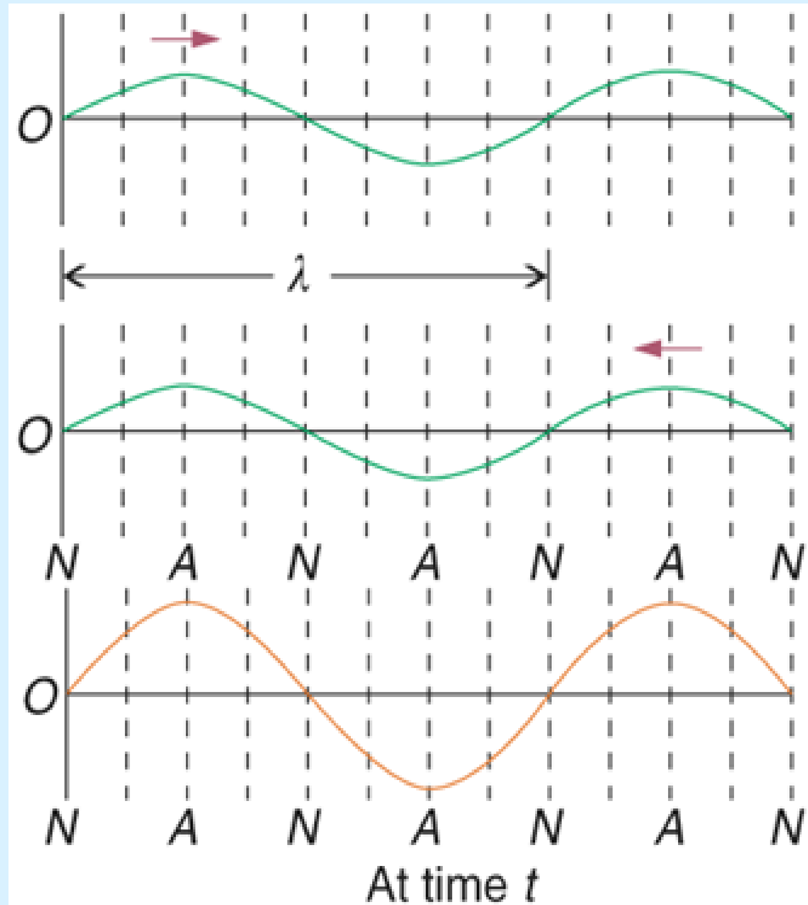
Introduction to stationary waves

(a) at time *t* (constructive interference)

(1) To right

(2) To left

(1) + (2) = (3)



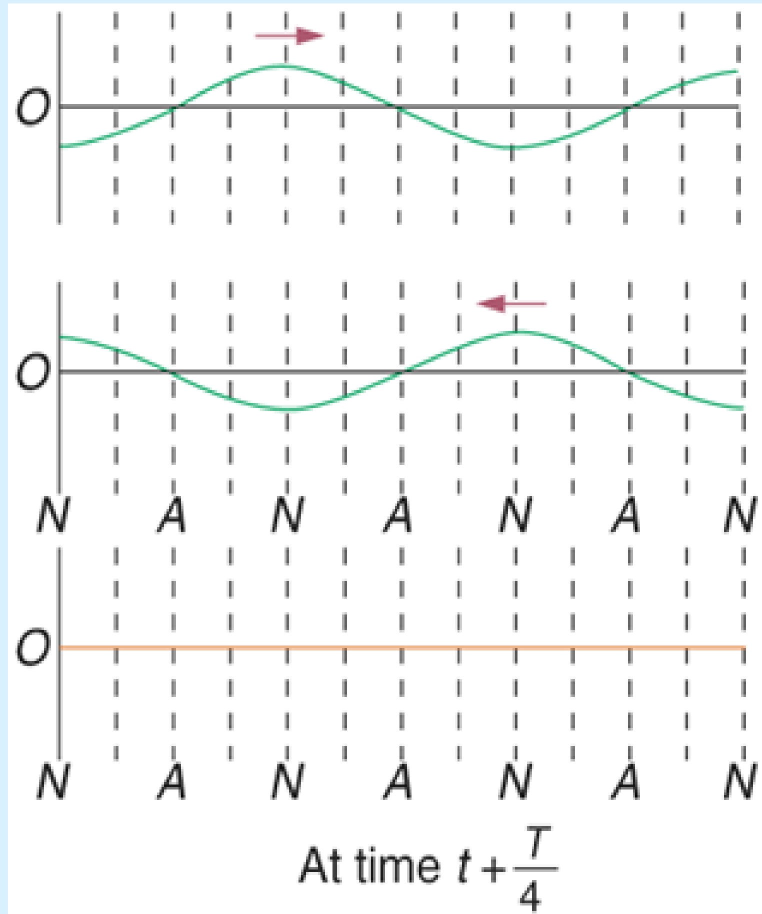
Introduction to stationary waves

(b) at time $t + T/4$ (destructive interference)

(1)

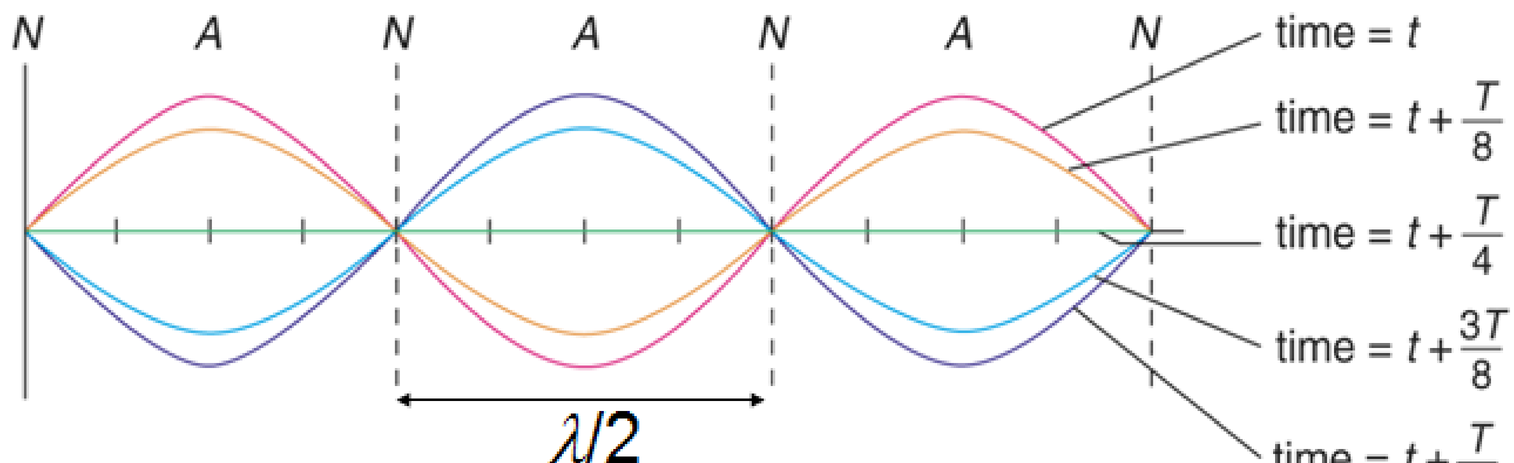
(2)

(1) + (2) = (3)



Introduction to stationary waves

Vibration of stationary wave at various instants



$$y_1 = a\sin\left(\omega t - \frac{2\pi x}{\lambda}\right), \quad y_2 = a\sin\left(\omega t + \frac{2\pi x}{\lambda}\right)$$

$$y = y_1 + y_2 = \left(2a\cos\frac{2\pi x}{\lambda}\right)\sin\omega t$$

Amplitude (A) of resultant stationary wave

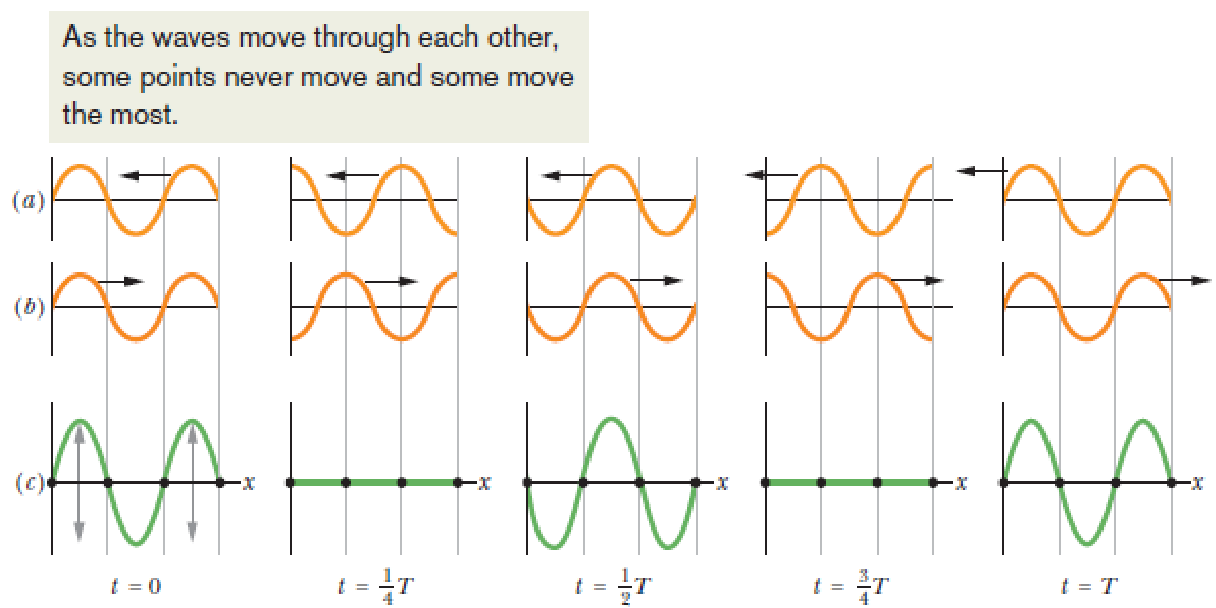
Standing Wave: Node and Antinode Points

Standing Waves

In Module 16-5, we discussed two sinusoidal waves of the same wavelength and amplitude traveling *in the same direction* along a stretched string. What if they travel in opposite directions? We can again find the resultant wave by applying the superposition principle.

Figure 16-17 suggests the situation graphically. It shows the two combining waves, one traveling to the left in Fig. 16-17*a*, the other to the right in Fig. 16-17*b*. Figure 16-17*c* shows their sum, obtained by applying the superposition

Figure 16-17 (*a*) Five snapshots of a wave traveling to the left, at the times t indicated below part (*c*) (T is the period of oscillation). (*b*) Five snapshots of a wave identical to that in (*a*) but traveling to the right, at the same times t. (*c*) Corresponding snapshots for the superposition of the two waves on the same string. At $t = 0, \frac{1}{2}T$, and T, fully constructive interference occurs because of the alignment of peaks with peaks and valleys with valleys. At $t = \frac{1}{4}T$ and $\frac{3}{4}T$, fully destructive interference occurs because of the alignment of peaks with valleys. Some points (the nodes, marked with dots) never oscillate; some points (the antinodes) oscillate the most.



Standing Wave: Node and Antinode Points

Finding the node and antinode points of different signals

If two sinusoidal waves of the same amplitude and wavelength travel in *opposite* directions along a stretched string, their interference with each other produces a standing wave.

Node and antinode from the following signals: $y_1(x, t) = y_m \sin(kx - \omega t)$

$$y_2(x, t) = y_m \sin(kx + \omega t).$$

To analyze a standing wave, we represent the two waves with the equations

$$y_1(x, t) = y_m \sin(kx - \omega t) \tag{16-58}$$

and

$$y_2(x, t) = y_m \sin(kx + \omega t). \tag{16-59}$$

The principle of superposition gives, for the combined wave,

$$y'(x, t) = y_1(x, t) + y_2(x, t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t).$$

Applying the trigonometric relation of Eq. 16-50 leads to Fig. 16-18 and

$$y'(x, t) = [2y_m \sin kx] \cos \omega t. \tag{16-60}$$

Standing Wave: Node and Antinode Points

This equation does not describe a traveling wave because it is not of the form of Eq. 16-17. Instead, it describes a standing wave.

The quantity $2y_m \sin kx$ in the brackets of Eq. 16-60 can be viewed as the amplitude of oscillation of the string element that is located at position x. However, since an amplitude is always positive and $\sin kx$ can be negative, we take the absolute value of the quantity $2y_m \sin kx$ to be the amplitude at x.

In a traveling sinusoidal wave, the amplitude of the wave is the same for all string elements. That is not true for a standing wave, in which the amplitude *varies with position*. In the standing wave of Eq. 16-60, for example, the amplitude is zero for values of kx that give $\sin kx = 0$. Those values are

$$kx = n\pi, \qquad \text{for } n = 0, 1, 2, \ldots . \tag{16-61}$$

Substituting $k = 2\pi/\lambda$ in this equation and rearranging, we get

$$x = n\frac{\lambda}{2}, \qquad \text{for } n = 0, 1, 2, \ldots \qquad \text{(nodes)}, \tag{16-62}$$

as the positions of zero amplitude—the nodes—for the standing wave of Eq. 16-60. Note that adjacent nodes are separated by $\lambda/2$, half a wavelength.

The amplitude of the standing wave of Eq. 16-60 has a maximum value of $2y_m$, which occurs for values of kx that give $|\sin kx| = 1$. Those values are

$$\begin{aligned} kx &= \tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \tfrac{5}{2}\pi, \ldots \\ &= (n + \tfrac{1}{2})\pi, \qquad \text{for } n = 0, 1, 2, \ldots . \end{aligned} \tag{16-63}$$

Standing Wave: Node and Antinode Points

Substituting $k = 2\pi/\lambda$ in Eq. 16-63 and rearranging, we get

$$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \quad \text{for } n = 0, 1, 2, \ldots \quad \text{(antinodes)}, \tag{16-64}$$

as the positions of maximum amplitude—the antinodes—of the standing wave of Eq. 16-60. Antinodes are separated by $\lambda/2$ and are halfway between nodes.

Ex-25:

Find out the resultant amplitude, node and antinode points in terms of λ of the following equations: $y_1 = A\cos(\frac{1}{3}kx + \omega t)$ and $y_2 = y_m\cos(\frac{1}{3}kx - \omega t)$.

The positions of antinodes

Progressive wave	Stationary wave
Energy is transferred along the direction of propagation.	No energy is transferred along the direction of propagation.
The wave profile moves in the direction of propagation.	The wave profile does not move in the direction of propagation.
Every point along the direction of propagation is displaced.	There are points known as nodes where no displacement occurs.
Every point has the same amplitude.	Points between two successive nodes have different amplitudes.
Neighbouring points are not in phase.	All points between two successive nodes vibrate in phase with one other.

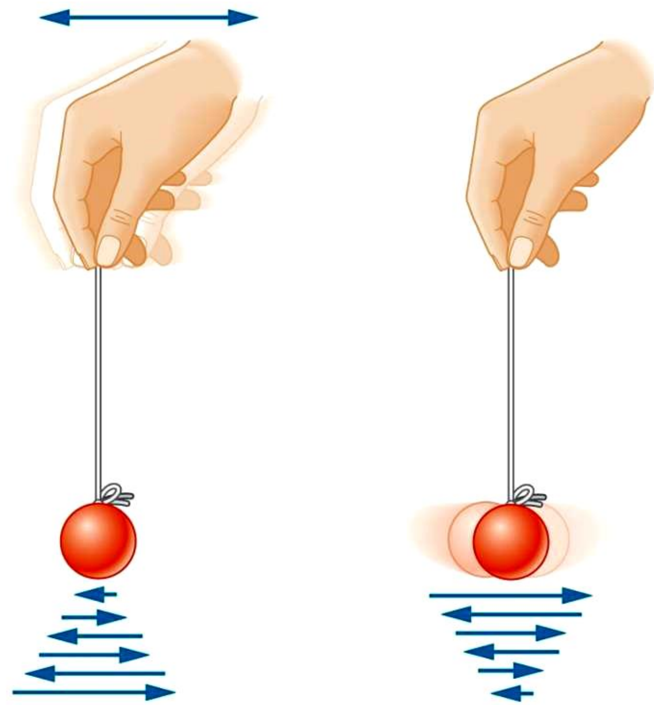
Forced Oscillations; Resonance

Forced vibrations occur when there is a periodic driving force. This force may or may not have the same period as the natural frequency of the system.

If the frequency is the same as the natural frequency, the amplitude can become quite large. This is called resonance.

Figure : Caption: (a) Large-amplitude oscillations of the Tacoma Narrows Bridge, due to gusty winds, led to its collapse (1940). (b) Collapse of a freeway in California, due to the 1989 earthquake.


m



Equation of Driven Harmonic Motion.

- Consider what happens when we apply a time-dependent force $F(t)$ to a system that normally would carry out SHM with an angular frequency ω_0.

- Assume the external force $F(\text{t}) = mF_0\sin(\omega t)$. The equation of motion can now be written as

$$\frac{d^2x}{dt^2} = -\omega_0{}^2 x + F_0 \sin\left(\omega t\right)$$

- The steady state motion of this system will be harmonic motion with an angular frequency equal to the angular frequency of the driving force.

- Consider the general solution

$$x(t) = A\cos(\omega t + \phi)$$

- The parameters in this solution must be chosen such that the equation of motion is satisfied. This requires that

$$-\omega^2 A\cos(\omega t + \phi) + \omega_0{}^2 A\cos(\omega t + \phi) - F_0 \sin(\omega t) = 0$$

- This equation can be rewritten as

$$\left(\omega_0{}^2 - \omega^2\right) A\cos(\omega t)\cos(\phi) -$$
$$\left(\omega_0{}^2 - \omega^2\right) A\sin(\omega t)\sin(\phi) - F_0 \sin(\omega t) = 0$$

- Our general solution must thus satisfy the following condition:

$$\left(\omega_0{}^2-\omega^2\right)A\cos\left(\omega t\right)\cos\left(\phi\right)-\left\{\left(\omega_0{}^2-\omega^2\right)A\sin\left(\phi\right)-F_0\right\}\sin\left(\omega t\right)=0$$

- Since this equation must be satisfied at all time, we must require that the coefficients of cos(ωt) and sin(ωt) are 0. This requires that

$$\left(\omega_0{}^2-\omega^2\right)A\cos\left(\phi\right)=0$$

and

$$\left(\omega_0{}^2-\omega^2\right)A\sin\left(\phi\right)-F_0=0$$

- The interesting solutions are solutions where $A \neq 0$ and $\omega \neq \omega_0$. In this case, our general solution can only satisfy the equation of motion if

$$\cos(\phi) = 0$$

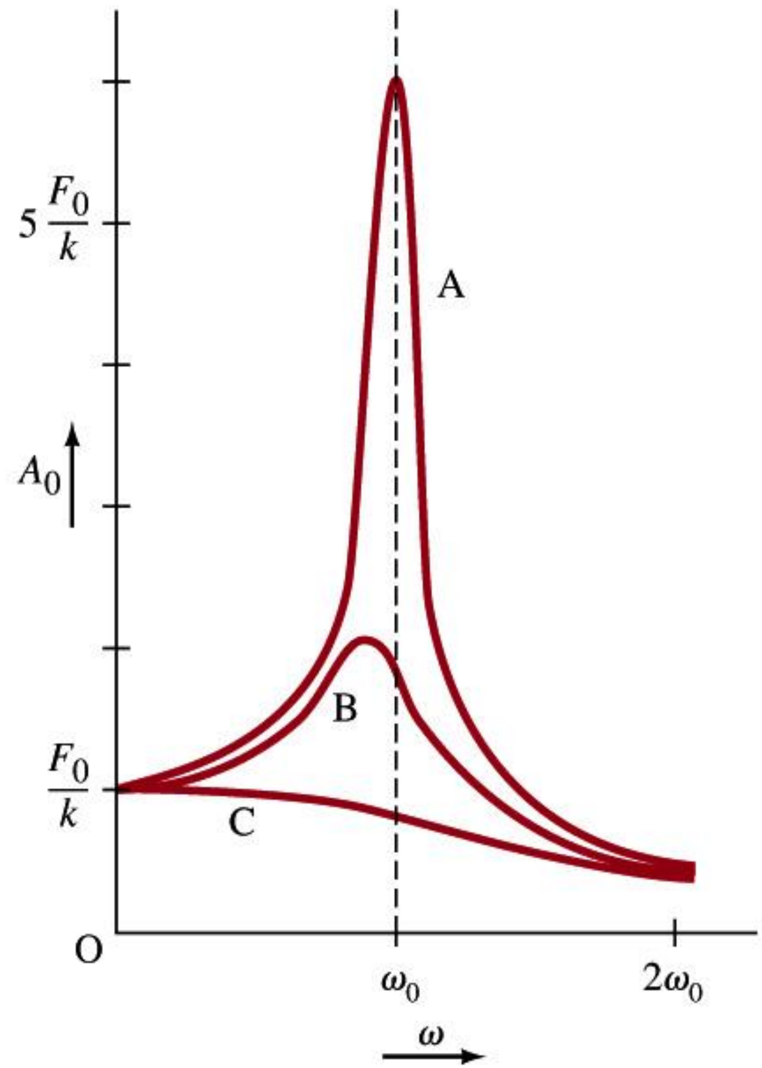
and

$$\left(\omega_0^{\,2} - \omega^2\right)A\sin(\phi) - F_0 = \left(\omega_0^{\,2} - \omega^2\right)A - F_0 = 0$$

- The amplitude of the motion is thus equal to

$$A = \frac{F_0}{\left(\omega_0^{\,2} - \omega^2\right)}$$

- If the driving force has a frequency close to the natural frequency of the system, the resulting amplitudes can be very large even for small driving amplitudes. The system is said to be in resonance.
- In realistic systems, there will also be a damping force. Whether or not resonance behavior will be observed will depend on the strength of the damping term.



$5\frac{F_0}{k}$
A_0
$\frac{F_0}{k}$
A
B
C
O
ω_0
$2\omega_0$
ω

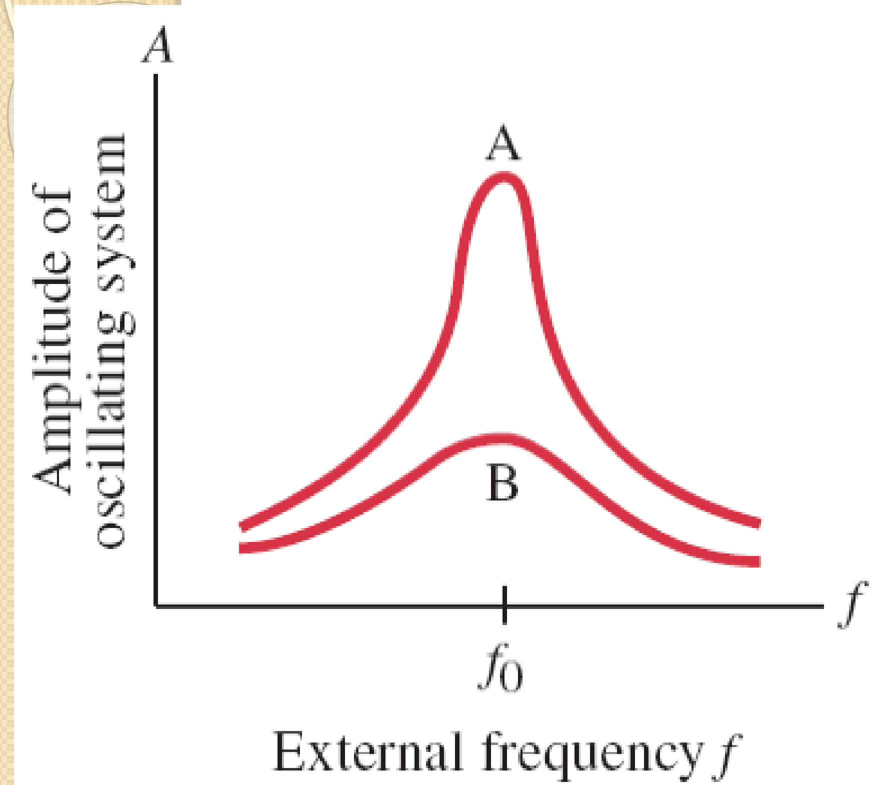
Driven Harmonic Motion.

Done for today!
Thursday: Temperature and Heat!

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

Unusually Strong Cyclone Off the Brazilian Coast: A lot of Rotational Motion!
Credit: Jacques Descloitres, MODIS Land Rapid Response Team, GSFC, NASA

Forced Oscillations; Resonance



The sharpness of the resonant peak depends on the damping. If the damping is small (A) it can be quite sharp; if the damping is larger (B) it is less sharp.

Like damping, resonance can be wanted or unwanted. Musical instruments and TV/radio receivers depend on it.

Forced Oscillations; Resonance

The equation of motion for a forced oscillator is:

$$\boxed{ma = -kx - bv + F_0 \cos \omega t.}$$

The solution is: $$x = A_0 \sin(\omega t + \phi_0),$$

where $$A_0 = \frac{F_0}{m\sqrt{(\omega^2 - \omega_0^2)^2 + b^2\omega^2/m^2}}$$

and

$$\phi_0 = \tan^{-1}\frac{\omega_0^2 - \omega^2}{\omega(b/m)}.$$

Forced Oscillations; Resonance

The width of the resonant peak can be characterized by the Q factor:

$$Q = \frac{m\omega_0}{b}.$$