Covariant Derivative Intrinsic Definition:

Why intrinsic Geometry?
↳ General Relativity, → intrinsicly curved 4-D Space.

In extrinsic geometry we define Basis vector as $\vec{e}_k = \frac{\partial \vec{R}}{\partial u^k}$, There is however no concept of specific origin in intrinsic geometry as the space is curved so the bases are defined in terms of derivative operators $\left(\frac{\partial}{\partial u^k} = e_k\right)$ themselves.

$\left(\vec{e}_1 = \frac{\partial}{\partial u^1}, \; \vec{e}_2 = \frac{\partial}{\partial u^2} \right)$

Now for intrinsic geometry there is no concept of normal vectors, because that would mean that we are pointing towards some outer space or in that outer space direction. and that does not exist in intrinsic geometry.

Now deriving the formula for the intrinsic part:

$$\nabla_{\frac{\partial}{\partial u^i}} \vec{v} = \frac{\partial \vec{v}}{\partial u^i} - \cancel{\vec{n}} \; (\text{DNE})$$

$$= \frac{\partial}{\partial u^i} (v^j \vec{e}_j)$$

$$= \left(\frac{\partial v^j}{\partial u^i} \vec{e}_j + \frac{\partial \vec{e}_j}{\partial u^i} v^j \right)$$

$$\frac{\partial \vec{e}_j}{\partial u^i} = \Gamma^1_{ij}\,\vec{e}_1 + \Gamma^2_{ij}\,\vec{e}_2 + \cancel{L_{ij}\,\hat{n}}$$

DNE for intrinsic geometry.

$$\frac{\partial \vec{e}_i}{\partial u^i} = \Gamma^k_{ij}\,\vec{e}_k$$

$$= \left(\frac{\partial v^k}{\partial u^i} + v^j \boxed{\Gamma^k_{ij}}\right)\vec{e}_k$$

↳ Need to find these,

as we cannot find it using the dot product method as that was easy to do in xyz cordinate which does not exist for intrinsic geometry.

(<u>New method to find</u> Γ^k_{ij} <u>for intrinsic geometry</u>)

$g_{ij} = g_{ji}$ means $e_i \cdot e_j = e_j \cdot e_i$

The metric Tensor is so important that we require $\vec{e}_i \cdot \vec{e}_j$ to be defined.

$$\frac{\partial}{\partial u^i}\vec{e}_j = \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}\right) \Rightarrow \text{commutative.}$$

$$\boxed{\Gamma^k_{ij} = \Gamma^k_{ji}}$$

New formula for Γ^{k}_{ij}

$$\Gamma^{m}_{jk} = \frac{1}{2} g^{im} \left(\frac{\partial g_{ij}}{\partial u^{k}} + \frac{\partial g_{ki}}{\partial u^{j}} - \frac{\partial g_{jk}}{\partial u^{i}} \right)$$

$$\frac{\partial \vec{v}}{\partial u^{i}} = \left(\frac{\partial v^{j}}{\partial u^{i}} + v^{j} \Gamma^{k}_{ij} \right) \vec{e}_{k}$$

where $\Gamma^{k}_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{li}}{\partial u^{j}} + \frac{\partial g_{jl}}{\partial u^{i}} - \frac{\partial g_{ij}}{\partial u^{l}} \right)$

as long as we are given the metric Tensor.

Parallel Transport vector along it self, $\nabla_{\vec{v}} \vec{v} = 0$, that means, the direction of the vector and its derivative direction are the same.

$\boxed{\nabla_{\vec{v}} \vec{v} = 0}$ → resulting curve is geodesic curve.

curve = some function of parameter λ,

Tangent vectors : $\frac{d}{d\lambda}$

$\boxed{\text{Geodesic : } \nabla_{\frac{d}{d\lambda}} \left(\frac{d}{d\lambda} \right) = 0}$

CamScanner

(Covariant Derivative Abstract Definition)

Basis vectors

$$\vec{e}_k \equiv \frac{\partial}{\partial u^k} \equiv \partial_k$$

$$\nabla_{\square} \square$$

→ input field

↑ direction

→ covariant derivative.

"Covariant derivative provides a connection b/w tangent spaces in a curved space."

$$\boxed{\frac{\partial}{\partial u^i} \vec{e}_j = \Gamma^k_{ij} \vec{e}_k},$$

abstraction version would be

$$\boxed{\nabla_{\vec{e}_i} \vec{e}_j = \Gamma^k_{ij} \vec{e}_k}$$

connection coefficient.

Torsion-Free property

$$\Rightarrow \nabla_{\vec{w}} \vec{v} = \nabla_{\vec{v}} \vec{w}$$

$$\Rightarrow \nabla_w (\vec{v} \cdot \vec{u}) = (\nabla_{\vec{w}} \vec{v}) \cdot \vec{u} + \vec{v} \cdot (\nabla_{\vec{w}} \vec{u})$$

↳ metric compatibility property.

$$\nabla_{\vec{w}} \vec{v} = \nabla_{\vec{w}} \vec{v} - \nabla_{\vec{v}} \vec{w} = [\vec{v}, \vec{w}]$$

$$[\vec{v}, \vec{w}] = \vec{v}\vec{w} - \vec{w}\vec{v}$$

↳ Lie Bracket

$$[\partial_i, \partial_j] = \partial_i \partial_j - \partial_j \partial_i = \frac{\partial^2}{\partial u^i \partial u^j} - \frac{\partial^2}{\partial u^j \partial u^i}$$

Solving for cristofel symbol using Torsion free, and metric compatibility property we get

$$\Gamma^m_{jk} = \frac{1}{2} g^{im} \left(\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk} \right)$$

(Fundamental theorem of Riemannian Geometry)

(Reimannian Manifold = curved Space with a metric)

This theorem says that:

There is a unique connection (Covariant Derivative)

that is:

- Torsion - Free
- Has metric Compatibility

This unique connection is called Levi-civita connection.

It's christoffel symbols (connection coefficients) are:

$$\Gamma^m_{jk} = \frac{1}{2} g^{im} \left(\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk} \right)$$

(31)

$$e^i(\vec{e}_i) = \delta^j_i$$

e^i → covector e^i

$\vec{e}_i$ → Basis vector $\vec{e}_i$

These e^i covectors acts as a dual space vector ~~can be~~ cause written as linear combination of e^i

$$\boxed{\alpha = \alpha_1 e^1 + \alpha_2 e^2}$$

→ Basis

Shortcut method for calculating Christofell symbol for orthogonal system.

(1) $$\Gamma^a_{ac} = \frac{1}{2} g^{aa} g_{aa,c}$$

(2) $$\Gamma^a_{aa} = 0$$

(3) $$\Gamma^a_{bc} = -\frac{1}{2} g^{aa} g_{bc,a}$$

These affine connections are called Levi-civita connections.

Covariant Derivative General formula:

$$A^{\alpha}_{\ B;\gamma} = \frac{\partial A^{\alpha}_{B}}{\partial x^{\gamma}} + \Gamma^{\alpha}_{c\gamma} A^{c}_{B} - \Gamma^{c}_{B\gamma} A^{\alpha}_{c}$$

General formula for covariant derivative

$$\boxed{\nabla_{\nu} V^{\mu} \equiv V^{\mu}_{\ ;\nu} = \partial_{\nu} V^{\mu} + \Gamma^{\mu}_{\nu\sigma} V^{\sigma}} \rightarrow \ast\ast\ast$$

Lie Bracket, Flow, Torsion Tensor:

Lie Bracket (Commutator) of vector fields

$[\vec{U}, \vec{V}] = \vec{U}(\vec{V}) - \vec{V}(\vec{U})$

(1) ↳ Required for understanding (RCT)

$R(\vec{U}, \vec{V}) = \nabla_{\vec{u}} \nabla_{\vec{v}} - \nabla_{\vec{v}} \nabla_{\vec{u}} - \nabla_{[\vec{u}, \vec{v}]}$ (RCT)

(2) Understand Torsion free connection.

Flow curve / Integral curve = a curve that is Tangent to all vectors in a vector field

and Flow curves are related to the Physical interpretation of Lie brackets.

Lie Bracket

$$[\vec{u}, \vec{v}] = \vec{u}(\vec{v}) - \vec{v}(\vec{u})$$

Derivative operators

Derivative of $\vec{v}$ in the direction of u.

* coordinate lines are just flow curves along basis vectors.

Lie Bracket / commutator = measures how much vector field flow curves fail to close.

Torsion Tensor :-

$$T(\vec{u}, \vec{v}) = \nabla_{\vec{u}} \vec{v} - [\vec{u}, \vec{v}] - \nabla_{\vec{v}} \vec{u}$$

Torsion Tensor $(T(u,v))$ gives us the Seperation b/w parralel transported vetor lines.

when zero, It means vetors are parralel Tranported.

Torsion-Free " means parrallel-Transported vectors close properly.

Torsion free is a property of connection and It does not depend on the vetor fields that we use.

$$T(\vec{u}, \vec{v}) = u^i v^j \left(\Gamma^k_{ij} - \Gamma^k_{ji} \right) \partial_k$$

If it is Torsion free connection then

$$\boxed{\Gamma^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} = 0} \Rightarrow \boxed{\Gamma^k_{ij} = \Gamma^k_{ji}}$$

(Reimann curvature Tensor, Holonomy + Geodesic Deviation)

How can we Tell if a Space is curved or flat?
(First Attempt)!

∴ When All $\Gamma^k_{ij} = 0$, a space is Flat.
↳ Geodesics are all straight lines.
↳ However, it is not always the case, a polar coordinate describes a flat space but not all the levicita connections are zero and not all geodesics are straight lines.

∴ The way we Tell if a Space is curved or not is given by Rieman Curvature Tensor.

* $R^d{}_{abc}$ (1 contra, 3 co)-Tensor

Two main ways to detect curvature;

① holonomy
② geodesic deviation

Riemann curvature Tensor Definition:-

$$R(\vec{u},\vec{v})\vec{w} = \nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} - \nabla_{v}\nabla_{u}\vec{w} - \nabla_{[\vec{u},\vec{v}]}\vec{w}$$

Lie Bracket is not Linear for each input

$$[a\vec{r},\vec{s}] \neq a[\vec{r},\vec{s}]$$

Riemann Normal coordinates at point p (Local inertial Frame)) can define in any Riemann manifold for a point.

- $g_{ij} = \delta_{ij}$ at point p.
- $\Gamma^{k}_{ij} = 0$ at point p.

Geodesic Deviation:-

Riemann curvature Tensor components + Symmetries:

- linear for all inputs:-

$$R(\vec{u},\vec{v})\vec{w} = u^i v^j w^k R(\vec{e_i},\vec{e_j})\vec{e_k}$$
$$= u^i v^j w^k R^m{}_{kij}\,\vec{e}_m$$

$$R^d{}_{cab} = \partial_a(\Gamma^d_{bc}) - \partial_b(\Gamma^d_{ac}) + \Gamma^i_{bc}\Gamma^d_{ai} - \Gamma^j_{ac}\Gamma^d_{bj}$$

Symmetries of Riemann Curvature Tensor

$R^{d}{}_{cab}$.

2 Dimensions $R^{d}{}_{cab}$ = 16 compont

4D $R^{d}{}_{cab}$ ↙ 256 componts.

<u>Symmetries</u>

$$R^{d}{}_{cba} = -R^{d}{}_{cab} \Rightarrow \text{34 symmetry}$$

$$R^{d}{}_{cab} + R^{d}{}_{bca} + R^{d}{}_{abc} = 0 \Rightarrow \text{Bianchi identity.}$$

→ cycle Basis term.

$$R_{bacd} = -R_{abcd}. \Rightarrow \text{12 symmetry}$$

$$R_{abcd} = R_{cdab} \Rightarrow \text{Flip Symmetry}$$

$$\boxed{\nabla_{\vec{e}_i} \vec{e}_j = \Gamma^{k}_{ij}}$$

In 2 Dimensions, the Riemann curvature Tensor only has one free parameter : R_{1212}

$$R_{1212} = -R_{1221} = R_{2121} = -R_{2112}$$

Riemann curvature Tensor for a Sphere.

(Geometric meaning of Ricci curvature Tensor and Ricci Scalar)

↳ Summarizing Reimann Tensor.

Ricci Tensor : Track " volume change " along geodesic.

two ways to understand this.

↳ sectional curvature

↳ volume element derivative.

Ricci Scalar : compare volume of a ball in curved Space vs flate space.

Reimann Tensor :)

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}$$

…ed indices
…ever, [illegible]
…ure can change.

Ricci Tensor

when we put a=c in Reimann curvature we get Ricci Tensor

$$R^a_{bad} = R_{bd} = \Gamma^a_{bd,a} - \Gamma^a_{ba,d} + \Gamma^a_{ea}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{ba}$$

Ricci Scalar:

$$R = g^{bd} R_{bd}$$

(Covariant Derivative)!

$$\boxed{V_{ab;c} = \nabla_c V_{ab} = V_{ab,c} - \Gamma^{d}_{ac} V_{bd} - \Gamma^{d}_{bc} V_{ad}}$$

; → covariant derivative

, → partial derivative

when the indexes are two, in this case it is a & b, then we use two christoffell symbols and so on ----

$$\boxed{\nabla_c V^{ab} = V^{ab}_{\ \ ;c} = V^{ab}_{\ \ ,c} + \Gamma^{a}_{dc} V^{db} + \Gamma^{b}_{dc} V^{da}}$$

Covariant derivative of contravariant tensor.

$$\boxed{\nabla_c V^{a}_{b} = V^{a}_{b;c} = V^{a}_{b,c} + \Gamma^{a}_{dc} V^{d}_{b} - \Gamma^{d}_{bc} V^{a}_{d}}$$

↳ Covariant derivative of a mixed tensor.

The Ricci Tensor is the contraction of Riemann Tensor

$$R_{uv} = R^{\rho}_{\rho u \rho v}$$

The Scalar curvature is the contraction of Ricci Tensor and it is written as R without subscripts or arguments,

$$R = g^{uv} R_{uv}.$$

$$\boxed{\partial_u A = A_{,u}}$$ → Representation of partial derivative.

How to find Reimann Tensor, Ricci Tensor, and Ricci Scalar?

Example:→ Find Reimann Tensor, Ricci Tensor and Ricci Scalar for the unit Sphere.

Solution

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$\theta = 1,$
$\phi = 2,$

$g_{11} = 1,$ $g_{22} = \sin^2\theta$

$g_{22} \to 1,$

$$\boxed{\Gamma^1_{22}, \Gamma^2_{12} = \Gamma^2_{21}}$$

(40)

$$\Gamma^{1}_{22} = \frac{1}{2} g^{1m} \left(g_{jm,k} + g_{km,j} - g_{kj,m} \right)$$

$$= \frac{1}{2} g^{11} \left(\cancel{g_{21,2}} + \cancel{g_{21,2}} - g_{22,1} \right)$$

$$= \frac{1}{2} \left(\frac{1}{\sin^2\theta} \right) 2\sin\theta\cos\theta$$

$$\boxed{\Gamma^{1}_{22} = \cancel{\tan\theta}} \quad \boxed{\Rightarrow -\sin\theta\cos\theta} = \Gamma^{1}_{22}$$

$$\Gamma^{2}_{12} = \frac{1}{2} g^{2m} \left(\cancel{g_{jm,k}} + g_{km,j} - \cancel{g_{kj,m}} \right)$$

$$\Gamma^{2}_{12} = \frac{1}{2} \frac{1}{\sin^2\theta} \left(2\sin\theta\cos\theta \right) = \frac{\cancel{\tan\theta}}{\cot\theta}$$

$$\boxed{\Gamma^{2}_{12} = \frac{\cancel{\tan\theta}}{\cot\theta} = \Gamma^{2}_{21}, \quad \Gamma^{1}_{22} = -\sin\theta\cos\theta}$$

Now finding Raimann Tensor):

$$R^{a}_{bcd} = \Gamma^{a}_{bd,c} - \Gamma^{a}_{bc,d} + \Gamma^{a}_{ec}\Gamma^{e}_{bd} - \Gamma^{a}_{ed}\Gamma^{e}_{bc}$$

as the metric tensor is (2×2) then the values of the indices a, b, c and d would vary from 1 to 2.

$$\begin{tikzcd}[column sep=small] & R^1_{bcd}, \arrow[dl] \arrow[dr] & & & \\ R^1_{1cd} \arrow[d] \arrow[dr] & & R^1_{2cd} \arrow[d] \arrow[dr] & & \\ R^1_{11d} & R^1_{12d} & R^1_{21d} \arrow[d] \arrow[dr] & R^1_{22d} & \\ & & R^1_{212} & R^1_{211} & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & R^2_{bcd}, \arrow[dl, dash] \arrow[dr, dash] & \\ R^2_{1cd} & & R^2_{2cd} \end{tikzcd}$$

Γ^a_{bc}

$$\boxed{\Gamma^2_{12}, \ \Gamma^2_{21} \quad \Gamma^1_{22}}$$

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}$$

$$R^1_{221} = \Gamma^1_{21,2} - \Gamma^1_{22,1} + \Gamma^1_{e2}\Gamma^e_{21} - \cancel{\Gamma^1_{e1}}\Gamma^1_{22}$$

$$= \cancel{\Gamma^1_{21,2}} - \Gamma^1_{22,1} + \Gamma^1_{22}\Gamma^2_{21}$$

$$= \frac{\partial}{\partial\theta} \sin\theta\cos\theta + (-\cancel{\sin\theta}\cos\theta)\frac{\cos\theta}{\cancel{\sin\theta}}$$

$$= \sin\theta - \sin\theta + \cos\theta\cos\theta + -(\cos^2\theta)$$

$$= -\sin^2\theta + \cancel{\cos^2\theta} - \cancel{\cos^2\theta}$$

$$\boxed{R^1_{221} = -\sin^2\theta} \qquad \boxed{R^1_{212} = \sin^2\theta}$$

Now finding Ricci Tensor:

$$R_{bd} = \Gamma^{a}_{bd,a} - \Gamma^{a}_{ba,d} + \Gamma^{a}_{ea}\Gamma^{e}_{bd} - \Gamma^{a}_{ed}\Gamma^{e}_{ba}$$

$R_{bd} \rightarrow R_{1d},\ R_{2d}$

$R_{1d} \rightarrow R_{11},\ R_{12}$; $R_{2d} \rightarrow R_{21},\ R_{22}$

$\boxed{R_{12}\quad R_{21}}$ → Diagonal matrix $= 0$

$R_{11},\ R_{22}$ → non-zero.

$$R_{11} = \cancel{\Gamma^{a}_{11,a}} - \Gamma^{a}_{1a,1} + \cancel{\Gamma^{a}_{ea}\Gamma^{e}_{1a}} - \Gamma^{a}_{e1}\Gamma^{e}_{1a}$$

$$= \cancel{\Gamma^{2}_{12,1} + \Gamma^{1}_{21}\Gamma^{2}_{12} - \Gamma^{1}_{21}\Gamma^{2}_{11}}$$

$$= -\Gamma^{a}_{1a,1} - \Gamma^{a}_{e1}\Gamma^{e}_{1a}$$

$$\boxed{R_{11} = -\Gamma^{2}_{12,1} - \Gamma^{2}_{21}\Gamma^{2}_{12}}$$

$$= -\frac{\partial}{\partial\theta}\cot\theta - \cot\theta\left(\cancel{\sin\theta}\ \cot\theta\right)$$

$$= +\operatorname{cosec}^2\theta - \cot^2\theta$$

$$\Rightarrow \operatorname{cosec}^2\theta - \cot^2\theta = 1$$

$$\Rightarrow \boxed{R_{11} = 1}$$

$$\boxed{R_{22} = \sin^2\theta}$$

(Ricci Scalar):-

$$R = R_{ab} g^{ab}$$

$$R = R_{11} g^{11} + \cancel{R_{12} g^{12}} + \cancel{R_{21} g^{21}} + R_{22} g^{22}$$

$$R = R_{11} g^{11} + R_{22} g^{22}$$

$$R = 1(1) + \sin^2\theta \frac{1}{\sin^2\theta}$$

$$\boxed{R = 2}$$

Now Finding Covariant Derivative)

Ex) Find divergence of a Tensor $V^u = (r\cos\theta, r\sin\theta)$

$$\nabla_{,u} V^u = ?$$

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \begin{matrix} 1 = r \\ 2 = \theta \end{matrix}$$

$\Gamma^1_{22} = -r$

$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$

Solution:

$$\nabla_{,u} V^u = V^u_{;u} = V^u_{,u} + \Gamma^u_{\alpha u} V^\alpha$$

$$V^u_{;u} = V^1_{,1} + V^2_{,2} + \Gamma^1_{\alpha 1} V^\alpha + \Gamma^2_{\alpha 2} V^\alpha$$

$$V^u_{;u} = V^1_{,1} + V^2_{,2} + \cancel{\Gamma^1_{11}} V^1 + \Gamma^1_{21} V^2 + \Gamma^2_{12} V^1 + \Gamma^2_{22} V^2$$

$$\Rightarrow \frac{\partial}{\partial r} r\cos\theta + \frac{\partial}{\partial \theta} r\sin\theta + \frac{1}{r} r\cos\theta$$

$$\Rightarrow \cos\theta + r\cos\theta + \cos\theta$$

$$\boxed{\nabla_u V^u = 2\cos\theta + r\cos\theta \Rightarrow \cos\theta(2+r)}$$

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

(Li derivative)

Li derivative is not with respect ~~to tensor~~ ~~or a vector~~ to an index like the usual one, it is always with respect to a tensor or a vector.

The tensor you are taking derivative with respect to should always be in the form of a (contra).

Let A^α and B_β are tensor, so, Lie derivative of B_β (w.r.s.to) A^α is given as

$$\mathcal{L}_A B_\beta, \quad \boxed{\mathcal{L}_A T^b = A^\alpha T^b_{,\alpha} - T^\alpha A^b_{;\alpha}}$$

index on A is not written

when contravariant tensor then (−) and when covariant tensor, then (+) sign.

$$B_\alpha A^\alpha_{,b}$$

$$\mathcal{L}_A B_\beta = A^\alpha B_{\beta,\alpha} + T_\alpha A^\alpha_{,\beta}$$

$$\mathcal{L}_A T^a_b = A^\alpha T^a_{b,\alpha} - A^a_{,\alpha} T^\alpha_b + A^\alpha_{,b} T^a_\alpha$$

$$d_A T^{ab} = A^c T^{ab}_{,c} - A^b_{,c} T^{ac} - A^a_{,c} T^{cb}$$

$$d_A T_{ab} = A^c T_{ab,c} + A^c_{,b} T_{ac} + A^c_{,a} T_{cb}$$

(Example)

$$T^a_b = \begin{pmatrix} pr & qr & 3q \\ p^2 & q^2 & r^2 \\ q^3/r & p^3/q & r^3/p \end{pmatrix} \begin{array}{l} 1-p \\ 2-q \\ 3-r \end{array}$$

$$A^c = \begin{pmatrix} p+q \\ q-r \\ r-p \end{pmatrix} \begin{array}{l} \to A^1 \\ \to A^2 \\ \to A^3 \end{array}$$

$$d_A T^a_b = A^c T^a_{b,c} - A^a_{,c} T^c_b + A^c_{,b} T^a_c$$

for $T^1_1 = ?$

$$d_A T^1_1 = A^c T^1_{1,c} - A^1_{,c} T^c_1 + A^c_{,1} T^1_c$$

$$= A^1 T^1_{1,1} + \cancel{A^2 T^1_{1,2}} + A^3 T^1_{1,3} - \cancel{A^1_{,1} T^1_1} - \cancel{A^2_{,1}} + A^2_{,1} T^1_2$$

$$A^1_{,2} T^2_1 - \cancel{A^1_{,2}} T^3_1 + \cancel{A^1_{,1}} T^1_1 +$$

$$+ A^3_{,1} T^1_3$$

$$d_A T^1_1 = \boxed{(p+q)r + (r-p)p - p^2 - Ipq}$$

General form of Geodesic Equation

$$\boxed{\frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0}$$