ADVANCED ENGINEERING MATHEMATICS

CYBER SEC. (III SEM)

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

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EULER’S METHOD

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$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

Q. 1. *Use Euler's method to solve the following differential equations*

$$\frac{dy}{dx} = xy \; ; \; y(0) = 1$$

to find the approximate value of y(0.4) and compare the result with exact value (take h = 0.1)

Sol. We are given that

$$\frac{dy}{dx} = xy = f(x, y)$$

From Euler's formula, we have

$$y_{n+1} = y_n + hf(x_n, y_n)$$

at $n = 0$

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1 + 0.1\,[(0)\,(1)] \\ &= 1 \end{aligned}$$

$n = 1$

$$\begin{aligned} y_2 &= y_1 + h\,f(x_1, y_1) \\ &= 1 + 0.1\,[(0.1)\,(1)] \\ &= 1 + 0.01 = 1.01 \end{aligned}$$

$n = 2$ $\qquad y_3 = y_2 + h\, f(x_2, y_2)$

$$= 1.01 + 0.1\, [(0.2)\,(1.01)]$$

$$= 1.01 + 0.0202 = 1.0302$$

$n = 3$ $\qquad y_4 = y_3 + h\, f(x_3, y_3)$

$$= 1.0302 + 0.1\, [(0.3)\,(1.0302)]$$

$$= 1.0302 + 0.030906 = 1.061106$$

and $\qquad y = e^{\frac{x^2}{2}}$

at $\qquad x = 0.4$ provides

$y = 1.083287$ as exact value

Q. 2. *Use Euler's method to solve the following differential equation*

$$\frac{dy}{dx} = \frac{y^2 - x}{y^2 + x} \; ; \; y(0) = 1$$

to find the approximate values of y for $x = 0.1, 0.2, 0.3, 0.4$ and 0.5.

Sol. Given that

$$\frac{dy}{dx} = f(x, y) = \frac{y^2 - x}{y^2 + x}, \; y(0) = 1$$

From Euler's formula

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$n = 0$ gives

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 1 + 0.1 \left[\frac{1-0}{1+0}\right]$$
$$= 1.1$$

$n = 1$ gives

$$y_2 = y_1 + h\, f(x_1, y_1)$$
$$= 1.1 + 0.1 \left[\frac{(1.1)^2 - .1}{(1.1)^2 + 0.1}\right]$$
$$= 1.1 + 0.0847328$$
$$= 1.1847$$

$n = 2$ gives $\quad y_3 = y_2 + h\, f(x_2, y_2)$

$$= 1.1847 + 0.1 \left[\frac{(1.1847)^2 - 0.2}{(1.1847)^2 + 0.2}\right]$$

$$= 1.2597$$

$n = 3$ gives $\quad y_4 = y_3 + h\, f(x_3, y_3)$

$$= 1.2597 + 0.1 \left[\frac{(1.2597)^2 - 0.3}{(1.2597)^2 + 0.3}\right]$$

$$= 1.3279$$

$n = 4$ gives $\quad y_5 = y_4 + h\, f(x_4, y_4)$

$$= 1.3279 + 0.1 \left[\frac{(1.3279)^2 - 0.4}{(1.3279)^2 + 0.4}\right]$$

$$= 1.3909$$

MODIFIED EULER’S METHOD

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$$\boxed{\begin{aligned} y_{n+1} &= y_n + h\, f(x_n, y_n) \\ y_{n+1}^{*} &= y_n + \frac{h}{2}\,[f(x_n, y_n) + f(x_{n+1}, y_{n+1})] \end{aligned}}$$

Q. 1. *Use modified Eulers method to solve the following differential equation*

$$\frac{dy}{dx} = y^2 - \frac{y}{x} \; ; \; y(1) = 1$$

to find the approximate values of y for x = 1.1 to 1.6 taking h = 0.1

Sol. Euler's modified method gives

$$y_1 = y_0 + h\, f(x_0, y_0)$$

and $$y_1^* = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1)]$$

Here, we are given that

$$\frac{dy}{dx} = f(x, y) = y^2 - \frac{y}{x} \; ; \; y(x = 1) = 1$$

$$\therefore \quad y_{1.1} = y_1 = 1 + 0.1\, f(1, 1) = 1 + 0.1\left[(1)^2 - \frac{1}{1}\right]$$

$$= 1$$

and $$y_{1.1}^* = y_1^* = y_0 + \frac{h}{2}\,[f(1, 1) + f(1.1, 1)]$$

$$= 1 + \frac{0.1}{2}\left[\left(1^2 - \frac{1}{1}\right) + \left\{(1.)^2 - \left(\frac{1.0}{1.1}\right)\right\}\right]$$

$$= 1.004545 \text{ approxi.}$$

Second approximation provides

$$y_{1.2} = y_2 = y_1 + h\, f(x_1, y_1) = 1.004545 + 0.1\, f(1.1,\ 1.004545)$$

$$= 1.004545 + 0.1\left[(1.004545)^2 - \left(\frac{1.004545}{1.1}\right)\right]$$

$$= 1.01413$$

and

$$y_{1.2}{}^{*} = y_2{}^{*} = y_1 + \frac{h}{2}\,[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 1.004545 + 0.05\left[\left\{(1.004545)^2 - \left(\frac{1.004545}{1.1}\right)\right\}\right] + \left[\left\{(1.01413)^2 - \left(\frac{1.01413}{1.2}\right)\right\}\right]$$

$$= 1.0185$$

Third approximation, provides

$$y_{1.3} = y_3 = y_2 + h\, f(x_2, y_2) = 1.0185 + 0.1\, f(1.2,\ 1.0185)$$

$$= 1.0185 + 0.1 \left[(1.0185)^2 - \left(\frac{1.0185}{1.2}\right)\right]$$

$$= 1.03736$$

and

$$y_{1.3}{}^* = y_3{}^* = y_2 + \frac{h}{2}\ [f(x_2, y_2) + f(x_3, y_3)]$$

$$= 1.0185 + 0.05 \left[\left\{(1.0185)^2 - \left(\frac{1.0185}{1.2}\right)\right\} + \left\{(1.03736)^2 - \left(\frac{1.03736}{1.3}\right)\right\}\right]$$

$$= 1.0420$$

Fourth approximation provides

$$y_{1.4} = y_4 = y_3 + h\, f(x_3, y_3) = 1.0420 + 0.1\, f(1.3,\ 1.0420)$$

$$= 1.0420 + 0.1 \left[(1.0420)^2 - \left(\frac{1.0420}{1.3}\right)\right]$$

$$= 1.0704$$

and

$$y_{1.4} = y_4^* = y_3 + \frac{h}{2}\,[f(x_3, y_3) + f(x_4, y_4)]$$

$$1.0420 + 0.05 \left[\left\{(1.0420)^2 - \left(\frac{1.0420}{1.3}\right)\right\} + \left\{(1.0704)^2 - \left(\frac{1.0704}{1.4}\right)\right\}\right]$$

$$= 1.0731$$

Fifth approximation provides

$$y_{1.5} = y_5 = y_4 + h\,f(x_4, y_4) = 1.0731 + 0.1\,f(1.4,\ 1.0731)$$

$$= 1.0731 + 0.1\left[(1.0731)^2 - \left(\frac{1.0731}{1.4}\right)\right]$$

$$= 1.1116$$

and

$$y_{1.5}^{*} = y_5 = y_4 + \frac{h}{2}\,[f(x_4, y_4) + f(x_5, y_5)]$$

$$= 1.0731 + 0.05\left[\left\{(1.0731)^2 - \left(\frac{1.0731}{1.4}\right)\right\} + \left\{(1.1116)^2 - \left(\frac{1.1116}{1.5}\right)\right\}\right]$$

$$= 1.1171$$

Sixth approximation provides

$$y_{1.6} = y_6 = y_5 + h\, f(x_5, y_5) = 1.1171 + 0.1\, f(1.5,\ 1.1171)$$

$$= 1.1171 + 0.1 \left[(1.1171)^2 - \left(\frac{1.1171}{1.5} \right) \right]$$

$$= 1.1674$$

and

$$y_{1.6}^* = y_6^* = y_5 + \frac{h}{2}\ [f(x_5, y_5) + f(x_6, y_6)]$$

$$= 1.1171 + 0.05 \left[\left\{ (1.1171)^2 - \left(\frac{1.1171}{1.5} \right) \right\} + \left\{ (1.1674)^2 - \left(\frac{1.1674}{1.6} \right) \right\} \right]$$

$$= 1.1739$$

Q. 2. *Use modified Eulers method to solve the following differential equation*

$$\frac{dy}{dx} = x^2 + y \ ; \ y(0) = 0.94$$

to find the approximate value of y(0.1).

Sol. Euler's modified method gives

$$y_1 = y_0 + h\, f(x_0, y_0)$$

and

$$y_1^* = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1)]$$

Here, we are given that

$$\frac{dy}{dx} = f(x, y) = x^2 + y \ ; \ y(x = 0) = 0.94$$

$$\therefore \quad y(0.1) = y_1 = 0.94 + 0.1\, f(0, 0.94)$$

$$= 0.94 + 0.1\,[(0)^2 + 0.94]$$

$$= 1.034$$

and

$$y_{(0.1)}^* = y_1^* = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 0.94 + 0.05\,[\{(0)^2 + 0.94\} + \{(0.1)^2 + (1.034)\}]$$

$$= 1.0392$$

MILNE’S P - C METHOD

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$$y_{k+1}^{(P)} = y_{k-3} + \frac{4h}{3} [2f_{k-2} - f_{k-1} + 2f_k]$$

$$y_{k+1}^{(c)} = y_{k-1} + \frac{h}{3} [f_{k-1} + 4f_k + f_{k+1}^{(p)}]$$

Q. 1. *Use Milne's method to solve the following differential equation*

$$\frac{dy}{dx} = x - y^2 \; ; \; y(0) = 0.0000, \; y(0.2) = 0.0200$$

$y(0.4) = (0.0795)$, $y(0.6) = (0.1762)$

to find the approximate value of $y(0.8)$

Sol. The predicted value of $y(0.8)$ is given by the predictor formula as

$$y_{0.8}^{(p)} = y_4^{(p)} = y_0 + \frac{4h}{3}\,[2f_1 - f_2 + 2f_3]$$

We are given that

$$h = 0.2 \text{ and } f(x, y) = x - y^2,$$

The functional values are computed at the starting point. These values are tabulated along with the starting points in the following table.

i	x_i	y_i	$f(x_i, y_i) = x_i - y_i^2 = f_i$
0	0.0	0.0000	0.0000
1	0.2	0.0200	0.1996
2	0.4	0.0795	0.39368
3	0.6	0.1762	0.56895

Now, $y^{(p)}_{(0.8)} = y_4^{(p)} = 0.0 + \dfrac{4(0.2)}{3}\ [2(0.1996) - .39368 + 2(0.56891)]$

$$= \frac{0.914736}{3} = 0.3049$$

Thereore,

$$f^{(p)}_{0.8} = f_4^{(p)} = f(0.8,\ 0.3049) = 0.8 - (0.3049)^2$$

$$= 0.707036 \text{ approx.}$$

To correct the predicted value of $y(0.8)$, we use the corrected formula as

$$y^{(c)}_{(0.8)} = y_4^{(c)} = y_2 + \frac{h}{3}\ [f_2 + 4f_3 + f_4^{(p)}]$$

$$= 0.0795 + \frac{0.2}{3}\ (0.39368 + 4(0.56895 + 0.707036)$$

$$= 0.3046$$

Q. 2. *Use Milne's method to solve the following differential equation*

$$\frac{dy}{dx} = x + y \; ; \; y(0) = 0$$

to find the approximate value of y for x = 0.4, 0.5, 0.6 (take h = 0.1)

Sol.

$$\frac{dy}{dx} = x + y \; ; \; y(x = 0) = 0$$

i.e.

$$\frac{dy}{dx} = f(x, y) = x + y \text{ and } x_0 = 0, y_0 = 0$$

Initially, we need to find $y(0.1)$, $y(0.2)$, $y(0.3)$ to apply Milne's method, for which we have to apply Euler's method as

$$y_{(0.1)} = y_1 = y_0 + hf(x_0, y_0) = 0 + 0.1\, f(0, 0) = 0.1(0 + 0) = 0$$

$$y_{(0.2)} = y_2 = y_1 + hf(x_1, y_1) = 0 + 0.1\, f(0.1, 0) = 0.1(0.1 + 0) = 0.01$$

$$y_{(0.3)} = y_3 = y_2 + hf(x_2, y_2) = 0.01 + 0.1\, f(0.2, 0.01)$$

$$= 0.01 + 0.1(0.2 + 0.01) = 0.031$$

Now, to proceed further, we coustruct the following table

i	x_i	y_i	$f(x_i, y_i) = x_i + y_i = f_i$
0	0.0	0.0000	0.0000
1	0.1	0.0000	0.1000
2	0.2	0.0100	0.2100
3	0.3	0.0310	0.3310

Now, $y^{(p)}_{(0.4)} = y_4^{(p)} = y_0 + \dfrac{4h}{3}\,[2f_1 - f_2 + 2f_3]$

$$= 0 + \frac{0.4}{3}\,[0.2 - 0.2100 + 0.6620] = 0.0869$$

therefore $\quad f^{(p)}_{0.4} = f_4^{(p)} = f(0.4, 0.0869) = 0.4 + 0.0869 = 0.4869$

To correct the predicted value of $y(0.4)$, we use the corrected formula as

$$y_{0.4}^{(c)} = y_4^{(c)} = y_2 + \frac{h}{3}\,[f_2 + 4f_3 + f_4^{(p)}]$$

$$= 0.01 + \frac{0.1}{3}\,[0.21 + 1.324 + 0.4869] = 0.0774$$

and $$y_{(0.5)}^{(p)} = y_5^{(p)} = y_1 + \frac{4h}{3}\,[2f_2 - f_3 + 2f_4]$$

$$= 0 + \frac{0.4}{3}\,[0.42 - 0.331 + 2f(0.4,\ 0.0774)]$$

$$= \frac{0.4}{3}\,[0.42 - 0.331 + 2(0.4 + 0.0774)]$$

$$= 0.1392$$

therefore $$f_{0.5}^{(p)} = f_5^{(p)} = f(0.5,\ 0.1392) = 0.5 + 0.1392 = 0.6392$$

To correct the predicted value of $y(0.5)$, we use the corrected formula as

$$y_{0.5}^{(c)} = y_5^{(c)} = y_3 + \frac{h}{3}[f_3 + 4f_4 + f_5^{(p)}]$$

$$= 0.031 + \frac{0.1}{3}[0.331 + 1.9096 + 0.6392] = 0.1270$$

and $$y_{(0.6)}^{(p)} = y_6^{(p)} = y_2 + \frac{4h}{3}[2f_3 - f_4 + 2f_5]$$

$$= 0.01 + \frac{0.4}{3}[0.662 - 0.4774 + 2f(0.5, 0.1270)]$$

$$= 0.01 + \frac{0.4}{3}[0.662 - 0.4774 + 2(0.5 + 0.1270)]$$

$$= 0.2018$$

therefore $$f_{0.6}^{(p)} = f_6^{(p)} = f(0.6, 0.2018) = 0.6 + 0.2018 = 0.8018$$

To correct the predicted value of $y(0.6)$, we use the corrected formula as

$$y_{0.6}^{(c)} = y_6^{(c)} = y_4 + \frac{h}{3}[f_4 + 4f_5 + f_6^{(p)}]$$

$$= 0.0774 + \frac{0.1}{3}[0.4774 + 2.508 + 0.8018] = 0.2036$$

Hence, $$y_{(0.4)} = 0.0774,\ y_{(0.5)} = 0.1270,\ y_{(0.6)} = 0.2036$$

RUNGE – KUTTA FOURTH ORDER METHOD

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$$k_1 = h\, f(x_n, y_n)$$

$$k_2 = h\, f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\, f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h\, f(x_n + h, y_n + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{n+1} = y_n + k$$

Q. 1. *Use Runge-Kutta method to solve the following differential equation*

$$\frac{dy}{dx} = x + y^2 \; ; \; y(0) = 1$$

to find the approximate value of y(0.2)

Sol. We have

$$f(x, y) = x + y^2 \; ; \; y(x = 0) = 1$$

To calculate $y(x = 0.2)$, we take $h = 0.2$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.2\, f(0, 1)$$
$$= 0.2\,[0 + 1^2]$$
$$= 0.2$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right] = 0.2\, f(0.1, 1.1)$$
$$= 0.2\,[0.1 + (1.1)^2]$$
$$= 0.262$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right] = 0.2\, f(0.1, 1.131)$$
$$= 0.2\,[0.1 + (1.131)^2]$$
$$= 0.2758$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2\, f(0.2, 1.2758)$$
$$= 0.2\,[0.2 + (1.2758)^2]$$
$$= 0.3655$$

Thus, $$k = \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}\,[0.2 + 0.524 + 0.5516 + 0.3655]$$

$$= \frac{1}{6}\,(1.6411) = 0.2735$$

Hence, $$y_{0.2} = y_1 = y_0 + k$$

or $$y_{0.2} = 1 + 0.2735 = 1.2735$$

Q. 2. *Use Runge-Kutta method to solve the following differential equation*

$$\frac{dy}{dx} = -2xy^2 \; ; \; y(0) = 1$$

to find the approximate value of y for x = 0.2, 0.4 and compare the results with exact values (take h = 0.2)

Sol. We have

$$f(x, y) = -2xy^2 \; ; \; y(x = 0) = 1$$

To calculate $y(x = 0.2)$, we take $h = 0.2$

Here $x_0 = 0, \; y_0 = 1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.2\, f(0, 1)$$
$$= 0.2\, [-2(0)\,(1)^2] = 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\, f(0.1, 1)$$
$$= 0.2\, [-2(0.1)\,(1)^2]$$
$$= -0.04$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2\, f(0.1, 0.98)$$
$$= 0.2\, [-2(0.1)\,(0.98)^2] = -0.0384$$

$$k_4 = hf\,[x_0 + h, y_0 + k_3] = 0.2\, f(0.2, 0.9616)$$
$$= 0.2\, [-2(0.2)\,(0.9616)^2]$$
$$= -0.074$$

Thus, $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}(0 - 0.08 - 0.0768 - 0.074)$$
$$= \frac{1}{6}(-0.2308)$$
$$= -0.0385$$

Hence, $y_{0.2} = y_1 = y_0 + k = 1 - 0.0385 = 0.9615$

and to calculate $y(x = 0.4)$, we have $x_1 = 0.2$ and $y_1 = 0.9615$.

$\therefore$ $$k_1 = h\,f(x_1, y_1) = 0.2\,f(0.2, 0.9615) = 0.2\,[-2(0.2)(0.9615)^2]$$
$$= -0.074$$
$$k_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.2\,f(0.3, 0.9245) = 0.2\,[-2(0.3)(0.9245)^2]$$
$$= -0.1026$$
$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.2\,f(0.3, 0.9102) = 0.2\,[-2(0.3)(0.9102)^2]$$
$$= -0.0994$$
$$k_4 = h\,f(x_1 + h, y_1 + k_3) = 0.2\,f(0.4, 0.8621)$$
$$= 0.2\,[-2(0.4)(0.8621)^2] = -0.1189$$

Thus, $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(-0.074 - 0.2052 - 0.1988 - 0.1189)$$

$$= \frac{1}{6}(-0.5969) = -0.0995$$

Hence, $$y_{0.4} = y_2 = y_1 + k = 0.9615 - 0.0995 = 0.862$$

EXERCISE

Use Euler's method to solve the following differential equations :

1. $\dfrac{dy}{dx} = y + x \; ; \; y(0) = 1$

to find the approximate value of $y(0.5)$

2. $\dfrac{dy}{dx} = x^2 + y^2 \; ; \; y(0) = 0$

to find the approximate values of y for $x = 0.1 \; (0.1) \; 0.5$

Use modified Euler's method to solve the following differential equations :

3. $\dfrac{dy}{dx} = 2xy \; ; \; y(0) = 1$

to find the approximate value of $y(0.25)$

4. $\dfrac{dy}{dx} = x + y \; ; \; y(0) = 1$

to find the approximate value of $y(1)$ (take $h = 0.2$).

Use Milne's method to solve the following differential equations :

5. $\dfrac{dy}{dx} = 1 + y^2$; $y(0) = 0$, $y(0.2) = 0.2027$, $y(0.4) = 0.4228$, $y(0.6) = 0.6841$
to find the approximate values of $y(0.8)$ and $y(1.0)$

6. $\dfrac{dy}{dx} = x^2 + y^2 - 2$; $y(0) = 1$
to find the approximate value of $y(0.4)$ (take $h = 0.1$).

Use Runge—Kutta fourth order method to solve the following differential equations :

7. $\dfrac{dy}{dx} = x + y$; $y(0) = 1$
to find the approximate value of $y(0.2)$

8. $\dfrac{dy}{dx} = \dfrac{1}{x + y}$; $y(0) = 1$
to find the approximate value of $y(1)$ (take $h = 0.5$)

ANSWERS

1. 1.72
2. 0, 0.001, 0.005, 0.014, 0.030
3. 1.0642
4. 3.2566
5. 1.0294, 1.5557
6. 0.6148
7. 1.2428
8. 1.5837

THANKS