Recurrence Relations

Recurrence Relations:-

Let $\{a_n\} = \{a_{0,}\ a_{1,} a_{2,\ldots,} a_{r,\ldots}\}$ be a sequence of real numbers. A relation that expresses a_n in terms of one or more of the previous terms i.e., $a_{n-1,}\ a_{n-2,\ldots} a_{0,}$ where n is a non-negative integer, is called a recurrence relation for the sequence $\{a_n\}$.

Example:- **(i)** $a_n = a_{n-1} - a_{n-2} + 2a_{n-3}, n \geq 3$ with initial conditions $a_0 = 1, a_1 = 3,\ a_2 = 5$ is a recurrence relation.

(ii) $a_n + a_{n-1} + a_{n-2} = 0$

(iii) $a_n^2 + a_{n-1} + 5a_{n-2} = 0$

(iv) $a_n + n\, a_{n-1} = n^2 + n + 1$

If a recurrence relation is satisfied by the terms of the sequence $\{a_n\}$ then a_n is called the solution of the recurrence relation. A recurrence relation is also known as a difference equation.

Example:- The Fibonacci sequence , i.e., 1,1,2,3,5,8,... can be defined by the recurrence relation $a_n = a_{n-1} + a_{n-2}$ with initial conditions $a_0 = 1, a_1 = 1$.

Order of a recurrence relation:- The difference of the greatest and the smallest subscript appearing in the recurrence relation is called its order.

Example:- (i). The order of the recurrence relation $a_n = a_{n-1} + a_{n-2}$ with initial conditions $a_0 = 1, a_1 = 1$ is:- $\mathbf{n} - (\mathbf{n} - \mathbf{2}) = \mathbf{2}$.

(ii). The order of the recurrence relation $a_n = a_{n-1} - a_{n-2} + 2a_{n-3}$ is:- $\mathbf{n} - (\mathbf{n} - \mathbf{3}) = 3$.

Linear Recurrence Relation :-

If the relation does not have a term which is a product of two or more terms of the sequence then the recurrence relation is called Linear and otherwise it will be a non-linear recurrence relation. Thus the relation

(i). $a_n = a_{n-1} + a_{n-2}$ is a Linear recurrence relation.

(ii). $a_n^2 + 2a_{n-1} = f(n)$ is a Non-Linear recurrence relation.

(iii). $a_n \ a_{n-1} + 2a_{n+2} = n$ is a Non-Linear recurrence relation.

Linear Recurrence Relations with constant coefficients :-

A linear recurrence relation of order $k \in Z^{+}$ with constant coefficients is an equation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n) \qquad (1)$$

where $c_0, c_1, \cdots, c_k$ are constants such that $c_0 \neq 0$ and $c_k \neq 0$ and $f(n)$ is a function of n alone.

[A] Linear Homogeneous Recurrence Relation:-

If in relation (1), $f(n) = 0$ then it is said to be **Linear Homogeneous Recurrence Relation of order $\boldsymbol{k}$.**

[B] Linear Non-homogeneous Recurrence Relation:-

If in relation (1), $f(n) \neq 0$ then it is said to be **Linear Non-homogeneous Recurrence Relation of order $\boldsymbol{k}$.**

The examples of different types of recurrence relations are:-

Ex: (i). $a_n = a_{n-1} + a_{n-2}$ is a Linear homogeneous recurrence relation of order 2 with constant coefficients.

(ii). $3a_n + 5a_{n-1} = 2^n$ is a Linear non- homogeneous recurrence relation of order 1 with constant coefficients.

(iii). $a_n = 2a_{n-1} + a_{n-2}^2$ is a non-linear(since power of all a_i's is not 1) homogeneous recurrence relation of order 2 with constant coefficients.

(iv). $a_n^2 + 2a_{n-1} + a_{n-2} = (1 + n)$is a non-homogenous type non-Linear recurrence relation with constant coefficients.

(v). $a_n + na_{n-1} = n^2 + n + 1$ is a linear non-homogenous recurrence relation of order 1 with variable coefficients.

Solution of a Linear Recurrence Relations with constant coefficients :-

Let $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$ (1)

be a linear recurrence relation of order $k \in Z^+$ with constant coefficients. Its solution will consist of Two Part

(i) Complementary Part

(ii) Particular solution Part

The complementary function part will be the solution (or **Homogeneous solution)** of the relation.

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \qquad (2)$$

Where the right member $f(n) = 0$. The particular solution part will be a particular solution of (1).

The Complete solution of equation (1) will be expressed as

$$\boldsymbol{a_n = (a_n)_{homogeneous\ solution} + (a_n)_{particular\ solution}}$$

[A] Homogeneous Part of solution of a Recurrence Relation:-

A linear homogeneous recurrence relation of order k with constant coefficients is the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \quad , \quad c_0 \neq 0 \text{ and } c_k \neq 0 \qquad (1)$$

Let the solution of (1) is of the form $a_n = r^n$ (OR b^n OR α^n),where r (OR b OR α) is a constant, then it must satisfy the given equation (1) i.e.,

$$c_0 r^n + c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} = 0$$

or $$(c_0 r^k + c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k) r^{n-k} = 0$$

or $$c_0 r^k + c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k = 0 \qquad (2)$$

This equation (2) is called the **Characteristic equation OR Auxiliary equation** for the relation (1) and the roots of this equation are called **Characteristic roots.**

Since equation (2) is of k^{th}degree so it has k roots, namely, $r_1, r_2, \cdots, r_k$. The solution of relation (1) depends on the nature of these roots.

For this we have to consider the following cases:-

Case 1. When the roots are real and distinct

If $r_{1,} r_2, \cdots, r_k$ are real and distinct roots of equation then the solution of relation is given as

$$a_n = C_1 r_1^n + C_2 r_2^n + \cdots + C_k r_k^n$$

Case 2. When the roots are real and equal

If two roots r_1 and r_2 are real and equal i,e., $r_1 = r_2 = r$ (say) and remaining roots are real and distinct then the solution of relation is given as

$$a_n = (C_1 + C_2 n) r^n + C_3 r_3^n + \cdots + C_k r_k^n$$

Similarly, If three roots r_1, r_2 and r_3 are real and equal i,e., $r_1 = r_2 = r_3 = r$ (say) and remaining roots are real and distinct then the solution of relation is given as

$$a_n = (C_1 + C_2 n + C_3 n^2) r^n + C_4 r_4^n + \cdots + C_k r_k^n$$

In general, if the root r_1 is repeated k times, then the solution of relation is given as

$$a_n = (C_1 + C_2 n + C_3 n^2 + C_4 n^3 + \cdots + C_k n^{k-1}) r^n$$

Case 3. When the roots are imaginary and distinct

If $\alpha \pm i\beta$ are two imaginary roots of equation then the solution of relation is given as

$$a_n = (\alpha^2 + \beta^2)^{\frac{n}{2}}(C_1 \cos n\theta + C_2 \sin n\theta);\ \ \theta = \tan^{-1}\frac{\beta}{\alpha}$$

Case 4. When the roots are imaginary and repeated

If $\alpha \pm i\beta$ and $\alpha \pm i\beta$ are four imaginary roots of equation then the solution of relation is given as

$$a_n = (\alpha^2 + \beta^2)^{\frac{n}{2}}\,[(C_1 + C_2 n)\cos n\,\theta + (C_3 + C_4 n)\sin n\,\theta];\ \ \theta = \tan^{-1}\frac{\beta}{\alpha}$$

Remark:- In above the four cases $C_1, C_2, \cdots, C_k$ are the constants whose values are determined by the given initial conditions.

Example:- Solve the recurrence relation $\boldsymbol{a_n + a_{n-1} - 6a_{n-2} = 0}$ where $\mathbf{n \geq 2}$ and $\boldsymbol{a_0 = 1, a_1 = 2}$.

Solution:- Put $\boldsymbol{a_n = r^n}$ in the given relation, then characteristic equation or Auxiliary equation is given by

$$r^2 + r - 6 = 0 \quad \Rightarrow \quad r = 2, -3 \text{ (real and distinct)}$$

$\therefore$ The solution is

$$a_n = C_1(2)^n + C_2(-3)^n \tag{1}$$

Now, given that $\boldsymbol{a_0 = 1, a_1 = 2}$, Put $n = 0$ in (1), we get

$$a_0 = C_1 + C_2 \quad \Rightarrow \quad C_1 + C_2 = 1 \tag{2}$$

Put n =1 in (1), we get

$$a_1 = 2C_1 - 3C_2 \quad \Rightarrow \quad 2C_1 - 3C_2 = 2 \tag{3}$$

On solving (2) and (3) we get $C_1 = 1, C_2 = 0$

Put the values of $C_1 = 1, C_2 = 0$ in equation (1)

$\therefore$ The required solution is $a_n = (2)^n$

Example : Solve the recurrence relation

$a_n = a_{n-1} + a_{n-2}$, $n \geq 2$ with initial conditions $a_0 = 0$ and $a_1 = 1$.

Solution:- Put $a_n = r^n$ in the given relation, then characteristic equation or Auxiliary equation is given by

$$r^2 - r - 1 = 0 \qquad \Rightarrow r = \frac{1 \pm \sqrt{1+4}}{2} \qquad = \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ The solution is $a_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ .....(1)

Now, given that $a_0 = 0$ and $a_1 = 1$ so (1) gives

$$a_0 = C_1 + C_2 \Rightarrow C_1 + C_2 = 0 \qquad .....(2)$$

and $a_1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right)$

$$\Rightarrow \quad C_1\left(\frac{1+\sqrt{5}}{2}\right)+C_2\left(\frac{1-\sqrt{5}}{2}\right)=1 \qquad .....(3)$$

$$\Rightarrow \quad C_1\left(\frac{1+\sqrt{5}}{2}\right)-C_1\left(\frac{1-\sqrt{5}}{2}\right)=1 \qquad \text{[using (2)]}$$

$$\Rightarrow \quad \frac{C_1}{2}\left[1+\sqrt{5}-1+\sqrt{5}\right]=1 \Rightarrow \frac{C_1}{2}\left(2\sqrt{5}\right)=1$$

$$\Rightarrow \quad C_1=\frac{1}{\sqrt{5}}$$

and then (2) $\Rightarrow$ $C_2=-C_1=-\dfrac{1}{\sqrt{5}}$

$\therefore$ The required solution is $\quad a_n=\dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n-\dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^n.$

Example Solve the recurrence relation $d_n = 2d_{n-1} - d_{n-2}$

with initial conditions $d_1 = 1.5$ and $d_2 = 3$.

Solution. The characteristic equation is

$$r^2 - 2r + 1 = 0 : \qquad \Rightarrow r = 1, 1$$

$\therefore$ The solution is $d_n = (C_1 + C_2 n)(1)^n = C_1 + C_2 n$ .....(1)

Given that $d_1 = 1.5$ and $d_2 = 3$ so (1) gives

$$d_1 = C_1 + C_2 \Rightarrow C_1 + C_2 = 1.5 \qquad .....(2)$$

and $d_2 = C_1 + 2C_2 \Rightarrow C_1 + 2C_2 = 3$ .....(3)

$(3) - (2) \Rightarrow C_2 = 1.5$

$\therefore$ $(2) \Rightarrow C_1 = 1.5 - C_2 = 1.5 - 1.5 = 0$

$\therefore$ The required solution is $d_n = 1.5n$.

Example . Solve the recurrence relation

$$a_{n+2} + a_n = 0,\ n \geq 0 \text{ with } a_0 = 0,\ a_1 = 3.$$

Solution. The characteristic equation is

$$r^2 + 1 = 0 \Rightarrow r = \pm i$$

$\therefore$ The solution is $\quad a_n = (1)^{\frac{n}{2}}(C_1 \cos n\theta + C_2 \sin n\theta)$

where $\theta = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

$$\therefore \quad a_n = C_1 \cos\frac{n\pi}{2} + C_2 \sin\frac{n\pi}{2} \qquad \text{.....(1)}$$

$$\left.\begin{array}{ll} \text{given} & a_0 = 0 \Rightarrow 0 = C_1 \Rightarrow C_1 = 0 \\ \text{also} & a_1 = 3 \Rightarrow 3 = C_2 \Rightarrow C_2 = 3 \end{array}\right\} \qquad \text{.....(2)}$$

using (2) in (1) we get the required solution as $\quad a_n = 3\sin\frac{n\pi}{2}$.

Solution of a Linear Non-Homogeneous Recurrence Relations with constant coefficients :-

A linear **Non-Homogeneous** recurrence relation of order $k \in Z^+$ with constant coefficients is an equation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n) \qquad (1)$$

where $c_0, c_1, \cdots, c_k$ are constants such that $c_0 \neq 0$ and $c_k \neq 0$ and $f(n)$ is a function of n alone.

The Complete solution of relation (1) is of the form $a_n = a_n^{(h)} + a_n^{(p)}$ (2)

Where $a_n^{(h)}$ is a solution of the following associated **homogeneous recurrence relation**

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \qquad (3)$$

and $a_n^{(p)}$ is particular solution of the relation (1).

There is no general method to find the particular solution of the recurrence relation (1). It depends on the nature (form) of $f(n)$. Let, in relation (1)

$$f(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$$

where $b_0, b_1, \cdots, b_t$ and s are numbers. Now there are two possibilities for s , as given below.

Case 1. When s is not a root of the Characteristic equation of the associated linear homogeneous recurrence relations

In this case we assume that the form of a particular solution is

$$a_n^{(p)} = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n \qquad (4)$$

Since $a_n^{(p)}$ is a particular solution of relation (1) so it must satisfy the relation (1). We substitute $a_n = a_n^{(p)}$ on the left hand side of (1) and then on comparing the coefficients on both sides we obtain the values of the constants $p_{0,} p_{1,\cdots,} p_t$ which on substitution in equation (4) give the desired particular solution.

Case 2. When s is a root of the Characteristic equation of the associated linear homogeneous recurrence relations with multiplicity m

In this case we assume that the form of a particular solution is

$$a_n^{(p)} = n^m(b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0)s^n \qquad (5)$$

Where $p_{0,} p_{1,\cdots,} p_t$ are real numbers determined by the same procedure as given in Case-1 and m is any positive integer.

Example Solve the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$.

Solution. The characteristic equation is

$$r^2 + 5r + 6 = 0 \quad \Rightarrow r = -2, -3$$

$$\therefore \quad a_n^{(h)} = C_1(-2)^n + C_2(-3)^n \qquad .....(1)$$

Now, here $F(n) = (3n^2 - 2n + 1)(1)^n$

The particular solution of the given relation is of the form

$$a_n^{(p)} = (p_2n^2 + p_1n + p_0) \qquad .....(2)$$

[Since here t = 2 (highest power of n) and s = 1 which is not a root of characteristic equation so we apply Case 1]

It must satisfy the given relation i.e., $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$.

$$(p_2n^2 + p_1n + p_0) + 5\{p_2(n-1)^2 + p_1(n-1) + p_0\} + 6\{p_2(n-2)^2 + p_1(n-2) + p_0\}$$
$$= 3n^2 - 2n + 1$$

$$\text{or, } n^2(p_2 + 5p_2 + 6p_2) + n(p_1 - 10p_2 + 5p_1 - 24p_2 + 6p_1)$$
$$+ (p_0 + 5p_2 - 5p_1 + 5p_0 + 24p_2 - 12p_1 + 6p_0) = 3n^2 - 2n + 1$$

$$\text{or, } \quad 12p_2n^2 + (12p_1 - 34p_2)n + (12p_0 - 17p_1 + 29p_2) = 3n^2 - 2n + 1$$

On comparing the coefficients of equal powers of n on both sides, we get

$$12p_2 = 3: \qquad \Rightarrow p_2 = \frac{1}{4} \qquad .....(3).$$

$$12p_1 - 34p_2 = -2: \qquad \Rightarrow 12p_1 = 34\left(\frac{1}{4}\right) - 2 = \frac{13}{2} \qquad \text{[using (3)]}$$

$$\text{or} \qquad p_1 = \frac{13}{24} \qquad .....(4)$$

$$\text{and } 12p_0 - 17p_1 + 29p_2 = 1 \qquad \Rightarrow 12p_0 - 17 \times \frac{13}{24} + 29 \times \frac{1}{4} = 1$$

$$\text{or,} \qquad 12p_0 = 1 + \frac{221}{24} - \frac{29}{4} = \frac{1}{24}(24 + 221 - 174) = \frac{71}{24}$$

$$\text{or,} \qquad p_0 = \frac{71}{288} \qquad \therefore \qquad (2) \Rightarrow a_n^{(p)} = \frac{1}{4}n^2 + \frac{13}{24}n + \frac{71}{288}$$

Thus, the complete solution of the given relation is $a_n = a_n^{(h)} + a_n^{(p)}$ i.e.,

$$a_n = C_1(-2)^n + C_2(-3)^n + \frac{1}{4}n^2 + \frac{13}{24}n + \frac{71}{288}$$

Example . Solve the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 42(4)^n$

Solution. Here the characteristic equation is

$$r^2 + 5r + 6 = 0 \Rightarrow r = -2, -3$$

$$\therefore \quad a_n^{(h)} = C_1(-2)^n + C_2(-3)^n \qquad \text{.....(1)}$$

Here $F(n) = 42(4)^n$ i.e.; $s = 4$ which is not a root of the characteristic equation so we assume that the particular solution be

$$a_n^{(p)} = p_0(4)^n \qquad \text{.....(2)} \qquad \text{[by putting } t = 0 \text{ in Case 1]}$$

It must satisfy the given relation i.e.,

$$p_0(4)^n + 5p_0(4)^{n-1} + 6p_0(4)^{n-2} = 42(4)^n$$

$$\text{or, } \left(p_0 + \frac{5}{4}p_0 + \frac{6}{16}p_0\right)(4)^n = 42(4)^n$$

on comparing the coefficients on both sides, we get

$$p_0 + \frac{5}{4} p_0 + \frac{6}{16} p_0 = 42$$

or, $\left(1 + \frac{5}{4} + \frac{3}{8}\right) p_0 = 42 \Rightarrow \frac{21}{8} p_0 = 42$

or, $\quad p_0 = 16$

$\therefore \quad (2) \Rightarrow a_n^{(p)} = 16(4)^n$

Hence the complete solution is

$$a_n = a_n^{(h)} + a_n^{(p)} \text{ i.e.,}$$

$$a_n = C_1(-2)^n + C_2(-3)^n + 16(4)^n.$$

Example Solve the recurrence relation: $a_n - 2a_{n-1} = 3(2)^n$.

Solution. The characteristic equation is

$$r - 2 = 0 \Rightarrow r = 2$$

$$\therefore \quad a_n^{(h)} = C(2)^n \qquad .....(1)$$

Here $F(n) = 3(2)^n$

[i.e., here $t = 0$ and $s = 2$ which is a root of C. equation with multiplicity 1]

Let the particular solution be

$$a_n^{(p)} = np_0 2^n \qquad .....(2)$$

[by putting $t = 0$, $m = 1$ and $s = 2$ in Case 2]

It must satisfy the given relation so

$$np_0 2^n - 2(n-1)p_0 2^{n-1} = 3(2)^n$$

or, $(p_0 - p_0)n\ 2^n + p_0\ 2^n = 3(2)^n$

or, $p_0\ 2^n = 3(2)^n$

$\Rightarrow \quad p_0 = 3$

$\therefore \quad (2) \Rightarrow a_n^{(p)} = 3n2^n$. .....(3)

Thus the complete solution is

$$a_n = a_n^{(h)} + a_n^{(p)} \text{ i.e.,}$$

$$a_n = C(2)^n + 3n(2)^n.$$