Binomial Theorem

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

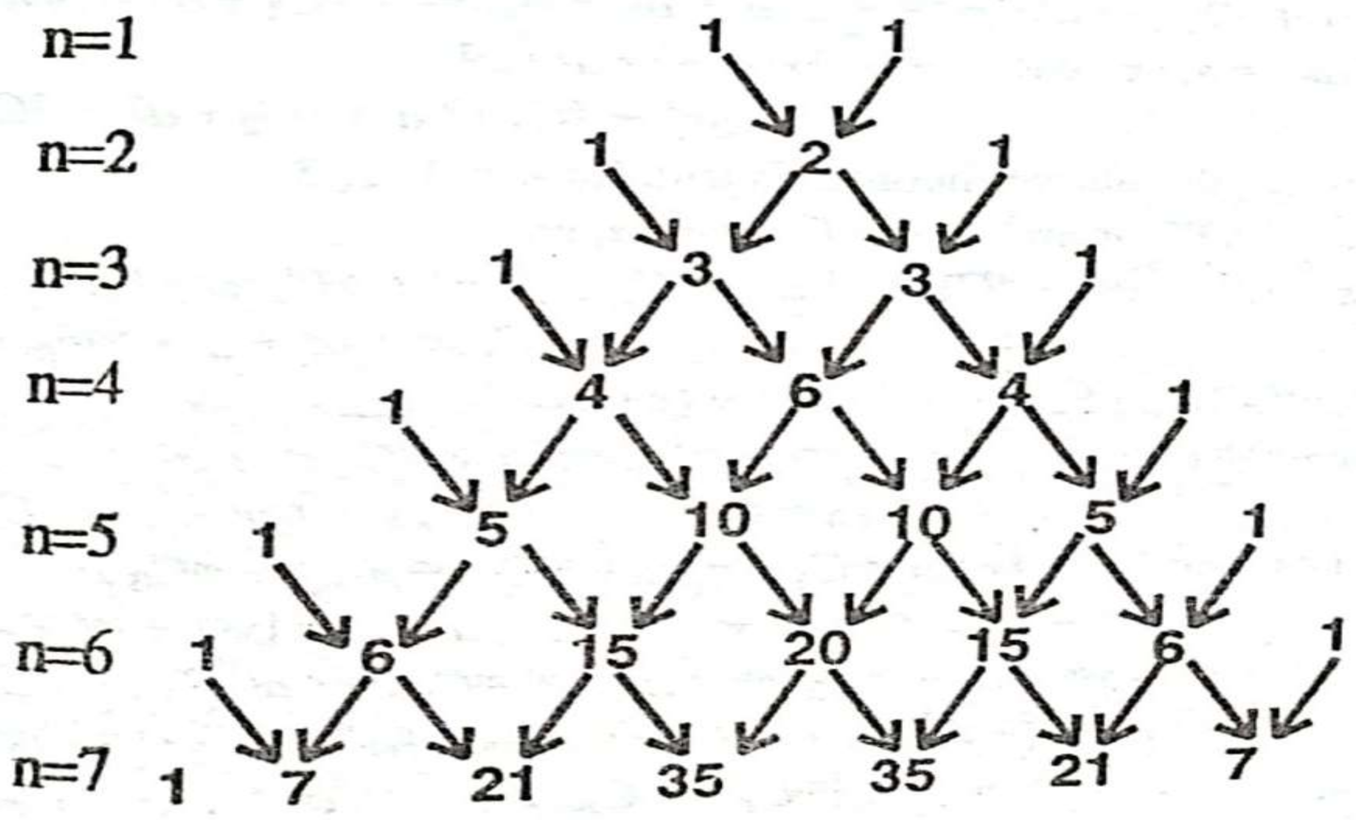
By actual multiplication we can also see that

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

Now we shall find the result for $(x+y)^n$.

Power

Pascal Triangle Coefficients



Binomial Theorem:-

If an expression has two terms it is called Binomial Expression. If the expansion of any index of the binomial expression is in the form of a series, it is called Binomial Theorem. It is as follows:

$$(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, a + {}^nC_2\, x^{n-2}\, a^2 + ... + {}^nC_n\, a^n$$

This series is called Binomial Expansion. The coefficients of the different powers of x are called ${}^nC_0$, ${}^nC_1$, ${}^nC_2$ ...${}^nC_n$ and the value of ${}^nC_0$ (and ${}^nC_n$) is one 1.

Binomial Theorem for Positive Index

$$(x+a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + {}^nC_3 x^{n-3} a^3 + \ldots + {}^nC_r x^{n-r} a^r + \ldots + {}^nC_n a^n$$

Other Forms of Binomial Theorem :

(i) on putting $-a$ for a,

$$(x-a)^n = x^n - {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 - {}^nC_3 x^{n-3} a^3 + \ldots + (-1)^r {}^nC_r x^{n-r} a^r + \ldots + (-1)^n a^n$$

(ii) on putting $x = 1$ and $a = x$ in the expansion of $(x+a)^n$,

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots {}^nC_r x^r + \ldots + {}^nC_n x^n$$

(iii) on putting $x = 1$ and $a = -x$ in the expansion of $(x+a)^n$,

$$(1-x)^n = 1 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots (-1)^r {}^nC_r x^r + \ldots (-1)^n {}^nC_n x^n$$

(iv)- $(x+a)^n = x^n\left(1+\frac{a}{x}\right)^n$

$$= x^n\left[1+{}^nC_1\left(\frac{a}{x}\right)+{}^nC_2\left(\frac{a}{x}\right)^2+\ldots+{}^nC_r\left(\frac{a}{x}\right)^r+\ldots+\left(\frac{a}{x}\right)^n\right]$$

Properties of Binomial Expansion :

1. The total number of terms in the expansion of $(x+a)^n$ is $(n+1)$.
2. In each term the sum of powers of x and a is same, and it is equal to n (the binomial index).
3. The coefficients of the terms equidistant from the beginning and end are equal i.e. ${}^nC_r = {}^nC_{n-r}, \quad 1 \le r \le n$

4. The $(r+1)^{th}$ term of the expansion is the general term. It is

represented by T_{r+1}. Hence the general term of $(x+a)^n$ is

$$T_{r+1} = {}^nC_r x^{n-r} a^r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}} x^{n-r} a^r$$

The general term in $(x-a)^n$ is $T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r} a^r$
The general term in $(1+x)^n$ is $T_{r+1} = {}^nC_r\, x^r$
The general term in $(1-x)^n$ is $T_{r+1} = (-1)^r\, {}^nC_r\, x^r$

5. The prefix of C is equal to the index of the Binomal and the suffix of C is one less than the number of the term.
6. The power of the first factor x is equal to the difference of prefix and suffix of C and the power of second factor a is equal to suffix of C.

- Middle term : As the number of ter[illegible] the expansion of $(x+a)^n$ is $(n+1)$, so if n is even, the middle term is $\left(\frac{n}{2}+1\right)^{th}$ term & if n is odd, the middle terms will be $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms. For ex. in $(x+a)^8$, the middle term is 5^{th} term but in $(x+a)^9$, the middle terms are 5^{th} and 6^{th} terms.

Particular Term in the Binomial Expansion :

If x^m occurs in T_{r+1} in the expansion of $\left(a.x^p \pm \frac{k}{x^q}\right)^n$, then

$$T_{r+1} = {}^nC_r (ax^p)^{n-r}\left(\pm\frac{k}{x^q}\right)^r = {}^nC_r\, a^{n-r} (\pm k)^r\, x^{np-r(p+q)}$$

$$\therefore\ np - r(p+q) = m$$

The particular term is obtained from this by calculating the value of r, r is always positive integer & so coefficient of x^m is

$${}^nC_r\, a^{n-r} (\pm k)^r$$

In the expansion for the term independent of x

$$np - r(p+q) = 0$$

$$\therefore\ r = \frac{np}{p+q}$$

Number of terms in the expansion of $(a+b+c)^n$:

$\because (a+b+c)^n = [(a+b)+c]^n$

$$= (a+b)^n + {}^nC_1\,(a+b)^{n-1}\,c + {}^nC_2\,(a+b)^{n-2}\,c^2 + \ldots + c^n$$

Number of terms in $(a+b+c)^n$ is

$= (n+1)$ terms $+ \; n$ terms $+ \; (n-1)$ terms $+ \ldots + \; 1$ term

$$= \frac{(n+1)(n+2)}{2} \text{ terms}$$

Ex. *Expand* $(3x + 2y)^6$.

Sol. $(3x+2y)^6$

$$= (3x)^6 + {}^6C_1\,(3x)^5\,(2y) + {}^6C_2\,(3x)^4\,(2y)^2$$

$$+ {}^6C_3\,(3x)^3\,(2y)^3 + {}^6C_4\,(3x)^2\,(2y)^4 + {}^6C_5\,(3x)\,(2y)^5 + (2y)^6$$

$$= 729x^6 + 2816\,x^5\,y + 4860x^4\,y^2 + 4320x^3y^3 + 2160x^2y^4$$

$$+ 576xy^5 + 64y^6$$

Ex. *Find the value of* $\left(1+\sqrt{5}\right)^5 + \left(1-\sqrt{5}\right)^5$

Sol. From Binomial theorem

$$\left(1+\sqrt{5}\right)^5 = 1 + {}^5C_1\left(\sqrt{5}\right) + {}^5C_2\left(\sqrt{5}\right)^2 + {}^5C_3\left(\sqrt{5}\right)^3 + {}^5C_4\left(\sqrt{5}\right)^4 + \left(\sqrt{5}\right)^5 \quad ..(i)$$

and

$$\left(1-\sqrt{5}\right)^5 = 1 - {}^5C_1\left(\sqrt{5}\right) + {}^5C_2\left(\sqrt{5}\right)^2 - {}^5C_3\left(\sqrt{5}\right)^3 + {}^5C_4\left(\sqrt{5}\right)^4 - \left(\sqrt{5}\right)^5 \quad ..(ii)$$

adding (i) and (ii)

$$\left(1+\sqrt{5}\right)^5 + \left(1-\sqrt{5}\right)^5 = 2\left[1 + {}^5C_2\left(\sqrt{5}\right)^2 + {}^5C_4\left(\sqrt{5}\right)^4\right]$$

$$= 2\,[\,1 + 50 + 125] = 2 \times 176 = 352$$

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Ex. ***Find the value of $(10.1)^5$ by Binomial theorem.***

Sol. $(10 \cdot 1)^5 = (10 + 0 \cdot 1)^5$

$$= (10)^5 + {}^5C_1 (10)^4 (\cdot 1) + {}^5C_2 (10)^3 (\cdot 1)^2 + {}^5C_3 (10)^2 (\cdot 1)^3 + {}^5C_4 (10) (\cdot 1)^4 + (\cdot 1)^5$$

$$= 100000 + 5 (10000) (\cdot 1) + 10 (1000) (\cdot 01) + 10 (100) (\cdot 001) + 5 (10) (\cdot 00001) + \cdot 00001$$

$$= 100000 + 5000 + 100 + 1 + \cdot 005 + \cdot 00001$$

$$= 105101 . 00501$$

Ex . *If in the expansion $(1+ax)^n$, the first three terms are $1+12x+64x^2$, then find the values of n & a.*

Sol. Given $(1+ax)^n = 1 + 2x + 64x^2 + \ldots$ ....(i)

$(1+ax)^n = 1 + {}^nC_1 (ax) + {}^nC_2 (ax)^2 + \ldots$..(by Binomial theorem)

$$= 1 + nax + \frac{n(n-1)}{\underline{|2}} a^2x^2 + \ldots \quad \text{.....(ii)}$$

Comparing the coefficients of power of x in (i) and (ii)

$$n\,a = 12 \qquad \ldots\text{(iii)} \qquad \frac{n(n-1)}{\underline{|2}}a^2 = 64 \qquad \ldots\text{(iv)}$$

dividing (iv) by the square of (iii)

$$\frac{n(n-1)}{2} \times \frac{1}{na^2} = \frac{64}{144}$$

$$\Rightarrow \quad \frac{n-1}{n} = \frac{2 \times 64}{144} = \frac{8}{9}$$

$$\Rightarrow 9n - 9 = 8n \qquad \Rightarrow 9n - 8n = 9$$

$$\therefore \ n = 9$$

$$\because n\,a = 12 \qquad \Rightarrow 9a = 12 \qquad \therefore a = \frac{12}{a} = \frac{4}{3}$$

Hence $n = 9$ and $a = \dfrac{4}{3}$

Ex. . ***Find the coefficient of x^4 in the expansion of*** $(1+x+x^2)^3$

Sol. $(1+x+x^2)^3 = [(1+x)+x^2]^3$

$= (1+x)^3 + {}^3C_1(1+x)^2(x^2) + {}^3C_2(1+x)(x^2)^2 + (x^2)^3$

$= (1+3x+3x^2+x^3) + 3(1+2x+x^2)x^2 + 3(1+x)x^4 + x^6$

$= 1+3x+3x^2+x^3+3x^2+6x^3+3x^4+3x^4+3x^5+x^6$

$= 1+3x+6x^2+7x^3+6x^4+3x^5+x^6$

Hence the coefficient of $x^4 = 6$.

Ex. . ***Find the 10^{th} term in*** $(2x^2 - y^3)^{12}$.

Sol. $\because\ r+1 = 10 \quad \Rightarrow r = 9$ and $n = 12$

General term $T_{r+1} = (-1)^r\, {}^nC_r\, (x)^{n-r}\, (a)^r$

$T_{10} = (-1)^9\, {}^{12}C_9\, (2x^2)^3\, (y^3)^9$

$= -(220)(8x^6)(y^{27})$

$= -1760\, x^6 y^{27}$

Ex. *Find the coefficient of x^{18} in the expansion of*

$$\left(x^2+\frac{3a}{x}\right)^{15}$$

Sol. Let x^{18} occur in T_{r+1}.

$$T_{r+1}={}^{15}C_r(x^2)^{15-r}\left(\frac{3a}{x}\right)^r={}^{15}C_r(3)^r a^r x^{30-3r}$$

$\because\ 30-3r=18,\Rightarrow 3r=12 \qquad \therefore r=4$

Hence the coefficient of $x^{18}={}^{15}C_r\,(3)^r\,a^r$

$$={}^{15}C_4\,(3)^4\,a^4=110565a^4$$

Ex. *Find the value of the term independent of x in the expansion of* $\left(2x^2 + \frac{1}{2x}\right)^9$

Sol. Let the $(r+1)^{th}$ term be independent of x

$$T_{r+1} = {}^9C_r (2x^2)^{9-r}\left(\frac{1}{2x}\right)^r$$

$$= {}^9C_r \frac{2^{9-r}}{2^r}\frac{(x^2)^{9-r}}{x^r} = {}^9C_r\, 2^{9-2r}\, x^{18-3r}$$

$\because$ for the term independent of x, the power of x is zero.

$\therefore 18 - 3r = 0 \Rightarrow 3r = 18 \quad \Rightarrow r = 6$

Hence 7^{th} term is independent of x.

$$T_7 = {}^9C_6\, 2^{9-12} = \frac{9\times 8\times 7}{1\times 2\times 3}\times\frac{1}{2^3} = \frac{21}{2} = 10\frac{1}{2}$$

Ex. *Find the middle term in the expansion of* $\left(\frac{2}{3}x^2 - \frac{3}{2x}\right)^{20}$

Sol. $\because$ n is even, so the middle term is $=\left(\frac{n}{2}+1\right)$ term

$$=\left(\frac{20}{2}+1\right)=11^{\text{th}} \text{ term.}$$

$$T_{11} = {}^{20}C_{10}\left(\frac{2}{3}x^2\right)^{10}\left(\frac{3}{2x}\right)^{10}$$

$$= {}^{20}C_{10}\, x^{10} = 184756\, x^{10}$$

Ex. *Prove that the middle term in the expansion of*

$$(1+x)^{2n} \text{ is } \frac{1.3.5....(2n-1)}{\lfloor n}2^n x^n$$

Sol. $\because$ $2n$ is even, so the middle term is

$$\left(\frac{2n}{2}+1\right) \text{ i.e. } (n+1)^{\text{th}} \text{ term.}$$

$$T_{n+1} = {}^{2n}C_n x^n$$

$$= \frac{\lfloor 2n}{\lfloor n \lfloor n} x^n = \frac{1.2.3.4.5.6....(2n-3)(2n-2)(2n-1)(2n)}{\lfloor n \lfloor n} x^n$$

$$= \frac{[1.3.5....(2n-3)(2n-1)][2.4.6...(2n-2)(2n)]}{\lfloor n \lfloor n} x^n$$

$$= \frac{[1.3.5....(2n-3)(2n-1)]\, 2^n [1.2.3...(n-1)(n)]}{\lfloor n \lfloor n} x^n$$

$$= \frac{1.3....(2n-3)(2n-1).2^n \lfloor n}{\lfloor n \lfloor n} x^n \qquad = \frac{1.3.5.....(2n-3)(2n-1)}{\lfloor n} 2^n x^n$$

Ex. *. If the coefficients of x^7 and of $\frac{1}{x^7}$ in the expansion of $\left(ax+\frac{1}{bx}\right)^{11}$ are equal, prove $a\,b = 1$.*

Sol. The general term in the expansion of $\left(ax+\frac{1}{bx}\right)^{11}$,

$$T_{r+1} = {}^{11}C_r (ax)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r \frac{a^{11-r}}{b^r} x^{11-2r}$$

So the coefficient of $x^{11-2r} = {}^{11}C_r \frac{a^{11-r}}{b^r}$

When $11 - 2r = 7$, $r = 2$

$\therefore$ coefficient of $x^7 = {}^{11}C_2 \frac{a^9}{b^2} = 55\frac{a^9}{b^2}$

When $11 - 2r = -7$, $r = 9$

$\therefore$ coefficient of $x^{-7} = {}^{11}C_9 \frac{a^2}{b^9} = 55\frac{a^2}{b^9}$

As both these coefficients are equal,

$$55\frac{a^9}{b^2} = 55\frac{a^2}{b^9} \Rightarrow a^7b^7 = 1$$

$$\Rightarrow (ab)^7 = 1$$

$$\Rightarrow ab = 1$$

Ex. *If the coefficients of 5^{th}, 6^{th} & 7^{th} terms in $(1+x)^n$ are in A.P., find the value of n.*

Sol. $\because$ In $(1+x)^n$, $T_{r+1} = {}^nC_r\, x^r$

$\therefore$ coefficient of $(r+1)^{th}$ term $= {}^nC_r$

$\therefore$ coeff. of 5^{th} term $= {}^nC_4$

coeff. of 6^{th} term $= {}^nC_5$

coeff. of 7^{th} term $= {}^nC_6$

${}^nC_4,\ {}^nC_5,\ {}^nC_6$ are in A.P., so

$2\,{}^nC_5 = {}^nC_4 + {}^nC_6$

$$\Rightarrow \frac{2\underline{|n}}{\underline{|5}\,\underline{|n-5}} = \frac{\underline{|n}}{\underline{|4}\,\underline{|n-4}} + \frac{\underline{|n}}{\underline{|n-6}\,\underline{|6}}$$

$$\Rightarrow \frac{1}{60\underline{|n-5}} = \frac{1}{24\underline{|n-4}} + \frac{1}{720\underline{|n-6}}$$

$$\Rightarrow \frac{12}{(n-5)\underline{|n-6}} = \frac{30}{(n-4)(n-5)\underline{|n-6}} + \frac{1}{\underline{|n-6}}$$

$$\Rightarrow \frac{12}{(n-5)} - 1 = \frac{30}{(n-4)(n-5)}$$

$$\Rightarrow \frac{12-(n-5)}{(n-5)} = \frac{30}{(n-4)(n-5)}$$

$\Rightarrow (17-n)(n-4)(n-5) = 30(n-5)$

$\Rightarrow (17-n)(n-4) = 30$

$\Rightarrow 21n-68-n^2 = 30$

$\Rightarrow n^2 - 21n + 98 = 0$

$\Rightarrow (n-14)(n-7) = 0 \Rightarrow n=7$ or 14

$\therefore n = 7$ or 14

Exercise

1. Expand the following expressions by binomial theorem :

(i) $(3x + 2y)^5$ (ii) $\left(x - \frac{1}{x}\right)^7$ (iii) $\left(ax - \frac{b}{x}\right)^6$

(iv) $(1 + x + x^2)^4$ (v) $\left(\sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}}\right)^6$ (vi) $\left(x - \frac{1}{3x}\right)^5$

2. Find the value of the following by binomial theorem .

(i) $\left(2 + \sqrt{3}\right)^7 + \left(2 - \sqrt{3}\right)^7$

(ii) $\left(x + \sqrt{x^2 + 1}\right)^5 + \left(x - \sqrt{x^2 + 1}\right)^5$

(iii) $\left(\sqrt{3} + 1\right)^5 - \left(\sqrt{3} - 1\right)^5$ (iv) $(.99)^5 + (1.01)^5$

3. Find the value of the following b[illegible]nomial theorem :

(i) $(99.9)^4$ 11^5

(iii) $(10.01)^3$

4. Find the required term in the following :

(i) 7^{th} term in $\left(3x^2 - \frac{1}{x^3}\right)^{10}$

(ii) 9^{th} term in $\left(\frac{x}{y} - \frac{3y}{x^2}\right)^{12}$

(iii) 8^{th} term in $\left(x^{3/2}y^{1/2} - x^{1/2}y^{3/2}\right)^{10}$

5. Find the coefficient of :

(i) x^{10} in $\left(2x^2 - \frac{1}{x}\right)^{20}$ (ii) x^6 in $\left(a - bx^2\right)^{10}$

(iii) x^{-15} in $\left(3x^2 - \frac{c}{3x^3}\right)^{10}$ (iv) x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^8$

Binomial Theorem for Any Index :

If n is a positive integer,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \ldots\ldots + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r} x^r + \ldots + x^n$$

when n is fraction or negative or some positive and $|x| \le 1$, then the expansion of $(1+x)^n$ is as follows :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r} x^r + \ldots$$

This is general form of binomial theorem and its expansion is called binomial series.

Important points of the expansion of $(1+x)^n$ for any index n.

1. The general form is similar to that of the binomial series for positive index.
2. If n is a fraction or negative quantity, ${}^nC_r$ is meaningless. So we should not write the coefficient of terms as ${}^nC_1$, ${}^nC_2$, ${}^nC_3$,.... and it should be written as above.
3. If n is a positive integer, then number of terms is $(n+1)$ & if n is fraction or negative, then the number of terms will be infinite.
4. In the binomial form $(1 \pm x)^n$, the value of $|x| \leq 1$.
5. In the expansion of $(x+a)^n$, if $x < a$, change $(x+a)^n = a^n\left(1+\frac{x}{a}\right)^n$.

   & then expand it. If $a < x$, $(x+a)^n = x^n\left[1+\frac{a}{x}\right]^n$ & then expand it.
6. The general term is $(r+1)^{th}$ term and its value is

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}x^r$$

Some Important Expansions :

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{\underline{\lfloor 2}}x^2 - \frac{n(n-1)(n-2)}{\underline{\lfloor 3}}x^3 + \ldots$$

$$+ (-1)^r \frac{n(n-1)(n-2)..(n-r+1)}{\underline{\lfloor r}}x^r + \ldots. \qquad \text{(i)}$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{\underline{\lfloor 2}}x^2 - \frac{n(n+1)(n+2)}{\underline{\lfloor 3}}x^3 + \ldots$$

$$+ (-1)^r \frac{n(n+1)(n+2)..(n+r-1)}{\underline{\lfloor r}}x^r + \ldots. \qquad \text{(ii)}$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{\underline{\lfloor 2}}x^2 + \frac{n(n+1)(n+2)}{\underline{\lfloor 3}}x^3 + \ldots$$

$$+ \frac{n(n+1)(n+2)..(n+r-1)}{\underline{\lfloor r}}x^r + \ldots. \qquad \text{(iii)}$$

putting $n = 1, 2, 3$ in series (ii) and (iii),

$$(1+x)^{-1} = 1 - x + x^2 - x^3 \ldots\ldots + (-1)^r x^r + \ldots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 \ldots\ldots + (-1)^r (r+1) x^r + \ldots$$

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 \ldots$$

$$+ (-1)^r \frac{(r+1)(r+2)}{\underline{|2}} x^r + \ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 \ldots\ldots + x^r + \ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 \ldots\ldots + (r+1) x^r + \ldots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 \ldots\ldots + \frac{(r+1)(r+2)}{\underline{|2}} x^r + \ldots$$

Ex. *Expand* $\left(1+\frac{2}{3}x\right)^{3/2}$ *upto 4 terms.*

Sol. $\left(1+\frac{2}{3}x\right)^{3/2}=1+\left(\frac{3}{2}\right)\left(\frac{2}{3}x\right)$

$$+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{\underline{|2}}\left(\frac{2}{3}x\right)^2+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{\underline{|3}}\left(\frac{2}{3}x\right)^3.....$$

$$=1+x+\frac{3}{2}.\frac{1}{2}.\frac{1}{2}.\frac{4}{9}x^2+\frac{3}{2}.\frac{1}{2}\left(-\frac{1}{2}\right).\frac{1}{6}\left(\frac{8}{27}\right)x^3+.....$$

$$=1+x+\frac{1}{6}x^2-\frac{1}{54}x^3+......$$

Ex. . *Expand* $(2+3x)^{-4}$ *upto 4 terms.*

Sol. $(2+3x)^{-4}=2^{-4}\left(1+\dfrac{3}{2}x\right)^{-4}$

$$=\frac{1}{2^4}\left[1-4\left(\frac{3}{2}x\right)+\frac{4\times5}{\underline{|2}}\left(\frac{3}{2}x\right)^2-\frac{4\times5\times6}{\underline{|3}}\left(\frac{3}{2}x\right)^3+\ldots.\right]$$

$$=\frac{1}{16}\left[1-6x+\frac{45}{2}x^2-\frac{135}{2}x^3+\ldots.\right]$$

Ex. . ***Find the general term in the expansion of*** $(1-2x)^{-3/2}$

Sol. The general term T_{r+1} of $(1-x)^{-n}$ is

$$=\frac{n(n+1)(n+2)\ldots(n+r-1)}{\underline{|r}}x^r$$

The general term of $(1-2x)^{-3/2}$ is

$$T_{r+1}=\frac{\frac{3}{2}\left(\frac{3}{2}+1\right)\left(\frac{3}{2}+2\right)\ldots\left(\frac{3}{2}+r-1\right)}{\underline{|r}}(2x)^r$$

$$=\frac{3.5.7\ldots.(r+1)}{2^r\underline{|r}}2^rx^r\quad=\frac{3.5.7\ldots\ldots(2r+1)}{\underline{|r}}x^r$$

Ex. . *Find the 8th term of* $\left(1+\frac{x}{3}\right)^{-5}$

Sol. $$T_{r+1} = (-1)^r \frac{n(n+1)(n+2)\ldots(n+r-1)}{\underline{|r}} x^r$$

Put $r = 7$ and n = 5

$$T_8 = (-1)^7 \frac{(5)(6)(7)(8)(9)(10)(11)}{\underline{|7}} \left(\frac{x}{3}\right)^7$$

$$= -\frac{5.6.7.8.9.10.11}{7.6.5.4.3.2.1} \frac{x^7}{3^7} = -\frac{110}{729} x^7$$

Ex. . ***Find the coefficient of*** x^3 ***in*** $\frac{(1+3x)^2}{1-2x}$

Sol. $\frac{(1+3x)^2}{1-2x} = (1+3x)^2(1-2x)^{-1}$

$= (1 + 6x + 9x^2)\,(1 + 2x + 4x^2 + 8x^3 \ldots)$

coefficient of x^3 is $= 8 + 24 + 18 + = 50$

Exercise

1. Expand upto 4 terms of the following binomial expressions :

   (i) $\left(2 - x^2\right)^{-2/3}$ (ii) $(1 + x)^{3/2}$

   (iii) $(4a - 8x)^{-1/2}$ (iv) $\dfrac{1}{\sqrt{5 + 4x}}$

   (v) $\dfrac{1}{(4 - 3x^2)^{1/3}}$ (vi) $\dfrac{1}{(2 - 3x)^3}$

2. Find the general term in the following :

   (i) $(3 - 2x^2)^{-2/3}$ (ii) $\left(4 - 5x^2\right)^{-1/2}$

   (iii) $\left(a^3 - x^3\right)^{2/3}$ (iv) $\dfrac{1}{\sqrt{1 - 4x}}$

3. Find the required term in the following expansions :

   (i) 7^{th} term of $(1 + x)^{5/2}$ (ii) 4^{th} term of $(1 - 2x)^{-1/2}$

   (iii) 5^{th} term of $(9 + 6x)^{-3/2}$

4. If $|x| < \dfrac{1}{3}$, find the coefficient of x^5 in the expansion o $(1 - 3x)^{-1/3}$

5. Find the coeff. of x^{10} in $(1 + x^2)^{-3}$

Ex. .. *If x is so small that its square and higher powers are neglected, then prove that :*

$$\frac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x}=\frac{1}{4}-\frac{17}{384}x$$

Sol. L.H.S.

$$=\frac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x}$$

$$=\frac{9^{1/2}\left(1+\frac{7}{9}x\right)^{1/2}-16^{1/4}\left(1+\frac{3}{16}x\right)^{1/4}}{4+5x}$$

$$=\frac{3\left(1+\frac{1}{2}\times\frac{7}{9}x\right)-2\left(1+\frac{1}{4}\times\frac{3}{16}x\right)}{4+5x}$$

$$=\frac{3+\frac{7}{6}x-2-\frac{3}{32}x}{4+5x}=\left(1+\frac{103}{96}x\right)(4+5x)^{-1}$$

$$=\left(1+\frac{103}{96}x\right).4^{-1}\left[1+\frac{5}{4}x\right]^{-1}$$

$$= \frac{1}{4}\left(1+\frac{103}{96}x\right)\left(1-\frac{5}{4}x\right) = \frac{1}{4}\left(1+\frac{103}{96}x-\frac{5}{4}x\right)$$

$$= \frac{1}{4}\left(1-\frac{17}{96}x\right) = \frac{1}{4}-\frac{17}{384}x = \text{R.H.S.}$$

Ex. . *Find the value of* $\sqrt[3]{126}$ *upto 5 decimals.*

Sol. $\sqrt[3]{126} = (126)^{1/3} = (125+1)^{1/3}$

$$= (125)^{1/3}\left[1+\frac{1}{125}\right]^{1/3} = 5\left[1+\frac{1}{5^3}\right]^{1/3}$$

$$= 5\left[1+\frac{1}{3}\left(\frac{1}{5^3}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{\lfloor 2}\left(\frac{1}{5^3}\right)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{\lfloor 3}\left(\frac{1}{5^3}\right)^3+\ldots\right]$$

$$= 5\left[1+\frac{1}{3}\cdot\frac{1}{5^3}+\frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{5^6}\right)+\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(\frac{1}{6}\right)\times\frac{1}{5^9}+\ldots\right]$$

$$= 5\left[1+\frac{1}{3}\cdot\frac{1}{5^3}-\frac{1}{9}\cdot\frac{1}{5^6}+\frac{1}{81}\cdot\frac{1}{5^9}+\ldots\ldots\right]$$

$$= 5+\frac{1}{3}\cdot\frac{1}{5^2}-\frac{1}{9}\cdot\frac{1}{5^5}+\frac{1}{81}\cdot\frac{1}{5^7}\ldots\ldots$$

$$= 5+.01333-.000035+\ldots\ldots = 5.01329 \cong 5.01330$$

Properties of Binomial Coefficients :

In the expansion of $(1+x)^n$, the coefficients of various powers of x are $^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$. They are called binomial coefficients, which are also represented as $C_0, C_1, C_2, \ldots, C_n$.

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n \qquad \ldots(i)$$

(i) put $x = 1$,

$$(1+1)^n = C_0 + C_1 + C_2 + C_3 + \ldots + C_n$$

i.e. $$C_0 + C_1 + C_2 + C_3 + \ldots + C_n = 2^n \qquad \ldots(ii)$$

(ii) put $x = -1$,

$$(1-1)^n = C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n C_n$$

$$\Rightarrow C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n C_n = 0 \qquad \ldots(iii)$$

(iii) $$C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1} \qquad \ldots(iv)$$

Ex. *Find the value of* $^{15}C_1 + {}^{15}C_2 + {}^{15}C_3 + \ldots\ {}^{15}C_{15}$

Sol. $\because\ {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$

putting $n = 15$,

$^{15}C_0 + {}^{15}C_1 + {}^{15}C_2 + {}^{15}C_3 + \ldots + {}^{15}C_{15} = 2^{15}$

$^{15}C_1 + {}^{15}C_2 + {}^{15}C_3 + \ldots + {}^{15}C_{15} = 2^{15} - 1$, $[\because\ {}^{15}C_0 = 1]$

Ex. *Prove* $C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n\,2^{n-1}$

Sol. L.H.S. $C_1 + 2C_2 + 3C_3 + \ldots + nC_n$

$$= n + 2.\frac{n(n-1)}{\underline{|2}} + 3.\frac{n(n-1)(n-2)}{\underline{|3}} + \ldots + n$$

$$= n\left[1 + (n-1) + \frac{(n-1)(n-2)}{\underline{|2}} + \ldots + 1\right]$$

$$= n\left[1 + (n-1) + \frac{(n-1)(n-2)}{\underline{|2}} + \ldots \frac{\underline{|n-1}}{\underline{|0}\ \underline{|n-1}}\right]$$

$$= n\left[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots\ldots {}^{n-1}C_{n-1}\right]$$

$$= n(1+1)^{n-1} = n\left[2^{n-1}\right] = n\ 2^{n-1} = \text{R.H.S.}$$

Ex ***If $C_0, C_1, C_2 \ldots\ldots\ C_n$ are the coefficients in the expansion of $(1+x)^n$, prove***

$$\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \ldots \quad n \text{ terms} = \frac{n(n+1)}{2}$$

Sol. L.H.S. $= \dfrac{C_1}{C_0} + 2\dfrac{C_2}{C_1} + 3\dfrac{C_3}{C_2} + \ldots\ n$ terms

$$= \frac{n}{1} + 2\frac{\frac{n(n-1)}{\lfloor 2}}{n} + 3\frac{\frac{n(n-1)(n-2)}{\lfloor 3}}{\frac{n(n-1)}{\lfloor 2}} + \ldots\ n \text{ terms}$$

$$= n + (n-1) + (n-2) + \ldots\ n \text{ terms}$$

$$= \frac{n(n+1)}{2} = \text{R.H.S.}$$

Sum of Series by Binomial Theorem :

To find the sum of a given binomial series, we compare its terms by the corresponding terms of the following standard binomial series :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \ldots$$

We make two equations on the basis of comparison & get the values of n and x. Then putting the values of n and x in $(1+x)^n$ we obtain the sum.

Ex. ***Find the sum of the following infinite series :***

$$1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \ldots.$$

Sol. Given series is $1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+....$

Standard series is

$$(1+x)^n = 1+nx+\frac{n(n-1)}{\underline{|2}}x^2+\frac{n(n-1)(n-2)}{\underline{|3}}x^3+.....$$

on comparing the terms of both the series

$$nx = \frac{3}{4} \qquad ...(i)$$

$$\frac{n(n-1)}{\underline{|2}}x^2 = \frac{3.5}{4.8} \qquad ...(ii)$$

squaring (i) and dividing in (ii)

$$\frac{n(n-1)}{\underline{|2}}\cdot\frac{x^2}{n^2x^2} = \frac{3.5}{4.8}\times\left(\frac{4}{3}\right)^2$$

$$\Rightarrow \frac{(n-1)}{n} = \frac{2.3.5.4.4}{4.8..3.3} \Rightarrow \frac{n-1}{n} = \frac{5}{3}$$

$$\Rightarrow 5n = 3n - 3 \qquad \Rightarrow 2n = -3 \qquad \therefore n = -3/2$$

putting the values of n in (i)

$$\left(-\frac{3}{2}\right)x = \frac{3}{4} \Rightarrow x = -\frac{1}{2}$$

Hence the sum of the series $= (1+x)^n = \left(1 - \frac{1}{2}\right)^{-3/2}$

$$= \left(\frac{1}{2}\right)^{-3/2} = (2)^{3/2} = 2\sqrt{2}$$

Ex. *Prove*

$$\frac{7}{5}\left\{1+\frac{1}{10^2}+\frac{1.3}{1.2}.\frac{1}{10^4}+\frac{1.3.5}{1.2.3}.\frac{1}{10^6}+\ldots\ldots\right\}=\sqrt{2}$$

Sol. Given series is

$$\frac{7}{5}\left\{1+\frac{1}{10^2}+\frac{1.3}{1.2}.\frac{1}{10^4}+\frac{1.3.5}{1.2.3}.\frac{1}{10^6}+\ldots\ldots\right\}=\sqrt{2}$$

$$\Rightarrow 1+\frac{1}{10^2}+\frac{1.3}{1.2}.\frac{1}{10^4}+\frac{1.3.5}{1.2.3}.\frac{1}{10^6}+\ldots\ldots=\frac{5\sqrt{2}}{7}$$

Standard series is

$$(1+x)^n=1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\frac{n(n-1)(n-2)}{\lfloor 3}x^3+\ldots$$

on comparing both the series

$$nx = \frac{1}{10^2} \qquad ..(\text{ i })$$

$$\frac{n(n-1)}{\underline{|2}} x^2 = \frac{1.3}{1.2} \cdot \frac{1}{10^4} \qquad ..(\text{ ii })$$

Divide (ii) by the square of (i)

$$\frac{n(n-1)}{\underline{|2}} x^2 \times \frac{1}{n^2 x^2} = \frac{1.3}{1.2} \times \frac{1}{10^4} \times 10^4$$

$$\Rightarrow \frac{n-1}{n} = 3 \qquad \Rightarrow 3n = n - 1$$

$$\Rightarrow 2n = -1 \qquad \therefore\ n = -\frac{1}{2}$$

putting the value of n in (i)

$$\left(-\frac{1}{2}\right)x = \frac{1}{10^2} \Rightarrow x = -\frac{2}{100} = -\frac{1}{50}$$

Hence the sum of the series

$$= (1+x)^n = \left(1 - \frac{1}{50}\right)^{-1/2} = \left(\frac{49}{50}\right)^{-1/2} = \left(\frac{50}{49}\right)^{1/2}$$

$$= \sqrt{\frac{25 \times 2}{49}} = \frac{5}{7}\sqrt{2}$$

Ex. *If* $y=\frac{2}{5}+\frac{1.3}{\lfloor 2}\left(\frac{2}{5}\right)^2+\frac{1.3.5}{\lfloor 3}\left(\frac{2}{5}\right)^3+....$ *then prove that*

$$y^2+2y-4=0$$

Sol. $\because \; y=\frac{2}{5}+\frac{1.3}{\lfloor 2}\left(\frac{2}{5}\right)^2+\frac{1.3.5}{\lfloor 3}\left(\frac{2}{5}\right)^3+...$

$$\Rightarrow 1+y=1+\frac{2}{5}+\frac{1.3}{\lfloor 2}\left(\frac{2}{5}\right)^2+\frac{1.3.5}{\lfloor 3}\left(\frac{2}{5}\right)^3+...$$

$$(1+x)^n=1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\frac{n(n-1)(n-2)}{\lfloor 3}x^3+...$$

comparing R.H.S., $nx=\frac{2}{5}$. ...(i)

$$\frac{n(n-1)}{\lfloor 2}x^2=\frac{1.3}{\lfloor 2}\left(\frac{2}{5}\right)^2 \qquad ...(ii)$$

dividing (ii) by the square of (i)

$$\frac{n(n-1)}{2}x^2 \times \frac{1}{n^2x^2} = \frac{3}{12}\left(\frac{2}{5}\right)^2 \times \left(\frac{5}{2}\right)^2$$

$$\Rightarrow \frac{(n-1)}{n} = 3 \qquad \Rightarrow 3n = n-1$$

$$\Rightarrow 2n = -1 \qquad n = -\frac{1}{2}$$

putting the value of n in (i)

$$\left(-\frac{1}{2}\right)x = \frac{2}{5} \qquad x = -\frac{4}{5}$$

so the sum of the R.H.S. of the series

$$= \left(1-\frac{4}{5}\right)^{-1/2} = \left(\frac{1}{5}\right)^{-1/2} = (5)^{1/2}$$

$\therefore \quad 1 + y = (5)^{1/2}$

squaring $\quad 1 + 2y + y^2 = 5$

$$\Rightarrow y^2 + 2y - 4 = 0$$

Ex. *Prove* $x^n = 1 + n\left(1-\frac{1}{x}\right) + \frac{n(n+1)}{\underline{|2}}\left(1-\frac{1}{x}\right)^2 + \ldots.$

Sol. R.H.S. $= 1 + n\left(1-\frac{1}{x}\right) + \frac{n(n+1)}{\underline{|2}}\left(1-\frac{1}{x}\right)^2 + \ldots.$

Let $1-\frac{1}{x} = y$

then R.H.S. $= 1 + ny + \frac{n(n+1)}{\underline{|2}} y^2 \ldots \qquad = (1-y)^{-n}$

(putting the value of y)

$$= \left[1-\left(1-\frac{1}{x}\right)\right]^{-n} = \left(1-1+\frac{1}{x}\right)^{-n}$$

$$= \left(\frac{1}{x}\right)^{-n} = x^n = \text{L.H.S.}$$