Example. *Using Gauss forward interpolation formulas, find the value of log 337.5 from the following table :*

x	:	310	320	330	340	350	360
$y_x = \log x$	:	2.4914	2.5051	2.5185	2.5315	2.5441	2.5563

Sol. Let $x_0 = 330$, then $p = \dfrac{x-330}{10}$ since $h = 10$. Now the forward difference table is :

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
310	−2	2.4914					
			0.0138				
320	−1	2.5052		−0.0005			
			0.0133		0.0002		
330	0	**2.5185**		**−0.0003**		**−0.0003**	
			0.0130		**−0.0001**		**0.0004**
340	1	2.5315		−0.0004		0.0001	
			0.0126		0.0000		
350	2	2.5441		−0.0004			
			0.0122				
360	3	2.5563					

Gauss's forward formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2} + \ldots\ldots$$

$\therefore p = \frac{x-x_0}{h}$, $x = x_0 + uh$

when $x = 337.5$, $\quad p = \dfrac{337.5-330}{10} = 0.75$, $y_0 = 2.5185$

$\Delta y_0 = 0.0130$, $\Delta^2 y_{-1} = -0.0003$, $\Delta^3 y_{-1} = -0.0001$,

$\Delta^4 y_{-2} = -0.0003$, $\Delta^5 y_{-2} = 0.0004$

$$\therefore \quad y_{0.75} = 2.5185 + (0.75)(0.0130) + \frac{(0.75)(0.75-1)}{2!}(-0.0003)$$

$$+ \frac{(0.75+1)(0.75)(0.75-1)}{3!}(-0.0001)$$

$$+ \frac{(0.75+1)(0.75)(0.75-1)(0.75-2)}{4!}(-0.0003)$$

$$+ \frac{(0.75+2)(0.75+1)(0.75)(0.75-1)(0.75-2)}{5!}(0.0004)$$

$$= 2.5185 + 9.75 \times 10^{-3} + 2.8125 \times 10^{-5} + 5.46875 \times 10^{-6} - 5.1269531 \times 10^{-6} + 3.7597656 \times 10^{-6}$$

$$= 2.5282822$$

$\therefore \quad \log 337.5 = 2.5283.$

Example. *Interpolate by means of Gauss's backward formula the sales of a concern for the year 1976 given that*

Year	:	*1940*	*1950*	*1960*	*1970*	*1980*	*1990*
Sales (in lakhs of Rs.)	:	*17*	*20*	*27*	*32*	*36*	*38*

Sol. Taking 1970 as the origin and $h = 10$ years as one unit the sales of the place is to be found for $p = \dfrac{x - 1970}{10}$. Now the forward difference table is :

x	p	y	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1940	−3	17					
			3				
1950	−2	20		4			
			7		−6		
1960	−1	27		−2		7	
			5		1		−9
1970	0	32		−1		−2	
			4		−1		
1980	1	36		−2			
			2				
1990	2	38					

Gauss's backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-3} + \ldots$$

$x = x_0 + u$

when $x = 1976,\ p = \dfrac{1976 - 1970}{10} = 0.6,\ y_0 = 32,\ \Delta y_{-1} = 5,$

$\Delta^2 y_{-1} = -1,\ \Delta^3 y_{-2} = 1,\ \Delta^4 y_{-2} = -2,\ \Delta^5 y_{-3} = -9$

$$y_{0.6} = 32 + (0.6)(5) + \frac{(0.6+1)(0.6)}{2!}(-1) + \frac{(0.6+1)(0.6)(0.6-1)}{3!}(1) + \frac{(0.6+2)(0.6+1)(0.6)(0.6-1)}{4!}(-2) + \frac{(0.6+2)(0.6+1)(0.6)(0.6-1)(0.6-2)}{5!}(-9)$$

$$= 32 + 3 - 4.8 - 0.064 + 0.0832 - 0.104832 = 30.114368$$

∴ The sales in the year 1976 is Rs 30.114368 lakhs.

22.9. (c) STIRLING'S FORMULA IS

$$\mathbf{y_x = y_0 + x \cdot \frac{1}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{x^2}{2!}\Delta^2 y_{-1} + \frac{x(x^2-1)}{3!} \cdot \frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{x^2(x^2-1)}{4!}\Delta^4 y_{-2} + \ldots}$$

$$f(x) = f(x_0) + (x - x_0) \underset{x_1}{\Delta} f(x_0) + (x - x_0)(x - x_1) \underset{x_1, x_2}{\Delta} f(x_0)$$

$$+ \ldots\ldots + (x - x_0)(x - x_1) \ldots\ldots (x - x_{n-1}) \underset{x_1, \ldots\ldots, x_n}{\Delta^n} f(x_0)$$

Example. *Using Newton's divided difference formula, evaluate f(8) and f(15), given*

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

Sol. The divided difference table is as follows :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
4	48			
		$\underset{5}{\Delta} f(4) = \dfrac{100-48}{5-4} = 52$		
5	100		$\underset{5,7}{\Delta^2} f(4) = \dfrac{97-52}{7-4} = 15$	
		$\underset{7}{\Delta} f(5) = \dfrac{294-100}{7-5} = 97$		$\underset{5,7,10}{\Delta^3} f(4) = \dfrac{21-15}{10-4} = 1$
7	294		$\underset{7,10}{\Delta^2} f(5) = \dfrac{202-97}{10-5} = 21$	
		$\underset{10}{\Delta} f(7) = \dfrac{900-294}{10-7} = 202$		$\underset{7,10,11}{\Delta^3} f(5) = \dfrac{27-21}{11-5} = 1$
10	900		$\underset{10,11}{\Delta^2} f(7) = \dfrac{310-202}{11-7} = 27$	
		$\underset{11}{\Delta} f(10) = \dfrac{1210-900}{11-10} = 310$		$\underset{10,11,13}{\Delta^3} f(7) = \dfrac{33-27}{13-7} = 1$
11	1210		$\underset{11,13}{\Delta^2} f(10) = \dfrac{409-310}{13-10} = 33$	
		$\underset{13}{\Delta} f(11) = \dfrac{2028-1210}{13-11} = 409$		
13	2028			

Newton's divided difference formula is

$$f(x) = f(4) + (x-4) \underset{5}{\Delta} f(4) + (x-4)(x-5) \underset{5,7}{\Delta^2} f(4)$$
$$+ (x-4)(x-5)(x-7) \underset{5,7,10}{\Delta^3} f(4) + \ldots\ldots$$

Putting $x = 8$, we get $f(8) = 48 + (8-4) \times 52 + (8-4)(8-5) \times 15 + (8-4)(8-5)(8-7) \times 1$
$$= 48 + 208 + 180 + 12 = 448$$

Putting $x = 15$, we get

$$f(15) = 48 + (15-4) \times 52 + (15-4)(15-5) \times 15 + (15-4)(15-5)(15-7) \times 1$$
$$= 48 + 572 + 1650 + 880 = 3150.$$

TEST YOUR KNOWLEDGE

1. The values of annuities for certain ages are given for the following ages. Find the annuity at age $27\frac{1}{2}$ using Gauss's forward interpolation formula

Age :	25	26	27	28	29
Annuity :	16.195	15.919	15.630	15.326	15.006