Permutations and Combinations

Fundamental Principle of counting:-

Rule of Sum : *If one experiment has m possible outcomes and another experiment has n possible outcomes, then there are $m + n$ possible outcomes when exactly one of these experiments takes place.*

In set-theoretic notations, sum rule states that if A and B are two finite sets such that $A \cap B = \phi$ then

$$|A \cup B| = |A| + |B|$$

where $|\ |$ denotes the number of elements in X.

Rule of Product : *If one experiment has m possible outcomes and another experiment has n possible outcomes, then there are $m \times n$ possible outcomes when both of these experiments takes place.*

In set-theoretic notations, product rule says that if A and B are finite sets, then

$$|A \times B| = |A| \times |B|$$

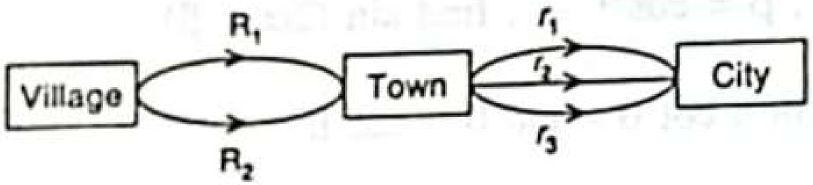
Note : *The above rule of product can be extended to more than two events.*

Q:- if we take first three letters of the English alphabets, *i.e.*, A, B and C and want to form different word consisting of two letters.

Sol:- A, B and C and want to form different words consisting of two letters.

These words can be formed as AB, AC, BA, BC, CA and CB.

If the repetition is allowed then the words are AB, AC, BA, BC, CA, CB, AA, BB and CC.

Q: 

Sol: The possible routes are R_1r_1, R_1r_2, R_1r_3, R_2r_1, R_2r_2 and R_2r_3.

Q:-

In a class there are 15 boys and 12 girls. Their mathematics teacher wants to select one boy and one girl to represent the class in a function. In how many different ways can he make the selection ?

Solution. One boy out of 15 boys can be selected in 15 ways.

One girl out of 12 girls can be selected in 12 ways.

$\therefore$ By multiplication principle,

Total number of ways $= 15 \times 12 = 180$

Q:-

There are 3 questions in a question paper. If the questions have 4, 3 and 2 solutions, find the total number of possible solutions.

Solution. Here question 1 has 4 solutions,

question 2 has 3 solutions

and question 3 has 2 solutions

$\therefore$ By multiplication principle,

Total number of solutions $= 4 \times 3 \times 2 = 24$

Q: *How many 2-digit even numbers can be formed by using 2, 4, 3, 5, 7 when :*

(i) the repetition of digits is allowed

(ii) the repetition of digits is not allowed ?

Solution. Since the number is to be even, so the unit's place can be occupied by either 2 or by 4 *i.e.*, in 2 ways.

(*i*) Since the repetition of digits is allowed so the ten's digit can be occupied by any one of the given digits 2, 4, 3, 5 or 7 *i.e.*, in 5 ways.

$\therefore$ The required number of 2-digit numbers $= 2 \times 5 = 10$

(*ii*) Since the repetition of digits is not allowed, so the ten's digit can be occupied by any one of the remaining 4 digits in 4 ways.

$\therefore$ The required number of 2 digits numbers $= 2 \times 4 = 8$

Permutations

An arrangement of a finite set of n objects in a given particular order is called a permutation of the objects (taken all at a time).

Any arrangement of any $r < n$ of these objects in a given order is called a r-permutations or a permutation of the n objects taken r at a time.

For example, If we have pens of three colours say Red (R), Blue (B) and Green (G), then out of these three (n) we can arrange two (r) pens as follows :

Permutations of 2 objects (pens)

(*i*) RB
(*ii*) RG
(*iii*) BR
(*iv*) BG
(*v*) GR
(*vi*) GB

Number of permutations

If there are n distinct objects and r is an integer, with $1 \le r \le n$, then by the principle of product, the number of permutation of size r taken out from n objects is

$$n \times (n-1) \times ..... \times (n-r+1) = \frac{(n)!}{(n-r)!}$$

It is denoted by $P(n, r)$ or ${}^{n}P_{r}$

So $\quad P(n, r) = \frac{(n)!}{(n-r)!}$ in which repeatior of numbers are not allowed

For $\quad r = 0, \; P(n, 0) = \frac{(n)!}{(n)!} = 1$

For $\quad r = n, \; P(n, n) = \frac{(n)!}{(0)!} = (n)!$

So the number of different arrangements of n given things taken all at a time is $(n)!$ (factorial n).

Number of Permutation when repeations are allowed.

If there are n distinct objects, then the number of permutations of r objects taken out from then, when repeations are allowed, is given by the principle of product.

$$\underbrace{n \times n \times n \ldots\ldots \times n}_{r \text{ times}} = n^r . \ n \geq 0$$

So there are n^r possible such arrangements.

Ex. .. A number of four digits is formed taking digits from the set [1,2,3,......,9]. Then how many such numbers can be formed, if repeation is allowed.

The answer is $9\times9\times9\times9 = 9^4$

Ex. The number of permutation of the letters in the word COMPANY is (7)!. If only four letters are used, the number of permutations of size four is

$$P(7,4) = \frac{(7)!}{(7-4)!} = 7\times6\times5\times4 = 840,$$

If repeation of letters, be allowed, the number of possible 4–letter sequence is $7\times7\times7\times7 = 7^4$.

Permutations with Repeatitions:

Theorem 2.3. *The number of permutations of n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike is :* $\dfrac{n!}{n_1!,\ n_2!, \ldots\ldots n_r!}$)

Ex. *There are 8 chairs in a room. In how many ways 5 students can sit on them ?*

Sol. $\because n = 8,$ and $r = 5$

$\therefore$ Number of permutation $P(n, r) = P(8, 5) = \dfrac{8!}{(8-5)!} = \dfrac{8!}{3!}$

or, number of ways of sitting $= \dfrac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$

$= 6{,}720$

Ex. How many distinct permutions of the letters in the following word are there

(i) BANANA

Sol. (i) There are 6 letters in the word BANANA, out of them 3 A, 2 N and 1 B.

So the distinct permutations will be

$$\frac{(6)!}{(3)!\,(2)!} = \frac{6\times5\times4}{2\times1} = 60$$

(ii) MISSISSIPPI.

(ii) Similar to the above part the number of distinct permutations are

$$\frac{(11)!}{(4)!\,(4)!\,(2)!} = 34650$$

Example . *Find r when* $5\,.\,{}^4P_r = 6\,.\,{}^5P_{r-1}$.

Solution. $5\,.\,{}^4\mathrm{P}_r = 6\,.\,{}^5\mathrm{P}_{r-1}$

$$\Rightarrow \quad 5\,.\,\frac{4!}{(4-r)!} = 6\,.\,\frac{5!}{\{5-(r-1)\}!}$$

$$\Rightarrow \quad \frac{5!}{(4-r)!} = \frac{6.5!}{(6-r)!}$$

$$\Rightarrow \quad \frac{1}{(4-r)!} = \frac{6}{(6-r)(5-r)(4-r)!}$$

$$\Rightarrow \quad (6-r)(5-r) = 6$$

$$\Rightarrow \quad r^2 - 11r + 30 = 6$$

$$\Rightarrow \quad r^2 - 11r + 24 = 0$$

$$\Rightarrow \quad (r-8)(r-3) = 0$$

$$\Rightarrow \quad r = 8 \text{ or } 3$$

But $\quad r \leq n$ and $n \leq 5$

$\therefore \quad r = 3$

Example *Find the value of n if 16 . $^{n}P_3 = 13 . ^{n+1}P_3$.*

Solution. We are given that

$$16.\ ^{n}P_3 = 13.\ ^{n+1}P_3$$

i.e., $$16 \cdot \frac{n!}{(n-3)!} = 13.\ \frac{(n+1)!}{(n+1-3)!}$$

or $$\frac{16.n!}{(n-3)!} = \frac{13.(n+1)\,n!}{(n-2)(n-3)!}$$

or $$16(n-2) = 13(n+1)$$

or $$16n - 32 = 13n + 13$$

or $$16n - 13n = 13 + 32$$

or $$3n = 45$$

$\therefore$ $$n = 15$$

Example . *If* $^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, *find r.*

Solution. $\dfrac{^{56}P_{r+6}}{^{54}P_{r+3}} = \dfrac{30800}{1}$

or $$\dfrac{\dfrac{56!}{(56-r-6)!}}{\dfrac{54!}{(54-r-3)!}} = \dfrac{30800}{1}$$

or $$\frac{56.55.54!}{(50-r)!} \times \frac{(51-r)!}{54!} = \frac{30800}{1}$$

or $$\frac{56.55.(51-r)(50-r)!}{(50-r)!} = \frac{30800}{1}$$

or $$56 \times 55 \times (51-r) = 30800$$

or $$51 - r = 10$$

or $$r = 41$$

Example *Show that* $^{n}P_r = {}^{n-1}P_r + r\,.\,{}^{n-1}P_{r-1}$

Solution. R.H.S. $= {}^{n-1}P_r + r\,.\,{}^{n-1}P_{r-1}$

$$= \frac{(n-1)!}{(n-1-r)!} + r\,.\,\frac{(n-1)!}{[n-1-(r-1)]!}$$

$$= \frac{(n-1)!}{(n-1-r)!} + r\,.\,\frac{(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + r\,.\,\frac{(n-1)!}{(n-r)(n-r-1)!}$$

Example . *How many 5 digit telephone numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7 if no digit is repeated.*

Solution. We have 7 digits and 5 are to be selected.

$\therefore$ Number of telephone numbers $= {}^{7}P_5$

$$= \frac{7!}{2!}$$

$$= 7 \times 6 \times 5 \times 4 \times 3$$

$$= 2520$$

Example . *It is required to seat 4 men and 5 women in a row so that the men occupy the even places. How many such arrangements are possible?*

Solution. In the row there are 9 places and 2nd, 4th, 6th and 8th places are even places. 4 men can be arranged on 4 places in ${}^4P_4$ ways.

Now 5 women can be arranged in the remaining 5 places in ${}^5P_5$ ways.

$\therefore$ By fundamental principle of counting, the required number of seating arrangements is

$$= {}^4P_4 \times {}^5P_5 = 4! \times 5!$$
$$= 24 \times 120$$
$$= 2880$$

Restricted Permutations

(i) The number of permutations of n different things taken r at a time in which p particular things do not occur is

$$^{n-p}P_r$$

(ii) The number of permutations of n different things taken r at a time in which p particular things are present is

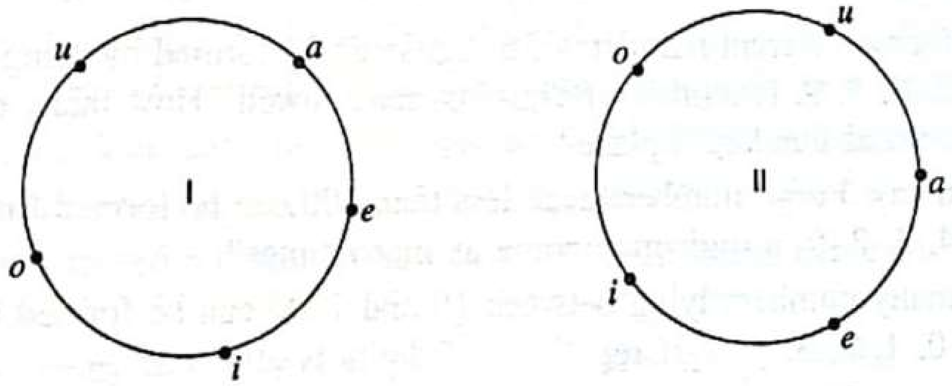
$$^{n-p}P_{r-p} \times {}^{r}P_p$$

Example . *How many 7 digit telephone numbers can be formed by using the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 if each number is to start with '572' and no digit is repeated.*

Solution. As the numbers are to begin with '572', we have to choose 4 digits from the remaining 6 digits.

$$\begin{aligned} \therefore \text{ Number of telephone numbers } &= {}^6P_4 \\ &= 6 \times 5 \times 4 \times 3 \\ &= 360 \end{aligned}$$

Circular Permutations



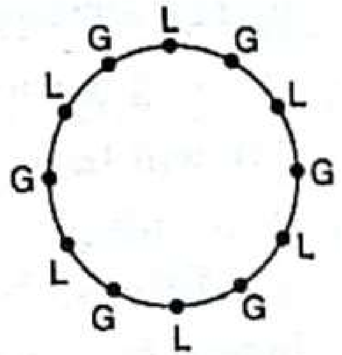
consider the relative position of the different objects.

As the circular permutations depend on the relative positions of the objects, we fix the position of one object and then arrange the remaining $(n - 1)$ objects in the possible ways. This we can do in $(n - 1)$!

Hence the number of circular permutations of n objects $= (n - 1)$!

Example . *In how many ways 6 gents and 5 ladies dine at a round table, if no two ladies are to sit together ?*

Solution.



As the ladies are not to sit together we shall first allot the seats to gents. Now 6 gents can have $(6-1)!$ circular permutations.

$\therefore$ Number of permutations in which gents can take their seats $= 5! = 120$

Now the 5 ladies can occupy seats marked (L). There are 6 such seats. This can be done in $^6P_5 = 720$ ways

$\therefore$ Required number of ways

$$= 120 \times 720 = 86400$$

In the arrangements of beads in a necklace, flowers in a garland the clockwise and anti–clockwise arangements are not distinct, then the number of circular permutation of n distinct items is $\frac{1}{2}((n-1)!)$.

Example *In how many ways can 5 beads of different colours form a necklace?*

Solution. 5 beads of different colours can have $(5-1)!$ circular permutations.

Thus there are $4!$, *i.e.*, 24 ways of forming circular permutations with 5 beads. But in necklace clockwise arrangements can be obtained from anti clockwise arrangements by turning round the necklace.

...[$\because$ $a\ b\ c\ d\ e$ and $a\ e\ d\ c\ b$ are not different if rotated in a circle]

$\therefore$ Number of ways of forming the necklace $= \frac{24}{2} = 12$

Combination

A group of some or all objects out of finite number of objects irrespective of their positions is called a combination.

For example, if we have pens of three colours say Red (R), Blue (B) and Green (G), then out of these three colours, we can arrange three (all) or two pens as follows :

Combinations of 2 colours (pens)

(*i*) R , B
(*ii*) R , G
(*iii*) B , G

Combinations of 3 colours (pens)

(*i*) R, B, G

Number of combinations of r objects out of n different objects :

$$^{n}C_{r} = C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,r) = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

Relation between Permutation and Combination :

$$\because \quad P(n,r) = n(n-1)(n-2)\ldots(n-r+1) \qquad (i)$$

and

$$C(n,r) = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \qquad (ii)$$

By (*i*) and (*ii*) ,

(1) $C(n,r) = \dfrac{P(n,r)}{r!}$

(2) *The number of combinations of r objects taking p objects at a time out of n different objects* $= C(n-p, r-p)$

(3) *The number of combinations of r objects when p objects are never selected out of n different objects* $= C(n-p, r)$

Complementary Combinations :

The number of combinations of r objects out of n different objects is equal to the number of combinations of remaining (n − r) objects out of n different objects.

Therefore these are called complementary combinations.

Symbolically:

(1) $C(n,r) = C(n, n-r)$

(3) ***The number of ways to place r balls of the same colour in n numbered boxes allowing as many balls in a box as we wish is :***

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-r)!}$$

. Total number of all possible combinations of different objects:

The total number of combinations of all from 1 to n out of n different objects or the total number of combinations of 1, 2, 3, 4 .... and n at a time out of n different objects

$$= C(n,1) + C(n,2) + C(n,3) + \ldots\ldots + C(n,n) = 2^n - 1 .$$

Example *Evaluate:* $^{12}C_8 + {}^{12}C_7$

Solution.

$$^{12}C_8 + {}^{12}C_7 = {}^{13}C_8 \quad \ldots[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$$

$$= {}^{13}C_5 \quad \ldots[{}^nC_r = {}^nC_{n-r}]$$

$$= \frac{13 \times 12 \times 11 \times 10 \times 9}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= 1287$$

Example *If* $^{12}C_{2n} = {}^{12}C_{2n-4}$, *find* $^{6}C_n$

Solution. We know that $\quad {}^nC_r = {}^nC_{n-r}$

$$\therefore \quad {}^{12}C_{2n} = {}^{12}C_{12-2n} \qquad \ldots(i)$$

We are also given that $\quad {}^{12}C_{2n} = {}^{12}C_{2n-4} \qquad \ldots(ii)$

$\therefore$ From (i) and (ii), we get

$$^{12}C_{2n-4} = {}^{12}C_{12-2n}$$

$$\text{or} \quad 2n - 4 = 12 - 2n$$

$$\text{or} \quad 4n = 16$$

$$\therefore \quad n = 4$$

$$\text{Now} \quad {}^6C_n = {}^6C_4 = {}^6C_2 = \frac{6 \times 5}{2 \times 1} = 15$$

Ex. *. Out of 10 flowers, in how many ways 6 flowers can be selected for the worship of the God?*

Sol. Total objects $(n) = 10$ flowers; selected objects $(r) = 4$ flowers

$$\therefore \text{ Total combinations } = C(10,4) = \frac{10!}{(10-4)!4!}$$

$$= \frac{10!}{6!4!}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1}$$

$= 210$ ways.

Ex. ***Out of 7 men and 4 women, how many committees of 6 members be formed having atleast two women?***

Sol. The committees can be constituted in following three ways having atleast three women :

(*i*) 2 women and 4 men (*ii*) 3 women and 3 men (*iii*) 4 women and 2 men

$\therefore$ The number of committees in the above three cases are

$$= [C(4,2)\times C(7,4)] + [C(4,3)\times C(7,3)] + [C(4,4)\times C(7,2)]$$

$$= \left(\frac{4\times 3}{2\times 1}\times\frac{7\times 6\times 5\times 4}{4\times 3\times 2\times 1}\right) + \left(\frac{4}{1}\times\frac{7\times 6\times 5}{3\times 2\times 1}\right) + \left(\frac{1}{1}\times\frac{7\times 6}{2\times 1}\right)$$

$$= 210 + 140 + 21$$

$$= 371$$

Ex. Out of 15 hockey players, in how many ways a team be formed, if :

(a) a team has 11 players.

(b) all the teams have the same captain

(c) two players are not included in any of the team.

Sol. Total players $n = 15$ and players in the team $r = 11$

$\therefore$ (a) Number of teams $= C(15,11) = \dfrac{15}{(15-11)!(11)!} = \dfrac{15!}{(4)!(11)!}$

$$= \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = 1365 \text{ ways}$$

(b) Since all the teams are having the same captain, so $n = 15 - 1 = 14$

The number of players to be selected $11 - 1 = 10$

$\therefore$Number of teams $= C(14,10) = \dfrac{14!}{(14-10)!(10)!} = \dfrac{14!}{(4)!(10)!}$

$$= \frac{14 \times 13 \times 12 \times 11}{4 \times 3 \times 2 \times 1} = 1{,}001 \text{ ways}$$

(c) Since two players are not to be included in any of the team,

so 11 players are to be selected out of the remaining $15-2=13$ players.

$\therefore$ Number of teams $= C(13,11)$

$$= \frac{(13)!}{(13-11)!(11)!}$$

$$= \frac{(13)!}{(2)!(11)!}$$

$$= \frac{13\times 12}{2\times 1} = 78$$

Ex. . *Find the number of different out comes when 3 dice are rolled.*

Sol. Here throwing 3 dice is equivalent to the selections of 3 digits out of 6 digits 1, 2, 3, 4, 5 and 6 where repeation is allowed.

Therefore the required number of different results

$$= C(6+3-1, 3) = C(8, 3) = 56$$

Ex. . ***A person has 6 friends. In how many ways he can invite one or more for the lunch?***

Sol. Out of 6 friends, 1,2,3,4,5 and 6 may be invited for the lunch . Therefore the number of combinations of 1 or more friends

$$= C(6,1) + C(6,2) + C(6,3) + C(6,4) + C(6,5) + C(6,6)$$

$$= 6 + 15 + 20 + 15 + 6 + 1 = 63$$

Using formula $(2^n - 1)$, no of combinations $= (2^6 - 1)$

$$= 64 - 1 = 63 .$$