Midterm 1
CS-1110/ MAT-2203-1: Discrete Math
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September 24, 2025 **Max Marks: 100** **Time: 1:20 hours**

Name: Gaurav **Major:** Math

Instructions

1. This test has 5 problems worth 20 marks each. Each part has equal weightage.
2. Write your answers in the space provided.
3. Carefully state any theorem that you use from the lectures when you use them.
4. Justify each step.
5. Read the entire paper once before starting, and do the problem you find easiest first.

∼ Use as a rough page ∼

Problem 1 ((20 = 5 × 4 marks). This problem is on the number 97. If you use a theorem, state it.

(a.) Find the remainder when 96! is divided by 97.

(b.) Find $\varphi(97)$.

(c.) Find the remainder when 79^{97} is divided by 97.

(d.) Find an $x \in \mathbb{Z}_{97}$ such that $2x \equiv 1 \pmod{97}$.

(e.) Find $\varphi(97^2)$. (Hint. It helps to think more generally.)

Note that 97 is a prime. To verify you need to divide by primes 2, 3, 5, 7.

(a) $96! \equiv -1 \pmod{97}$ By Wilson's thm $(p-1)! \equiv -1 \pmod{p}$ iff p is a prime.

(b) $\phi(97) = 96$ (1, 2, ..., 96 are all relatively prime to 97)

(c) $79^{97} \equiv 79 \pmod{97}$ (By Fermet's little thm, $a^{p-1} \equiv 1 \pmod{p}$ for p a prime, $(a,p)=1$)

$\| \\ (79)^{96} \cdot 79$

(d) Note: $2 \cdot \frac{97+1}{2} \equiv 1 \pmod{97}$

$x = \frac{97+1}{2} = 49$

(e) $\phi(p^2)$ = # of numbers relatively prime to p^2

Total $= p^2$. div by p = $p, 2p, \ldots, (p-1)p$
= p of them

So $\phi(p^2) = p^2 - p$

$\phi(97^2) = 97^2 - 97 = 97(96)$.

Problem 2 $(20 = 5 \times 4)$**.** Prove the following

(a.) $\displaystyle\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$.

(b.) $\displaystyle\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$.

(c.) The converse of: If $B \subset A$, then $A \cap B = B$.

(d.) The contrapositive of: If $B \subset A$, then $A \cap B = B$.

(e.) In a large meeting, the number of delegates who shake hands an odd number of times is even.

(a) Let $S = \sum_{k=0}^{n} q^k$. $S(1-q) = (1+q+\dots+q^n)(1-q)$

$= 1 - \cancel{q} + \cancel{q} - \cancel{q^2} + \dots + q^n - q^{n+1}$

$= 1 - q^{n+1}$ (by telescoping)

$\Rightarrow S = \dfrac{1-q^{n+1}}{1-q}$. □

(b) We prove as f.p.s.

$\left(\sum_{k=0}^{\infty} q^k\right)(1-q)$: Coeff of $q^0 = 1$

$(1+q+q^2+\dots)(1-q)$

Coeff of $q^m = 1-1=0$ for $m>0$

$\Rightarrow \left(\sum_{k=0}^{\infty} q^k\right)(1-q) = 1$ (as f.p.s.)

$\Rightarrow \sum_{k=0}^{\infty} q^k = \dfrac{1}{1-q}$. □

(c) Let $A \cap B = B$. We n.t.s. if $x \in B$ then $x \in A$.
Let $x \in B = A \cap B \Rightarrow x \in A$ and $x \in B$. In particular $x \in A$. □.

(d) Contrapositive has same truth value as statement. The contrapositive of $P \Rightarrow Q$ is not $Q \Rightarrow$ not P. (But we can show $P \Rightarrow Q$ too.) Assume $B \subset A$.
Sp $A \cap B \neq B$. n.t.s $B \not\subset A$.
Let $x \in B \Rightarrow x \in A \Rightarrow x \in A \cap B$. So $B \subset A \cap B$ (*)
Of course if $x \in A \cap B$ then $x \in A$ and $x \in B$ so in particular $x \in B$
So $A \cap B \subset B$. (**)
thus (*) + (**) $\Rightarrow A \cap B = B$ $\rightarrow\leftarrow$. □.

(e) Let $v_1, \dots, v_n$ represent delegates. If v_i, v_j shake hands, put edge between v_i & v_j. degree$(v_i) = d_i$ = # of people whose hands v_i shakes.

Count pairs (v, e):

v-fixed $\sum d_i$ = # of total handshakes

e fixed : $2 \sum_{e \in \text{Edge set}} 1 = 2\#E$.

Since they are equal $\sum d_i = 2\#E$.

Look at this mod 2.

$\sum_{d_i \text{ odd}} 1 = 0 \pmod 2$ ↳ # has to be even. If odd $\sum 1 = 1 \pmod 2$

Problem 3 (20 = 4 × 5 marks). This problem is about Pascal's triangle which contains the values of $\binom{n}{k}$.

(a.) Make a table with the first seven rows of the Pascal's triangle,for $n = 0, 1, 2, 3, 4, 5, 6$ and $k = 0, 1, 2, \ldots$.

(b.) What does $\binom{n}{k} = \binom{n}{n-k}$ mean in the table? Explain with examples and prove this symmetry of the binomial coefficients.

(c.) Add each row. What do you get? Find a pattern and express it as a sum. Prove this pattern.

(d.) Alternatively add and subtract elements in a row. For example: $1 = 1$, $1 - 1 = 0$, $1 - 2 + 1 = ?$. Find a pattern and express as a sum, and prove it.

(a)

n \ k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	1	2	1	0	0	0	0
3	1	3	3	1	0	0	0
4	1	4	6	4	1	0	0
5	1	5	10	10	5	1	0
6	1	6	15	20	15	6	1

(b) $\binom{n}{k} = \binom{n}{n-k}$ means they are symmetric across middle.

$\binom{n}{k} \leftarrow$ kth from left $=$ kth from right $= \binom{n}{n-k}$

Proof: $\binom{n}{k} = \#$ of binary strings of length n with exactly k 1's.

$\binom{n}{n-k} = \# \ldots$ k 0's

interchange 1's $\leftrightarrow$ 0's to find a 1-to-1 correspondence. So they are equal.

(c)

n	sum	pattern
0	1	2^0
1	2	2^1
2	4	2^2
3	8	2^3
4	16	2^4
5	32	2^5
6	64	2^6

Sum

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ put $x = 1$ to get $2^n = \sum_{k=0}^{n} \binom{n}{k}$. $\square$

(d) We get $1, 0, 0, 0, \ldots$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

Proof: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. $n = 0$: get 1 on both sides

For $n > 0$: $x = -1$ to get $0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$. $\square$

Problem 4 (20 = 5 × 4 marks)**.** This problem is about the falling factorial $P(x,n)$, defined for $n = 0, 1, \ldots,$ as: $P(x,0) := 1$, and

$$P(x,n) := x(x-1)\cdots(x-n+1).$$

(a.) Argue that $P(x,n)$ is a polynomial. What is the degree? Let $s(n,k)$ be the coefficient of x^k in this polynomial.

(b.) Show that:
$$s(0,0) = 1; s(n,0) = 0 \text{ for } n > 0; \text{ and, } s(n,k) = 0 \text{ for } k > n.$$

(c.) Show that, for $k > 0$:
$$s(n+1,k) = s(n,k-1) - ns(n,k),$$

(d.) Make a table of $s(n,k)$ for $n, k \leq 4$.

(a). $P(x,0) = 1$ (empty product)

$P(x,1) = x$, $P(x,2) = x(x-1)$, $P(x,3) = x(x-1)(x-2)$

$P(x,n)$ is a product of n polynomials $x, x-1, \ldots, x-(n-1)$ each of degree 1.

So is a polynomial of degree n.

$P(x,n) = \sum_{k=0}^{n} s(n,k)x^k$

(b) $s(0,0) = 1$ (since $P(x,0) = 1$)

$s(n,0) = 0$ $P(x,n) = x(x-1)\cdots$ always there, so constant term is 0.

$s(n,k) = 0$ for $k > n$, since degree is n

(c) $P(x,n+1) = (x-n)P(x,n)$

$\sum_{k=0}^{n+1} s(n+1,k)x^k = x\sum_{k=0}^{n} s(n,k)x^k - n\sum_{k=0}^{n} s(n,k)x^k$

Take coeff of x^k for $k = 1, 2, \ldots, n$ on both sides, to get

$s(n+1,k) = s(n,k-1) - n\,s(n,k)$. $\square$

(d)

n\k	0	1	2	3	4
0	1	0	0	0	0
1	0	1	0	0	0
2	0	−1	1	0	0
3	0	2	−3	1	0
4	0	−6	11	−6	1

$11 = 2 - (3)(-3)$ ← n

Problem 5 (20 = 2 × 10). This problem is about an island containing Knights and Knaves. Knights always tell the truth and Knaves always lie.

(a.) You meet three people, A, B and C, each either a Knight or a Knave. A and B make the following statements.

A: All of us are Knaves.
B: Exactly one of us is a Knight.

What are A, B and C?

(b.) This problem is about an island whose inhabitants are either Knights or Knaves. Knights always tell the truth and Knaves always lie. You meet three people, A, B and C. You ask A, "Are you a Knight or a Knave?" A answers something but you cannot hear what he said. Then you ask B, "What did A say?" B replies, "A said that he is a Knave." But now C chimes in, and says, "Don't believe B; he is a Knave!". What are B and C?

(a) A cannot be true (since A true ⇒ A is a knave)
So A is a knave. ⇒ statement of A is false. i.e there is at least one knight. If B is false, only C is only knight making B true. So B is true. This means C has to be a knave.
↳ knight.

Ans: A knave B knight C knave.

(b) B says A is a knave. But A cannot have said that. (If A knave → he would have lied and said he's a knight
If A knight → he would have told truth and said he's a knight.)

So B is a knave.
Since C's statement is correct, C is a knight.
Ans. B is knave
C is knight.
(Cannot say anything about A.).