Lecture 22 — Bell #s.

We consider a pending classic problem with a view to give another technique for generating functions.

Recall: Stirling #s of the second kind.

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$$

$$S(n,0) = 0 \quad \text{for } n>0$$

$$S(0,0) := 1, \quad S(n,k) = 0 \quad \text{for } k>n$$

$S(n,k) =$ # of set partitions of I_n into k parts.

For generating functions, there are 2 possibilities

$$F_k(x) = \sum_{n=0}^{\infty} S(n,k)\,x^n$$

$$G_n(x) = \sum_{k=0}^{\infty} S(n,k)\,x^k.$$

We try $F_k(x)$.

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$$

mult by x^n and sum over $n \geq 1$

$$\sum_{n=1}^{\infty} S(n,k)x^n = x\sum_{n=1}^{\infty} S(n-1,k-1)x^{n-1} + kx\sum_{k=1}^{\infty} S(n-1,k)$$

$$\Rightarrow F_k(x) - F_k(0) = x\,F_{k-1}(x) + kx\,F_k(x)$$

$F_k(x) = S(0,k) + S(1,k)x + \cdots$; $F_k(0) = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{if } k>0. \end{cases}$

$$\Rightarrow F_k(x)(1-kx) = x\,F_{k-1}(x) \quad (\text{for } k>0).$$

$$\Rightarrow F_k(x) = \frac{x}{1-kx}F_{k-1}(x)$$

$$= \frac{x^2}{(1-kx)(1-(k-1)x)}F_{k-2}(x)$$

$$= \cdots$$

$$= \frac{x}{1-x}\cdot\frac{x}{1-2x}\cdot\ \cdots\ \cdot\frac{x}{1-kx}F_0(x)$$

So $$F_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

Note: In this case, g.f satisfies a 2-term recurrence so can be solved easily.

The answer is a rational function, so we go to partial fractions.

Sp $$\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} = \frac{A_1}{1-x} + \frac{A_2}{1-2x} + \cdots + \frac{A_k}{1-kx}$$

mult by $1-rx$ and take $x = 1/r$

RHS: A_r

LHS: $$\frac{1}{r^k}\,\frac{1}{(1-1/r)}\cdot\frac{1}{(1-2/r)}\cdots\frac{1}{(1-\frac{r-1}{r})}\cdot\frac{1}{(1-\frac{r+1}{r})}\cdots\frac{1}{(1-\frac{k}{r})}$$

$$= \frac{1}{r(r-1)(r-2)\cdots(r-r+1)}\cdot\frac{1}{(-1)(-2)\cdots(r-k)}$$

$$\Rightarrow \frac{(-1)^{k-r}}{r!\,(k-r)!} = A_r$$

$$F_k(x) = \sum_{r=1}^{k}\frac{(-1)^{k-r}}{r!\,(k-r)!}\,\frac{1}{1-rx}$$

$$= \sum_{r=1}^{k}\frac{(-1)^{k-r}}{r!\,(k-r)!}\sum_{n=0}^{\infty}(rx)^n$$

$$= \sum_{n=0}^{\infty}x^n\left(\sum_{r=1}^{k}\frac{(-1)^{k-r}r^n}{r!\,(k-r)!}\right)$$

So $$S(n,k) = \sum_{r=1}^{k}\frac{(-1)^{k-r}r^n}{r!\,(k-r)!}.$$

Example: $S(n,2) =$ # of ways of partitioning an n-set into 2 parts

$$= \frac{(-1)^1 1^n}{1!\,1!} + \frac{2^n}{2!} = 2^{n-1}-1$$

check in Table.

An explicit formula, but not so nice.

- alternating signs: since a PIE argument.

Bell #'s.

$b(n) :=$ # of ways to partition I_n

$$= \sum_{k=0}^{n} S(n,k)$$

$b(n)$: 1, 1, 2, 5, 15, ...

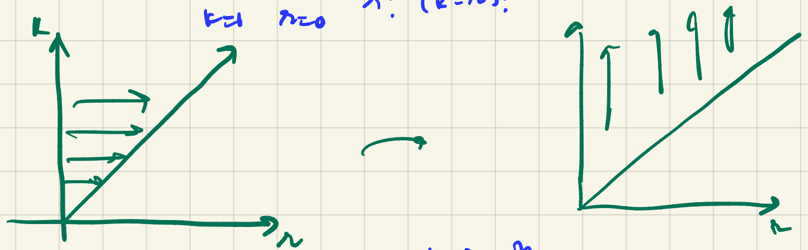
Prop. (a) $$b(n) = e^{-1}\sum_{r=0}^{\infty}\frac{r^n}{r!}$$

(b) let $$B(x) = \sum_{n=0}^{\infty} b(n)\frac{x^n}{n!} \quad (\text{exp g.f.})$$ then $B(x) = e^{e^x-1}$

(c) $$b(n+1) = \sum_{k=0}^{n}\binom{n}{k}b(k), \quad n\geq 0,\ b(0)=1$$

Proof: $$b_n = \sum_{k=0}^{\infty} S(n,k) \quad (\text{since } S(n,k)=0 \text{ for } k>n)$$

$$= \sum_{k=1}^{\infty}\sum_{r=0}^{k}\frac{(-1)^{k-r}r^n}{r!\,(k-r)!}$$



$$= \sum_{r=0}^{\infty}\sum_{k=r}^{\infty}\frac{(-1)^{k-r}\,r^n}{r!\,(k-r)!}$$

$$= \sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\,r^n}{r!\,k!} \quad (\text{shifting index } k \mapsto k+r)$$

$$= \sum_{r=0}^{\infty}\frac{r^n}{r!}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}$$

$$= e^{-1}\sum_{r=0}^{\infty}\frac{r^n}{r!} = e^{-1}\left(0 + \frac{1}{1!} + \frac{2^n}{2!} + \cdots\right).$$

(Needs interpretation $0^0 = 1$!)

Note: $$b(0) = e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right) = e^{-1}\cdot e = 1\ ✓.$$

Proof of (b).

$$B(x) = \sum_{n=0}^{\infty} e^{-1}\sum_{r=0}^{\infty}\frac{r^n}{r!}\,\frac{x^n}{n!}$$

$$= e^{-1}\cdot\sum_{r=0}^{\infty}\frac{1}{r!}\sum_{n=0}^{\infty}\frac{r^n x^n}{n!} =$$

$$= e^{-1}\sum_{r=0}^{\infty}\frac{e^{rx}}{r!} = e^{-1}e^{e^x} = e^{e^x-1}.$$

part (c) [New trick: GF $\to$ recurrence].

$$\log B(x) = e^x - 1$$

$$\frac{d}{dx}\log B(x) = (e^x)$$

$$\Rightarrow \frac{B'(x)}{B(x)} = e^x$$

$$\Rightarrow \sum_{n=1}^{\infty} n\,b(n)\frac{x^{n-1}}{n!} = (e^x)\left(\sum_{n=0}^{\infty} b_n\frac{x^n}{n!}\right)$$

$$\Rightarrow \sum_{n=0}^{\infty} b(n+1)\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{n=0}^{\infty} b_n\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}b_k\frac{x^n}{n!} \quad (\text{Ex}).$$

$$\Rightarrow b(n+1) = \sum_{k=0}^{n}\binom{n}{k}b_k \cdot \left(\text{coeff of } \frac{x^n}{n!}\right)$$

Ex: Use this to generate 5 values by hand and 50 by Sage. Verify from OEIS and prebuilt Sage function.

Remarks ① GF $\to$ recurrence — log derivative technique. V. imp.