Lecture 16: Hall's Marriage Problem

G: Simple, bipartite graph. $(V = X \cup Y, E)$
↳ disjoint.

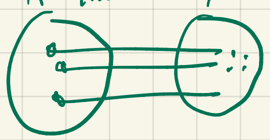
Notation: $a \in V$, $\Gamma(a) :=$ set of vertices adjacent to a

Let $A \subset V$: $\Gamma(A) := \bigcup_{a \in A} \Gamma(a)$ = set of vertices adjacent to vertices in A

Def (Matching) A matching is a subset $M \subseteq E$ s.t. no vertex is incident with more than one edge in M.

— A complete matching from X to Y is a matching s.t. every vertex in X is incident with an edge in M.

X: Girls Y = Boys



Boys may be leftover.

Thm (Hall's thm). Let G be a simple, bipartite graph. Then there is a complete matching from X to Y if and only if for every $A \subseteq X$;

$$\# \Gamma(A) \geq \# A.$$

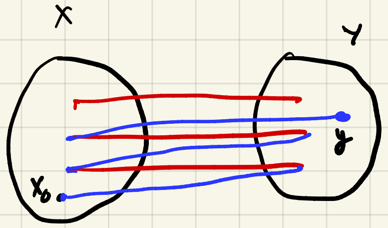
Proof: ($\Rightarrow$) if there is a complete matching, then clearly $\# \Gamma(A) \geq \# A$ for any $A \subseteq X$.

($\Leftarrow$) Let $\# X = n$, $m < n$ and suppose there is a matching M with m edges. We show there is a larger matching with $m+1$ edges. Let edges of M be red and all others be blue.

Let $x_0 \in X$ be a vertex in X which is not incident with an edge of M.

Claim: There is a simple path, of odd length

- starting from x_0
- uses blue and red edges alternately
- terminates with a blue edge and a vertex $y \in Y$



If we find such a path, we are done, because we can interchange red and blue in this path and get a larger matching.

Proof of claim: Since $\# \Gamma(\{x_0\}) \geq 1$, there is a $y_1 \in Y$ adjacent to x_0. y_1 is either on red edge or not. If not, we are done, since $x_0 - y_1$ is a blue edge and can be added to M to get a larger matching.

Otherwise, y_1 is on red edge, and other side is x_1.

Consider $|\Gamma(\{x_0, x_1\})| \geq 2$
↳ $\# \Gamma(\ \ )$

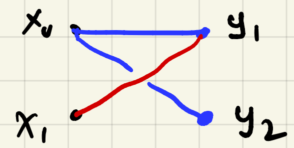
There is a y_2 diff from y_1 s.t. it is adjacent to at least one vertex in $\{x_0, x_1\}$. If y_2 is not in a red edge, stop. Else there is an x_2 st. $x_2 - y_2$ is red edge. Continue by induction. If $\{x_0, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$ have been defined

$$|\Gamma(\{x_0, \dots x_k\})| \geq k+1 \text{ there is a } y_{k+1}.$$

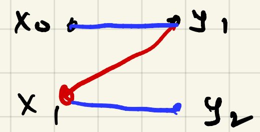
Not incident with red → stop.

So procedure terminates with y_{k+1}. It is incident with a blue edge. Let $x_i \in \{x_0, \dots, x_k\}$ be other end. Add a blue edge y_i and carry on until you reach x_0. This generates the required path.

- ends in Y so, odd length
- alt blue/red
- 1 extra blue ☐

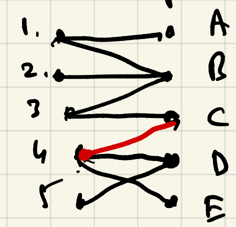


OR



← When returning back we retrace steps.

Ex. Follow steps in proof.



begin with M=4-C
$x_0 = 1$.

Cor. A finite regular bipartite graph has a perfect matching.
↳ all vertices have same degree. ↳ every vertex is in matching.

Pf: — if regular, $|X| = |Y|$

$$d|X| = \text{edges incident with } X$$
$$= \text{edges incident with } Y$$
$$= d|Y|.$$
$$\Rightarrow |X| = |Y|$$

Thus any matching is a perfect matching.

— $d|A| \geq |A|$ for any $A \subset X$. Hall's condn is satisfied. ☐.

" Hall's marriage problem " X = set of girls, Y = boys | edges g-b if g likes b.

$|X| = |Y| \rightarrow$ perfect matching if Hall's condn satisfied.

Remark. This is a fundamental thm with many formulations

1. (S.D.R.) Let $A_1, \dots, A_n$ be subsets of a finite set S.

Then $(a_i)_{i=1}^n$ form a system of distinct representatives (SDR) if $a_i \in A_i$ and a_i's are distinct.

Hall's condn: for any subset $J \subseteq I_n$,

$$\left| \bigcup_{i \in J} A_i \right| \geq |J|$$

$\Longleftrightarrow A_1, \dots, A_n$ have an S.D.R.

2. König's Theorem. A = 0-1 matrix (entries 0 or 1)

A line is a row or column.

Thm: The min number of lines of A that contain all of 1's in A is equal to the maximum # of 1's in A no two on a line.

↳ max ind set min-cover.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

c_1 c_2 cover

Pf. (will outline in Ex.).

3. König's in terms of graphs

- A vertex cover in a graph is a set of vertices that includes at least one end point in each edge
- A minimal cover if no other cover has fewer vertices

Thm (König) Let G be a finite bipartite graph. The # of edges in a maximal matching equals the # of vertices in a minimal vertex cover.

4. Min Cut – Max Flow (Ford-Fulkerson). (Suban has papers in this area with Maheshwari (legendary IITD prof)).

— Directed network

↳ $c(e)$ ← each edge has capacity

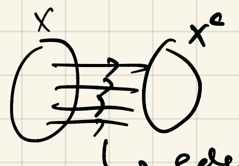
There is a flow through network. $f(e) \leq c(e)$

Flow of Network: Flows out of source = flows into sink.

Inside network: flow into vertex = flow out of vertex (Conservation law).

Cut: X = set of vertices which contain source

X^c = . . . sink.



↳ edges from X to X^c. If you cut them the flow will stop. Called Cut.

Capacity of Cut = Sum of capacities of Cut.

Thm: Min Cut = Max Flow.

eg. to Hall's theorem.. Proof is the algo usually implemented to find SDR.