Lecture 14. Partitions: "picture writing": DIWALI SPECIAL

Permutations: write $\sigma \in S_n$ as a product of disjoint cycles.
Say $\sigma = T_1 T_2 T_3 \dots T_k$, $T_i = \lambda_i$ cycle. $\lambda_i \geq 1$
then $n = \lambda_1 + \lambda_2 + \dots + \lambda_k$ WLOG $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$.

Recall: Def: (partition) A partition of n is a way of writing n as an unordered sum of numbers.

Def: Let σ_1, σ_2 be two permutations. We say σ_1 is a conjugate of σ_2 if there is a permutation τ s.t. $\sigma_2 = \tau \sigma_1 \tau^{-1}$.

Ex 1. $\sigma = (i_1, i_2, \dots, i_n)$. Then $\tau\sigma\tau^{-1} = (\tau(i_1), \tau(i_2), \dots, \tau(i_n))$
ie $\tau\sigma\tau^{-1}$ takes $\tau(i)$ to $\tau(\sigma(i))$ for $i = 1, 2, \dots, n$.
2. Conjugation is an equivalence relation
3. σ and τ are conjugates iff they have same cycle structure.
So # of conjugacy classes = # of partitions of n.

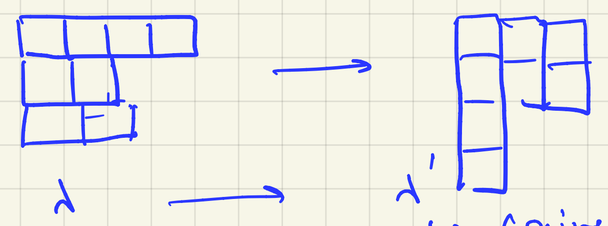
Question: $p(n) \overset{?}{\gtreqless} n!$ Ans. $<$ since $n!$ ← all permutations.

Notation: $\lambda = (\lambda_1, \lambda_2, \dots)$; $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$
$n = |\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_k$. A partition of n into k parts.
$\lambda \vdash n$ ← λ is a partition of n. $p(n) = $ # of partitions of n. $p(0) := 1$.

Ex: Pictures

Prop 1 # of partitions of n with largest part $\leq k$ = # of partitions of n with at most k parts.

Proof:



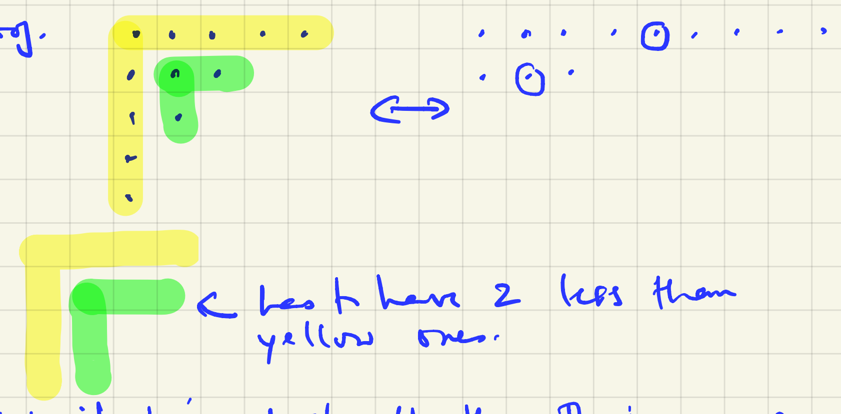
↳ Conjugate of λ.
λ' = Young diagram of the transpose of λ.
Any partition of n with largest part $\leq k$ is in 1-to-1 correspondence with a partition with $\leq k$ parts.
Note: ∵ largest $= k$ = ∵ $= k$ parts □

Notation. $p(n \mid \text{largest part} \leq k) = p(n \mid \text{at most } k \text{ parts})$
# of partitions of n — some condn.

Note:
$\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$
$\lambda'_i = \# \{ i \mid \lambda_i \geq i \}$ ← conjugate
$i = 1, 2, \dots, k$.

Prop 2 $p(n \mid \text{self conjugate partitions}) = p(n \mid \text{odd parts that are distinct})$.

Proof.



Take 'hooks' and straighten them. The inverse mapping also works because odd → can break into 2 with middle shared; distinct → differ by at least 2.

Generating function (Euler).

Prop: $\sum_{n=0} p(n) q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\dots}$

Proof: RHS $= (1+q+q^2+\dots)(1+q^2+q^4+q^6+\dots)\dots (1+q^{\square}+q^{\boxplus}+\dots)(1+q^{\square}+q^{\boxplus}+\dots)$ $\sum q^{\boxplus}$

$= \sum_{\substack{\text{sum over} \\ \text{finite sets of } f_i\text{'s}}} q^{1 f_1 + 2 f_2 + 3 f_3 + \dots}$

$= \sum_{n=0}^{\infty} p(n) \, q^n$ (collect powers of n).

partition of n: $n = 1 f_1 + 2 f_2 + \dots$
$= \underbrace{1+1+\dots+1}_{f_1} + \underbrace{2+2+\dots+2}_{f_2} + \dots$
f_i = frequency of occurrence of i's in partition.

Ex. Polya's coin changing ("On picture writing" article by Polya)
$\frac{1}{1-q} \cdot \frac{1}{1-q^5} \cdot \frac{1}{1-q^{10}} \cdot \frac{1}{1-q^{25}} \cdot \frac{1}{1-q^{50}} = \sum * q^n$
Find $*$ when $n = 100$.

Prop. $p(n \mid \text{odd parts}) = p(n \mid \text{distinct parts})$
Proof: Note that
g.f. of $p(n \mid \text{odd parts}) = \frac{1}{(1-q)} \cdot \frac{1}{(1-q^3)} \cdot \frac{1}{(1-q^5)} \cdots$
g.f. of $p(n \mid \text{distinct parts}) = (1+q)(1+q^2)(1+q^3)\dots$

Note that
$\frac{1}{(1-q)(1-q^3)\dots} = \frac{(1-q^2)(1-q^4)(1-q^6)\dots}{(1-q)(1-q^3)(1-q^5)\dots}$
↑ g.f of LHS
$= \frac{(1-q)(1+q)(1-q^2)(1+q^2)(1-q^3)(1+q^3)\dots}{(1-q)(1-q^3)(1-q^5)\dots}$
$= (1+q)(1+q^2)(1+q^3)\dots$ □
↑ g.f of RHS.

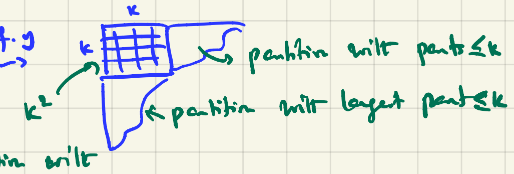
Ex. find explicit bijection (Prove by pictures).

Note: G.f of partitions with parts $\leq k$
$= \frac{1}{(1-q)(1-q^2)\dots(1-q^k)} = \frac{1}{(q;q)_k}$ ↵ notation of q-Pochhammer factorial

Recall: $(A;q)_k = (1-A)(1-Aq)\dots(1-Aq^{k-1})$.

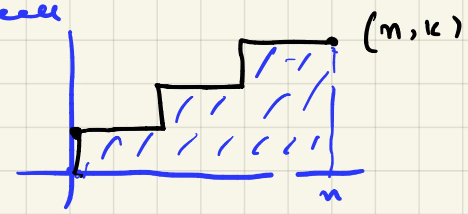
Prop: $\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k^2} = \frac{1}{(q;q)_\infty}$ (As f.p.s).

$\frac{q^{k^2}}{[(1-q)(1-q^2)\dots(1-q^k)]^2}$ ↳ g.f of a partition with a square of size k.



$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k^2}$ = g.f of all partitions
$= \frac{1}{(1-q)(1-q^2)\dots} = \frac{1}{(q;q)_\infty}$. □.

Recall



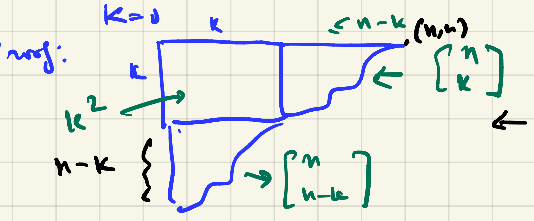
path from $(0,0)$ to (n,k)
$n+k$ steps
k horizontal steps
$W(P)$ = area under path

G.f $:= \sum_{P:\text{path}} q^{W(P)} = \begin{bmatrix} n+k \\ n \end{bmatrix}_q = \begin{bmatrix} n+k \\ k \end{bmatrix}_q$

path ⟷ Young diagram (upside down in a mirror)
area = # boxes ⟷ n ↳ in box
So G.f = # of partitions whose Young diagram fits into rectangle with dimensions $k \times n$.

Thm: $\sum_{k=0}^{\infty} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}^2 = \begin{bmatrix} 2n \\ n \end{bmatrix}$ ⟷ $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$
→ "q-analogue" of

Proof:



← Summing g.f we get all partitions in box of dim $n \times n$
□