Lecture 21 : Probability (cont.)

Philosophy: Whenever $\sum a_n = S$, $a_n \geq 0$, $\sum p_n = 1$ with $p_n = a_n/S$. ← Counting problem.

Get a discrete prob distribution.

Continuous distributions : From $\int_a^b f(x)\,dx = S$, $f(x) \geq 0$, $(S > 0.)$

Last time : Geometric Distribution ; Binomial distribution.

Ex 3 (Poisson distribution). Recall $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$

$$P_k = P(X = k) := \frac{\lambda^k}{k!} e^{-\lambda}$$

Then $\sum_{k=0}^{\infty} P_k = 1$

λ = parameter of Poisson distribution.

$$E(X) = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \frac{\lambda}{e^{\lambda}} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \frac{\lambda}{e^{\lambda}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda,$$

- Can obtain as a limit of Binomial distributions $(p = \lambda/n, n \to \infty)$
- Interpretation → # of 'rare events' (Prob Stats course)

Ex 4 (Hypergeometric distribution)

Recall Chu-Vandermonde identity. (Prob 63 (b))

$$\sum_{k=0}^{j} \binom{m}{k}\binom{n}{j-k} = \binom{m+n}{j}$$

proof: $(1+x)^{m+n} = (1+x)^m(1+x)^n$

choose coeff of x^j on both sides

Natural limits $\sum_{k=1}^{n} (\ )$ too : $\binom{n}{k} = 0$ $k > m$

$\binom{n}{j-k} = 0$ $j-k<0$

Probability interpretation: Rewrite as $(n \mapsto n-m)$

$$\sum_{k=0}^{j} \binom{m}{k}\binom{n-m}{j-k} = \binom{n}{j}.$$

n balls are there of which m are red. In a sample of j balls, what is the probability that k are red?

Total : $\binom{n}{j}$ samples.

Choose k red from m in $\binom{m}{k}$ ways

From $n-m$ choose $j-k$ balls.

Total # of ways : $\binom{m}{k}\binom{n-m}{j-k}$

:

$$P_k = \frac{\binom{m}{k}\binom{n-m}{j-k}}{\binom{n}{j}}.$$

Remark: Think of red balls as defective items. You are sampling j, and finding k defective.

Other enumeration problems lead to elementary probability problems

Ref: Kai Lai Chung. (P. Ch 1-4 can read).

Remarks? Generating functions are useful in probability. Let

$Q(t) = \sum_{n=0}^{\infty} p_n t^n$. (May be a finite sum).

Then $Q(1) = 1$ ← sum of probabilities.

$$E(X) = t\frac{d}{dx}Q(t)\Big|_{x=1} = \sum_{n=0}^{\infty} n p_n t^n \Big|_{x=1}$$

- Sp $Q(t)$ is prob generating function for random variables X_1 and X_2 (independent samples). What is gen function of $P(X_1 + X_2 = n)$?

  prob $= P_n = \sum_{k=0}^{n} P(X_1 = k)\cdot P(X_2 = n-k)$

  G.f: $= Q(t)^2$

- Similarly $P(X_1 + X_2 + \cdots + X_k = n) \leftarrow Q(t)^k$.

- $E(X+Y) = E(X) + E(Y)$ (Ex).

- If X, Y are independent, then $E(X \cdot Y) = E(X)E(Y)$.

  Ind : $P(X = x, Y = y) = P(X = x)\cdot P(Y = y)$

  Exp: $\sum_{\substack{\omega \in \Omega \\ \omega = xy}} xy\, P(X = x, Y = y) = \sum_{xy} xy\, P(X = x) P(Y = y)$

  $= \sum_x x P(X = x) \sum_y P(Y = y)$

  $= E(X) E(Y)$.

$Q_X(t)$ = g.f for X

$Q_Y(t)$ = g.f for Y

$$Q_{X+Y}(t) = Q_X(t)\, Q_Y(t)$$

Example (Derangement problem). Montmort (1708) Probleme des Rencontres (Problem of Coincidences). An urn contains n balls labelled $1, 2, 3, \ldots, n$. A ball is drawn and its number noted (so permutation). A coincidence occurs if k^{th} ball is labelled k.

Find prob of no coincidences: ie, a derangement — a permutation with no fixed points.

Take $D_0 = 1$, D_n = # of derangements of I_n.

Total # of permutations $= n!$

Reqd probability $= D_n/n!$

Prop (1) $n! = \sum_{k=0}^{n} \binom{n}{k} D_k$

(2) $\frac{D_n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$

$= \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

Proof: $n! = \sum_{k=0}^{n} \binom{n}{k} D_k$ ← # of derangements on k letters.

↳ choose $n-k$ points that are fixed $= \binom{n}{n-k}$

- Count pairs (A, B) A = subset of I_n, B = derangement on $I_n \setminus A$.

- Total pairs : $n!$

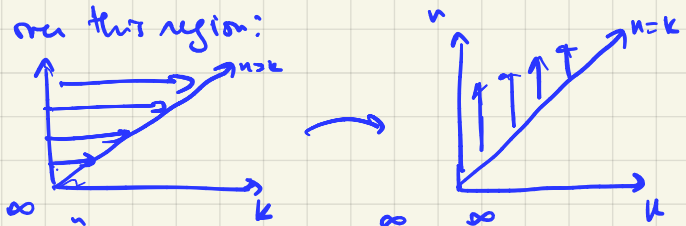
$$= \sum_k \sum_{|A| = n-k} D_k = \sum_{k=0}^{n} \binom{n}{n-k} D_k = \sum_{k=0}^{n} \binom{n}{k} D_k.$$

Part 2. Let $D(t) = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}$. By recurrence

$$\sum_{n=0}^{\infty} \frac{n!\,t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} D_k \frac{t^n}{n!}$$

$$\frac{1}{1-t}$$

Sum is over this region:



$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(n,k) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} A(n,k)$$

$$= \sum_{k=0}^{\infty} \sum_{n=}^{\infty} A(n+k, k)$$

So

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\cancel{n!}}{k!(n-k)!} \frac{D_k t^n}{\cancel{n!}} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{D_k t^{n+k}}{k!\, n!}$$

$$= \sum_{k=0}^{\infty} \frac{D_k t^k}{k!} \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t D(t).$$

$$\Rightarrow D(t) = \frac{e^{-t}}{1-t} = \sum_{k=0}^{\infty} t^k \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!}$$

$$\Rightarrow \frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \quad \left(\text{product of series} = \sum_{k=0}^{n} a_k b_{n-k}\right)$$

Remark: n envelopes, n letters. Secy randomly sends letter. Prob of at least one reaching right person.

Ans: $1 - \frac{D_n}{n!} \xrightarrow[n \to \infty]{} 1 - \frac{1}{e} = 0.6321.$

Ex. Do this problem by inclusion - exclusion.

Remarks 1. - When you see $\binom{n}{k}$ in convolution, think exp g.f.

Ex: $\sum_{k=0}^{\infty} \frac{a_k}{k!} \sum_{k=0}^{\infty} \frac{b_k}{k!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{x^n}{n!}$

2. When you see alternating signs → think exclusion - inclusion.