Discrete Math : Lecture 1

Grading: Weekly Quiz Score : $10 \times 20 = 200$ (Drop 1) — M − 20, W − x } → Scale to 20

Midterms : $2 \times 100 = 200$

Final : $1 \times 100 = 100$

500

A's → 90, B's → 80s etc. } ← plan like this.

PS → Weekend

Monday Quiz (lectures + PS)

xxx

1. SETS.

Set: A well-defined collection of objects. "Think of" a set like this.
↳ we can decide whether an element a belongs to set A or not. Symbols: $a \in A$ a is in A; $a \notin A$ a is not in A.

Concept of: ϕ ← Empty set
Universal set ← sets context

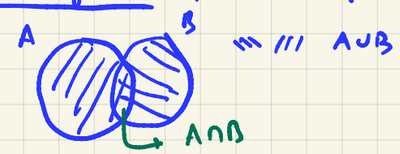
Def: ① $A \subset B$: A is a subset of B (A ⊆ B is also used)
if $x \in A$ then $x \in B$ is true for all $x \in A$.

② $A = B$: if $A \subset B$ and $B \subset A$

③ $A \cup B := \{x \in A \text{ OR } x \in B\}$
↑ union ↑ is defined as ↳ allows $x \in A$ and B

④ $A \cap B := \{x \in A \text{ AND } x \in B\}$

Venn diagrams pictorial rep



/// $A \cup B$

⑤ $B - A := \{x \in B \text{ and } x \notin A\}$

$A^c := \{x \notin A\}$ ($A^c = X - A$) ← universal set

Remark:
An empty set is a subset of every set.
Argument: Let A be set. There is no element of A which is not in ϕ.
Another way: If $x \in \phi$ then $x \in A$ is vacuously true.

Most of this comes a about the set
$\mathbb{N} := \{1, 2, 3, 4, \dots\}$ 'and so on'.
or $\mathbb{N} \cup \{0\} = \mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$

Example: To give you a flavor of how one can develop $\mathbb{N}$ from sets, let us construct a sequence of sets with labels 0, 1, 2

$\phi = \{\ \}$ ⟷ 0
$\{\phi\}$ ⟷ 1
$\{\phi, \{\phi\}\}$ ⟷ 2
$\{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$ ⟷ 3
⋮

Note the following:
1. Each label $0, 1, \dots, n \in n+1$
2. For each n, there is a 'successor' n+1
3. 0 is not the successor to any n.
4. We can define (say) 5+2 as successor of successor of 5.

Remark: We will regard $\mathbb{N}$ as 'god-given'. We will take the following principle for granted as one of the axioms of $\mathbb{N}$.

Principle of Mathematical Induction (PMI): Let $S \subseteq \mathbb{N}$ be s.t.
(i.) $1 \in S$
(ii) if $k \in S$ then $k+1 \in S$.
Then $S = \mathbb{N}$.

A similar statement can be made for $\mathbb{N}_0$, which also we refer to as PMI.

We will use the notation $I_n := \{1, 2, \dots, n\}$. This is a typical set with n elements. $I_0 := \phi$

Example: (set of subsets). $P(X) :=$ set of subsets of X

	$P(I_k)$	#
k=0 :	$\{\phi\}$	1
k=1 :	$\{\phi, \{1\}\}$	2
k=2 :	$\{\phi, \{1\}, \{2\}, \{1,2\}\}$	4
k=3 :	$\{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$	8
k=4 :	$\{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{4\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$	16.

insert 4 in each!
Do we get all?

Now there are several things we can do.
1. Guess from pattern: 1, 2, 4, 8, 16, 32, 64, ...
Ans: # of subsets of an n-set is 2^n.
2. What if pattern not obvious? Go to oeis.org and see if this sequence is known. (Try it now!).
3. Recurrence: Sp # is P_k. Once we listed $P(I_k)$, we get $P(I_{k+1})$ by inserting k+1 in each set in $P(I_k)$. So $P_{k+1} = 2 \cdot P_k$, $P_0 = 1$

So
$$P_n = 2 \cdot P_{n-1} = 2 \cdot 2 \cdot P_{n-2} = \dots = 2^n P_0 = 2^n$$

4. Proof: We can convert the above into a formal proof by induction.

Proof: $S = \{n \in \mathbb{N}_0 \mid \#\text{ of elements in } P(I_n) = 2^n\}$

$n = 0$: $P(I_0) = \{\phi\}$ $\# = 2^0 = 1$. So $0 \in S$.
Sp true for n, ie $\# P(I_n) = 2^n$; ie sp. $n \in S$.
Then claim: $P(I_{n+1})$ contains all sets in $P(I_n)$ and all sets of the form $A \cup \{n+1\}$, where $A \in P(I_n)$.
Any set in $P(I_{n+1})$ either contains n+1 or not.
if it doesn't it is in $P(I_n)$. if it does, deleting n+1 gives a set in $P(I_n)$.
So $P_{n+1} = 2P_n = 2 \cdot 2^n = 2^{n+1}$. Thus $n+1 \in S$.
By PMI, $S = \mathbb{N}_0$.

5. Generating Function: We can consider the sum
$$P(z) = \sum_{n=0}^{\infty} P_n z^n = \sum_{n=0}^{\infty} 2^n z^n = 1 + 2z + (2z)^2 + \dots$$
$$= \frac{1}{1-2z}.$$
This simple expression contains all the information of the sequence. We can get the coeff of z^n on demand, when required.

Def: (Cartesian product). Let A, B be non-empty sets
$A \times B := \{(a, b) \mid a \in A, b \in B\}$
= set of 'ordered pairs' or 'list of length 2'
$(a_1, b_1) = (a_2, b_2)$ iff. $a_1 = a_2$ and $b_1 = b_2$

Example: $I_2 = \{1, 2\}$ $I_3 = \{1, 2, 3\}$
$I_2 \times I_3 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

Problem: Find # of binary strings of length n, the set $\{0,1\}^n = \{0,1\} \times \{0,1\} \times \dots \times \{0,1\}$ (n copies)

Def: Two sets A and B have equal # of elements if there is a one-to-one correspondence between them.

6. We connect with a problem already solved. Claim. # of binary strings = # of subsets of set I_n of length n

1 0 1 0 1 1 ⟷ $A = \{1, 3, 5, 6\} \subset I_6$
if it place is 1, $i \in A$
0, $i \notin A$
Clearly, it's a 1-to-1 correspondence.

So # of such binary strings $= 2^n$.

Notation: 2^X: Power set of X.