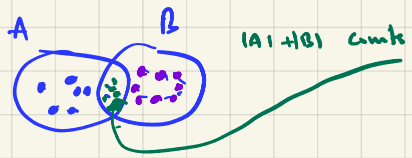
Lec 6: Set theory and logic

We have got some experience now with proofs. This lecture we will do some proofs for set theory and examine them with a view to solidify our understanding of logic. Along with Smullyan's puzzles, this will lead you how to influence people (and lose friends) for the rest of your life.

Theorem: Let A, B be sets. Then
$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof: Pic:



$|A| + |B|$ counts $A \cap B$ twice.

Recall: $|A \cup B| = |A| + |B|$ if $A \cap B = \phi$.

- $A \cup B = A \cup (B \setminus A)$
  ↳ $B \setminus A$
- $A \cap (B \setminus A)$ is disjoint
- So $|A \cup B| = |A \cup (B \setminus A)| = |A| + |B \setminus A|$.
- show $(B \setminus A) \cup (A \cap B) = B$,
  and $(B \setminus A) \cap (A \cap B) = \phi$.

So $|B \setminus A| + |A \cap B| = |B|$.

To complete proof n.t.s

(1) $A \cup B = A \cup (B \setminus A)$

How to show 2 sets are equal?

Recall Def: $A = B$ if $A \subset B$ and $B \subset A$.

(a) $A \cup B \subset A \cup (B \setminus A)$

We need to show: if $x \in A \cup B$ then $x \in A \cup (B \setminus A)$

ie if $x \in A \cup B$ then $x \in A$ OR $x \in B \setminus A$

- meaning of OR → P OR Q if either P is true, or Q is true or P and Q are both true.
- to show P OR Q, assume P is not true and show Q is true.

Let $x \in A \cup B$ and sp $x \notin A$.
$\Rightarrow x \in B$ ie $x \in B$ and $x \notin A \Rightarrow x \in B \setminus A$.

(b) n.t.s. $A \cup (B \setminus A) \subset A \cup B$

Let $x \in A \cup (B \setminus A) \Rightarrow x \in A$ or $x \in B \setminus A$

Again sp $x \notin A$. Then $x \in B \setminus A = \{b \in B \mid b \notin A\}$
$\Rightarrow x \in B$
ie $x \in A \cup B$.

(a) and (b) together imply $A \cup B = A \cup (B \setminus A)$

Remark: We have shown
Let $x \in \ldots \Rightarrow x \in \ldots$

The conclusion is: For all x : $x \in \ldots \Rightarrow x \in \ldots$
prove for one x, but it is valid for all.

(c) $A \cap (B \setminus A) = \phi$

Let $x \in A \cap (B \setminus A) \Rightarrow x \in A$ and $x \in B \setminus A$ (def of $\cap$)
$\Rightarrow x \in B \setminus A$ (And → both are true.)
$\Rightarrow x \in B$ and $x \notin A$ (def of $B \setminus A$)
$\Rightarrow x \notin A$ →← (to $x \in A$).
So there is no x in $A \cap (B \setminus A)$. So it is empty.

(d) $(B \setminus A) \cup (A \cap B) = B$

$\subseteq$: Let $x \in (B \setminus A) \cup (A \cap B)$
$\Rightarrow x \in B \setminus A$ OR $x \in A \cap B$
OR: at least one of 2 have to be true

if $x \in B \setminus A \Rightarrow x \in B$ and $x \notin A$
$\Rightarrow x \in B$
if $x \in A \cap B \Rightarrow x \in B$ and $x \in A$
$\Rightarrow x \in B$
So $x \in B$.

$\supseteq$ Let $x \in B$. n.t.s $x \in B \setminus A$ or $x \in B \cap A$.
Sp $x \notin B \cap A \Rightarrow x \notin B$ OR $x \notin A$.
NOT (P and Q) is NOT P or NOT Q.
Now we are assuming $x \in B$. So $x \notin A \Rightarrow x \in B$ and $x \notin A$
ie $x \in B \setminus A$. ✓

e. $(B \setminus A) \cap (A \cap B) = \phi$

Let $x \in (B \setminus A) \cap (A \cap B)$
$\Rightarrow x \in B \setminus A$ AND $x \in A \cap B$
$\Rightarrow x \notin A$ AND $x \in A$
→←.

Note: $x \in A$ and $x \notin A$ cannot both be true. This is essentially law of excluded middle. Both P and not P cannot be true.

All steps of proof have been filled in. □

Prop. If $A \subset B$. Then $A \cup B = B$.

Proof: $B \subset A \cup B$: This is easy: Let $x \in B \Rightarrow x \in B$ or $x \in A \Rightarrow x \in A \cup B$
↑ true because at least one is true.

Proof of Converse: $A \cup B \subset B$

if P then Q. Converse is if Q then P.

We will show contrapositive: If not Q then not P.

sp $x \notin B$. then $x \notin A \cup B = \{x \in A \text{ or } x \in B\}$.
we need to show $x \notin A$.
sp $x \in A$. Then since $A \subset B$, $x \in B$. →←.
So $x \notin A$.

Remark: Direct argument is easy too.

Prop: Converse of previous one:
Suppose $A \cup B = B$ then $A \subset B$.

Proof: What is the negation of $A \subset B$?
$A \subset B$ is: If $x \in A$ then $x \in B$.
Negation: there is an $x \in A$ s.t. $x \notin B$.
One counterexample is enough to disprove a statement.

Sp $A \subset B$ is not true. Then there is an $x \in A$ s.t. $x \notin B$.
$\Rightarrow x \in A \cup B$ but $x \notin B$.
$\Rightarrow A \cup B \neq B$.
So again we showed NOT $(A \subset B) \Rightarrow$ NOT $(A \cup B = B)$.

Prop: (de Morgan's laws).

(1) $(A \cup B)^c = A^c \cap B^c$

(2) $(A \cap B)^c = A^c \cup B^c$

NOT (P OR Q) = NOT P AND NOT Q

NOT (P AND Q) = NOT P OR NOT Q

Proof: (1) Let $x \in (A \cup B)^c$
$\Rightarrow x \notin A \cup B$

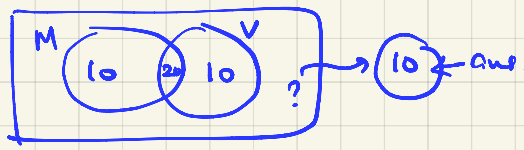
Claim $\Rightarrow x \notin A$ AND $x \notin B$.
sp $x \in A \Rightarrow x \in A \cup B$ →←. Similarly $x \in B$ leads to a →←
So $x \in A^c \cap B^c$

Conversely: sp $x \in A^c \cap B^c \Rightarrow x \notin A$ and $x \notin B$.
sp $x \in A \cup B \Rightarrow x \in A$ or $x \in B$. neither can be covered.

(2) Ex.

Example. (of theorem)

At a party: 50 total children
30 ate a mango pastry
30 had vanilla ice-cream
20 had both. How many had neither?



← one.

You can use theorem, but actually we just make a pic

Example: (The Barber Paradox)

A barber shaves all who don't shave themselves. Question:
Does the barber shave himself?
- if he does → →← to definition
- if not → shaves himself by def →←

Russell's Paradox:

Let X be the set of all sets which DO NOT contain themselves.
if $X \in X$ → then $X \notin X$ by def of set X.
if $X \notin X$ → then $X \in X$ by def.

So we cannot decide whether $X \in X$ or not.

Led to "well-defined" ← which we began with in lecture 1.

Clarified set theory.

Reading: Diophantus Gödel, Escher, Bach: An eternal golden braid. by Douglas Hofstadter. Discusses ideas of importance to Computer Scientists.