Lecture 9 — Newton's Binomial Theorem

We have shown
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad (1)$$
where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $\quad (0 \le k \le n) \quad (2)$

or (Recurrence Relation): $\binom{n}{0} = 1$; $\binom{n}{k} = 0$ for $k > n$; $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ $\quad (*)$

(*) is better for calculation (because $n!$ becomes too large!)
- Induction proof it determines $\binom{n}{k}$ for all n, k. (Ex)
- Ex: Show RHS of (2) satisfies (*). Thus they must be equal.

Remark: We can simply write it equivalently as
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad (3)$$
- take $y \to x$, $x = 1$ in (1) to get (3).
- take $x \to y/x$ in (3), multiply by x^n to get (1)

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Generating function ideology - Wilf.

Let (a_n) be a sequence (sequence: $a : \mathbb{N}_0 \to \mathbb{R}/\mathbb{C}/\ldots$)

Ordinary Generating Function (OGF): $\sum_{n=0}^{\infty} a_n z^n$

Exponential Generating function (EGF): $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$

Dirichlet g.f.: $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ (s usually complex)

Examples: $a_n = 1$ for all n.
$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$$
$$\sum_{n=0}^{\infty} \frac{q^n}{n!} = e^q$$
$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \leftarrow \text{Riemann zeta function.}$$

Remarks: (1) Most of the time we require them as f.p.s. In combinatorics $a_n \in \mathbb{N}_0$ and count something.

(2) To estimate a_n we use these series as complex functions "analytic". Most theorems to estimate a_n taught in advanced complex analysis courses / physics w/out proof / applied math contexts.

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Binomial Series

How to guess/discover/

We are given a finite sum
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \quad \text{where } n = 0, 1, 2, \ldots$$
We wish to extend to case when n is not a number.
$$(1+x)^n = \sum_{k=0}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} x^k$$
$$= \sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} x^k$$
(since $n(n-1)\cdots = 0$ when $k > n$ — $\binom{n}{k} = 0$ for $k > n$.)

Guess: $(1+x)^a = \sum_{k=0}^{\infty} \frac{a(a-1)(a-2)\cdots(a-k+1)}{k!} x^k$.

Fact: This series converges for $|x| < 1$.

Proof 1 (Formal power series)

$f(x) = (1+x)^a$

$f'(x) = a(1+x)^{a-1}$, $\quad f'(0) = a$

$f''(x) = a(a-1)(1+x)^{a-2}$, $\quad f''(0) = a(a-1)$

$\vdots$

$f^{(k)}(x) = a(a-1)(a-2)\cdots(a-k+1)(1+x)^{a-k}$

$f^{(k)}(0) = a(a-1)\cdots(a-k+1)$

By Taylor Thm for fps
$$f(x) = \sum_{k=0}^{\infty} \frac{a(a-1)\cdots(a-k+1)}{k!} x^k \quad (\text{as fps}).$$

Proof 2 (As analytic functions)

requires some facts which we state w/out proof: Proofs come in Calc/Real analysis/Complex analysis/Diff eqns.

First of all we prefer to write it a bit differently:

Notation: $(a)_k := \begin{cases} 1 & \text{if } k = 0 \\ \underbrace{a(a+1)\cdots(a+k-1)}_{k \text{ terms}} & \text{if } k > 0 \end{cases}$

Rising factorial

Note: (1) $(1)_k = k!$

(2) $(-a)_k = (-a)(-a+1)\cdots(-a+k-1) = (-1)^k a(a-1)\cdots(a-k+1) = (-1)^k P(a,k)$

Thm: (Binomial Theorem (Newton))
$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} x^k = (1-x)^{-a} \quad \text{for } |x| < 1.$$
- Replaced $x \to -x$ and $a \to -a$ to get rid of minus signs
- Note: $x = q$, $a = 1$ gives $\frac{1}{1-q} = 1 + q + \cdots$

Proof (outline) Let $f_a(x) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} x^k$.

By ratio test, $f_a(x)$ converges absolutely and uniformly in $|x| \le \epsilon$, for any ϵ between 0 and 1. This allows us to differentiate, integrate etc.

Step 1: Find a differential eqn.
$$f_a'(x) = \sum_{k=1}^{\infty} \frac{k(a)_k}{k!} x^{k-1}$$
$$= \sum_{k=0}^{\infty} \frac{(k+1)(a)_{k+1}}{(k+1)!} x^k$$
$$= \sum_{k=0}^{\infty} \frac{a(a+1)_k}{k!} x^k = a f_{a+1}(x).$$

Step 2 Find a difference eqn.

Claim: $f_a(x) - f_{a+1}(x) = -x f_{a+1}(x)$

$$\text{LHS} = \sum_{k=1}^{\infty} \left(\frac{(a)_k x^k}{k!} - \frac{(a+1)_k x^k}{k!} \right) = \sum_{k=1}^{\infty} (\quad)$$
$k=0$ terms are $1 - 1$
$$= \sum_{k \ge 1} \frac{x^k}{k!} (a+1)_{k-1} (a - (a+k)) = \sum_{k \ge 1} \frac{(-k)(a+1)_{k-1}}{k!} x^k$$
$$= -\sum_{k \ge 1} \frac{x^k}{(k-1)!} (a+1)_{k-1} = -x f_{a+1}(x).$$

From step 1 and 2 we get
$$f_a'(x) = \frac{a}{1-x} f_a(x). \quad \text{Note } f_a(0) = 1$$

Step 3: Solve this initial value theorem.
$$\frac{f_a'(x)}{f_a(x)} = \frac{a}{1-x}$$
$$\int \frac{f_a'(x)}{f_a(x)} dx = -a \log(1-x) + c$$
$$\Rightarrow \log f_a(x) = \log (1-x)^{-a} + c$$
$$\Rightarrow f_a(x) = c(1-x)^{-a} \quad \text{for some } c.$$
$$f_a(0) = 1 \Rightarrow c = 1$$
So $f_a(x) = (1-x)^{-a}$ for $|x| < 1$.

Note: Does $(1-x)^{-a}$ make sense for x, a complex. Half of math is about answering such a question, filling gaps and extending ideas here to different contexts.

Ex: (How did Newton check his work?)
$$(1-x)^{1/2}(1-x)^{1/2} \stackrel{?}{=} (1-x)$$
$$\left(1 - \tfrac{1}{2}x + \tfrac{1}{2}(\tfrac{1}{2}-1)\tfrac{(-x)^2}{2!} + \cdots\right)\left(1 - \tfrac{1}{2}x + \tfrac{1}{2}(\tfrac{1}{2}-1)\tfrac{(-x)^2}{2!} + \cdots\right)$$
$$= 1 + x\left(-\tfrac{1}{2} - \tfrac{1}{2}\right) + x^2\left((\tfrac{-1}{2})(\tfrac{-1}{2}) - \tfrac{1}{4 \cdot 2!} - \tfrac{1}{4 \cdot 2!}\right) + \cdots$$

Ex: - Find coeff of x^3 by hand.
- Coeff until x^{10} by Sage.

(Can generate Taylor series in Sage).