Lecture 25: Three negative statements: Beyond CS.

Thm. $\sqrt{2}$ is not rational. (i.e. $\sqrt{2}$ is irrational)

Recall: Rational $= p/q$, $p, q \in \mathbb{Z}$, $q \neq 0$.

Proof: suppose $\sqrt{2} = a/b$, with $(a,b) = 1$. (so $\sqrt{2}\, b = a$)

Then $\exists\, x, y \in \mathbb{Z}$ s.t. $ax + by = 1$

$$\Rightarrow \sqrt{2} = \sqrt{2}\,ax + \sqrt{2}\,by$$
$$= 2bx + ay \in \mathbb{Z}. \quad \rightarrow\leftarrow$$

since $\sqrt{2}$ is not an integer. (why?)

$1^2 = 1$, $2^2 = 4$

so $1 < \sqrt{2} < 2$

Thm Let $P = \{p_1, p_2, \dots\}$ be the set of primes. Then P is not finite (i.e. infinite)

Proof (Euler) Sp $p_1, p_2, \dots, p_k$ are all the primes.

Consider

$$P = \frac{1}{(1 - 1/p_1)(1 - 1/p_2)\cdots(1 - 1/p_k)}$$

$$= \left(1 + \frac{1}{p_1} + \frac{1}{p_1^2} + \cdots\right)\left(1 + \frac{1}{p_2} + \frac{1}{p_2^2} + \cdots\right)\cdots\left(1 + \frac{1}{p_k} + \frac{1}{p_k^2} + \cdots\right)$$

$$= \sum \frac{1}{p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}} \qquad a_1, \dots, a_k \in \mathbb{N}_0$$

Now all numbers $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, by unique factorization

so $P = \sum_{n=1}^{\infty} \frac{1}{n}$.

However we saw $H_k = \sum_{n=1}^{k} \frac{1}{n} \sim \Theta(\lg k)$

so $\lim_{k\to\infty} H_k = \lim_{k\to\infty} \log k = \infty$.

But P is a finite product. $\rightarrow\leftarrow$.

Thm Let $B = \{$set of infinite binary strings$\}$. Then B is not countable.

Remark: Recall ① S is countable if it can be put into 1-1 correspondence with a subset of $\mathbb{N}$. ② By base representation theorem, all numbers can be represented as binary strings of finite length. Can do same with rational #s (p, q).

Proof (Cantor's diagonalization)

Sp there is a 1-to-1 correspondence $x = 001\dots$

$1 \longleftrightarrow 1001100\dots$

$2 \longleftrightarrow 01\,011001001\dots$

$3 \longleftrightarrow 11000\dots$

Claim: all strings are not covered. Let x be st that i'th bit is different from i'th bit of string i

x is not in list. Sp x is n'th entry. But n'th entry of list differs from x in n'th place. $\rightarrow\leftarrow$

So B is uncountable $\square$.

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Overview / Summary

"Its combinatorics that counts"