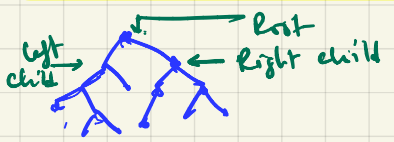
Lecture 17.

TWO TREE COUNTING PROBLEMS

ROOTED BINARY TREES.



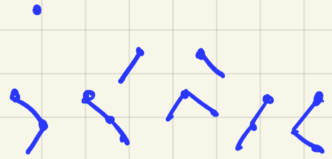
T_n = # of rooted binary trees on n vertices.

$T_0 := 1$

$T_1 = 1$

$T_2 = 2$

$T_3 = 5$



Sp vertices are $n+1$. For some k, left child = k vertices # T_k, right child = $n-k$ vertices # T_{n-k}.

So $$T_{n+1} = \sum_{k=0}^{n} T_k T_{n-k} \qquad (*)$$

Claim: $$T_n = \frac{1}{n+1}\binom{2n}{n}$$

Consider $$T(t) = \sum_{n=0}^{\infty} T_n t^n = \text{g.f. of } T_n\text{'s}.$$

Multiply $(*)$ by t^{n+1} and sum over n

$$\sum_{n=0}^{\infty} T_{n+1} t^{n+1} = t \sum_{n=0}^{\infty} \sum_{k=0}^{n} T_k T_{n-k} t^n = t\,(T(t))^2$$

$$\Rightarrow \quad T(t) - T_0 = T(t)^2$$

$$\Rightarrow \quad tT^2 - T + 1 = 0$$

$$\Rightarrow \quad T = \frac{1 \pm \sqrt{1-4t}}{2}$$

ie $$T(t) = \frac{1+\sqrt{1-4t}}{2t} \quad \text{or} \quad \frac{1-\sqrt{1-4t}}{2t}. \quad \leftarrow \lim_{t\to 0}(\;) = 1.$$

But $T(0) = T_0 = 1$ so reject.

Recall: $$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k = \sum_{k=0}^{\infty} \frac{a(a-1)\cdots(a-k+1)}{k!} x^k$$

So $$T(t) = \frac{1}{2t}\left(1 - \sum_{k=0}^{\infty} (1/2)(\tfrac{1}{2}-1)\cdots(\tfrac{1}{2}-k+1)\frac{(-4)^k t^k}{k!}\right)$$

$$= \frac{1}{2t}\left(-\sum_{k=1}^{\infty} (\tfrac{1}{2})(-\tfrac{1}{2})(-\tfrac{3}{2})\cdots(-k+\tfrac{3}{2})\frac{(-4)^k t^k}{k!}\right)$$

$$= -\frac{1}{2}\sum_{k=0}^{\infty} (\tfrac{1}{2})(-\tfrac{1}{2})\cdots(-k+1/2)\frac{(-4)^{k+1} t^k}{(k+1)!}$$

So $$T_n = (-\tfrac{1}{2})(\tfrac{1}{2})(-\tfrac{1}{2})\cdots(-n+1/2)\frac{(-4)^{n+1}}{(n+1)!}$$

$$= (-\tfrac{1}{2})\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 2^n}(-1)^n(-1)^{n+1}\frac{2^{2n+2}}{(n+1)!}$$

$$= \frac{2^n}{(n+1)!}\cdot 1\cdot 3\cdot 5\cdots(2n-1)\cdot\frac{2\cdot 4\cdots 2n}{2^n(1\cdot 2\cdots n)}$$

$$= \frac{1}{(n+1)!}\,\frac{(2n)!}{n!} = \frac{1}{n+1}\binom{2n}{n}.$$

So $T_n = \frac{1}{n+1}\binom{2n}{n}$ ← Catalan's #.

Thm. (1) $T_0 := 1$, $T_{n+1} = \sum_{k=0}^{n} T_k T_{n-k}$

(2) $T_n = \frac{1}{n+1}\binom{2n}{n}$.

Remarks: (logs)

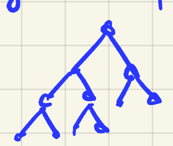
	1	2	2^2	2^3	2^4	2^5	2^6
exponents	0	1	2	3	4	5	6

x

+

Notation If $b^e = a$ we write $\log_b a = e$

log is exponent.



1

2

2^2

⋮

2^h

Sp h levels/depth

# of vertices $= 1+2+\cdots+2^h$

$$= \frac{2^{h+1}-1}{2-1} = 2^{h+1}-1$$

$$\approx 2^h$$

For n vertices,

ht of tree $h = \log_2(n+1)-1$

$$\approx \log_2 n = \lg n$$

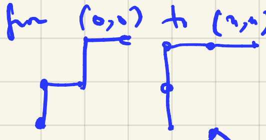
sometimes used in CS.

Recall $\lg 2^{64} = 64$ ↳ much smaller than 2^{64}

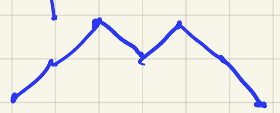
Remarks (Catalan Counting). Catalan #s $C_n\ (= T_n) = \frac{1}{n+1}\binom{2n}{n}$ count many, many things. Stanley's book Enumerative Combinatorics lists 214 items. #s: 1, 1, 2, 5, 14, 42, 132, 429, 1430, ...

(1) Paths from $(0,0)$ to (n,n), N and E steps, lying above $y=x$.

Eg

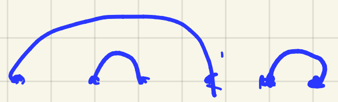


2. Dyck paths



$(0,0) \to (2n,0)$, never below x axis

3.

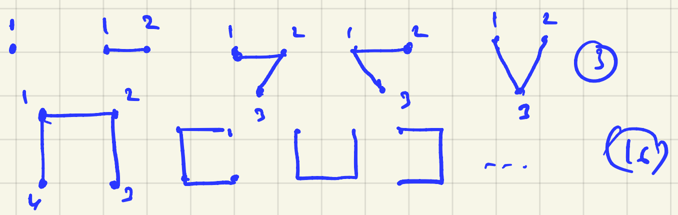


non-intersecting arcs.

4. $(x_1\ (x_2 x_3))\ (x_4 x_5)$ ← legal brackets ($n+1$ symbols).

labelled trees

Thm (Cayley) The # of labelled trees on n vertices is n^{n-2}. (for $n \geq 2$)



Recall: $$(a_1+\cdots+a_m)^n = \sum_{\sum k_i = n} \binom{n}{k_1,\ldots,k_m} a_1^{k_1} a_2^{k_2}\cdots a_m^{k_m} \qquad (1)$$

Prob 79. Consider

$$(a_1+\cdots+a_m)^n = (a_1+\cdots+a_m)^{n-1}(a_1+\cdots+a_m)$$

Compare coeff of $a_1^{k_1}\cdots a_m^{k_m}$ to get

$$\binom{n}{k_1,\ldots,k_m} = \sum_{j=1}^{m} \binom{n-1}{k_1,k_2,\ldots,k_j-1,k_{j+1},\ldots,k_m} \qquad (2)$$

Let $t(n; d_1,\ldots,d_n)$ = # of labelled trees with vertex degrees $d_1, d_2, \ldots, d_n$. Let $n \geq 2$.

On deleting a pendant vertex, we get one vertex with degree $d_i - 1$.

$$\Rightarrow \quad t(n; d_1,\ldots,d_n) = \sum_{i=1}^{n} t(n-1; d_1,\ldots,d_i-1,\ldots,d_n) \qquad (3)$$

Claim: $$t(n; d_1,\ldots,d_n) = \binom{n-2}{d_1-1, d_2-1, \ldots, d_n-1} \qquad (4)$$

For $n=3$ $$\binom{1}{0,0,1} = \frac{1!}{0!\,0!\,1!} = 1$$

degrees: 1, 1, 2.

$t(3;1,1,2) = t(3;1,2,1) = t(3;2,1,1) = 1$ → all match.

- By (3) both sides of (4) satisfy (2) with $n \to n-2$, $m \to n$. They satisfy the initial conditions for $n=3$.

Now from multinomial theorem; $a_1 = a_2 = \cdots = 1$, $n \to n-2$, $m \to n$ to get

$$n^{n-2} = \sum_{\sum k_i = n-2} \binom{n-2}{k_1,\ldots,k_n}$$

Note sum of degrees of tree

$= \sum d_i = 2(n-1)$ ↳ # of edges

$$\Rightarrow \sum (d_i - 1) = 2(n-1) - n = n-2.$$

So $$\sum_{\substack{\sum(d_i-1)\\ = n-2}} t(n; d_1,\ldots,d_n) = \sum_{\sum(d_i-1) = n-2} \binom{n-2}{d_1-1,\ldots,d_n-1} = n^{n-2}.$$

$\square$.