Lecture 12: Enumeration (cont.)

8. Set partitions. We continue our balls into boxes analogy

- I_n ← labelled
- Boxes ← unlabelled.
- Order within boxes does not matter

$\boxed{1\ 3}\ \boxed{4}\ \boxed{5\ 2}$

Def (Set Partitions) A set partition of I_n into k parts is a collection $B_1, B_2, \dots, B_m$ of subsets of I_n s.t.

(1) $\bigcup_{k=1}^{m} B_k = I_n$

(2) $B_i \cap B_j = \phi$ for $i \neq j$, $1 \le i, j \le m$

$S(n,m) :=$ # of partitions of n into m parts.
= Stirling numbers of the second kind.

$B_i \neq \phi$ except one exceptional case $n=0, k=0$.

Example

n\k	0	1	2	3
0	{}			
1	x	{1}		
2	x	{1,2}	{1}∪{2}	
3	x	{1,2,3}	{1,2}∪{3}, {1}∪{2,3}, {1,3}∪{2}	{1}∪{2}∪{3}

k	0	1	2	3	4
4	x	{1,2,3,4} (1)	{1,2,3}∪{4}, {1,2,4}∪{3}, {1,2}∪{3,4}, {1,4}∪{2,3}, {1}∪{2,3,4}, {1,3,4}∪{2}, {1,3}∪{2,4} (7)	{1,2}∪{3}∪{4}, {1}∪{2,3}∪{4}, {1,3}∪{2}∪{4}, {1,4}∪{2}∪{3}, {1}∪{2,4}∪{3}, {1}∪{2}∪{3,4} (6)	{1}∪{2}∪{3}∪{4}

Prop: $S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$
$S(n,0) = 0$ for $n>0$
$S(0,0) := 1$
$S(n,k) = 0$ for $k>n$

Proof: Set partitions into k parts

- Subset 1: $\{n\}$ is a part by itself → $S(n-1,k-1)$
- Deletion of n gives a partition of I_{n-1} into k parts → $k\,S(n-1,k)$ □

$S(n,k)$: table.

n\k	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	1				
3	0	1	3	1			
4	0	1	7	6	1		
5	0	1	15	25	10	1	
6	0	1	31	90	65	15	1

xxx

Def (Bell #s) $b(n) = \sum_{k=0}^{n} S(n,k)$

Named after E.T. Bell. = # of ways of partitioning a set
$b(0)=1,\ b(1)=1,\ b(2)=2,\ b(3)=5,\ b(4)=15$

Formulas for $b(n)$ → later.

Thm The number of surjections (i.e. onto functions) from $I_n \to I_m$ is $m!\,S(n,m)$.

Proof: Count pairs (ϕ, A) where $\phi: I_n \to I_m$ is a surjection and A is the set partition of I_n made of $(\phi^{-1}(1), \phi^{-1}(2), \dots, \phi^{-1}(m))$.
$\phi^{-1}(k) = \{t \in I_n \mid \phi(t) = k\}$.

Given ϕ: unique set partition. So count = # of surjections
Given set partition: there are $m!$ corresponding surjections.
∴ count = $m!\,S(n,m)$ □.

Ex. ϕ: $1\to 3$, $2\to 1$, $3\to 1$, $4\to 2$, $5\to 2$

i	$\phi^{-1}(i)$
1	{2,3}
2	{4,5}
3	{1}

Def. Let $\sim$ be an equivalence relation on a set A. Let, for $a \in A$, $[a] := \{x \in A \mid x \sim a\}$ be the equivalence class of a.

Prop: Let $\sim$ be an equivalence relation. The equivalence classes form a partition of A. That is

(a) $A = \bigcup_{a\in A} [a]$

(b) $[a] \cap [b] = \phi$ or $[a] = [b]$

Proof: (a) Clearly $a \in [a]$. So $[a] \subset A$ for all a
$\Rightarrow \bigcup_{a\in A}[a] \subset A$
Conversely $\forall b \in A$, $b \in [b] \Rightarrow b \in \bigcup_{a\in A}[a]$.

(b) Sp $c \in [a]\cap[b]$
$\Rightarrow c \sim a$ and $c \sim b$
$\Rightarrow a \sim b \Rightarrow b \in [a]$ and $a \in [b]$
$\Rightarrow [a] \subset [b]$ ($c \in [a] \Rightarrow c \sim b \Rightarrow c \in [b]$)
and $[b] \subset [a]$
i.e. $[a] = [b]$

So either $[a]$ and $[b]$ are disjoint, or equal. □

Thus, equivalence classes partition a set. Vice versa, a set partition corresponds to an equivalence class, and if they belong to same set.

Cor: # of equivalence classes on I_n := $S(n,1) + S(n,2) + \dots + S(n,n) = b(n)$

Ex: $\mathbb{Z}$, $\equiv$, is an equivalence class mod m
$[0] = \{0, \pm m, \pm 2m, \dots\}$
$[1] = \{1, \pm(m+1), \dots\}$
$\vdots$
$[m-1] = \{m-1, \pm 2m-1, \pm 3m-1, \dots\}$

9. Labelled balls, unlabelled boxes, some boxes empty (not surjective)
$\# = S(n,0) + S(n,1) + S(n,2) + \dots + S(n,m)$

10. Integer partitions.
unlabelled balls, unlabelled boxes!

Def: (partition) A partition of n is a way of writing n as an unordered sum of numbers.

Ex: $0 = 0$ (Def), $1 = 1$ | $2 = 1+1$, $3 = 2+1 = 1+1+1$, $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$

$p(n,k)$ = # of partitions with k parts
$p(n)$ = # of partitions of n. } No easy formula here.

Ex (1) List partitions of 5, 6.
(2) Use Sage to list all the objects mentioned.