Lecture 18: Big Oh Calculations

Objective: Sometime we cannot count exact number. Even if we can we cannot tell if something is bigger or smaller. So asymptotic analysis helps decide what to do.

Defn: ① (Big O notation) We say $f(n) = O(g(n))$ as $n \to \infty$, if there is $n_0 \in \mathbb{N}$ s.t.

$$|f(n)| \le C|g(n)| \text{ for } n \ge n_0,\ C > 0.$$

- Can also have $f(x) = O(g(x))$ as $x \to a$. There is $\epsilon > 0$, s.t. $|f(x)| \le C|g(x)|$ as if $|x-a| < \epsilon$

② We say $f(n) \sim g(n)$ if $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 1$

③ We say $f(n) = o(g(n))$ (small o) if $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$

④ $f(n) = \Omega(g(n))$ if $|f(n)| \ge c|g(n)|$

⑤ $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

⑥ We use $\prec, \preceq$ to order by small o. (Not standard)

- NOTATIONS ONLY COME ON RHS.

Facts (From Calculus)

1. $\lim\limits_{n\to\infty} x^n = \begin{cases} \infty & (x > 1) \\ 0 & 0 < x < 1 \end{cases}$ So $e^n \to \infty$, $e^{-n} \to 0$

2. $\lim\limits_{n\to\infty} n^k = \begin{cases} +\infty, & k > 0 \\ 0 & k < 0 \end{cases}$

So $x^n = o(x^m)$ for $m > n$, as $x \to \infty$

$x^n = O(x^n)$ as $x \to \infty$

3. $\frac{n^p}{e^n} \to 0$ if $p > 0$

4. $\frac{(\log n)^p}{n^a} \to 0$ for $p > 0, a > 0$

5. $\sum\limits_{n\ge 0}^{\infty} \frac{x^n}{n!} < \infty \implies \frac{x^n}{n!} \to 0$ as $n \to \infty$ $(x \in \mathbb{R})$

6. $|f(n)| < |g(n)| \iff \log|f(n)| < \log|g(n)|$.

7. $\frac{\log_b a}{\log_b c} = \log_c a$ ($a, c > 1$ make sense). So $\log_e x = \Theta(\log_{10} x) = \Theta(\lg x)$

8. $\sum \frac{1}{n^p} < \infty$ for $p > 1$

So $\sum \frac{1}{n^p} = O(1) \leftarrow$ constant

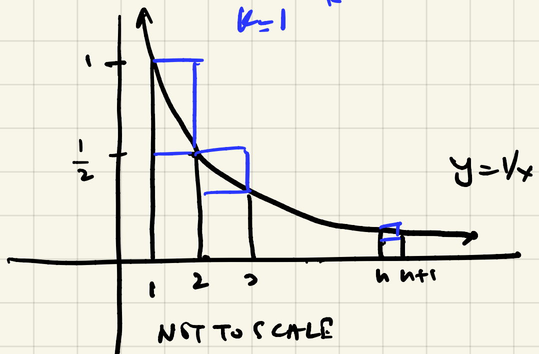
9. $(1+x)^a = \sum\limits_{k=0}^{\infty} \binom{a}{k} x^k < \infty$ for $|x| < 1$

$$= 1 + ax + \frac{a(a-1)}{2!}x^2 + O(x^3) \text{ as } x \to 0.$$

- Can truncate any convergent power series.

- $\lg(1+x) = x + O(x^2)$ as $x \to 0$.

10. $H_n = \sum\limits_{k=1}^{n} \frac{1}{k}$: Harmonic #s.



Area under curve $= \int_1^{n+1} \frac{1}{x}\,dx = \lg(n+1)$

Area of upper Δ's $1 + \frac{1}{2} + \cdots + \frac{1}{n} = H_n$

Lower Δ's : $\frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n+1} = H_{n+1} - 1$

$$g_n = \sum_{k=1}^{n} \frac{1}{k} - \lg(n+1) \le \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \cdots + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) = 1 - \tfrac{1}{n+1} \le 1$$

- g_n increasing sequence, $0 < g_n < 1$ (bounded) so converges.

Say $g_n \to \gamma = \lim\limits_{n\to\infty} \left(H_n - \lg(n+1)\right)$

$= \lim\limits_{n\to\infty} H_n - \lg(n)$ (Ex.)

$\gamma = 0.5772\ldots$ = Euler-Mascheroni Constant

So $\log(n+1) < H_n < \lg(n) + \gamma$

So $H_n - \lg n = O(1)$, $H_n = \lg n + O(1)$

$H_n - \lg n \sim \gamma$, etc.

Fact: $H_n = \lg n + \gamma + O(1/n)$

11. (Stirling's Formula).

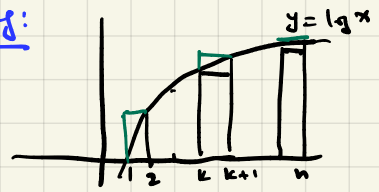
Thm (Weaker version of Stirling's formula) For $n \ge 2$,

$$n\lg n - n < \lg(n!) < n\lg n$$

So $\lg(n!) \sim n\lg n$

Most of the time this is enough in C.S.

Proof:



$\lg n! = \lg n + \lg(n-1) + \cdots + \lg 2 + \lg 1$ (with $\lg 1 = 0$)

$> \int_1^n \lg x\,dx = x\lg x - x \Big|_1^n$

$= n\lg n - n$

$\lg n! = \lg n + \lg(n-1) + \cdots$

$\le n\lg n$ (easy).

Fact: Stirling's Formula (w/out proof)

$$n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}$$

$\lg n! = n\lg n - n + \frac{1}{2}\lg n + \frac{1}{2}\lg(2\pi) + \epsilon_n$

When $\epsilon_n \to 0$ as $n \to \infty$.

More precisely,

$$n! = \sqrt{2\pi}\, n^{n+1/2} e^{-n} e^{\lambda_n}$$

where $\frac{1}{12n+1} \le \lambda_n \le 1/12n$. (other bounds can be given).

Application: $\binom{n}{k}$ $k = 0, 1, \ldots, n$

largest is in middle. When $k = \lfloor \frac{n}{2} \rfloor$.

$$\binom{n}{k} \le \frac{n!}{\lfloor \frac{n}{2} \rfloor! \lfloor \left(\frac{n}{2}\right) \rfloor!} \sim \frac{\sqrt{2\pi}\, n^{n+1/2} e^{-n}}{(\sqrt{2\pi})^2 \left[\left(\frac{n}{2}\right)^{\frac{n}{2}+1/2}\right]^2 e^{-\frac{n}{2}-\frac{n}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2^{n+1}}{\sqrt{n}} = 2^n\sqrt{\frac{2}{n\pi}}$$

Now we can put almost everything in order. (except 2 items)

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$a_n = 1, n^2, 2n+3, n\lg n, e^n, 2^n, \lg n, (\lg n)^2, n^5, H_n = \sum\limits_{k=1}^{n} \frac{1}{k}$

$\frac{n}{\lg n}, \frac{e^n}{n^3}, \frac{1}{n}, e^{-n}, \lg(\lg n), n!, n^{n/2}, \binom{n}{\lfloor n/2 \rfloor}, b(n), S(n, \lfloor n/2 \rfloor)$.

$e^{-n}, 1/n, 1$ — $O(1)$ /constant.

$\log\log n$ — $\to \infty$ but very, very slowly.

$H_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ $\log n$, $(\lg n)^2$,

$n/\log n$

$2n+3$, n^2, n^5 polynomial — linear

$n\log n \leftarrow$ nearly linear.

$e^n/n^3 \leftarrow$ exponential

2^n e^n $\binom{n}{\lfloor n/2 \rfloor}$

$n^{n/2}$ $n!$

Need to look and decide which is better!