Lecture 11 **Enumeration: Counting**

"It's combinatorics that counts"

(Lecture 10 (Midterm))

Some formulas we already know:

1. Words / lists of length n, with alphabet of length m, repetition allowed
   - $=$ $f : I_n \to I_m$
   - $\# = m^n$
   - Recurrence: $N(n) = m\,N(n-1)$
   - In ex → alternate ways of looking at it.

2. Words of length n, alphabet of length m, no repetition allowed
   - $=$ $f : I_n \to I_m$ injective
     $=$ permutations
   - $\# = P(m,n) = \begin{cases} m(m-1)(m-2)\cdots(m-n+1) & (m \geq n) \\ 0 & \text{otherwise} \end{cases}$
   - Also denoted $m^{\underline{n}}$ in some books
     ↳ falling factorial. $= (-1)^n (-m)_n$ ← rising factorial,
   - $P(m,n) = (m-n+1)\,P(m,n-1)$
   - Coeff are $s(n,k)$ ← Stirling #s of first kind.

3. Subsets: Can be regarded as selecting k objects from an alphabet of n. (= paths / binary strings of length n w k 1's, etc).
   - Same as permutations, but order does not matter.
   - $\#: \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$
   - $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
   - Also called # of Combinations.

Next

Think of permutations / functions as "Balls into boxes"
put i in box a_i

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

distinguished objects (labelled)

↳ boxes (labelled by alphabet $A \supseteq \{a_1, \dots, a_n\}$)

Ex $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & b & a & c & a \end{pmatrix}$ 1, 3, 5 in a; 2, in b; 4 in c

functions: order of placement in box doesn't matter
permutation: No more than one item in a box.
subset: put in 2 boxes < k in one, $n-k$ in rest; order does not matter.

4. distribution in order

Thm: The number of distributions of n distinct objects into m distinct boxes, in order, is $(m)_n$.

↳ Recall: $\begin{cases} m(m+1)\cdots(m+n-1); & n>0 \\ 1; & n=0. \end{cases}$

Proof: Let $t(m,n) = \#$ of arrangements of elements of I_n into m-boxes where order in boxes matters.

Sp $1, 2, \dots, n-1$ have been placed and n has to be placed. In the following, $*$ = elements of I_{n-1} already placed; $|$ demarcation into boxes. 0: empty slots

$0 * 0 \,|\, 0 * 0 * 0 * 0 \,|\, 0 * 0 * 0 \,|\, 0 * \cdots * 0 * 0 \,|\, 0 \,|$

For a given arrangement of I_{n-1} into m boxes there are
- $n-2$ places between $n-1$ already placed
- 2 places at ends
- $m-1$ places to choose the box (last in box or first in box)

So $t(m,n) = (n+m-1)\,t(m,n-1)$

Further, $t(m,1) = m$.

So
$$\begin{aligned} t(m,n) &= (n+m-1)\,t(m,n-1) \\ &= (n+m-1)(n+m-2)\,t(m,n-2) \\ &= \cdots \\ &= (n+m-1)(n+m-2)\cdots(m+1)\,t(m,1) \\ &= (m)_n \qquad \square \end{aligned}$$

5. Objects being placed in boxes are not labelled.

ex. $\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ E & E & D & L & H & E & H & L \end{pmatrix}$

$A = \{D, E, H, L, I\}$
- The word EEDLHEHL = DEEEHHLL
- These are called n-combinations with repetitions.
- Equivalent to # of Solutions of
  $x_1 + \cdots + x_m = n$, where $x_k \geq 0$, ($x_k \in \mathbb{N}_0$)
  $x_k \simeq$ # of objects in box k.

Thm: Let $m, n \in \mathbb{N}$. The # of nonnegative integral solutions of $x_1 + \cdots + x_m = n$ are:
$$\frac{(m)_n}{n!} = \binom{m+n-1}{n} \left(= \binom{m+n-1}{m-1} \right)$$

Proof: We count pairs (A, π) where A is a distribution of n unlabelled objects into m boxes and π is a permutation of I_n.
- Fix dist: Label objects and take permutation. $n!$ possible π : # pairs: $\sum_{A \in A} n! = \#A \cdot n!$
- fix permutation: # of distributions $= (m)_n$

So $\#A = \dfrac{(m)_n}{n!} = \dfrac{(m)(m+1)\cdots(m+n-1)}{n!} = \cdots$

6. unlabelled objects placed in boxes, each box must get one object
   $=$ # of solutions of $x_1 + \cdots + x_m = n$, but $x_k > 0$.

Def: (composition of n). A composition of $n \in \mathbb{N}$ is an ordered sum of positive integers. An m-composition of n is an ordered sum of n into m parts.

ex:
1 : 1
2 : 2 = 1+1 (1, 1)
3 : 3 = 2+1 = 1+2 = 1+1+1 (1, 2, 1)
4 : 4 = 3+1, 1+3, 2+2 = 2+1+1, 1+2+1, 1+1+2 = 1+1+1+1
(1, 3, 3, 1)

Thm: # of m-compositions of n $= \binom{n-1}{m-1}$.

Proof: We want the # of solutions of $x_1 + \cdots + x_m = n$, where now $x_i > 0$.
Let $y_i = x_i - 1$. So equivalently we want # of solutions of $y_1 + \cdots + y_m = n+m$, where $y_i \geq 0$.
This equals $\binom{n+m-m-1}{n+m} = \binom{n-1}{m-1}$ as reqd. $\square$

Ex: 1. Think of compositions as bars are boxes. obtain formula from this.
$\cdots | \cdots | \cdots |$
↳ $\# = x_k$

2. $\binom{n-1}{m-1} = \#$ of $m-1$ subsets of I_{n-1}.
Give 1-to-1 correspondence of compositions with subsets.

7. Extend $\binom{n}{k}$ = # of ways of putting I_n into 2 boxes / categories, with k in one box, $n-k$ in other
   - order does not matter, but both are labeled

Sp we have m boxes. $C_1, C_2, \dots, C_m$. Sp n elements are placed in $C_1, C_2, \dots, C_m$, $\# C_i = k_i$. k_i's add up to n, so $k_1 + \cdots + k_m = n$. This # is denoted
$$\binom{n}{k_1, \dots, k_m} \leftarrow \text{Multinomial coefficient.}$$

Note: $\binom{0}{0, \dots, 0} := 1$.

Thm. Let $n, k_1, \dots, k_m$ be positive integers, such that $k_1 + \cdots + k_m = n$. then
$$\binom{n}{k_1, \dots, k_m} = \frac{n!}{k_1! \cdots k_m!}.$$

Note: works for $k_i = 0$ too, but need special argument

Proof: Count pairs $(\pi, (S_1, \dots, S_m))$ where π is a permutation of n, S_1 = Set with 1st k_1 numbers of π, S_2 = next k_2 …
Fix π: there is one $(S_1, \dots, S_m)$: $\# = n!$
Fix $(S_1, \dots, S_m)$: there are $k_1!\,k_2! \cdots k_m!$ permutations.
So $\binom{n}{k_1, \dots, k_m} k_1! \cdots k_m! = n!$ $\square$