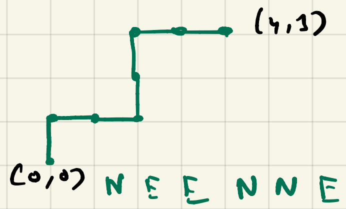
LECTURE 8 The non-commutative q-binomial theorem.

Today we will solve the following problem: Count the # of lattice paths made of North and East steps, starting from (0,0).
- n total steps
- k vertical / North steps



$n = 7$, $k = 3$ — path is from $(0,0)$ to $(n-k, k)$

(0,0) N E E N N E E

$\updownarrow$

1 0 0 1 1 0 0

$\updownarrow$

$\{1,4,5\}$

$\updownarrow$

y x x y y x x

a term in $(x+y)^7 = (x+y)\cdots(x+y)$

1-to-1 correspondence with

Binary strings of length n with exactly k 0's

$\updownarrow$

k subsets of $I_n = \{1,\dots,n\}$ i.e. subset of i = ith element in that is 1

string of y's and x

Let's say $(x+y)^n = \sum \binom{n}{k} x^{n-k} y^k$

$\binom{n}{k}$ = # of strings/paths/etc.

Read "n choose k".

Question: How to calculate $\binom{n}{k}$?

Base Cases:

$\binom{0}{0} := 1$ "empty path"

$\binom{n}{0} = 1$ $n = 1, 2, \dots$

$\binom{n}{k} = 0$ if $k > n$

Prop: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ for $k > 0$.

(Three term recurrence)

Proof: Take all paths of length n+1 with k horizontal steps. Put them in two subsets

last step is — : $\# = \binom{n}{k}$

last step is | : $\# = \binom{n}{k-1}$

so $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. □

Ex: Argue by induction that we can compute $\binom{n}{k}$ from these conditions.

From these we can create a table.

n \ k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7								

$\bigcirc \leftarrow$ 0's here.

$\leftarrow$ Ex.

Induction on? $n \leftarrow$ rows.

Ex (Class 11) Count k-permutations on I_n in two ways

Count pairs (A, B) $A \to$ k subset

$B \to$ permutation on k-letters.

A fixed: $\sum_{k\text{-subsets}} k! = \binom{n}{k}\cdot k!$

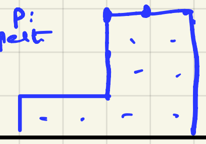
B fixed: $\sum_{\text{permutations}} 1 = P(n,k) = n(n-1)\cdots(n-k+1)$

So: $\binom{n}{k} = \dots$

We will 'refine' this result by giving a weight to paths.

Let $\mathcal{P}$ = paths from $(0,0)$ to $(n-k, k)$

P: path

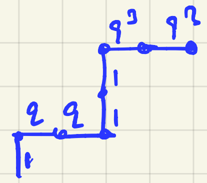


Area under path = 8

$W(P) = q^8$

Put a q in each box and multiply

OR



$w(e)$: $(n-1,k) \xrightarrow{q^k} (n,k)$

weight of an edge

put weight q^k on edge (E)

1 on N.

$W(P) = q\cdot q\cdot q^3\cdot q^3\cdot = q^8$

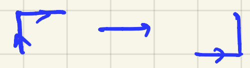
$W(P)$ = product of $w(e)$ for edges on path

We consider the GENERATING FUNCTION

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \sum_{P\in\mathcal{P}} q^{W(P)}$$

P is path from $(0,0)$ to $(n-k, k)$.

How to calculate? We invent an algebra which represents the weight calculation.

Note:  Reduces area by 1

$yx \longrightarrow xy$

So to compute weight, we use $yx = qxy$

q-commutes with x, y

ex: $y\,x\,x\,y\,y\,x\,x$ $\leftarrow$ 'word' representing path

$= q\,xy\,xyyxx$

$= q^2 x^2 y^2 x x$

$= q^2 x^2 (\cancel{x} y^3)\, q^3\, x$

$= q^{2+3}\, x^3\, q^3\, x y^3$

$= q^{2+3+3}\, x^4 y^3$ $= q^8\, x^4 y^3$.

Area under path

So we begin with

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^k$$

Examples

$(x+y)^0 := 1$

$(x+y)^1 = x+y$

$(x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2$

$= x^2 + (1+q)xy + y^2$

$(x+y)^3 = (x^2 + (1+q)xy + y^2)(x+y)$

$= x^3 + (1+q)xyx + y^2x +$

$x^2y + (1+q)xy^2 + y^3$ (Do now)

$= x^3 + (1+q+q^2)x^2y + (1+q+q^2)xy^2 + y^3$

observe:

— $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a poly in q

— when $q=1$, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$

— $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ — $\begin{bmatrix} n \\ n \end{bmatrix}_q = 1$

Thm. Let $\begin{bmatrix} n \\ k \end{bmatrix}_q$ be as above. Then

(a) $\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$

(b) $\phantom{\begin{bmatrix} n+1 \\ k \end{bmatrix}_q} = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$

Proof:

$(x+y)^{n+1} = (x+y)^n (x+y)$

$\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q x^{n+1-k} y^k = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^k (x+y)$

$= \sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k+1} y^k + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^{k+1}$

$k \le n$: Compare coeff of $x^{n-k+1} y^k$

for $\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$

$k = n+1$: $\begin{bmatrix} n+1 \\ n+1 \end{bmatrix} = 0 + \begin{bmatrix} n \\ n \end{bmatrix}$ ✓ works (Ex: Induction $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q := 1$)

(2) $(x+y)^{n+1} = (x+y)(x+y)^n = (x+y)\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^k$

$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n+1-k} y^k + \sum_{k=0}^{n} q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^{k+1}$

. . . Fill in details.

Ex. Combining (1) and (2) we get

$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{1-q^{n+1-k}}{1-q^k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q = \dots =$

$= \frac{(1-q)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)\cdot(1-q)\cdots(1-q^{n-k})} \cdot \begin{bmatrix} n \\ 0 \end{bmatrix}_q$ $\leftarrow = 1.$

Note: $q = 1$ $\frac{1-q^A}{1-q} = 1 + q + q^2 + \dots + q^{A-1}\Big|_{q=1} = A.$

$= [A]_q \leftarrow$ q-number.

(Schutzenberger - Non commutative binomial thm).

Thm. Let $qxy = xy$. Then $(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^k$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} 1 & \text{when.} \\ \dots & 0 < k \le n \end{cases}$