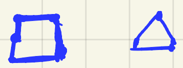
Lecture 15: GRAPH THEORY: BASIC THEOREMS

Recall: Graph $G = (V, E)$ V = Vertex set, E = edges

E is a collection of pairs of vertices. If e joins u and v then we say u and v are adjacent to each other.

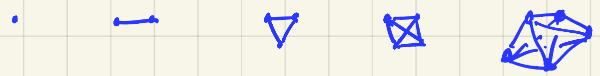
- A loop is an edge
- Parallel edges u v
- A simple graph: one with no loops or parallel edges.

Examples: ①



circuit/cycle

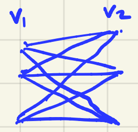
② K_n complete graph. Every pair is an edge.



Defns:
- A walk: a sequence of edges in a graph $v_0 \to v_1 \to v_2 \to \dots \to v_n$
- path: all vertices except possibly v_0 and v_n are distinct
- trail: all edges are distinct.
- cycle: path from $v_0 - v_1 - v_2 - \dots - v_n = v_0$.

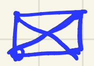
- A graph is connected if for every pair of vertices u, v, there is a path from u to v.
- A connected component is a maximal connected part/subgraph of a graph.
- A simple, connected graph with no cycles is called a tree.
- A bipartite graph is one where $V = V_1 \cup V_2$ (disjoint) and edges are only between V_1 and V_2.

eg $K_{3,3}$



complete bipartite graph.

- A graph is planar if it can be drawn on the plane without any edge crossing.

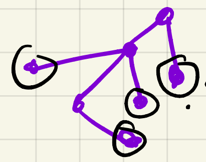


← Is this planar?

← can be drawn on plane/sphere too

Ex: A connected graph with n vertices is a tree iff it has $n-1$ edges.

Prop A tree with more than 1 vertex has to have at least 2 pendant vertices
↳ a vertex of degree 1.



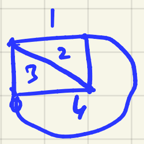
← pendant vertices.

Proof: Let # of vertices be n, where $n \geq 2$.
Any path in tree has to have length $\leq n$.
Let P: $v_1 \to \dots \to v_k$ be the longest path.
Claim: v_1 and v_k are pendant vertices. Sp v_1 is not pendant. Then there is a $v \in V$ s.t. $v - v_1$ is an edge.
But $v \notin \{v_2, v_3, \dots, v_k\}$, else there is a cycle in the tree.
So $v \to v_1 \to v_2 \to \dots \to v_k$ is a path. ⇒⇐ to P being longest.
Similarly, v_k has to be pendant too.

Prop: # of edges in a tree with n vertices is $n-1$.
Proof: Ex: Prove by induction on n.

Thm (Euler) Let G be a connected planar graph.
Let $v = \#V$ = # of vertices
$e = \#E$ = ... edges
$f = \#F$ = # of faces (regions of plane)
Then $v - e + f = 2$.

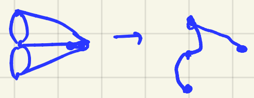
Ex K_4



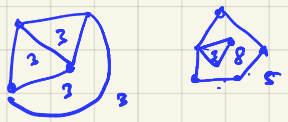
$v = 4$ $e = 6$
$f = 4$
$v - e + f = 4 - 6 + 4 = 2$ ✓.

Proof: If there is a cycle, we can remove one edge. One face will also reduce, so $v-e+f$ will not change. Continue until all cycles are removed. Then $f = 1$ and we obtain a tree. So $e = v - 1$.
and $v - e + f = v - v + 1 + 1 = 2$. □

Ex.



Defn: Let the degree of a region = # of edges bordering that region.



Thm: Sum of degrees of regions $= 2|E|$.
(dual of Hand shaking lemma)
Pf: Consider 'dual' graph. Put vertex in each face, and edge if the faces share edge.

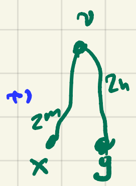
Thm: A graph is bipartite iff it contains no cycle of odd length.
Proof: ⇒ Claim: If a graph is bipartite, then all its cycles are of even length.

Proof of claim: Sp $u_1 - u_2 - \dots u_k - u_{k+1} = u_1$ is a cycle of odd length
with $\{u_1, u_3, \dots\} \in X$
$\{u_2, u_4, \dots\} \in Y$, where $V = X \cup Y$ is a bipartition ($X \cap Y = \emptyset$).
length of cycle is k. If k is odd, $u_{k+1} \in Y$. But $u_{k+1} = u_1 \in X$. ⇒⇐.

⇐ Sp G contains no odd cycles. We show how to color its vertices B and R. Choose any vertex $v \in G$ and define:

$B = \{u \in V \mid$ shortest path from v to u is of even length$\}$
$R = \{u \in V \mid$ " " odd length$\}$.

All vertices are here because G is connected.
Note: $v \in B$. We have to check there is no edge xy s.t. $x, y \in B$ or $x, y \in R$.
Sp $x - y$ is an edge, with $x, y \in B$. Let:
$P(v,x)$ = shortest path from v to x length $= 2m$
$P(v,y)$ = " " v to y length $= 2n$
Since $v - x - y$ is a path from v to y
$\Rightarrow 2n \leq 2m+1$. Similarly $2m \leq 2n+1$
$\Rightarrow \boxed{n = m}$



Now let w be the last vertex in common in the 2 paths.

The # edges in path w to x and w to y are same (say k).
Then the cycle $x - w - y - x$ has length $2k+1$, ⇒⇐ hypothesis.

Similarly for $x, y \in R$ we can show its not poss (Ex).
□.

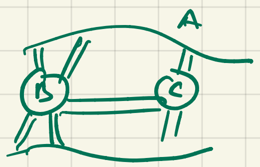
Cor: Every tree is bipartite (vacuously true).

Ex. Let T be a simple graph with p vertices.
TFAE
(1) T is a tree
(2) T has $p-1$ edges and no cycle
(3) T has $p-1$ edges and is connected.

Def: Forest: A graph with every connected component a tree. (Disconnected), locally.

Def. An Eulerian circuit is a closed trail/~~walk~~ which contains every edge of graph.

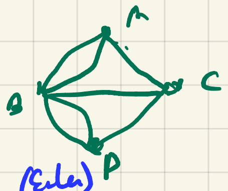
Konigsberg Bridge Problem



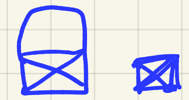
Traverse all seven bridges and return to starting point without going over any bridge twice.

$\Leftrightarrow$

Draw graph without lifting pen from paper.



(Euler)

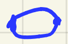


Thm: Let G be a connected graph. Then G is Eulerian (ie has an Eulerian circuit) iff every vertex has even degree.

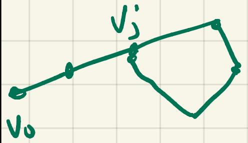
We first prove a useful proposition.

Prop: Let G be a graph in which every vertex has an even degree. Then the edge set of G is a disjoint union of cycles.

Proof: (by induction) e = # of edges

$e = 2$  ✓

Sp G has k edges and sp prop is true for all graphs with $e < k$. Take any vertex V_0 and take a walk from V_0, continuing until a vertex is visited a second time, say V_j. Then walk from V_j to V_j is a cycle C.



Delete C from the graph to get a graph H with $< k$ edges, where every vertex is of even degree. By induction it must be an edge disjoint union of edges. $\square$.

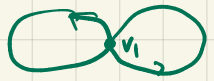
IDEA.

← cannot happen.

Proof of Thm. $\Rightarrow$ Obvious (if you enter a vertex you have to exit it).

$\Leftarrow$ Sp every vertex is of even degree. By prop, G is an edge disjoint union of cycles. Take any one cycle, say C_1. If all edges of G are in C_1, then we are done. Otherwise there is a vertex v_1 in C_1 and $v_2 \in G \setminus C_1$ s.t. $v_1 - v_2$ is an edge. Now $v_1 - v_2$ is in some cycle C_2.

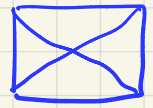
If $G = C_1 \cup C_2$, we are done.



Else there is a w_1 on C_2 and $w_2 \notin (C_1 \cup C_2)$ s.t. $w_1 w_2$ is an edge. Continue until Graph is traversed.

Remark: ① Can start with any vertex

② If exactly 2 vertices v_1 and v_2 have an odd degree then place an edge between them to get a walk. Begin from v_1 and traverse and delete the last edge from v_2 to v_1 to get a walk from v_1 to v_2.



← impossible.