



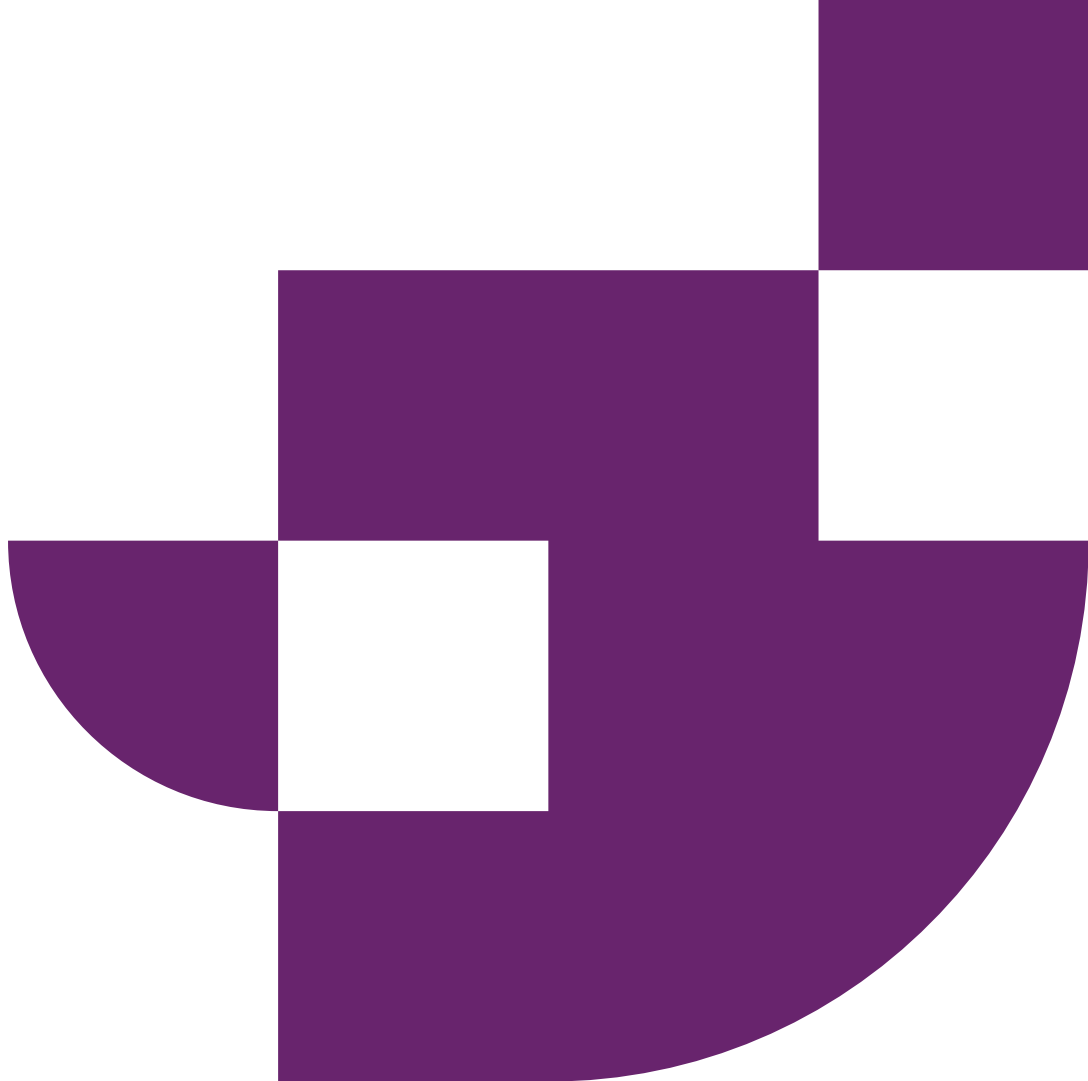
Durham
University

Introduction to Music Computing

Models of Tonal Space

Dr Robert Lieck

robert.lieck@durham.ac.uk



Tonnetz

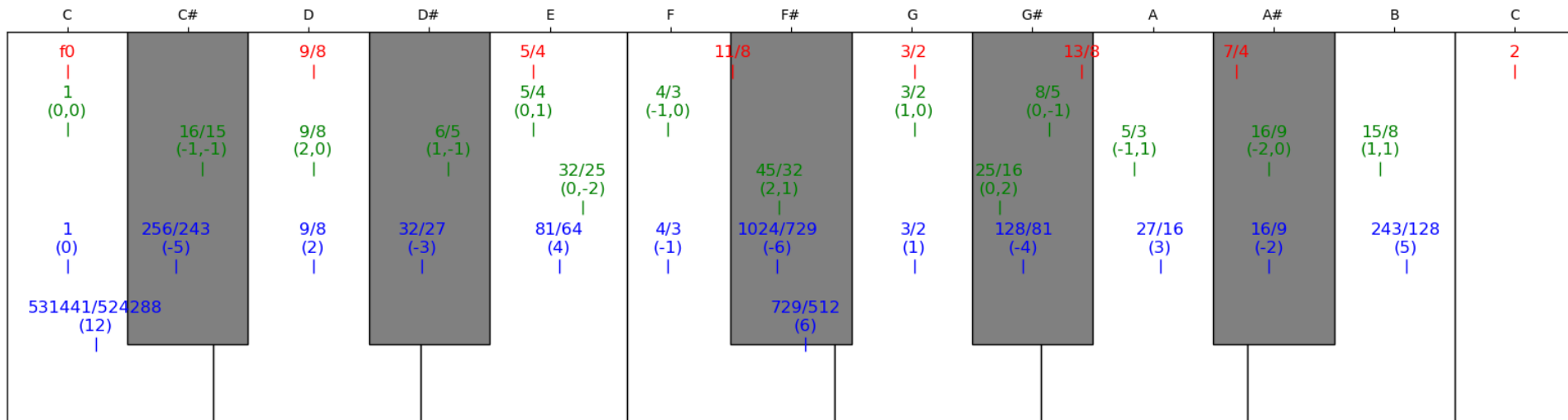
Tuning Systems

Equal Temperament

Overtone

Just Intonation: (Fifths, Thirds)

Pythagorean Tuning: (Fifths)



Just Intonation

- Tones are tuned in
 - **perfect fifths**: third harmonic / factor 3
 - **major thirds**: fifth harmonic / factor 5

above the fundamental.

- Each note can be specified by a triplet of integers

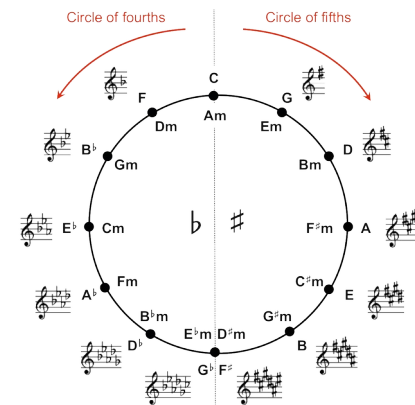
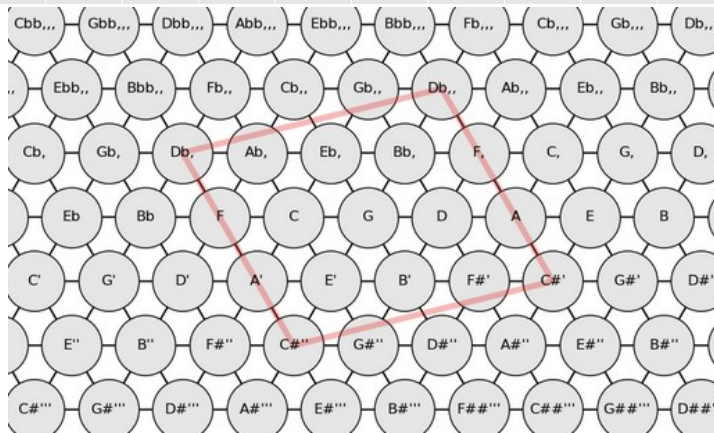
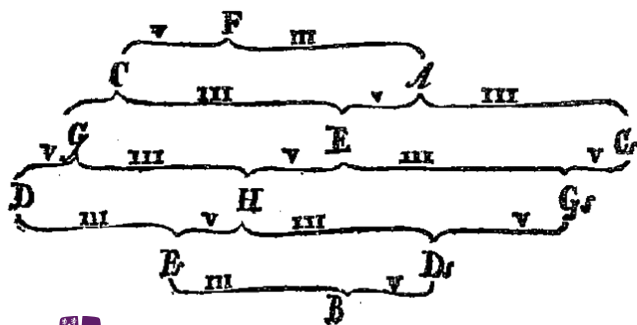
$$(\text{octaves, fifths, thirds}) \rightarrow f_0 \cdot 2^{\text{octaves}} \cdot 3^{\text{fifths}} \cdot 5^{\text{thirds}}$$

specifying the number of steps of the respective intervals.

- In many cases, we ignore the octave and only consider the **pitch class**.

Just Intonation → Tonnetz

	Line of Fifths																							
Major Thirds	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
-2	Ebbb,,	Bbbb,,	Fbb,,	Cbb,,	Gbb,,	Dbb,,	Abb,,	Ebb,,	Bbb,,	Fb,,	Cb,,	Gb,,	Db,,	Ab,,	Eb,,	Bb,,	F,,	C,,	G,,	D,,	A,,	E,,	B,,	
-1	Gbb,	Dbb,	Abb,	Ebb,	Bbb,	Fb,	Cb,	Gb,	Db,	Ab,	Eb,	Bb,	F,	C,	G,	D,	A,	E,	B,	F#,	C#,	G#,	D#,	
0	Bbb	Fb	Cb	Gb	Db	Ab	Eb	Bb	F	C	G	D	A	E	B	F#	C#	G#	D#	A#	E#	B#	F##	
1	Db'	Ab'	Eb'	Bb'	F'	C'	G'	D'	A'	E'	B'	F#'	C#'	G#'	D#'	A#'	E#'	B#'	F##'	C##'	G##'	D##'	A##'	
2	F''	C''	G''	D''	A''	E''	B''	F#''	C#''	G#''	D#''	A#''	E#''	B#''	F##''	C##''	G##''	D##''	A##''	E##''	B##''	F###''	C###''	
3	A'''	E'''	B'''	F#'''	C#'''	G#'''	D#'''	A#'''	E#'''	B#'''	F##'''	C##'''	G##'''	D##'''	A##'''	E##'''	B##'''	F###'''	C###'''	G###'''	D###'''	A###'''	E###'''	

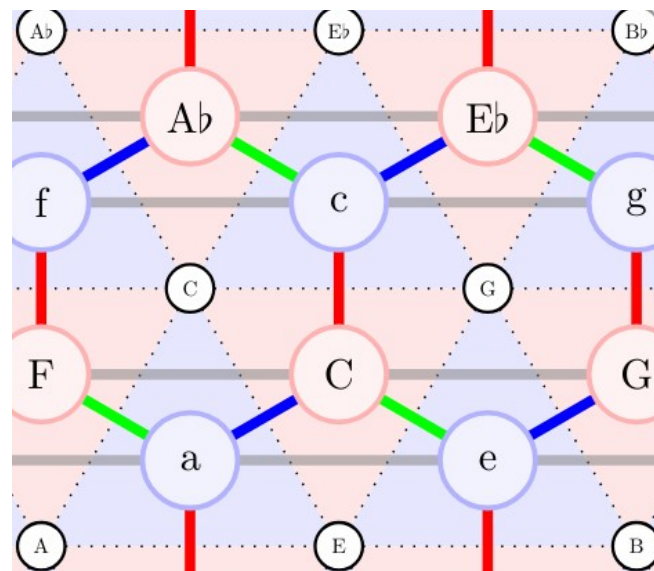


Just Intonation → Tonnetz

- Diatonic Scales
 - segments of length 7 along the line of fifths
 - different tonics possible (church modes: Ionian/major, Dorian, Phrygian, Lydian, Mixolydian, Aeolian/minor, Locrian)
- Spelled Pitch / Western Pitch Notation
 - basic names alphabetic according to Aeolian/minor
 - when shifting, cycle basic names and add \sharp/\flat
- Pitches in just intonation are compact region around fundamental
 - but different “unit cells” containing all 12 enharmonically equivalent pitches are possible
- Grid can be made more compact by going from rectangular to triangular
 - two possibilities, but minor third is more consonant than minor second

Tonnetz

- Two versions (equivalent/dual)
 - **pitch-based:** pitches as hexagonal faces (triads at corners)
 - **triad-based:** triads as triangular faces (pitches at corners)
- Neo-Riemannian operations with minimal voice-leading
 - Relative major/minor
C major (C, E, G) \leftrightarrow a minor (A, C, E)
 - Parallel major/minor
C major (C, E, G) \leftrightarrow c minor (C, Eb, G)
 - Leading-Tone Exchange
C major (C, E, G) \leftrightarrow e minor (E, G, B)

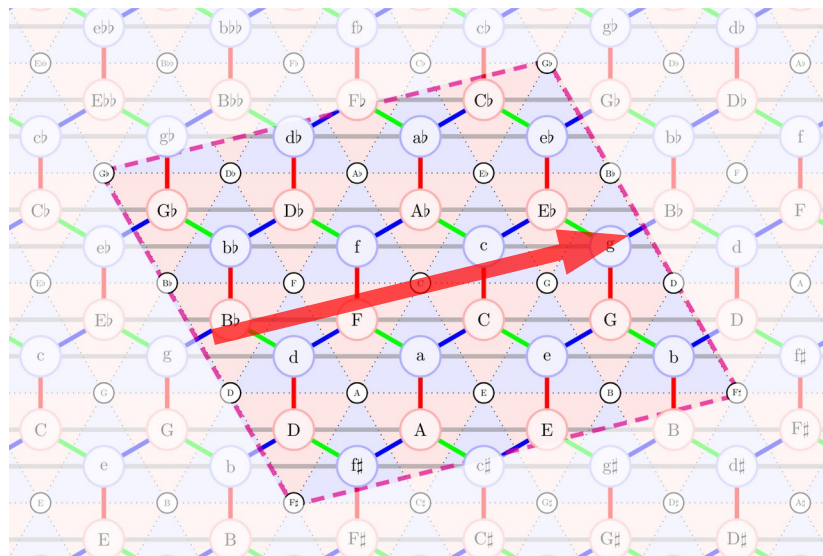


Pythagorean Tuning

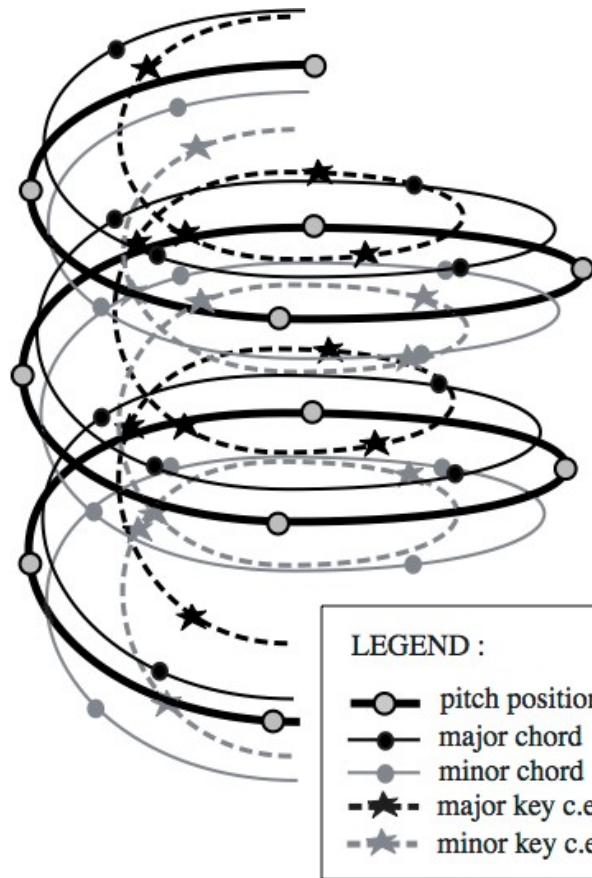
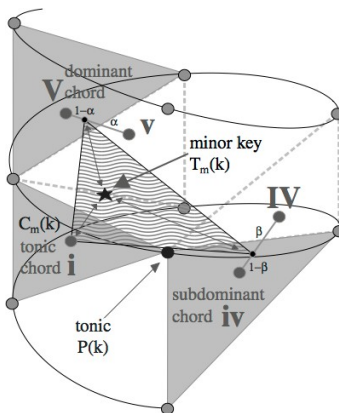
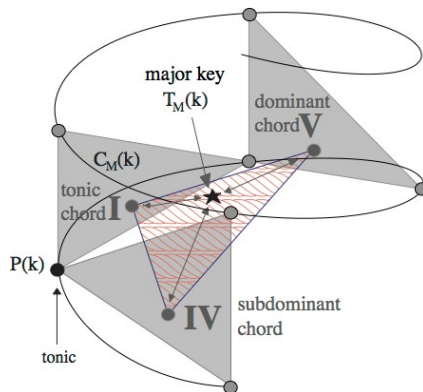
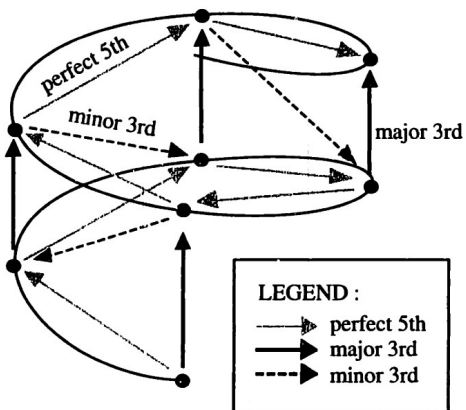
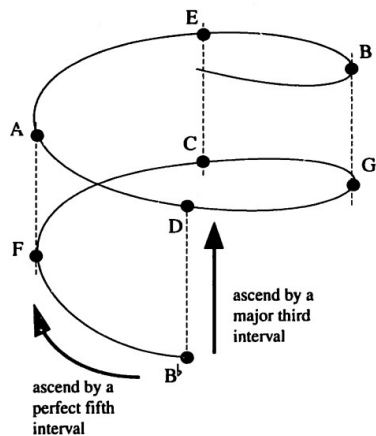
- Tones are tuned in **perfect fifths** above/below the fundamental.
- Each note can be specified by a pair of integers

$$(\text{octaves}, \text{fifths}) \rightarrow f_0 \cdot 2^{\text{octaves}} \cdot 3^{\text{fifths}}$$

- **Pitch classes** are defined by the number of steps along the line of fifths.
- **Ignore syntonic comma**
→ roll up to match pitches that differ by syntonic comma



Spiral Array (Chew 2000)



Spiral Array (Chew 2000)

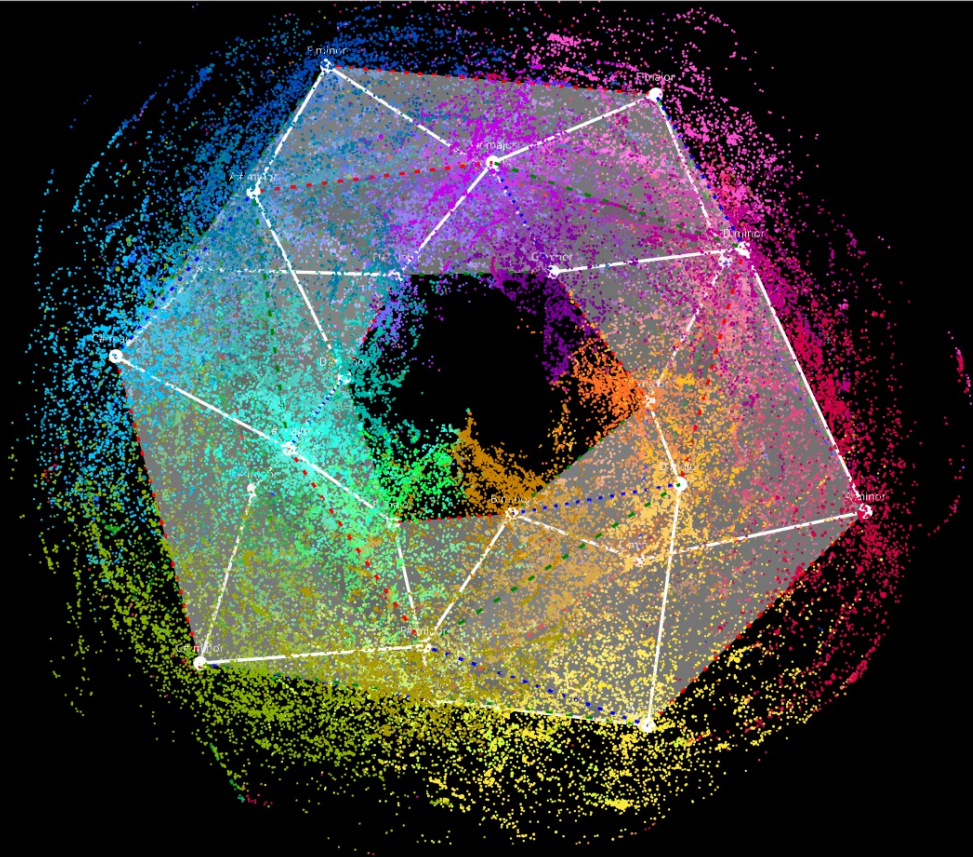
- Line of fifths is a spiral around the tube's axis.
- Pitch classes along the axis differ by a major third.
- Spiral of major/minor triads as means of the involved pitch classes.
- Spiral of major/minor keys as means of tonic, sub-dominant, and dominant chords:
 - major: (I, IV, V)
 - minor: (i, iv/IV, v/V) with parameters α and β interpolating between v/V and iv/IV, respectively

Equal Temperament

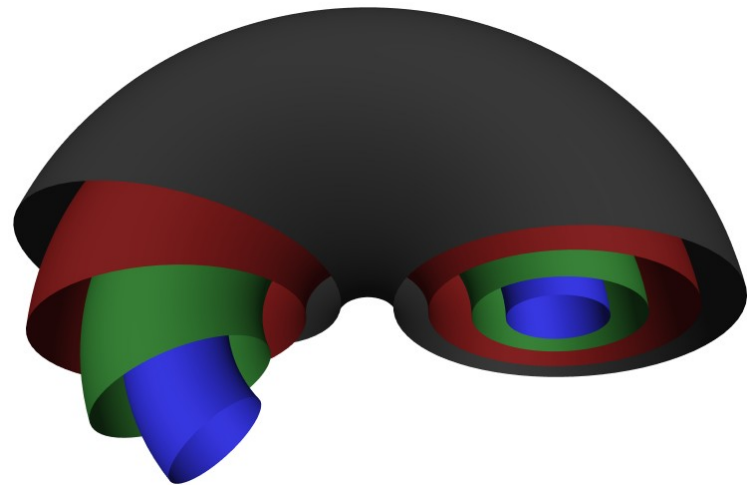
- **Ignore Pythagorean comma**
→ roll up to match pitches that differ by Pythagorean comma
- This gives us a toroidal shape:



Torus of Pitch-Class Distributions



- pitch classes
- triads
- keys
- key signatures

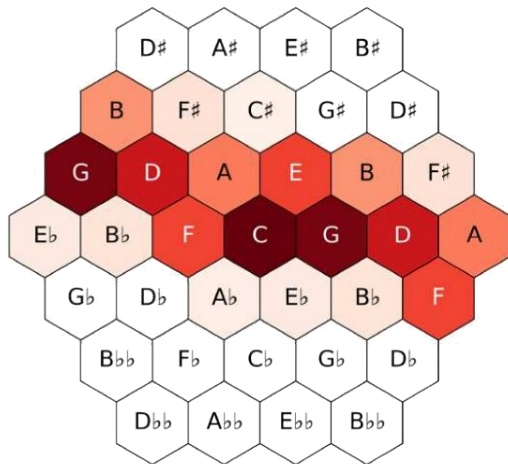


Tonal Diffusion Model

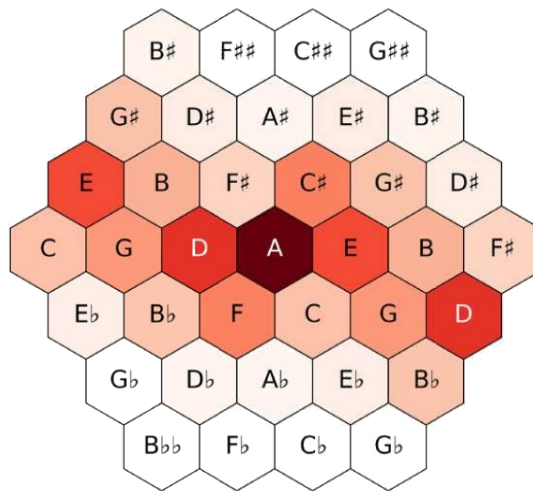
(Lieck et al. 2020, Lieck et al. 2024)

Tonal Diffusion Model

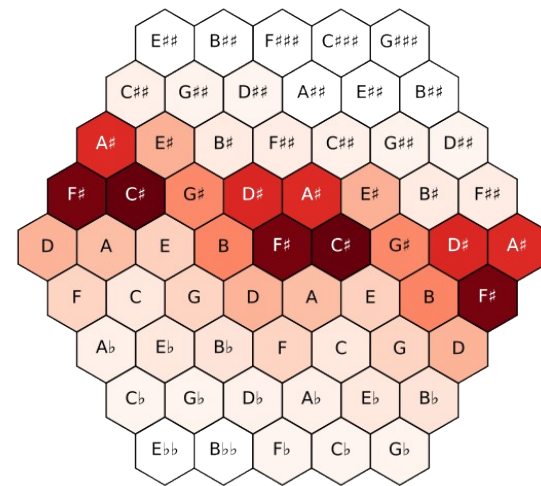
- We can use the Tonnetz to visualise (tonal/spelled) pitch class distributions
- Most pieces look like they have a **centre** from which probability **diffuses**



(a) Johann Sebastian Bach, Prelude in C major, BWV 846 (1722, Baroque).



(b) Ludwig van Beethoven, Sonata op. 31, no. 2 in D minor 'Tempest', 1st mov. (1802, Classical).

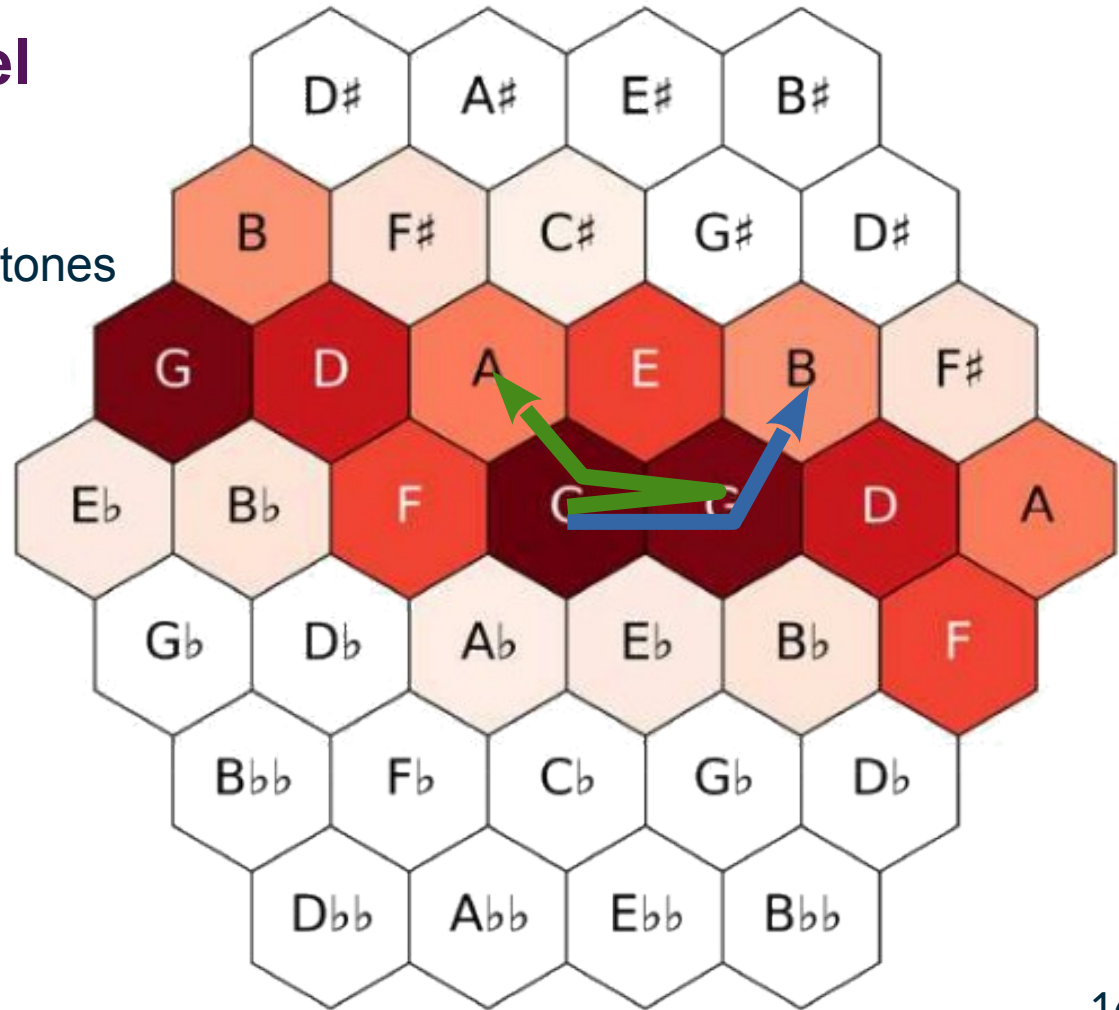


(c) Franz Liszt, *Bénédiction de Dieu dans la Solitude*, S. 173/3 (1853, Romantic).

Tonal Diffusion Model

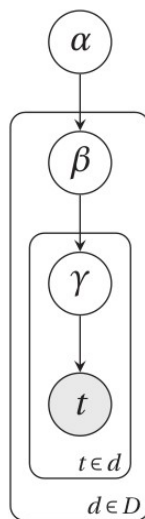
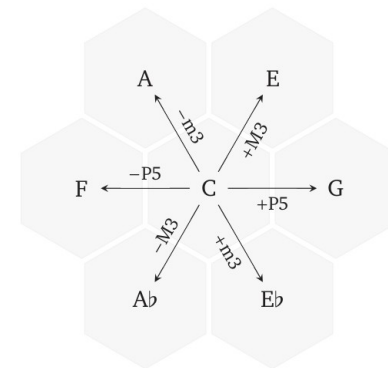
Basic Idea

- look at global distribution (“tones in a bag”)
- tones are generated by starting at the centre and taking a number of steps on the Tonnetz
- this generative process defines (with specific parameters) a distribution on the Tonnetz



Tonal Diffusion Model – Topic Models

- Parameters/variables on different levels
→ corpus (α), piece (β), event (γ)
 - learn: corpus/piece parameters (α/β)
 - marginalise out: event variables (γ)



corpus-level parameters (α)

$H_c, \alpha_c, H_w, \alpha_w, h_\lambda$

piece-level variables (β)

c : distribution over tonal centers

w : distribution over intervals

λ : distribution over path lengths

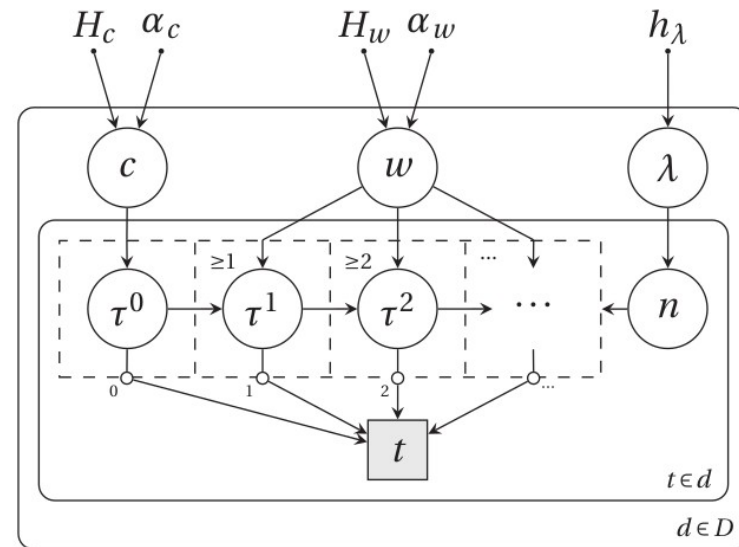
latent variables (γ)

τ^i : intermediate tones

n : number of steps

observed variables

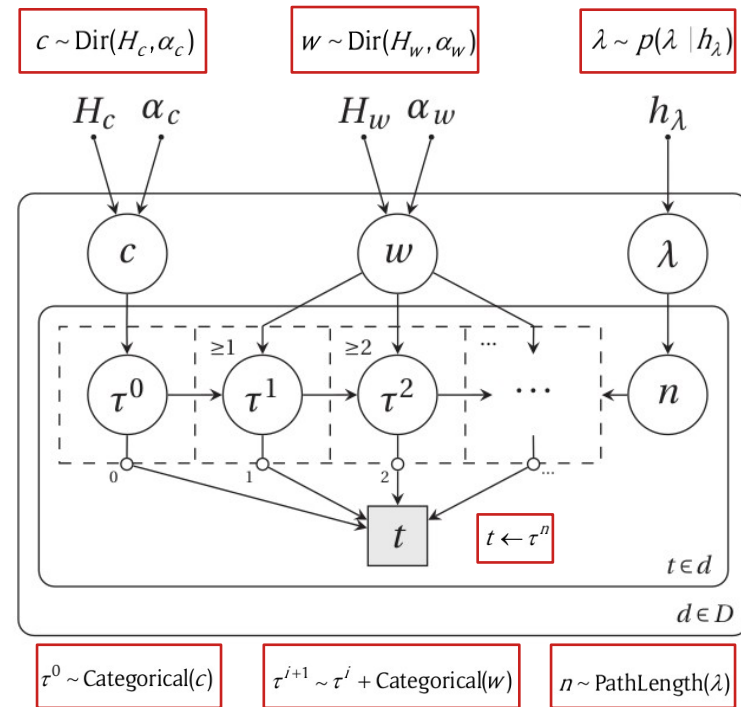
t : tones in a piece



Tonal Diffusion Model

Compute pitch-class distribution as follows

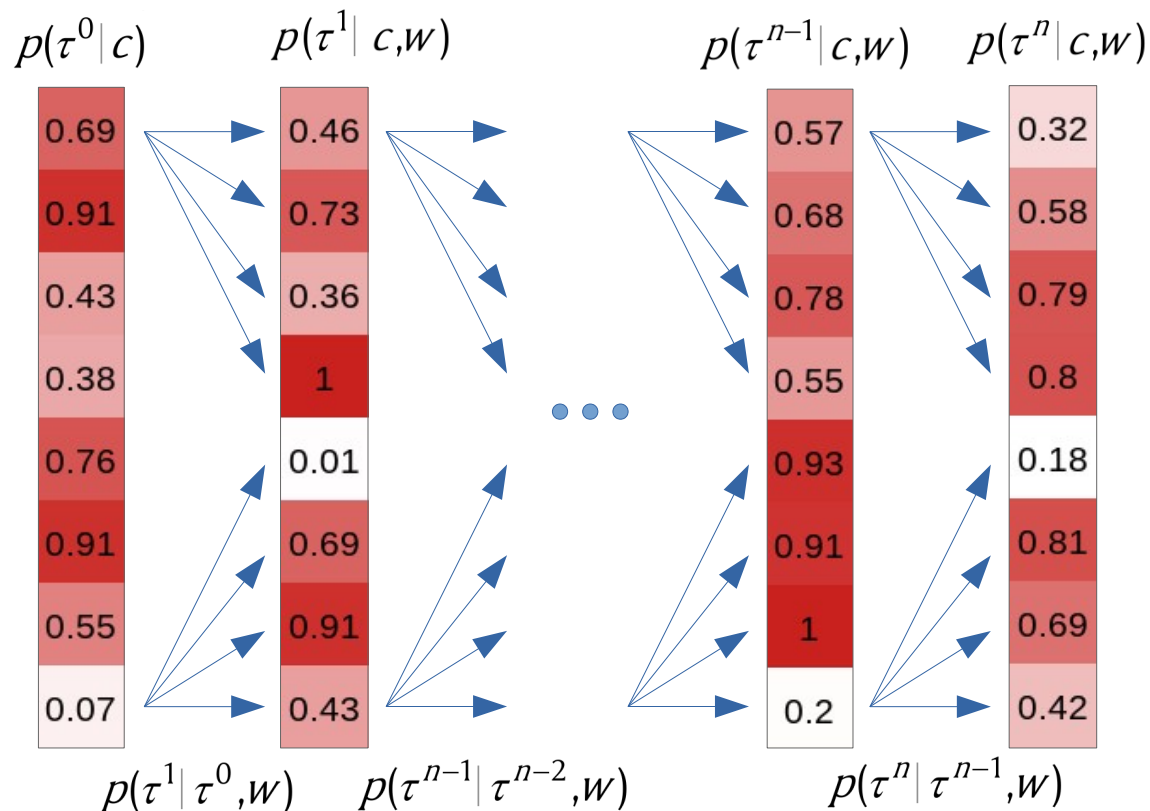
- for fixed parameters c, w, γ
 - get initial distribution over τ^0
- iteratively compute distribution for k^{th} step
 - start at τ^0
 - compute transition probabilities using w
 - iterate using dynamic programming
- simultaneously sum over n .



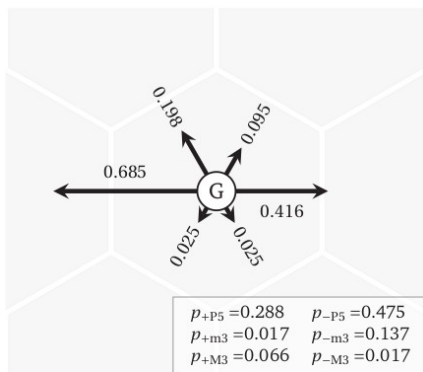
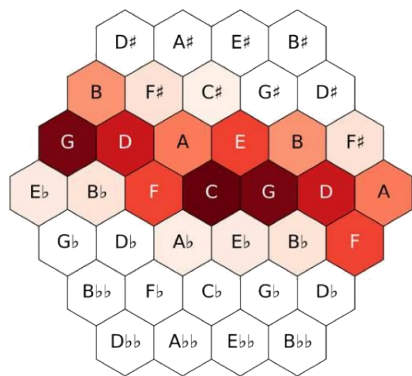
→ Implement in PyTorch and backprop to optimise parameters

$$p(t | c, w, \lambda) = \sum_{n=0}^{\infty} p(n | \lambda) \underbrace{\sum_{\tau^{n-1}} p(\tau^n | \tau^{n-1}, w) \sum_{\tau^{n-2}} p(\tau^{n-1} | \tau^{n-2}, w) \cdots \sum_{\tau^0} p(\tau^1 | \tau^0, w) p(\tau^0 | c)}_{p(\tau^n | c, w)} p(\tau^n | c, w)$$

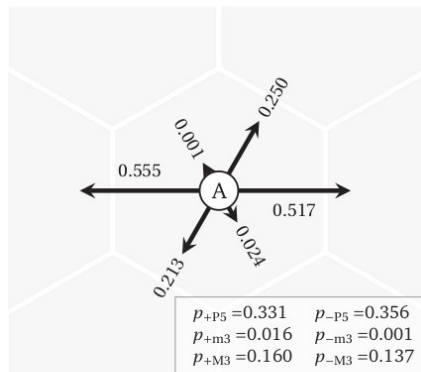
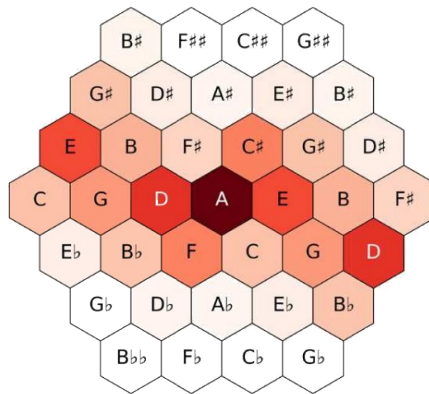
Tonal Diffusion Model



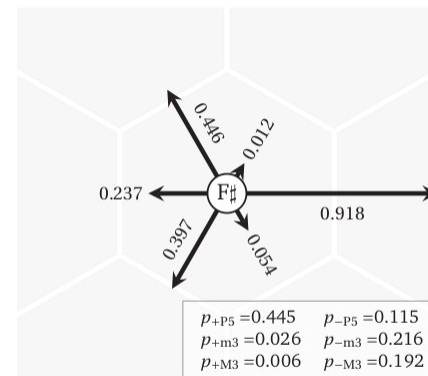
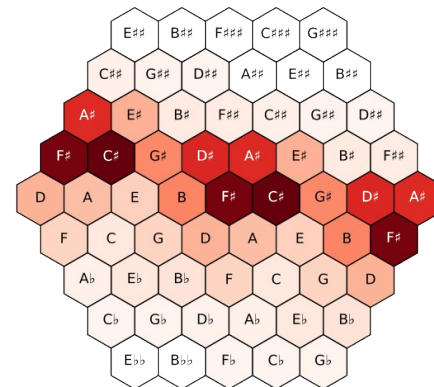
Tonal Diffusion Model



(a) Bach, C major Prelude
($\mu = 1.44, \sigma = 0.69$).



(b) Beethoven, 'Tempest' Sonata
($\mu = 1.56, \sigma = 0.76$).

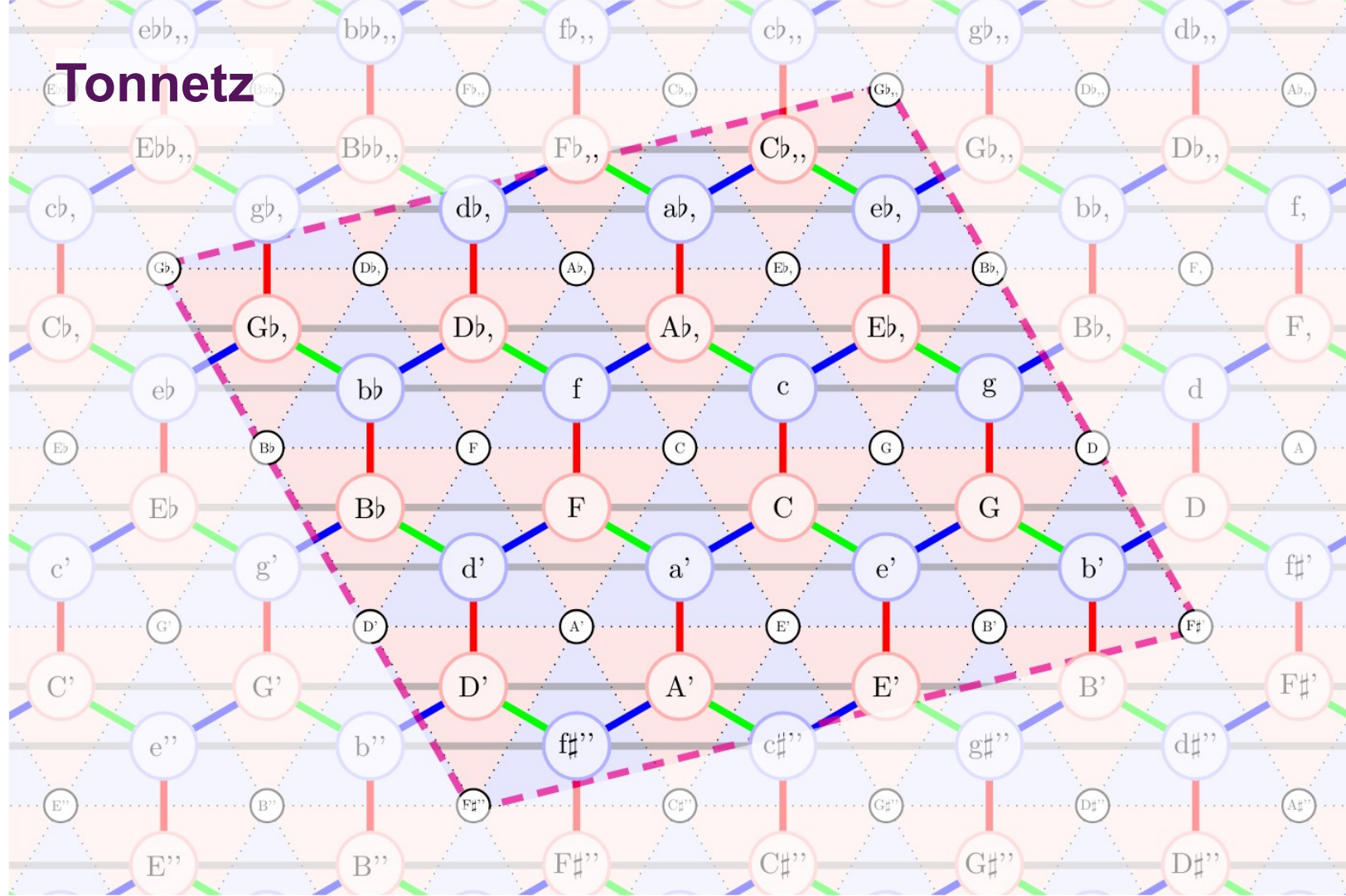


(c) Liszt, *Bénédiction de Dieu dans la Solitude* ($\mu = 2.06, \sigma = 1.25$).

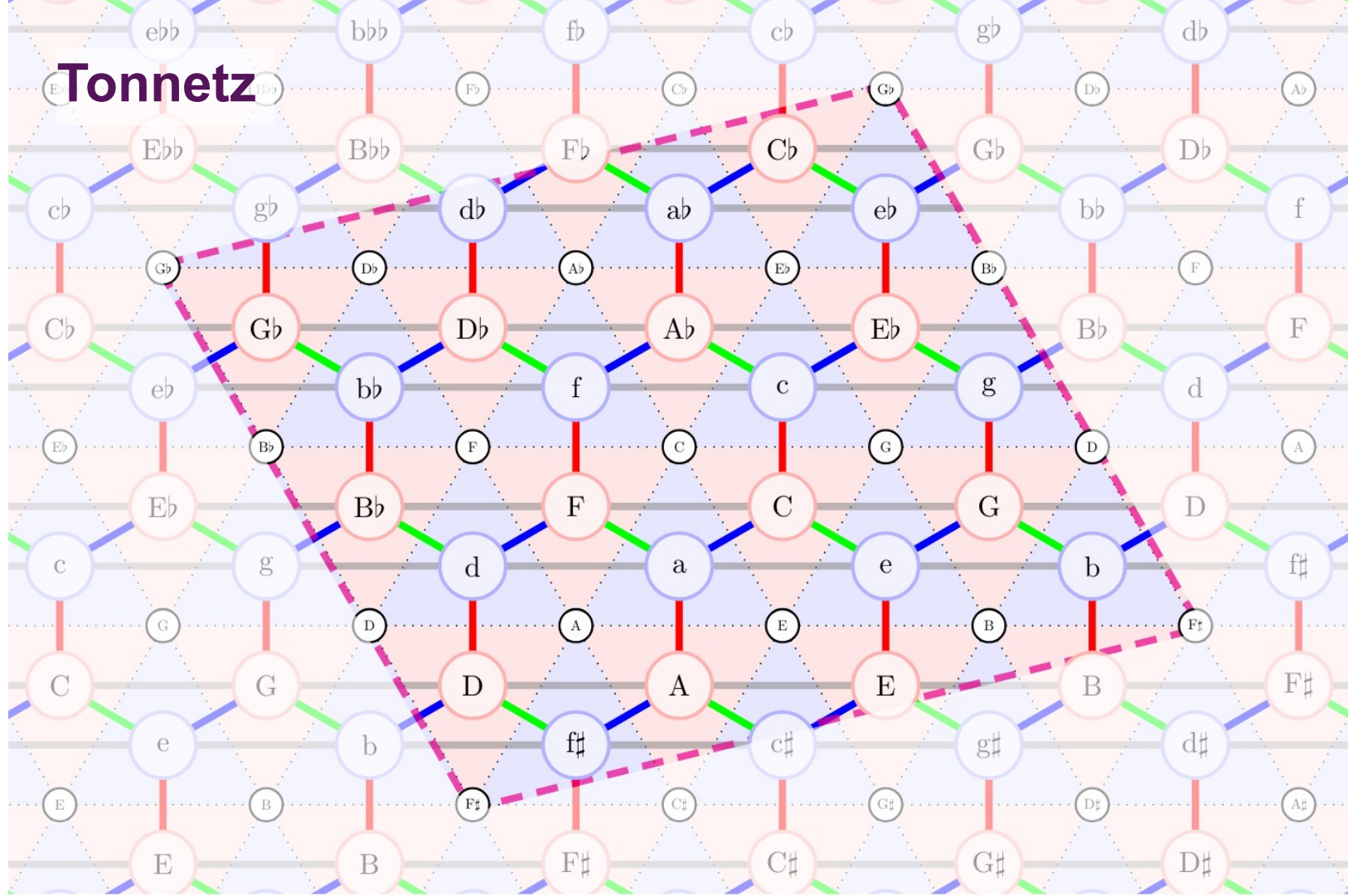
References

- Euler L (1739) Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae. Ex Typographia Academiae Scientiarum, St. Petersburg
- Chew E (2000) Towards a mathematical model of tonality. PhD Thesis, Massachusetts Institute of Technology
- Blei DM, Ng AY, Jordan MI (2003) Latent dirichlet allocation. Journal of machine Learning research 3:993–1022
- Lieck R, Moss FC, Rohrmeier M (2020) The Tonal Diffusion Model. Transactions of the International Society for Music Information Retrieval 3:153–164. <https://doi.org/10.5334/tismir.46>
- Moss FC, Lieck R, Rohrmeier M (2024) Computational modeling of interval distributions in tonal space reveals paradigmatic stylistic changes in Western music history. Humanit Soc Sci Commun 11:1–11. <https://doi.org/10.1057/s41599-024-03168-1>

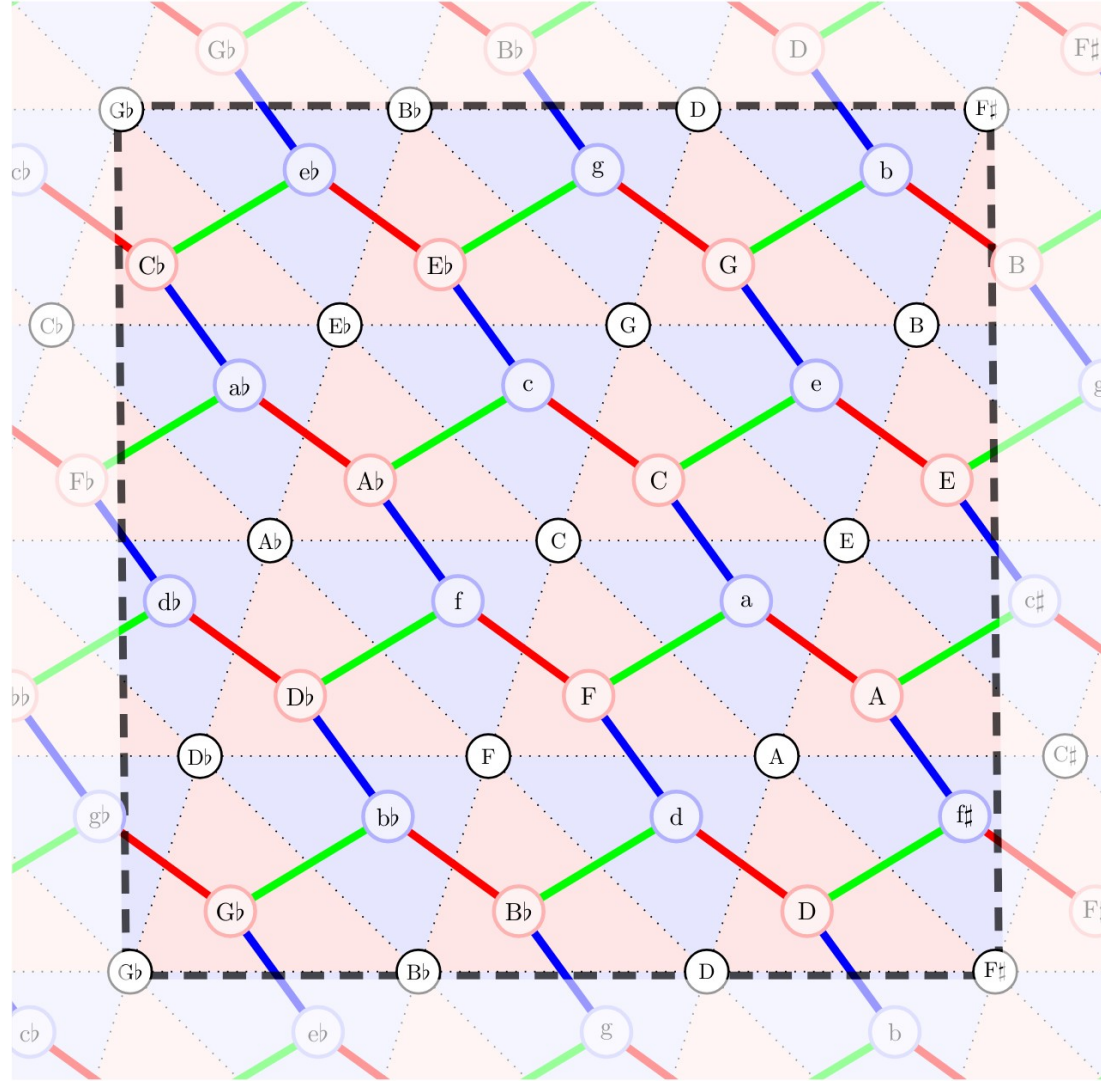
Tonnetz



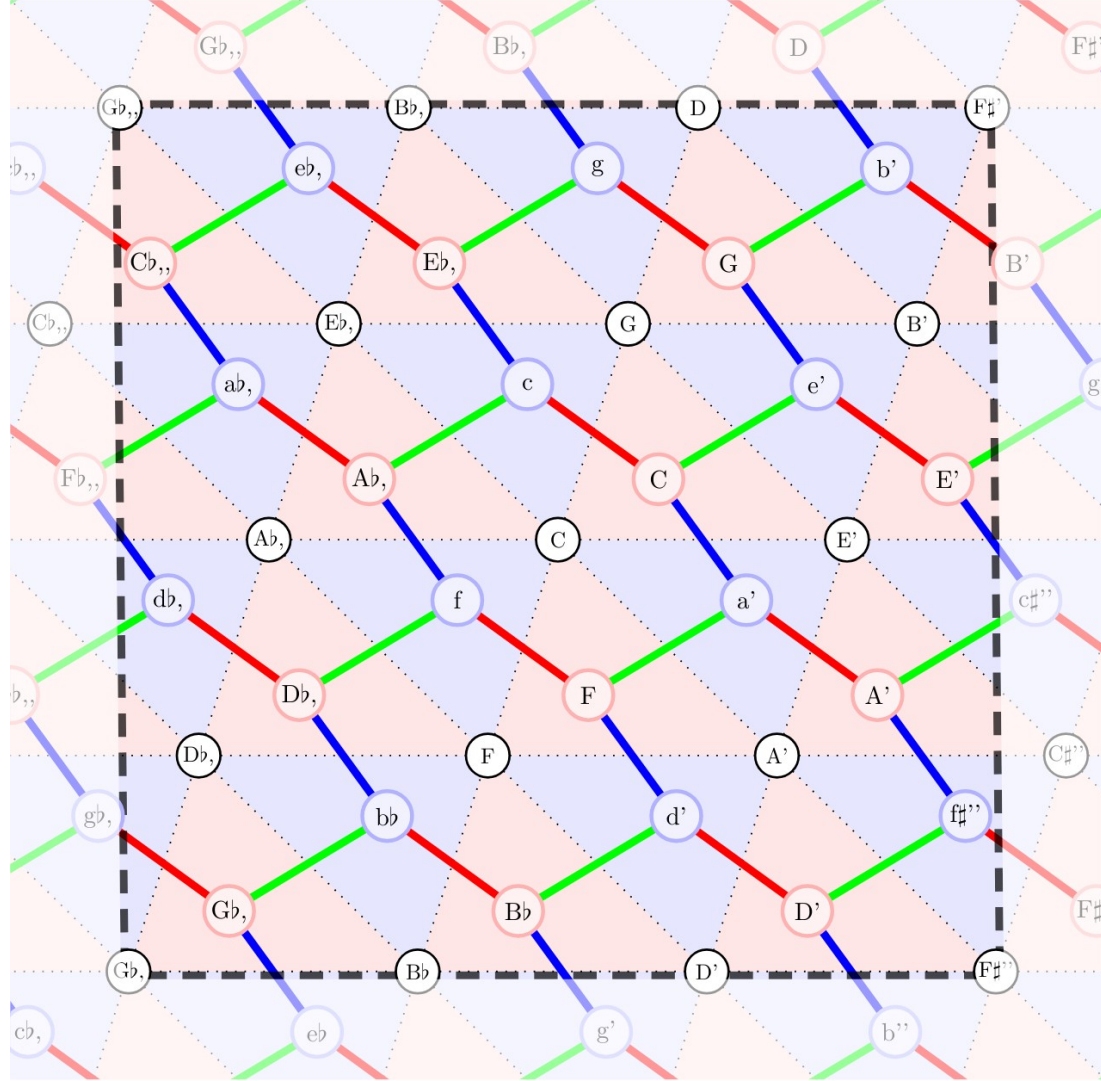
Tonnetz



Tonnetz



Tonnetz



Tonnetz

