The purpose of this report is to show how to obtain an expression for F, the amplitude of sinusoidal forcing, such that a given threshold steady-state amplitude of a Hopf oscillator is achieved in a system of two oscillators. In this system, the forcing is given to a frequency-scaled linear oscillator which is connected with unilateral coupling to a frequency-scaled Hopf oscillator. The linear oscillator is intended to represent a place on the basilar membrane (BM), and the Hopf oscillator is intended to represent an amplifying bundle of outer hair cells on the organ of Corti (OC). This expression for F will be obtained as a function of the oscillator and input parameters in the following system of two oscillators:

$$\tau \dot{z}_{bm} = z_{bm}\mu_{bm} + Fe^{\mathrm{i}2\pi f_0 t} \tag{1}$$
$$\tau \dot{z}_{oc} = z_{oc}(\mu_{oc} + \xi|z_{oc}|^2) + Az_{bm} \tag{2}$$

where

$$\begin{aligned}
\tau &\in \mathbb{R} = 1/f \\
f &\in \mathbb{R} \text{ is the natural frequency of the oscillators} \\
z_{bm} &\in \mathbb{C} \text{ is the state of the BM oscillator} \\
z_{oc} &\in \mathbb{C} \text{ is the state of the OC oscillator} \\
\{\mu_{bm}, \mu_{oc}, \xi\} &\in \mathbb{C} \text{ are parameters of the oscillators} \\
f_0 &\in \mathbb{R} \text{ is the frequency of the stimulus} \\
A &\in \mathbb{R} \text{ is the connectivity coefficient from BM to OC} \\
F &\in \mathbb{R} \text{ is the forcing amplitude of the stimulus}
\end{aligned}$$

To prepare for a polar transformation of equations (1) and (2), we can separate the real and imaginary parts of the parameters thus:

$$\tau \dot{z}_{bm} = z_{bm}(\alpha_{bm} + \mathrm{i}2\pi) + Fe^{\mathrm{i}2\pi f_0 t} \tag{3}$$
$$\tau \dot{z}_{oc} = z_{oc}(\alpha_{oc} + \mathrm{i}2\pi + (\beta + \mathrm{i}\delta)|z_{oc}|^2) + Az_{bm} \tag{4}$$

with $\{\alpha_{bm}, \alpha_{oc}, \beta, \delta\} \in \mathbb{R}$. We will now begin obtaining an expression for the steady-state amplitude r^*_{bm} of the linear oscillator z_{bm}. This expression will enable us to solve for F in the z_{oc} equation. We begin with the fact that $z = re^{\mathrm{i}\phi}$ where $\{r, \phi\} \in \mathbb{R}$ are amplitude and phase, respectively. Using the product rule for differentiation,

$$\dot{z} = \dot{r}e^{\mathrm{i}\phi} + r\mathrm{i}\dot{\phi}e^{\mathrm{i}\phi} \tag{5}$$

so that

$$e^{\mathrm{i}\phi_{bm}}(\dot{r}_{bm} + \mathrm{i}r_{bm}\dot{\phi}_{bm}) = \{r_{bm}e^{\mathrm{i}\phi_{bm}}(\alpha_{bm} + \mathrm{i}2\pi) + Fe^{\mathrm{i}2\pi f_0 t}\}f$$

Simplifying,

$$\dot{r}_{bm} + \mathrm{i}r_{bm}\dot{\phi}_{bm} = \{r_{bm}(\alpha_{bm} + \mathrm{i}2\pi) + Fe^{\mathrm{i}(2\pi f_0 t - \phi_{bm})}\}f$$

Separating the real and imaginary parts to transform to polar coordinates, and using Euler's formula $e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$, we have

$$\begin{aligned}
\dot{r}_{bm} &= f\alpha_{bm}r_{bm} + fF\cos(2\pi f_0 t - \phi_{bm}) \\
\dot{\phi}_{bm} &= 2\pi f + \frac{fF}{r_{bm}}\sin(2\pi f_0 t - \phi_{bm})
\end{aligned}$$

We can define a phase difference $\psi_{bm} = \phi_{bm} - 2\pi f_0 t$ so that

$$\dot{\psi}_{bm} = \dot{\phi}_{bm} - 2\pi f_0 = 2\pi f - 2\pi f_0 + \frac{fF}{r_{bm}}\sin(-\psi) \tag{6}$$

We can then define $\Omega = 2\pi(f - f_0)$, and use the properties $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$ to get our revised system of equations:

$$\dot{r}_{bm} = f\alpha_{bm}r_{bm} + fF\cos(\psi_{bm}) \tag{7}$$
$$\dot{\psi}_{bm} = \Omega - \frac{fF}{r_{bm}}\sin(\psi_{bm}) \tag{8}$$

With r^*_{bm} the steady-state amplitude of the BM oscillator and ψ^*_{bm} the steady-state phase difference between the BM oscillator and the input sinusoid, we have the following steady-state equations:

$$0 = \alpha_{bm}r^*_{bm} + F\cos(\psi^*_{bm}) \tag{9}$$
$$0 = \Omega - \frac{fF}{r^*_{bm}}\sin(\psi^*_{bm}) \tag{10}$$

We must now find a way to eliminate ψ^*_{bm} if we want to solve for r^*_{bm} independently. Using the property $\sin^2(x) + \cos^2(x) = 1$, we know that

$$\cos(\psi^*_{bm}) = \sqrt{1 - \sin^2(\psi^*_{bm})} \tag{11}$$

as long as $\cos(\psi^*_{bm}) \geq 0$. We know also from equation (10) that

$$\sin\psi^*_{bm} = \frac{r^*_{bm}\Omega}{fF} \tag{12}$$

Plugging in the equality from (12) into (11) and plugging that equality into (9) we have an identity independent from ψ^*_{bm}:

$$0 = \alpha_{bm}r^*_{bm} + F\sqrt{1 - \left(\frac{r^*_{bm}\Omega}{fF}\right)^2}$$

Rearranging terms and squaring both sides,

$$\left(\frac{\alpha_{bm}r^*_{bm}}{F}\right)^2 = 1 - \left(\frac{r^*_{bm}\Omega}{fF}\right)^2$$

It is now apparent that we can solve for r^*_{bm}. Skipping some intermediate steps, we reach an expression with only one r^*_{bm}:

$$\alpha^2_{bm} = \frac{F^2}{r^{*2}_{bm}} - \frac{\Omega^2}{f^2}$$

Rearranging terms, we reach a formula for r^*_{bm}:

$$\frac{F}{\sqrt{\alpha^2_{bm} + \left(\frac{\Omega}{f}\right)^2}} = r^*_{bm} \tag{13}$$

We now turn back to equation (4) for the Hopf oscillator receiving input from z_{bm}. Again using (5), (4) becomes

$$e^{\mathrm{i}\phi_{oc}}(\dot{r}_{oc} + \mathrm{i}r\dot{\phi}_{oc}) = \{r_{oc}e^{\mathrm{i}\phi_{oc}}(\alpha_{oc} + \mathrm{i}2\pi + (\beta + \mathrm{i}\delta)|r_{oc}e^{\mathrm{i}\phi_{oc}}|^2) + Ar_{bm}e^{\mathrm{i}\phi_{bm}}\}f$$

Simplifying,

$$\dot{r}_{oc} + \mathrm{i}r_{oc}\dot{\phi}_{oc} = fr_{oc}\alpha_{oc} + \mathrm{i}fr_{oc}2\pi + f\beta r^3_{oc} + \mathrm{i}f\delta r^3_{oc} + fAr_{bm}e^{\mathrm{i}(\phi_{bm} - \phi_{oc})}$$

Separating the real and imaginary parts and again using Euler's formula,

$$\dot{r}_{oc} = f r_{oc}\alpha_{oc} + f\beta r_{oc}^3 + fAr_{bm}\cos(\phi_{bm} - \phi_{oc}) \tag{14}$$

$$\dot{\phi}_{oc} = 2\pi f + f\delta r_{oc}^2 + \frac{fAr_{bm}}{r_{oc}}\sin(\phi_{bm} - \phi_{oc}) \tag{15}$$

We then again define a phase difference, this time between the two oscillators, $\psi_{oc} = \phi_{oc} - \phi_{bm}$, so that $\dot{\psi}_{oc} = \dot{\phi}_{oc} - \dot{\phi}_{bm}$. By equation (6) we know then that

$$\dot{\psi}_{oc} = 2\pi f + f\delta r_{oc}^2 - \frac{fAr_{bm}}{r_{oc}}\sin(\psi_{oc}) - 2\pi f_0$$

and we note the presence of $\Omega = 2\pi(f - f_0)$ here as well, so our revised system of equations is

$$\dot{r}_{oc} = f r_{oc}\alpha_{oc} + f\beta r_{oc}^3 + fAr_{bm}\cos(\psi_{oc})$$

$$\dot{\psi}_{oc} = \Omega + f\delta r_{oc}^2 - \frac{fAr_{bm}}{r_{oc}}\sin(\psi_{oc})$$

Thus our steady-state equations are

$$0 = r_{oc}^*\alpha_{oc} + \beta r_{oc}^{*3} + Ar_{bm}^*\cos(\psi_{oc}^*) \tag{16}$$

$$0 = \frac{\Omega}{f} + \delta r_{oc}^{*2} - \frac{Ar_{bm}^*}{r_{oc}^*}\sin(\psi_{oc}^*) \tag{17}$$

If we rearrange (17) and again utilize the property $\sin^2(x) + \cos^2(x) = 1$, we get

$$\cos\psi_{oc}^* = \sqrt{1 - \left(\frac{r_{oc}^*\left(\frac{\Omega}{f}\right) + \delta r_{oc}^{*3}}{Ar_{bm}^*}\right)^2}$$

and our new single steady-state equation, having eliminated ψ_{oc}^*, is

$$0 = r_{oc}^*\alpha_{oc} + \beta r_{oc}^{*3} + Ar_{bm}^*\sqrt{1 - \left(\frac{r_{oc}^*\left(\frac{\Omega}{f}\right) + \delta r_{oc}^{*3}}{Ar_{bm}^*}\right)^2}$$

We will move step by step from here to the end, to be explicit. We have our formula (13) for r_{bm}^*, but first we simplify a bit. We move the first two terms to the other side and then square both sides,

$$r_{oc}^{*2}\alpha_{oc}^2 + 2r_{oc}^{*4}\alpha_{oc}\beta + r_{oc}^{*6}\beta^2 = A^2 r_{bm}^{*2}\left(1 - \frac{\left(r_{oc}^*\left(\frac{\Omega}{f}\right) + \delta r_{oc}^{*3}\right)^2}{A^2 r_{bm}^{*2}}\right)$$

multiply $A^2 r_{bm}^{*2}$ into the parentheses, and cancel factors:

$$r_{oc}^{*2}\alpha_{oc}^2 + 2r_{oc}^{*4}\alpha_{oc}\beta + r_{oc}^{*6}\beta^2 = A^2 r_{bm}^{*2} - \left(r_{oc}^*\left(\frac{\Omega}{f}\right) + \delta r_{oc}^{*3}\right)^2$$

At this point we plug in the formula for r_{bm}^* from (13) to solve for F:

$$r_{oc}^{*2}\alpha_{oc}^2 + 2r_{oc}^{*4}\alpha_{oc}\beta + r_{oc}^{*6}\beta^2 = \frac{A^2F^2}{\left(\alpha_{bm}^2 + \left(\frac{\Omega}{f}\right)^2\right)} - \left(r_{oc}^*\left(\frac{\Omega}{f}\right) + \delta r_{oc}^{*3}\right)^2$$

Adding the last term to both sides and expanding it,

$$r_{oc}^{*2}\alpha_{oc}^2 + 2r_{oc}^{*4}\alpha_{oc}\beta + r_{oc}^{*6}\beta^2 + \frac{\Omega^2 r_{oc}^{*2}}{f^2} + \frac{2\Omega\delta r_{oc}^{*4}}{f} + \delta^2 r_{oc}^{*6} = \frac{A^2F^2}{\left(\alpha_{bm}^2 + \left(\frac{\Omega}{f}\right)^2\right)}$$

Solving for F, we then obtain the lengthy expression

$$\frac{\sqrt{\left(\alpha_{bm}^2 + \left(\frac{\Omega}{f}\right)^2\right)\left(r_{oc}^{*2}\alpha_{oc}^2 + 2r_{oc}^{*4}\alpha_{oc}\beta + r_{oc}^{*6}\beta^2 + \frac{\Omega^2 r_{oc}^{*2}}{f^2} + \frac{2\Omega\delta r_{oc}^{*4}}{f} + \delta^2 r_{oc}^{*6}\right)}}{A} = F$$

Simplifying slightly by removing an r_{oc}^* from the radical and grouping terms, we obtain a final solution for F:

$$\frac{r_{oc}^*\sqrt{\alpha_{bm}^2 + \left(\frac{\Omega}{f}\right)^2}\sqrt{(\beta^2+\delta^2)r_{oc}^{*4} + 2\left(\delta\left(\frac{\Omega}{f}\right) + \alpha_{oc}\beta\right)r_{oc}^{*2} + \alpha_{oc}^2 + \left(\frac{\Omega}{f}\right)^2}}{A} = F \tag{18}$$