Connection Types in GrFNN Toolbox

For notational simplicity, the zdot and cdot equations for internal connections are given here but the same equations can be used for a connection between two different networks. The frequency-scaled versions of the equations (for log-frequency networks) are shown here.

1freq:

$$\frac{1}{f_i}\dot{z}_i = z_i\left(\alpha + \mathrm{i}2\pi + (\beta_1 + \mathrm{i}\delta_1)\,|z_i|^2 + \frac{\epsilon\,(\beta_2 + \mathrm{i}\delta_2)\,|z_i|^4}{1-\epsilon|z_i|^2}\right) + \sum_{j\neq i} c_{ij} z_j$$

$$\frac{2}{f_i + f_j}\dot{c}_{ij} = c_{ij}\left(\lambda_1 + \mu_1|c_{ij}|^2 + \frac{\epsilon_c\mu_2|c_{ij}|^4}{1-\epsilon_c|c_{ij}|^2}\right) + \kappa z_i \bar{z}_j$$

2freq:

$$\frac{1}{f_i}\dot{z}_i = z_i\left(\alpha + \mathrm{i}2\pi + (\beta_1 + \mathrm{i}\delta_1)\,|z_i|^2 + \frac{\epsilon\,(\beta_2 + \mathrm{i}\delta_2)\,|z_i|^4}{1-\epsilon|z_i|^2}\right) + \sum_{j\neq i} \epsilon^{\frac{n_{ij}+d_{ij}-2}{2}} c_{ij} z_j^{n_{ij}} \bar{z}_i^{d_{ij}-1}$$

$$\frac{n_{ij} + d_{ij}}{n_{ij} f_j + d_{ij} f_i}\dot{c}_{ij} = c_{ij}\left(\lambda_1 + \mu_1|c_{ij}|^2 + \frac{\epsilon_c\mu_2|c_{ij}|^4}{1-\epsilon_c|c_{ij}|^2}\right) + \epsilon_c^{\frac{n_{ij}+d_{ij}-2}{2}} \kappa z_i^{d_{ij}} \bar{z}_j^{n_{ij}}$$

where $n_{ij} f_j \approx d_{ij} f_i$ (n_{ij} and d_{ij} are positive integers).

3freq:

$$\begin{aligned}\frac{1}{f_i}\dot{z}_i &= z_i\left(\alpha + \mathrm{i}2\pi + (\beta_1 + \mathrm{i}\delta_1)\,|z_i|^2 + \frac{\epsilon\,(\beta_2 + \mathrm{i}\delta_2)\,|z_i|^4}{1-\epsilon|z_i|^2}\right) \\ &\quad + \sum_{j} \epsilon^{\frac{|n_{ij1}|+|n_{ij2}|+d_{ij}-2}{2}} c_{ij} y_{ij1}^{|n_{ij1}|} y_{ij2}^{|n_{ij2}|} \bar{z}_i^{d_{ij}-1}\end{aligned}$$

$$\begin{aligned}\frac{|n_{ij1}| + |n_{ij2}| + d_{ij}}{|n_{ij1}| f_{ij1} + |n_{ij2}| f_{ij2} + d_{ij} f_i}\dot{c}_{ij} &= c_{ij}\left(\lambda_1 + \mu_1|c_{ij}|^2 + \frac{\epsilon_c\mu_2|c_{ij}|^4}{1-\epsilon_c|c_{ij}|^2}\right) \\ &\quad + \epsilon_c^{\frac{|n_{ij1}|+|n_{ij2}|+d_{ij}-2}{2}} \kappa z_i^{d_{ij}} \bar{y}_{ij1}^{|n_{ij1}|} \bar{y}_{ij2}^{|n_{ij2}|}\end{aligned}$$

where $n_{ij1} f_{ij1} + n_{ij2} f_{ij2} \approx d_{ij} f_i$ (n_{ij1} and n_{ij2} are positive or negative integers and d_{ij} is a positive integer). y_{ijk} is z_{ijk} (if $n_{ijk} > 0$) or $\bar{z}_{ijk}$ (if $n_{ijk} < 0$) where $k = 1, 2$. Here the subscript j indexes significant three-frequency resonant relationships existing among oscillator frequencies.

All2freq:

$$\begin{aligned}\frac{1}{f_i}\dot{z}_i &= z_i\left(\alpha + \mathrm{i}2\pi + (\beta_1 + \mathrm{i}\delta_1)\,|z_i|^2 + \frac{\epsilon\,(\beta_2 + \mathrm{i}\delta_2)\,|z_i|^4}{1-\epsilon|z_i|^2}\right) \\ &\quad + \sum_{j\neq i} c_{ij} \frac{z_j}{1-\sqrt{\epsilon} z_j} \cdot \frac{1}{1-\sqrt{\epsilon}\bar{z}_i}\end{aligned}$$

$$
\begin{aligned}
\frac{f_i+f_j}{2f_if_j}\dot{c}_{ij} = c_{ij}&\left(\lambda_1+\mu_1|c_{ij}|^2+\frac{\epsilon_c\mu_2|c_{ij}|^4}{1-\epsilon_c|c_{ij}|^2}\right)\\
&+\kappa\frac{z_i}{1-\sqrt{\epsilon_c}z_i}\cdot\frac{\bar{z}_j}{1-\sqrt{\epsilon_c}\bar{z}_j}
\end{aligned}
$$

The scaling factor in the cdot equation is an approximation based on that for two-frequency resonance. If we want to remove 1:1 and subsequent $n:n$ monomials from the cdot equation (i.e., when no11 $=1$), we use instead

$$
\begin{aligned}
\frac{f_i+f_j}{2f_if_j}\dot{c}_{ij} = c_{ij}&\left(\lambda_1+\mu_1|c_{ij}|^2+\frac{\epsilon_c\mu_2|c_{ij}|^4}{1-\epsilon_c|c_{ij}|^2}\right)\\
&+\kappa\left(\frac{z_i}{1-\sqrt{\epsilon_c}z_i}\cdot\frac{\bar{z}_j}{1-\sqrt{\epsilon_c}\bar{z}_j}-\frac{z_i\bar{z}_j}{1-\epsilon_c z_i\bar{z}_j}\right).
\end{aligned}
$$

Note that the ratio for the geometric series that is subtracted is ϵ_c, not $\sqrt{\epsilon_c}$, since $z_i\bar{z}_j+\epsilon_c z_i^2\bar{z}_j^2+\epsilon_c^2 z_i^3\bar{z}_j^3+\cdots$ needs to be subtracted from $(z_i+\sqrt{\epsilon_c}z_i^2+\epsilon_c z_i^3+\cdots)\cdot(\bar{z}_j+\sqrt{\epsilon_c}\bar{z}_j^2+\epsilon_c\bar{z}_j^3+\cdots)$.

Allfreq:

$$
\begin{aligned}
\frac{1}{f_i}\dot{z}_i = z_i&\left(\alpha+\mathrm{i}2\pi+(\beta_1+\mathrm{i}\delta_1)|z_i|^2+\frac{\epsilon(\beta_2+\mathrm{i}\delta_2)|z_i|^4}{1-\epsilon|z_i|^2}\right)\\
&+\sum_{j\neq i}c_{ij}\frac{z_j}{1-\sqrt{\epsilon}z_j}\cdot\frac{1}{1-\sqrt{\epsilon}\bar{z}_j}\cdot\frac{1}{1-\sqrt{\epsilon}\bar{z}_i}
\end{aligned}
$$

$$
\begin{aligned}
\frac{f_i+f_j}{2f_if_j}\dot{c}_{ij} = c_{ij}&\left(\lambda_1+\mu_1|c_{ij}|^2+\frac{\epsilon_c\mu_2|c_{ij}|^4}{1-\epsilon_c|c_{ij}|^2}\right)\\
&+\kappa\frac{z_i}{1-\sqrt{\epsilon_c}z_i}\cdot\frac{\bar{z}_j}{1-\sqrt{\epsilon_c}\bar{z}_j}\cdot\frac{1}{1-\sqrt{\epsilon_c}z_j}
\end{aligned}
$$

When no11 $=1$, use

$$
\begin{aligned}
\frac{f_i+f_j}{2f_if_j}\dot{c}_{ij} = c_{ij}&\left(\lambda_1+\mu_1|c_{ij}|^2+\frac{\epsilon_c\mu_2|c_{ij}|^4}{1-\epsilon_c|c_{ij}|^2}\right)\\
&+\kappa\left(\frac{z_i}{1-\sqrt{\epsilon_c}z_i}\cdot\frac{\bar{z}_j}{1-\sqrt{\epsilon_c}\bar{z}_j}\cdot\frac{1}{1-\sqrt{\epsilon_c}z_j}-\frac{z_i\bar{z}_j}{1-\epsilon_c z_i\bar{z}_j}\cdot\frac{1}{1-\epsilon_c|z_j|^2}\right)
\end{aligned}
$$

because now $z_i(\bar{z}_j+\epsilon_c|z_j|^2\bar{z}_j+\epsilon_c^2|z_j|^4\bar{z}_j+\cdots)+\sqrt{\epsilon_c}z_i^2(\sqrt{\epsilon_c}\bar{z}_j^2+\epsilon_c\sqrt{\epsilon_c}|z_j|^2\bar{z}_j^2+\epsilon_c^2\sqrt{\epsilon_c}|z_j|^4\bar{z}_j^2+\cdots)+\cdots=(z_i\bar{z}_j+\epsilon_c z_i^2\bar{z}_j^2+\cdots)\cdot(1+\epsilon_c|z_j|^2+\epsilon_c^2|z_j|^4+\cdots)$ needs to be subtracted.