

Musical Informatics: Music Alignment



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Outline

- Music Alignment
 - Applications and Examples
 - Music Representation and Features
 - Dynamic Time Warping
- Automatic Accompaniment demo

Music Alignment

Same music, different interpretations

- Different versions of the same piece/song that correspond to the same musical work!



Symbolic Recordings of Piano Performances

- Late 19th Century: Player Pianos!
 - Pianolas and piano rolls
 - Pneumatic mechanism
 - Music recorded on perforated paper: Piano-rolls
- 21st Century: Computer Controlled Grand Pianos
 - MIDI and derived formats
 - Yamaha, Bösendorfer, Steinway!
 - Competitions
 - Datasets



A player piano roll being played by Draconichiaro - Own work, CC BY-SA 4.0, <https://commons.wikimedia.org/w/index.php?curid=82604752>

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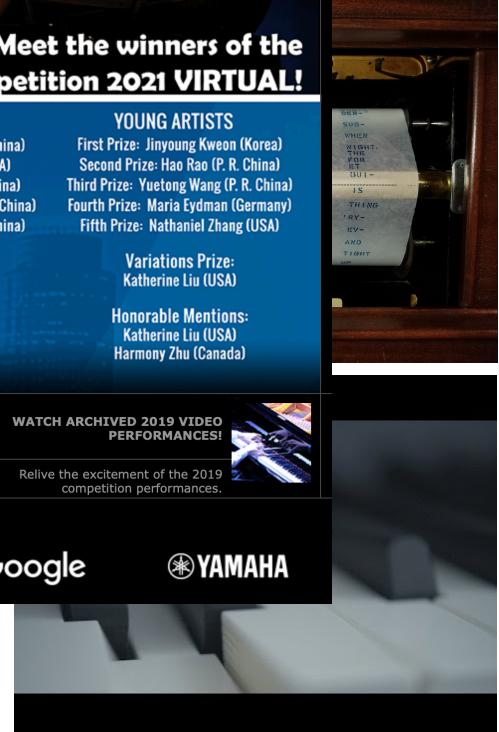
Symbolic Recordings

- Late 19th Century: Player Pianos
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The screenshot shows the homepage of the e-Piano Junior Competition 2021 VIRTUAL website. At the top, there's a banner for the competition with the text "e-Piano Junior Competition 2021 VIRTUAL! Minneapolis, MN. USA". To the right, it says "SCHOOL OF MUSIC UNIVERSITY OF MINNESOTA Driven to Discover". Below the banner, there are navigation links: HOME, GENERAL INFORMATION, CONTESTANT INFORMATION, COMPETITION HISTORY, CURRENT COMPETITION, and LIVESTREAM!. A video player in the center displays a young pianist performing. To the right of the video, the text "Meet the winners of the e-Piano Junior Competition 2021 VIRTUAL!" is displayed. The page is divided into several sections listing award winners:

- SPARKS**
 - First Prize: Taige Wang (USA)
 - Second Prize: Thannapas Luangpitpong (Thailand)
 - Third Prize: Sihong Li (P. R. China)
 - Fourth Prize: Isabel Feng (USA)
 - Fifth Prize: Masanobu Pires (USA)
- ASPIRES**
 - First Prize: Yanyan Bao (P. R. China)
 - Second Prize: Xinran Shi (USA)
 - Third Prize: Jiayou Xu (P. R. China)
 - Fourth Prize: Xuanxiang Wu (P. R. China)
 - Fifth Prize: Zhiwei Chen (P. R. China)
- YOUNG ARTISTS**
 - First Prize: Jinyoung Kweon (Korea)
 - Second Prize: Hao Rao (P. R. China)
 - Third Prize: Yuetong Wang (P. R. China)
 - Fourth Prize: Maria Eydmann (Germany)
 - Fifth Prize: Nathaniel Zhang (USA)
- Mendelssohn Prize:** Taige Wang (USA)
- Schubert Prize:** Yanyan Bao (P. R. China)
- Honorable Mentions:** Zhiyu Fang (P. R. China), Papitcha Vejmongkolkorn (Thailand)
- Honorable Mentions:** Edison Chen (USA), Seokyung Hong (Korea)
- Variations Prize:** Katherine Liu (USA)
- Honorable Mentions:** Katherine Liu (USA), Harmony Zhu (Canada)

At the bottom, there are links to "CONGRATULATIONS TO THE WINNERS OF THE 2021 COMPETITION!", "WATCH ARCHIVED 2019 VIDEO PERFORMANCES!", and "See the entire list of participants here." Logos for the School of Music, International piano-e-competition, Google, and Yamaha are also present.



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How do machines listen?: Humans vs. Computers

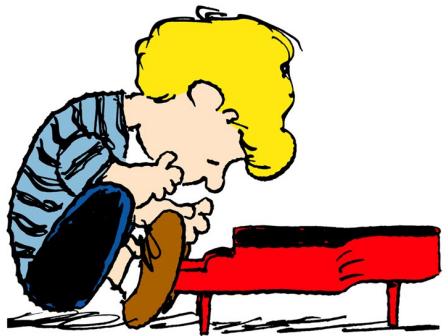
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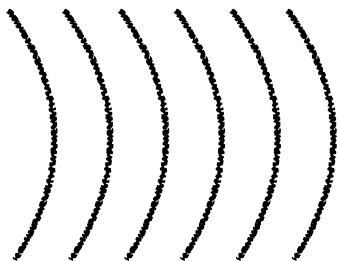
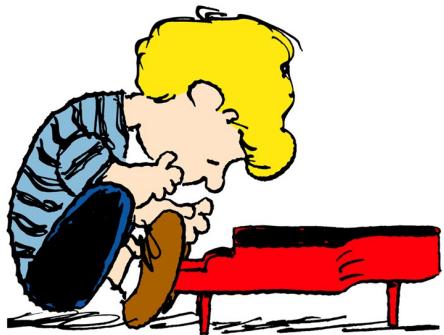
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 - Human brains are marvelous information processing machines that allow us to **make sense** of auditory signals (almost) effortlessly

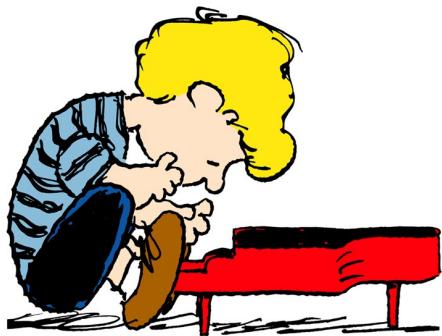
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- Computers struggle performing basic tasks!

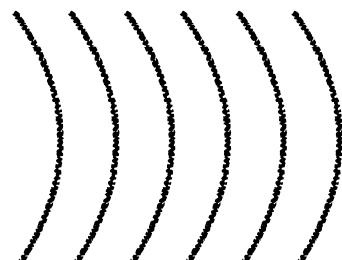


Pressure Wave





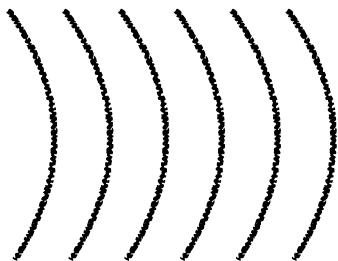
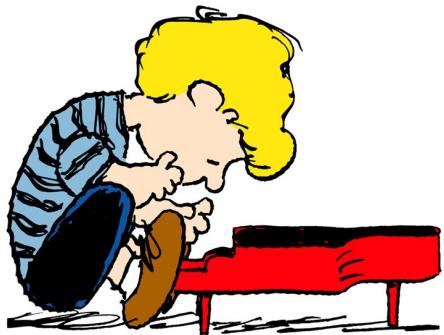
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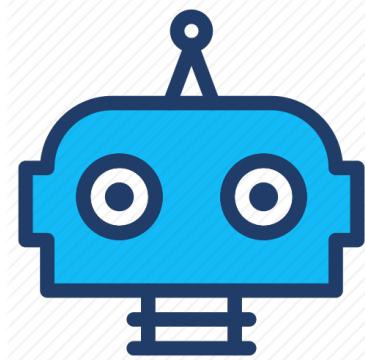
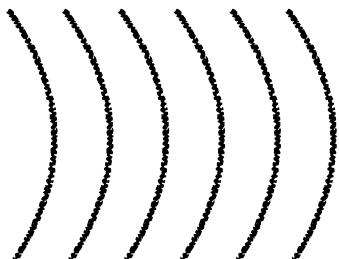
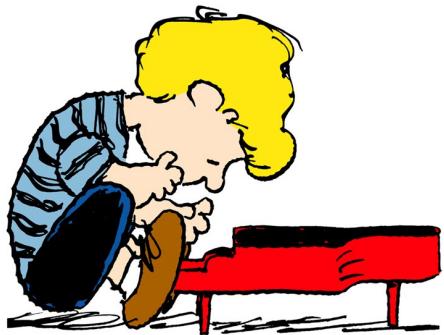
Music



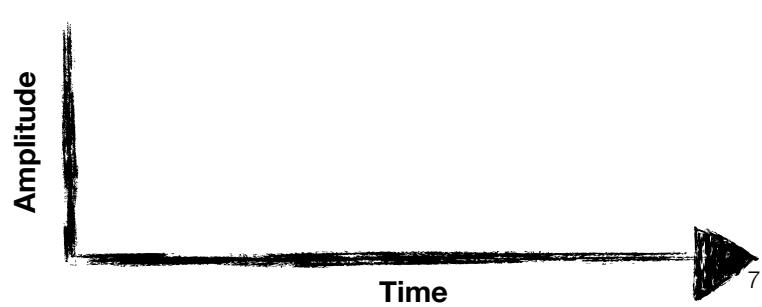
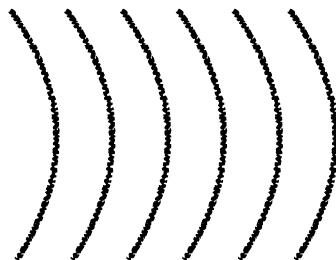
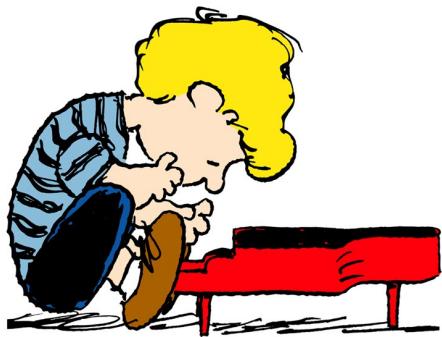
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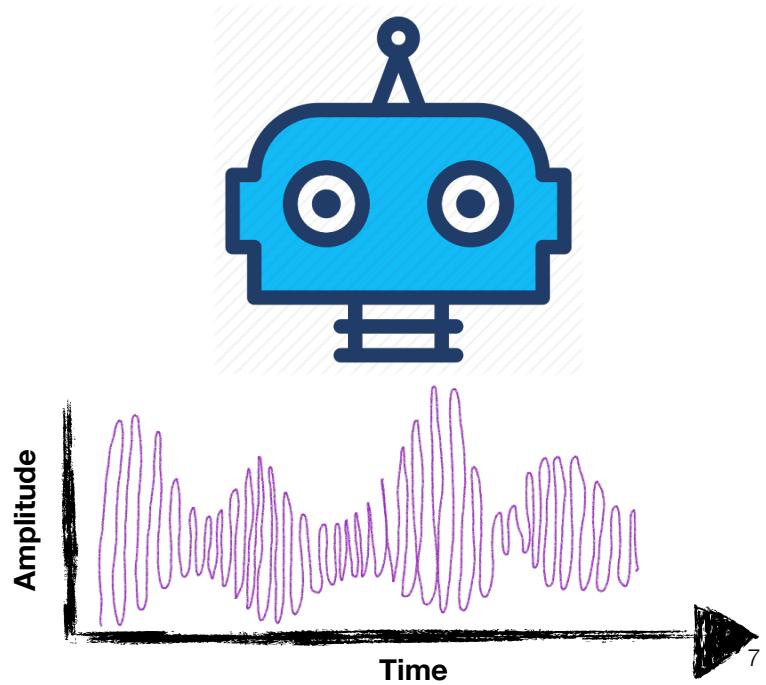
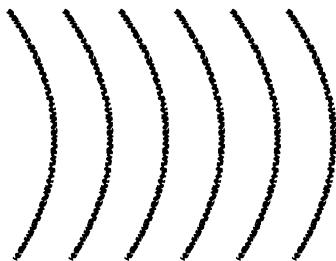
Pressure Wave



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 - Ideally, we would like to develop **computational models** that can identify or **predict some of the same kinds of patterns** in music that (expert) **human listeners** would perceive.
 - Developing such models will require us to understand more about **human music perception**.
 - [Herrera et al. 2009]: “We will only develop music understanding systems by means of understanding music understanding”.

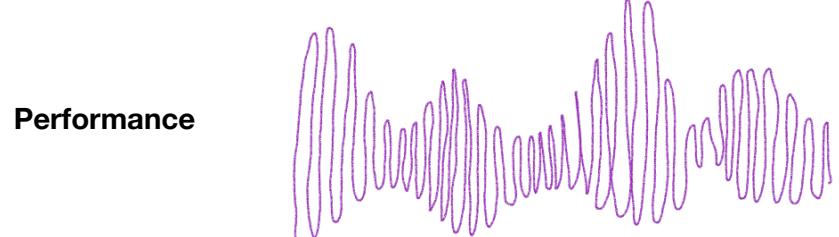
Automatic Music Alignment

Score

Tb. (sehr ausdrucksvoll)

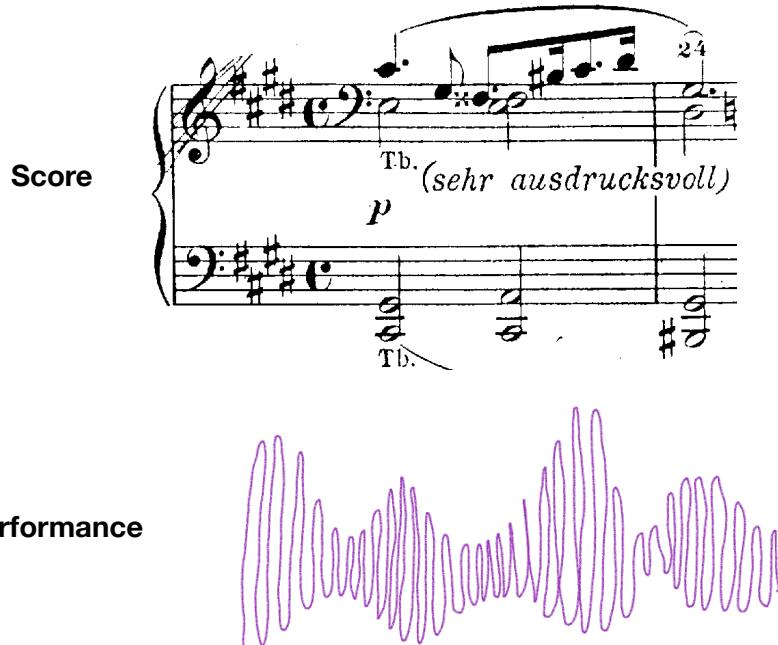
p

24



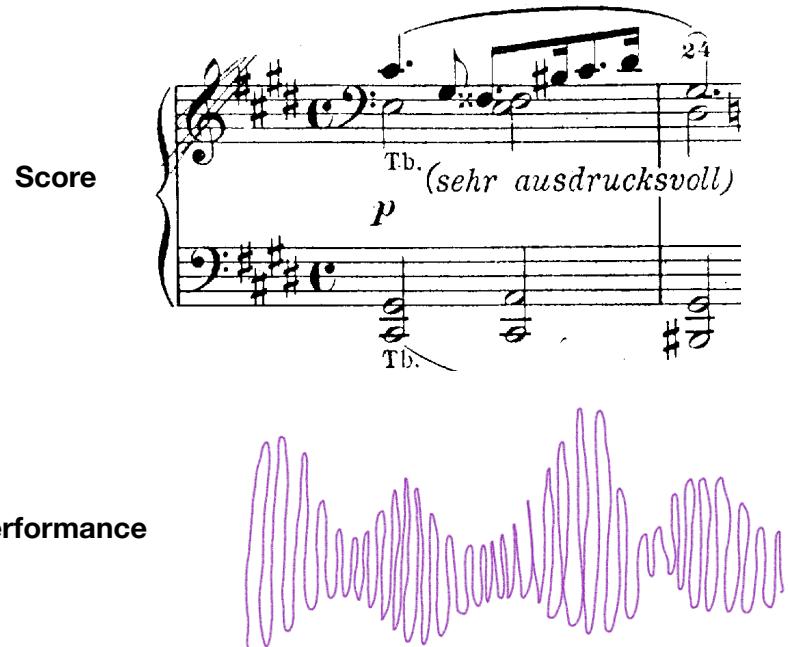
Automatic Music Alignment

- Matching **musical signals** of the **same piece**



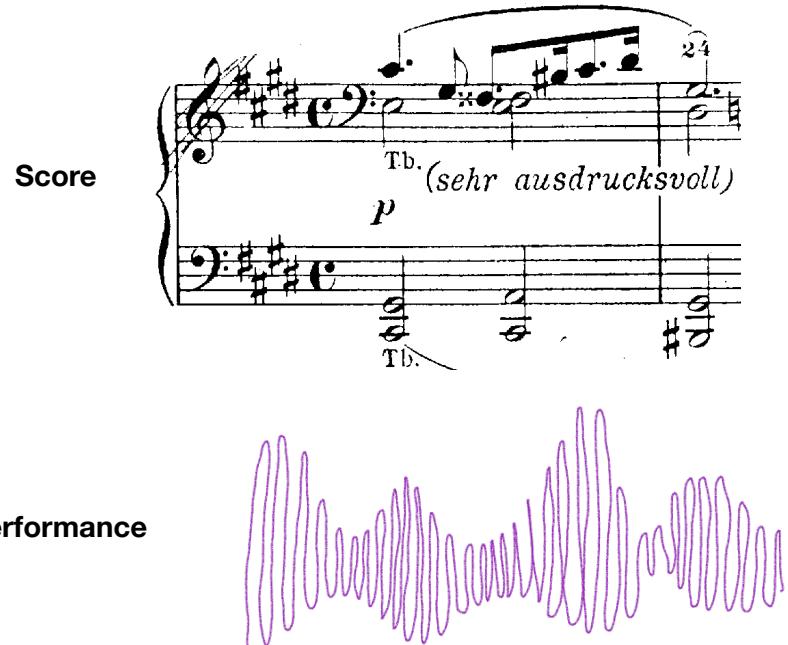
Automatic Music Alignment

- Matching **musical signals** of the **same piece**
 - Different performances of the same piece



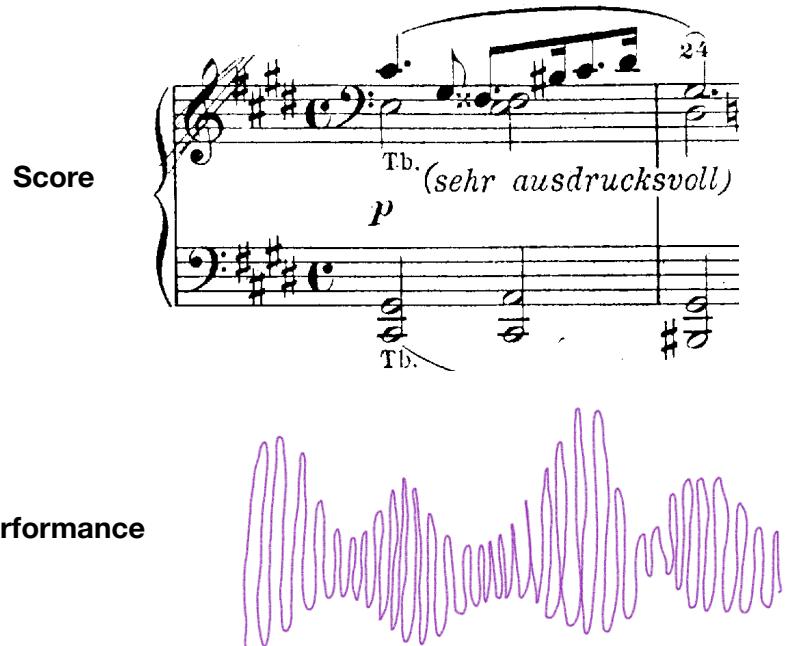
Automatic Music Alignment

- Matching **musical signals** of the **same piece**
 - Different performances of the same piece
 - A performance with its score



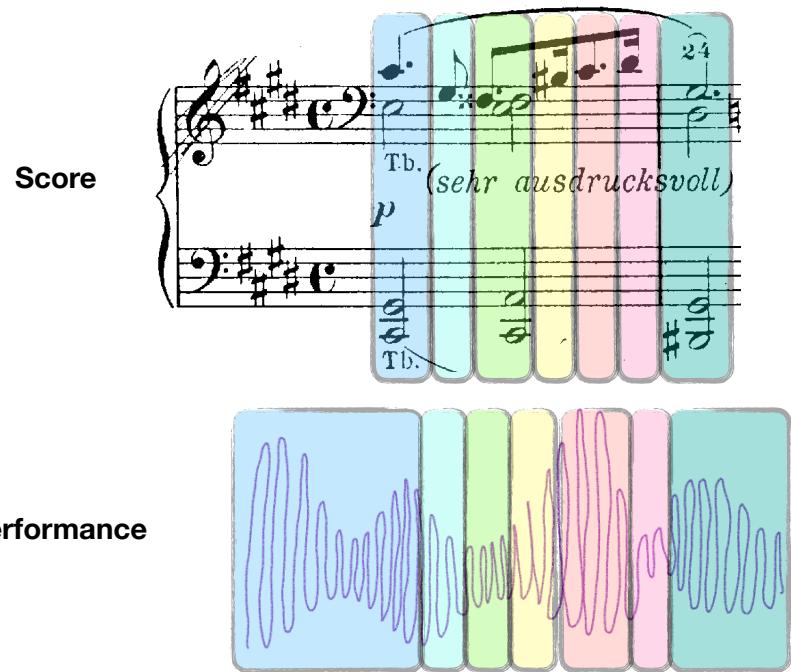
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- Matching **musical signals** of the **same piece**
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 - A performance with its score
- If it is in real time, it is referred to as **score following**

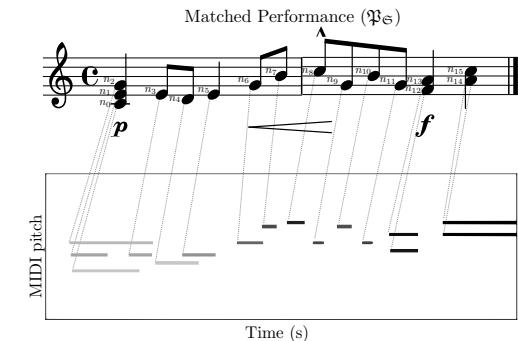
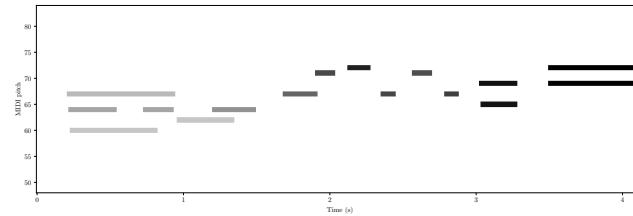
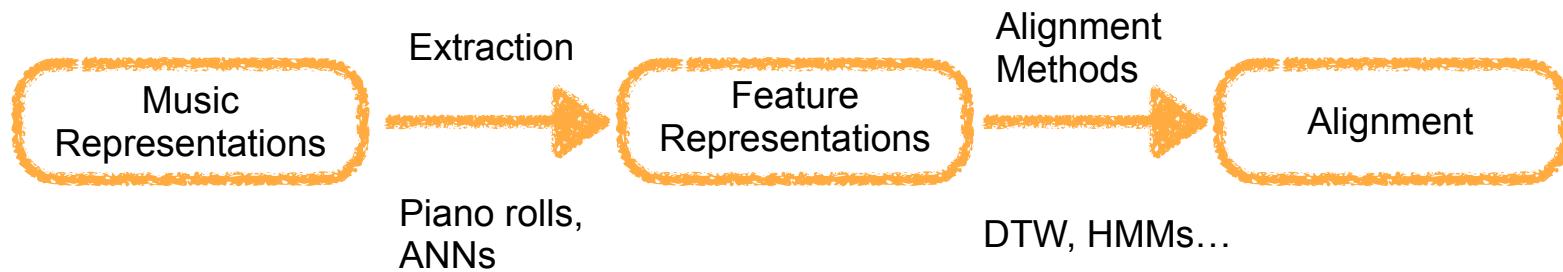


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Music Alignment Pipeline



MELD: Music Encoding and Linker

TROMPA

Piano

var I

VAR. I

Pno.

14

Page 1 / 3

Show alignment confidence Show errors



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TROMPA

Piano

var I

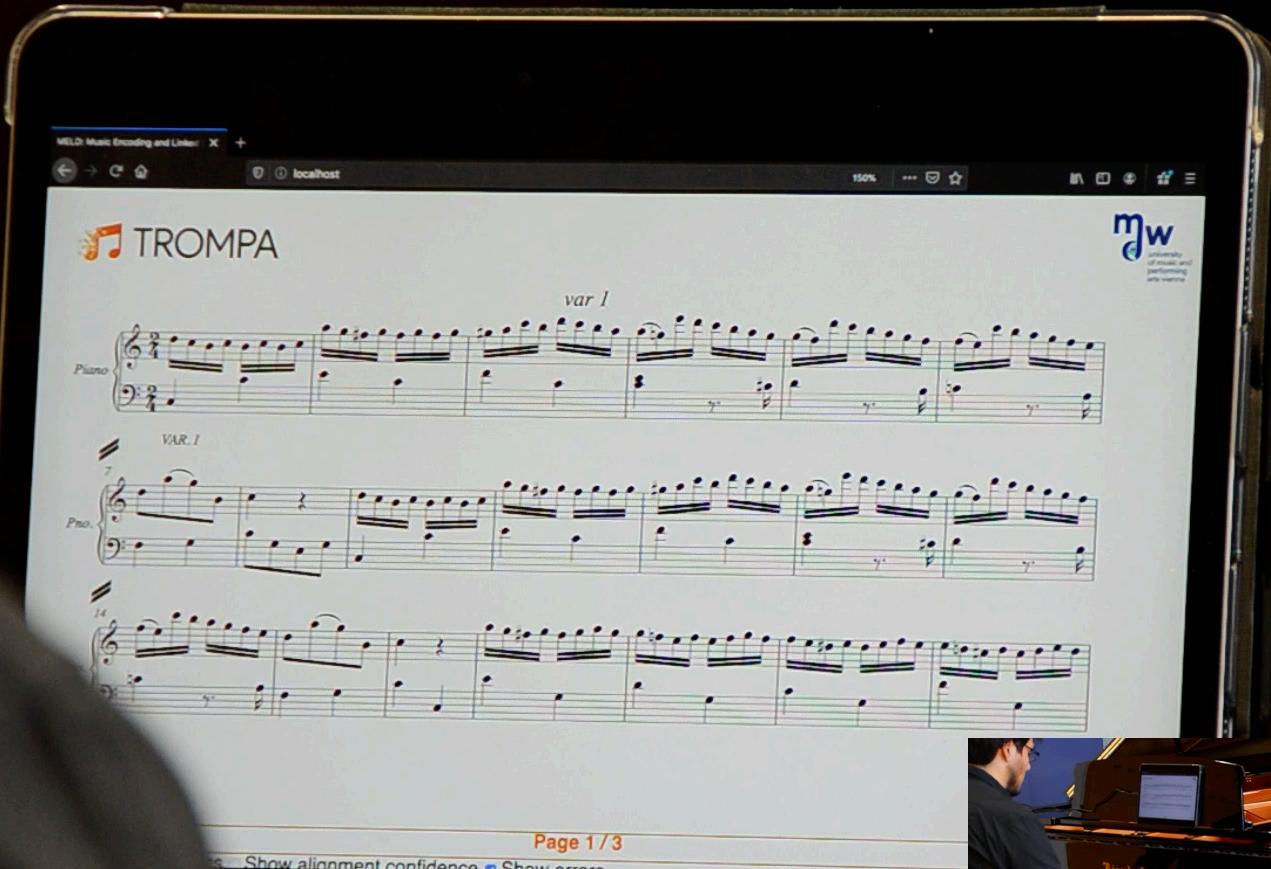
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Page 1 / 3

Show alignment confidence Show errors



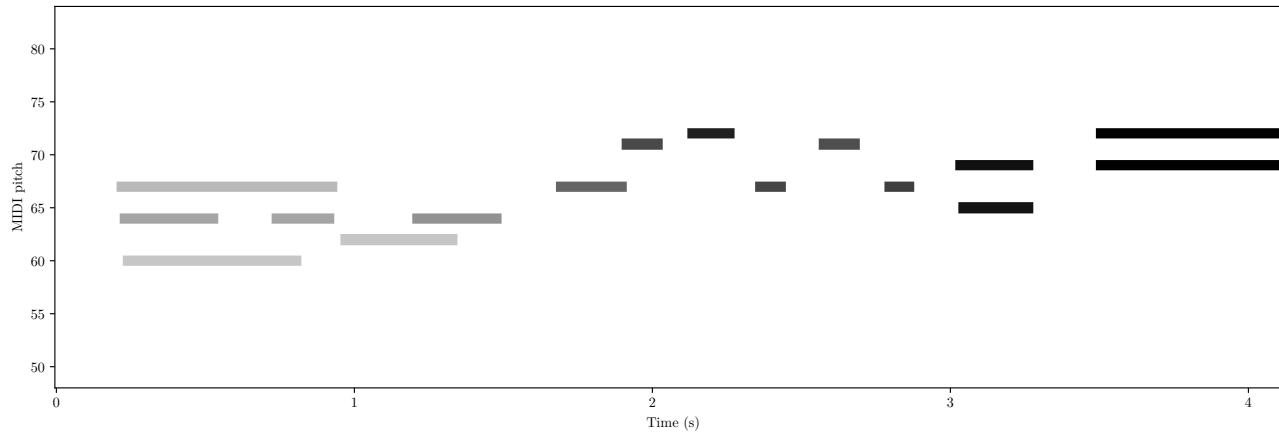
Music Representation and Features

Features

- To make data comparable and algorithmically accessible, the first step is to extract suitable features that capture relevant aspects while suppressing irrelevant details

The Piano Roll

- A 2D representation of (MIDI) pitch and time
- We can extract piano rolls from MIDI and MusicXML files with Partitura!



Other Features

- Piano rolls are not the only (or the best) features for alignment
 - Neural networks: Features learned from Autoencoders (Lattner et al., 2018) that are transposition invariant
 - Pitch Class distribution
 - and many more!

Dynamic Time Warping

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 - for other applications, sequences could be discrete signals, feature sequences, sequences of characters
- **Feature Space:** $\mathbf{x}_n, \mathbf{y}_m \in \mathcal{F}$ for $n \in [1 : N]$ and $m \in [1 : M]$
- **Local cost measure:** $c : \mathcal{F} \times \mathcal{F} \mapsto \mathbb{R}$
 - Basic intuition: $c(\mathbf{x}, \mathbf{y})$ is small if \mathbf{x} and \mathbf{y} are similar

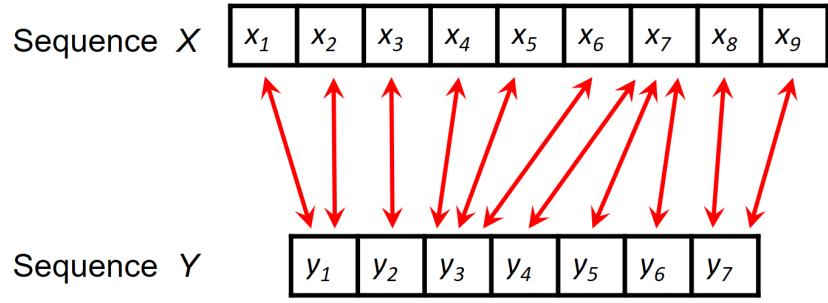
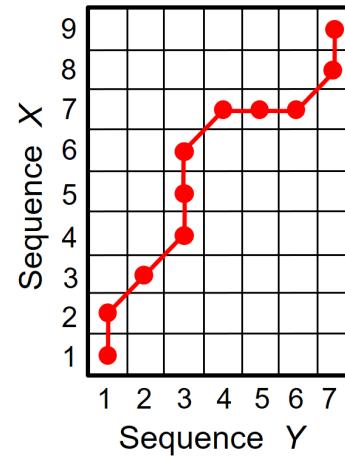


Figure 3.12 from [Müller, FMP, Springer 2015]



Cost Matrix

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- The **choice** of the local cost measure **matters!**

- Euclidean Distance: $c(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_i (x_i - y_i)^2}$
- Manhattan Distance: $c(\mathbf{x}, \mathbf{y}) = \sum_i |x_i - y_i|$
- Cosine Distance: $c(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{x}^T \mathbf{y}}{||\mathbf{x}|| \cdot ||\mathbf{y}||}$
- Others?

Warping Path

- An (N, M) -**warping path** of length $L \in \mathbb{N}$ is a sequence

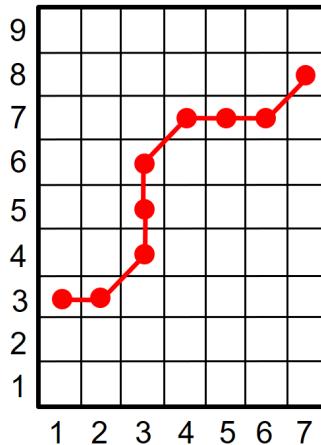
$$P = (p_1, \dots, p_l)$$

with $p_l = (n_l, m_l) \in [1 : N] \times [1 : M]$ that defines an alignment between sequences $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ and $\mathbf{Y} = \{\mathbf{y}_1, \dots, \mathbf{y}_M\}$

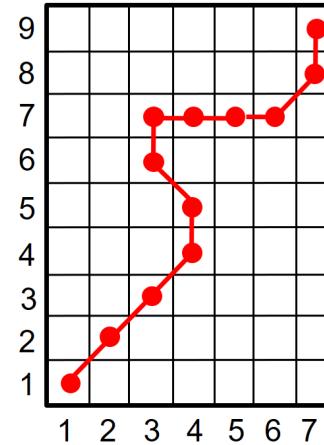
- Three conditions:
 - **Boundary condition:** $p_1 = (1,1)$ and $p_L = (N, M)$
 - **Monotonicity condition:** $n_1 \leq n_2 \leq \dots \leq n_L$ and $m_1 \leq m_2 \leq \dots \leq m_L$
 - **Step size condition:** $p_{l-1} - p_l \in \{(1,0), (0,1), (1,1)\}$

Warping Path

Violates Boundary condition



Violates Monotonicity



Violates Step Size Condition

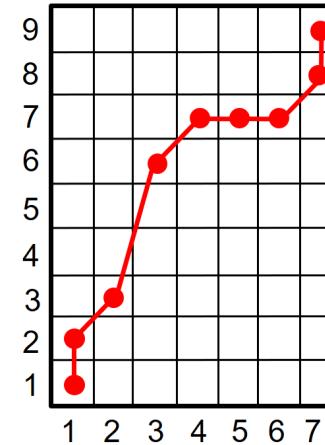


Figure 3.13 from [Müller, FMP, Springer 2015]

Optimal Warping Path

- Total cost of a warping path:

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$$\text{DTW}(\mathbf{X}, \mathbf{Y}) = c_{P^*}(\mathbf{X}, \mathbf{Y})$$

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- **DTW(\mathbf{X}, \mathbf{Y}): DTW distance**

- Not a real **distance**: is symmetric, but not positive definite

- $\text{DTW}(\mathbf{X}, \mathbf{Y}) = 0$ does not necessarily mean that $\mathbf{X} = \mathbf{Y}$!

Dynamic Programming Algorithm

- Dynamic Programming: Break down a problem into simpler subproblems and then combine the solution!

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- Idea: Find optimal warping paths for subsequences of **X** and **Y**, and then combine the solutions recursively!

Dynamic Programming Algorithm

- Dynamic Programming: Break down a problem into simpler subproblems and then combine the solution!
- Idea: Find optimal warping paths for subsequences of \mathbf{X} and \mathbf{Y} , and then combine the solutions recursively!
- Accumulated Cost Matrix
 - $\mathbf{D}(n, m) = \text{DTW}(\mathbf{X}(1 : n), \mathbf{Y}(1 : m))$

Algorithm

Algorithm: DTW

Table 3.2 from [Müller, FMP, Springer 2015]

Input: Cost matrix \mathbf{C} of size $N \times M$

Output: Accumulated cost matrix \mathbf{D}
Optimal warping path P^*

Procedure: Initialize $(N \times M)$ matrix \mathbf{D} by $\mathbf{D}(n, 1) = \sum_{k=1}^n \mathbf{C}(k, 1)$ for $n \in [1 : N]$ and $\mathbf{D}(1, m) = \sum_{k=1}^m \mathbf{C}(1, k)$ for $m \in [1 : M]$. Then compute in a nested loop for $n = 2, \dots, N$ and $m = 2, \dots, M$:

$$\mathbf{D}(n, m) = \mathbf{C}(n, m) + \min \{\mathbf{D}(n - 1, m - 1), \mathbf{D}(n - 1, m), \mathbf{D}(n, m - 1)\}.$$

Set $\ell = 1$ and $q_\ell = (N, M)$. Then repeat the following steps until $q_\ell = (1, 1)$:

Increase ℓ by one and let $(n, m) = q_{\ell-1}$.

If $n = 1$, then $q_\ell = (1, m - 1)$,

else if $m = 1$, then $q_\ell = (n - 1, m)$,

else $q_\ell = \operatorname{argmin} \{\mathbf{D}(n - 1, m - 1), \mathbf{D}(n - 1, m), \mathbf{D}(n, m - 1)\}$.

(If ‘ argmin ’ is not unique, take lexicographically smallest cell.)

Set $L = \ell$ and return $P^* = (q_L, q_{L-1}, \dots, q_1)$ as well as \mathbf{D} .

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$\mathbf{C} = \left[\begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right]$$

Pen and Paper Example!

Inputs

$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$\mathbf{C} = \begin{bmatrix} 0 \\ \vdots \end{bmatrix}$$

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$$C_{12} = |x_1 - y_2| = |3 - 1| = 2$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 \\ \end{bmatrix}$$

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$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

$$C_{12} = |x_1 - y_2| = |3 - 1| = 2$$

$$C_{13} = |x_1 - y_3| = |3 - 9| = 6$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 \\ \end{bmatrix}$$

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

$$C_{12} = |x_1 - y_2| = |3 - 1| = 2$$

$$C_{13} = |x_1 - y_3| = |3 - 9| = 6$$

$$C_{14} = |x_1 - y_4| = |3 - 9| = 6$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 \end{bmatrix}$$

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

$$C_{12} = |x_1 - y_2| = |3 - 1| = 2$$

$$C_{13} = |x_1 - y_3| = |3 - 9| = 6$$

$$C_{14} = |x_1 - y_4| = |3 - 9| = 6$$

$$C_{15} = |x_1 - y_5| = |3 - 7| = 4$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 \end{bmatrix}$$

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

$$C_{12} = |x_1 - y_2| = |3 - 1| = 2$$

$$C_{13} = |x_1 - y_3| = |3 - 9| = 6$$

$$C_{14} = |x_1 - y_4| = |3 - 9| = 6$$

$$C_{15} = |x_1 - y_5| = |3 - 7| = 4$$

$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \end{bmatrix}$$

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

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$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & & & & & \end{bmatrix}$$

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

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$$C_{15} = |x_1 - y_5| = |3 - 7| = 4$$

$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 \end{bmatrix}$$

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

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$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

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$$C_{15} = |x_1 - y_5| = |3 - 7| = 4$$

$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & & & \end{bmatrix}$$

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

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$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & & \end{bmatrix}$$

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

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$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

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$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 \end{bmatrix}$$

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

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$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

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$$C_{13} = |x_1 - y_3| = |3 - 9| = 6$$

$$C_{14} = |x_1 - y_4| = |3 - 9| = 6$$

$$C_{15} = |x_1 - y_5| = |3 - 7| = 4$$

$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \end{bmatrix}$$

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

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$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 \end{bmatrix}$$

Pen and Paper Example!

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$$C_{15} = |x_1 - y_5| = |3 - 7| = 4$$

$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & & & & \end{bmatrix}$$

Pen and Paper Example!

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$$\mathbf{X} = \{3, 9, 8\}$$

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$$C_{15} = |x_1 - y_5| = |3 - 7| = 4$$

$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & & & \end{bmatrix}$$

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

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$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

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$$C_{15} = |x_1 - y_5| = |3 - 7| = 4$$

$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & & \end{bmatrix}$$

Pen and Paper Example!

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$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

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$$C_{15} = |x_1 - y_5| = |3 - 7| = 4$$

$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Pen and Paper Example!

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$C_{11} = |x_1 - y_1| = |3 - 3| = 0$$

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$$C_{15} = |x_1 - y_5| = |3 - 7| = 4$$

$$C_{16} = |x_1 - y_6| = |3 - 8| = 5$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1} \quad D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{D} = \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right]$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1} \quad D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 \\ 6 \\ 11 \end{bmatrix}$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1}$$

$$D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & & & & & \\ 11 & & & & & \end{bmatrix}$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$D_{22} = C_{22} + \min(D_{2-1,2-1}, D_{2-1,2}, D_{2,2-1}) = 8 + \min(0, 2, 6) = 8$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1}$$

$$D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & & & & \\ 11 & & & & & \end{bmatrix}$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$D_{22} = C_{22} + \min(D_{2-1,2-1}, D_{2-1,2}, D_{2,2-1}) = 8 + \min(0, 2, 6) = 8$$

$$D_{23} = C_{23} + \min(D_{2-1,3-1}, D_{2-1,3}, D_{2,3-1}) = 0 + \min(2, 8, 8) = 2$$

Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1}$$

$$D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & & & \\ 11 & & & & & \end{bmatrix}$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$D_{22} = C_{22} + \min(D_{2-1,2-1}, D_{2-1,2}, D_{2,2-1}) = 8 + \min(0, 2, 6) = 8$$

$$D_{23} = C_{23} + \min(D_{2-1,3-1}, D_{2-1,3}, D_{2,3-1}) = 0 + \min(2, 8, 8) = 2$$

$$D_{24} = C_{24} + \min(D_{2-1,4-1}, D_{2-1,4}, D_{2,4-1}) = 0 + \min(8, 14, 2) = 2$$

Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1}$$

$$D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & & \\ 11 & & & & & \end{bmatrix}$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1}$$

$$D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$D_{22} = C_{22} + \min(D_{2-1,2-1}, D_{2-1,2}, D_{2,2-1}) = 8 + \min(0, 2, 6) = 8$$

$$D_{23} = C_{23} + \min(D_{2-1,3-1}, D_{2-1,3}, D_{2,3-1}) = 0 + \min(2, 8, 8) = 2$$

$$D_{24} = C_{24} + \min(D_{2-1,4-1}, D_{2-1,4}, D_{2,4-1}) = 0 + \min(8, 14, 2) = 2$$

$$D_{25} = C_{25} + \min(D_{2-1,5-1}, D_{2-1,5}, D_{2,5-1}) = 2 + \min(14, 18, 2) = 4$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & \\ 11 & & & & & \end{bmatrix}$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1}$$

$$D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$D_{22} = C_{22} + \min(D_{2-1,2-1}, D_{2-1,2}, D_{2,2-1}) = 8 + \min(0, 2, 6) = 8$$

$$D_{23} = C_{23} + \min(D_{2-1,3-1}, D_{2-1,3}, D_{2,3-1}) = 0 + \min(2, 8, 8) = 2$$

$$D_{24} = C_{24} + \min(D_{2-1,4-1}, D_{2-1,4}, D_{2,4-1}) = 0 + \min(8, 14, 2) = 2$$

$$D_{25} = C_{25} + \min(D_{2-1,5-1}, D_{2-1,5}, D_{2,5-1}) = 2 + \min(14, 18, 2) = 4$$

$$D_{26} = C_{26} + \min(D_{2-1,6-1}, D_{2-1,6}, D_{2,6-1}) = 1 + \min(18, 23, 4) = 5$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & & & & & \end{bmatrix}$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1}$$

$$D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$D_{22} = C_{22} + \min(D_{2-1,2-1}, D_{2-1,2}, D_{2,2-1}) = 8 + \min(0, 2, 6) = 8$$

$$D_{23} = C_{23} + \min(D_{2-1,3-1}, D_{2-1,3}, D_{2,3-1}) = 0 + \min(2, 8, 8) = 2$$

$$D_{24} = C_{24} + \min(D_{2-1,4-1}, D_{2-1,4}, D_{2,4-1}) = 0 + \min(8, 14, 2) = 2$$

$$D_{25} = C_{25} + \min(D_{2-1,5-1}, D_{2-1,5}, D_{2,5-1}) = 2 + \min(14, 18, 2) = 4$$

$$D_{26} = C_{26} + \min(D_{2-1,6-1}, D_{2-1,6}, D_{2,6-1}) = 1 + \min(18, 23, 4) = 5$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & & & & \end{bmatrix}$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1}$$

$$D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$D_{22} = C_{22} + \min(D_{2-1,2-1}, D_{2-1,2}, D_{2,2-1}) = 8 + \min(0, 2, 6) = 8$$

$$D_{23} = C_{23} + \min(D_{2-1,3-1}, D_{2-1,3}, D_{2,3-1}) = 0 + \min(2, 8, 8) = 2$$

$$D_{24} = C_{24} + \min(D_{2-1,4-1}, D_{2-1,4}, D_{2,4-1}) = 0 + \min(8, 14, 2) = 2$$

$$D_{25} = C_{25} + \min(D_{2-1,5-1}, D_{2-1,5}, D_{2,5-1}) = 2 + \min(14, 18, 2) = 4$$

$$D_{26} = C_{26} + \min(D_{2-1,6-1}, D_{2-1,6}, D_{2,6-1}) = 1 + \min(18, 23, 4) = 5$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & & & \end{bmatrix}$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

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Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1}$$

$$D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$D_{22} = C_{22} + \min(D_{2-1,2-1}, D_{2-1,2}, D_{2,2-1}) = 8 + \min(0, 2, 6) = 8$$

$$D_{23} = C_{23} + \min(D_{2-1,3-1}, D_{2-1,3}, D_{2,3-1}) = 0 + \min(2, 8, 8) = 2$$

$$D_{24} = C_{24} + \min(D_{2-1,4-1}, D_{2-1,4}, D_{2,4-1}) = 0 + \min(8, 14, 2) = 2$$

$$D_{25} = C_{25} + \min(D_{2-1,5-1}, D_{2-1,5}, D_{2,5-1}) = 2 + \min(14, 18, 2) = 4$$

$$D_{26} = C_{26} + \min(D_{2-1,6-1}, D_{2-1,6}, D_{2,6-1}) = 1 + \min(18, 23, 4) = 5$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & & \end{bmatrix}$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1}$$

$$D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$D_{22} = C_{22} + \min(D_{2-1,2-1}, D_{2-1,2}, D_{2,2-1}) = 8 + \min(0, 2, 6) = 8$$

$$D_{23} = C_{23} + \min(D_{2-1,3-1}, D_{2-1,3}, D_{2,3-1}) = 0 + \min(2, 8, 8) = 2$$

$$D_{24} = C_{24} + \min(D_{2-1,4-1}, D_{2-1,4}, D_{2,4-1}) = 0 + \min(8, 14, 2) = 2$$

$$D_{25} = C_{25} + \min(D_{2-1,5-1}, D_{2-1,5}, D_{2,5-1}) = 2 + \min(14, 18, 2) = 4$$

$$D_{26} = C_{26} + \min(D_{2-1,6-1}, D_{2-1,6}, D_{2,6-1}) = 1 + \min(18, 23, 4) = 5$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & \end{bmatrix}$$

Pen and Paper! Example II

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

Initialization

$$D_{i,1} = \sum_{k=1}^i C_{k,1}$$

$$D_{1,i} = \sum_{k=1}^i C_{1,k}$$

Iteration

$$D_{ij} = C_{ij} + \min(D_{j-1,i-1}, D_{i-1,j}, D_{i,j-1})$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & 6 & 6 & 4 & 5 \\ 6 & 8 & 0 & 0 & 2 & 1 \\ 5 & 7 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$D_{22} = C_{22} + \min(D_{2-1,2-1}, D_{2-1,2}, D_{2,2-1}) = 8 + \min(0, 2, 6) = 8$$

$$D_{23} = C_{23} + \min(D_{2-1,3-1}, D_{2-1,3}, D_{2,3-1}) = 0 + \min(2, 8, 8) = 2$$

$$D_{24} = C_{24} + \min(D_{2-1,4-1}, D_{2-1,4}, D_{2,4-1}) = 0 + \min(8, 14, 2) = 2$$

$$D_{25} = C_{25} + \min(D_{2-1,5-1}, D_{2-1,5}, D_{2,5-1}) = 2 + \min(14, 18, 2) = 4$$

$$D_{26} = C_{26} + \min(D_{2-1,6-1}, D_{2-1,6}, D_{2,6-1}) = 1 + \min(18, 23, 4) = 5$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3,9,8\}$$

$$\mathbf{Y} = \{3,1,9,9,7,8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$q_1 = (3,6)$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3,9,8\}$$

$$\mathbf{Y} = \{3,1,9,9,7,8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$q_1 = (3,6)$$

$$q_2 = \arg \min(D_{3-1,6-1}, D_{3,6-1}, D_{3-1,6}) = (3,5)$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3,9,8\}$$

$$\mathbf{Y} = \{3,1,9,9,7,8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$q_1 = (3,6)$$

$$q_2 = \arg \min(D_{3-1,6-1}, D_{3,6-1}, D_{3-1,6}) = (3,5)$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3,9,8\}$$

$$\mathbf{Y} = \{3,1,9,9,7,8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$q_1 = (3,6)$$

$$q_2 = \arg \min(D_{3-1,6-1}, D_{3,6-1}, D_{3-1,6}) = (3,5)$$

$$q_3 = \arg \min(D_{3-1,5-1}, D_{3,5-1}, D_{3-1,5}) = (2,4)$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3,9,8\}$$

$$\mathbf{Y} = \{3,1,9,9,7,8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$q_1 = (3,6)$$

$$q_2 = \arg \min(D_{3-1,6-1}, D_{3,6-1}, D_{3-1,6}) = (3,5)$$

$$q_3 = \arg \min(D_{3-1,5-1}, D_{3,5-1}, D_{3-1,5}) = (2,4)$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$q_1 = (3, 6)$$

$$q_4 = \arg \min(D_{2-1,4-1}, D_{2,4-1}, D_{2-1,4}) = (2, 3)$$

$$q_2 = \arg \min(D_{3-1,6-1}, D_{3,6-1}, D_{3-1,6}) = (3, 5)$$

$$q_3 = \arg \min(D_{3-1,5-1}, D_{3,5-1}, D_{3-1,5}) = (2, 4)$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$q_1 = (3, 6)$$

$$q_4 = \arg \min(D_{2-1,4-1}, D_{2,4-1}, D_{2-1,4}) = (2, 3)$$

$$q_2 = \arg \min(D_{3-1,6-1}, D_{3,6-1}, D_{3-1,6}) = (3, 5)$$

$$q_3 = \arg \min(D_{3-1,5-1}, D_{3,5-1}, D_{3-1,5}) = (2, 4)$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3, 9, 8\}$$

$$\mathbf{Y} = \{3, 1, 9, 9, 7, 8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$q_1 = (3, 6)$$

$$q_2 = \arg \min(D_{3-1,6-1}, D_{3,6-1}, D_{3-1,6}) = (3, 5)$$

$$q_3 = \arg \min(D_{3-1,5-1}, D_{3,5-1}, D_{3-1,5}) = (2, 4)$$

$$q_4 = \arg \min(D_{2-1,4-1}, D_{2,4-1}, D_{2-1,4}) = (2, 3)$$

$$q_5 = \arg \min(D_{2-1,3-1}, D_{2,3-1}, D_{2-1,3}) = (1, 2)$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3,9,8\}$$

$$\mathbf{Y} = \{3,1,9,9,7,8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$q_1 = (3,6)$$

$$q_2 = \arg \min(D_{3-1,6-1}, D_{3,6-1}, D_{3-1,6}) = (3,5)$$

$$q_3 = \arg \min(D_{3-1,5-1}, D_{3,5-1}, D_{3-1,5}) = (2,4)$$

$$q_4 = \arg \min(D_{2-1,4-1}, D_{2,4-1}, D_{2-1,4}) = (2,3)$$

$$q_5 = \arg \min(D_{2-1,3-1}, D_{2,3-1}, D_{2-1,3}) = (1,2)$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3,9,8\}$$

$$\mathbf{Y} = \{3,1,9,9,7,8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$q_1 = (3,6)$$

$$q_2 = \arg \min(D_{3-1,6-1}, D_{3,6-1}, D_{3-1,6}) = (3,5)$$

$$q_3 = \arg \min(D_{3-1,5-1}, D_{3,5-1}, D_{3-1,5}) = (2,4)$$

$$q_4 = \arg \min(D_{2-1,4-1}, D_{2,4-1}, D_{2-1,4}) = (2,3)$$

$$q_5 = \arg \min(D_{2-1,3-1}, D_{2,3-1}, D_{2-1,3}) = (1,2)$$

$$q_6 = (1, q_5[1] - 1) = (1,1)$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3,9,8\}$$

$$\mathbf{Y} = \{3,1,9,9,7,8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$q_1 = (3,6)$$

$$q_2 = \arg \min(D_{3-1,6-1}, D_{3,6-1}, D_{3-1,6}) = (3,5)$$

$$q_3 = \arg \min(D_{3-1,5-1}, D_{3,5-1}, D_{3-1,5}) = (2,4)$$

$$q_4 = \arg \min(D_{2-1,4-1}, D_{2,4-1}, D_{2-1,4}) = (2,3)$$

$$q_5 = \arg \min(D_{2-1,3-1}, D_{2,3-1}, D_{2-1,3}) = (1,2)$$

$$q_6 = (1, q_5[1] - 1) = (1,1)$$

Pen and Paper Example III

Inputs

$$\mathbf{X} = \{3,9,8\}$$

$$\mathbf{Y} = \{3,1,9,9,7,8\}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 2 & 8 & 14 & 18 & 23 \\ 6 & 8 & 2 & 2 & 4 & 5 \\ 11 & 13 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$q_1 = (3,6)$$

$$q_2 = \arg \min(D_{3-1,6-1}, D_{3,6-1}, D_{3-1,6}) = (3,5)$$

$$q_3 = \arg \min(D_{3-1,5-1}, D_{3,5-1}, D_{3-1,5}) = (2,4)$$

$$q_4 = \arg \min(D_{2-1,4-1}, D_{2,4-1}, D_{2-1,4}) = (2,3)$$

$$q_5 = \arg \min(D_{2-1,3-1}, D_{2,3-1}, D_{2-1,3}) = (1,2)$$

$$q_6 = (1, q_5[1] - 1) = (1,1)$$

$$\mathbf{P}^* = \{(1,1), (1,2), (2,3), (2,4), (3,5), (3,6)\}$$