Minimal Surface

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Contents

This is the course notes of 18.137 *Topics in Geometric PDEs* taught by William Minicozzi in spring 2016.

1 What is a minimal surface?

In this section, without specific mention, all the domains, functions, etc. are as regular as we want.

1.1 Graph

Suppose Ω is a domain in $\mathbb{R}^3$, and $u : \Omega \to \mathbb{R}$ is a function, and G_u is the graph of u.

Then the tangent space to G_u is spanned by two vectors $\partial_x = (1, 0, u_x)$ and $\partial_y = (0, 1, u_y)$.

The normal vector is defined as

$$\mathbf{N} = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}}$$

The area of G_u is by definition

$$\int_\Omega \sqrt{1 + |\nabla u|^2} dxdy$$

Question When dos G_u have the least area for its boundary?

1.2 Variation

We consider the variation of this integral. Consider $u + t\eta$, where $\eta : \Omega \to \mathbb{R}$, and vanishes on boundary. Then we define:

$$A(t) = \int_\Omega \sqrt{1 + |\nabla u|^2 + 2t\langle \nabla u, \nabla \eta \rangle + t^2 |\nabla \eta|^2}$$

Take derivative we have

$$A'(t) = \int_\Omega \frac{\langle \nabla u, \nabla \eta \rangle}{\sqrt{1 + |\nabla u \nabla|^2}} = \int_\Omega \eta div(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}})$$

It should be 0 for any η, if G_u is minimal. So we get the **minimal surface equation (MSE)**:

$$div(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}})$$

We call the solution to this equation is **minimal surface**.

remark 1.1. *When $|\nabla u|$ is bounded, this equation is uniformly elliptic equation, so we can use some classical theory;*

When $|\nabla u| \to 0$, Then this equation is in some sense linearization of zero solution. Which means the equation is close to Laplacian equation.

remark 1.2. *The solution to the MSE includes the planes: $u = ax + by + c$*

1.3 Area Minimizing Property of G_u

If G_u is area minimizing, then it satisfies the MSE. How about the converse case? Note if G_u satisfies MSE, it is just a "critical solution", not a "minimal solution"

lemma 1.1. *If G_u is minimal and $\Sigma \subset \Omega \times \mathbb{R}$ with $\partial\Sigma = \partial G_u$, then $Area(G_u) \leq Area(\Sigma)$*

So the lemma tells us "critical is minimal"

Proof. Recall $\mathbf{N} = \frac{(-u_x, -u_y, 1)}{\sqrt{1+|\nabla u|^2}}$ is the normal vector on G_u. Extend it to $\Omega \times \mathbb{R}$, which is independent of z.

By MSE,

$$div_{\mathbb{R}^3}(\mathbf{N}) = div_{\mathbb{R}^2}(\frac{-\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0$$

By stokes theorem, we have

$$\int_{G_u} \mathbf{N} \cdot normal_{G_u} - \int_{\Sigma} \mathbf{N} \cdot normal_{\Sigma} = \int_{volume\ between\ \Omega\ and\ \Sigma} div_{\mathbb{R}^3}(\mathbf{N}) = 0$$

$$\Rightarrow Area(G_u) = \int_{\Sigma} \mathbf{N} \cdot normal_{\Sigma} \leq Area(\Sigma)$$

□

Note that this lemma is not true if $\Sigma \not\subseteq \Omega \times \mathbb{R}$. To see this, see picture 1.

However, this is true if Ω is convex. We can then replace $\Sigma - \Omega \times \mathbb{R}$ by its projection to the boundary, hits is area non-increasing. (WHY?)

remark 1.3. *The idea of the proof of the lemma is called* ***calibration****.*

In general, for a p-dimensional hypersurface, we define the calibrate form ω as a p-form, which equal to the volume form on the hypersurface, and for q any p-vector, we have $\omega(q) \leq 1$

Corollary 1.1. *Suppose G_u is minimal, $D_r \subset \Omega$ is a disk of radius r, then $Area(B_r \cap G_u) \leq 2\pi r^2$, here B_r is the ball with radius r, with center has the same x, y coordinates with the center of D_r*

Proof. We compare the minimal surface with sphere. By Sard theorem, for ϵ arbitrary small, we can find a sphere $B_{r-\epsilon'}$ which intersects G_u transversally, where $\epsilon' < \epsilon$. Then G_u split this sphere into at least two parts. Choose the part which has less area, and use the lemma above, and let $\epsilon \to 0$ we then prove the theorem. $\square$

2 Geometry of submanifold

2.1 Notation and Definition Review

Suppose Σ is a hypersurface, in the ambient space M. Let $e_1, e_2 \cdots, e_n$ is the orthonormal frame of Σ

Let V is a vector along Σ, then we can divided it into two parts

$$V = V^\top + V^\perp$$

Which are the tangential part and the vertical part (sometimes we write it as V^N).

Suppose X, Y are vector fields along Σ, both are tangential, then

$$\nabla_X Y = \nabla_X^\top Y + \nabla_X^\perp Y$$

The first part is called the induced connection on Σ, and the second part is called the **second fundamental forms**, which denoted by $A(X, Y)$

We define **Mean Curvature** $H = tr(A) = \sum_{i=1}^{n} A(e_i, e_i)$ and we define the norm of A by $|A|^2 = \sum_{i,j=1}^{n} |A(e_i, e_j)|^2$

We define **Divergence along** Σ by $div_\Sigma(X) = \sum_{i=1}^{n} \langle \nabla_{e_i} X, e_i \rangle$

A fact:

$$div_\Sigma(Y^N) = \sum_{i=1}^{n} \langle \nabla_{e_i} Y^N, e_i \rangle = \sum_{i=1}^{n} [e_i \langle Y^N, e_i \rangle - \langle Y^N, \nabla_{e_i} e_i \rangle]$$

$$= -\langle Y^N, \sum_{i=1}^{n} \nabla_{e_i} e_i \rangle = -\langle Y^N, \sum_{i=1}^{n} \nabla_{e_i}^\perp e_i \rangle = -\langle Y, H \rangle$$

Which means calculate the divergence of a vector fields on a submanifold is exactly the same to calculate the inner product of it with mean curvature.

2.2 Variation

Consider $F : \Sigma \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ is a family of maps parameterized by t, with boundary fixed at any time t. $F(x, 0)$ is just identity map to the original Σ. Choose a local orthonormal frame $\partial_i = \partial_{x_i}$, and push it to the frame on the image denoted by $F_i = F_{x_i}$.

Write $g_{ij} = \langle F_i, F_j \rangle$ is the metric. Then we can write down the volume at time t as:

$$vol(t) = \int_\Sigma \sqrt{detg_{ij}(t)}dx = \int_\Sigma \frac{\sqrt{detg_{ij}(t)}}{\sqrt{detg_{ij}(0)}}dV_\Sigma$$

Note that for the later part of the above formula, $V(x,t) = \frac{\sqrt{detg_{ij}(t)}}{\sqrt{detg_{ij}(0)}}$ is independent with the coordinates at fixed point. Thus we can choose good local frame to calculate. Choose our coordinates such that $g_{ij}(x,0) = \delta_{ij}$, which means F_i are orthonormal frames at $(x,0)$

$$vol'(0) = \int_\Sigma V'(x,0)dV_\Sigma$$

We calculate V':

$$v'(x,0) = \frac{1}{2}[detg_{ij}(x,t)]'|_{t=0}$$

$$= \frac{1}{2}Tr(g'_{ij}(x,0)) = \sum_{i=1}^{n}\langle F_{x_i,t}, F_{x_i}\rangle = \sum_i \langle \nabla_{F_i} F_t, F_{x_i}\rangle = div_\Sigma(F_t)$$

So if Σ is minimal, we must have $div_\Sigma(F_t) = 0$ for any variation V with compact support.

In particular, if F_t is normal, then by previous formula, we have

$$div_\Sigma(F_t) = -\langle F_t, H\rangle$$

Thus Σ is critical point of area is equivalent to $H = 0$.

2.3 Harmonic Coordinate

lemma 2.1. *Coordinate functions are harmonic on minimal Σ*

This is interesting, because it tells us that an extrinsic function (coordinate functions are natural for the ambient space, but not natural for the minimal surface).

Proof. We calculate the Laplacian of coordinate functions explicitly:

$$\nabla_\Sigma x_i = \nabla^\top x_i = \partial_i^\top$$

$$\Delta_\Sigma x_i = div_\Sigma(\nabla_\Sigma x_i) = div_\Sigma(\partial_i - \partial_i^\perp)$$

Note ∂_i is a constant vector field, so

$$= -div_\Sigma(\partial_i^\perp) = \langle H, \partial_i^\perp\rangle = 0$$

Since Σ is minimal.

$\square$

The most basic fact about harmonic functions is the maximal principle. By using maximal principle we can get the following corollary:

Corollary 2.1. *Given any coordinate function u on Σ, u must has its maximum and minimum on the boundary of Σ.*

Moreover, by this fact, if Σ is minimal, then $\Sigma \subset conv(\partial\Sigma)$, where $conv(X)$ is the convex hull of set X.

Proof. Suppose there is a point x on Σ which doesn't lies in $conv\partial\Sigma$. Then there is a hyperplane such that x belongs to it but it doesn't intersect with $conv\partial\Sigma$. To see this first note $\partial\Sigma$ is compact, that means $conv\partial\Sigma$ is closed. Then we can choose the plane orthogonal to the project segment from x to $conv\partial\Sigma$.

Then we choose the coordinate function which is the height function vertical to this hyperplane. By the lemma above we get a contradiction. □

Let's see an application in topology of minimal submanifold. We just see the simplest case:

Corollary 2.2. *(Monotonicity of Topology) Suppose $\Sigma \subset \mathbb{R}^3$ is minimal and simply connected, compact with boundary $\partial\Sigma$. Suppose B_R is a ball disjoint from $\partial\Sigma$, then $B_R \cap \Sigma$ is a union of disks.*

Proof. Since Σ is simply connected, then for any closed curve $\gamma \subset B_R \cap \Sigma$, γ must bound a disk $\Gamma \subset \Sigma$.

By previous corollary, this minimal surface Γ lies inside $conv(\gamma) \subset B_R$. Which implies the corollary. □

3 Scale Invariant Monotonicity Quantity

Next, we want to find a scale invariant monotonicity quantity. This is an idea in geometric problem and differential equation theory.

Note scale invariant is important, because that helps us consider the problem really geometrically (But not depends on the parameters).

3.1 Volume Density

Consider $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold. Let us consider the quantity:

$$\Theta_R = \frac{vol(B_R \cap \Sigma)}{R^k}$$

(Compare to Bishop-Gromov theorem)

It is an obvious scale invariant.

3.2 Baby Case

Let us first consider the most simple case: $\Sigma = \mathbb{R}^k$ is the hyperplane.

Then $\Delta_{\mathbb{R}^k}|x|^2 = 2k$, so we have:

$$V(R) = vol(B_R \cap \mathbb{R}^k) = \frac{1}{2k}\int_{B_R} \Delta_{\mathbb{R}^k|x|^2}$$

$$= \frac{1}{2k}\int_{\partial B_R} 2\langle(x_1, x_2, \cdots, x_k, 0, \cdots, 0), outer\ normal\ to\ B_R)\rangle = \frac{1}{2k}\int_{\partial B_R} R$$

So we have

$$V = \frac{R}{k}V' \Rightarrow \log(\frac{V}{R^k}) = 0$$

So we find the quantity is nondecreasing (in fact, a constant).

3.3 Coarea Formula

Hope I can fulfill this part in few weeks

Theorem 3.1.

$$\int_{\{f \le t\}} h|\nabla f| = \int_{-\infty}^{t}\int_{f=s} hds$$

3.4 General Case

let $V(R) = vol(B_R \cap \Sigma)$. We have

$$V(R) = \int_{|x| \le R} |\nabla_\Sigma |x||^{-1} |\nabla_\Sigma |x|| = \int_0^R \int_{\partial B_R \cap \Sigma} |\nabla_\Sigma |x||^{-1}$$

so

$$V'(R) = \int_{\partial B_R \cap \Sigma} |\nabla_\Sigma |x||^{-1}$$

Note, since Σ^k is minimal, we have

$$\Delta_\Sigma |x|^2 = div_\Sigma(2x^\top)$$

(This step is because $\nabla_\Sigma |x|^2 = \nabla^\top |x|^2 = 2x^\top$)

$$= 2div\Sigma(x - x^\top) = 2div_\Sigma(x) = 2k$$

By Stokes theorem, we have

$$V = \frac{1}{2k}\int_{\partial B_R \cap \Sigma} \nabla^\top |x|^2 \cdot normal = \frac{R}{k}\int_{\partial B_R \cap \Sigma} |\nabla^\top |x||$$

In the class Bill gave two proof of monotonicity.
Method 1: just note since $|\nabla^{\top}|x|| \leq 1 \Rightarrow V' \geq \int_{\partial B_R \cap \Sigma} 1$
Also notice

$$V \leq \frac{R}{k} \int_{\partial B_R \cap \Sigma} 1$$

$$\Rightarrow \frac{V'}{V} \geq \frac{k}{R} \Rightarrow \frac{V}{R^k} \geq 0$$

We can see when we have equality. Equality holds when:

$$|\nabla^{\top}|x|| = 1$$

That means x always tangent to Σ. Thus, Σ is invariant under dilation, means Σ is a minimal cone.

Method 2: We directly calculate the derivative.

$$(\frac{V}{R^k})' = \frac{V'}{R^k} - k\frac{V}{R^{k+1}} = \int_{\partial B_R \cap \Sigma} [\frac{|\nabla_{\Sigma}|x||^{-1}}{R^k} - \frac{|\nabla_{\Sigma}|x||}{R^k}]$$

$$= \int_{\partial B_R \cap \Sigma} |x|^{-k-1} \frac{|x^{\perp}|^2}{|x^{\top}|}$$

$$\Rightarrow (\frac{V}{R^k})' = \int_{\partial B_R \cap \Sigma} |x|^{-k-2} |x^{\perp}|^2 (\frac{|x|}{|x^{\top}|}))$$

Here we want to use coarea formula (Note $\nabla_{\Sigma}|x| = \frac{x^{\top}}{|x|}$)
Then we integrate both part, by coarea formula, we have

$$\frac{V(t)}{t^k} - \frac{V(s)}{s^k} = \int_{B_t \cap \Sigma - B_s} \frac{|x^{\perp}|^2}{|x|^{k+2}} \geq 0$$

remark 3.1. *We have monotonicity about the quantity $\Theta(R)$. Then we can well define the quantity **Density** at point 0 by*

$$\Theta_0 = \lim_{R \to 0} \frac{V(R)}{R^k} \geq 0$$

(Or we divide it by a dimensional constant)

We can also prove the monotonicity of the quantity in some special case in a simpler way:

Let $\sigma = \partial B_R \cap \Sigma$, and let C is the cone on σ. Then we have

$$vol(C) = \frac{R}{k}|\sigma| \geq vol(B_R \cap \Sigma)$$

$$\Rightarrow \frac{R}{k}|vol(\partial B_R \cap \Sigma)| \geq vol(B_R \cap \Sigma)$$

This is again the same differential equation we get before.

4 Density

Intuitively, density describe the local behavior of the point. Since every smooth surface Σ is locally $\mathbb{R}^2$, we can image at point $x_0 \in \Sigma$ we must have $\Theta(x_0) = \pi$

This is proved by Allard:

Theorem 4.1. *For $\Sigma^2 \subset \mathbb{R}^3$ minimal, then we have:*

1. $\Theta = \pi$ at all smooth points;

2. $\Theta > \pi + \epsilon$ at all singular points, for some $\epsilon > 0$

remark 4.1. *This type of "ϵ-regularity" theorems also appear in many other problems. The main spirit of this type of theorem is that if the scale invariant monotonicity quantity close to its minimal value, then the underlying space should be smooth.*

We come back to Allard theorem later. Now let's discuss a property of Θ. Let Θ_x denote the density at point x:

lemma 4.1. *Θ_y is upper semi-continuous in y. (i.e., if $y_i \to y$, then $\Theta_y \geq \limsup \Theta_{y_i}$)*

The by Allard theorem, we can get a corollary: singular points are closed.

Proof. Choose r small enough such that

$$|\frac{vol(B_r(y) \cap \Sigma)}{r^k} - \Theta_y| < \epsilon$$

Then if we choose y_i close to y, we have:

$$\Theta_{y_i} \leq \frac{vol(B_{r-|y-y_i|}(y_i) \cap \Sigma)}{(r - |y - y_i|)^k} \leq \frac{(B_r(y) \cap \Sigma)}{(r - |y - y_i|)^k}$$

Assume $|y - y_i < \epsilon r$, then we have

$$\Theta_{y_i} \leq \frac{(B_r(y) \cap \Sigma)}{(1 - \epsilon)^k r^k}$$

Let $\epsilon \to 0$ we get the required estimate.

□

4.1 Mean Value Inequality

We define density to be an quantity related to area, which is integral over the manifold with integrate function 1. How about a general function?

Suppose $\Sigma^k \subset \mathbb{R}^n$ is minimal as before. f is a function over Σ

Let $I(R) = \int_{B^r} \cap \Sigma$. If $f \equiv 1$ this is just volume as before.

Theorem 4.2.

$$\frac{I(t)}{t^k} - \frac{I(s)}{s^k} = \int_{B_t - B_s} f \frac{|x^\perp|^2}{|x|^{2+k}} + \frac{1}{2}\int_s^t r^{-k-1}[\int_{B_r \cap \Sigma} (r^2 - |x|^2)\Delta_\Sigma f]$$

Corollary 4.1. *If $\Delta_\Sigma f \geq 0$, then for $t \geq s$*

$$\frac{I(t)}{t^k} \geq \frac{I(s)}{s^k} \geq \lim_{s\to 0} \frac{I(s)}{s^k}$$

If Σ is smooth, by Allard theorem the quantity satisfies:

$$= f(0) vol(B_1 \subset \mathbb{R}^k)$$

Then we have

$$f(0) \leq \frac{\int_{B_t \cap \Sigma} f}{vol(B_t \subset \mathbb{R}^k)}$$

Proof. Let's prove mean value inequality.

Just as previous proof of density monotonicity, recall $\Delta_\Sigma |x|^2 = 2k$. Then we write

$$I(R) = \frac{1}{2k} \int_{B_R \cap \Sigma} f \Delta_\Sigma (|x|^2 - R^2)$$

Note

$$div_\Sigma (f \nabla_\Sigma (|x|^2 - R^2) - (|x|^2 - R^2) \nabla_\Sigma f)$$

$$= \nabla_\Sigma f \cdot 2x^\top + f \Delta_\Sigma (|x|^2 - R^2) - 2x^\top \cdot \nabla_\Sigma f - (|x|^2 - R^2) \Delta_\Sigma f$$

So by Stokes theorem:

$$I(R) = \frac{1}{2k} [\int_{B_R \cap \Sigma} \Delta_\Sigma f + \int_{\partial B_R \cap \Sigma} 2f |x^\top|]$$

By coarea formula, we have

$$I'(R) = \int_{\partial B_R \cap \Sigma} f \frac{R}{|x^\top|}$$

Then we can calculate:

$$(R^{-k} I)' = R^{-k} \int_{\partial B_R \cap \Sigma} f \frac{R}{|x|^\top} - kR^{-k-1} [\frac{1}{2k} \int_{B_R \cap \Sigma} (|x|^2 - R^2) \Delta_\Sigma f + \frac{1}{k} \int_{\partial B_R \cap \Sigma f |x^\top|}]$$

$$= \frac{1}{2} R^{-k-1} \int_{B_R \cap \Sigma} (R^2 - |x|^2) \Delta_\Sigma f + R^{-k} \int_{\partial B_R \cap \Sigma} f [\frac{R}{|x^\top|} - \frac{|x^\top|}{R}]$$

$$= \frac{1}{2} R^{-k-1} \int_{B_R \cap \Sigma} (R^2 - |x|^2) \Delta_\Sigma f + \int_{\partial B_R \cap \Sigma} f \frac{|x^\perp|^2}{|x|^{k+2}} \frac{|x|}{|x^\top|}$$

Then apply the coarea formula we get what we want.

∎

Corollary 4.2. *Suppose $\Delta_\Sigma f \geq -\lambda f$ and $f \geq 0$, and $0 \in \Sigma$. Then we have*

$$f(0) \leq e^{\frac{\lambda}{2}} \frac{\int_{B_1 \cap \Sigma} f}{vol(B_1 \subset \mathbb{R}^k)}$$

Proof. Note if $\lambda = 0$ then is just subharmonic case as before.

Let $g(t) = t^{-k} \int_{B_t \cap \Sigma} f$ for $t \leq 1$

Then we have

$$g'(t) \geq \frac{1}{2} t^{-k-1} \int_{B_t} (t^2 - |x|^2)(-\lambda f) \geq -\frac{\lambda}{2} t^{-k+1} \int_{B_t} f \geq \frac{\lambda}{2} tg \geq -\frac{\lambda}{2} g$$

By the differential inequality we can get the result.

□

4.2 $\Delta f \geq -f^2$?

We have discussed the mean value property for subharmonic function and sub-eigenvalue functions. How about f satisfies $\Delta f \geq -f^2$? Does the previous things work for this f?

The answer if **NO**. In fact, we can find f such that $\Delta f \geq -f^2$, and $\int_{B_1 \cap \Sigma} f = 1$, but $f(0)$ can be arbitrary large.

This is a bad news for minimal surface. In fact, Simon inequality:

$$\Delta |A|^2 \geq -2|A|^4$$

Is in this form. Thus, many estimate should first assume the second fundamental form is bounded.

5 Strong Maximal Principle

Recall the strong maximal principle for elliptic pdes:

Theorem 5.1. *Suppose w is a function over a domain Ω, satisfies equation:*

$$div(A_{ij} \partial_j w) = 0$$

where $A_{ij}(x) = A_{ji}(x)$ is positive definite smooth function on Ω. Then w can not have interior minimum value unless w is a constant.

This is the form we want to analysis minimal surface. We want to show the following property of minimal surfaces:

Theorem 5.2. *Suppose Σ and Γ are two smooth compact connected minimal surface and $\partial\Sigma \cap \Gamma = \emptyset, \partial\Gamma \cap \Sigma = \emptyset$. If Σ and Γ intersect but do not cross each other, then they coincide.*

Observe that suppose p is such a point that two minimal surface intersect but do not cross each other, then locally these two surfaces are graph over the tangent space at p. Thus, we only need to consider the problem under the assumption that Σ and Γ are graph of function u and v.

Proof. set $w = u - v$. By MSE, we have

$$div(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = div(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}) = 0$$

Define $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F(a) = \frac{a}{\sqrt{1+|a|^2}}$, then we have

$$div(F(\nabla u) - F(\nabla v))$$

we compute

$$dF_Y(Z) = \frac{Z}{\sqrt{1+|Y|^2}} - \frac{\langle Y, Z\rangle Y}{(1+|Y|^2)^{3/2}}$$

Then we have

$$\langle dF_Y(Z), Z\rangle = \frac{|Z|^2}{\sqrt{1+|Y|^2}} - \frac{\langle Y, Z\rangle^2}{(1+|Y|^2)^{3/2}} > 0$$

if $Z \neq 0$
So dF is positive definite. Thus

$$F(\nabla u) - F(\nabla v) = \int_0^1 dF_{(\nabla v + t\nabla w)}(\nabla w)dt = 0$$

is an elliptic equation in our previous maximal principle setting. So w satisfies maximal principle, which means if $w = 0$ in the interior some where then we must have $u = v$, which means two surface coincident. □

5.1 Rado-Schoen theorem

This section needs a lot of pictures, so I update it later.

6 Second Variation Formula

In 1968, J.Simons gave a fundamental variation formula for minimal surface.

Suppose $F : \Sigma \times (-\epsilon, \epsilon) \to M^{n+1}$ satisfies the following conditions:

(1)Σ is minimal and two sided (here two sided means Σ has trivial normal bundle);

(2)$F(\cdot, 0)$ is identity map;

(3)F_t has compact support;

(4)$F_t^\top = 0$

Then we have the second variation formula:

$$vol''(0) = -\int_\Sigma \langle F_t, LF_t \rangle$$

here L is an operator

$$L = \Delta_\Sigma^\perp + Ric_M(n,n) + |A|^2$$

Note the Δ_Σ is the non-positive term, and $Ric_M(n,n) + |A|^2$ is non-negative term if $Ric \geq 0$. This is the requirement always appear in minimal surface theory.

In the rest of this section, we will prove the second variation formula.

Proof. Let us choose local coordinates x_i on Σ, which induce metric $g_{ij}(x,t)$

Define

$$V(x,t) = \frac{\sqrt{detg_{ij}(x,t)}}{\sqrt{detg_{ij}(x,0)}}$$

We want to calculate the second derivative.

By computation of first order derivative, we have

$$V' = \frac{1}{2} V trace(g^{-1}g')$$

This can be derived from directed computation. Next we calculate the second variation of V:

$$V''(0) = \frac{1}{2}V'Trace(g^{-1}g') + \frac{1}{2}VTrace((g^{-1})'g') + \frac{1}{2}VTrace(g^{-1}g'')$$

For the rest of the proof, let us fix a point x and choose a local coordinate such that $g_{ij}(x,0) = \delta_{ij}$. Also note that $g^{-1}g = \mathbf{1}$, take derivative we get $(g^{-1})' = -g^{-1}g'g^{-1}$

So in this setting, note at time $t = 0$, we have $V = 1, V' = 0$:

$$V''(0) = \frac{1}{2}Trace((g')^2) + \frac{1}{2}Trace(g'')$$

Since g' is a symmetric matrix, we have $Trace((g')^2) = |g'|^2$

Now we need two lemma:

lemma 6.1.

$$Trace((g')^2)|^2 = 4|\langle A, F_t \rangle|^2$$

Proof.

$$g_{ij} = \langle F_{x_i}, F_{x_j} \rangle$$

So we have

$$g'ij = \langle F_{tx_i}, F_{x_j} \rangle + \langle F_{x_i}, F_{tx_j} \rangle$$

Since F_t is in the normal direction

$$= -\langle F_t, \nabla_{F_{x_i}} F_{x_j} \rangle - \langle F_t, \nabla_{F_{x_j}} F_{x_i} \rangle$$

$$= -\langle F_t, \nabla^{\perp}_{F_{x_i}} F_{x_j} \rangle - \langle F_t, \nabla^{\perp}_{F_{x_j}} F_{x_i} \rangle$$

So we have the result. □

lemma 6.2.

$$Trace(g'') = 2|\langle A, F_t \rangle|^2 + 2|\nabla^{\perp} F_t|^2 - 2Ric(F_t, F_t) + 2div_{\Sigma}(F_{tt})$$

Proof. Note

$$Trace(g''(0)) = 2\langle F_{x_i t}, F_{x_i t} \rangle + 2\langle F_{x_i tt}, F_{x_i}$$

We compute each part. The first part is:

$$\langle F_{x_i t}, F_{x_i t} \rangle = |\nabla_{F_{xi}} F_t|^2 = |\nabla_{\Sigma} F_t|^2 = |\nabla^{\perp}_{\Sigma} F_t|^2 + |\nabla^{\top}_{\Sigma} F_t|^2$$

$$= |\nabla^{\perp}_{\Sigma} F_t|^2 + |\langle A, F_t \rangle|^2$$

Here note that

$$|\nabla F^{\top}_{x_i} F_t|^2 = \langle \nabla_{F_{x_i} F_t, F_{x_j}} \rangle^2 = \langle F_t, \nabla_{F_{x_i}} F_{x_j} \rangle$$

Here we take the summation of i, j.
The second part is:

$$\langle \nabla_{F_t} \nabla_{F_t} F_{x_i}, F_{x_i} \rangle = \langle \nabla_{F_t} \nabla_{F_{x_i}} F_t, F_{x_i} \rangle = \langle R(F_{x_i}, F_t) F_t, F_{x_i} \rangle + \langle \nabla_{F_{x_i}} \nabla_{F_t} F_t, F_{x_i} \rangle$$

$$= -\langle R(F_t, F_{x_i}) F_t, F_{x_i} \rangle + div_{\Sigma}(F_{tt}) = -Ric(F_t, F_t) + div_{\Sigma}(F_{tt})$$

Combined these two formula we get the result. □

Now, we take the integral of $V''(x, o)$, we have

$$Vol''(0) = -\int_{\Sigma} |\langle A, F_t \rangle|^2 - |\nabla^{\perp}_{\Sigma} F_t|^2 + Ric(n, n)\langle F_t, F_t \rangle - div_{\Sigma}(F_{tt})$$

Note by Stokes theorem and minimality, we have the integral of the last term is zero. Then we finish the proof. □

6.1 Stability

We can a minimal surface is **stable** is $V''(0) \geq 0$ for any compact supported variation F_t.

Stability involves in many interesting problems in minimal surface theory. For instance: if Σ is stable and 2-sided, then we can write down the second variation formula in the following form:

$$\int_\Sigma fLf \leq 0$$

Here $F_t = fn$ is the variation. This formula is exactly:

$$\int_\Sigma f\Delta_\Sigma f + Ric_M(n,n)f^2 + |A|^2 f^2 \leq 0$$

i.e.

$$\int_\Sigma (|A|^2 + Ric_M(n,n))f^2 \leq \int_\Sigma |\nabla_\Sigma f|^2$$

We can get many result from this solution. For instance, For any ambient manifold with Ricci curvature greater than 0, if we choose $f = 1$ over the minimal closed subsurface, we know that there is no stable 2-sided closed minimal surfaces in this manifold.

There also many other properties concerning stability. Maybe add some here later.

6.2 More about Stability

We continue considering properties of stable minimal surface. The starting points are always second variational formula. If the operator L is related to some stable minimal surface, then sometimes we also call L is stable.

lemma 6.3. *L is stable if there exists a function $u > 0$ on Σ such that $Lu = 0$*

Intuitively, this can be viewed from the perspective of spectrum theory. Consider that L is symmetric, then we can diagonalize it to get all eigenvalues λ_i and eigenfunctions u_i. i.e.

$$Lu_j = -\lambda_j u_j$$

by spectrum theory

$$\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

if we can prove that $0 = \lambda_1$ is the lowest eigenvalue, then we done. Note if $u_1 > 0$, in fact we get λ_1 is multiplicity 1. (Why?)

Proof. Take f is a compactly supported function on Σ, then we have

$$div(f^2 \frac{\nabla u}{u}) = 2f\langle \nabla f, \frac{\nabla u}{u}\rangle + f^2 \frac{\Delta u}{u} - f^2 \frac{|\nabla u|^2}{u^2}$$

Define

$$q = |A|^2 + Ric_M(n,n)$$

Note by $Lu = 0$, we have

$$\frac{\Delta u}{u} = -q$$

So

$$\int f^2 q = \int -f^2 \frac{|\nabla u|^2}{u^2} + 2f\langle \nabla f, \frac{\nabla u}{u} \rangle$$

$$= \int |\frac{f\nabla u}{u} - \nabla f|^2 + |\nabla f|^2 \leq \int |\nabla f|^2$$

This means for any f it satisfies the stability condition.

□

As a corollary, we can show minimal graphs are always stable:

Corollary 6.1. *Minimal graphs in $\mathbb{R}^3$ are stable.*

Proof. let ∂z is the constant vector field point to the z- direction everywhere in $\mathbb{R}^3$. Let $f = \langle \partial z, n \rangle$. Since Σ is a graph, $f > 0$. so we by the lemma above we only need to prove that $Lf = 0$.

Note the ambient space is flat, so we do not need to consider the curvature term.

Let us compute $L\langle \partial z, n \rangle$ terms by terms. Let us choose a local geodesic coordinate $\{x_i\}$ on the surface.

$$\Delta_\Sigma \langle \partial z, n \rangle = \nabla_i \nabla_i \langle \partial z, n \rangle = \langle \partial z, \nabla_i \nabla_i n \rangle$$

$$= \langle \partial z, \nabla_i(-A_{ji}\partial j) \rangle = \langle \partial z, (-A_{ji,i})\partial j \rangle = \langle \partial z, -A_{ji}\nabla_i \partial j \rangle$$

By Bianchi identity, $A_{ji,i} = A_{ii,j}$, by minimality of the surface, $A_{ii,j} = 0$. By the selection of local geodesic frame we have at one point x, we have $\nabla_i^\Sigma \partial j = 0$, i.e. $\nabla_i \partial j = A_{ij}$

So we have:

$$\Delta_\Sigma \langle \partial z, n \rangle = -\langle \partial z, n \rangle |A|^2$$

Combined with the other term of L we have the result. □

6.3 Parabolic

Definition 6.1. *We call a complete non-compact manifold Σ is parabolic if there exist a sequence of functions u_j with compact support such that:*

(1)$u_j \to 1$ on compact sets of Σ

(2)$\int_\Sigma |\nabla u_j|^2 \to 0$ as $j \to +\infty$

Parabolic is a useful property when we analyze stable minimal surfaces.

Theorem 6.1. *Suppose Σ is a 2-sided stable parabolic minimal hypersurface of M, while M has non-negative Ricci curvature.*

Then we have Σ is totally geodesic.

Proof. By stability we have the inequality:

$$\int \Sigma(|A|^2 + Ric_M(n,n))u_j^2 \leq \int_\Sigma |\nabla u_j|^2$$

Note by definition of parabolic, the right hand side will goes to 0 when $j \to \infty$, meanwhile the left hand side will goes to $\int_\Sigma |A|^2$. Hence A vanishes on Σ everywhere. Thus, Σ is totally geodesic. $\square$

Theorem 6.2. *suppose Σ is a complete hypersurface with no boundary, for any $r > 0$ Area of $B_r \leq Cr^2$. Then Σ is parabolic.*

Before we prove this property, we first see an example: how to show that $\mathbb{R}^2$ is a parabolic manifold.

Let us construct the functions

$$u_j(x) = \begin{cases} 1 & \text{on } B_{e^j} \\ 2 - \frac{\log r}{j} & \text{on } B_{e^{2j}} - B_{e^j} \\ 0 & \text{outside} \end{cases}$$

Notice the construction use the Newtonian potential in $\mathbb{R}^2$. Then we have $\int |\nabla u_j|^2 = \frac{2\pi}{j}$, so $\mathbb{R}^2$ is parabolic by definition.

Now we prove the theorem:

Proof. Similarly construction. We construct a sequence of functions as:

$$u_j(x) = \begin{cases} 1 & \text{on } B_{e^j} \\ 2 - \frac{\log r}{j} & \text{on } B_{e^{2j}} - B_{e^j} \\ 0 & \text{outside} \end{cases}$$

Note here the balls are the intrinsic ball over the manifold.

Though we still have $|\nabla u_j| = \frac{1}{jr}|\nabla r| \leq \frac{1}{jr}$ in the non-constant part of the manifold, we can not just prove the estimate as in the example because now the manifold is no longer homogeneous. However we can estimate the integration in bands, i.e.:

$$\int_{B_{e^{j+1}}-B_{e^j}} |\nabla u_j|^2 \leq \frac{1}{j^2}\frac{1}{e^{2j}} Area(B_{e^{j+1}})$$

by the assumption we have

$$\int_{B_{e^{j+1}}-B_{e^j}} |\nabla u_j|^2 \leq C\frac{e^2}{j^2}$$

Then we integrate $\int_{B_{e^{2j}}-B_{e^j}}$ into these pieces, we have

$$\int_{B_{e^{2j}}-B_{e^j}} |\nabla u_j|^2 \leq C\frac{e^2}{j}$$

So we proved that Σ is parabolic.

$\square$

Corollary 6.2. *(Bernstein Theorem) Any entire minimal graph in $\mathbb{R}^2$ is a plane.*

Proof. Since the minimal graph satisfies the local minimal property:

$$Area(B_r) \leq 2\pi r^2$$

So by the lemma the minimal surface is parabolic. Since it is also 2-sided and stable, it is totally geodesic so it is a plane. $\square$

Finally we show one more property related to parabolic.

lemma 6.4. *Suppose Σ is parabolic, and u is a positive superharmonic over Σ, then u is a constant.*

Proof. By computation we have:

$$div(u_j^2\frac{\nabla u}{u}) = 2u_j\langle\nabla u_j, \frac{\nabla u}{u}\rangle + u_j^2\frac{\Delta u}{u} - u_j^2\frac{|\nabla u|^2}{u^2}$$

By integration we have:

$$\int u_j^2\frac{|\nabla u|^2}{u^2} \leq \int 2\langle\nabla u_j, \frac{u_j\nabla u}{u}\rangle\frac{1}{2}\int u_j\frac{|\nabla u|^2}{u^2} + \int |\nabla u_j|^2$$

Then we have

$$\int u_j^2\frac{|\nabla u|^2}{u^2} \leq 4\int |\nabla u_j|^2 \to 0$$

Let $j \to \infty$ we have:

$$|\nabla u| = 0$$

So u is constant.

$\square$

7 Simons Inequality

In 1968, Simons discovered the important inequality for second fundamental proof. This is one of the most important technique in minimal surface theory.

Theorem 7.1. *Suppose Σ is minimal, then we have*

$$\Delta_\Sigma A = -|A|^2 A$$

From this equality we have:

$$\frac{1}{2}\Delta_\Sigma |A|^2 = |A|^4 + |\nabla A|^2$$

Hence we have

$$\frac{1}{2}\Delta_\Sigma |A|^2 \geq -|A|^4$$

The last inequality we will use in the curvature estimate.

Before we prove the equality, let us first recall the Codazzi equation. Codazzi equation is about the calculation of the second fundamental forms:

$$(\nabla_U A)(V,W) = \nabla_U(A(V,W)) - A(\nabla_U^\top V, W) - A(V, \nabla_U^\top W)$$

$$= U\langle \nabla_V W, n\rangle - \langle \nabla_{\nabla_U^\top V} W, n\rangle - \langle \nabla_V \nabla_U^\top W, n\rangle$$

$$= -U\langle W, \nabla_V n\rangle + \langle W, \nabla_{\nabla_U^\top V} n\rangle + \langle \nabla_U^\top W, \nabla_V n\rangle$$

Note $\nabla_X n$ is tangential, we have

$$= -\langle \nabla_U W, \nabla_V n\rangle - \langle W, \nabla_U \nabla_V n\rangle + \langle W, \nabla_{\nabla_U^\top V} n\rangle + \langle \nabla_V n, \nabla_U W\rangle$$

Thus by the definition of Riemannian tensor we know that

$$(\nabla_U A)(V,W) = (\nabla_V A)(U,W)$$

So the derivative of the second fundamental forms are fully symmetric.

Now we prove Simons inequality. We show the general case:

Theorem 7.2. *Consider $\Sigma^n \to \mathbb{R}^{n+1}$, here Σ is not necessary minimal. Then we have*

$$\Delta_\Sigma A = -Hess_H - |A|^2 A - HA^2$$

Just a caution: note here the different definition of H (the symbol) will influence the formulate of the result. We define H such that S^n has $H > 0$

Proof. Let us first accept a lemma here:

$$A_{ij,kl} - A_{ij,lk} = R^{\Sigma}_{lkim} A_{mj} + R^{\Sigma}_{lkjm} A_{mi}$$

It is easy, just directly computation. So we can compute:

$$\Delta A(e_j, e_j) = A_{ij,kk}$$

By Codazzi equation:

$$= A_{ik,jk} = A_{ik,kj} + R_{kjim} A_{mk} + R_{kjkm} A_{mi}$$

Note the first term is $A_{kk,ij}$ by Codazzi equation, which is the Hessian term. This term vanishes in minimal case. The rest term, by Gauss equation:

$$R_{ijkl} = A_{ik} A_{jl} - A_{il} A_{jk}$$

So we have

$$\Delta_{\Sigma} A = -Hess_H - |A|^2 A - H A^2$$

Then we can naturally get Simons inequality. □

Here in fact we reduce the difficulty in general case, since the ambient space is flat. If the ambient space is non-flat, then we can get a more general Simons inequality:

$$\frac{1}{2} \Delta_{\Sigma} |A|^2 \geq -\lambda |A|^4 + |\nabla A|^2$$

Where λ depends on the curvature and the derivative of the curvature of the ambient space.

7.1 Curvature Estimate by Choi and Schoen

Choi and Schoen used Simons inequality showed the following result (we consider the easiest case):

Theorem 7.3. *Suppose $\Sigma^2 \subset B_R \subset \mathbb{R}^3, \partial\Sigma \subset \partial B_R$, here Σ is a minimal surface. Then there exists $\epsilon > 0$ such that if $\int_\Sigma |A|^2 < \epsilon$, then*

$$(R - |x|)^2 |A|^2(x) \leq 1$$

What's more, $|A|^2(0) \leq \frac{1}{R^2}$

The key idea is to find a scaling invariant quantity. Note the condition $\int_\Sigma |A|^2 < \epsilon$ itself is a scaling invariant quantity in dimension 2.

Proof. Let us define $F(x) = (R - |x|)^2|A|^2(x)$. Fix $p \in \Sigma$ with $F(p) = \max_{x \in B_R} F(x)$. Since F vanishes on the boundary, p must in the interior of Σ. We want to show $F(p) \leq 1$. Show by contradiction: suppose $F(p) > 1$. Define $\sigma > 0$ such that $\sigma^2|A|^2(p) = \frac{1}{4}$. Then we have:

$$\frac{1}{4\sigma^2} = |A|^2(p) = \frac{F(p)}{(R - |p|)^2} > \frac{1}{(R - |p|)^2}$$

Hence we have

$$\sigma \leq \frac{R - |p|}{2}$$

on the ball $B_\sigma(p)$, by triangle inequality we have

$$\frac{1}{2} \leq \frac{R - |x|}{R - |p|} \leq 2$$

Then we have

$$(R - |p|)^2|A|^2(x) \leq 4(R - |x|)^2|A|^2(x) = 4F(x) \leq 4F(p) = 4|A|^2(p)(R - |p|)^2$$

So on the ball $B_\sigma(p)$ we have

$$|A|^2(x) \leq 4|A|^2(p)$$

Next we define $\tilde{\Sigma}$ to be the rescaled surface which is Σ rescale by $\frac{1}{\sigma}$ around p. Then $\tilde{\Sigma}$ is still minimal. After the rescaling we have:

$$\sup_{B_1(p) \cap \tilde{\Sigma}} |\tilde{A}|^2 \leq 4|\tilde{A}|^2(p) = 4|A|^2(p)\sigma^2 = 1$$

i.e.$|\tilde{A}|^2 \leq 1$ on this rescaling ball. So by Simons inequality we have

$$\Delta|\tilde{A}|^2 \geq -2|\tilde{A}|^2|\tilde{A}|^2 \geq -2|\tilde{A}|^2$$

Then we applied mean value property we get:

$$|\tilde{A}|^2(p) \leq \frac{e}{\pi} \int_{B_1(p) \cap \tilde{\Sigma}} |\tilde{A}|^2 = \frac{e}{\pi} \int_{B_\sigma(p) \cap \Sigma} |A|^2$$

So if we choose $\epsilon \leq \frac{\pi}{4e}$ we will get a contradiction. Thus we finish the proof. $\square$

There is a similar curvature estimate of minimal graph by Heinz, which is the following theorem:

Theorem 7.4. *(Heinz) Suppose Σ is a minimal graph in $\mathbb{R}^3$ over $D_R \subset \mathbb{R}^2$, then*

$$|A|^2(\textit{point above } 0) \leq \frac{C}{R^2}$$

The proof of the theorem uses Choi-Schoen estimate once if we have a curvature integral $\int |A|^2$ bound. To do this, we need to notice that graph is stable with quadratic area growth, hence we can apply stability equation.

remark 7.1. *It is interesting to ask the question: why the condition in Choi-Schoen is a bound of* $\int_{B_R \cap \Sigma} |A|^2$*?*

There are several ways to see it is a natural bound in minimal surface theory. First thing is the stability inequality. Suppose our minimal surface is stable, then we can get a bound of $\int_{B_R \cap \Sigma} |A|^2$*. Another thing is Gauss-Bonnet formula, by Gauss-Bonnet we can also get a bound of* $\int_{B_R \cap \Sigma} |A|^2$*, notice we use Gauss equation here.*

This is one reason why minimal surface in higher dimension is more difficult, because $\int_{B_R \cap \Sigma} |A|^2$ *is no longer a scaling invariant in higher dimension, but the quantity we can get from stability inequality is still* $\int_{B_R \cap \Sigma} |A|^2$*.*

8 Curvature Estimate for Intrinsic Ball

When we study a submanifold, we always need to aware that the intrinsic geometry and the extrinsic geometry of the submanifold may be different. So sometimes we also need intrinsic estimate for the minimal surface.

From now on we will use $B_r(p)$ to denote the extrinsic ball, and $\mathcal{B}_r(p)$ to denote the intrinsic ball on Σ, i.e. $\mathcal{B}_r(p) = \{x \in \Sigma | dist_\Sigma(x,p) \le r\}$.

We need several lemmas to proceed the curvature estimate for intrinsic balls. First of all, we want a locally graphic lemma:

lemma 8.1. *Suppose* $|A| \le 1$ *on* $\mathcal{B}_1(p)$*, then we have* $\mathcal{B}_{\frac{1}{2}}(p)$ *is a graph over* $T_p\Sigma$ *with the gradient* $|\nabla u| \le 1$*, here we assume* u *is the graph function. Moreover,* $\partial \mathcal{B}_{\frac{1}{2}}(p) \cap B_{\frac{1}{4}}(p) = \emptyset$

This fact can be viewed intuitively: suppose the second fundamental does not change a lot near some point, then in a small neighborhood of that point the normal vector will also does not change a lot, so locally it looks like a graph.

Proof. Let $d(t) = dist_{S^2}(n(p), n(\gamma(t))$, if we can show this distance is no more than $\frac{\pi}{4}$ then it is locally a graph. Note d is a Lipschitz function, hence we can take differentiation:

$$|d'(t)| \le |\nabla_{\gamma'} n| \le |A| \le 1$$

Hence we know $d(t) \le t \le \frac{1}{2}$ for $t \le \frac{1}{2}$. So $\mathcal{B}_{\frac{1}{2}}(p)$ is locally a graph with gradient no more than 1.

Next we show that $\partial \mathcal{B}_{\frac{1}{2}}(p) \cap B_{\frac{1}{4}}(p) = \emptyset$. To show this, we only need to prove that any geodesic γ starting from p will leave $B_{\frac{1}{4}}(p)$ after time $\frac{1}{2}$. Set $f(t) = \langle \gamma(t), \gamma'(0) \rangle$. Then we only need to show $f(\frac{1}{2}) > \frac{1}{4}$, which means this geodesic at least in $\gamma'(0)$ direction can go out of the ball.

Note $f(0) = 0$, and take derivative:

$$f'(t) = \langle \gamma'(t), \gamma'(0) \rangle$$

$$f''(t) = \langle \nabla_{\gamma'} \gamma', \gamma'(0) \rangle = \langle A(\gamma', \gamma')n, \gamma'(0) \rangle$$

so $f'(0) = 1$, $|f''| \leq 1$. By integration we have $f(t) \geq t - \frac{1}{2}t^2$, hence we get the result. $\square$

The next theorem concerning the estimate from Gauss-Bonnet theorem. Recall

Theorem 8.1. *(Gauss-Bonnet) Suppose Σ is a two dimensional surface with possible boundary $\partial\Sigma$ which is piecewise smooth. Then we have:*

$$\int_{\Sigma} K + \int_{\partial\Sigma} k_g + \sum angles = 2\pi\chi(\Sigma)$$

Here K is the Gauss curvature, k_g is the geodesic curvature and χ is the Euler characteristic.

From Gauss-Bonnet theorem, we can get calculate the area of surfaces. Let $l(r) = |\partial\mathcal{B}_r(p)|$ and $A_r = |\mathcal{B}_r(p)|$. Then take derivative we have

$$A'(r) = l(r), l'(r) = \int_{\partial\mathcal{B}_r} k_g = 2\pi - \int_{\mathcal{B}_r} K$$

Now assume Σ is minimal, then we have $K = -\frac{1}{2}|A|^2$, and then

$$l'(r) = 2\pi + \frac{1}{2}\int_{\mathcal{B}_r} |A|^2$$

Then take integration we have

$$l(r) = 2\pi r + \frac{1}{2}\int_0^r \int_{\mathcal{B}_s} |A|^2 ds$$

$$A(r) = \pi r^2 + \frac{1}{2}\int_0^r \int_0^t \int_{\mathcal{B}_s} |A|^2 ds$$

Corollary 8.1. *Suppose Σ is a minimal disk and $\mathcal{B}_R \cap \partial\Sigma = \emptyset$. Then we have*

$$2(A(R) - \pi R^2) = \int_0^R \int_0^t \int_{\mathcal{B}_s} |A|^2 = \frac{1}{2}\int_{\mathcal{B}_R} |A|^2 (R-r)^2$$

Here r is the intrinsic distance to the center of the ball.

To prove the corollary, we only need to do integration by part of two functions $f(t) = \int_0^t \int_{\mathcal{B}_s} |A|^2 ds$ and $g(t) = \frac{1}{2}(R-t)^2$.

By this formula we can derive the following area bound:

Theorem 8.2. *(Colding-Minicozzi) Suppose Σ is a stable two sided minimal disk, $\mathcal{B}_R \cap \partial\Sigma = \emptyset$, then we have $A(R) \leq \frac{4\pi}{3}R^2$*

Proof. We use $R-r$ as a cut-off function in stability inequality, then we have

$$\int_{\mathcal{B}_R} (R-r)^2|A|^2 \leq \int_{\mathcal{B}_R} |\nabla(R-r)|^2$$

Since the left hand side is $4A(R) - 4\pi R^2$ by previous formula, and the right hand side is $A(R)$, we get the result.

□

Corollary 8.2. *Let Σ a stable 2 sided complete minimal surface in $\mathbb{R}^3$, then Σ is flat.*

Only note from previous theorem if Σ is a disk, then it has quadratic area growth, which means it is parabolic. Hence it is flat. If Σ is not a disk, consider its universal cover, which is still stable, then from disk case we can get the result.

Similar to this result, Schoen conjectured that: If $\Sigma^3 \subset \mathbb{R}^4$ is a complete 2-sided stable minimal hypersurface, then it is flat. In 1975, Schoen-Simon-Yau proved a special case: when $Area(B_R \cap \Sigma) \leq CR^3$, then the conjecture holds.

8.1 Colding-Minicozzi Estimate for Embedded Minimal Disk

Theorem 8.3. *Given constant C_1, there exists C_2 such that if Σ is an embedded minimal disk in $\mathbb{R}^3$, then if $mathcal B_{2s} \subset \Sigma - \partial\Sigma$ and $\int_{\mathcal{B}_{2s}} |A|^2 \leq C_1$, we have $\sup\limits_{\mathcal{B}_s} |A|^2 \leq \frac{C_2}{s^2}$, here C_2 only depends on C_1.*

Note this theorem implies the classical Bernstein theorem for embedding minimal surface with integration bounded for curvature.

Before prove the theorem, we prove a key lemma.

lemma 8.2. *Given $C_0 > 0$ there exists a constant $\epsilon > 0$ such that if Σ is a minimal disk, $\mathcal{B}_{9r} \subset \Sigma - \partial\Sigma$ and $\int_{\mathcal{B}_{9r}} |A|^2 \leq C$, and $\int_{\mathcal{B}_{9r}-\mathcal{B}_r} |A|^2 \leq \epsilon$, then we have $\sup\limits_{\mathcal{B}_r} |A|^2 \leq \frac{C}{r^2}$ (In fact, we can show $\sup\limits_{\mathcal{B}_r} |A|^2 \leq \frac{1}{r^2}$)*

Before we prove this lemma, let us see what's the difference between this key lemma and the Choi-Schoen estimate. For Choi-Schoen, we need a global tiny bound for the integration of curvature. Here we only need an upper bound of integration of curvature in the whole space, and only a tiny bound in the annulus around the ball we want to estimate.

From a certain perspective, this means if the area around the ball has bounded curvature integration, then the ball it self can not have pointwise large curvature.

Proof. Let $Q = \mathcal{B}_{8r} - \mathcal{B}_{2r}$. Then for any $p \in Q$, we have $\int_{\mathcal{B}_r(p)} |A|^2 \leq \epsilon$. So we can apply Choi-Schoen estimate in this ball, hence

$$|A|^2(p) \leq \frac{C\epsilon}{r^2}$$

For any $p \in Q$. Also, our previous locally graphic lemma can applied to this small ball. Now we show that Q is locally a graph over a plane and the projection down will cover the annulus $D_{3r} - D_{2r}$ of the plane.

Now fix $p \in \partial\mathcal{B}_{2s}$, and let $T_p\Sigma$ is horizontal. Then given any $q \in Q$, we can join p, q by a path in Σ with length at most $6r + \frac{1}{2}|\partial\mathcal{B}_{2r}|$. To construct such a path, we only first connected q to $\partial\mathcal{B}_{2r}$ by a segment with length no more than $6r$, then we use the path on the boundary to connected that point to p.

By the length formula before, we have

$$l(R) = 2\pi R + \frac{1}{2}\int_0^R \int_{\mathcal{B}_t} |A|^2 dt$$

So we can estimate

$$l(2r) \leq 4\pi r + r\int_{\mathcal{B}_{2r}} |A|^2 \leq (4\pi + C_0)r$$

Then we get a path connected p, q with length lies in Q no more than $C_1 r$. So we have

$$|n(p) - n(q)| \leq C_1 r \sup_Q |A| \leq C_1 r \frac{C\sqrt{\epsilon}}{r} = C_2\sqrt{\epsilon}$$

Then if we take ϵ small enough we can bound $|n(p) - n(q)|$ as small as we want. So we proved that Q is locally graphic by previous locally graphic lemma.

Next, we fix $x \in \partial\mathcal{B}_{2s}$, and let γ_x is the geodesic go out of x which is perpendicular to $\partial\mathcal{B}_{2s}$. Then we have

$$|\gamma_x''| = |A(\gamma_x', \gamma_x')| \leq \frac{C\sqrt{\epsilon}}{r}$$

We integrate to get

$$|\gamma_x'(0) - \gamma_x'(t)| \leq \frac{C\sqrt{\epsilon}}{r}t$$

Integrate $\langle\gamma_x'(t), \gamma_x'(0)\rangle = \langle\gamma_x'(t) - \gamma_x'(0) + \gamma_x'(0), \gamma_x'(0)\rangle$, we get the estimate that

$$\langle\gamma_x(6r) - \gamma_x(0), \gamma_x'(0)\rangle > 6r(1 - C\sqrt{\epsilon})$$

So it means $\gamma_x(6r)$ is outside the cylinder over the plane of radius $3r$. Then we proved that the projection of Q covers the annulus of the plane.

Let's now consider the cylinder of radius $2r$ intersects Q, which is a collection of closed embedded and graphical curves over ∂D_{2s}. Those intersections are curves, i.e. transversal intersection with Q is because Q is locally a graph (or we can use Sard type theorem to assert with a different cylinder with a little bit different radius). These curves are closed because of our previous conclusion, because the projection of Q covers $D_{3r} - D_{2r}$.

Each curve bounds a disk in Σ because Σ is a disk and we have convex hull theorem in previous sections. Let us choose one such that it bounds $\mathcal{B}_r$, we call this curve σ and the disk it bounds by Γ, $\mathcal{B}_r \subset \Gamma$. Since $\partial\Gamma = \sigma$ is a graph over the boundary of a convex domain, by Rado theorem we have Γ is a graph over the disk D_{2r}

Then we use Heinz theorem we know that

$$|A|^2 \leq \frac{C}{r^2}$$

on $\mathcal{B}_r$. Then we get the result.

□

By this lemma, we can prove the theorem in the beginning of this section:

Proof. Let us fix ϵ in the lemma. For given C_1, we can choose N large enough such that $\frac{C_1}{N} \leq \epsilon$

Then for any $p \in \mathcal{B}_s$ we consider the annulus $D_i = \mathcal{B}_{9^{-i+1}s}(p) - \mathcal{B}_{9^{-i}s}(p)$, we have

$$D_1 \cup D_2 \cup \cdots \cup D_N \cup \mathcal{B}_{9^{-N}s}(p) \subset \mathcal{B}_{2s}$$

And they do not intersect each other. So at least one of these components have integration curvature no more than ϵ. So we can apply the lemma and we can assert that

$$|A|^2(p) \leq \frac{C}{9^{-2N-2}s^2} \leq \frac{\tilde{C}}{s^2}$$

So we finish the proof.

□

8.2 Schoen-Simon-Yau (SSY) Estimate

In this section we discuss the famous Schoen-Simon-Yau estimate:

Theorem 8.4. *Suppose $\Sigma^{n-1} \subset \mathbb{R}^n$ is a 2-sided stable minimal hypersurface. Then for $p \in [2, 2+\sqrt{\frac{2}{n-1}}]$ and ϕ a cut-off function, we have*

$$\int_\Sigma |A|^{2p}\phi^{2p} \leq C_{n,p} \int_\Sigma |\nabla\phi|^{2p}$$

One perspective concerning this estimate is the following: from Sobolev embedding theorem, we know that (roughly) $\int_\Sigma |\nabla u|^{2p}$ can control the oscillation of u when $2p > n-1$. So if $4 + 2\sqrt{\frac{2}{n-1}} > n-1$ we can bound the L^∞ norm of A by standard elliptic estimate. So if $n = 2, 3, 4, 5, 6$, from the SSY estimate we can get a good bound for $|A|_{L^\infty}$, then we can applied Colding-Minicozzi estimate in the previous section to get a Bernstein type theorem.

When $n \geq 8$ we know Bernstein theorem fails. What about $n = 7$? We need more specific analysis to show Bernstein theorem.

Proof. Choose $\eta = |A|^{1+q} f$, let $0 \leq q \leq \sqrt{\frac{2}{n-1}}$. Then we have

$$\nabla \eta = (1+q)|A|^q f \nabla |A| + |A|^{1+q} \nabla f$$

$$|\nabla \eta|^2 = (1+q)|A|^2 q f^2 |\nabla |A||^2 + |A|^{2+2q} |\nabla f|^2 + 2(1+q)|A|^{1+2q} f \langle \nabla f, \nabla |A| \rangle$$

So by stability inequality we have

$$\int |A|^{4+2q} f^2 \leq (1+q)^2 \int f^2 |A|^{2q} |\nabla |A||^2 + \int |A|^{2+2q} |\nabla f|^2 + 2(1+q) \int |A|^{1+2q} f \langle \nabla f, \nabla |A| \rangle$$

By Simons inequality we have

$$|A| \nabla |A| + |A|^4 \geq \frac{2}{n-1} |\nabla |A||^2$$

then we multiply by $f^2 |A|^{2q}$ and integrate to get

$$\frac{2}{n-1} \int f^2 |A|^{2q} |\nabla |A||^2 \leq \int f^2 |A|^{1+2q} \Delta |A| + \int f^2 |A|^{4+2q}$$

integration by part:

$$= \int f^2 |A|^{4+2q} - \int 2 \langle \nabla f, \nabla |A| \rangle f |A|^{1+2q} - (1+2q) \int f^2 |A|^{2q} |\nabla |A||^2$$

combined these two inequality we have

$$(\frac{2}{n-1} - q^2) \int f^2 |A|^{2q} |\nabla |A||^2 \leq \int |A|^{2q+2} |\nabla f|^2 + 2q \int f |A|^{1+2q} |\nabla f| |\nabla |A||$$

note

$$f |A|^{1+2q} |\nabla f| |\nabla |A|| = (f |A|^q |\nabla |A||)(|A|^{1+q} |\nabla f|)$$

So by standard Holder inequality method we have

$$(\frac{2}{n-1} - q^2 - \epsilon q) \int f^2 |A|^{2q} |\nabla |A||^2 \leq (1 + \frac{q}{\epsilon}) \int |A|^{2+2q} |\nabla f|^2$$

If $q^2 < \frac{2}{n-1}$, we can choose ϵ small enough such that $q^2 + \epsilon q < \frac{2}{n-1}$
Plug into the stability inequality again we have

$$\int |A|^{4+2q} f^2 \leq 2(1+q)^2 \int f^2 |A|^{2q} |\nabla |A||^2 + 2 \int |A|^{2+2q} |\nabla f|^2$$

$$\leq \left(\frac{2 + 2(1+q)^2(q + \frac{1}{\epsilon})}{\frac{1}{n-1} - q^2 - \epsilon q}\right) \int |A|^{2+2q} |\nabla f|^2$$

Let $f = \phi^p$ for $p = 2 + q$, we get the result.

□

Notice that if $\int |\nabla \phi|^{2p} \to 0$ but in a large range $\phi \geq 1$, we can get Bernstein theorem. This can be done if $vol(\mathcal{B}_r) \leq r^a$ for $a < 2p$, since we can applied the log cut-off to get $\int |\nabla \phi|^{2p} \leq (2r)^a r^{-2p}$

8.3 How to Find Stable Items?

In order to applied the stability inequality, we always want to have some stable guys. There are several ways to find stable minimal items:
(1)Plateau problems
(2)Hoffman-Meeks
(3)Min-Max
(4)Disjoint guys are close to some limit

9 From Helicoid to Multi-graph

The first thing we do to study something is to look at some examples. Besides plane, the simplest example of embedded minimal surface in $\mathbb{R}^3$ is helicoid.

Definition 9.1. Helicoid *is the minimal surface in* $\mathbb{R}^3$ *defined by the equation*

$$\tan z = \frac{y}{x}$$

.

Helicoid is a ruled surface, i.e. at each point on the surface there is a straight line through this point complete lies in the surface.

We can compute that on the helicoid, on the z-axis $|A| = 1$, while in the most of rest part of helicoid, $|A|$ is very tiny. And the whole helicoid spiral along the z-axis, turns out to be a multi-graph besides z-axis. This is one property we hope to study for general minimal surface.

We want to show the following results:

1. If $|A|$ is large somewhere, then the spiraling happens. i.e. we can find local multi-graph;

2. once spiraling happens, the spiraling will keep going.

Roughly speaking, we want to get the following theorem:

Theorem 9.1. *(Colding-Minicozzi) Let Σ be an embedded minimal disk, $\Sigma \subset B_R$, $\partial\Sigma \subset \partial B_R$, and $|A|$ is large in B_1. Then Σ contains a multi-value graph in $B_{R/C}$ for C is a fixed constant independent of Σ.*

We haven't define multi-value graph (In later context, multi-graph in abbreviation), and we also do not explicit say how large $|A|$ is in the unit ball. We will say more about them in further context.

9.1 Graph Over a Surface

Image we have a minimal surface Σ, and there is another one Σ' very close to it, let's say it looks like a graph over it. Then Σ' can be viewed "almost a Jacobi field" over Σ. Since we know positive Jacobi field indicates stability, we hope this Σ' can indicate some "almost stability" of Σ. This is what we are going to study in this section.

Suppose Σ is a surface in $\mathbb{R}^3$, e_i's are orthonormal frame on Σ. Let n be the normal vector to Σ.

Now let u be a function on Σ, we construct a graph $\Sigma_u = \{p + u(p)n(p) : p \in \Sigma)\}$. Note if $u \cong 0$, then $\Sigma_u = \Sigma$.

For a special case $u \cong s$ is a constant function. Then if $\gamma(t)$ is a curve in Σ, $\gamma(t) + sn(\gamma(t))$ is a curve in Σ_s. Note $\nabla_{\gamma'} n(\gamma(t)) = -A(\gamma')$. So we can define a map

$$B = Id - sA$$

, which is the differential of the curve in Σ_s.

For general Σ_u, we have new local frame

$$e_i \longrightarrow B(p,u)e_i + u_i(p)n$$

Here $B(p,u) = Id - u(p)A(p)$. The new normal vector is

$$n \longrightarrow n - B^{-1}(p,u)(\nabla u)$$

Then we can compute the new metric over Σ_u is

$$g = B^2 + \nabla u \otimes \nabla u$$

Then we can further compute the mean curvature

$$H_u = -\Delta u - |A|^2 u + Q$$

where Q is quadratic in $uA, \nabla u$.

9.2 1/2-stability

Let us recall the definition of stability: We say a 2-sided minimal surface Σ is stable if for all ϕ has compact support,

$$\int |A|^2\phi^2 \leq \int |\nabla\phi|^2$$

Definition 9.2. *We call Σ is* **1/2-stable** *if for all ϕ has compact support,*

$$\int |A|^2\phi^2 \leq 2\int |\nabla\phi|^2$$

Although it is not as meaningful as stable from geometric view, it is reasonable in the following sense. Recall Corollary 8.1, we get an estimate of area:

$$4A(R) - 4\pi R^2 = \int_{\mathcal{B}_R} |A|^2(R-r)^2$$

Suppose the RHS satisfies the estimate

$$\int_{\mathcal{B}_R} |A|^2(R-r)^2 \leq C\int_{\mathcal{B}_R} |\nabla(R-r)|^2 = CA(R)$$

Here if Σ is stable, then the estimate holds for $C = 1$. Note we can get an area bound for $C < 4$. In particular, if we assume 1/2-stable, we get the above estimate for $C = 2$.

So 1/2-stable in fact give us the quadratic area growth for minimal surface, and as a result, we obtain the curvature estimate $\sup_{\mathcal{B}_{\mathcal{R}/\in}} |A|^2 \leq C/R^2$ for C independent of R from Choi-Scheon estimate.

Next lemma will help us to show 1/2-stablity:

lemma 9.1. *There is exist a constant $\delta > 0$, such that if Σ_u is a minimal graph over Σ with $u > 0$, and $|\nabla u| + |u||A| < \delta$, then Σ is 1/2-stable.*

Proof. Let $w = \log u$, then plug it into the minimal surface equation for graphs over minimal surface, we get

$$\Delta w = -|\nabla w|^2 + \operatorname{div}(a\nabla w) + \langle\nabla w, a\nabla w\rangle + \langle b, \nabla w\rangle + (c-1)|A|^2$$

where a is an operator, b is a vector, c is a function satisfies

$$\begin{aligned} &|a|, |c| \leq 3|A||u| + |\nabla u| \\ &|b| \leq 2|A||\nabla u| \end{aligned}$$

as long as $|u||A| + |\nabla u| \leq \delta$ for δ small.

Now suppose ϕ is any cut-off function on Σ, apply Stokes theorem to $\operatorname{div}(\phi^2\nabla w - \phi^2 a\nabla w)$, we can get 1/2-stability. (Come back if we need more details) $\square$

9.3 Analysis of Sectors

Now let us go back to intrinsic geometry of minimal embedded disk Σ. Let r be the intrinsic distance function to some fixed point in Σ. Note Σ has non positive curvature, so r is smooth (in fact, the exponential map is a diffeomorphism).

Let $L_t = |\partial\mathcal{B}_t|$ is the length of boundary of intrinsic ball, then by calculus of variation we have

$$L'(t) = \int_{\partial\mathcal{B}_t} k_g = 2\pi - \int_{\mathcal{B}_t} K$$

by Gauss-Bonnet theorem. By minimality, $K = 1/2|A|^2$

Suppose σ is a part of $\partial\mathcal{B}_t$. We define the **sector** $S_t(\sigma)$ to be the union of geodesic go out of normal directions on σ with length $\leq t$. You can image it is really a sector in the plane.

To show it is a well-posed sector, there are two things need to be ruled out.

- some of the geodesics hit σ twice
- two of the geodesics intersect (in particular, one geodesic hit itself)

All these are not hard to be shown by non positive curvature property and Gauss-Bonnet theorem, so we omit the proof here.

Now let $L(t)$ be the length of outer boundary of S_t. Then by co-area formula we can check

$$A(t) := Area(S_t) = \int_0^t L(s)ds$$

and by first variational formula, we have

$$L'(t) = \int_{\text{outer boundary}} k_g = \int_\sigma k_g + \frac{1}{2}\int_{S_t} |A|^2 = L'(0) + \frac{1}{2}\int_{S_t} |A|^2$$

Integrate both sides

$$L(t) = L(0) + tL'(0) + \frac{1}{2}\int_0^t \int_{S_s} |A|^2 ds$$

Combine all the formula above, we get the formula for the area of sectors

$$A(R) = RL(0) + \frac{1}{2}R^2 L'(0) + \frac{1}{2}\int_0^R \int_0^t \int_{S_s} |A|^2 ds$$

By the same trick as we get area-curvature estimate before (Corollary 8.1), we get the following co-area formula:

$$A(R) = RL(0) + \frac{1}{2}R^2 L'(0) + \frac{1}{4}\int_{S(R)} (R-\rho)^2 |A|^2 \tag{1}$$

Here ρ is the distance function to σ.

Sectors play an important role in the proof. They are the small unit where we want to analyze the behavior of embedded minimal surface.

Let us first explain how to use sectors to find multi-graph. Image we have a lot of sectors, by embeddedness they will not intersect, then we can find many of them close to each other. If two of them are close to each other, and they have bounded $|A|$, then they will have 1/2-stability. Then we can show they are super flat, hence they form some multi-graphs.