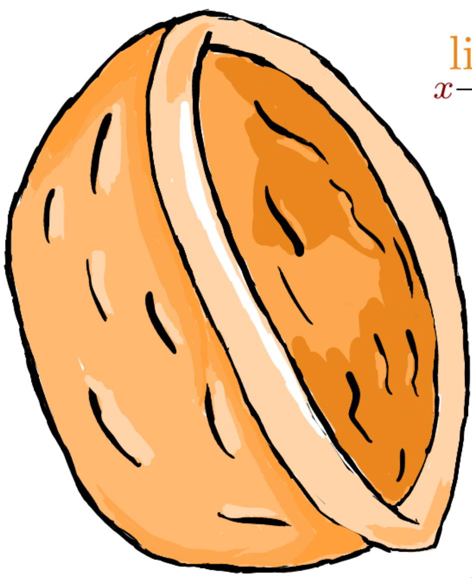
Real Analysis

– in a nutshell –

This book was created and used for the lecture *Mathematical Analysis* at Hamburg University of Technology in the summer term 2019 for General Engineering Science and Computer Science students.

Hamburg, 16th January 2022, Version 1.3



$$\lim_{x \to x_0} f(x)$$

$$\sum_{k=0}^{\infty} a_k$$

$$\int_a^b \phi(x)\, dx$$

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- to all students who pointed out typos and other problems in this script.

Hamburg, 16th January 2022 J.P.G.

Contents

Some words

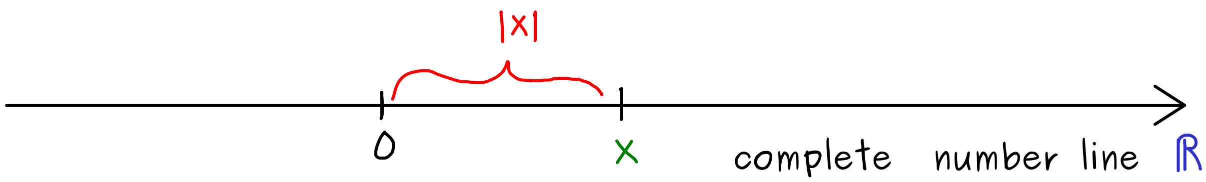
This text should help you to understand the course Real Analysis. It goes hand in hand with a video course you find on YouTube. Therefore, you will always find links and QR codes that bring you to the corresponding videos.

The whole playlist can be found here::
https://jp-g.de/bsom/rab/ra-list/

To expand your knowledge even more, you can look into the following books:

- Jonathan Lewin: *An Interactive Introduction to Mathematical Analysis*,
- A. N. Kolmogorov: *Introductory Real Analysis*,
- Claudio Canute, Anita Tabacco: *Mathematical Analysis I*.
- Vladimir A. Zorich: *Mathematical Analysis I*.

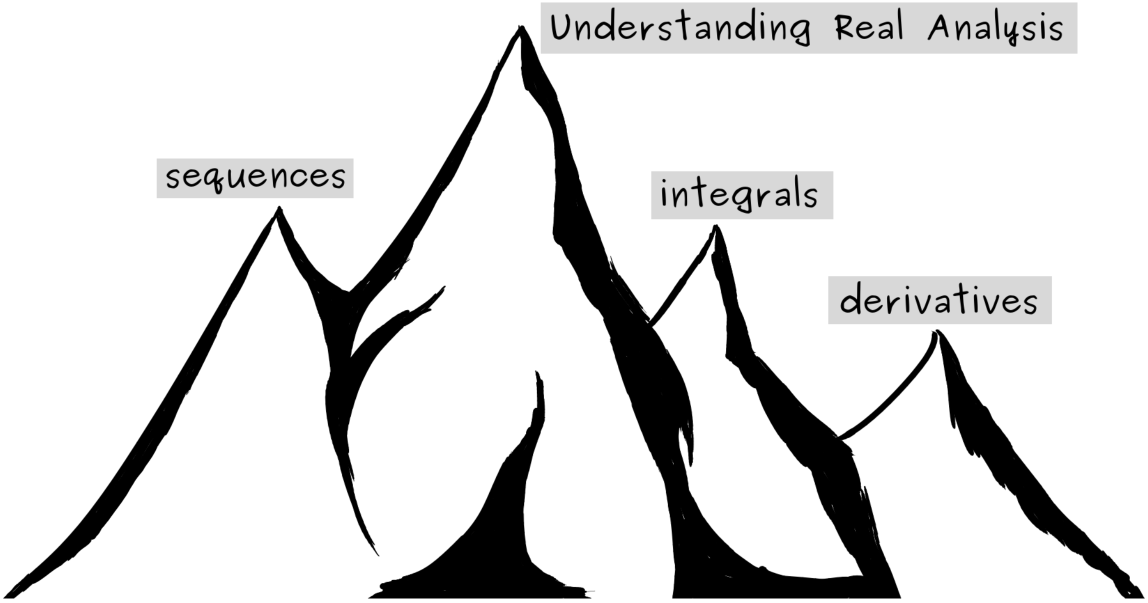
Real Analysis is also known as Calculus with real numbers. It is needed for a lot of other topics in mathematics and the foundation of every new career in mathematics or in fields that need mathematics as a tool. We[1] discuss simple examples later. Some important bullet points are *limits*, *continuity*, *derivatives* and *integrals*. In order to describe these things, we need a good understanding of the real numbers. They form the foundation of a real analysis course.



For this reason, the first step in a *Real Analysis* is to define the real number line. After this, we will be able to work with these numbers and understand the field as a whole. Of course, this is not an easy task and it will be a hiking tour that we will do together. The

[1] In mathematical texts, usually, the first-person plural is used even if there is only one author. Most of the time it simply means "we" = "I (the author) and the reader".

summit and goal is to understand why working with real numbers is indeed a meaningful mathematical theory.



We start in the valley of mathematics and will shortly scale the first hills. Always stay in shape, practise and don't hesitate to ask about the ways up. It is not an easy trip but you can do it. Maybe the following tips can guide you:

- You will need a lot of time for this course if you really want to *understand* everything you learn. Hence, make sure that you have enough time each week to do mathematics and keep these time slots clear of everything else.
- Work in groups, solve problems together and discuss your solutions. Learning mathematics is not a competition.
- Explain the content of the lectures to your fellow students. Only the things you can illustrate and explain to others are really understood by you.
- Learn the Greek letters that we use in mathematics:

α	alpha	β	beta	γ	gamma	Γ	Gamma
δ	delta	ϵ	epsilon	ε	epsilon	ζ	zeta
η	eta	θ	theta	Θ	Theta	ϑ	theta
ι	iota	κ	kappa	λ	lambda	Λ	Lambda
μ	mu	ν	nu	ξ	xi	Ξ	Xi
π	pi	Π	Pi	ρ	rho	σ	sigma
Σ	Sigma	τ	tau	υ	upsilon	Υ	Upsilon
ϕ	phi	Φ	Phi	φ	phi	χ	chi
ψ	psi	Ψ	Psi	ω	omega	Ω	Omega

  This video may help you there:
  https://jp-g.de/bsom/la/greek/

- Choosing a book is a matter of taste. Look into different ones and choose the book that really convinces you.

- Keep interested, fascinated and eager to learn. However, do not expect to understand everything at once.

DON'T PANIC J.P.G.

1 Sequences and Limits

I'm a gym member. I try to go four times a week, but I've missed the last twelve hundred times.

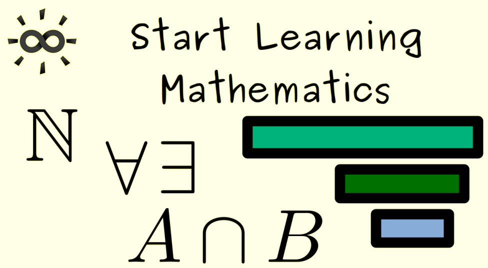
Chandler Bing

1.1 Just Numbers

Before we start with the Real Analysis course, we need to lie down some foundations. You only need some knowledge about working with sets and maps to get started. From this, we will introduce all the number sets we will need in this course in a quick way. Hence, we quickly have the *real numbers* $\mathbb{R}$ we work with throughout this course.

However, if you interested in a more detailed discussion, I can recommend you my video series about the foundations of mathematics:

Video: Start Learning Mathematics



https://jp-g.de/bsom/la/slm/

In order to construct the real number line, we need to generalise the equality sign. We get a more abstract notion that we can use for sets to put similar elements into the same box. In the end, we want to calculate with these boxes.

It turns out that we just need three properties from the equality sign to get the general concept of an equivalence relation.

Definition 1.1. Equivalence relation

Let X be a set. A subset $R_\sim \subset X \times X$ is called a relation on X. We write $x \sim y$ if $(x, y) \in R_\sim$. A relation $R_\sim$ is called an equivalence relation if it satisfies the following:

(a) $x \sim x$ for all $x \in X$. *(reflexive)*

(b) If $x \sim y$, then also $y \sim x$. *(symmetric)*

(c) If $x \sim y$ and $y \sim z$, then also $x \sim z$. *(transitive)*

By having this, we now can put equivalent elements in the corresponding boxes. These boxes ared called *equivalent classes*.

Proposition & Definition 1.2. Equivalent classes

An equivalence relation $\sim$ on X gives a partition of X into disjoint subsets. For all $a \in X$, we define

$$[a]_\sim := \{x \in X : x \sim a\}$$

and call it an equivalent class. We have the disjoint union:

$$X = \bigcup_{a \in X} [a]_\sim .$$

In the same way as we generalised the equality sign, we can also generalise the *greater or equal* sign you might have seen often for numbers. It turns out that we just need some defining properties there to get an abstract notion of such an ordering.

Definition 1.3. Ordering

Let X be a set. Let $R_\leq \subset X \times X$ be a relation on X where we write $x \leq y$ if $(x, y) \in R_\leq$. A relation $R_\leq$ is called a partial order if it satisfies the following:

(a) $x \leq x$ for all $x \in X$. *(reflexive)*

(b) If $x \leq y$ and $y \leq x$, then $x = y$. *(antisymmetric)*

(c) If $x \leq y$ and $y \leq z$, then also $x \leq z$. *(transitive)*

If, in addition,

(d) for all $x, y \in X$, we have either $x \leq y$ or $y \leq x$. *(total)*

then $\leq$ is called a total order or chain.

Remark: Notation

If one has an ordering relation $\leq$, one usually also defines the following symbols:

$$x \geq y :\Longleftrightarrow y \leq x$$

$$x < y :\Longleftrightarrow \big(x \leq y \text{ and } x \neq y\big) .$$

Reminder: Natural numbers

The natural numbers with zero $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ *originate from counting. There is an operation* $+$ *that is associative and commutative, called addition. Then we can define a total order* $\leq$*. Instead of* $3+3+3+3+3$*, one writes* $5 \cdot 3$*. This defines also an associative and commutative operation on* $\mathbb{N}_0$*, called multiplication.*
The element 0 *is the neutral element with respect to the addition* $+$ *and the element* 1 *is the neutral element with respect to the multiplication* $\cdot$*.*

Example 1.4. Sequence of numbers

An ordered infinite list of numbers is called a sequence. We will define it later in more detail. For example:

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ 256,\ 512,\ 1024,\ \ldots$$

Here, one would shorten that to $(a_n)_{n \in \mathbb{N}_0}$ *with* $a_n = 2^n$*.*
What can we do with this? Fibonacci numbers! Collatz conjecture!

Box 1.5. Construction of $\mathbb{Z}$

We do not have an element $x \in \mathbb{N}_0$ *that fulfils* $x + 5 = 0$*. This is what we would call the inverse of* 5 *with respect to the addition. This element is what we recognise inside the following elements of* $\mathbb{N}_0 \times \mathbb{N}_0$*:*

$$(0,5) \quad (1,6) \quad (101,106) \quad (56,61) \quad (77,82) \quad (91,96)$$

We define an equivalence relation on the set $\mathbb{N}_0 \times \mathbb{N}_0$ *by setting:*

$$(a,b) \sim (c,d) :\Longleftrightarrow\ a + d = b + c$$

The equivalent classes are excactly what we need:

$$[(a,b)] := \{(x,y) : (x,y) \sim (a,b)\} .$$

We can also define an addition:

$$[(a,b)] + [(c,d)] := [(a+c, b+d)] .$$

It is well-defined, associative and commutative and $[(0,0)]$ *defines the neutral element. The set of all equivalence classes with this new addition is called the integers and denoted by* $\mathbb{Z}$*.*

Also extend the multiplication to $\mathbb{Z}$. Moreover, we can also define a multiplication:

$$[(a,b)] \cdot [(c,d)] := [(ac + bd, ad + bc)] .$$

Definition 1.6. Commutative ring

The integers $\mathbb{Z}$ form a so-called commutative ring. This means:

(A) Addition

- *(A1) associative:* $x + (y + z) = (x + y) + z$
- *(A2) neutral element: There is a (unique) element* 0 *with* $x + 0 = x$ *for all* x.
- *(A3) inverse element: For all* x *there is a (unique)* y *with* $x + y = 0$. *We write for this element simply* $-x$.
- *(A4) commutative:* $x + y = y + x$

(M) Multiplication

- *(M1) associative:* $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- *(M2) neutral element: There is a (unique) element* 1 *with* $1 \cdot x = x$ *for all* x.
- *(M3) commutative:* $x \cdot y = y \cdot x$

(D) Distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$.

Box 1.7. Construction of $\mathbb{Q}$

We do not have an element $x \in \mathbb{Z} \setminus \{0\}$ *that fulfils* $x \cdot 5 = 1$. *This is what we would call the inverse of* 5 *with respect to the multiplication. This element is what we recognise inside the following elements of* $\mathbb{Z} \setminus \{0\} \times \mathbb{Z} \setminus \{0\}$:

$$(1,5) \quad (2,10) \quad (101,505) \quad (50,250) \quad (11,55) \quad (500,2500)$$

We define an equivalence relation on the set $\mathbb{Z} \setminus \{0\} \times \mathbb{Z} \setminus \{0\}$ *by setting:*

$$(a,b) \sim (c,d) :\Longleftrightarrow a \cdot d = b \cdot c$$

The equivalent classes are excactly what we need:

$$[(a,b)] := \{(x,y) : (x,y) \sim (a,b)\}.$$

We can also define a multiplication:

$$[(a,b)] \cdot [(c,d)] := [(a \cdot c, b \cdot d)].$$

It is well-defined, associative and commutative. The set of all equivalence classes with this new multiplication is called the rational numbers (without zero).

$$[(a,b)] + [(c,d)] := [(ad + bc, bd)]$$

We can visualise these numbers in the number line:

Definition 1.8. Real numbers

The real numbers are a non-empty set $\mathbb{R}$ *together with the operations* $+ : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *and* $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and an ordering relation* $\leq : \mathbb{R} \times \mathbb{R} \to \{True, False\}$ *that fulfil the following rules*

(A) Addition

- *(A1) associative:* $x + (y + z) = (x + y) + z$
- *(A2) neutral element: There is a (unique) element* 0 *with* $x + 0 = x$ *for all* x.
- *(A3) inverse element: For all* x *there is a (unique)* y *with* $x + y = 0$. *We write for this element simply* $-x$.
- *(A4) commutative:* $x + y = y + x$

(M) Multiplication

- *(M1) associative:* $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- *(M2) neutral element: There is a (unique) element* $1 \neq 0$ *with* $x \cdot 1 = x$ *for all* x.
- *(M3) inverse element: For all* $x \neq 0$ *there is a (unique)* y *with* $x \cdot y = 1$. *We write for this element simply* x^{-1}.
- *(M4) commutative:* $x \cdot y = y \cdot x$

(D) Distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$.

(O) Ordering

- *(O1)* $x \leq x$ *is true for all* x.
- *(O2) If* $x \leq y$ *and* $y \leq x$, *then* $x = y$.
- *(O3) transitive:* $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$.
- *(O4) For all* $x, y \in X$, *we have either* $x \leq y$ *or* $y \leq x$.
- *(O5)* $x \leq y$ *implies* $x + z \leq y + z$ *for all* z.
- *(O6)* $x \leq y$ *implies* $x \cdot z \leq y \cdot z$ *for all* $z \geq 0$.
- *(O7)* $x > 0$ *and* $\varepsilon > 0$ *implies* $x < \varepsilon + \cdots + \varepsilon$ *for sufficiently many summands.*

(C) Let $X, Y \subset \mathbb{R}$ *be two non-empty subsets with the property* $x \leq y$ *for all* $x \in X$ *and* $y \in Y$. *Then there is a* $c \in \mathbb{R}$ *with* $x \leq c \leq y$ *for all* $x \in X$ *and* $y \in Y$.

Remark:

We will later reformulate the completeness axiom with the help of sequences. Then it sounds like:
Completeness: Every sequence $(a_n)_{n\in\mathbb{N}}$ with the property *[For all* $\varepsilon > 0$ *there is an* $N \in \mathbb{N}$ *with* $|a_n - a_m| < \varepsilon$ *for all* $n, m > N$*]* has a limit.

Exercise 1.9.

Use the axioms to show:

$(1): \ 0 \cdot x = 0$

(2) : $-x = (-1) \cdot x$

(3) : $(-1) \cdot (-1) = 1$

(4) : $1 > 0$

(1): $0 \cdot x \overset{(A2)}{=} (0+0)x \overset{(D)}{=} 0x + 0x \overset{(A2)}{\Longrightarrow} 0x = 0$

(2): $-x \overset{(*)}{=} 0x + (-x) \overset{(A3)}{=} (1+(-1))x + (-x) \overset{(D)}{=} x + (-1)x + (-x) \overset{(A2-4)}{=} (-1)x$

(3): $x(-1)(-1) \overset{(**)}{=} -(-x) \overset{(A3)}{=} x \overset{(M2)}{\Longrightarrow} (-1) \cdot (-1) = 1$

(4): Exercise!

Definition 1.10. Absolute value for real numbers

The absolute value of a number $x \in \mathbb{R}$ is defined by

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Proposition 1.11. Two important properties

For any two real numbers $x, y \in \mathbb{R}$, one has

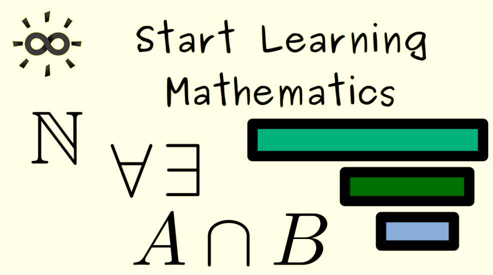
(a) $|x \cdot y| = |x| \cdot |y|$, *($|\cdot|$ is multiplicative),*

(b) $|x + y| \leq |x| + |y|$, *($|\cdot|$ fulfils the triangle inequality).*

1.2 Convergence of Sequences

Now we start with sequences and the important definition of convergence.

Video: Start Learning Mathematics



https://jp-g.de/bsom/la/slm/

Definition 1.12.

Let M be a set. A sequence in M is a map $a : \mathbb{N} \to M$ or $a : \mathbb{N}_0 \to M$.

We use the following symbols for sequences:

$$(a_n)_{n\in\mathbb{N}}, \qquad (a_n), \qquad (a_n)_{n=1}^{\infty}, \quad (a_1, a_2, a_3, \dots).$$

Remark:

M is usually a real subset ($M \subset \mathbb{R}$), but M can also be a complex subset ($M \subset \mathbb{C}$) or a subset of some normed space (or $M = \mathbb{R}$, $M = \mathbb{C}$, M a normed space itself).

Example 1.13. (a) $a_n = (-1)^n$, then $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}} = (-1, 1, -1, 1, -1, 1, \dots)$;

(b) $a_n = \frac{1}{n}$, then $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots)$;

(c) $a_n = \mathrm{i}^n$ (i is the imaginary unit), then $(a_n)_{n\in\mathbb{N}} = (\mathrm{i}^n)_{n\in\mathbb{N}} = (\mathrm{i}, -1, -\mathrm{i}, 1, \mathrm{i}, -1, \dots)$;

(d) $a_n = \frac{1}{2^n}$, then $(a_n)_{n\in\mathbb{N}} = (\frac{1}{2^n})_{n\in\mathbb{N}} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots)$;

(e) **Approximation of π:**
Consider a circle with radius $r = 1$.

The area is given by $A = \pi r^2 = \pi$.

Now: Approximation of the circle by a regular polygon (all edges have equal length):
Area of a "piece of cake": $A_{cn} = \sin(\frac{\pi}{n})\cos(\frac{\pi}{n}) = \frac{1}{2}\sin(\frac{2\pi}{n})$ (the latter equation holds true due to the general equality $\sin(2x) = 2\sin(x)\cos(x)$ (will be treated later)).
The area of the polygon is therefore given by

$$A_n = n \cdot A_{cn} = \frac{n}{2} \cdot \sin\left(\frac{2\pi}{n}\right).$$

Now consider the sequence $(A_n)_{n\in\mathbb{N}}$. Some values for A_n are listed in the following table:

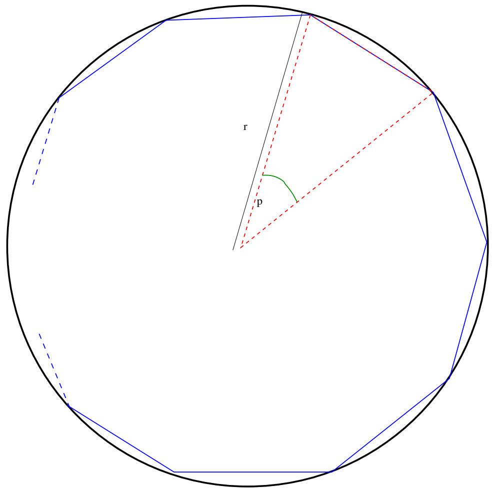


Figure 1.1: Circle approximated by a polygon

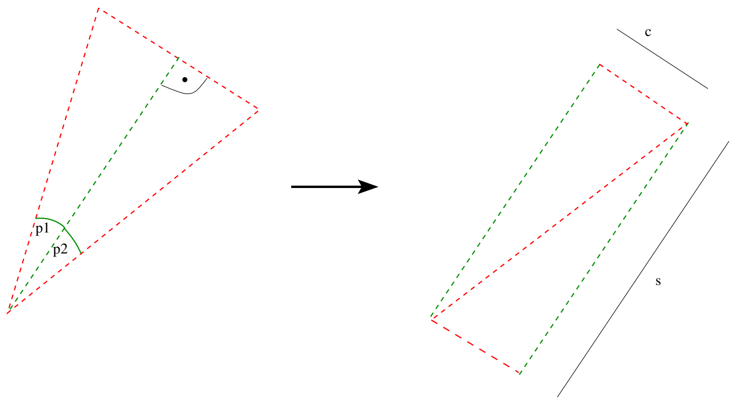


Figure 1.2: "Piece of cake"

n	A_n	$\pi - A_n$
3	1.299038	1.84255
6	2.598076	0.54351
12	3.000000	0.14159
3072	3.14159046	0.00000219
50331648	3.141592653589	$8.16 \cdot 10^{-16}$

(f) **Some linear algebra:**
Consider the linear system

$$A\mathbf{x} = \mathbf{b}, \quad A \in \mathbb{R}^{n\times n}, \quad \mathbf{b} \in \mathbb{R}^n.$$

Then $\mathbf{o} = -A\mathbf{x} + \mathbf{b}$ and thus

$$\mathbf{x} = \mathbf{x} - A\mathbf{x} + \mathbf{b} = (\mathbb{1} - A)\mathbf{x} + \mathbf{b}.$$

Consider the iteration

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{o} \\ \mathbf{x}_{n+1} &= (\mathbb{1} - A)\mathbf{x}_n + \mathbf{b} \text{ for } n \geq 1 \end{aligned} \qquad \text{“Richardson iteration”}$$

The sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a sequence of approximate solutions. The method is "good" if $(\mathbf{x}_n)_{n\in\mathbb{N}}$ "converges".

Since most (but not all) results stay the same in the real and complex case, we make the following definition:

Definition 1.14.

The symbol $\mathbb{F}$ stands for either the real or complex numbers, i.e., $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Next we define the notions of convergence and limits:

Definition 1.15. Convergence/divergence of sequences

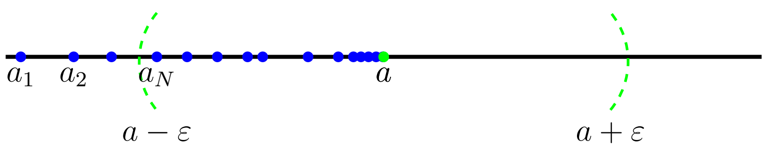
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{F}$. We say that

- *$(a_n)_{n\in\mathbb{N}}$ is convergent to $a \in \mathbb{F}$ if for all $\varepsilon > 0$ there exists some $N = N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N$ holds $|a_n - a| < \varepsilon$. In this case, we write*

$$\lim_{n\to\infty} a_n = a.$$

- *$(a_n)_{n\in\mathbb{N}}$ is divergent if it is not convergent, i.e., for all $a \in \mathbb{F}$ holds: There exists some $\varepsilon > 0$ such that for all N there exists some $n > N$ with $|a_n - a| \geq \varepsilon$.*

Convergence for real sequences means that if you give any small distance ε, one finds that all sequence members a_n lie in the interval $(a-\varepsilon, a+\varepsilon)$ with the exception of only *finitely* many.



Example 1.16. • Show: $(a_n)_{n\in\mathbb{N}}$ with $a_n = (1/n)$ is convergent with limit 0.

- Show: $(b_n)_{n\in\mathbb{N}}$ with $b_n = (1/\sqrt{n})$ is convergent with limit 0.

Proof. Let $\varepsilon > 0$. Choose $N > \frac{1}{\varepsilon^2}$. Then for all $n \geq N$, we have

$$|b_n - 0| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon$$

This means b_n is arbitrarily close to 0, eventually. □

Remark:

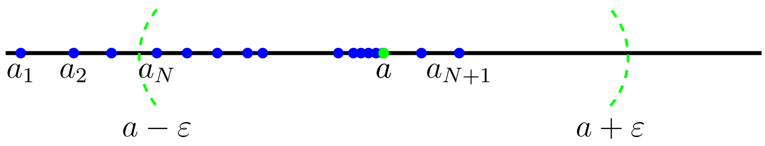
(a) It can be shown that for a complex sequence $(a_n)_{n\in\mathbb{N}}$, convergence to $a \in \mathbb{C}$ holds true if, and only if, $(\operatorname{Re}(a_n))_{n\in\mathbb{N}}$ converges to $\operatorname{Re}(a)$ and $(\operatorname{Im}(a_n))_{n\in\mathbb{N}}$ converges to $\operatorname{Im}(a)$

(b) In fact, convergence can also be defined for sequences in some arbitrary normed

vector space V (for the definition of a normed space, e.g. consult the linear algebra script). Then one has to replace the absolute value by the norm (e.g., "$\|a_n - a\| < \varepsilon$".)

(c) Due to the fact that for any $N \in \mathbb{N}$, we can find some $x \in \mathbb{R}$ with $x > N$, we can equivalently reformulate the convergence definition as follows: "$(a_n)_{n\in\mathbb{N}}$ is convergent to $a \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists some $N = N(\varepsilon) \in \mathbb{R}$ such that for all $n \geq N$ holds $|a_n - a| < \varepsilon$". In the following, we just write "there exists some N".

What does convergence mean?



Outside any ε-neighbourhood of a only finitely many elements of the sequence exist.

Example 1.17. (a) The real sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ converges to 0.
Proof: Let $\varepsilon > 0$ (be arbitrary): Choose $N = \frac{1}{\varepsilon} + 1 = \frac{1+\varepsilon}{\varepsilon}$.
Then for all $n \geq N$ holds

$$\frac{1}{n} \leq \frac{1}{N} = \frac{\varepsilon}{1+\varepsilon} < \varepsilon.$$

Therefore

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon.$$

(b) The real sequence $((-1)^n)_{n\in\mathbb{N}}$ is divergent.
Proof by contradiction:
Assume that $((-1)^n)_{n\in\mathbb{N}}$ is convergent to $a \in \mathbb{R}$. Then take e.g. $\varepsilon = \frac{1}{10}$. Due to convergence, we should have some N such that for all $n \geq N$ holds

$$|(-1)^n - a| < \tfrac{1}{10}.$$

Thus we have that $|-1-a| < \frac{1}{10}$ and $|1-a| < \frac{1}{10}$. As a consequence,

$$2 = |1 + a - a + 1| \leq |1 + a| + |-a + 1| = |1 + a| + |a - 1| < \tfrac{1}{10} + \tfrac{1}{10} = \tfrac{1}{5}.$$

This is a contradiction. □

(c) For $q \in \mathbb{C}\backslash\{0\}$ with $|q| < 1$ the complex sequence $(q^n)_{n\in\mathbb{N}}$ converges to 0.
Proof: $|q| < 1$ gives rise to $\frac{1}{|q|} > 1$, whence $\frac{1}{|q|} - 1 > 0$. Therefore, we are able to apply Bernoulli's inequality (see tutorial) in the following way:

$$\frac{1}{|q|^n} = \left(1 + \left(\frac{1}{|q|} - 1\right)\right)^n = \left(1 + \left(\frac{1-|q|}{|q|}\right)\right)^n \geq 1 + n \cdot \left(\frac{1-|q|}{|q|}\right),$$

and thus

$$|q|^n \leq \frac{1}{1 + n \cdot \left(\frac{1-|q|}{|q|}\right)} = \frac{|q|}{|q| + n \cdot (1 - |q|)}.$$

Now let $\varepsilon > 0$ (be arbitrary):

Choose

$$N = \frac{|q|}{\varepsilon \cdot (1 - |q|)} - \frac{|q|}{1 - |q|} + 1$$

Then for all $n \geq N$ holds

$$n > \frac{|q|}{\varepsilon \cdot (1 - |q|)} - \frac{|q|}{1 - |q|}$$

and thus

$$n \cdot (1 - |q|) > \frac{|q|}{\varepsilon} - |q|.$$

This leads to

$$|q| + n \cdot (1 - |q|) > \frac{|q|}{\varepsilon},$$

whence

$$\frac{|q|}{|q| + n \cdot (1 - |q|)} < \varepsilon.$$

The above calculations now imply

$$|q^n - 0| = |q|^n \leq \frac{|q|}{|q| + n \cdot (1 - |q|)} < \varepsilon.$$

□

Remark:

The choice of the N often seems "to appear from nowhere". However, there is a systematic way to formulate the proof. For instance in a), we need to end up with the equation $|\frac{1}{n} - 0| < \varepsilon$ or, equivalently, $\frac{1}{n} < \varepsilon$. Inverting this expression leads to $n > \frac{1}{\varepsilon}$. Therefore, if N is chosen as $N = \frac{1}{\varepsilon} + 1 = \frac{1+\varepsilon}{\varepsilon}$, the desired statement follows. If one has to formulate such a proof (for instance, in some exercise), then first these above calculations have to be done "on some extra sheet" and then formulate the convergence proof in the style as in a) or c).

Definition 1.18. Boundedness of sequences

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{F}$. Then $(a_n)_{n\in\mathbb{N}}$ is called

- *bounded if there exists some $c \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ holds $|a_n| \leq c$;*
- *unbounded if it is not bounded, i.e., for all $c \in \mathbb{R}$, there exists some $n \in \mathbb{N}$ with $|a_n| > c$.*

Theorem 1.19.

Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{F}$. Then $(a_n)_{n\in\mathbb{N}}$ is bounded.

Proof. Suppose that $\lim_{n\to\infty} a_n = a$. Take $\varepsilon = 1$. Then there exists some N such that for all $n \geq N$ holds $|a_n - a| < 1$. Thus, for all $n \geq N$ holds

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a|.$$

Now choose

$$c = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, |a| + 1\}$$

and consider some arbitrary sequence element a_k.
If $k < N$, then $|a_k| \leq \max\{|a_1|, |a_2|, \dots, |a_{N-1}|\} \leq c$.
In the case $k \geq N$, the above calculations lead to $|a_k| < |a| + 1 \leq c$.
Altogether, this implies that $|a_k| \leq c$ for all $k \in \mathbb{N}$, so $(a_n)_{n\in\mathbb{N}}$ is bounded by c. □

Remark:

For a convergent sequence $(a_n)_{n\in\mathbb{N}}$ that is bounded by c, we can also deduce from the above argumentation that for the limit a holds $|a| \leq c$:
Suppose that $\lim_{n\to\infty} a_n = a \in \mathbb{F}$, then for an arbitrary $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a - a_n| < \varepsilon$ for all $n \geq N$. Hence $|a| = |a - a_n + a_n| \leq |a - a_n| + |a_n| \leq \varepsilon + c$. Since $\varepsilon > 0$ can be chosen arbitrarily small this implies $|a| \leq c$.

Theorem 1.20. Uniqueness of the limit of a convergent sequence

*Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{F}$. Then there exists only **one** limit.*

Proof by contradiction: Let $a, b \in \mathbb{F}$ be two distinct ($a \neq b$) limits of $(a_n)_{n\in\mathbb{N}}$. Then we have $|a - b| > 0$ and for $\varepsilon = \frac{1}{4}|a - b|$ the following statements are fulfilled:
There exists some N_1 such that for all $n \geq N_1$ holds $|a_n - a| < \varepsilon$.
There exists some N_2 such that for all $n \geq N_2$ holds $|a_n - b| < \varepsilon$.
Let $n \geq \max\{N_1, N_2\}$. Then

$$\begin{aligned} |a - b| &= |a - a_n + a_n - b| \\ &\leq |a - a_n| + |a_n - b| \leq \varepsilon + \varepsilon = 2\varepsilon = \frac{1}{2} \cdot |a - b| \end{aligned}$$

Thus, $|a - b| \leq \frac{1}{2}|a - b|$. However, this implies $|a - b| \leq 0$ which is only fulfilled, if $|a - b| = 0$, or, equivalently $a = b$. This is a contradiction to the initial assumption. □

In the following, we present some results that allow us to determine some further limits.

Theorem 1.21. Formulae for convergent sequences

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be a convergent sequences in $\mathbb{F}$. Then the following holds true:

(i) $(a_n + b_n)_{n\in\mathbb{N}}$ *is convergent with*

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

(ii) $(a_n \cdot b_n)_{n\in\mathbb{N}}$ *is convergent with*

$$\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$$

(iii) If $\lim_{n\to\infty} b_n \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then the sequence $(\frac{a_n}{b_n})_{n\in\mathbb{N}}$ is convergent with

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}.$$

Proof. Let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$.

(i) Let $\varepsilon > 0$ be arbitrary. Then
there exists some N_1 such that for all $n \geq N_1$ holds $|a - a_n| < \frac{\varepsilon}{2}$, and
there exists some N_2 such that for all $n \geq N_2$ holds $|b - b_n| < \frac{\varepsilon}{2}$.
Now choose $N = \max\{N_1, N_2\}$. Then for all $n \geq N$ holds

$$\begin{aligned} &|(a+b) - (a_n + b_n)| \\ &= |(a - a_n) + (b - b_n)| \leq |a - a_n| + |b - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

(ii) Due to Theorem 1.19, both sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded. Choose numbers $c_1, c_2 > 0$ such that $|a_n| < c_1$ and $|b_n| < c_2$ for all $n \in \mathbb{N}$. Define

$$c = \max\{c_1, c_2\}.$$

Let $\varepsilon > 0$. Convergence of $(a_n)_{n\in\mathbb{N}}$ to a implies that there exists some N_1 such that $|a - a_n| < \frac{\varepsilon}{2c}$ for all $n \geq N_1$. Furthermore, the convergence of $(b_n)_{n\in\mathbb{N}}$ to b implies that there exists some N_2 such that $|b - b_n| < \frac{\varepsilon}{2c}$ for all $n \geq N_2$. Now define $N = \max\{N_1, N_2\}$. Then for $n \geq N$ holds

$$\begin{aligned} &|ab - a_n b_n| \\ &= |(ab - a_n b) + (a_n b - a_n b_n)| \leq |ab - a_n b| + |a_n b - a_n b_n| \\ &= |a - a_n| \cdot |b| + |a_n| \cdot |b - b_n| \leq |a - a_n| \cdot c + c \cdot |b - b_n| \\ &< \frac{\varepsilon}{2c} \cdot c + c \cdot \frac{\varepsilon}{2c} = \varepsilon. \end{aligned}$$

(iii) First we specialize to the case where $(a_n)_{n\in\mathbb{N}}$ is the constant sequence $(a_n)_{n\in\mathbb{N}} = (1, 1, 1, 1, \ldots)$. Due to $b > 0$, we have the existence of some N_1 such that for all $n \geq N_1$ holds

$$|b_n - b| < \frac{|b|}{2}.$$

This just follows by an application of the "ε-criterion" to $\varepsilon = \frac{|b|}{2}$. In particular, this leads to $|b| \leq |b_n - b| + |b_n| < \frac{|b|}{2} + |b_n|$ and thus $|b_n| > \frac{|b|}{2}$.
Let $\varepsilon > 0$. Let N_2 such that for all $n \geq N_2$ holds

$$|b_n - b| < \frac{\varepsilon \cdot |b|^2}{2}.$$

Then for $n \geq N := \max\{N_1, N_2\}$ holds

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{1}{|b_n| \cdot |b|} \cdot |b_n - b| < \frac{2}{|b|^2} \cdot |b_n - b| < \frac{2}{|b|^2} \cdot \frac{\varepsilon \cdot |b|^2}{2} = \varepsilon.$$

So far, we have shown that for some sequence $(b_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} b_n \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$ holds

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\lim_{n\to\infty} b_n}.$$

The general statement for the sequence $(\frac{a_n}{b_n})_{n\in\mathbb{N}}$ follows by rewriting

$$\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}} = \left(a_n \cdot \frac{1}{b_n}\right)_{n\in\mathbb{N}}$$

and applying the multiplication rule (ii).

□

Remark:

(a) *Since the constant sequence $(a)_{n\in\mathbb{N}} = (a, a, a, \ldots)$ is, of course, convergent to a, statement (ii) also implies the formula*

$$\lim_{n\to\infty}(a \cdot b_n) = a \cdot \lim_{n\to\infty} b_n.$$

(b) *For $k \in \mathbb{N}$, a k-times application of statement (ii) yields that for some convergent sequence $(a_n)_{n\in\mathbb{N}}$, also the sequence $(a_n^k)_{n\in\mathbb{N}}$ is convergent with*

$$\lim_{n\to\infty} a_n^k = \left(\lim_{n\to\infty} a_n\right)^k.$$

Theorem 1.22. Monotonicity of limits

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two convergent real sequences with

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b.$$

Further, assume that for all $n \in \mathbb{N}$ holds $a_n \le b_n$. Then the following holds true:

(i) *$a \le b$;*

(ii) *If $a = b$ and $(c_n)_{n\in\mathbb{N}}$ is another sequence with $a_n \le c_n \le b_n$ for all $n \in \mathbb{N}$, then $(c_n)_{n\in\mathbb{N}}$ is convergent with*

$$\lim_{n\to\infty} c_n = a = b.$$

(Sandwich-Theorem)

Proof. (i) Consider the sequence of differences between b_n and a_n, i.e., $(b_n - a_n)_{n\in\mathbb{N}}$. By Theorem 1.21, it suffices to show that

$$b - a = \lim_{n\to\infty}(b_n - a_n) \ge 0.$$

Assume the converse statement, i.e., $b - a < 0$. Then, we have that both numbers $a - b$ and $b_n - a_n$ are positive and thus

$$|a - b - (a_n - b_n)| = a - b + (b_n - a_n) > a - b.$$

In particular, there exists no $n \in \mathbb{N}$ such that $|a-b-(a_n-b_n)| < \varepsilon$ for $\varepsilon = a-b > 0$. This is a contradiction to $\lim_{n\to\infty}(b_n - a_n) = b - a$.

(ii) Again consider the sequence $(b_n - a_n)_{n\in\mathbb{N}}$ which is tending to zero according to Theorem 1.21. Further, consider the sequence $(c_n - a_n)_{n\in\mathbb{N}}$. Then we have for all $n \in \mathbb{N}$ that $0 \le c_n - a_n \le b_n - a_n$. Let $\varepsilon > 0$. Since $b_n - a_n$ is tending to zero, there exists some N such that for all $n \ge N$ holds $|b_n - a_n - 0| < \varepsilon$. Due to $0 \le c_n - a_n \le b_n - a_n$, we can conclude that for $n \ge N$ holds

$$|c_n - a_n - 0| = c_n - a_n \le b_n - a_n = |b_n - a_n - 0| < \varepsilon.$$

This implies that $(c_n - a_n)_{n\in\mathbb{N}}$ is convergent with $\lim_{n\to\infty}(c_n - a_n) = 0$. Hence $a = 0 + a = \lim_{n\to\infty}(c_n - a_n) + \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$.

□

Remark:

Since the modification of finitely many sequence elements does not change the limits (take a closer look at Definition 1.15), the statements of Theorem 1.22 can be slightly generalised by only claiming that there exists some n_0 such that for all $n \geq n_0$ holds $a_n \leq b_n$ (resp. for all $n \geq n_0$ holds $a_n \leq c_n \leq b_n$ in (ii)). In the proof of (i), one has to replace the words "there exists no $n \in \mathbb{N}$ such that" by "there exists no $n \geq n_0$ such that" and in the proof of (ii) the number N has to be replaced by $\max\{N, n_0\}$.

Attention!

From the fact that we have the strict inequality $a_n < b_n$, we cannot conclude that the limits satisfy $a < b$. To see this, consider the sequences $(a_n)_{n\in\mathbb{N}} = (0, 0, 0, \dots)$ and $(b_n)_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}}$. In this case, we have $a = b = 0$ though the strict inequality $a_n = 0 < \frac{1}{n} = b_n$ holds true for all $n \in \mathbb{N}$.

Example 1.23. a) Consider $(\frac{1}{n^k})_{n\in\mathbb{N}}$ for some $k \in \mathbb{N}$. We state two alternative ways to show that this sequence tends to zero. The first possibility is, of course, an argumentation as in statement (b) in Remark on page 18. The second way to treat this problem is making use of the inequality

$$\frac{1}{n} \geq \frac{1}{n^k} > 0.$$

Since we know from Example 1.17 a) that the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ tends to zero, statement (ii) of Theorem 1.22 directly leads to the fact that $(\frac{1}{n^k})_{n\in\mathbb{N}}$ also tends to zero.

b) Consider $(a_n)_{n\in\mathbb{N}}$ with

$$a_n = \frac{2n^2 + 5n - 1}{-5n^2 + n + 1}.$$

Rewriting

$$a_n = \frac{2 + \frac{5}{n} - \frac{1}{n^2}}{-5 + \frac{1}{n} + \frac{1}{n^2}},$$

and using that both $(\frac{1}{n})_{n\in\mathbb{N}}$ and $(\frac{1}{n^2})_{n\in\mathbb{N}}$ tend to zero, we can apply Theorem 1.21 to obtain that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2 + 5n - 1}{-5n^2 + n + 1} = \lim_{n\to\infty} \frac{2 + \frac{5}{n} - \frac{1}{n^2}}{-5 + \frac{1}{n} + \frac{1}{n^2}} = -\frac{2}{5}.$$

c) Consider $(a_n)_{n\in\mathbb{N}}$ with $a_n = \sqrt{n^2 + 1} - n$. At first glance, none of the so far presented results seem to help to analyse convergence of this sequence. However, we can compute

$$\begin{aligned} a_n &= \sqrt{n^2 + 1} - n = \frac{(\sqrt{n^2 + 1} - n)(\sqrt{n^2 + 1} + n)}{\sqrt{n^2 + 1} + n} \\ &= \frac{n^2 + 1 - n^2}{\sqrt{n^2 + 1} + n} = \frac{1}{\sqrt{n^2 + 1} + n} < \frac{1}{n}. \end{aligned}$$

By Theorem 1.22, we now get that $\lim_{n\to\infty} a_n = 0$.

Next we introduce some further properties of real sequences. In particular, we declare what the phrase “the sequence tends to infinity (∞)” means.

Definition 1.24. Monotonicity, boundedness, divergence to $\pm\infty$

*A **real** sequence $(a_n)_{n\in\mathbb{N}}$ is called*

(a) *monotonically increasing if for all $n \in \mathbb{N}$ holds $a_n \leq a_{n+1}$.*

(b) *strictly monotonically increasing if for all $n \in \mathbb{N}$ holds $a_n < a_{n+1}$.*

(c) *monotonically decreasing if for all $n \in \mathbb{N}$ holds $a_n \geq a_{n+1}$.*

(d) *strictly monotonically decreasing if for all $n \in \mathbb{N}$ holds $a_n > a_{n+1}$.*

(e) *bounded from above if there exists some $c \in \mathbb{R}$ with $a_n \leq c$ for all $n \in \mathbb{N}$.*

(f) *bounded from below if there exists some $c \in \mathbb{R}$ with $a_n \geq c$ for all $n \in \mathbb{N}$.*

(g) *divergent to ∞ if for all $c \in \mathbb{R}$ there exists some N with $a_n \geq c$ for all $n \geq N$. In this case, we write*
$$\lim_{n\to\infty} a_n = \infty.$$

(h) *divergent to $-\infty$ if for all $c \in \mathbb{R}$ there exists some N with $a_n \leq c$ for all $n \geq N$. In this case, we write*
$$\lim_{n\to\infty} a_n = -\infty.$$

Remark:

*It can be readily seen from the definition that a sequence is bounded if and only if it is **both** bounded from above **and** bounded from below.*

Example 1.25. (a) For $k \in \mathbb{N}$, the sequence $(\frac{1}{n^k})_{n\in\mathbb{N}}$ is strongly monotonically decreasing due to $\frac{1}{n^k} > \frac{1}{(n+1)^k}$ and, moreover, both bounded from above and bounded from below.

(b) The sequence $(n^3)_{n\in\mathbb{N}}$ is bounded from below and divergent to ∞.
Proof: The fact that this sequence is bounded from below directly follows from $n^3 \geq 0$ for all $n \in \mathbb{N}$. To show that this sequence is divergent to ∞, let $c \in \mathbb{R}$ be arbitrary and choose
$$N = \begin{cases} \sqrt[3]{c} + 1 & : \text{ if } c \geq 0, \\ 0 & : \text{ else.} \end{cases}$$
Then for $n \geq N$, we have that $n^3 > c$ and thus, the sequence $(a_n)_{n\in\mathbb{N}}$ tends to ∞.

Reminder:

Definition of intervals:
$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$$
$$[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}$$
$$(a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}$$

$$[a, b] := \{x \in \mathbb{R} \mid a \le x \le b\}$$
$$(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$$
$$(-\infty, b] := \{x \in \mathbb{R} \mid x \le b\}$$
$$(a, \infty) := \{x \in \mathbb{R} \mid a < x\}$$
$$[a, \infty) := \{x \in \mathbb{R} \mid a \le x\}$$

Definition 1.26. Upper and lower bounds for sets

Let $M \subset \mathbb{R}$ be any subset of real numbers. A real number b is called an upper bound of M if $x \le b$ for all $x \in M$. Analogously, $a \in \mathbb{R}$ is called a lower bound of M if $a \le x$ for all $x \in M$.

Definition 1.27. Bounded from below or above for sets

Let $M \subset \mathbb{R}$. If there is an upper bound for M, then one calls the set bounded from above. If there is a lower bound for M, then one calls the set bounded from below. If both properties hold, we call the set simply bounded.

Definition 1.28. Maximal and minimal element of a set

Let $M \subset \mathbb{R}$. An element $d \in M$ is called maximal if $x \le d$ for all $x \in M$. An element $c \in M$ is called minimal if $x \ge c$ for all $x \in M$. If these numbers exist, one write $\max M = d$ and $\min M = c$.

Definition 1.29. Supremum and infimum

Let $M \subset \mathbb{R}$ be a set.

(a) A real number s is called the supremum of M if:

- $x \le s$ *for all* $x \in M$,
- *for all* $\varepsilon > 0$ *there is an* $x \in M$ *with* $s - \varepsilon < x$.

In this case we write $s = \sup M$.

(b) A real number l is called the infimum of M if:

- $x \ge l$ *for all* $x \in M$,
- *for all* $\varepsilon > 0$ *there is an* $x \in M$ *with* $l + \varepsilon > x$.

In this case we write $l = \inf M$.

(c) We further define

- $\sup M = \infty$ *if M is not bounded from above;*
- $\inf M = -\infty$ *if M is not bounded from below;*
- $\sup \varnothing = -\infty$;
- $\inf \varnothing = \infty$.

To remember: Sup and Inf

The infimum is the greatest lower bound and the supremum is the lowest upper bound.

Example 1.30. (a) $\sup[0,1] = 1, \quad \inf[0,1] = 0;$

(b) $\sup(0,1) = 1, \quad \inf(0,1) = 0;$

(c) $\sup\{\frac{1}{n} \,:\, n \in \mathbb{N}\} = 1, \qquad \inf\{\frac{1}{n} \,:\, n \in \mathbb{N}\} = 0;$

(d) $\sup\{x \in \mathbb{Q} \,:\, x^2 < 2\} = \sqrt{2}, \qquad \inf\{x \in \mathbb{Q} \,:\, x^2 < 2\} = -\sqrt{2};$

Remark: Difference between sup and max (resp. inf and min)

In contrast to the maximum, the supremum does not need to belong to the respective set. For instance, we have $1 = \sup(0,1)$*, but* $\max(0,1)$ *does not exist. The analogous statement holds true for* inf *and* min*. However, we can make the following statement: If* $\max M$ $(\min M)$ *exists, then* $\max M = \sup M$ $(\min M = \inf M)$*.*

The next result concerns the special property of the real numbers that supremum and infimum are defined for all subsets of the real numbers. This theorem goes back to Julius Wilhelm Richard Dedekind (1831–1916). It follows from the completeness axiom (C):

Theorem 1.31. Dedekind's Theorem

Every non-empty bounded set $M \subset \mathbb{R}$ *has a supremum and an infimum with* $\sup M, \inf M \in \mathbb{R}$*.*

We make essential use of Dedekind's theorem to prove the following result:

Theorem 1.32. Convergence of bounded and monotonic sequences

Let $(a_n)_{n\in\mathbb{N}}$ *be a real sequence that has one of the following properties:*

- $(a_n)_{n\in\mathbb{N}}$ *is monotonically increasing and bounded from above;*
- $(a_n)_{n\in\mathbb{N}}$ *is monotonically decreasing and bounded from below;*

Then $(a_n)_{n\in\mathbb{N}}$ *is convergent.*

Proof: Let us first assume that $(a_n)_{n\in\mathbb{N}}$ is monotonically increasing and bounded from above. Define the set $M = \{a_n \,:\, n \in \mathbb{N}\}$. Since M is bounded, Dedekind's theorem implies that there exists some $K \in \mathbb{R}$ such that

$$K = \sup M.$$

We show that K is indeed the limit of the sequence $(a_n)_{n\in\mathbb{N}}$.
Let $\varepsilon > 0$. By the definition of the supremum, we have that $a_n \leq K$ for all $n \in \mathbb{N}$ and there exists some $N \in \mathbb{N}$ such that $a_N > K - \varepsilon$. The monotonicity of $(a_n)_{n\in\mathbb{N}}$ implies that for all $n \geq N$ holds $a_N \leq a_n$. Altogether, we have

$$K - \varepsilon < a_N \leq a_n \leq K$$

and thus $|K - a_n| = K - a_n < \varepsilon$ for all $n \geq N$. This implies convergence to K.
To prove that convergence is also guaranteed in the case where $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing and bounded from below, we consider the sequence $(-a_n)_{n\in\mathbb{N}}$, which is now bounded from above and monotonically increasing. By the (already proven) first statement of this theorem, the sequence $(-a_n)_{n\in\mathbb{N}}$ is convergent, whence $(a_n)_{n\in\mathbb{N}}$ is convergent as well. □

Remark:

By the same argumentation as in Remark from page 19, the monotone increase (decrease) of $(a_n)_{n\in\mathbb{N}}$ can be slightly relaxed by only claiming that $a_n \leq a_{n+1}$ ($a_n \geq a_{n+1}$) for all $n \geq n_0$ for some n_0 in $\mathbb{N}$. In such a case, the limit of the sequence is then given by $\sup\{a_n : n \geq n_0\}$ (resp. $\inf\{a_n : n \geq n_0\}$).

Example 1.33. a) Consider the sequence $(a_n)_{n\in\mathbb{N}}$ which is recursively defined via $a_1 = 1$ and

$$a_{n+1} = \frac{a_n + \frac{2}{a_n}}{2} \quad \text{for } n \geq 1.$$

We now prove that this sequence is convergent by showing that it is bounded from below and for all $n \geq 2$ holds $a_{n+1} \leq a_n$.

Proof: To show boundedness from below, we use the inequality $\sqrt{xy} \leq \frac{x+y}{2}$ for all nonnegative $x, y \in \mathbb{R}$. This inequality is a consequence of

$$0 \leq \frac{(\sqrt{x} - \sqrt{y})^2}{2} = \frac{x+y}{2} - \sqrt{xy}.$$

The first inequality is a consequence of the fact that squares of real numbers cannot be negative.
Using this inequality, we obtain for $n \geq 1$

$$a_{n+1} = \frac{a_n + \frac{2}{a_n}}{2} \geq \sqrt{a_n \cdot \frac{2}{a_n}} = \sqrt{2}.$$

Thus, (a_n) is bounded from below. For showing monotonicity, we consider

$$a_{n+1} - a_n = \frac{a_n + \frac{2}{a_n}}{2} - a_n = \frac{1}{2a_n}(2 - a_n^2).$$

In particular, if $n \geq 2$, we have that $a_n > 0$ and $2 - a_n^2 \leq 0$. Thus, $a_{n+1} - a_n \leq 0$ for $n \geq 2$. An application of Theorem 1.31 (resp. the slight generalisation in Remark from above) now leads to the existence of some $a \in \mathbb{R}$ with $a = \lim_{n\to\infty} a_n$.

To compute the limit, we make use of the relation $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ (follows directly from Definition 1.15) and the formulae for limits in Theorem 1.21. This yields

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n + \frac{2}{a_n}}{2} = \frac{a + \frac{2}{a}}{2}.$$

This relation leads to the equation $2 - a^2 = 0$, i.e., we either have $a = \sqrt{2}$ or $a = -\sqrt{2}$. However, the latter solution cannot be a limit since all sequence elements are positive. Therefore, we have

$$\lim_{n\to\infty} a_n = \sqrt{2}.$$

b) Let $x \in \mathbb{R}$ with $x > 1$. Consider the sequence $(\sqrt[n]{x})_{n\in\mathbb{N}}$. It can be directly seen that $(\sqrt[n]{x})_{n\in\mathbb{N}}$ is monotonically decreasing and bounded from below by one. Therefore, the limit

$$a = \lim_{n\to\infty} \sqrt[n]{x}$$

exists with $a \geq 1$. To show that $a = 1$, we assume that $a > 1$ and lead this to a contradiction.
The equation $a > 1$ leads to the existence of some $n \in \mathbb{N}$ with $a^n > x$, and thus $a > \sqrt[n]{x}$. On the other hand, the monotone decrease of $(\sqrt[n]{x})_{n\in\mathbb{N}}$ implies that

$$a = \lim_{n\to\infty} \sqrt[n]{x} = \inf\{\sqrt[n]{x} \,:\, n \in \mathbb{N}\} < a,$$

which is a contradiction.

c) Let $x \in \mathbb{R}$ with $0 < x < 1$. Consider the sequence $(a_n)_{n\in\mathbb{N}} = (\sqrt[n]{x})_{n\in\mathbb{N}}$. Then we have by Example b) and Theorem 1.21 that

$$\lim_{n\to\infty} \sqrt[n]{x} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{1}{x}}} = \frac{1}{1} = 1.$$

d) Let (a_n) be a nonnegative sequence with $a_n \to a$ and $k \in \mathbb{N}$. Then for all $\varepsilon > 0$ there exists $N > 0$ such that $|a_n - a| < \varepsilon^k$. From this it follows that

$$|\sqrt[k]{a_n} - \sqrt[k]{a}| \leq \sqrt[k]{|a_n - a|} < \varepsilon.$$

Thus $(\sqrt[k]{a_n})$ is convergent with limit $\sqrt[k]{a}$.

e) The sequence $(a_n)_{n\in\mathbb{N}}$ defined as $a_n := \left(1 + \frac{1}{n}\right)^n$ is convergent.

Remark:

The limit of the sequence

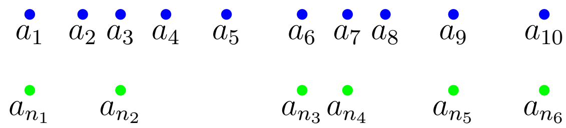
$$(a_n)_{n\in\mathbb{N}} = \left(\left(1 + \frac{1}{n}\right)^n\right)_{n\in\mathbb{N}},$$

i.e. $e := \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ *is well known as Euler's number. Later on we will define the exponential function* exp*. It holds that* $e = \exp(1) \approx 2.7182818...$ *. Indeed, we will show later on that* $e^z = \lim_{n\to\infty}\left(1 + \frac{z}{n}\right)^n = \exp(z)$*.*

1.3 Subsequences and accumulation values

Definition 1.34. Subsequence

Let $(a_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{F}$*. Let* $(n_k)_{k\in\mathbb{N}}$ *be a strongly monotonically increasing sequence with* $n_k \in \mathbb{N}$ *for all* $k \in \mathbb{N}$*. Then* $(a_{n_k})_{k\in\mathbb{N}}$ *is called a* subsequence.



Example 1.35. Consider the sequence $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}}$. Then some subsequences are given by

- $(a_{n_k})_{k\in\mathbb{N}} = (a_{2k})_{k\in\mathbb{N}} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots)$;
- $(a_{n_k})_{k\in\mathbb{N}} = (a_{k^2})_{k\in\mathbb{N}} = (1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots)$;
- $(a_{n_k})_{k\in\mathbb{N}} = (a_{2^k})_{k\in\mathbb{N}} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots)$;
- $(a_{n_k})_{k\in\mathbb{N}} = (a_{k!})_{k\in\mathbb{N}} = (1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}, \ldots)$.

Theorem 1.36. Convergence of subsequences

Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{F}$ with $\lim_{n\to\infty} a_n = a$. Then all subsequences $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ are also convergent with

$$\lim_{k\to\infty} a_{n_k} = a.$$

Proof: Since $1 \le n_1 < n_2 < n_3 < \ldots$ and $n_k \in \mathbb{N}$ for all $k \in \mathbb{N}$, we have that $n_k \ge k$ for all $k \in \mathbb{N}$. Let $\varepsilon > 0$. By the convergence of $(a_n)_{n\in\mathbb{N}}$, there exists some N such that $|a_k - a| < \varepsilon$ for all $k \ge N$. Due to $n_k \ge k$, we thus also have that $|a_{n_k} - a| < \varepsilon$ for all $k \ge N$. □

Attention!

The existence of a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ does in general not imply the convergence of $(a_n)_{n\in\mathbb{N}}$. For instance, consider $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$. Both subsequences

$$\begin{aligned}(a_{2k})_{k\in\mathbb{N}} &= ((-1)^{2k})_{k\in\mathbb{N}} &&= (1, 1, 1, 1, \ldots)\\ (a_{2k+1})_{k\in\mathbb{N}} &= ((-1)^{2k+1})_{k\in\mathbb{N}} &&= (-1, -1, -1, -1, \ldots)\end{aligned}$$

are convergent though $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$ is divergent (see Example 1.17 b)).

However, we can "rescue" this statement by additionally claiming that $(a_n)_{n\in\mathbb{N}}$ is monotonic.

Theorem 1.37. Subsequences of monotonic sequences

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If $(a_n)_{n\in\mathbb{N}}$ is monotonic and there exists a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$, then $(a_n)_{n\in\mathbb{N}}$ is convergent with

$$\lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{n_k}.$$

Proof: Denote $a = \lim_{k\to\infty} a_{n_k}$. We just consider the case where $(a_n)_{n\in\mathbb{N}}$ is monotonically increasing (the remaining part can be done analogously to the argumentations at the end of the proof of Theorem 1.31). Since $(a_{n_k})_{k\in\mathbb{N}}$ is also monotonically increasing, we have that $a = \sup\{a_{n_k} : k \in \mathbb{N}\}$.
Let $\varepsilon > 0$. Due to the convergence and monotonicity of $(a_{n_k})_{k\in\mathbb{N}}$, there exists some $K \in \mathbb{N}$

such that for all $k \geq K$ holds

$$a - \varepsilon < a_{n_k} \leq a.$$

Now assume that $n \geq N = n_K$. Monotonicity then implies that $a - \varepsilon < a_{n_K} \leq a_n \leq a_{n_n} \leq a$. In particular, we have that

$$|a - a_n| = a - a_n < \varepsilon.$$

□

Next we present the famous Theorem of Bolzano-Weierstraß.

Theorem 1.38. Theorem of Bolzano-Weierstraß

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{F}$. Then there exists some convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$.

Proof: First we consider the case $\mathbb{F} = \mathbb{R}$. Since $(a_n)_{n\in\mathbb{N}}$ is bounded, there exist some $A, B \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ holds $A \leq a_n \leq B$. We will now successively construct subintervals $[A_n, B_n] \subset [A, B]$ which still include infinitely many sequence elements of $(a_n)_{n\in\mathbb{N}}$.

Inductively define $A_0 = A$, $B_0 = B$ and for $k \geq 1$,

a) $A_k = A_{k-1}$, $B_k = \frac{A_{k-1}+B_{k-1}}{2}$, if the interval $[A_{k-1}, \frac{A_{k-1}+B_{k-1}}{2}]$ contains infinitely many sequence elements of $(a_n)_{n\in\mathbb{N}}$, and

b) $A_k = \frac{A_{k-1}+B_{k-1}}{2}$, $B_k = B_{k-1}$, else.

By the construction of A_k and B_k, we have that each interval $[A_k, B_k]$ has infinitely many sequence elements of $(a_n)_{n\in\mathbb{N}}$. We furthermore have $B_1 - A_1 = \frac{1}{2}(B - A)$, $B_2 - A_2 = \frac{1}{4}(B - A)$, ..., $B_k - A_k = \frac{1}{2^k}(B - A)$. Moreover, the sequence $(A_n)_{n\in\mathbb{N}}$ is monotonically increasing and bounded from above by B, i.e., it is convergent by Theorem 1.32. The relation $B_k - A_k = \frac{1}{2^k}(B - A)$ moreover implies that $(B_n)_{n\in\mathbb{N}}$ is also convergent and has the same limit as $(A_n)_{n\in\mathbb{N}}$. Denote

$$a = \lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n.$$

Define a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ by $n_1 = 1$ and n_k with $n_k > n_{k-1}$ and $a_{n_k} \in [A_k, B_k]$ (which is possible since $[A_k, B_k]$ contains infinitely many elements of $(a_n)_{n\in\mathbb{N}}$). Then $A_k \leq a_{n_k} \leq B_k$. Theorem 1.22 then implies that

$$a = \lim_{k\to\infty} a_{n_k}.$$

Finally we consider the case $\mathbb{F} = \mathbb{C}$. Write $a_n = b_n + ic_n$ where i is the imaginary unit, $b_n := \operatorname{Re}(a_n)$ denotes the real part and $c_n := \operatorname{Im}(a_n)$ denotes the imaginary part of a_n. Since $|a_n| = \sqrt{b_n^2 + c_n^2} \geq \max\{|b_n|, |c_n|\} \geq 0$, the boundedness of the complex sequence $(a_n)_{n\in\mathbb{N}}$ implies the boundedness of both real sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$. Then, by the previous, we now that $(b_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(b_{n_k})_{k\in\mathbb{N}}$. Since the subsequence $(c_{n_k})_{k\in\mathbb{N}}$ of the bounded sequence $(c_n)_{n\in\mathbb{N}}$ is also bounded, it also has a convergent subsequence $(c_{n_{k_m}})_{m\in\mathbb{N}}$. The subsequence $(b_{n_{k_m}})_{m\in\mathbb{N}}$ of the convergent sequence $(b_{n_k})_{k\in\mathbb{N}}$ also converges. Hence $(a_{n_{k_m}})_{m\in\mathbb{N}} = (b_{n_{k_m}} + ic_{n_{k_m}})_{m\in\mathbb{N}}$ is a convergent subsequence of $(a_n)_{n\in\mathbb{N}}$ with $\lim_{m\to\infty} a_{n_{k_m}} = \lim_{m\to\infty} b_{n_{k_m}} + i \cdot \lim_{m\to\infty} c_{n_{k_m}}$. □

Definition 1.39. Accumulation value

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{F}$. Then $a \in \mathbb{F}$ is called accumulation value if there exists some subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with

$$a = \lim_{k\to\infty} a_{n_k}.$$

Attention! Names

Accumulation values are often called by other names, like accumulation points, limits points *or* cluster points.

Proposition 1.40.

$a \in \mathbb{F}$ is an accumulation value if and only if in every ε-neighbourhood of a, there are infinitely many elements of the sequence $(a_n)_{n\in\mathbb{N}}$.

Definition 1.41. Accumulation values $\pm\infty$

A real sequence $(a_n)_{n\in\mathbb{N}}$ is said to have the (improper) accumulation value ∞ if it is not bounded from above. Analogously, we define the (improper) accumulation value $-\infty$ if it is not bounded from below.

Definition 1.42. Limit superior - limit inferior

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. A number $a \in \mathbb{R} \cup \{\infty, -\infty\}$ is called

- *limit superior of $(a_n)_{n\in\mathbb{N}}$ if a is the largest accumulation value of $(a_n)_{n\in\mathbb{N}}$. In this case, we write*
$$a = \limsup_{n\to\infty} a_n.$$
- *limit inferior of $(a_n)_{n\in\mathbb{N}}$ if a is the smallest accumulation value of $(a_n)_{n\in\mathbb{N}}$. In this case, we write*
$$a = \liminf_{n\to\infty} a_n.$$

Remark:

Almost needless to say, we define the ordering between infinity and real numbers by $-\infty < a < \infty$ for all $a \in \mathbb{R}$. It can be shown that (in contrast to the limit) the limit superior and limit inferior always exist for any real sequence. This will follow from the subsequent results.

Lemma 1.43.

Let $(a_n)_{n\in\mathbb{R}}$ be a real sequence. Then the following statements hold

a) $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \,|\, k \geq n\}$

b) $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k \,|\, k \geq n\}$

c) *A sequence is convergent if and only if* $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n \notin \{\pm\infty\}$. *In this case holds* $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

d) *A sequence is divergent to* ∞ *if and only if* $\liminf_{n\to\infty} a_n = \infty$. *In this case also holds* $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \infty$.

e) *A sequence is divergent to* $-\infty$ *if and only if* $\limsup_{n\to\infty} a_n = -\infty$. *In this case also holds* $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = -\infty$.

Proof:

a) If (a_n) is not bounded from below, then, by Definition 1.41, $-\infty$ is an accumulation value of (a_n) which necessarily must be the smallest one. By Definition 1.42 $\liminf a_n = -\infty$. On the other hand, the unboundedness from below of (a_n) implies $s_n := \inf\{a_k \mid k \geq n\} = -\infty$ for all $n \in \mathbb{N}$ and therefore also $\lim_{n\to\infty} s_n = -\infty$. Note that formally we only defined limits for sequences with values in $\mathbb{R}$ and not with values in $\mathbb{R} \cup \{-\infty, \infty\}$. Here we implicitely used the obvious extension, namely we said that the limit of the sequence (s_n) which is constantly $-\infty$ has the limit $-\infty$.

Next we consider the case where (a_n) is divergent to $+\infty$. In particular, (a_n) is not bounded from above and therefore $+\infty$ is an accumulation value by Definition 1.41. This is also the only accumulation value, since each subsequence of (a_n) also diverges to $+\infty$. Hence, by Definition 1.42, $\liminf a_n = +\infty$. On the other hand for each $c > 0$ there is an $N \in \mathbb{N}$ such that $a_n \geq c$ for all $n \geq N$. Therfore $s_n = \inf\{a_k \mid k \geq n\} \geq c$ for all $n \geq N$ which shows that also (s_n) diverges to $+\infty$, i.e. $\lim_{n\to\infty} s_n = +\infty$.

Finally we consider the remaining case where (a_n) is bounded from below and not divergent to $+\infty$. Then there exist constants $c_1, c_2 \in \mathbb{R}$ such that $c_1 \leq a_n$ for all $n \in \mathbb{N}$ and $a_n \leq c_2$ for infinitely many $n \in \mathbb{N}$. This implies

$$c_1 \leq s_n = \inf\{a_k \mid k \geq n\} \leq c_2$$

for all $n \in \mathbb{N}$, i.e. (s_n) is bounded. Since (s_n) is also monotonically increasing as

$$s_{n+1} = \inf\{a_k \mid k \geq n+1\} \geq \min\{\inf\{a_k \mid k \geq n+1\}, a_n\} = \inf\{a_k \mid k \geq n\} = s_n \ ,$$

it must be convergent. Set $s := \lim_{n\to\infty} s_n$. We can recursively define a subsequence (a_{n_k}) of (a_n) with $n_1 = 1$ and $n_{k+1} > n_k$ such that

$$s_{(n_k+1)} = \inf\{a_m \mid m \geq n_k + 1\} \leq a_{n_{k+1}} \leq s_{(n_k+1)} + \frac{1}{k} \ .$$

Since the right- and left-hand sides of this inequality converge to s for $k \to \infty$, we also have $\lim_{k\to\infty} a_{n_k} = s$ which shows that s is an accumulation value of (a_n). On the other hand, if x is any other accumulation value of (a_n) and if (a_{j_k}) is a corresponding subsequence such that $\lim_{k\to\infty} a_{j_k} = x$, then

$$s_{j_k} = \inf\{a_m \mid m \geq j_k\} \leq a_{j_k}$$

shows that $s = \lim_{k\to\infty} s_{j_k} \leq \lim_{k\to\infty} a_{j_k} = x$ which means that s is indeed the smallest accumulation value of (a_n), that is $\liminf a_n = s$.

b) Analogous to a).

c) "$\Rightarrow$": Since the sequence (a_n) is convergent every subsequence is convergent with the same limit. By Definition 1.39 there exists only one accumulation value and thus $\liminf a_n = \limsup a_n$.
"$\Leftarrow$": Let $s := \liminf a_n = \limsup a_n$. Then for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $s - \varepsilon < a_n < s + \varepsilon$. This implies convergence of $(a_n)_{n\in\mathbb{N}}$ to s.

d) Let $s_n := \inf\{a_k : k \geq n\}$.
"$\Rightarrow$": We have for any $c > 0$ an $N \in \mathbb{N}$ such that $a_n > c+1$ for all $n \geq N$. Thus $s_n > c$ for all $n \geq N$.
"$\Leftarrow$": By definition of s_n we have $a_n \geq s_n$. Thus $a_n \to \infty$ since $s_n \to \infty$.

e) Analogous to d).

□

Example 1.44. (a) $(a_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$. Then ∞ is the only accumulation value and consequently $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \infty$.

(b) $(a_n)_{n\in\mathbb{N}} = ((-1)^n n)_{n\in\mathbb{N}} = (-1, 2, -3, 4, -5, 6, \ldots)$. Then ∞ and $-\infty$ are the only accumulation values and consequently $\limsup a_n = \infty$ and $\liminf a_n = -\infty$.

(c) $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$. Then 1 and -1 are the only accumulation values and consequently $\limsup a_n = 1$ and $\liminf a_n = -1$.

(d) $(a_n)_{n\in\mathbb{N}}$ with

$$a_n = \begin{cases} (-1)^n & : \text{ if } n \text{ is divisible by } 3, \\ n & : \text{ else.} \end{cases}$$

Then we have $(a_n)_{n\in\mathbb{N}} = (1, 2, -1, 4, 5, 1, 7, 8, -1, 9, 10, \ldots)$ and the set of accumulation values is given by $\{-1, 1, \infty\}$. Thus, we have $\limsup a_n = \infty$ and $\liminf a_n = -1$.

1.4 Cauchy Sequences

Definition 1.45. Cauchy sequences

A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ is called Cauchy sequence if for all $\varepsilon > 0$, there exists some N such that for all $n, m \geq N$ holds

$$|a_n - a_m| < \varepsilon.$$

Remark:

By the expression "$n, m \geq N$", we mean that both n and m are greater or equal than N, i.e., $n \geq N$ and $m \geq N$.

Now we show that convergent sequences are indeed Cauchy sequences.

Theorem 1.46.

Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence. Then $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof: Let $a = \lim_{n\to\infty} a_n$ and $\varepsilon > 0$. Then there exists some N such that for all $k \geq N$ holds $|a - a_k| < \frac{\varepsilon}{2}$. Hence, for all $m, n \geq N$ holds

$$|a_n - a_m| = |(a_n - a) + (a - a_m)| \leq |a_n - a| + |a - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

The following theorem is closely related to Theorem 1.19.

Theorem 1.47. Cauchy sequences are bounded

Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then $(a_n)_{n\in\mathbb{N}}$ is bounded.

Proof: Take $\varepsilon = 1$. Then there exists some N such that for all $n, m \geq N$ holds $|a_n - a_m| < 1$. Thus, for all $n \geq N$ holds

$$|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| < 1 + |a_N|.$$

Now choose

$$c = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, |a_N| + 1\}$$

and consider some arbitrary sequence element a_k.
If $k < N$, we have that $|a_k| \leq \max\{|a_1|, |a_2|, \dots, |a_{N-1}|\} \leq c$.
If $k \geq N$, we have, by the above calculations, that $|a_k| < |a_N| + 1 \leq c$.
Altogether, this implies that $|a_k| \leq c$ for all $k \in \mathbb{N}$, so $(a_n)_{n\in\mathbb{N}}$ is bounded by c. □

Now we show that Cauchy sequences in $\mathbb{F}$ are even convergent:

Theorem 1.48.

Every Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ converges.

Proof: By Theorem 1.47, $(a_n)_{n\in\mathbb{N}}$ is bounded. By Theorem 1.38 of Bolzano-Weierstraß it has a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$. Set $a := \lim_{k\to\infty} a_{n_k}$. For given $\varepsilon > 0$ there exist $N_1, N_2 \in \mathbb{N}$ such that $|a_{n_k} - a| < \varepsilon/2$ for all $k \geq N_1$ and $|a_n - a_m| < \varepsilon/2$ for all $n, m \geq N_2$. Thus for $n \geq N := \max\{N_1, N_2\}$ holds $n_n \geq n \geq N$ and

$$|a_n - a| \leq |a_n - a_{n_n} + a_{n_n} - a| \leq |a_n - a_{n_n}| + |a_{n_n} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon\ .$$ □

Theorem 1.48 is not true for arbitrary normed $\mathbb{F}$-vector spaces. Those normed $\mathbb{F}$-vector spaces $(V, ||\cdot||)$ for which every Cauchy sequence has a limit in V are called complete or Banach spaces (in honour of the Polish mathematician Stefan Banach). Without proof we state that all finite dimensional normed $\mathbb{F}$-vector spaces are Banach spaces.

1.5 Bounded, Open, Closed and Compact Sets

Next, we define some particular sets and special properties of sets.

Definition 1.49. ε-neighbourhood

Let $x \in \mathbb{F}$. Then for $\varepsilon > 0$, the ε-neighbourhood of x is defined by the set

$$B_\varepsilon(x) = \{y \in \mathbb{F} : |x - y| < \varepsilon\}.$$

A set $M \subset \mathbb{F}$ is called neighbourhood of x, if there exists some $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subset M.$$

Example 1.50. (a) If $\mathbb{F} = \mathbb{R}$, then the ε-neighbourhood of $x \in \mathbb{R}$ is given by the interval

$$B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon).$$

(b) If $\mathbb{F} = \mathbb{C}$, $\varepsilon > 0$, then the ε-neighbourhood of $x \in \mathbb{C}$ consists of all complex numbers being in the interior of a circle in the complex plane with midpoint x and radius ε.

(c) $[0,1]$ is a neighbourhood of $\frac{1}{2}$ (also of $\frac{3}{4}$, $\frac{1}{\sqrt{2}}$ etc.), but it is not a neighbourhood of 0 or 1.

Definition 1.51. Bounded, Open, closed, compact sets

Let $M \subset \mathbb{F}$. Then M is called

(i) bounded if there exists some $c \in \mathbb{R}$ such that for all $x \in M$ holds: $|x| \le c$.

(ii) open if for all $x \in M$ holds: M is a neighbourhood of x.

(iii) closed if for all convergent sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in M$ for all $n \in \mathbb{N}$ holds: $\lim_{n\to\infty} a_n = a \in M$.

(iv) compact if for all sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in M$ for all $n \in \mathbb{N}$ holds: There exists some convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} a_{n_k} = a \in M$.

Example 1.52. (a) The interval $(0,1)$ is open.
Proof: Consider $x \in (0,1)$. Then for $\varepsilon = \min\{x, 1-x\}$ holds $\varepsilon > 0$ and $B_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) \subset (0,1)$. □

(b) The interval $(0,1)$ is not closed.
Proof: Consider the sequence $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n+1})_{n\in\mathbb{N}}$. Clearly, for all $n \in \mathbb{N}$ holds $a_n = \frac{1}{n+1} \in (0,1)$, but $(a_n)_{n\in\mathbb{N}}$ converges to $0 \notin (0,1)$. □

(c) The interval $(0,1)$ is not compact.
Proof: Again consider the sequence $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n+1})_{n\in\mathbb{N}}$ in $(0,1)$. The convergence of $(a_n)_{n\in\mathbb{N}}$ to $0 \notin (0,1)$ also implies that this holds true for any subsequence $(a_{n_k})_{k\in\mathbb{N}}$ (see Theorem 1.36). Hence, any subsequence of the above constructed one is not convergent to some value in $(0,1)$. □

(d) The interval $(0,1]$ is neither open nor closed.
Proof: The closedness can be disproved by considering again the sequence $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n+1})_{n\in\mathbb{N}}$, whereas the non-openness follows from the fact that $(0,1]$ is not a neighbourhood of 1. □

(e) The set $\mathbb{R}$ is open and closed but not compact.
Proof: Openness and closedness are easy to verify. To see that this set is not compact, consider the sequence $(a_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$ (which is of course in $\mathbb{R}$). It can be readily verified that any subsequence $(a_{n_k})_{k\in\mathbb{N}} = (n_k)_{k\in\mathbb{N}}$ is unbounded, too. Therefore, arbitrary subsequences $(a_{n_k})_{k\in\mathbb{N}} = (n_k)_{k\in\mathbb{N}}$ cannot converge. □

(f) The empty set $\varnothing$ is open, closed and compact.
Proof: $\varnothing$ is a neighbourhood of all $x \in \varnothing$ (there is none, but the statement "for all $x \in \varnothing$" holds then true more than ever). By the same kind of argumentation, we can show that this set is compact and closed. The non-existence of a sequence in $\varnothing$ implies that every statement holds true for them. In particular, all sequences $(a_n)_{n\in\mathbb{N}}$ in $\varnothing$ converge to some $x \in \varnothing$ and have a convergent subsequence with limit in $\varnothing$. □

Next we relate these three concepts to each other.

Theorem 1.53.

For a set $C \subset \mathbb{F}$, the following statements are equivalent:

(i) C is open;

(ii) $\mathbb{F}\backslash C$ is closed.

Proof:
"(i)⇒(ii)": Let C be open. Consider a convergent sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{F}\backslash C$. We have to show that for $a = \lim_{n\to\infty} a_n$ holds $a \in \mathbb{F}\backslash C$. Assume the converse, i.e., $a \in C$. Since C is open, we have that $B_\varepsilon(a) \subset C$ for some $\varepsilon > 0$. By the definition of convergence, there exists some N such that for all $n \geq N$ holds $|a - a_n| < \varepsilon$, i.e.,

$$a_n \in B_\varepsilon(a) \subset C.$$

However, this is a contradiction to $a_n \in \mathbb{F}\backslash C$.

"(ii)⇒(i)": Let $\mathbb{F}\backslash C$ be closed. We have to show that C is open. Assume the converse, i.e., C is not open. In particular, this means that there exists some $a \in C$ such that for all $n \in \mathbb{N}$ holds $B_{\frac{1}{n}}(a) \not\subset C$. This means that for all $n \in \mathbb{N}$, we can find some $a_n \in \mathbb{F}\backslash C$ with $a_n \in B_{\frac{1}{n}}(a)$, i.e., $|a - a_n| < \frac{1}{n}$. As a consequence, for the sequence $(a_n)_{n\in\mathbb{N}}$ holds that

$$\lim_{n\to\infty} a_n = a \in C,$$

but $a_n \in \mathbb{F}\backslash C$ for all $n \in \mathbb{N}$. This is a contradiction to the closedness of $\mathbb{F}\backslash C$. □

Now we present the connection between compactness, closedness and boundedness of subsets of $\mathbb{F}$. Note that these results hold as well in the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$.

Theorem 1.54. Theorem of Heine-Borel

For a subset $C \subset \mathbb{F}$, the following statements are equivalent:

(i) C is compact;

(ii) C is bounded and closed.

Proof:
"(i)⇒(ii)": Let C be compact.

Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{F}$ with $a_n \in C$ and $a := \lim_{n\to\infty} a_n \in \mathbb{F}$. Since C is compact, there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $b := \lim_{k\to\infty} a_{n_k} \in C$. By Theorem 1.36 we have $a = b \in C$.

Now assume that C is unbounded. Then for all $n \in \mathbb{N}$, there exists some $a_n \in C$ with $|a_n| \geq n$. Consider an arbitrary subsequence $(a_{n_k})_{k\in\mathbb{N}}$. Due to $|a_{n_k}| \geq n_k \geq k$, we have that $(a_{n_k})_{k\in\mathbb{N}}$ is unbounded, i.e., it cannot be convergent. This is also a contradiction to compactness.

"(ii)⇒(i)": Let C be closed and bounded. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in C. The boundedness of C then implies the boundedness of $(a_n)_{n\in\mathbb{N}}$. By the Theorem of Bolzano-Weierstraß, there exists a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$, i.e.,

$$\lim_{k\to\infty} a_{n_k} = a$$

for some $a \in \mathbb{F}$. For compactness, we now have to show that $a \in C$. However, this is guaranteed by the closedness of C. □

Remark:

Taking a closer look to the proof "(i)⇒(ii)', we did not explicitly use that we are dealing with one of the spaces $\mathbb{R}$ or $\mathbb{C}$. Indeed, the implication that compact sets are bounded and closed holds true for all normed spaces. However, "(ii)⇒(i)" does not hold true in arbitrary normed spaces. Indeed, there are examples of normed spaces that have bounded and closed subsets which are not compact.

Definition 1.55. Interior, closure, boundary

For $C \subset \mathbb{F}$, we define the

(i) interior of C by the set

$$\mathring{C} = \{x \in C : \text{ there exists some } \varepsilon > 0 \text{ such that } B_\varepsilon(x) \subset C\}.$$

The elements of $\mathring{C}$ are called inner points of C.

(ii) closure of C by the set

$$\overline{C} = \{x \in \mathbb{F} : \text{ there exists a sequ. } (a_n)_{n\in\mathbb{N}} \text{ in } C \text{ with } \lim_{n\to\infty} a_n = x\}.$$

The elements of $\overline{C}$ are called osculation points of C.

(iii) boundary of C by the set

$$\partial C = \overline{C} \setminus \mathring{C}.$$

The elements of ∂C are called boundary points of C.

Remark:

The relation $\mathring{C} \subset C \subset \overline{C}$ holds true for arbitrary subsets $C \subset \mathbb{F}$. The first inclusion holds true by definition of $\mathring{C}$. To verify $C \subset \overline{C}$ we take an arbitrary $x \in C$ and consider the constant sequence $(x)_{n\in\mathbb{N}}$. Since this sequence is completely contained in C and converges to $x \in C$, we must have that $x \in \overline{C}$.
It can be shown that for all sets C, $\mathring{C}$ is always open and $\overline{C}, \partial C$ are always closed sets. In particular, if C is open (closed), then we have $\mathring{C} = C$ (resp. $\overline{C} = C$)

Example 1.56. These are examples in the case $\mathbb{F} = \mathbb{R}$:

(a) $C = [0, 1]$, then $\mathring{C} = (0, 1)$, $\overline{C} = [0, 1]$ and $\partial C = \{0, 1\}$;

(b) $C = \{\frac{1}{n} : n \in \mathbb{N}\}$, then $\mathring{C} = \varnothing$ and $\overline{C} = \partial C = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.

2 Infinite Series

> Truth hurts. Maybe not as much as jumping on a bicycle with the seat missing, but it hurts.
>
> Lt. Frank Drebin (Leslie Nielsen)

The topic of this part are "infinite sums" of the form

$$\sum_{k=1}^{\infty} a_k$$

for some sequence $(a_n)_{n\in\mathbb{N}}$. Before we present a mathematically precise definition, we present "a little paradoxon" that aims to show that one really has to be careful with series.
Consider the case where $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$. On the one hand we can compute

$$\begin{aligned}\sum_{k=1}^{\infty}(-1)^k &= -1+1-1+1-1+1-1+1-1+1-\dots\\ &=(-1+1)+(-1+1)+(-1+1)+(-1+1)+(-1+1)+\dots\\ &=0+0+0+0+0+\dots=0\end{aligned}$$

and on the other hand

$$\begin{aligned}\sum_{k=1}^{\infty}(-1)^k &= -1+1-1+1-1+1-1+1-1+1-\dots\\ &= -1+(1-1)+(1-1)+(1-1)+(1-1)+(1-1)+\dots\\ &= -1+0+0+0+0+0+\dots=-1.\end{aligned}$$

This is a very dramatic contradiction! To exclude such awkward phenomena, we have to use a precise mathematical definition of "infinite sums".

2.1 Basic Definitions, Convergence Criteria and Examples

Definition 2.1. Infinite series

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{F}$. Then the sequence $(s_n)_{n\in\mathbb{N}}$ defined by

$$s_n := \sum_{k=1}^{n} a_k$$

is called infinite series (or just "series"). The sequence element s_n is called n-th partial sum of $(a_n)_{n\in\mathbb{N}}$. The series is called convergent if $(s_n)_{n\in\mathbb{N}}$ is convergent. In this case, we write

$$\sum_{k=1}^{\infty} a_k := \lim_{n\to\infty} s_n.$$

Remark:

In the literature, the symbol $\sum_{k=1}^{\infty} a_k$ is also called series. So this symbol has a two-fold meaning, namely the limit of the series (if existent) and the series itself. At the accordant places of this manuscript, the concrete meaning will be clear from the context.
The above definition formally does not include infinite sums of kind

$$\sum_{k=0}^{\infty} a_k, \qquad \sum_{k=2}^{\infty} a_k, \qquad \textit{or} \qquad \sum_{k=n_0}^{\infty} a_k \ \textit{ for some } n_0 \in \mathbb{N}.$$

However, their meaning is straightforward to define and we will call these expressions infinite series, too.

Before we give some criteria for the convergence of series, we first present the probably most important series and analyze their convergence.

Example 2.2. (a) For $q \in \mathbb{F}$, the *geometric series*

$$\sum_{k=0}^{\infty} q^k$$

is convergent if and only if $|q| < 1$. *Proof:* We can show that the n-th partial sum is given by

$$s_n = \sum_{k=0}^{n} q^k = \begin{cases} \frac{1-q^{n+1}}{1-q} & : \text{ if } q \neq 1, \\ n+1 & : \text{ if } q = 1. \end{cases}$$

Hence, $(s_n)_{n\in\mathbb{N}}$ is convergent if and only if $|q| < 1$. In this case we have

$$\sum_{k=0}^{\infty} q^k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q}.$$

(b) The *harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

is divergent to $+\infty$.
Proof: If we construct some unbounded subsequence $(s_{n_l})_{l\in\mathbb{N}}$, the divergence of the harmonic series is proven (since it is monotonically increasing). Indeed, we now show the unboundedness of the subsequence $(s_{2^l})_{l\in\mathbb{N}}$: First, observe that

$$s_{2^l} = s_1 + (s_2 - s_1) + (s_4 - s_2) + (s_8 - s_4) + \ldots + (s_{2^l} - s_{2^{l-1}}) = s_1 + \sum_{j=1}^{l} (s_{2^j} - s_{2^{j-1}}).$$

Now we take a closer look to the number $s_{2^j} - s_{2^{j-1}}$: By definition of s_n, we have

$$s_{2^j} - s_{2^{j-1}} = \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} > \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j} = 2^{j-1}\frac{1}{2^j} = \frac{1}{2}.$$

The inequality in the above formula holds true since every summand is replaced by the smallest summand $\frac{1}{2^j}$. The second last equality sign then comes from the fact that the number $\frac{1}{2^j}$ is summed up 2^{j-1}-times. Now using this inequality together with the above sum representation for s_{2^l}, we obtain

$$s_{2^l} = s_1 + \sum_{j=1}^{l} (s_{2^j} - s_{2^{j-1}}) > 1 + \sum_{j=1}^{l} \frac{1}{2} = 1 + \frac{l}{2}.$$

As a consequence, the subsequence $(s_{2^l})_{l\in\mathbb{N}}$ is unbounded. □

(c) For $\alpha > 1$, the sequence

$$\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$$

is convergent.
Proof: The sequence of partial sums is strictly monotonically increasing due to

$$s_{n+1} - s_n = \frac{1}{(n+1)^\alpha} \geq 0.$$

Therefore, by Theorem 1.37 and Theorem 1.32, the convergence of $(s_n)_{n\in\mathbb{N}}$ is shown if we find some bounded subsequence $(s_{n_j})_{j\in\mathbb{N}}$. Again we use the representation for $s_{2^j} - s_{2^{j-1}}$ as in example b). We can estimate

$$\begin{aligned} s_{2^j} - s_{2^{j-1}} &= \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k^\alpha} < \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{(2^{j-1}+1)^\alpha} \\ &= \frac{2^{j-1}}{(2^{j-1}+1)^\alpha} < \frac{2^{j-1}}{(2^{j-1})^\alpha} = \left(\frac{2}{2^\alpha}\right)^{j-1} = \left(\frac{1}{2^{\alpha-1}}\right)^{j-1}, \end{aligned}$$

so we have $s_{2^j} - s_{2^{j-1}} < q^{j-1}$ for $q = \frac{1}{2^{\alpha-1}}$ and, due to $\alpha > 1$, it holds that $0 < q < 1$. Using that $s_1 = 1 = q^0$, we obtain

$$s_{2^l} = s_1 + \sum_{j=1}^{l} (s_{2^j} - s_{2^{j-1}}) < 1 + \sum_{j=0}^{l-1} q^j = 1 + \frac{1-q^l}{1-q} < 1 + \frac{1}{1-q}.$$

Hence, the sequence $(s_{2^l})_{l\in\mathbb{N}}$ is bounded. This implies the desired result. □

Remark:

Except for the first example, we have not computed the limits of the other stated convergent series. We only proved existence or non-existence of limits. Indeed, the computation of limits of series is, in general, a very difficult issue and is not possible in many cases.
The function

$$\zeta(\alpha) = \sum_{k=1}^{\infty} \frac{1}{k^\alpha}$$

is very popular in analytic number theory under the name Riemann Zeta Function. In b) and c), we have implicitly proven that $\zeta(\cdot)$ is defined on the interval $(1, \infty)$ and has a pole at 1. This function is subject of the Riemann hypothesis *which is one of the most important unsolved problems in modern mathematics. Some known values of the Zeta function are (without proof)*

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2) = \frac{\pi^2}{6}, \qquad \sum_{k=1}^{\infty} \frac{1}{k^4} = \zeta(4) = \frac{\pi^4}{90}, \qquad \sum_{k=1}^{\infty} \frac{1}{k^6} = \zeta(6) = \frac{\pi^6}{945}.$$

Next we consider sums of convergent series and multiplication of series by some scalar variable. The proof just consists of a straightforward application of Theorem 1.21 and is therefore skipped.

Theorem 2.3. Formulae for convergent series

Let $\lambda \in \mathbb{F}$ and

$$\sum_{k=1}^{\infty} a_k, \qquad \sum_{k=1}^{\infty} b_k$$

be two convergent series in $\mathbb{F}$. Then

(i)

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k;$$

(ii)

$$\sum_{k=1}^{\infty} (\lambda a_k) = \lambda \sum_{k=1}^{\infty} a_k.$$

Theorem 2.4. Cauchy Criterion

A series $\sum_{k=1}^{\infty} a_k$ in $\mathbb{F}$ is convergent if and only if for all $\varepsilon > 0$, there exists some N such that for all $n \geq m \geq N$ holds

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon.$$

Proof: By Theorem 1.46 and Theorem 1.48, a series converges if and only if the sequence

$(s_n)_{n\in\mathbb{N}}$ of partial sums is a Cauchy sequence.
On the other hand, for $n \geq m$, we have

$$|s_n - s_{m-1}| = \left|\sum_{k=m}^{n} a_k\right|.$$

Therefore, the Cauchy criterion is really equivalent to the fact that $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{F}$. □

Remark:

Reconsidering the example at the very beginning of this chapter, the divergence of this sequence can be directly verified be employing the Cauchy criterion.

As a corollary, we can formulate the following criterion.

Theorem 2.5. Necessary criterion for convergence of series

Let

$$\sum_{k=1}^{\infty} a_k$$

be a convergent series in $\mathbb{F}$. Then $(a_n)_{n\in\mathbb{N}}$ is convergent with

$$\lim_{n\to\infty} a_n = 0.$$

Proof: Since the series converges, the Cauchy criterion implies that for all $\varepsilon > 0$, there exists some N such that for all $n \geq m \geq N$ holds

$$\left|\sum_{k=m}^{n} a_k\right| < \varepsilon.$$

Now considering the special case $n = m$, we have that for all $n \geq N$ holds

$$|a_n| < \varepsilon.$$

However, this is nothing but convergence of $(a_n)_{n\in\mathbb{N}}$ to zero. □

Remark:

The zero convergence of $(a_n)_{n\in\mathbb{N}}$ is by far not sufficient for convergence. We have already seen that the harmonic series diverges though the sequence $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}}$ converges to zero.

For some particular cases, we nevertheless are able to show convergence.

Theorem 2.6. Leibniz Convergence Criterion

Let $(a_n)_{n\in\mathbb{N}}$ be a monotonically decreasing real sequence with

$$\lim_{n\to\infty} a_n = 0.$$

Then the alternating sequence

$$\sum_{k=1}^{\infty}(-1)^k a_k$$

converges.

Proof: Since $(a_n)_{n\in\mathbb{N}}$ is convergent to zero and monotonically decreasing, we have $a_n \geq 0$ for all $n \in \mathbb{N}$. Let $(s_n)_{n\in\mathbb{N}}$ be the corresponding sequence of partial sums. Then we have for all $l \in \mathbb{N}$ that

$$s_{2l+2} - s_{2l} = -a_{2l+1} + a_{2l+2} \leq 0, \quad s_{2l+3} - s_{2l+1} = a_{2l+2} - a_{2l+3} \geq 0,$$

i.e., the subsequence $(s_{2l})_{l\in\mathbb{N}}$ is monotonically decreasing and $(s_{2l+1})_{l\in\mathbb{N}}$ is monotonically increasing. Furthermore, due to $s_{2l+1} - s_{2l} = -a_{2l+1} \leq 0$ holds

$$s_{2l+1} \leq s_{2l}.$$

Altogether, the $(s_{2l})_{l\in\mathbb{N}}$ is monotonically decreasing and bounded from below, and $(s_{2l+1})_{l\in\mathbb{N}}$ is monotonically increasing and bounded from above. By Theorem 1.32, both subsequences are convergent. Due to

$$\lim_{l\to\infty}(s_{2l+1} - s_{2l}) = \lim_{l\to\infty} a_{2l+1} = 0,$$

an application of Theorem 1.21 yields that both subsequence have the same limit, i.e.,

$$\lim_{l\to\infty} s_{2l+1} = \lim_{l\to\infty} s_{2l} = s$$

for some $s \in \mathbb{R}$. Let $\varepsilon > 0$. Then there exists some N_1 such that for all $l \geq N_1$ holds $|s_{2l} - s| < \varepsilon$. Furthermore, there exists some N_2 such that for all $l \geq N_2$ holds $|s_{2l+1} - s| < \varepsilon$. Now choosing $N = \max\{2N_1, 2N_2 + 1\}$, we can say the following for some $m \geq N$:

In the case where m is even, we have some $l \in \mathbb{N}$ with $m = 2l$. By the choice of N, we also have $l \geq N_1$ and thus

$$|s_m - s| = |s_{2l} - s| < \varepsilon.$$

In the case where m is odd, we have some $l \in \mathbb{N}$ with $m = 2l + 1$. By the choice of N, we also have $l \geq N_2$ and thus

$$|s_m - s| = |s_{2l+1} - s| < \varepsilon.$$

□

Example 2.7. (a) The *alternating harmonic series*

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}.$$

converges. The limit is (without proof) $\log(2)$.

(b) The *alternating Leibniz series*

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}.$$

converges. The limit is (without proof) $\frac{\pi}{4}$.

(c) The series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}.$$

converges. The limit is not expressible in a closed form.

We will later treat the topic of *Taylor series*. Thereafter we will be able to determine some further limits of sequences.

2.2 Absolute Convergence and Criteria

Definition 2.8. Absolute convergence

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{F}$. Then the series $\sum_{k=1}^{\infty} a_k$ is called absolutely convergent if the real series $\sum_{k=1}^{\infty} |a_k|$ converges.

Remark:

We will see that absolute convergence is really a stronger requirement than convergence. However, for real series $\sum_{k=1}^{\infty} a_k$ with $a_k \geq 0$ for all $k \in \mathbb{N}$, absolute convergence and convergence are equivalent. This is a direct consequence of the fact that $a_k \geq 0$ implies $|a_k| = a_k$.

Theorem 2.9. Absolute convergence implies convergence

Let $\sum_{k=1}^{\infty} a_k$ be an absolutely convergent series in $\mathbb{F}$. Then the series is also convergent.

Proof: Let $\varepsilon > 0$. By the absolute convergence of the series, the necessity of Cauchy's convergence criterion implies that there exists some N such that for all $n \geq m \geq N$ holds

$$\sum_{k=m}^{n} |a_k| < \varepsilon.$$

A use of the triangular inequality gives

$$\left|\sum_{k=m}^{n} a_k\right| \leq \sum_{k=m}^{n} |a_k| < \varepsilon.$$

Then the sufficiency of Cauchy's convergence criterion implies the convergence of $\sum_{k=1}^{\infty} a_k$.

Example 2.10. The alternating harmonic series is convergent as we have seen in in Example 2.7 a), but it is not absolutely convergent, since the series of absolute values is the harmonic series (see Example 2.2 b)).

The following criterion can be seen as a "series version" of the comparison criterion for sequences presented in Theorem 1.22.

Theorem 2.11. Majorant criterion

Let $\sum_{k=1}^{\infty} a_k$ be a series in $\mathbb{F}$. Moreover, let $n_0 \in \mathbb{N}$ and let $\sum_{k=1}^{\infty} b_k$ be a real convergent series such that $|a_k| \leq b_k$ for all $k \geq n_0$. Then $\sum_{k=1}^{\infty} a_k$ converges absolutely.

Proof: Let $\varepsilon > 0$. By the Cauchy criterion applied to $\sum_{k=1}^{\infty} b_k$ there is an $N \geq n_0$ such that for all $n \geq m \geq N$ holds

$$0 \leq \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k = \left| \sum_{k=m}^{n} b_k \right| < \varepsilon.$$

The Cauchy criterion now implies that $\sum_{k=1}^{\infty} |a_k|$ converges. □

Remark:

The series $\sum_{k=1}^{\infty} b_k$ with the properties as stated in Theorem 2.11 is called a major-ant of $\sum_{k=1}^{\infty} a_k$.

Now we present a kind of *reversed majorant criterion* that gives us a sufficient criterion for divergence.

Theorem 2.12. Minorant criterion

Let $\sum_{k=1}^{\infty} a_k$ be a real series. Moreover, let $n_0 \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ be a divergent series such that $a_k \geq b_k \geq 0$ for all $k \geq n_0$. Then $\sum_{k=1}^{\infty} a_k$ diverges.

Proof: We prove the result by contradiction: Let $\sum_{k=1}^{\infty} b_k$ be divergent. Assume that $\sum_{k=1}^{\infty} a_k$ converges. Then, due to $a_k \geq b_k \geq 0$, the majorant criterion implies the convergence of $\sum_{k=1}^{\infty} b_k$, too. This is a contradiction to our assumption. □

Remark:

The series $\sum_{k=1}^{\infty} b_k$ with the properties as stated in Theorem 2.12 is called a minorant of $\sum_{k=1}^{\infty} a_k$.

Theorem 2.13. Quotient criterion

Let $n_0 \in \mathbb{N}$ and let $\sum_{k=1}^{\infty} a_k$ be a series in $\mathbb{F}$ with the following properties:

– *$a_k \neq 0$ for all $k \geq n_0$;*

– *there exists some $q \in (0, 1)$ such that for all $k \geq n_0$ holds*

$$\frac{|a_{k+1}|}{|a_k|} \leq q.$$

Then $\sum_{k=1}^{\infty} a_k$ converges absolutely.

Proof: We inductively obtain that for all $k \geq n_0$ holds

$$|a_k| \leq q^{k-n_0} |a_{n_0}| \ .$$

Therefore, the series $\sum_{k=1}^{\infty} q^{k-n_0}|a_{n_0}|$ is a majorant of $\sum_{k=1}^{\infty} a_k$. However, the majorant is convergent due to

$$\sum_{k=1}^{\infty} q^{k-n_0}|a_{n_0}| = \frac{|a_{n_0}|}{(1-q)q^{n_0-1}} \qquad \text{(see Example 2.2 a)).}$$

The majorant criterion then implies convergence. □

Remark:

Note that the quotient criterion is different from claiming $\frac{|a_{k+1}|}{|a_k|} < 1$ *(which is for instance fulfilled by the divergent harmonic series). There has to exist some* $q < 1$ *such that the quotient is below* q.
The quotient criterion is only sufficient for convergence and indeed, there are examples of absolutely convergent series that do not fulfill the quotient criterion. For instance, consider the absolutely convergent series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Observing that

$$\frac{|a_{k+1}|}{|a_k|} = \left|\frac{k}{k+1}\right|^2,$$

the fact that this expression converges to 1 *implies that there does not exist some* $q < 1$ *for which the quotient criterion is fulfilled. However, this series is convergent as we have proven in Example 2.2 c).*
We could also formulate "an alternative quotient criterion" that gives us a sufficient criterion for divergence. Namely, consider a real series $\sum_{k=1}^{\infty} a_k$ *with positive* a_k *and assume that* $\frac{a_{k+1}}{a_k} \geq 1$ *for all* $k \geq n_0$ *for some fixed* $n_0 \in \mathbb{N}$*. This gives us* $0 < a_k \leq a_{k+1}$*, i.e., the sequence* $(a_n)_{n\in\mathbb{N}}$ *is positive and monotonically increasing. Such a sequence cannot converge to zero and thus,* $\sum_{k=1}^{\infty} a_k$ *is divergent.*

Now we present a "limit form" of the quotient criterion:

Theorem 2.14. Quotient criterion (limit form)

Let $\sum_{k=1}^{\infty} a_k$ *be a series in* $\mathbb{F}$ *and assume that there exists some* $n_0 \in \mathbb{N}$ *such that* $a_k \neq 0$ *for all* $k \geq n_0$*. If*

$$\limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} < 1,$$

then $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*

Proof: Set $c := \limsup \frac{|a_{k+1}|}{|a_k|} < 1$. Since $\limsup$ is defined to be the largest accumulation point of a sequence, we have for every $\varepsilon > 0$ that $\frac{|a_{k+1}|}{|a_k|} \geq c + \varepsilon$ holds true for at most finitely many $k \in \mathbb{N}$. Hence, for $\varepsilon := \frac{1-c}{2}$, there exists some $N \in \mathbb{N}$ such that for all $k \geq N$ holds

$$\frac{|a_{k+1}|}{|a_k|} < c + \varepsilon = c + \frac{1-c}{2} = \frac{1+c}{2}.$$

Thus, the quotient criterion holds true for $q := \frac{1+c}{2}$ which satisfies $q < 1$ due to $c < 1$.

□

Remark:

Since, in case of convergence of $\frac{|a_{k+1}|}{|a_k|}$*, the limit and limes superior coincide, the criterion*

$$\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|}<1$$

is also sufficient for absolute convergence of $\sum_{k=1}^{\infty} a_k$*. However, this criterion requires the convergence of the quotient sequence and is therefore weaker than the above one.*

Example 2.15. For $x \in \mathbb{F}$, consider the series

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

Then an application of the limit form of the quotient criterion yields

$$\begin{aligned}\limsup_{k\to\infty}\frac{\left|\frac{x^{k+1}}{(k+1)!}\right|}{\left|\frac{x^k}{k!}\right|} &= \limsup_{k\to\infty}\frac{|x|^{k+1}}{|x|^k}\frac{k!}{(k+1)!}\\ &= \limsup_{k\to\infty}|x|\frac{1}{k+1} = \lim_{k\to\infty}|x|\frac{1}{k+1} = 0<1.\end{aligned}$$

Hence, the series converges absolutely. The series $\sum_{k=0}^{\infty}\frac{x^k}{k!}$ is called *exponential series* and we will indeed define $\exp(x)$ by this expression.

Theorem 2.16.

Let $x \in \mathbb{F}$*. Then*

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=\sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

Proof: First we show that for fixed $k \in \mathbb{N}_0$ holds

$$\lim_{n\to\infty}\frac{n!}{(n-k)!n^k}=1. \tag{2.1}$$

Note that $\frac{n!}{(n-k)!n^k}$ is well defined for $n \geq k$. For such an $n \in \mathbb{N}$ we have

$$1 \geq \prod_{j=0}^{k-1}\frac{n-j}{n} = \frac{n!}{(n-k)!n^k} \geq \frac{(n-k+1)^k}{n^k} = (1-\frac{k-1}{n})^k\ . \tag{2.2}$$

Since the right-hand side converges to 1 for $n\to\infty$, (2.1) follows.

Now let $\varepsilon>0$ and let $K\in\mathbb{N}$ be big enough such that

$$\sum_{k=K}^{\infty}\frac{|x|^k}{k!}<\frac{\varepsilon}{3}.$$

By (2.1) there is an $N \geq K$ such that for all $n \geq N$ also

$$\sum_{k=0}^{K-1} \left| \frac{n!}{(n-k)!n^k} - 1 \right| \frac{|x|^k}{k!} < \frac{\varepsilon}{3} \, .$$

For $n \geq N$ we estimate

$$\begin{aligned}
\left| \left(1 + \frac{x}{n}\right)^n - \sum_{k=0}^{\infty} \frac{x^k}{k!} \right| &= \left| \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{n^k} - \sum_{k=0}^{\infty} \frac{x^k}{k!} \right| \\
&\leq \sum_{k=0}^{K-1} \left| \binom{n}{k} \frac{x^k}{n^k} - \frac{x^k}{k!} \right| + \sum_{k=K}^{n} \binom{n}{k} \frac{|x|^k}{n^k} + \underbrace{\sum_{k=K}^{\infty} \frac{|x|^k}{k!}}_{<\varepsilon/3} . \\
&= \underbrace{\sum_{k=0}^{K-1} \left| \frac{n!}{(n-k)!n^k} - 1 \right| \frac{|x|^k}{k!}}_{<\varepsilon/3} + \underbrace{\sum_{k=K}^{n} \underbrace{\frac{n!}{(n-k)!n^k}}_{\leq 1} \frac{|x|^k}{k!}}_{<\varepsilon/3} + \underbrace{\sum_{k=K}^{\infty} \frac{|x|^k}{k!}}_{<\varepsilon/3} \\
&< \varepsilon.
\end{aligned}$$

□

The following criterion of Raabe refines the quotient criterion.

Theorem 2.17. Raabe criterion

Let $(a_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{F}$.

a) If there is a $k_0 \in \mathbb{N}$ and a $\beta \in (1, \infty)$ such that $a_k \neq 0$ and

$$\frac{|a_{k+1}|}{|a_k|} \leq 1 - \frac{\beta}{k}$$

for all $k \geq k_0$, then the series $\sum_{k=1}^{\infty} a_k$ converges absolutely.

b) If $\mathbb{F} = \mathbb{R}$ and if there is a $k_0 \in \mathbb{N}$ such that $a_k \neq 0$ and

$$\frac{a_{k+1}}{a_k} \geq 1 - \frac{1}{k}$$

for all $k \geq k_0$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Proof: a) For $k \geq k_0$ holds $\frac{|a_{k+1}|}{|a_k|} \leq \frac{k-\beta}{k}$. Thus $k|a_{k+1}| \leq (k-\beta)|a_k|$ and therefore $\beta|a_k| \leq k|a_k| - k|a_{k+1}|$. Since $\beta > 1$ and $|a_k| > 0$ subtracting $|a_k|$ from both sides of this inequality yields

$$0 < (\beta - 1)|a_k| \leq (k-1)|a_k| - k|a_{k+1}| =: b_k \, . \tag{2.3}$$

In particular, this shows that $(k|a_{k+1}|)_{k\geq k_0-1}$ is monotonically decreasing. Since it is bounded from below by zero, it must converge to some limit $s := \lim_{k\to\infty} k|a_{k+1}|$. Thus also the telescoping series $\sum_{k=k_0}^{\infty} b_k$ converges to $\sum_{k=k_0}^{\infty} (k-1)|a_k| - k|a_{k+1}| = (k_0-1)|a_{k_0}| -$

s. From (2.3) we conclude that $\frac{1}{\beta-1}\sum_{k=k_0}^{\infty} b_k$ is a convergent majorant of $\sum_{k=k_0}^{\infty}|a_k|$. By the majorant criterion, $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

b) We may assume that $k_0 > 1$. For $k \geq k_0$ holds $0 \neq \frac{a_{k+1}}{a_k} \geq \frac{k-1}{k} > 0$. Since the right-hand side is nonnegative, a_k and a_{k+1} must have the same sign (either + or -). Since for the same reason a_{k+1} and a_{k+2} must also have the same sign we see that all a_k have the same sign for $k \geq k_0$. Without loss of generality we may therefore assume that $a_k > 0$ for all $k \geq k_0$. (Otherwise consider the sequence $(-a_k)_{k\in\mathbb{N}}$.) Then $ka_{k+1} \geq (k-1)a_k$ shows that $ka_{k+1} \geq (k_0-1)a_{k_0} =: \alpha > 0$ for all $k \geq k_0$, that is $a_{k+1} \geq \frac{\alpha}{k}$ for $k \geq k_0$. Consequently, $\sum_{k=k_0}^{\infty}\frac{\alpha}{k}$ is a divergent minorant for the series $\sum_{k=k_0}^{\infty} a_{k+1}$ so that the minorant criterion implies the divergence of $\sum_{k=1}^{\infty} a_k$. □

We want to remark that the assumption $\mathbb{F} = \mathbb{R}$ in b) is not essential. The assertion also holds for $\mathbb{F} = \mathbb{C}$. In this case we have to argue that for $k \geq k_0$ all $a_k = |a_k|e^{i\varphi_k}$ must have the same argument $\varphi_k \in [0, 2\pi)$. Then again without loss of generality we may assume that $\varphi_k = 0$ for all $k \geq k_0$.

Theorem 2.18. Root criterion

Let $n_0 \in \mathbb{N}$ and let $\sum_{k=1}^{\infty} a_k$ be a series in $\mathbb{F}$ and assume that there exists some $q \in (0,1)$ such that for all $k \geq n_0$ holds

$$\sqrt[k]{|a_k|} \leq q.$$

Then $\sum_{k=1}^{\infty} a_k$ converges absolutely.

Proof: Taking the k-th power of the inequality $\sqrt[k]{|a_k|} < q$, we obtain that for all $k \geq n_0$ holds

$$|a_k| < q^k$$

Therefore, the convergent geometric series $\sum_{k=1}^{\infty} q^k$ is a majorant of $\sum_{k=1}^{\infty} a_k$ and thus, we have absolute convergence. □

Theorem 2.19. Root criterion (limit form)

Let $\sum_{k=1}^{\infty} a_k$ be a series in $\mathbb{F}$ and assume that

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} < 1.$$

Then $\sum_{k=1}^{\infty} a_k$ converges absolutely.

Proof: The argumentation is analogous as in the proof of Theorem 2.14. Let $c := \limsup \sqrt[k]{|a_k|} < 1$. Then for $\varepsilon := \frac{1-c}{2}$, there exists some $N \in \mathbb{N}$ such that for all $k \geq N$ holds

$$\sqrt[k]{|a_k|} < c + \varepsilon = c + \frac{1-c}{2} = \frac{1+c}{2} < 1.$$

The root criterion with $q := \frac{1+c}{2} < 1$ now implies convergence. □

Example 2.20. Consider the series

$$\sum_{k=1}^{\infty} \frac{k^5}{3^k}.$$

Then we have

$$\limsup_{k\to\infty} \sqrt[k]{\left|\frac{k^5}{3^k}\right|} = \limsup_{k\to\infty} \frac{\sqrt[k]{k}^5}{3}$$

Since we know (from the tutorial) that $\sqrt[k]{k}$ converges to 1, the whole expression converges to $\frac{1}{3} < 1$. Hence, the series converges.

We will now state two convergence criteria for series of the form $\sum_{k=1}^{\infty} a_k b_k$. They are easily deduced from the following lemma.

Lemma 2.21. Abel's partial sums

For $n \in \mathbb{N}$ and $a_1, ..., a_n, b_1, ..., b_{n+1} \in \mathbb{F}$ holds

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}) \ ,$$

where $A_k := \sum_{i=1}^{k} a_i$ for $k \in \{1, ..., n\}$.

Proof: If we additionally define $A_0 := 0$, then

$$\begin{aligned} \sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n} (A_k - A_{k-1}) b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_{k-1} b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n-1} A_k b_{k+1} \\ &= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1} = \sum_{k=1}^{n} A_k (b_k - b_{k+1}) + A_n b_{n+1} \ . \end{aligned}$$

Theorem 2.22. Abel criterion

If the series $\sum_{k=1}^{\infty} a_k$ in $\mathbb{F}$ converges and if the real sequence $(b_k)_{k\in\mathbb{N}}$ is monotonic and bounded, then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

Proof: Set $A_k := \sum_{i=1}^{k} a_i$. By assumption both sequences $(A_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ converge so that $(A_k b_{k+1})_{k\in\mathbb{N}}$ converges also. Since $(b_k)_{k\in\mathbb{N}}$ is monotonic, the telescoping series $\sum_{k=1}^{\infty} (b_k - b_{k+1})$ converges absolutely as

$$\sum_{k=1}^{n} |b_k - b_{k+1}| = |\sum_{k=1}^{n} (b_k - b_{k+1})| = |b_1 - b_{n+1}| \xrightarrow{n\to\infty} |b_1 - \lim_{k\to\infty} b_k| \ .$$

Since $(A_k)_{k\in\mathbb{N}}$ is bounded, $\sum_{k=1}^{\infty} A_k (b_k - b_{k+1})$ is also absolutely convergent. Summing up, Lemma 2.21 implies that the series $\sum_{k=1}^{\infty} a_k b_k$ converges to the limit

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} A_n b_{n+1} + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}) \ .$$

$\square$

Theorem 2.23. Dirichlet criterion

If the series $\sum_{k=1}^{\infty} a_k$ in $\mathbb{F}$ is bounded and if the real sequence $(b_k)_{k\in\mathbb{N}}$ converges monotonically to zero, then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

Proof: Set $A_k := \sum_{i=1}^{k} a_i$. By assumption $(A_k)_{k\in\mathbb{N}}$ is bounded. Hence $(A_k b_{k+1})_{k\in\mathbb{N}}$ converges to zero. By the same argument as in the proof of Theorem 2.22, the series $\sum_{k=1}^{\infty} A_k(b_k - b_{k+1})$ is absolutely convergent. Lemma 2.21 again implies that the series $\sum_{k=1}^{\infty} a_k b_k$ converges to the limit

$$\sum_{k=1}^{\infty} a_k b_k = \underbrace{\lim_{n\to\infty} A_n b_{n+1}}_{=0} + \sum_{k=1}^{\infty} A_k(b_k - b_{k+1}) \ .$$

$\square$

Note that the Leibniz criterion 2.6 follows from the Dirichlet criterion by taking $a_k := (-1)^k$.

Definition 2.24. Reordering

Let $\sum_{k=1}^{\infty} a_k$ be a series in $\mathbb{F}$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a bijective mapping. Then the series

$$\sum_{k=1}^{\infty} a_{\tau(k)}$$

is called a reordering of $\sum_{k=1}^{\infty} a_k$.

Theorem 2.25. Reordering of absolutely convergent series

Let $\sum_{k=1}^{\infty} a_k$ be an absolutely convergent series in $\mathbb{F}$ and let $\sum_{k=1}^{\infty} a_{\tau(k)}$ be a reordering. Then $\sum_{k=1}^{\infty} a_{\tau(k)}$ is also absolutely convergent. Moreover,

$$\sum_{k=1}^{\infty} a_{\tau(k)} = \sum_{k=1}^{\infty} a_k.$$

Proof: Let $a = \sum_{k=1}^{\infty} a_k$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be bijective. Let $\varepsilon > 0$. Since we have absolute convergence, there exists some N_1 such that

$$\sum_{k=N_1}^{\infty} |a_k| < \frac{\varepsilon}{2}$$

and thus

$$\left| a - \sum_{k=1}^{N_1-1} a_k \right| = \left| \sum_{k=N_1}^{\infty} a_k \right| \leq \sum_{k=N_1}^{\infty} |a_k| < \frac{\varepsilon}{2}.$$

Now choose $N := \max\{\tau^{-1}(1), \tau^{-1}(2), ..., \tau^{-1}(N_1 - 1)\}$.Then

$$\{1, 2, \ldots, N_1 - 1\} \subset \{\tau(1), \tau(2), \ldots, \tau(N)\}.$$

Then, for all $n \geq N$ holds

$$\left| a - \sum_{k=1}^{n} a_{\tau(k)} \right| \leq \left| a - \sum_{k=1}^{N_1-1} a_k \right| + \left| \sum_{k=1}^{N_1-1} a_k - \sum_{k=1}^{n} a_{\tau(k)} \right| < \frac{\varepsilon}{2} + \sum_{k=N_1}^{\infty} |a_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The absolute convergence of $\sum_{k=1}^{\infty} a_{\tau(k)}$ can be shown by an application of the above argumentation to the series $\sum_{k=1}^{\infty} |a_k|$ and $\sum_{k=1}^{\infty} |a_{\tau(k)}|$. □

2.3 The Cauchy Product of Series

Definition 2.26. Cauchy product of sequences

Let $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$ be two series in $\mathbb{F}$. Then the Cauchy product *of $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$ is given by*

$$\sum_{k=0}^{\infty} c_k \quad \text{with } c_k = \sum_{l=0}^{k} a_l b_{k-l}.$$

Remark:

Note that we considered series with lower summation index 0. The Cauchy product can be also defined for sequences $\sum_{k=n_0}^{\infty} a_k$, $\sum_{k=n_1}^{\infty} b_k$ with arbitrary $n_0, n_1 \in \mathbb{N}$ (or even $n_0, n_1 \in \mathbb{Z}$). In this case, the Cauchy product is given by

$$\sum_{k=n_0+n_1}^{\infty} c_k \quad \text{with } c_k = \sum_{l=n_0}^{k-n_1} a_l b_{k-l}.$$

In order to "keep the set of indices manageable", this is not further treated here. Note that the following result about convergence properties of the Cauchy product still hold true in this above mentioned more general case.

The following theorem justifies the name "product" in the above definition.

Theorem 2.27. Convergence of the Cauchy product

Let $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$ be series in $\mathbb{F}$. Assume that $\sum_{k=0}^{\infty} a_k$ is absolutely convergent and $\sum_{k=0}^{\infty} b_k$ is convergent. Then the Cauchy product $\sum_{k=0}^{\infty} c_k$ is absolutely convergent. Moreover, the limit satisfies

$$\sum_{k=0}^{\infty} c_k = \left(\sum_{k=0}^{\infty} a_k\right) \cdot \left(\sum_{k=0}^{\infty} b_k\right).$$

Proof: Denote the sequence of partial sums of $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$ and $\sum_{k=0}^{\infty} c_k$ by $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$, respectively. Moreover, set $a := \sum_{k=0}^{\infty} a_k$ and $b = \sum_{k=0}^{\infty} b_k$. Then we have

$$C_n = \sum_{k=0}^{n} c_k = \sum_{k=0}^{n}\sum_{l=0}^{k} a_l b_{k-l} = \sum_{k=0}^{n} a_{n-k} \sum_{l=0}^{k} b_l = \sum_{k=0}^{n} a_{n-k} B_k.$$

Using this expression, we obtain

$$C_n = \sum_{k=0}^{n} a_{n-k}(B_k - b) + \sum_{k=0}^{n} a_{n-k} b = \sum_{k=0}^{n} a_{n-k}(B_k - b) + A_n b.$$

Let $\varepsilon > 0$. Since $\sum_{k=0}^{\infty} a_k$ converges absolutely, there exists some N_0 such that for all $n \geq N_0$ holds

$$|B_n - b| < \frac{\varepsilon}{4(\sum_{k=0}^{\infty} |a_k| + 1)}.$$

Since $(a_n)_{n\in\mathbb{N}}$ converges to zero (see Theorem 2.5), there exists some N_1 such that for all $n \geq N_1$ holds

$$|a_n| < \frac{\varepsilon}{4N_0\ (\sup\{|b - B_l|\ :\ l \in \mathbb{N}\} + 1)}.$$

Also there exists some N_2 such that for all $n \geq N_2$ holds

$$|A_n - a| < \frac{\varepsilon}{2(|b| + 1)}.$$

Therefore, with $N = \max\{N_0 + N_1, N_2\}$, we have that for all $n \geq N$ holds

$$\begin{aligned}
|C_n - ab| &= \left|\sum_{k=0}^{n} a_{n-k}(B_k - b) + b(A_n - a)\right| \leq \sum_{k=0}^{n} |a_{n-k}||B_k - b| + |b||A_n - a| \\
&= \sum_{k=0}^{N_0-1} \underbrace{|a_{n-k}|}_{<\frac{\varepsilon}{4N_0\ (\sup\{|b-B_l|\,:\,l\in\mathbb{N}\}+1)}} |B_k - b| + \sum_{k=N_0}^{n} |a_{n-k}| \underbrace{|B_k - b|}_{<\frac{\varepsilon}{4(\sum_{k=0}^{\infty}|a_k|+1)}} + |b| \underbrace{|A_n - a|}_{<\frac{\varepsilon}{2(|b|+1)}} \\
&< \sum_{k=0}^{N_0-1} \frac{\varepsilon|B_k - b|}{4N_0\ (\sup\{|b - B_l|\,|\,l \in \mathbb{N}\} + 1)} + \sum_{k=N_0}^{n} \frac{\varepsilon|a_{n-k}|}{4(\sum_{k=0}^{\infty} |a_k| + 1)} + \frac{\varepsilon|b|}{2(|b| + 1)} \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Example 2.28. Let $x, y \in \mathbb{F}$ and consider the series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \sum_{k=0}^{\infty} \frac{y^k}{k!}$$

which are absolutely convergent as shown in Example 2.15. Then the Cauchy product of both series is given by $\sum_{k=0}^{\infty} c_k$ with

$$c_k = \sum_{l=0}^{k} \frac{x^l}{l!} \frac{y^{k-l}}{(k-l)!} = \frac{1}{k!} \cdot \sum_{l=0}^{k} \binom{k}{l} x^l y^{k-l}.$$

By the Binomial Theorem (see tutorial), we obtain

$$\sum_{l=0}^{k} \binom{k}{l} x^l y^{k-l} = (x + y)^k.$$

Hence, by Theorem 2.27, we have

$$\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) \cdot \left(\sum_{k=0}^{\infty} \frac{y^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{(x + y)^k}{k!}.$$

Altogether, this means that the function

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

fulfills $f(x + y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$ (and even $x, y \in \mathbb{C}$). This property is for instance fulfilled by the exponential function. Indeed, we have that f as defined above fulfills $f(x) = e^x$.

3 Continuous Functions

Now do you think you can beat the champ?

I can take him blindfolded.

What if he's not blindfolded?

Police Squad!

The central notion of this chapter is *continuity* that is a property of functions. Very very roughly speaking, it means that the function has no jumps. This is again a very abstract concept. However, the remaining parts of the lecture are getting easier!

3.1 Bounded Functions, Pointwise and Uniform Convergence

Definition 3.1. Bounded functions

Let I be a set. Then we call a function $f : I \to \mathbb{F}$ bounded, if

$$\sup\{|f(x)| \,:\, x \in I\} < \infty.$$

In the following, we consider sequences of functions and introduce some convergence concepts.

Definition 3.2. Pointwise convergence

A sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : I \to \mathbb{F}$ is called pointwisely convergent to $f : I \to \mathbb{F}$ if for all $x \in I$ holds

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Using logical quantifiers this reads:

$$\forall x \in I \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad : \quad |f_n(x) - f(x)| < \varepsilon \,. \tag{3.1}$$

Pointwise convergence means nothing else but that for all $x \in I$, the sequence $(f_n(x))_{n\in\mathbb{N}}$

in $\mathbb{F}$ converges to $f(x)$.

We now present an alternative convergence concept for functions:

Definition 3.3. Uniform convergence

A sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : I \to \mathbb{F}$ is called uniformly convergent to $f : I \to \mathbb{F}$ if for all $\varepsilon > 0$ there exists some N such that for all $n \geq N$ and $x \in I$ holds

$$|f(x) - f_n(x)| < \varepsilon.$$

Using logical quantifiers this reads (in contrast to (3.1)):

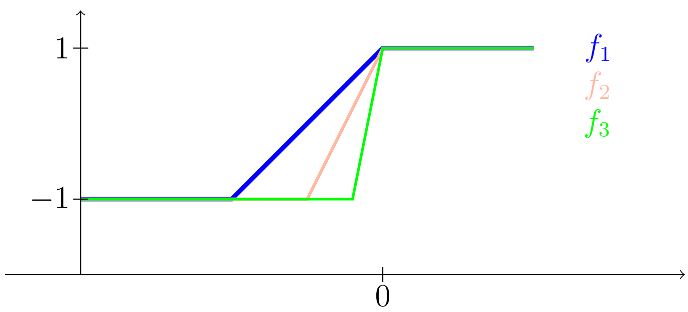
$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad \forall x \in I \quad : \quad |f_n(x) - f(x)| < \varepsilon \; . \tag{3.2}$$

As a rule of thumb, you can think of pushing one quantifier to the right but, of course, this will change a lot. The interpretation is that we can measure the distance between two functions f und g as the largest distance between the two graphs, that means the distance you can measure at a given point:

$$\sup_{x\in I} |f(x) - g(x)|$$

We have the uniform convergence if this measured distance between f_n and f is convergent to zero. (See below.)

Look at the following example:



One sees that the functions f_n pointwisely converges to a limit function. We also see that we can build a jump by increasing n for the limit function. The distance between the limit function to each member of the sequence is indeed

$$\sup_{x\in\mathbb{R}} |f(x) - f_n(x)| = 2$$

since there is always an $x \in \mathbb{R}$ which can be chosen close enough to 0 to get an approximation of this distance. This means that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ does not converges uniformly in spite of being pointwisely convergent. The uniform convergence is in fact a much stronger notion.

We will now see that uniform convergence is a stronger property than pointwise convergence.

Theorem 3.4. Uniform convergence implies pointwise convergence

Let a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n : I \to \mathbb{F}$ be uniformly convergent to $f : I \to \mathbb{F}$. Then $(f_n)_{n\in\mathbb{N}}$ is also pointwisely convergent to f.

Proof: Let $\varepsilon > 0$. Then there exists some N such that for all $n \geq N$ and $x \in I$ holds

$$|f(x) - f_n(x)| < \varepsilon.$$

In particular, for some arbitrary $x \in I$ holds

$$|f(x) - f_n(x)| < \varepsilon.$$

Hence, the sequence $(f_n(x))_{n\in\mathbb{N}}$ in $\mathbb{F}$ converges to $f(x)$. □

Remark:

Uniform convergence to $f : I \to \mathbb{F}$ means that for all $\varepsilon > 0$ holds that all (except finitely many) functions f_n are "inside some ε-stripe around f" (see Fig. 3.1).

Theorem 3.5.

A sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : I \to \mathbb{F}$ converges uniformly to $f : I \to \mathbb{F}$, if, and only if,

$$\lim_{n\to\infty} ||f_n - f||_\infty = \lim_{n\to\infty} \sup\{|f(x) - f_n(x)| \,:\, x \in I\} = 0. \tag{3.3}$$

This means that uniform convergence is nothing but convergence with respect to the infinity norm $||\cdot||_\infty$.

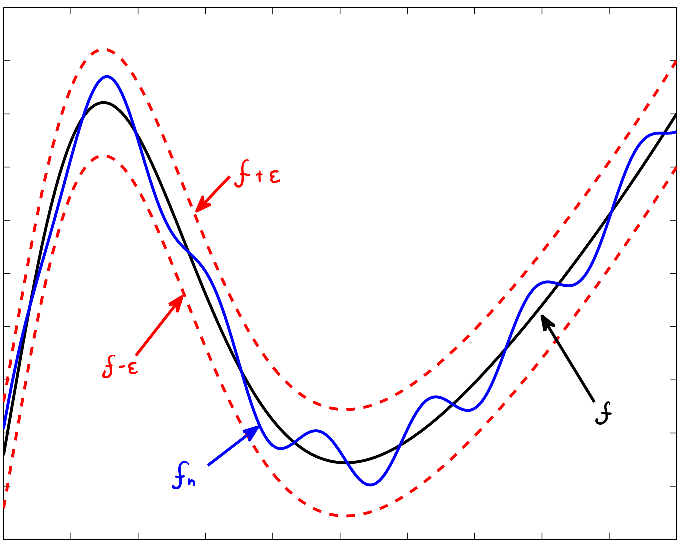


Figure 3.1: Uniform convergence graphically illustrated

Proof. Assume that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. Let $\varepsilon > 0$. Then there exists some N such that for all $n \geq N$ and $x \in I$ holds

$$|f(x) - f_n(x)| < \frac{\varepsilon}{2}.$$

Therefore, for all $n \geq N$, we have

$$\sup\{|f(x) - f_n(x)| \ : \ x \in I\} \leq \frac{\varepsilon}{2} < \varepsilon,$$

and thus, (3.3) holds true.
Conversely, assuming that (3.3) holds true, we obtain that for $\varepsilon > 0$, there exists some N with the property that for all $n \geq N$ holds

$$\sup\{|f(x) - f_n(x)| \ : \ x \in I\} < \varepsilon.$$

This means that for all $n \geq N$ and $x \in I$, there holds that

$$|f(x) - f_n(x)| < \varepsilon.$$

However, this statement is nothing but uniform convergence of $(f_n)_{n\in\mathbb{N}}$ towards f. $\square$

Example 3.6. a) Let $I = [0,1]$ and consider the sequence $f_n(x) = x^n$. Then we have the pointwise limit

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & , \text{ if } x \in [0,1), \\ 1 & , \text{ if } x = 1. \end{cases}$$

Is $(f_n)_{n\in\mathbb{N}}$ also uniformly convergent to f?
The answer is no, since for $x_n = 1/\sqrt[n]{2}$, there holds

$$|f(x_n) - f_n(x_n)| = \left|0 - \tfrac{1}{2}\right| = \tfrac{1}{2}.$$

b) We now consider the same sequence on the smaller interval $[0, \frac{1}{2}]$. The pointwise limit is now $f = 0$. For $n \in \mathbb{N}$, we have

$$\sup\left\{|f(x) - f_n(x)| \ : \ x \in [0, \tfrac{1}{2}]\right\} = \sup\left\{x^n : x \in [0, \tfrac{1}{2}]\right\} = \frac{1}{2^n}.$$

Therefore

$$\lim_{n\to\infty} \sup\left\{|f(x) - f_n(x)| \ : \ x \in [0, \tfrac{1}{2}]\right\} = 0$$

and hence, we have uniform convergence.

c) Define the function $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n^2 x(1 - nx) & , \text{ if } x \in [0, \frac{1}{n}[, \\ 0 & , \text{ if } x \in [\frac{1}{n}, 1]. \end{cases}$$

Then for all $x \in [0,1]$ holds

$$\lim_{n\to\infty} f_n(x) = 0$$

since $f_n(0) = 0$ and $f_n(x) = 0$ if $x > \frac{1}{n}$. The sequence $(f_n)_{n\in\mathbb{N}}$ is however not uniformly convergent to $f = 0$, since

$$\sup\{|f_n(x) - 0| \ : \ x \in [0,1]\} \geq \left|f_n\left(\tfrac{1}{2n}\right)\right| = \frac{n}{4}.$$

Theorem 3.7.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of bounded functions $f_n : I \to \mathbb{F}$. Assume that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f : I \to \mathbb{F}$. Then f is bounded.

Proof: For $\varepsilon = 1$, there exists some N such that for all $n \geq N$ and $x \in I$ holds

$$|f_n(x) - f(x)| < 1.$$

In particular, we have $|f_N(x) - f(x)| < 1$ for all $x \in I$. This consequences that for all $x \in I$, there holds

$$|f(x)| < |f_N(x)| + 1.$$

The boundedness of f_N then implies the boundedness of f. $\square$

Remark:

Note that the assumption of uniform convergence is essential for the boundedness of f. For instance, consider the sequence $(f_n)_{n\in\mathbb{N}}$ of bounded functions $f_n : [0,\infty) \to \mathbb{R}$ with

$$f_n(x) = \begin{cases} x : & \text{if } x < n \\ 0 : & \text{else.} \end{cases}$$

First we argument that $(f_n)_{n\in\mathbb{N}}$ converges pointwisely to $f : [0,\infty) \to \mathbb{R}$ with $f(x) = x$: Let $x \in [0,\infty)$. Then there exists some $N \in \mathbb{N}$ with $x < N$. Hence, for all $n \geq N$, we have $f_n(x) = x$. This implies convergence to $f : [0,\infty) \to \mathbb{R}$ with $f(x) = x$.
Second we state that each f_n is bounded: This is a consequence of the fact that, by the definition of f_n, there holds $f_n(x) < n$ for all $x \in [0,\infty)$.
Altogether, we have found a sequence of bounded functions pointwisely converging to some unbounded function. Hence, Theorem 3.7 is no longer valid, if we replace the phrase "uniformly convergent" by "pointwisely convergent".

3.2 Continuity

Now we begin to introduce the concept of continuity.

Definition 3.8.

Let $I \subset \mathbb{F}$, let $f : I \to \mathbb{F}$ be a function, and let $x_0 \in I$. Then we define

*(i) the limit of f as x tends to x_0 by $c \in \mathbb{F}$ if **for all** sequences $(x_n)_{n\in\mathbb{N}}$ in $I\backslash\{x_0\}$ with $\lim_{n\to\infty} x_n = x_0$ holds $\lim_{n\to\infty} f(x_n) = c$. In this case, we write*

$$\lim_{x\to x_0} f(x) = c$$

*(ii) the limit of f as x tends from the left to x_0 by $c \in \mathbb{F}$ if $I \subset \mathbb{R}$ and if **for all** sequences $(x_n)_{n\in\mathbb{N}}$ in $\{x \in I : x < x_0\}$ with $\lim_{n\to\infty} x_n = x_0$ holds $\lim_{n\to\infty} f(x_n) = c$. In this case, we write*

$$\lim_{x\nearrow x_0} f(x) = c.$$

(iii) *the limit of f as x tends from the right to x_0 by $c \in \mathbb{F}$ if $I \subset \mathbb{R}$ and if* ***for all*** *sequences $(x_n)_{n\in\mathbb{N}}$ in $\{x \in I : x > x_0\}$ with $\lim_{n\to\infty} x_n = x_0$ holds $\lim_{n\to\infty} f(x_n) = c$. In this case, we write*

$$\lim_{x\searrow x_0} f(x) = c.$$

In all three cases we assume that at least one sequence $(x_n)_{n\in\mathbb{N}}$ with the stated property exists.

Remark:

From the above definition, we can also conclude that $\lim_{x\to x_0} f(x)$ exists in the case $I \subset \mathbb{R}$ if and only if $\lim_{x\nearrow x_0} f(x)$ and $\lim_{x\searrow x_0} f(x)$ exist and are equal. In this case, there holds

$$\lim_{x\nearrow x_0} f(x) = \lim_{x\searrow x_0} f(x) = \lim_{x\to x_0} f(x).$$

Though not explicitly introduced in the above definition, it should be intuitively clear what is meant by the following expressions

$$\lim_{x\to\infty} f(x) = y, \qquad \lim_{x\to-\infty} f(x) = y, \qquad \lim_{x\to x_0} f(x) = \infty, \qquad \lim_{x\to x_0} f(x) = -\infty.$$

Example 3.9. a) Consider the *Heaviside function* $H : \mathbb{R} \to \mathbb{R}$ with

$$H(x) = \begin{cases} 1 & , \text{ if } x \geq 0, \\ 0 & , \text{ if } x < 0. \end{cases}$$

Then we have $\lim_{x\nearrow 0} H(x) = 0$, since for all $x_n \in \mathbb{R}$ with $x_n < 0$ holds $H(x_n) = 0$. Further, $\lim_{x\searrow 0} H(x) = 1$, since for all $x_n \in \mathbb{R}$ with $x_n > 0$ holds $H(x_n) = 1$. The limit $\lim_{x\to 0} H(x)$ does not exist. E.g., take the sequence $x_n = \frac{(-1)^n}{n}$. Then

$$H(x_n) = \begin{cases} 1 : & \text{if } n \text{ is even}, \\ 0 : & \text{if } n \text{ is odd}. \end{cases}$$

Hence, $(H(x_n))_{n\in\mathbb{N}}$ is divergent.

b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1 & , \text{ if } x = 0, \\ 0 & , \text{ if } x \neq 0. \end{cases}$$

Then for all sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\setminus\{0\}$ holds that $(f(x_n))_{n\in\mathbb{N}}$ is a constant zero sequence. Hence, $\lim_{x\to 0} f(x) = 0$.

c) Consider a *polynomial* $p : \mathbb{R} \to \mathbb{R}$ with $p(x) = a_n x^n + \ldots + a_1 x + a_0$ for some given $a_0, \ldots, a_n \in \mathbb{R}$. Let $x_0 \in \mathbb{R}$. By Theorem 1.21, we have that for all real sequences $(x_n)_{n\in\mathbb{N}}$ converging to x_0 holds that $p(x_n)$ converges to $p(x_0)$, i.e.,

$$\lim_{x\to x_0} p(x) = p(x_0).$$

Definition 3.10. Continuity

Let $I \subset \mathbb{F}$ and let $f : I \to \mathbb{F}$ be a function. Then f is called continuous in $x_0 \in I$ *if*

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Moreover, f is called continuous on I *if it is continuous in x_0 for all $x_0 \in I$.*

Remark:

Sometimes we will just say $f : I \to \mathbb{F}$ is continuous whereby we mean it is continuous on I.

Example 3.11. a) The *constant function* $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = c$ for some $c \in \mathbb{R}$ is continuous on $\mathbb{R}$.

b) The Heaviside function (Example 3.9 a)) is discontinuous at $x_0 = 0$, but continuous everywhere else.

c) The function $f : \mathbb{R} \to \mathbb{R}$ as in Example 3.9 b) is discontinuous at $x_0 = 0$, but continuous everywhere else.

d) Polynomials (Examples 3.9 d)) are continuous on $\mathbb{R}$.

e) *Rational functions* $f : I \to \mathbb{F}$ with $f(x) = \frac{p(x)}{q(x)}$ for some polynomials p, q (q is not the zero polynomial) are defined on $I = \{x \in \mathbb{R} : q(x) \neq 0\}$ and are continuous on I (due to Theorem 1.21).

f) The *absolute value function* $|\cdot| : \mathbb{R} \to \mathbb{R}$, i.e.,

$$|x| = \begin{cases} x & , \text{ if } x \geq 0, \\ -x & , \text{ if } x < 0. \end{cases}$$

is continuous on $\mathbb{R}$.

g) The function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1 & , \text{ if } x \in \mathbb{Q}, \\ 0 & , \text{ if } x \notin \mathbb{Q} \end{cases}$$

is everywhere discontinuous.
Proof: Let $x_0 \in \mathbb{R}$:
First case: $x_0 \in \mathbb{Q}$. Then take a sequence $(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = x_0$ and $x_n \in \mathbb{R}\backslash\mathbb{Q}$ (for instance, $x_n = x_0 + \frac{\sqrt{2}}{n}$). Then $f(x_n) = 0$ for all $n \in \mathbb{N}$ and thus $\lim_{n\to\infty} x_n = 0 \neq f(x_0) = 1$.
Second case: $x_0 \in \mathbb{R}\backslash\mathbb{Q}$. Then take a sequence $(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = x_0$ and $x_n \in \mathbb{Q}$ (this exists since every real number can be approximated by a rational number in arbitrary good precision). Then $f(x_n) = 1$ for all $n \in \mathbb{N}$ and thus $\lim_{n\to\infty} x_n = 1 \neq f(x_0) = 0$.

Theorem 3.12. ε-δ criterion for continuity

Let $I \subset \mathbb{F}$ and $f : I \to \mathbb{F}$ be a function. Let $x_0 \in I$. Then the following two statements are equivalent.

(i) f is continuous in x_0;

(ii) For all $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $x \in I$ with $|x - x_0| < \delta$ holds

$$|f(x) - f(x_0)| < \varepsilon.$$

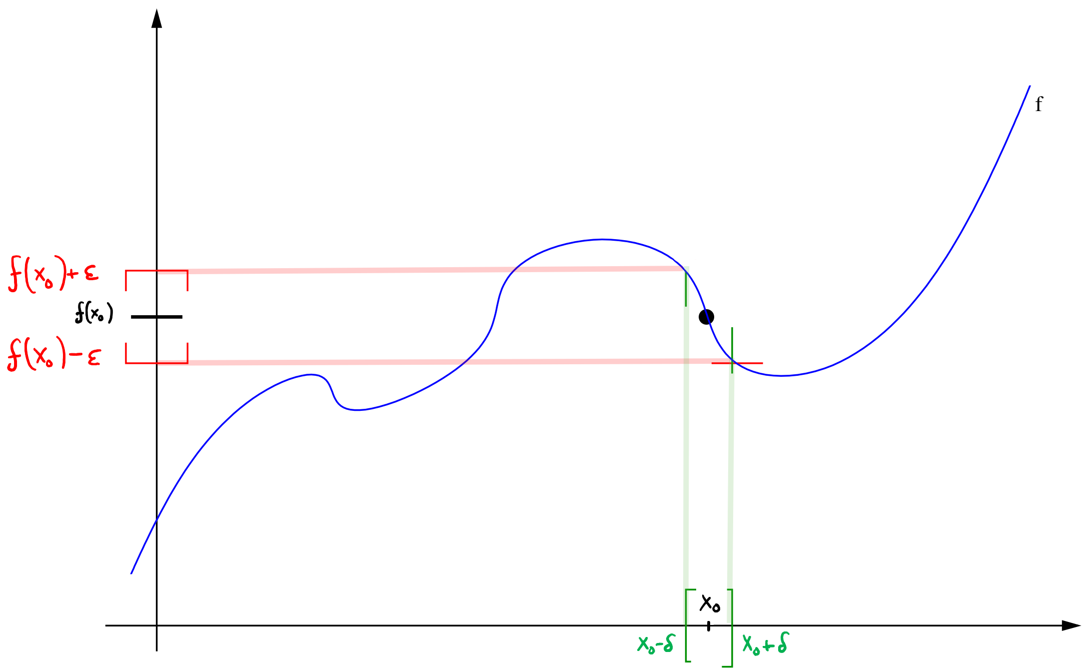


Figure 3.2: ε-δ criterion graphically illustrated

Proof:
"(i)⇒(ii)": Assume that (ii) is not fulfilled, i.e., there exists some $\varepsilon > 0$, such that for all $\delta > 0$, there exists some $x \in I \setminus \{x_0\}$ with $|x - x_0| < \delta$ and $|f(x) - f(x_0)| > \varepsilon$. As a consequence, for all $n \in \mathbb{N}$, there exists some $x_n \in I \setminus \{x_0\}$ with

$$|x_0 - x_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(x_0)| > \varepsilon.$$

Therefore, the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_0, but $|f(x_n) - f(x_0)| > \varepsilon$, i.e., $f(x_n)$ is not converging to $f(x_0)$.
"(ii)⇒(i)": Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $I\setminus\{x_0\}$ that converges to x_0. Let $\varepsilon > 0$. Then there exists some $\delta > 0$ such that for all $x \in I$ with $|x - x_0| < \delta$ holds $|f(x) - f(x_0)| < \varepsilon$. Since $(x_n)_{n\in\mathbb{N}}$ converges to x_0, there exists some N such that for all $n \geq N$ holds $|x_n - x_0| < \delta$. By the ε-δ-criterion, we have then for all $n \geq N$ that

$$|f(x_n) - f(x_0)| < \varepsilon.$$

Hence, $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x_0)$. $\square$

Next we give a result on the continuity of sums, products and quotients of functions. This looks very similar to Theorem 1.21. Indeed, those results on sums, products and quotients of sequences are the "main ingredients" for the proof (which is therefore skipped).

Theorem 3.13. Continuity of sums, products and quotients of functions

Let $I \subset \mathbb{F}$ and let $f, g : I \to \mathbb{F}$ be continuous in $x_0 \in I$. Then also $f + g$ and $f \cdot g$ are continuous in x_0. Furthermore if $g(x_0) \neq 0$, then also $\frac{f}{g}$ is continuous in x_0.

Now we consider the *composition of functions f and g* $(f \circ g)$ which is defined by the formula $(f \circ g)(x) = f(g(x))$.

Theorem 3.14. Continuity of compositions of functions

Let $I_1, I_2 \subset \mathbb{F}$ and $f : I_1 \to \mathbb{F}$, $g : I_2 \to \mathbb{F}$ be functions with

$$g(I_2) = \{g(x) \,:\, x \in I_2\} \subset I_1.$$

Assume that g is continuous in $x_0 \in I_2$ and f is continuous in $g(x_0) \in I_1$. Then also $f \circ g$ is continuous in x_0.

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in I_2 with $\lim_{n\to\infty} x_n = x_0$. By the continuity of g holds $\lim_{n\to\infty} g(x_n) = g(x_0)$. Then, by the continuity of f holds

$$\lim_{n\to\infty} f(g(x_n)) = f(g(x_0)).$$

$\square$

In the following, we collect very important properties of continuous functions. The first result is that continuous functions are bounded as far as they are defined on some compact set. Thereafter, we present the famous *Intermediate Value Theorem* which basically states that continuous functions attain every value between $f(x_0)$ and $f(x_1)$ for some arbitrary $x_0, x_1 \in I$. This result leads us to think about continuous functions as "those functions whose graph can be drawn without putting down the pencil".

Theorem 3.15. Continuous functions defined on a compact set

Let $I \subset \mathbb{F}$ be compact and let $f : I \to \mathbb{F}$ be continuous. Then $f(I)$ is compact. In particular, by the Theorem of Heine-Borel, $f(I)$ is bounded and closed. If further $f(I) \subset \mathbb{R}$, so that there exist $x^+, x^- \in I$ such that

$$f(x^+) = \max\{f(x) \,:\, x \in I\}, \qquad f(x^-) = \min\{f(x) \,:\, x \in I\}.$$

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $f(I)$. Then for each $n \in \mathbb{N}$ there is an $x_n \in I$ such that $y_n = f(x_n)$. Since I is compact, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to some $x \in I$. Now, since f is continuous, we have

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(x) =: y \in f(I).$$

Hence we found a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ that converges in $f(I)$. Therefore $f(I)$ is compact. $\square$

Now we show that a uniformly convergent sequence of continuous functions has to converge to a continuous function.

Theorem 3.16.

Let $I \subset \mathbb{F}$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n : I \to \mathbb{F}$ that uniformly converges to some $f : I \to \mathbb{F}$. Then f is continuous.

Proof: Let $\varepsilon > 0$ and let $x_0 \in I$. Since $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f : I \to \mathbb{F}$, there exists some N such that for all $n \geq N$ and $x \in I$ holds

$$|f(x) - f_n(x)| < \frac{\varepsilon}{3}.$$

Since f_n is continuous on I, there exists some $\delta > 0$ such that for all $x \in I$ with $|x-x_0| < \delta$ holds

$$|f_n(x_0) - f_n(x)| < \frac{\varepsilon}{3}.$$

Altogether, we then have

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore, f is continuous by the ε-δ criterion. □

Remark:

The above result also gives a sufficient characterisation for a sequence of pointwisely convergent functions $(f_n)_{n\in\mathbb{N}}$ that are not uniformly convergent: If $(f_n)_{n\in\mathbb{N}}$ converges to a discontinuous function, then this sequence cannot be uniformly convergent. For instance, consider the sequence $(f_n)_{n\in\mathbb{N}}$ in Examples 3.6 a) with $f_n : [0,1] \to \mathbb{R}$, $x \mapsto x^n$. The pointwise limit is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & , \text{ if } x \in [0,1), \\ 1 & , \text{ if } x = 1, \end{cases}$$

which is discontinuous at $x = 1$ though each f_n is continuous. Therefore, this sequence cannot be uniformly convergent.

Theorem 3.17. Intermediate Value Theorem

Let $a, b \in \mathbb{R}$ with $a < b$. Moreover, let $f : [a,b] \to \mathbb{R}$ be continuous. Let $x_0, x_1 \in [a,b]$ and let $y \in \mathbb{R}$ be between $f(x_0)$ and $f(x_1)$. Then there exists some $\hat{x}$ between x_0 and x_1 such that $y = f(\hat{x})$.

Proof. Without loss of generality, we assume that $x_0 < x_1$. Moreover, we can assume without loss of generality that $y = 0$ (otherwise, consider the function $f(x) - y$ instead of f). Furthermore, we can assume without loss of generality that $f(x_0) \leq 0$ and $f(x_1) \geq 0$ (otherwise, consider $-f$ instead of f).

We will construct $\hat{x}$ by nested intervals (compare the proof of Theorem 1.38).

Inductively define $A_0 = x_0$, $B_0 = x_1$ and for $k \geq 1$,

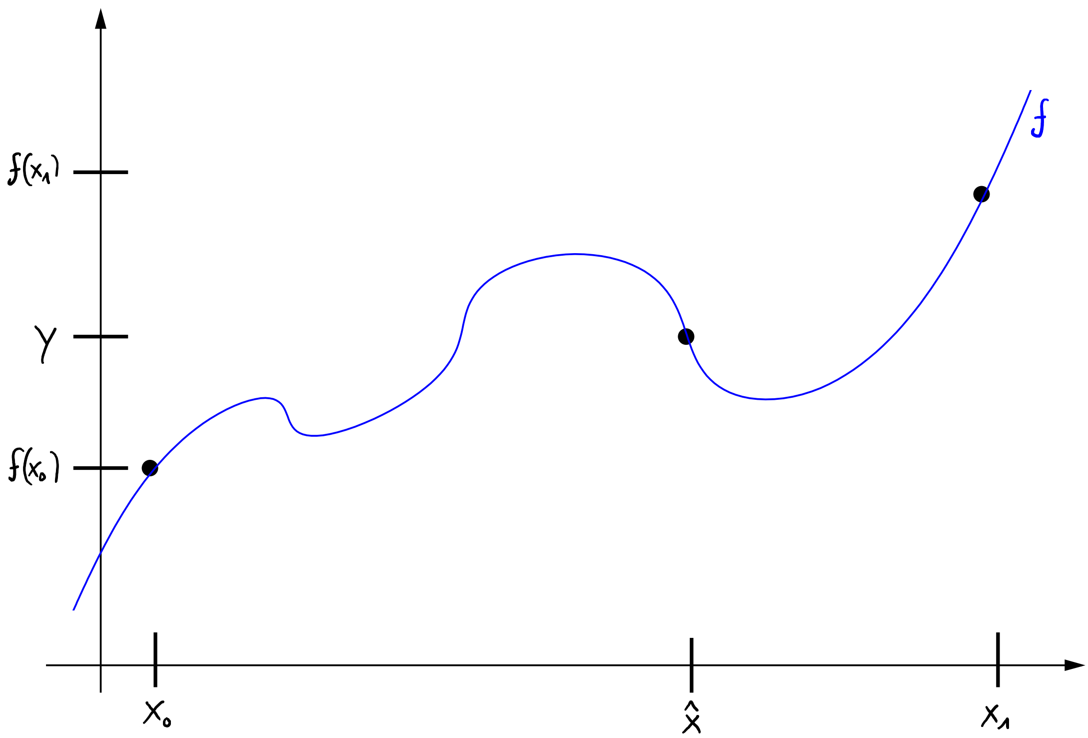


Figure 3.3: Intermediate Value Theorem graphically illustrated

a) $A_k = A_{k-1}$, $B_k = \frac{A_{k-1}+B_{k-1}}{2}$, if $f(\frac{A_{k-1}+B_{k-1}}{2}) \geq 0$, and

b) $A_k = \frac{A_{k-1}+B_{k-1}}{2}$, $B_k = B_{k-1}$, if $f(\frac{A_{k-1}+B_{k-1}}{2}) \leq 0$.

Then we have

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n =: \hat{x}.$$

By the continuity of f and the fact that $f(A_n) \leq 0$, $f(B_n) \geq 0$ for all $n \in \mathbb{N}$, we have

$$0 \geq \lim_{n\to\infty} f(A_n) = f(\hat{x}) = \lim_{n\to\infty} f(B_n) \geq 0$$

and thus $f(\hat{x}) = 0$. □

(∗) Supplementary details: Metric spaces and topologies

We close this chapter with some remarks on generalisations. As already mentioned in Chapter 1, almost all of the introduced concepts can be stated for general normed $\mathbb{F}$-vector spaces $(V, ||\cdot||)$ instead of the special one-dimensional Euclidean/unitary case $(\mathbb{F}, |\cdot|)$. In those cases where explicitly completeness properties of $(\mathbb{F}, |\cdot|)$ are used, $(V, ||\cdot||)$ must be taken as a complete normed $\mathbb{F}$-VS, i.e. as a Banach space, and in those situations where the Theorem of Heine-Borel is applied, $(V, ||\cdot||)$ can for instance be taken as a finite dimensional normed $\mathbb{F}$-vector space. A further generalization is that to metric spaces (X, d) introduced in the exercises, where the vector space V is replaced by an arbitrary set and the norm $||\cdot||$ is replaced by an abstract distance measurement, a metric d. The formal definition is: A metric on a set X is a function $d : X \times X \to [0, \infty)$ that has the following properties:

a) $d(x, y) = 0 \;\Leftrightarrow x = y$

b) $d(x, y) = d(y, x)$ (symmetry)

c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangle inequality).

Note that a normed vector space $(V, ||\cdot||)$ is a special case of a metric space since we can define $X := V$ and $d_{||\cdot||}(v,w) := ||v-w||$ for all $v, w \in V$ to obtain a metric space $(X, d_{||\cdot||})$ with the same so-called *topological* properties. Note also that a metric space has no algebraic structure, that is there is no summation or scalar multiplication defined on X. In particular, this means that generalizations of introduced concepts and theorems to metric spaces only make sense in those cases where no algebraical properties are needed.
For example, the (ε-δ)-criterion for continuity generalizes to a function $f : X_1 \to X_2$ from one metric space (X_1, d_1) to another metric space (X_2, d_2) in the following way: f is continuous if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in X_1$ with $d_1(x,y) < \delta$ holds $d_2(f(x), f(y)) < \varepsilon$.

Finally we mention that metric spaces are still not the end of the road for possible generalizations. Even more general so-called topological spaces (X, τ) can be considered, where $X \neq \emptyset$ is a set and τ is a set of subsets of X, i.e. each $U \in \tau$ is a subset of X. Then τ is called a *topology* on X if the following properties are satisfied:

a) $\emptyset, X \in \tau$

b) The union of each subset $\mathcal{W}$ of τ is again an element of τ, that is $\cup\mathcal{W} := \{x \in X \mid \exists W \in \mathcal{W} : x \in W\} \in \tau$.

c) The intersection of each finite subset $\mathcal{W}$ of τ is again an element of τ, that is, if $|\mathcal{W}| \in \mathbb{N}$, then $\cap\mathcal{W} := \{x \in X \mid \forall W \in \mathcal{W} : x \in W\} \in \tau$.

The elements of τ are called *open sets* and the complements of open sets $X \backslash U$ where $U \in \tau$ are called *closed* sets. A metric space (X, d) is special case of a topological space. If we set

$$\tau_d := \{U \subset X \mid \forall x \in U \ \exists \varepsilon > 0 : B_\varepsilon(x) \subset U\},$$

where $B_\varepsilon(x) := \{y \in X \mid d(x,y) < \varepsilon\}$ is the open d-ball with radius ε around x, then (X, τ_d) is a topological space with the same topological properties as (X, d).

Now, for example, a function $f : X_1 \to X_2$ from one topological space (X_1, τ_1) to another topological space (X_2, τ_2) is continuous, if for each $U_2 \in \tau_2$ holds $f^{-1}(U_2) \in \tau_1$, which means that each preimage of an open set in X_2 is an open set in X_1. Indeed, this generalizes the (ε-δ)-criterion for continuity for metric spaces to general topological spaces.
Good references to metric and topological spaces are the books “Mengentheoretische Topologie” of B.v. Querenburg, and “General Topology” of R. Engelking.

4 Elementary Functions

Her lips said no, but her eyes said 'Read my lips'!

Niles Crane

Here we will introduce some important functions like polynomials, rational functions, exp, sin, cos, log, sinh, cosh, power series etc. We will also consider the extension of some of the aforementioned functions to the complex plane.

4.1 Exponential Function

Definition 4.1.
The exponential function $\exp : \mathbb{F} \to \mathbb{F}$ *is defined as*

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The number

$$e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$$

is called Euler's number.

For Euler's number holds

$$e \approx 2.718281828459046.$$

We already know from Example 2.15 that the above defined series converges for all $x \in \mathbb{F}$. Now we present an estimate for $\exp(x)$ if the series is replaced by a finite sum.

Theorem 4.2.

For $n \in \mathbb{N}$ and $x \in \mathbb{F}$ with $|x| \leq 1 + \frac{n}{2}$ holds

$$\exp(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + r_n(x) \qquad \textit{with } |r_n(x)| \leq 2\frac{|x|^{n+1}}{(n+1)!}.$$

Proof:

$$\begin{aligned}
|r_n(x)| &= \left| \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \right| \leq \sum_{k=n+1}^{\infty} \frac{|x|^k}{k!} \\
&= \frac{|x|^{n+1}}{(n+1)!} \cdot \sum_{k=0}^{\infty} \frac{|x|^k}{(n+2)\cdot(n+3)\cdots(n+1+k)} \\
&\leq \frac{|x|^{n+1}}{(n+1)!} \cdot \sum_{k=0}^{\infty} \frac{|x|^k}{(n+2)^k} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{1}{1-\frac{|x|}{n+2}}
\end{aligned}$$

Since $|x| \leq \frac{n}{2} + 1 = \frac{n+2}{2}$, we have

$$|r_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{1}{1-\frac{1}{2}} = 2\frac{|x|^{n+1}}{(n+1)!}$$

□

Theorem 4.3. Properties of the Exponential Function

(i) *For all $x, y \in \mathbb{C}$ holds $\exp(x+y) = \exp(x)\exp(y)$.*

(ii) *For all $x \in \mathbb{C}$ holds $\exp(\overline{x}) = \overline{\exp(x)}$ ($\overline{y}$ denotes the complex conjugate of $y \in \mathbb{C}$).*

(iii) *For all $x \in \mathbb{C}$ holds $\exp(x) \neq 0$ and*

$$\exp(-x) = \frac{1}{\exp(x)}.$$

(iv) *For $x \in \mathbb{R}$ with $x \geq 0$ holds $\exp(x) \geq 1$. Where $\exp(x) = 1 \Leftrightarrow x = 0$.*

(v) *For $x \in \mathbb{R}$ with $x < 0$ holds $0 < \exp(x) < 1$.*

(vi) *$\exp$ in continuous.*

(vii) *$\exp : \mathbb{R} \to \mathbb{R}$ is strictly monotonically increasing.*

(viii) *$\lim_{x\to\infty} \exp(x) = \infty$ and $\lim_{x\to-\infty} \exp(x) = 0$.*

(ix) *$\exp : \mathbb{R} \to (0, \infty)$ is bijective.*

(x) *For all $x \in \mathbb{C}$ holds $|\exp(x)| = \exp(\mathrm{Re}(x))$.*

Proof:

(i) Already shown in Example 2.28.

(ii) Since we have $\overline{x_1 + x_2} = \overline{x_1} + \overline{x_2}$ and $\overline{x_1 x_2} = \overline{x_1} \cdot \overline{x_2}$ for all $x_1, x_2 \in \mathbb{C}$, we have

$$\exp(\bar{x}) = \sum_{k=0}^{\infty} \frac{\bar{x}^k}{k!} = \overline{\sum_{k=0}^{\infty} \frac{x^k}{k!}} = \overline{\exp(x)}.$$

(iii) By definition of exp, we have $\exp(0) = 1$. From (i), we get $\exp(x) \cdot \exp(-x) = \exp(x - x) = \exp(0) = 1$. Then the statement follows.

(iv) From $x \geq 0$, we get

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!} \geq 1.$$

It is immediately clear that equality only holds for $x = 0$.

(v) If $x < 0$, we get from (iii) that $\exp(-x) > 1$. Then, $\exp(x) = \frac{1}{\exp(-x)} < 1$.

(vi) First we show that exp is continuous at 0. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence converging to 0. By Theorem **??** holds for (small enough) x_n that $|\exp(x_n) - 1| = |r_0(x_n)|$ with

$$|r_0(x_n)| \leq 2\frac{|x_n|}{1!} = 2|x_n|.$$

Therefore,

$$\lim_{n\to\infty} |\exp(x_n) - \exp(0)| = \lim_{n\to\infty} |\exp(x_n) - 1| = \lim_{n\to\infty} |r_0(x_n)| = 0$$

which proves the continuity at 0. In order to show continuity on the whole real axis, we assume that $(x_n)_{n\in\mathbb{N}}$ converges to $x_0 \in \mathbb{R}$ and make use of

$$\lim_{n\to\infty} \exp(x_n) = \exp(x_0) \lim_{n\to\infty} \exp(x_n - x_0).$$

Since $(x_n - x_0)_{n\in\mathbb{N}}$ converges to 0, the above limit on the right hand side converges to 1 (due to the continuity in 0). Therefore, we have

$$\lim_{n\to\infty} \exp(x_n) = \exp(x_0)$$

which proves continuity of exp at x_0.

(vii) Let $x_1, x_2 \in \mathbb{R}$ with $x_1 > x_2$. Then we have $x_1 - x_2 > 0$ and thus, by (iv), $\exp(x_1 - x_2) > 1$. Since $\exp(x_2) > 0$, we have

$$\exp(x_1) = \exp(x_2)\exp(x_1 - x_2) > \exp(x_2).$$

(viii) The fact $\lim_{x\to\infty} \exp(x) = \infty$ follows, since we have for $x > 0$ that

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \sum_{k=2}^{\infty} \frac{x^k}{k!} > 1 + x.$$

The statement (iii) then directly implies $\lim_{x\to-\infty} \exp(x) = 0$.

(ix) The injectivity of $\exp : \mathbb{R} \to (0, \infty)$ follows from (vii). It remains to show surjectivity. Let $y \in (0, \infty)$.
Since $\lim_{x\to-\infty} \exp(x) = 0$, there exists some $x_0 \in \mathbb{R}$ such that $\exp(x_0) < y$.
Since $\lim_{x\to\infty} \exp(x) = \infty$, there exists some $x_1 \in \mathbb{R}$ such that $\exp(x_1) > y$.
Now, by the Mean Value Theorem, (remember that exp is continuous), there exists some $x \in \mathbb{R}$ with $\exp(x) = y$.

(x) Let $x = x_1 + ix_2$ with $x_1 = \mathrm{Re}(x)$, $x_2 = \mathrm{Im}(x)$. Then

$$|\exp(x)| = |\exp(x_1 + ix_2)| = |\exp(x_1)\exp(ix_2)| = |\exp(x_1)||\exp(ix_2)|.$$

If we now show that $|\exp(ix_2)| = 1$, the result is proven: Making use of (ii), we obtain

$$|\exp(ix_2)|^2 = \exp(ix_2) \cdot \overline{\exp(ix_2)} = \exp(ix_2 - ix_2) = \exp(0) = 1.$$

$\square$

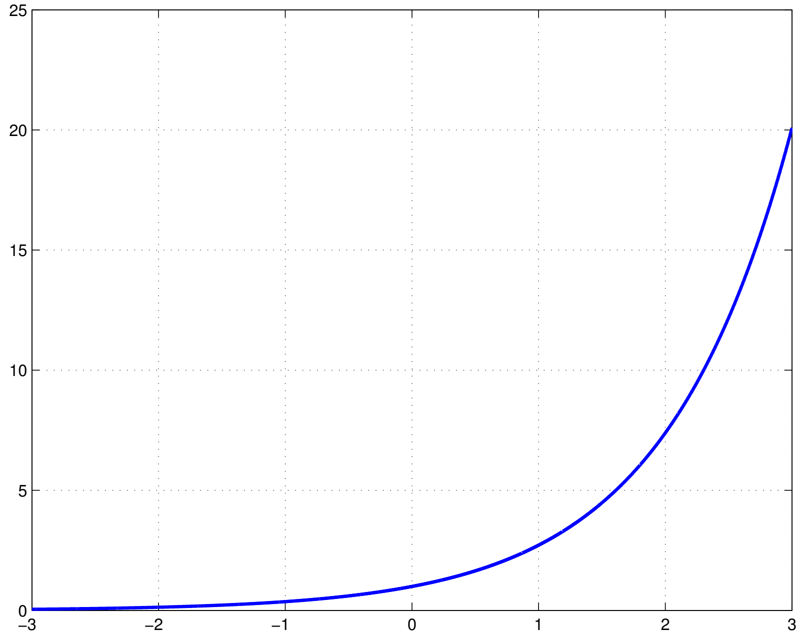


Figure 4.1: Graph of the exponential function

4.2 Logarithm

We will now define the logarithm as the inverse function of the exponential function. As we have seen, the exponential function is bijective as a map from $\mathbb{R}$ to $(0, \infty)$. This justifies the following definition.

Definition 4.4.

The (natural) logarithm $\log : (0, \infty) \to \mathbb{R}$ *is defined as the inverse function of* $\exp : \mathbb{R} \to (0, \infty)$, *i.e., for all* $x \in \mathbb{R}$ *holds* $\log(\exp(x)) = x$ *and for all* $y \in (0, \infty)$ *holds* $\exp(\log(y)) = y$.

In many books, the above defined function is also denoted by *logarithmus naturalis* ln. Before we collect some properties, a general result about continuity of inverse functions is presented.

Theorem 4.5. Continuity of the inverse function

Let $I, J \subset \mathbb{R}$ be open intervals and let $f : I \to J$ be a continuous, bijective and strictly monotonically increasing (or decreasing) function. Then the inverse function $f^{-1} : J \to I$ is continuous and strictly monotonically increasing (resp. decreasing).

Proof: First we show strict monotonic increase. Let $y_1, y_2 \in J$ with $y_1 < y_2$. Then

$$f(f^{-1}(y_1)) = y_1 < y_2 = f(f^{-1}(y_2))$$

shows that, since f is strictly monotonically increasing, $f^{-1}(y_1) \geq f^{-1}(y_2)$ cannot hold, so that $f^{-1}(y_1) < f^{-1}(y_2)$. But this means that f^{-1} is strictly monotonically increasing.

Now we show continuity. Let $\varepsilon > 0$ and $y_0 \in J = f(I)$. Set $x_0 := f^{-1}(y_0) \in I$. Since I is an open interval, there is an $\varepsilon' > 0$ with $\varepsilon' < \varepsilon$ such that $[x_0 - \varepsilon', x_0 + \varepsilon'] \subset I$. Since f is strictly monotonically increasing,

$$\delta := \min\{f(x_0 + \varepsilon') - y_0, y_0 - f(x_0 - \varepsilon')\} > 0.$$

Then for $y \in J$ with $|y - y_0| < \delta$ holds $f(x_0 - \varepsilon') < y < f(x_0 + \varepsilon')$ and the intermediate value theorem yields

$$x := f^{-1}(y) \in [x_0 - \varepsilon', x_0 + \varepsilon'] \subset \ (x_0 - \varepsilon, x_0 + \varepsilon) \ = \ (f^{-1}(y_0) - \varepsilon, f^{-1}(y_0) + \varepsilon) \ ,$$

i.e. $|f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$. By the ε-δ criterion this means that f^{-1} is continuous in y_0. □

We want to remark that Theorem 4.5 also holds for intervals I and J which are not open. In this case the proof of continuity of f^{-1} in $y_0 \in J$ has to be slightly adapted for boundary points $x_0 := f^{-1}(y_0)$. This was dropped for simplicity reasons.

Theorem 4.6. Properties of the Logarithm

(i) For all $x, y \in \ (0, \infty)$ holds $\log(x \cdot y) = \log(x) + \log(y)$.

(ii) $\log : \ (0, \infty) \ \to \ \mathbb{R}$ is strictly monotonically increasing.

(iii) $\log : \ (0, \infty) \ \to \ \mathbb{R}$ is continuous.

Proof:

(i) Define $x_1 = \log(x)$ and $y_1 = \log(y)$. Then, using $x = \exp(x_1)$ and $y = \exp(y_1)$, we obtain

$$\log(x \cdot y) = \log(\exp(x_1) \cdot \exp(y_1)) = \log(\exp(x_1 + y_1)) = x_1 + y_1 = \log(x) + \log(y).$$

(ii) and (iii) follow from Theorem 4.5. □

Remark:

Without further ado we introduce the logarithm for complex numbers $z = r\exp(i\varphi)$ in polar coordinates with $r > 0$ and $\varphi \in [0, 2\pi)$ by

$$\log(r\exp(i\varphi)) = \log(r) + \log(\exp(i\varphi)) = \log(r) + i\varphi \ .$$

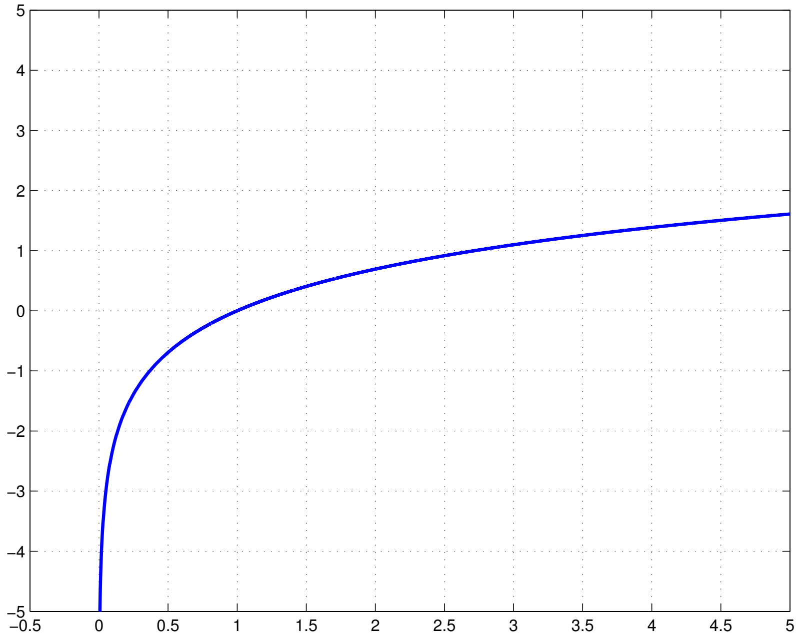


Figure 4.2: Graph of the logarithm

However, formally this is a quite delicate issue and not treated in this lecture in much detail. For further information on complex logarithms we refer to books on Complex Analysis *(German:* Funktionentheorie*).*

By means of the exponential function, the *general power* a^x (for $a > 0$, $x \in \mathbb{C}$) can be defined as follows:

$$a^x := \exp(\log(a) \cdot x).$$

This definition indeed makes sense as $a^0 = \exp(\log(a) \cdot 0) = 1$, $a^1 = \exp(\log(a) \cdot\ 1) = a$ and (for $n \in \mathbb{N}$)

$$a^n = \exp(\underbrace{\log(a) + \ldots + \log(a)}_{n-\text{times}}) = \exp(\log(a))^n.$$

It can further be seen that this definition implies $a^{\frac{1}{n}} = \sqrt[n]{a}$. The definition of the general power also justifies the notion $\exp(x) = e^x$.
A remaining question is how to solve the equation

$$a^x = y$$

for given $a > 0$, $y \in \mathbb{R}$ and unknown $x \in \mathbb{R}$. Using the definition of the general power, this equation becomes

$$\exp(\log(a) \cdot x) = y.$$

Performing the logarithm on both sides of this equation, we obtain $\log(a) \cdot x = \log(y)$ and thus

$$x = \frac{\log(y)}{\log(a)}.$$

In some literature, this expression is known as the *logarithm of y to the basis a* and abbreviated by

$$\log_a(y) := \frac{\log(y)}{\log(a)}.$$

By definition, we have $\log(x) = \log_e(x)$.

Remark: The pocket calculator and high school maths

In most of high school literature the logarithm (as we have defined it) is called natural logarithm *and is abreviated by* $\ln$. *However, in mathematical literature the symbol* $\log$ *is also commonly used for the inverse function of* $\exp$.
Furthermore, note that for many pocket calculators, the button `log` *stands for* $\log_{10}$, *the so-called* decimal logarithm, *whereas pushing the button* `ln` *gives the logarithm. Note that the decimal logarithm is given by*

$$\log_{10}(x) = \frac{\log(x)}{\log(10)}.$$

4.3 Hyperbolic and Trigonometric Functions

4.3.1 Hyperbolic Functions

Definition 4.7. Hyperbolic Sine and Hyperbolic Cosine

The hyperbolic sine (sinus hyperbolicus) and hyperbolic cosine (cosinus hyperbolicus) $\sinh : \mathbb{C} \to \mathbb{C}$ *and* $\cosh : \mathbb{C} \to \mathbb{C}$ *are defined as*

$$\sinh(x) = \frac{1}{2}\left(\exp(x) - \exp(-x)\right), \qquad \cosh(x) = \frac{1}{2}\left(\exp(x) + \exp(-x)\right).$$

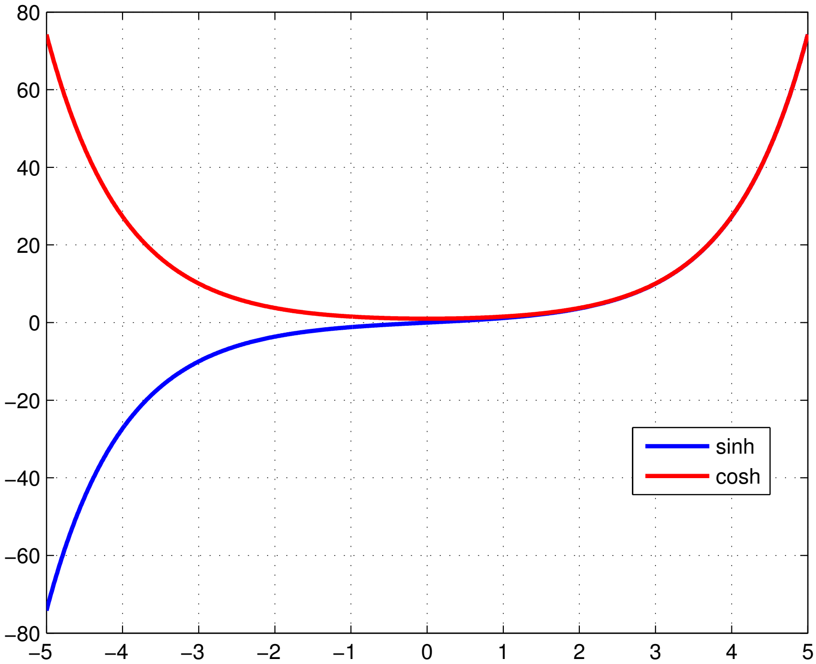


Figure 4.3: Graph of the hyperbolic functions

Theorem 4.8. Properties of the Hyperbolic Functions

(i) *For all* $x \in \mathbb{C}$ *holds* $\sinh(\overline{x}) = \overline{\sinh(x)}$, $\cosh(\overline{x}) = \overline{\cosh(x)}$.

(ii) $\sinh$ *and* $\cosh$ *are continuous.*

(iii) *For all* $x \in \mathbb{C}$ *holds* $\cosh^2(x) - \sinh^2(x) = 1$.

(iv) *For all* $x \in \mathbb{C}$ *holds*

$$\cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \qquad \sinh(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

(v) $\sinh : \mathbb{R} \to \mathbb{R}$ *is strictly monotonically increasing.*

(vi) $\cosh : \mathbb{R} \to \mathbb{R}$ *is strictly monotonically increasing on* $[0, \infty)$ *and strictly monotonically decreasing on* $(-\infty, 0]$.

Proof. (i) and (ii) follow from Theorem 4.3.
(iii):

$$\begin{aligned}\cosh^2(x) - \sinh^2(x) &= \frac{1}{4}\left((e^x)^2 + 2e^x e^{-x} + (e^{-x})^2 - (e^x)^2 + 2e^x e^{-x} - (e^{-x})^2\right) \\ &= \frac{1}{4} \cdot 4e^x e^{-x} = 1.\end{aligned}$$

(iv): Using the series representation for exp we have

$$\begin{aligned}\sinh(x) &= \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!} - \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\right)\\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!} - \sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!}\right)\\ &= \frac{1}{2}\left(2\sum_{k=1,3,5,\dots}\frac{x^k}{k!}\right) = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}.\end{aligned}$$

The series representation for cosh can be derived analogously.
(v): Let $x \in \mathbb{R}$ and $a > 0$. Then $e^a > 1$, $0 < e^{-a} < 1$ and

$$\sinh(x+a) = \tfrac{1}{2}(e^{x+a} - e^{-(x+a)}) = \tfrac{1}{2}(e^a e^x - e^{-a}e^{-x}) > \tfrac{1}{2}(e^x - e^{-x}) = \sinh(x).$$

(vi): First of all it holds for $x \in \mathbb{R}$

$$\cosh(-x) = \frac{1}{2}(e^{-x} + e^x) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x).$$

Now let $x \geq 0$. From (iii) we have $\cosh^2(x) = 1 + \sinh^2(x)$ and since $\cosh(x) \geq 1$ and $\sinh(x) \geq 0$ for $x \geq 0$ it follows directly from (v) that cosh is strictly increasing on $[0, \infty[$. Now let $x \leq 0$. Since $\cosh(-x) = \cosh(x)$ it follows from the first part that cosh is strictly decreasing on $[-\infty, 0)$. □

Remark:

The definition of the hyperbolic functions imply that $\sinh(-x) = -\sinh(x)$ *(resp.* $\cosh(-x) = \cosh(x)$*). A function with this property is called* odd *(resp. even).*
The monotonicity property of cosh *together with the fact that* $\cosh(0) = 1$ *imply that* cosh *does not have any real zeros. The hyperbolic sine function has only one zero at the origin.*

Remark:

Why are these functions called hyperbolic functions*?*

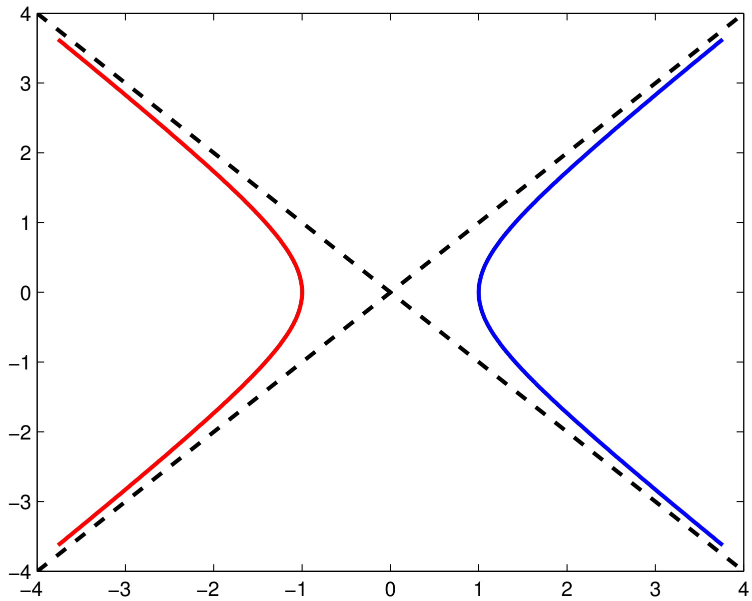


Figure 4.4: Hyperbola

Since $\cosh^2(x) - \sinh^2(x) = 1$, *the curve*

$$\{(-\cosh(t), \sinh(t)) \mid t \in \mathbb{R}\} \cup \{(\cosh(t), \sinh(t)) \mid t \in \mathbb{R}\}$$

describes a hyperbola see Figure 4.4. (In analogy the curve $\{(\cos(t), \sin(t)) \mid t \in \mathbb{R}\}$ *describes the unit circle.)*

Definition 4.9. Hyperbolic Tangent

The hyperbolic tangent (tangens hyperbolicus) $\tanh : \{x \in \mathbb{C} : \cosh(x) \neq 0\} \to \mathbb{C}$ *is defined as*

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

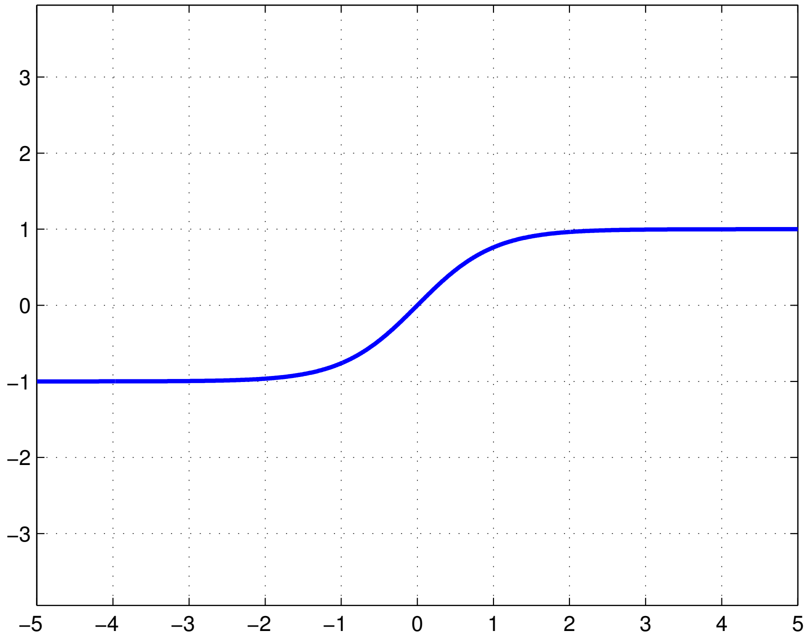


Figure 4.5: Graph of the hyperbolic tangent

Remark:

Since sinh *and* cosh *map real numbers to real numbers and, moreover,* cosh *has no zero in* $\mathbb{R}$*, the hyperbolic tangent is defined on the whole real axis. Furthermore, it can be seen that* tanh *is continuous, strictly monotonically increasing and*

$$\lim_{x\to\infty} \tanh(x) = 1, \qquad \lim_{x\to-\infty} \tanh(x) = -1.$$

4.3.2 Area Functions

We already know that $\sinh : \mathbb{R} \to \mathbb{R}$, $\cosh : [0,\infty) \to [1,\infty)$, $\tanh : \mathbb{R} \to (-1,1)$ are strictly monotonically increasing. They possess inverse functions defined on $\mathbb{R}$ (resp. $[1,\infty)$, $(-1,1)$).

Definition 4.10. Area Functions

(i) *The area hyperbolic sine or the area sinus hyperbolicus* $\operatorname{arsinh} : \mathbb{R} \to \mathbb{R}$ *is defined as the inverse function of* sinh.

(ii) *The area hyperbolic cosine or the area cosinus hyperbolicus is denoted by* $\operatorname{arcosh} : [1,\infty) \to [0,\infty)$ *is defined as the inverse function of* cosh.

(iii) *The area hyperbolic tangent or the area tangens hyperbolicus is denoted by* $\operatorname{artanh} : (-1,1) \to \mathbb{R}$ *is defined as the inverse function of* tanh.

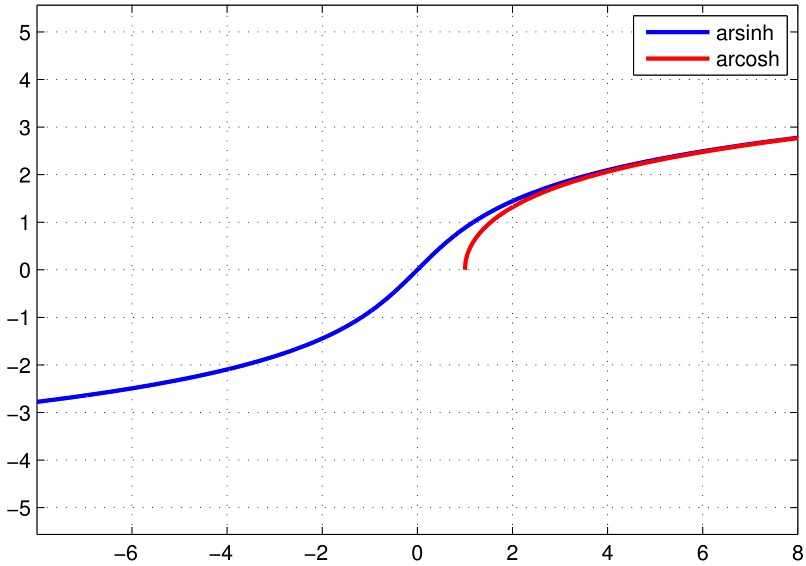


Figure 4.6: Graph of arsinh and arcosh

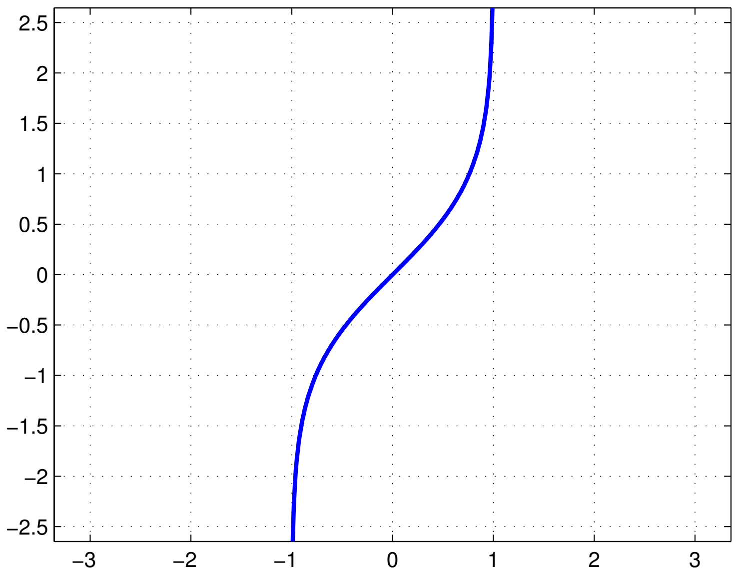


Figure 4.7: Graph of artanh

4.3.3 Trigonometric Functions

Definition 4.11. Sine and Cosine

The sine (sinus) $\sin : \mathbb{C} \to \mathbb{C}$ *and* cosine (cosinus) $\cos : \mathbb{C} \to \mathbb{C}$ *are defined as*

$$\sin(x) := \frac{1}{2i}(\exp(ix) - \exp(-ix)), \qquad \cos(x) := \frac{1}{2}(\exp(ix) + \exp(-ix)).$$

Theorem 4.12. Properties of $\sin$ and $\cos$

(i) For all $x \in \mathbb{C}$ holds $\sin(\overline{x}) = \overline{\sin(x)}$, $\cos(\overline{x}) = \overline{\cos(x)}$.

(ii) $\sin$ and $\cos$ are continuous.

(iii) For all $x \in \mathbb{C}$ holds $\sin(x) = -i\sinh(ix)$, $\cos(x) = \cosh(ix)$.

(iv) For all $x \in \mathbb{C}$ holds $\cos^2(x) + \sin^2(x) = 1$.

(v) For all $x \in \mathbb{C}$ holds

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \qquad \sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Proof. (i): Assume that $x = x_1 + ix_2$ with $x_1, x_2 \in \mathbb{R}$ and compute

$$\begin{aligned}
\overline{\sin(x)} = \overline{\sin(x_1 + ix_2)} &= \overline{\frac{1}{2i}(\exp(ix_1 - x_2) - \exp(-ix_1 + x_2))} \\
&= \frac{1}{-2i}(\overline{\exp(ix_1 - x_2)} - \overline{\exp(-ix_1 + x_2)}) \\
&= \frac{1}{2i}(\overline{\exp(-ix_1 + x_2)} - \overline{\exp(ix_1 - x_2)}) \\
&= \frac{1}{2i}(\exp(ix_1 + x_2) - \exp(-ix_1 - x_2)) \\
&= \frac{1}{2i}(\exp(i\overline{x}) - \exp(-i\overline{x})) \\
&= \sin(\overline{x})
\end{aligned}$$

The relation for $\cos(\overline{x})$ is analogous.
(ii): Continuity follows from that of the exponential function.
(iii): Follows by definition (and taking into account that $\frac{1}{i} = -i$).
(iv): Follows from (iii) and Theorem 4.8 (iii).
(v): The series representations can be obtained by using (iii) and the series representations of $\sinh(x)$, $\cosh(x)$. ☐

Remark:

By the above definition, we have that for all $x \in \mathbb{C}$ holds

$$\exp(ix) = \cos(x) + i\sin(x).$$

For $x \in \mathbb{R}$ this gives rise to

$$\cos(x) = \mathrm{Re}(\exp(ix)), \quad \sin(x) = \mathrm{Im}(\exp(ix)).$$

In particular, the equation $\cos^2(x) + \sin^2(x) = 1$ implies for $x \in \mathbb{R}$ that $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$.

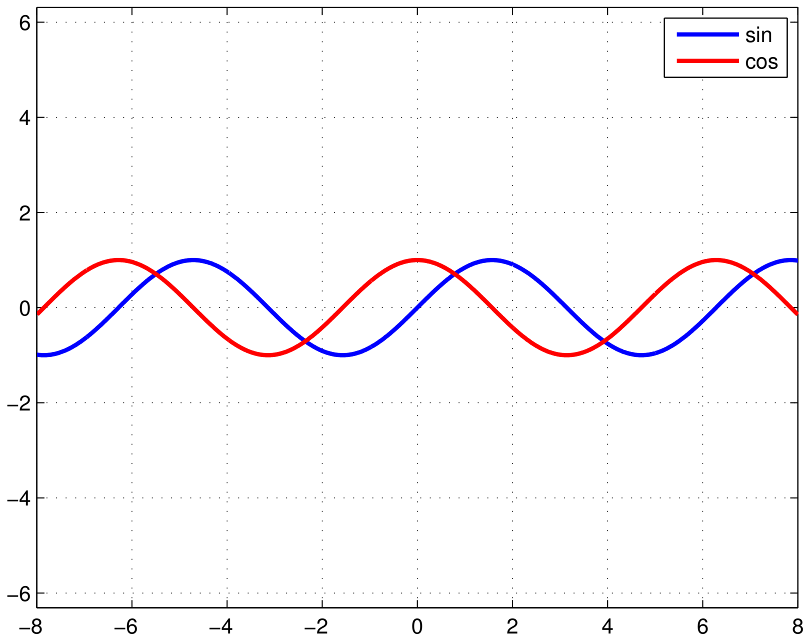


Figure 4.8: Graph of sin and cos

The following result gives formulas for sine and cosine applied to sums of (complex) numbers. These results can be readily verified by making use of Definition 4.11 and the equation $\exp(x_1 + x_2) = \exp(x_1)\exp(x_2)$.

Theorem 4.13. Trigonometric identities

For arbitrary $x, y \in \mathbb{C}$ the trigonometric functions fulfill

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y), \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y). \end{aligned}$$

Proof. This follows by

$$\begin{aligned}
&\sin(x)\cos(y)+\cos(x)\sin(y),\\
&=\frac{1}{2i}(\exp(ix)-\exp(-ix))\cdot\frac{1}{2}(\exp(iy)+\exp(-iy))\\
&\quad+\frac{1}{2}(\exp(ix)+\exp(-ix))\cdot\frac{1}{2i}(\exp(iy)-\exp(-iy))\\
&=\frac{1}{4i}(\exp(i(x+y))-\exp(i(y-x))+\exp(i(x-y))-\exp(-i(x+y))\\
&\quad+\exp(i(x+y))+\exp(i(y-x))-\exp(i(x-y))-\exp(-i(x+y)))\\
&=\frac{1}{4i}(2\exp(i(x+y))-2\exp(-i(x+y)))\\
&=\frac{1}{2i}(\exp(i(x+y))-\exp(-i(x+y)))\\
&=\sin(x+y)
\end{aligned}$$

and

$$\begin{aligned}
&\cos(x)\cos(y)-\sin(x)\sin(y),\\
&=\frac{1}{2}(\exp(ix)+\exp(-ix))\cdot\frac{1}{2}(\exp(iy)+\exp(-iy))\\
&\quad-\frac{1}{2i}(\exp(ix)-\exp(-ix))\cdot\frac{1}{2i}(\exp(iy)-\exp(-iy))\\
&=\frac{1}{2}(\exp(ix)+\exp(-ix))\cdot\frac{1}{2}(\exp(iy)+\exp(-iy))\\
&\quad+\frac{1}{2}(\exp(ix)-\exp(-ix))\cdot\frac{1}{2}(\exp(iy)-\exp(-iy))\\
&=\frac{1}{4}(\exp(i(x+y))+\exp(i(y-x))+\exp(i(x-y))+\exp(-i(x+y))\\
&\quad+\exp(i(x+y))-\exp(i(y-x))-\exp(i(x-y))+\exp(-i(x+y)))\\
&=\frac{1}{4}(2\exp(i(x+y))+2\exp(-i(x+y)))\\
&=\frac{1}{2}(\exp(i(x+y))-\exp(-i(x+y)))\\
&=\cos(x+y).
\end{aligned}$$

□

We now define the famous number π by the double of the first positive zero of the cosine function. The following result shows that this definition indeed makes sense.

Theorem 4.14. First positive zero of $\cos$, definition of π

The function $\cos:\mathbb{R}\to\mathbb{R}$ *has exactly one zero in the interval* $[0,2]$. *This zero is called* $\frac{\pi}{2}$.

The proof for this is not presented here. It basically consists of three steps: The first step consists of showing that $\cos(2)<0$. This can be shown by using the series representation in Theorem 4.12 (v). In the second step we have to show that cos is strictly monotonically decreasing in the interval $[0,2]$. This can be achieved by showing that sin is positive on the interval $[0, 2]$ and making use of the equation

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right),$$

which follows from the trigonometric identities. Altogether, we then have that $\cos(0) = 1 > 0$, $\cos(2) < 0$ and cos is strictly monotonically decreasing on $[0, 2]$. The intermediate value theorem implies that there exists a zero in $(0, 2)$. The strict monotonic decrease gives rise to the uniqueness of this zero.
Note that the positivity of sin on the interval $[0, 2]$ implies that $\sin(\frac{\pi}{2}) = 1$, since $\cos(\frac{\pi}{2}) = 0$ and $\sin^2(\frac{\pi}{2}) + \cos^2(\frac{\pi}{2}) = 1$.
Now we present some further fundamental properties of the trigonometric functions. These for instance include 2π-periodicity.

Theorem 4.15. Further properties of the trigonometric functions

For all $x \in \mathbb{C}$ holds:

(i) $\cos(x) = \sin(x + \frac{\pi}{2})$;

(ii) $\sin(x) = -\cos(x + \frac{\pi}{2})$;

(iii) $\sin(x) = -\sin(x + \pi)$;

(iv) $\cos(x) = -\cos(x + \pi)$;

(v) $\sin(x) = \sin(x + 2\pi)$;

(vi) $\cos(x) = \cos(x + 2\pi)$.

Proof. (i) follows from the trigonometric identity together with $\sin(\frac{\pi}{2}) = 1$ and $\cos(\frac{\pi}{2}) = 0$, namely

$$\sin\left(x + \frac{\pi}{2}\right) = \sin(x)\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} + \cos(x)\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} = \cos(x).$$

Item (ii) can be shown analogously.
(iii) is a consequence of

$$\sin(x + \pi) = \cos(x + \frac{\pi}{2}) = -\sin(x),$$

(iv) is analogous.
(v) and (vi) follow by a double application of (iii) (resp. (iv)). □

Definition 4.16. Tangent

The tangent $\tan : \{x \in \mathbb{C} : \cos(x) \neq 0\} \to \mathbb{C}$ *is defined by*

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

One can show that the set of zeros of the cosine function is $\{\frac{2n+1}{2}\pi \ : \ n \in \mathbb{Z}\}$. Therefore, the tangent is defined on

$$\mathbb{C} \backslash \left\{\tfrac{2n+1}{2}\pi \ : \ n \in \mathbb{Z}\right\}.$$

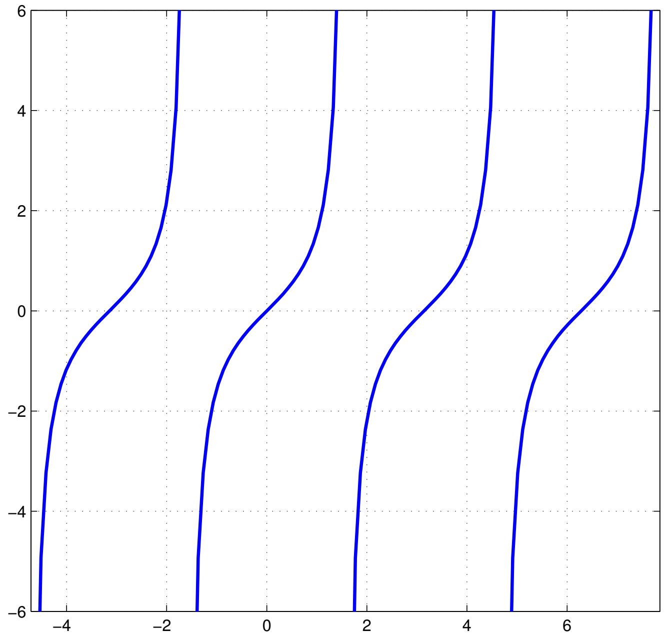


Figure 4.9: Graph of tan

4.4 Arcus functions

From the previous result, it is possible to derive that

a) sin is strictly monotonically increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin(-\frac{\pi}{2}) = -1$, $\sin(\frac{\pi}{2}) = 1$;

b) cos is strictly monotonically decreasing on $[0, \pi]$ with $\cos(0) = 1$, $\cos(\pi) = -1$;

c) tan is strictly monotonically increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$ with

$$\lim_{x \searrow -\frac{\pi}{2}} \tan(x) = -\infty, \qquad \lim_{x \nearrow \frac{\pi}{2}} \tan(x) = \infty.$$

Since sin, cos and tan are furthermore continuous, we can apply Theorem 4.5 to see that the following definition makes sense:

Definition 4.17.

(i) The inverse sine or arcus sinus $\arcsin : [-1, 1] \to \mathbb{R}$ *is defined as the inverse function of* $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$.

(ii) The inverse cosine or arcus cosinus $\arccos : [-1, 1] \to \mathbb{R}$ *is defined as the inverse function of* $\cos : [0, \pi] \to [-1, 1]$.

(iii) The inverse tangent or arcus tangens $\arctan : \mathbb{R} \to \mathbb{R}$ *is defined as the inverse function of* $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$.

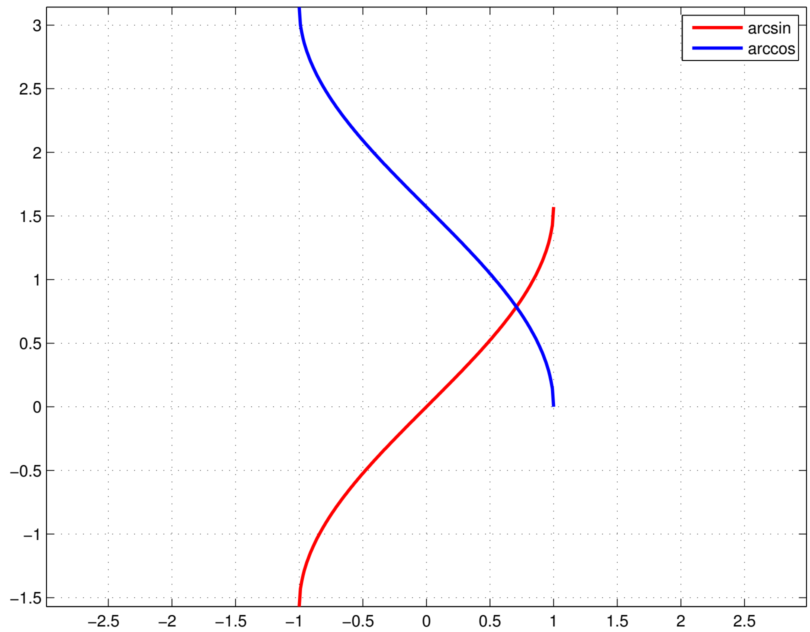


Figure 4.10: Graph of arcsin and arccos

4.5 Polynomials and Rational Functions

4.5.1 Polynomials

Definition 4.18. Polynomials

Let $a_0, \dots, a_n \in \mathbb{F}$. *Then a real (complex) polynomial is a function* $p : \mathbb{F} \to \mathbb{F}$ *with*

$$p(x) = \sum_{k=0}^{n} a_k x^k.$$

If $a_n \neq 0$, *then* a_n *is called leading coefficient and* n *is called degree of* p. *We write* $n =: \deg p$. *If* p *is the zero polynomial, we set* $\deg p := -\infty$.
The set of polynomials in $\mathbb{F}$ *is denoted by* $\mathbb{F}[x]$. *Moreover, we set*

$$\mathbb{F}_n[x] := \{p \in \mathbb{F}[x] \mid \deg p \leq n\}.$$

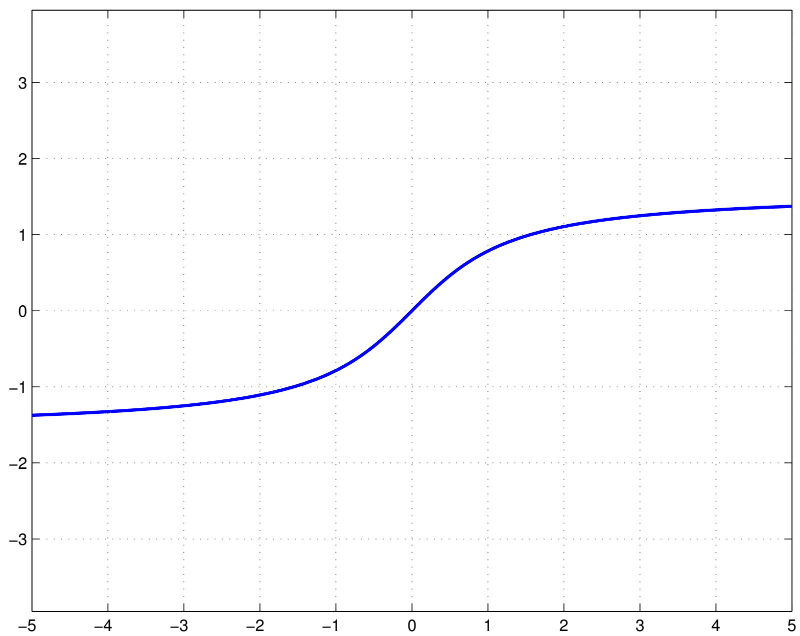


Figure 4.11: Graph of arctan

Remark:

Since sums and scalar multiples of polynomials are again polynomials, they form a vector space.

Theorem 4.19. Rules for the degree

For $p, q \in \mathbb{F}[x]$ holds

$$\deg(p \cdot q) = \deg(p) + \deg(q), \qquad \deg(p+q) \le \max\{\deg(p), \deg(q)\}.$$

Proof. Let $p(x) = \sum_{k=0}^{n} a_k x^k$, $q(x) = \sum_{k=0}^{m} b_k x^k$ with $a_n \neq 0$, $b_m \neq 0$. The formula for $\deg(p \cdot q)$ follows from

$$p(x) \cdot q(x) = \sum_{k=0}^{n+m} c_k x^k \quad \text{for } c_k = \sum_{l=0}^{k} a_l b_{k-l} \ ,$$

where $a_r := 0 =: b_s$ for $r \notin \{0, ..., n\}$ and $s \notin \{0, ..., m\}$. For the proof of the formula $\deg(p+q)$, we assume without loss of generality that $n \ge m$. Then

$$p(x) + q(x) = \sum_{k=0}^{m} (a_k + b_k) x^k + \sum_{k=m+1}^{n} a_k x^k.$$

As a consequence, we have $\deg(p+q) \le n = \max\{\deg(p), \deg(q)\}$. □

Remark:

As the example $p(x) = x$ and $q(x) = -x + 1$ shows, it may indeed happen that $\deg(p+q) < \max\{\deg(p), \deg(q)\}$.
Since $\deg(0 \cdot p) = \deg(0) = -\infty = -\infty + \deg(p)$ *and* $\deg(0 + p) = \deg(p) = \max\{-\infty, \deg(p)\}$, *the choice of* $\deg p = -\infty$ *makes indeed sense for preserving the above formulas. However, this belongs to the "not so important facts" of mathematical analysis.*

Next we consider the evaluation of polynomials at some point x_0. A closer look at the expression $p(x) = \sum_{k=0}^{n} a_k x^k$ yields that a "stupid" determination of $p(x_0)$ requires n summations and $1+2+\ldots+(n-1) = \frac{n(n-1)}{2}$ multiplications. With the following method, called *Horner scheme*, we can evaluate polynomials with effort that is only linear in n. The "trick" behind this method is an alternative representation, i.e we rewrite p as

$$p(x) = (x - x_0)(b_{n-1}x^{n-1} + \ldots - b_1 x + b_0) + c.$$

for some constants $b_{n-1}, \ldots, b_0, c$. It can be directly seen that $p(x_0) = c$. Collecting powers of x yields

$$\begin{aligned} p(x) &= b_{n-1}x^n + \ldots + b_1 x^2 + b_0 x - b_{n-1}x_0 x^{n-1} - \ldots - b_1 x_0 x - b_0 x_0 + c \\ &= b_{n-1}x^n + (b_{n-2} - b_{n-1}x_0)x^{n-1} + (b_{n-3} - b_{n-2}x_0)x^{n-2} \\ &\quad + \ldots + (b_0 - b_1 x_0)x + (c - b_0 x_0). \end{aligned}$$

By a comparison of coefficients, we obtain the system of linear equations

$$\begin{aligned} b_{n-1} &= a_n, \\ b_{n-2} - b_{n-1}x_0 &= a_{n-1}, \\ &\vdots \\ b_0 - b_1 x_0 &= a_1, \\ c - b_0 x_0 &= a_0. \end{aligned}$$

This system can be recursively solved as

$$\begin{aligned} b_{n-1} &= a_n, \\ b_{n-2} &= a_{n-1} + b_{n-1}x_0, \\ &\vdots \\ b_0 &= a_1 + b_1 x_0, \\ c &= a_0 + b_0 x_0. \end{aligned}$$

The following schematic representation summarizes the above introduced approach.

Remark:

If a power is missing in the polynomial, then in the Horner scheme the corresponding coefficient has to be set to 0. *For instance, the polynomial* $x^2 + 1$ *has to be written as* $x^2 + 0x + 1$.

	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_1	a_0
		$x_0 b_{n-1}$	$x_0 b_{n-2}$	$\cdots$	$x_0 b_1$	$x_0 b_0$
(mult. with x_0)		$\nearrow$ $\downarrow$	$\nearrow$ $\downarrow$	$\nearrow$ $\cdots$	$\nearrow$ $\downarrow$	$\nearrow$ $\downarrow$
	b_{n-1}	b_{n-2}	b_{n-3}		b_0	c

Table 4.1: Horner Scheme

Example 4.20.

Consider the polynomial $p(x) = x^3 - 6x^2 + 7x$ *and determine* $p(2)$*.*

	1	-6	7	0
		2	-8	-2
(mult. with 2*)*		$\nearrow$ $\downarrow$	$\nearrow$ $\downarrow$	$\nearrow$ $\downarrow$
	1	-4	-1	-2

For integers, "division with remainder" is well known. The same procedure can now be applied for polynomials and is called *polynomial division*. First we present an existence result. Afterwards, we will present a method to compute the polynomials in question.

Theorem 4.21. Polynomial Division

Let $p, q \in \mathbb{F}[x]$ *be given. Furthermore, assume that* q *is not the zero polynomial. Then there exist polynomials* g *and* r *with* $\deg r < \deg q$*, such that*

$$p = q \cdot g + r. \tag{4.1}$$

The polynomial r *is called* remainder*. If* r *is the zero polynomial, we say that* p *is divisible by* q*.*

To compute the polynomials g and r as in Theorem 4.21, we use the method of *polynomial division*. This method is not displayable by a mathematical theorem but only explainable by means of concrete examples. Polynomial division is of great importance, in particular for some special representations of rational functions.

Example 4.22. a) Let $p(x) = x^3 + 3x^2 - 8x - 4$ and $q(x) = x - 2$.

$$\begin{array}{rrrrl}
(x^3 & +3x^2 & -8x & -4 &) : (x-2) \;=\; x^2+5x+2 \qquad \text{Remainder: } 0 \\
-(x^3 & -2x^2) & & & \\
\hline
 & 5x^2 & -8x & & \\
 & -(5x^2 & -10x) & & \\
\hline
 & & 2x & -4 & \\
 & & -(2x & -4) & \\
\hline
 & & & 0 &
\end{array}$$

Test: We have: $(x^2+5x+2)(x-2) = x^3+3x^2-8x-4$.
This means that x^3+3x^2-8x-4 is divisible by $x-2$.

b) Let $p(x) = x^4+4x+5$ and $q(x) = x^2+x-1$.

$$\begin{array}{rrrrrl}
(x^4 & & & +4x & +5 &) : (x^2+x-1) \;=\; x^2-x+2 \qquad \text{Remainder: } x+7 \\
-(x^4 & +x^3 & -x^2) & & & \\
\cline{1-3}
 & -x^3 & +x^2 & +4x & & \\
 & -(-x^3 & -x^2 & +x) & & \\
\cline{2-4}
 & & 2x^2 & +3x & +5 & \\
 & & -(2x^2 & +2x & -2) & \\
\cline{3-5}
 & & & x & +7 & .
\end{array}$$

Test: We have: $(x^2-x+2)(x^2+x-1)+(x+7) = x^4+4x+5$.

c) Let $p(x) = x^4+4x+5$ and $q(x) = x^2+1$.

$$\begin{array}{rrrrl}
(x^4 & & +4x & +5 &) : (x^2+1) \;=\; x^2-1 \qquad \text{Remainder: } 4x+6 \\
-(x^4 & +x^2) & & & \\
\cline{1-2}
 & -x^2 & +4x & +5 & \\
 & -(-x^2 & & -1) & \\
\cline{2-4}
 & & +4x & +6 & .
\end{array}$$

Test: We have: $(x^2+1)(x^2-1)+(4x+6) = x^4+4x+5$.

Note that a polynomial p is divisible by $x-x_0$ if and only if x_0 is a zero of p, i.e., $p(x_0)=0$. This follows from the fact that the division with remainder theorem implies that there exists some polynomial $q \in \mathbb{F}[x]$ and a constant $c \in \mathbb{F}$ (i.e., a polynomial whose degree is less than 1), such that

$$p(x) = (x-x_0)q(x) + c.$$

In particular, we have that $p(x_0) = 0$ if and only if $c = 0$ if and only if p is divisible by $x - x_0$. This leads to the following definition:

Definition 4.23.

Let $p \in \mathbb{F}[x]$ be given with zero $x_0 \in \mathbb{C}$. Then the order (also called: multiplicity) of the zero x_0 is the greatest number $n \in \mathbb{N}$ such that p is divisible by $(x-x_0)^n$.

For sake of completeness, we say that the “order of the zero x_0 is zero, if x_0 is not a zero of p”.

As the following picture shows, a zero of even order touches the x-axis, while a zero of odd order crosses the x-axis. Finally, we present a result which is classical in polynomial algebra. It states that any nonconstant complex polynomial can be represented as a product of linear factors. Algebraists call this property the *algebraic closedness of* $\mathbb{C}$.

Theorem 4.24. Fundamental Theorem of Algebra

Any nonconstant $p \in \mathbb{C}[x]$ with $p(x) = \sum_{k=0}^{n} a_k x^k$, $n \geq 1$, $a_n \neq 0$ has a representa-

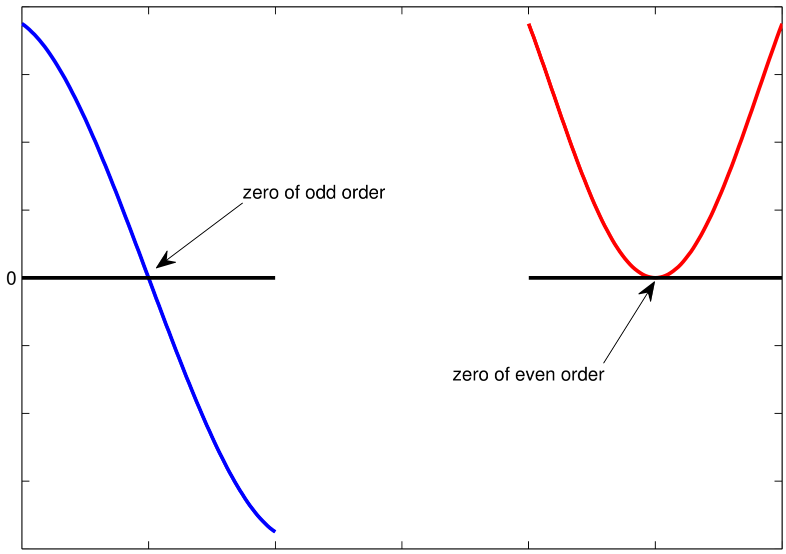


Figure 4.12: Qualitative behavior of even/odd order zeros of polynomials

tion

$$p(x) = a_n \prod_{k=1}^{n} (x - c_k)$$

for some $c_1, \ldots, c_n \in \mathbb{C}$.

The central question about polynomials is:

How can we find (compute) zeros?

This question is by far not simple for arbitrary polynomials, since there is no "universal strategy" for this. However, for polynomials of degree at most two, we can give simple explicit formulas:

- **Polynomials of degree 1:** For $p(x) = a_1 x + a_0$ with $a_1 \neq 0$, the only zero is obviously given by $x_1 = -\frac{a_0}{a_1}$.
- **Polynomials of degree 2:** For $p(x) = a_2 x + a_1 x + a_0$ with $a_2 \neq 0$, the zeros $x_1, x_2 \in \mathbb{C}$ are given by

$$\begin{aligned} x_1 &= \frac{-a_1 + \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}, \\ x_2 &= \frac{-a_1 - \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}. \end{aligned}$$

Note that for $p(x) = x^2 + px + q$, the expressions for the zeros read

$$x_{1/2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

In high school mathematics, this has the pictorial name of "*pq* Formula".
Note that in the case where $a_1^2 - 4a_0a_2$ is a negative real number, the square root has to be understood as the complex number(s) whose squaring gives $a_1^2 - 4a_0a_2$. For instance, the zeros of the polynomial $p(x) = x^2 + 1$ are given by

$$x_{1/2} = \pm\sqrt{-1} = \pm i.$$

Indeed, we have the factorization

$$x^2 + 1 = (x + i)(x - i).$$

- **Polynomials of degree higher than 2:** For polynomials of degree 3 or 4, there exist formulas for the roots (*Cardano's formulas*). However, these are really complicated and hardly applicable in concrete cases. For polynomials of degree higher than 4, it can be even shown that there do not exist any explicit formulas for the zeros.
  How can we nevertheless find zeros of polynomials of higher degree?
  Let us start with an example $p(x) = x^3 + x^2 - 6x + 4$. First we just try "some canonical candidates" for zeros, such as $-1, 0, 1, 2, \ldots$. Plugging this into p, we see that $p(1) = 0$, i.e., $x_1 = 1$ is a zero. As a consequence, p admits a factorization $p(x) = (x - 1) \cdot q(x)$ for some polynomial q. By the formulas for the degree of products of polynomials, we get $\deg q = 2$. The polynomial q can now be obtained by polynomial division, i.e.,

$$\begin{array}{rrrrl}
(x^3 & +x^2 & -6x & +4 &) : (x-1) \quad = \quad x^2 + 2x - 4 \qquad \text{Remainder: } 0 \\
x^3 & -x^2 & & & \\
\hline
& 2x^2 & -6x & & \\
& 2x^2 & -2x & & \\
\hline
& & -4x & +4 & \\
& & -4x & +4 & \\
\hline
& & & 0. &
\end{array}$$

Altogether, we now have $p(x) = (x^2 + 2x - 4) \cdot (x - 1)$. The zeros of p are therefore given by $x_1 = 1$ and the set of zeros of the polynomial $q(x) = x^2 + 2x - 4$. The latter ones can now be obtained by the *pq* formula, i.e.,

$$x_{2/3} = -1 \pm \sqrt{1 + 4} = -1 \pm \sqrt{5}.$$

The set of zeros is therefore given by

$$x_1 = 1, \qquad x_2 = -1 + \sqrt{5}, \qquad x_3 = -1 - \sqrt{5}.$$

The above strategy can be used for arbitrary polynomials as long as one has enough successful guesses for zeros. In exercises (or examinations), there is oftentimes a hint given for such guesses.

4.5.2 Rational Functions

Definition 4.25.

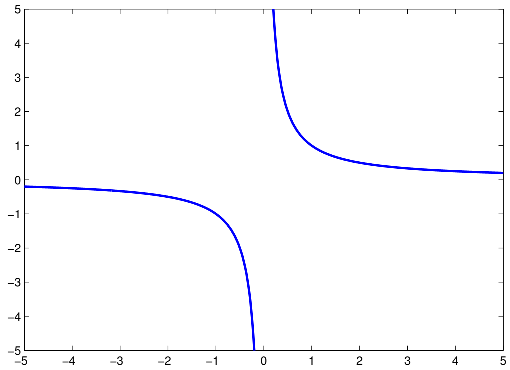
Let $p, q \in \mathbb{F}[x]$, where q is not the constant zero polynomial. Then the function $f : D(f) = \{x \in \mathbb{F} : q(x) \neq 0\} \to \mathbb{F}$ with

$$f(x) = \frac{p(x)}{q(x)}$$

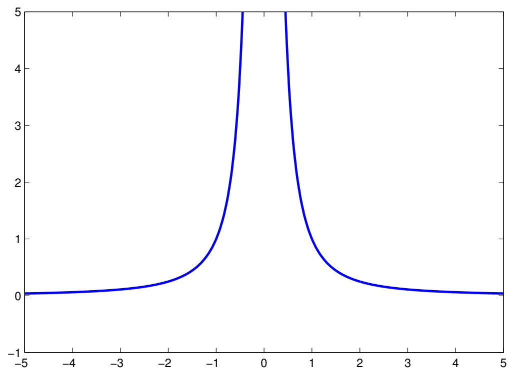
is called rational function.

It can be verified that for rational functions f, g, the functions $f + g$, $f - g$, $f \cdot g$, $\frac{f}{g}$ are again rational.

Example 4.26. a) $f(x) = \frac{1}{x}$



b) $f(x) = \frac{1}{x^2}$



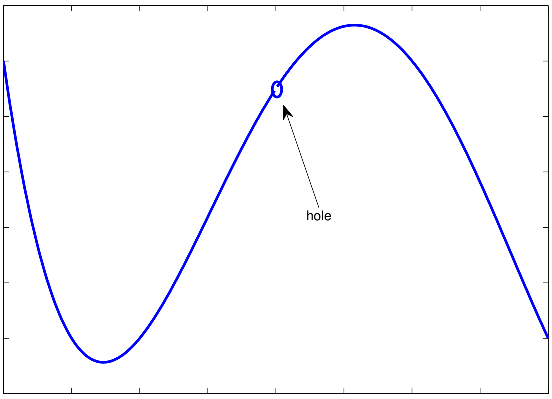
We will now take a closer look at the places where f is not defined, i.e., the zeros of the denominator polynomial.
Let $f = \frac{p}{q}$ be given, let x_0 be a zero of q and let $r \in \mathbb{N}_0$, $s \in \mathbb{N}$ be the multiplicities of the zero x_0 of p and q, respectively. This means that we have factorizations

$$p(x) = (x - x_0)^r p_1(x), \qquad q(x) = (x - x_0)^s q_1(x)$$

for some $p_1, q_1 \in \mathbb{F}[x]$ with $p_1(x_0) \neq 0$ and $q_1(x_0) \neq 0$. This means that for $x \in D(f)$ we have

$$f(x) = \frac{p(x)}{q(x)} = \frac{(x-x_0)^r p_1(x)}{(x-x_0)^s q_1(x)} = (x-x_0)^{r-s}\frac{p_1(x)}{q_1(x)}.$$

Next we distinguish between several cases for s and r:
1. Case: $r \geq s$:
"f has a hole at x_0"



2. Case: $r < s$:
f has a *pole of order* $s - r$ at x_0.

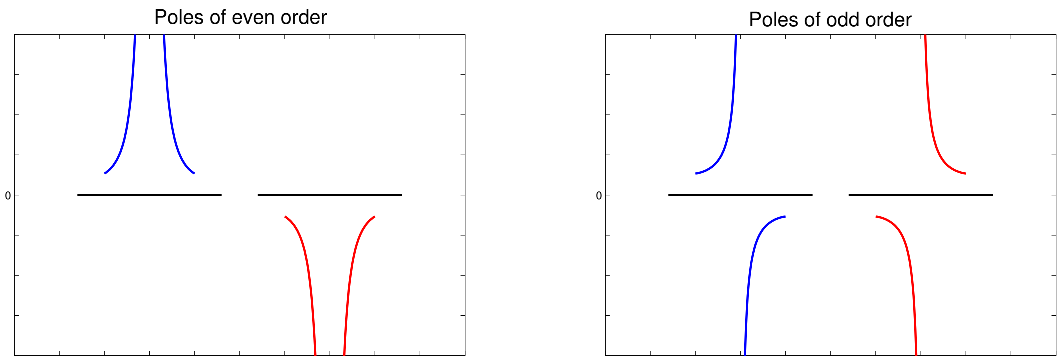


Figure 4.13: Qualitative behavior of even/odd order poles of rational functions

Example 4.27. a) The rational function $f(x) = \frac{x^2}{x}$, defined on $\mathbb{R}\backslash\{0\}$, has a hole at $x_0 = 0$.

b) The rational function $f(x) = \frac{x}{x^2}$, defined on $\mathbb{R}\backslash\{0\}$, has a pole of first order at $x_0 = 0$.

Definition 4.28.

A rational function $f = \frac{p}{q}$ *is called*

- *proper, if* $\deg p \leq \deg q$*;*
- *strictly proper, if* $\deg p < \deg q$.

Properness of a rational function can be equivalently characterized via the existence of the limit

$$\lim_{x\to\infty} f(x).$$

This limit is furthermore zero if and only if f is strictly proper.

Theorem 4.29.

Any rational function can be represented as a sum of a polynomial and a strictly proper rational function.

Proof: Let $f = \frac{p}{q}$. Applying polynomial division, we obtain that there exist $g, r \in \mathbb{F}[x]$ with $\deg r < \deg q$ and $p = qg + r$. Division by q gives

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)}.$$

Since the latter addend is a strictly proper rational function, the result is proven.

□

Next we consider a special representation of rational functions called *partial fraction decomposition*. This will be useful in particular for integration of rational functions.

Definition 4.30.

For $A, x_0 \in \mathbb{C}$, $k \in \mathbb{N}$, *the rational function*

$$\frac{A}{(x - x_0)^k}$$

is called partial fraction.

Next we present that any strictly proper rational function has a representation as sum of partial fractions. Note that for rational functions which are not strictly proper, we first have to perform an additive decomposition into a polynomial and a strictly proper rational function according to Theorem 4.29.

Theorem 4.31.

Let polynomials $p(x) = a_n x^n + \ldots + a_1 x + a_0$, $q(x) = b_m x^m + \ldots + b_1 x + b_0$ *with* $\deg(p) = n < m = \deg(q)$ *be given. Assume that* q *has a representation*

$$q(x) = b_m \prod_{j=0}^{k} (x - x_j)^{k_j}, \qquad \sum_{j=0}^{k} k_j = m$$

for some pairwise distinct x_j *(i.e.,* $x_i \neq x_j$ *for* $i \neq j$*).*

Then the rational function $f(x) = \frac{p(x)}{q(x)}$ has a representation

$$f(x) = \sum_{j=0}^{k} \sum_{l=1}^{k_j} \frac{A_{jl}}{(x - x_j)^l}.$$

This representation is called partial fraction decomposition.

Note that a combination of the above result with Theorem 4.29 yields that any rational function can be represented as the sum of a polynomial and some partial fractions.

The above theorem looks more complicated then it actually is. The polynomial q has oftentimes only simple multiplicities, i.e., $k_j = 1$ for all j. In that case, the double sum becomes a single sum of the form

$$f(x) = \sum_{j=0}^{k} \frac{A_{j1}}{x - x_j}.$$

We will give some examples. After that, we discuss how to compute the coefficients A_{lj}.

Example 4.32. (a)

$$f(x) = \frac{x+1}{x^2+1}$$

Since $x^2 + 1 = (x+i)(x-i)$, we have a partial fraction decomposition of the form

$$f(x) = \frac{A_{11}}{x+i} + \frac{A_{21}}{x-i}.$$

(b)

$$f(x) = \frac{1}{x^3 - 3x - 2}.$$

By the determination of the zeros of the denominator polynomial, we obtain a factorization $x^3 - 3x - 2 = (x+1)^2(x-2)$. Therefore, there exists a partial fraction decomposition of the form

$$f(x) = \frac{A_{11}}{x+1} + \frac{A_{12}}{(x+1)^2} + \frac{A_{21}}{x-2}.$$

For the computation of the coefficients, we make use of the fact that two polynomials coincide if and only if all their coefficients coincide. This technique is called *comparison of coefficients*.

Using this, we can make the following ansatz in Example a):

$$\frac{x+1}{x^2+1} = \frac{A_{11}}{x+i} + \frac{A_{21}}{x-i}.$$

Multiplying this equation by $(x+i)(x-i)$ from both sides gives

$$x + 1 = A_{11}(x-i) + A_{21}(x+i) = (A_{11} + A_{21})x + (-iA_{11} + iA_{21}).$$

A comparison of coefficients then leads to the following system of linear equations

$$\begin{aligned} A_{11} + A_{21} &= 1, \\ -iA_{11} + iA_{21} &= 1, \end{aligned}$$

i.e.

$$\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This leads to the solution $A_{11} = \frac{1+i}{2}$, $A_{21} = \frac{1-i}{2}$. Therefore, we have a partial fraction decomposition

$$\frac{x+1}{x^2+1} = \frac{\frac{1+i}{2}}{x+i} + \frac{\frac{1-i}{2}}{x-i}.$$

For the second example, we get

$$\frac{1}{(x+1)^2(x-2)} = \frac{A_{11}}{x+1} + \frac{A_{12}}{(x+1)^2} + \frac{A_{21}}{x-2}$$

and thus

$$\begin{aligned} 1 =& A_{11}(x+1)(x-2) + A_{12}(x-2) + A_{21}(x+1)^2 \\ =& (A_{11} + A_{21})x^2 + (-A_{11} + A_{12} + 2A_{21})x + (-2A_{11} - 2A_{12} + A_{21}) \end{aligned}$$

and thus we get the linear system

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{12} \\ A_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since the solution is given by $A_{11} = -\frac{1}{9}$, $A_{12} = -\frac{1}{3}$, $A_{21} = \frac{1}{9}$, we have

$$\frac{1}{(x+1)^2(x-2)} = -\frac{\frac{1}{9}}{x+1} - \frac{\frac{1}{3}}{(x+1)^2} + \frac{\frac{1}{9}}{x-2}.$$

In the following, we present a nice trick to compute some of the coefficients without solving a linear system of equations.

Theorem 4.33.

Let the assumptions of Theorem 4.31 be valid. Then the coefficients A_{i,k_i} are given by

$$A_{i,k_i} = \frac{p(x_i)}{b_m \prod\limits_{j=0, j\neq i}^{k} (x_i - x_j)^{k_j}}$$

Again, this formula looks more complicated than it really is. The determination of A_{i,k_i} can be done as follows:

- Keep the factor $(x - x_i)^{k_i}$ shut.
- Plug x_i into the remaining part.

In german, this method is known as "Zuhaltemethode". A native english speaking professor told me that there is no translation of this word.
As a consequence, we do not have to solve a linear system in the case where all zeros of the denominator polynomial have order one. However, even the order of the zeros are higher it is at least possible to reduce the order of the resulting linear system, since the coefficients belonging to the partial fractions involving the biggest powers can be calculated by this method. We now apply this method to the above two examples.

Example 4.34. (a)

$$f(x)=\frac{x+1}{x^2+1}=\frac{x+1}{(x+i)(x-i)}=\frac{A_{11}}{x+i}+\frac{A_{21}}{x-i}.$$

The determination of A_{11} can be done by

$$A_{11}=\left.\frac{x+1}{\cancel{(x+i)}\,(x-i)}\right|_{x=-i}=\frac{-i+1}{-i-i}=\frac{1+i}{2}$$

and A_{21} is given by

$$A_{21}=\left.\frac{x+1}{(x+i)\,\cancel{(x-i)}}\right|_{x=i}=\frac{i+1}{i+i}=\frac{1-i}{2}.$$

This is the same result that we obtained by solving the associated linear system.

(b)

$$f(x)=\frac{1}{(x+1)^2(x-2)}=\frac{A_{11}}{x+1}+\frac{A_{12}}{(x+1)^2}+\frac{A_{21}}{x-2}.$$

We now compute the coefficients A_{12} and A_{21} by the formula given in Theorem 4.33.

$$\begin{aligned}
A_{12}&=\left.\frac{1}{\cancel{(x+1)^2}\,(x-2)}\right|_{x=-1}&&=-\frac{1}{3},\\
A_{21}&=\left.\frac{1}{(x+1)^2\,\cancel{(x-2)}}\right|_{x=2}&&=\frac{1}{9}.
\end{aligned}$$

For the coefficient A_{11} now we only need to solve the reduced system

$$1+\frac{1}{3}(x-2)-\frac{1}{9}(x+1)^2=A_{11}(x+1)(x-2).$$

4.6 Power Series

Very roughly speaking, power series are "infinite polynomials". A precise definition is the following:

Definition 4.35.

Let a sequence $(a_k)_{k\in\mathbb{N}}$ in $\mathbb{F}$ be given and let $x_0 \in \mathbb{F}$. Then the function $f : D(f) \to \mathbb{F}$ defined by the series

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

is called power series.
The set

$$D(f) := \left\{ x \in \mathbb{F} \ : \ \sum_{k=0}^{\infty} a_k (x - x_0)^k \text{ is convergent} \right\}$$

is called domain of convergence. The domain of convergence at least includes x_0 since $\sum_{k=0}^{\infty} a_k (x_0 - x_0)^k = a_0$.

We have already seen several examples of power series in this chapter.

Example 4.36. a) The exponential function is defined via the power series

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

i.e., $(a_k)_{k\in\mathbb{N}} = (\frac{1}{k!})_{k\in\mathbb{N}}$ and $x_0 = 0$. Here $D(f) = \mathbb{C}$.

b) The sine function is defined via the power series

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

i.e., $(a_k)_{k\in\mathbb{N}} = (0, \frac{1}{1!}, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, 0, -\frac{1}{7!}, \ldots)_{k\in\mathbb{N}}$ and $x_0 = 0$. Again $D(f) = \mathbb{C}$.

c) cos, cosh, sinh are defined via the power series...

d) The function

$$f(x) = \sum_{k=1}^{\infty} \frac{(x-1)^k}{k}$$

is a power series.

Next we characterise the domain of convergence.

Theorem 4.37. Theorem of Cauchy-Hadamard

Let a power series

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

be given. Let

$$r := \frac{1}{\limsup\limits_{k\to\infty} \sqrt[k]{|a_k|}},$$

where we formally define $1/\infty := 0$ and $1/0 := \infty$. Then for all $x \in \mathbb{F}$ with $|x - x_0| < r$ holds $x \in D(f)$. Furthermore, for all $x \in \mathbb{F}$ with $|x - x_0| > r$ holds

$x \notin D(f)$.
The number r as defined above is called the radius of convergence*.*

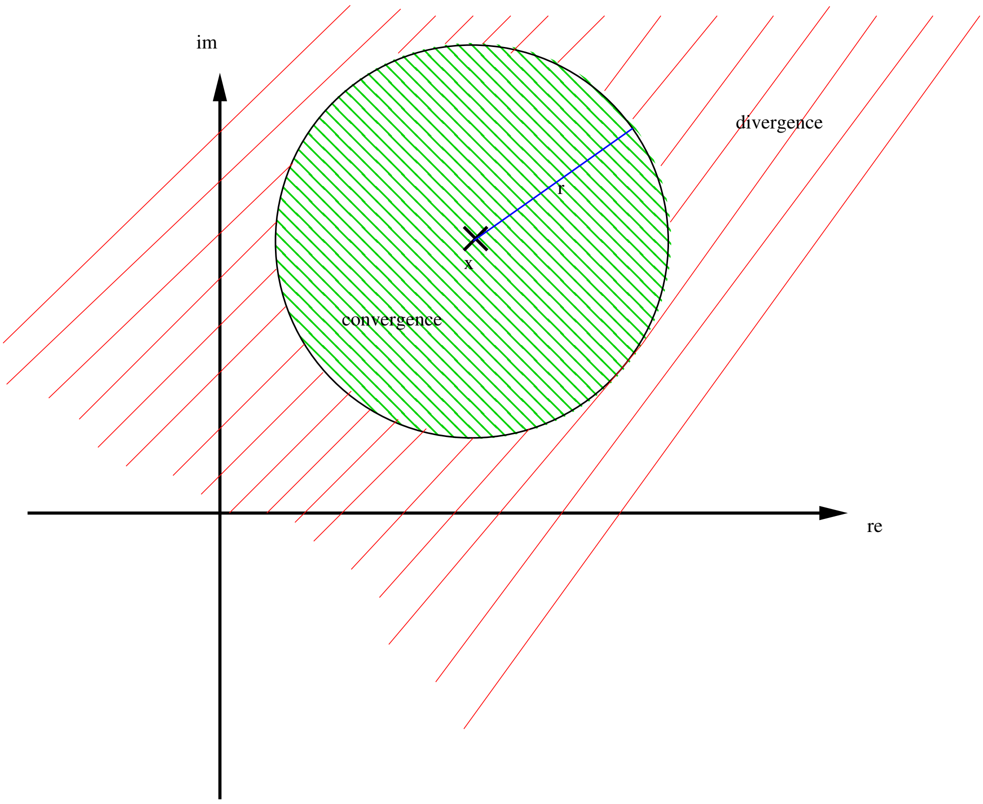


Figure 4.14: Domain of convergence

Proof: We have to show the following two statements:

(i) For all $x \in \mathbb{F}$ with $|x - x_0| \cdot \limsup_{k\to\infty} \sqrt[k]{|a_k|} < 1$, the power series is convergent.

(ii) For all $x \in \mathbb{F}$ with $|x - x_0| \cdot \limsup_{k\to\infty} \sqrt[k]{|a_k|} > 1$, the power series is divergent.

Statement (i) just follows from the limit form of the root criterion (Theorem 2.19), namely

$$\limsup_{k\to\infty} \sqrt[k]{|(x - x_0)^k a_k|} = |x - x_0| \cdot \limsup_{k\to\infty} \sqrt[k]{|a_k|} < 1.$$

For showing (ii), we also make use of the formula

$$\limsup_{k\to\infty} \sqrt[k]{|(x - x_0)^k a_k|} = |x - x_0| \cdot \limsup_{k\to\infty} \sqrt[k]{|a_k|} > 1.$$

This implies that the sequence $(x - x_0)^k a_k$ does not converge to 0 and therefore, the power series cannot converge. □

Geometrically, the above result implies that for all x inside a circle with midpoint x_0 and radius r, the series is convergent and outside this circle, we have divergence.
The Cauchy-Hadamard Theorem characterizes convergence/divergence of the power series in dependence of x whether it is inside or outside the circle around x_0 with radius r. In the case $|x - x_0| = r$, this result does not tell us anything. Indeed, we may have points on the circle with $x \in D(f)$ and also points on the circle with $x \notin D(f)$.

To see this, let us reconsider Example 4.36 d):
The radius of convergence is given by

$$r = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|\frac{1}{k}|}} = 1.$$

So we have convergence for all $x \in (0,2)$ and divergence for all $x \in (-\infty, 0) \cup (2, \infty)$. The remaining real points which are not characterized by the Theorem of Cauchy-Hadamard are $x = 0$ and $x = 2$. In the case $x = 0$, we obtain the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k}$$

which is convergent by the Leibniz criterion. Plugging in $x = 2$, the power series becomes a harmonic series

$$\sum_{k=0}^{\infty} \frac{1}{k}$$

that is well-known to be divergent.

Example 4.38. a) For the power series defined by the exponential function, we have

$$(a_k)_{k\in\mathbb{N}} = \left(\frac{1}{k!}\right)_{k\in\mathbb{N}}, \qquad x_0 = 0.$$

The radius of convergence is then given by

$$r = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|\frac{1}{k!}|}} = \frac{1}{0} = \infty.$$

As a consequence, the series converges for every $x \in \mathbb{C}$. The same holds true for the series of sin, cos, sinh, cosh.

b) As we have already seen above, the radius of convergence of the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k}$$

is $r = 1$.

c) Consider the power series

$$f(x) = \sum_{k=0}^{\infty} k! x^k.$$

The radius of convergence is given by

$$r = \frac{1}{\limsup\limits_{k\to\infty} \sqrt[k]{|k!|}} = \frac{1}{\infty} = 0.$$

So this series is divergent for any $x \neq 0$.

Sometimes the computation of the radius of convergence $r = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ of a power series $\sum_{n=1}^{\infty} a_n (x - x_0)^n$ is quite difficult. In such cases the following theorem might be better suited, which follows from the quotient criterion and is stated without proof.

Theorem 4.39.

Suppose that $\sum_{n=1}^{\infty} a_n(x-x_0)^n$ *is a power series with coefficients* $a_n \in \mathbb{F}$ *such that* $a_n \neq 0$ *for all* $n \geq N$ *with fixed* $N \in \mathbb{N}$. *If* $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$ *exists in* $\mathbb{R} \cup \{+\infty\}$ *than it is the radius of convergence of the power series.*

5 Differentiation of Functions

> For instance, on the planet Earth, man had always assumed that he was more intelligent than dolphins because he had achieved so much—the wheel, New York, wars and so on—while all the dolphins had ever done was muck about in the water having a good time. But conversely, the dolphins had always believed that they were far more intelligent than man—for precisely the same reasons.
>
> Douglas Adams, The Hitchhiker's Guide to the Galaxy

5.1 Differentiability and Derivatives

To motivate the problem, consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ and $g(x) = |x|$. As we already know, both these functions are continuous. Let us now focus

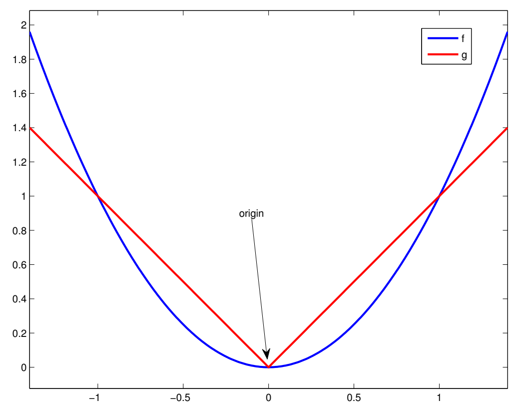


Figure 5.1: Domain of convergence

on the qualitative behavior of the functions at the origin.

f	g
smooth	sharp bend
there is a unique tangent	no unique tangent

Table 5.1: Qualitative behavior of f and g at the origin

A straight line $y(t)$ going through the points $(x_0, f(x_0))$ and $(x, f(x))$ is called *secant* [1] of f through these points. It is given by

$$y(t) = f(x_0) + \frac{f(x_0) - f(x)}{x_0 - x}(t - x_0).$$

In particular, the *slope* of y is the *difference quotient*

$$\frac{f(x_0) - f(x)}{x_0 - x} = \frac{f(x) - f(x_0)}{x - x_0}.$$

If we now let x tend to x_0, we obtain a tangent of f at x_0. This leads to the following

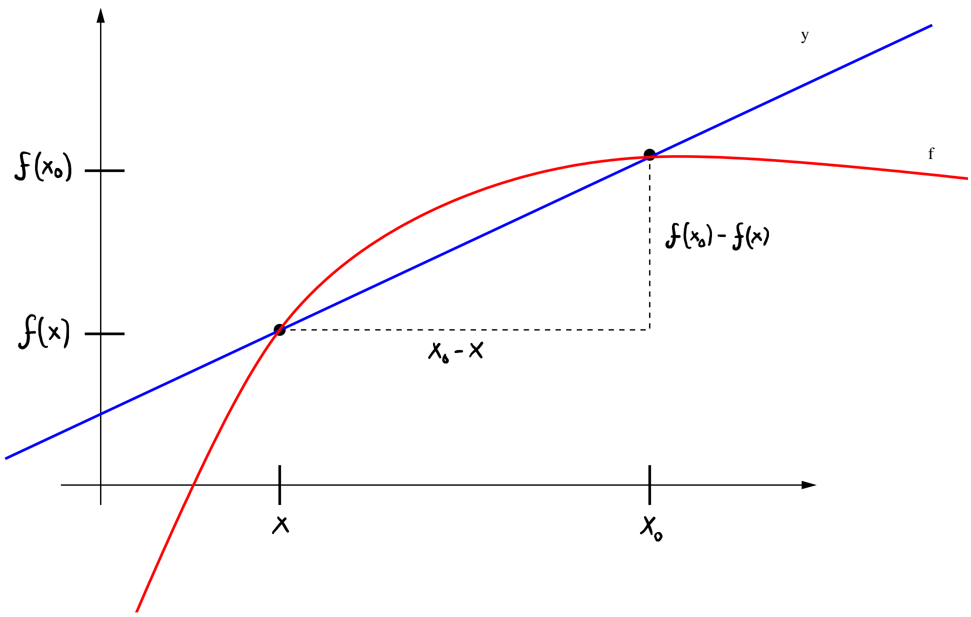


Figure 5.2: Secant of a function

definition.

Definition 5.1.

Let $I \subset \mathbb{R}$ be an interval with more than one point or an open set. Let $f : I \to \mathbb{R}$ be a function. Then f is called differentiable at $x_0 \in I$ if there exists a function

[1] from the Latin word *secare* = "to cut"

$\Delta_{f,x_0} : I \to \mathbb{R}$ that is continuous in x_0 and, moreover, for all $x \in I$ holds

$$f(x) = f(x_0) + (x - x_0) \cdot \Delta_{f,x_0}(x).$$

The number $\Delta_{f,x_0}(x_0)$ is called derivative of f at x_0.
The function f is called differentiable in I if it is differentiable at all $x_0 \in I$.

By solving the above equation for $\Delta_{f,x_0}(x)$, we get for $x \neq x_0$ that

$$\Delta_{f,x_0}(x) = \frac{f(x) - f(x_0)}{x - x_0},$$

i.e., it is the difference quotient. Continuity of $\Delta_{f,x_0}(x)$ at x_0 is therefore equivalent to the existence of the limit

$$\lim_{x \to x_0} \Delta_{f,x_0}(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =: f'(x_0).$$

Also the following notation is used in the literature for $f'(x_0)$:

$\frac{d}{dx} f(x_0)$, $\frac{\partial}{\partial x} f(x_0)$, $\frac{df}{dx}|_{x=x_0}$, $\frac{\partial f}{\partial x}|_{x=x_0}$, $\partial_x f(x_0)$.

The next result states that differentiability is a stronger property than continuity.

Theorem 5.2.

Let $f : I \to \mathbb{R}$ be differentiable at $x_0 \in I$. Then f is continuous in x_0.

Proof: By writing

$$f(x) = f(x_0) + (x - x_0) \cdot \Delta_{f,x_0}(x),$$

the continuity of Δ_{f,x_0} at x_0 implies the continuity of f at x_0. □

As the following example shows, the opposite implication cannot be made, i.e., not every continuous function is differentiable.

Example 5.3. Consider the absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$. We already know that it is continuous. For the analysis of differentiability, we distinguish between three cases:
1st Case: $x_0 > 0$.
Then we have that $|x_0| = x_0$ and, moreover, for x in some neighbourhood of x_0 holds $|x| = x$. Therefore, we have

$$\lim_{x \to x_0} \frac{|x| - |x_0|}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1.$$

2nd Case: $x_0 < 0$.
Then we have that $|x_0| = -x_0$ and, moreover, for x in some neighbourhood of x_0 holds $|x| = -x$. Therefore, we have

$$\lim_{x \to x_0} \frac{|x| - |x_0|}{x - x_0} = \lim_{x \to x_0} \frac{-x + x_0}{x - x_0} = -1.$$

3rd Case: $x_0 = 0$.
Then the two sequences $(x_n)_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} = (-\frac{1}{n})_{n\in\mathbb{N}}$ both tend to $x_0 = 0$. However, we have

$$\lim_{n\to\infty} \frac{|x_n| - |x_0|}{x_n - x_0} = \lim_{n\to\infty} \frac{|\frac{1}{n}| - |0|}{\frac{1}{n} - 0} = 1$$

and

$$\lim_{n\to\infty} \frac{|y_n| - |x_0|}{y_n - x_0} = \lim_{n\to\infty} \frac{|-\frac{1}{n}| - |0|}{-\frac{1}{n} - 0} = -1.$$

Therefore, the limit

$$\lim_{x\to 0} \frac{|x| - |0|}{x - 0}$$

does not exist. Thus, $|\cdot|$ is not differentiable at $x_0 = 0$.

Note that, by a substitution $h := x - x_0$, the difference quotient can be reformulated as

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

The derivative $a := f'(x_0)$ of f in x_0 can be interpreted in the following way: The linear mapping $\varphi : \mathbb{R} \to \mathbb{R},\ x \mapsto ax$ fulfills

$$\lim_{h\to 0} \frac{|f(x_0 + h) - (f(x_0) + \varphi(h))|}{|h|} = \lim_{h\to 0} \left| \frac{f(x_0 + h) - f(x_0)}{h} - a \right| = 0.$$

This means that the affine linear mapping $t(h) := f(x_0) + \varphi(h) = f(x_0) + ah$, which actually is the tangent of f at x_0, approximates $f(x)$ linearly in a neighbourhood of x_0 in a best possible way.

This “local linearisation” can be generalised to functions between arbitrary normed $\mathbb{R}$-vector spaces.

Definition 5.4. Total Derivative

Let $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ be two normed $\mathbb{R}$-vector spaces and let U be an open subset of E. Then a function $f : U \to F$ is said to be differentiable in a point $x_0 \in U$ if there is a continuous linear function $\varphi : E \to F$ such that

$$\lim_{x\to x_0} \frac{||f(x) - f(x_0) - \varphi(x - x_0)||_F}{||x - x_0||_E} = 0$$

or equivalently if

$$\lim_{h\to 0} \frac{||f(x_0 + h) - f(x_0) - \varphi(h)||_F}{||h||_E} = 0.$$

In this case φ is called the (total) derivative of f in x_0 which is denoted by $f'(x_0)$. If f is differentiable in all points of U then f is called differentiable in U or just differentiable and $f' : U \to \mathcal{L}(E, F),\ x \mapsto f'(x)$ is called the derivative of f.

Note carefully that the total derivative $f'(x_0)$ is a linear function. If E and F are finite dimensional $\mathbb{R}$-vector spaces, say $E = \mathbb{R}^m$, $F = \mathbb{R}^n$ for some $m, n \in \mathbb{N}$, then $f'(x_0)$ can be identified with its matrix representation $M \in \mathbb{R}^{n,m}$ with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$. The matrix $M =: J_f(x_0)$ is called the Jacobian matrix of f in x_0. In the one-dimensional case $m = 1 = n$, $M \in \mathbb{R}^{1,1}$ is a real number which is given by

$$M = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \Delta_{f,x_0}(x_0).$$

This clearifies the connection between Definition 5.1 and Definition 5.4. Another special case is $E = \mathbb{R}$ and $F = \mathbb{C} \cong \mathbb{R}^2$. Then, for

$$f : I \to \mathbb{C},\ x \mapsto \mathrm{Re}(f(x)) + i\,\mathrm{Im}(f(x)) \cong (\mathrm{Re}(f(x)), \mathrm{Im}(f(x))^T$$

with $I \subset \mathbb{R}$ open, we have $M = (\mathrm{Re}(f(x))', \mathrm{Im}(f(x))')^T \in \mathbb{R}^{2,1}$ and like in the one-dimensional case we identify $f'(x_0)$ with M and therefore write

$$f'(x_0) := \mathrm{Re}(f(x))' + i\,\mathrm{Im}(f(x))' \cong (\mathrm{Re}(f(x))', \mathrm{Im}(f(x))')^T.$$

We remark that linear mappings between finite-dimensional normed $\mathbb{R}$-vector spaces are automatically continuous so that the continuity assumption of φ in Definition 5.4 can be dropped in this case. Without proof we state that, if the total derivative $f'(x_0)$ of f in x_0 exists, then it is uniquely determined and its existence also implies continuity of f in x_0 like it was shown in Theorem 5.2 for the one-dimesional case.

Many of the subsequent results carried out for the cases $E = \mathbb{R} = F$ or $E = \mathbb{R}$ and $F = \mathbb{C}$ can be generalised to arbitrary E and F in a straight forward manner and the proofs are analogous and sometimes become even clearer in terms of total derivatives. This will be part of the exercises.

Now we consider some examples of differentiable functions.

Example 5.5. a) Given is some constant $c \in \mathbb{R}$. Consider the constant function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = c$ for all $x \in \mathbb{R}$. Then for all $x_0 \in \mathbb{R}$ holds

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{c - c}{x - x_0} = 0.$$

b) Given is some constant $c \in \mathbb{R}$. Consider the linear function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = cx$ for all $x \in \mathbb{R}$. Then for all $x_0 \in \mathbb{R}$ holds

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{cx - cx_0}{x - x_0} = \lim_{x\to x_0} c\frac{x - x_0}{x - x_0} = c.$$

c) For determining the derivatives of exp, sinh, cosh, sin and cos, we first determine the following limit for $\lambda \in \mathbb{C}$:

$$\lim_{h\to 0} \frac{\exp(\lambda h) - 1}{h}.$$

By Theorem **??**, we know that for $h \in \mathbb{R}$ with $|\lambda h| < 2$

$$\exp(\lambda h) = 1 + \lambda h + r_2(\lambda h)$$

with $|r_2(\lambda h)| \leq |\lambda h|^2$. Therefore,

$$\lim_{h\to 0} \frac{\exp(\lambda h) - 1}{h} = \lim_{h\to 0} \frac{1 + \lambda h + r_2(\lambda h) - 1}{h} = \lim_{h\to 0} \left(\lambda + \frac{r_2(\lambda h)}{h}\right) = \lambda.$$

We can further conclude

$$\lim_{h\to 0} \frac{\exp(\lambda(x_0+h)) - \exp(\lambda x_0)}{h} = \exp(\lambda x_0) \cdot \lim_{h\to 0} \frac{\exp(\lambda h) - 1}{h} = \lambda \exp(\lambda x_0).$$

This has manifold consequences for the derivatives of exponential, hyperbolic and trigonometric functions:

$$\exp'(x_0) = \lim_{h\to 0} \frac{\exp(x_0+h) - \exp(x_0)}{h} = \exp(x_0),$$

i.e., $\exp' = \exp$.
We can further conclude that

$$\begin{aligned}
\sinh'(x_0) =& \lim_{h\to 0} \frac{\sinh(x_0+h) - \sinh(x_0)}{h} \\
=& \frac{1}{2} \lim_{h\to 0} \left(\frac{\exp(x_0+h) - \exp(x_0)}{h} + \frac{-\exp(-(x_0+h)) + \exp(-x_0)}{h} \right) \\
=& \frac{1}{2}(\exp(x_0) + \exp(-x_0)) = \cosh(x_0).
\end{aligned}$$

Analogously, we can show that $\cosh' = \sinh$. Now consider the trigonometric functions:

$$\begin{aligned}
\sin'(x_0) =& \lim_{h\to 0} \frac{\sin(x_0+h) - \sin(x_0)}{h} \\
=& \frac{1}{2i} \lim_{h\to 0} \left(\frac{\exp(i(x_0+h)) - \exp(ix_0)}{h} + \frac{-\exp(-i(x_0+h)) + \exp(-ix_0)}{h} \right) \\
=& \frac{1}{2i}(i\exp(ix_0) + i\exp(-ix_0)) = \cos(x_0)
\end{aligned}$$

and

$$\begin{aligned}
\cos'(x_0) =& \lim_{h\to 0} \frac{\cos(x_0+h) - \cos(x_0)}{h} \\
=& \frac{1}{2} \lim_{h\to 0} \left(\frac{\exp(i(x_0+h)) - \exp(ix_0)}{h} + \frac{\exp(-i(x_0+h)) - \exp(-ix_0)}{h} \right) \\
=& \frac{1}{2}(i\exp(ix_0) - i\exp(-ix_0)) \\
=& -\frac{1}{2i}(\exp(ix_0) - \exp(-ix_0)) = -\sin(x_0).
\end{aligned}$$

Now we consider rules for the derivatives of sums, products and quotients of functions.

Theorem 5.6. Summation Rule, Product Rule, Quotient Rule

Let $f, g : I \to \mathbb{R}$ be differentiable in $x_0 \in I$.

(i) Then $f + g$ is differentiable in x_0 with $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

(ii) Then $f \cdot g$ is differentiable in x_0 with $(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$.

(iii) If $g(x_0) \neq 0$, then $\frac{f}{g}(x_0)$ is differentiable in x_0 with

$$\left(\frac{f}{g}(x_0)\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.$$

Proof: Let $f(x) = f(x_0) + (x - x_0) \cdot \Delta_{f,x_0}(x)$, $g(x) = g(x_0) + (x - x_0) \cdot \Delta_{g,x_0}(x)$. Then

(i)
$$(f+g)(x) = f(x) + g(x) = (f+g)(x_0) + (x - x_0) \cdot (\Delta_{f,x_0}(x) + \Delta_{g,x_0}(x)).$$

(ii)
$$\begin{aligned}(f \cdot g)(x) =& (f(x_0) + (x - x_0) \cdot \Delta_{f,x_0}(x_0))(g(x_0) + (x - x_0) \cdot \Delta_{g,x_0}(x)) \\ =& f(x_0)g(x_0) + (x - x_0)(\Delta_{f,x_0}(x_0)g(x_0) + \Delta_{g,x_0}(x)f(x_0)) \\ &+ (x - x_0)^2\Delta_{f,x_0}(x_0)\Delta_{g,x_0}(x).\end{aligned}$$

Thus,

$$\begin{aligned}&\lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\ =& \lim_{x\to x_0} \frac{(x - x_0)(\Delta_{f,x_0}(x)g(x_0) + \Delta_{g,x_0}(x)f(x_0)) + (x - x_0)^2\Delta_{f,x_0}(x)\Delta_{g,x_0}(x)}{x - x_0} \\ =& \lim_{x\to x_0} ((\Delta_{f,x_0}(x_0)g(x_0) + \Delta_{g,x_0}(x)f(x_0) + (x - x_0)\Delta_{f,x_0}(x)\Delta_{g,x_0}(x)) \\ =& \Delta_{f,x_0}(x)g(x_0) + \Delta_{g,x_0}(x)f(x_0)\end{aligned}$$

(iii) For convenience, we assume that $f \equiv 1$ (the general result follows by an application of the product rule). Then

$$\frac{1}{g(x)} - \frac{1}{g(x_0)} = \frac{g(x_0) - g(x)}{g(x_0)g(x)} = -\frac{(x - x_0)\Delta_{g,x_0}(x)}{g(x_0)g(x)}$$

and thus

$$\lim_{x\to x_0} \frac{1}{x - x_0}\left(\frac{1}{g(x)} - \frac{1}{g(x_0)}\right) = -\frac{g'(x_0)}{g^2(x_0)}.$$

□

Example 5.7. a) For a constant c and a differentiable function $f : I \to \mathbb{R}$ we have

$$(cf)'(x) = c'f(x) + cf'(x) = cf'(x).$$

b) Let $n \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^n$. Then $f'(x) = nx^{n-1}$.
Induction basis: $n = 1$. $f'(x) = (1 \cdot x)' \underset{a)}{=} 1 = 1 \cdot x^{1-1}$.
Induction step: $n > 1$: Using the product rule we immediately get

$$f'(x) = (x^n)' = (x \cdot x^{n-1})' = x'(x^{n-1}) + x(x^{n-1})' \underset{\text{Ind. Hyp.}}{=} x^{n-1} + x \cdot (n-1)x^{n-2} = nx^{n-1}$$

c) $f(x) = x^{-n} = \frac{1}{x^n}$, $n \in \mathbb{N}$. Then for $x \in \mathbb{R}\backslash\{0\}$

$$f'(x) = \frac{-nx^{n-1}}{x^{2n}} = -n\frac{1}{x^{n+1}} = -nx^{-n-1}$$

is true.

d) $f(x) = x \cdot \exp(x)$. Then for $x \in \mathbb{R}$

$$f'(x) = 1 \cdot \exp(x) + x \cdot \exp(x) = (1+x)\exp(x).$$

e) For a real polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ holds

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2 x + a_1.$$

Now we introduce differentiation rules for composition of functions and inverse functions.

Theorem 5.8. Differentiation of inverse functions

Let $I, J \subset \mathbb{R}$ be intervals and let $f : I \to J$ be bijective. If f is differentiable in $x_0 \in I$ with $f'(x_0) \neq 0$ and if $f^{-1} : J \to I$ is continuous in $y_0 = f(x_0)$, then the inverse function f^{-1} is differentiable in $y_0 := f(x_0)$ with

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(f^{-1}(y_0))}.$$

Proof: Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in J with $y_n \neq y_0$ and $\lim_{n\to\infty} y_n = y_0$. Define $x_n := f^{-1}(y_n)$. Since f^{-1} is continuous, we have $\lim_{n\to\infty} x_n = x_0$ and also $x_n \neq x_0$ by bijectivity of f. Thus we conclude

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

□

Example 5.9. a) $g(x) = \sqrt[n]{x} = x^{\frac{1}{n}}$ is the inverse function of $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = x^n$. Then for $x > 0$ holds

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n(g(x))^{n-1}} = \frac{1}{n\sqrt[n]{x}^{n-1}} = \frac{1}{n}x^{-\frac{n-1}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

b) $\log : (0,\infty) \to \mathbb{R}$ is the inverse function of exp. Then for $x > 0$ holds

$$\log'(x) = \frac{1}{\exp'(\log(x))} = \frac{1}{\exp(\log(x))} = \frac{1}{x}.$$

c) $\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the inverse function of sin. Then for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ holds

$$\arcsin'(x) = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))}.$$

Since $\sin^2(y) + \cos^2(y) = 1$ and $\cos(y) \geq 0$ for $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have

$$\cos(y) = \sqrt{1 - \sin^2(y)}.$$

Therefore

$$\arcsin'(x) = \frac{1}{\sqrt{1 - \sin^2(\arcsin(x))}} = \frac{1}{\sqrt{1-x^2}}.$$

Theorem 5.10. Chain rule

Let I, J be intervals and $f : J \to \mathbb{R}$ and $g : I \to J$ be given. Assume that for $x_0 \in I$, g is differentiable in x_0 and f is differentiable in $g(x_0)$. Then the composition $f \circ g$ ($(f \circ g)(x_0) = f(g(x_0))$) is differentiable in x_0 with

$$(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0).$$

Proof: By assumption there are functions $\Delta_{g,x_0}(x) : I \to \mathbb{R}$ and $\Delta_{f,g(x_0)}(y) : J \to \mathbb{R}$ which are continuous in x_0 and $g(x_0)$ respectively such that

$$\begin{aligned} g(x) &= g(x_0) + (x - x_0)\Delta_{g,x_0}(x) \\ f(y) &= f(g(x_0)) + (y - g(x_0))\Delta_{f,g(x_0)}(y) \end{aligned}$$

for all $x \in I$ and $y \in J$. Thus

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) = f(g(x_0)) + (g(x) - g(x_0))\Delta_{f,g(x_0)}(g(x)) \\ &= f(g(x_0)) + (x - x_0)\Delta_{g,x_0}(x)\Delta_{f,g(x_0)}(g(x_0) + (x - x_0)\Delta_{g,x_0}(x)) \end{aligned}$$

for all $x \in I$. Hence the function

$$\Delta_{f\circ g}(x) := \Delta_{g,x_0}(x)\Delta_{f,g(x_0)}(g(x_0) + (x - x_0)\Delta_{g,x_0}(x))$$

fulfills

$$(f \circ g)(x) = (f \circ g)(x_0)) + (x - x_0)\Delta_{f\circ g}(x)$$

for all $x \in I$. As a composition of functions that are continuous in x_0 it is also continuous in x_0 with $\Delta_{f\circ g}(x_0) = \Delta_{g,x_0}(x_0)\Delta_{f,g(x_0)}(g(x_0)) = g'(x_0)f'(g(x_0))$. □

Example 5.11. a) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \sin(x^2)$, then $f'(x) = \cos(x^2) \cdot 2x$.

b) Let $a \in \mathbb{R}$. Consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = a^x = \exp(x \log(a))$, then

$$f'(x) = \exp(x \log(a)) \cdot \log(a) = a^x \cdot \log(a).$$

c) Let $a \in \mathbb{R}$. Consider $f : \mathbb{R}^+ \to \mathbb{R}$ with $f(x) = x^a = \exp(a \log(x))$, then

$$f'(x) = \exp(a \log(x)) \cdot \frac{a}{x} = x^a \cdot \frac{a}{x} = a \cdot x^{a-1}.$$

d) Consider $f : \mathbb{R}^+ \to \mathbb{R}$ with $f(x) = x^x = \exp(x \log(x))$, then

$$f'(x) = \exp(x \log(x)) \cdot \left(\frac{x}{x} + \log(x)\right) = x^x \cdot (1 + \log(x)).$$

5.2 Mean Value Theorems and Consequences

Definition 5.12.

Let I be an interval and $f : I \to \mathbb{R}$ be a function. Then $x_0 \in I$ is called local maximum (local minimum) if there exists some neighbourhood U of x_0 such that

$$f(x_0) = \max\{f(x) : x \in I \cap U\} \qquad (f(x_0) = \min\{f(x) : x \in I \cap U\}).$$

A number $x_0 \in I$ is called local extremum if it is a local maximum or a local minimum.

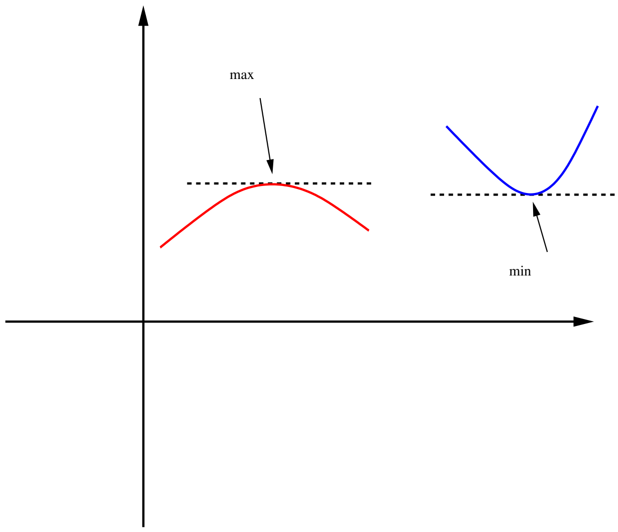


Figure 5.3: Local extrema

Theorem 5.13.

Let I be an interval and $f : I \to \mathbb{R}$ be a function that is differentiable in $x_0 \in I$. Assume that x_0 is an interior point of I and that x_0 is a local extremum. Then $f'(x_0) = 0$.

Proof: We assume that x_0 is a local maximum (the case of minimum is shown analogously). Let U be a neighbourhood of x_0 with $U \subset I$ and $f(x_0) = \max\{f(x) : x \in U\}$. Let

$$f(x) = f(x_0) + (x - x_0) \cdot \Delta_{f,x_0}(x).$$

Assume that $f'(x_0) = \Delta_{f,x_0}(x_0) > 0$. Since Δ_{f,x_0} is continuous in x_0, then there exists a neighbourhood $V \subset U$ of x_0 such that $\Delta_{f,x_0}(x) > 0$ for all $x \in V$. Then for all $x_1 \in V$ with $x_1 > x_0$ holds $f(x_1) = f(x_0) + (x_1 - x_0) \cdot \Delta_{f,x_0}(x_1) > f(x_0)$. This is a contradiction. On the other hand, assume that $f'(x_0) = \Delta_{f,x_0}(x_0) < 0$. Since Δ_{f,x_0} is continuous in x_0, then there exists a neighbourhood $V \subset U$ such that $\Delta_{f,x_0}(x) < 0$ for all $x \in V$. Then for all $x_1 \in V$ with $x_1 < x_0$ holds $f(x_1) = f(x_0) + (x_1 - x_0) \cdot \Delta_{f,x_0}(x_1) > f(x_0)$. This is also a contradiction. □

As a consequence, we will formulate the following result stating that derivatives of functions with equal boundary conditions have at least one zero.

Theorem 5.14. Theorem of Rolle

Let $f : [a,b] \to \mathbb{R}$ be differentiable in $[a,b]$ and let $f(a) = f(b)$. Then there exists some $x \in (a,b)$ such that $f'(x) = 0$.

Proof: If f is constant, the statement is clear (since then $f'(x) = 0$ for all $x \in (a,b)$). If f is not constant, consider the maximum and the minimum of f on $[a,b]$ (we know by Theorem 3.15 that they exist). So, let $x_-, x_+ \in [a,b]$ such that

$$f(x_+) = \max\{f(x) \; : \; x \in [a,b]\}, \qquad f(x_-) = \min\{f(x) \; : \; x \in [a,b]\}.$$

Then we have that $x_+ \in (a,b)$ or $x_- \in (a,b)$ since, otherwise, $f(x_+) = f(x_-)$ (constant). Then $f'(x_-) = 0$ or $f'(x_+) = 0$. □

As a corollary, we have that for a function f differentiable in some interval I, the following holds: Between two zeros of f, there always exists some point x_0 with $f'(x_0) = 0$.

Now we present the famous mean value theorem.

Theorem 5.15.

Let $f : [a,b] \to \mathbb{R}$ be differentiable. Then there exists some $\hat{x} \in (a,b)$ such that

$$f(b) - f(a) = f'(\hat{x}) \cdot (b - a).$$

Before the proof is presented, we give some graphical interpretation: A division of the above equation by $b - a$ gives

$$\frac{f(b) - f(a)}{b - a} = f'(\hat{x}).$$

The quantity on the left hand side is equal to the slope of the secant of f through a and b, whereas $f'(\hat{x})$ corresponds to the slope of tangent of f at $\hat{x}$. Therefore, the secant of f through a and b is parallel to a tangent of f.

Proof: Consider the function $F : [a,b] \to \mathbb{R}$ with

$$F(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a).$$

Then we have $F(a) = F(b) = 0$. By Rolle's Theorem, we get that there exists some $\hat{x} \in (a,b)$ with

$$0 = F'(\hat{x}) = f'(\hat{x}) - \frac{f(b) - f(a)}{b - a}$$

and thus

$$f'(\hat{x}) = \frac{f(b) - f(a)}{b - a}.$$

□

The mean value theorem leads us to determine monotonicity properties of a function by means of its derivative.

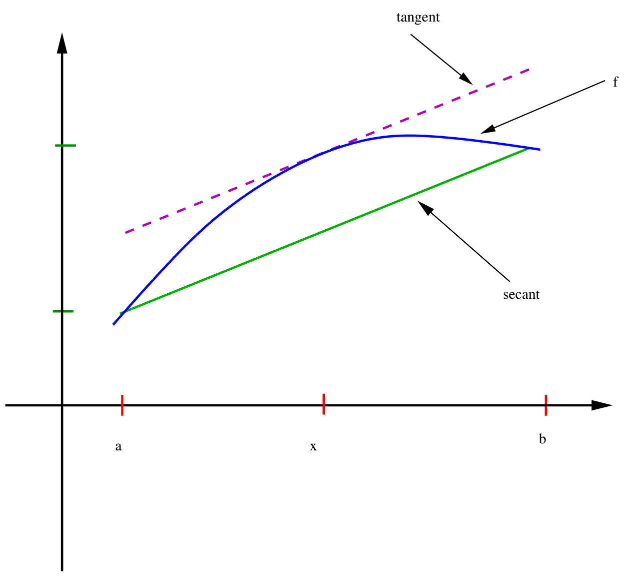


Figure 5.4: Mean value theorem

Theorem 5.16.

Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Then the following holds true.

(i) If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly monotonically increasing.

(ii) If $f'(x) < 0$ for all $x \in (a,b)$, then f is strictly monotonically decreasing.

(iii) If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is monotonically increasing.

(iv) If $f'(x) \leq 0$ for all $x \in (a,b)$, then f is monotonically decreasing.

Proof: (i) By the mean value theorem we have that for $x_1, x_2 \in (a,b)$ with $x_1 < x_2$, there exists some $\hat{x} \in (x_1, x_2)$ with

$$f(x_2) - f(x_1) = f'(\hat{x}) \cdot (x_2 - x_1) > 0.$$

The results (ii)-(iv) can be proven analogously. □

Now we consider a generalisation of the mean value theorem. This gives us a completely new tool for the determination of limits.

Theorem 5.17. Generalised mean value theorem

Let $f, g : [a,b] \to \mathbb{R}$ be differentiable. Assume that g' has no zero in (a,b). Then there exists some $\hat{x} \in (a,b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\hat{x})}{g'(\hat{x})}.$$

Proof: By introducing the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot (g(x) - g(a)),$$

we get $F(a) = F(b) = 0$. Now using the Theorem of Rolle, the result follows immediately. $\square$

Theorem 5.18. Theorem of l'Hospital

Let I be an interval and let $f, g : I \to \mathbb{R}$ be differentiable. Let $x_0 \in I$ and assume that $f(x_0) = g(x_0) = 0$ and there exists some neighbourhood $U \subset I$ of x_0 such that $g'(x) \neq 0$ for all $x \in U \backslash \{x_0\}$. Then, if $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists, then also $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $\lim_{n\to\infty} x_n = x_0$ and $x_n \neq x_0$ for all $n \in \mathbb{N}$. Then, by the generalised mean value theorem, there exists a sequence $(\hat{x}_n)_{n\in\mathbb{N}}$ with $\hat{x}_n$ between x_0 and x_n such that

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(x_0)}{g(x_n) - g(x_0)} = \frac{f'(\hat{x}_n)}{g'(\hat{x}_n)}.$$

In particular, since $(\hat{x}_n)_{n\in\mathbb{N}}$ converges to x_0, we have

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

$\square$

Example 5.19. a) Let $a \in \mathbb{R}$.

$$\lim_{x\to 0, ax>0} \frac{\log(1+ax)}{x} = \lim_{x\to 0, ax>0} \frac{a}{1+ax} = a.$$

b)

$$\lim_{x\to 0, ax>0} (1+ax)^{\frac{1}{x}} = \lim_{x\to 0, ax>0} \exp\left(\frac{\log(1+ax)}{x}\right) = \exp\left(\lim_{x\to 0, ax>0} \frac{\log(1+ax)}{x}\right) = \exp(a).$$

In particular, for $x = \frac{1}{n}$, we have

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = \exp(a).$$

c)

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{\sin(x)}{2x} = \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{1}{2}.$$

Theorem 5.20. Generalisations of l'Hospital's Theorem

- ***Expressions of type $\frac{\infty}{\infty}$, Limit as $x \to x_0$***
  Let $f, g : I\backslash\{x_0\} \to \mathbb{R}$ be differentiable functions with $\lim_{x\to x_0} f(x) = \infty$, $\lim_{x\to x_0} g(x) = \infty$. Then, if $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists, then also $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists

with

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}.$$

- ***Expressions of type $\frac{0}{0}$, Limit as $x\to\infty$***
Let $f,g:[t_0,\infty)\to\mathbb{R}$ be differentiable functions with $\lim_{x\to\infty}f(x)=0$, $\lim_{x\to\infty}g(x)=0$. Assume that there exists some $x_1\in\mathbb{R}$ such that $g'(x)\neq 0$ for all $x>x_1$. Then, if $\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$ exists, then also $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ exists with

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

- ***Expressions of type $\frac{\infty}{\infty}$, Limit as $x\to\infty$***
Let $f,g:[t_0,\infty)$ be differentiable functions with $\lim_{x\to\infty}f(x)=\infty$, $\lim_{x\to\infty}g(x)=\infty$. Then, if $\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$ exists, then also $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ exists with

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

Proof: The first result follows by an application of l'Hospital's Theorem to

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}=\lim_{x\to x_0}\frac{\frac{g'(x)}{(g(x))^2}}{\frac{f'(x)}{(f(x))^2}}=\frac{1}{\lim_{x\to x_0}\frac{f'(x)}{g'(x)}}\left(\lim_{x\to x_0}\frac{f(x)}{g(x)}\right)^2.$$

The second and third statement follow by a substitution $y=\frac{1}{x}$ and the consideration of

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{y\to 0}\frac{f\left(\frac{1}{y}\right)}{g\left(\frac{1}{y}\right)}=\lim_{y\to 0}\frac{-\frac{1}{y^2}f'\left(\frac{1}{y}\right)}{-\frac{1}{y^2}g'\left(\frac{1}{y}\right)}=\lim_{y\to 0}\frac{f'\left(\frac{1}{y}\right)}{g'\left(\frac{1}{y}\right)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

□

Note that we can also treat expressions of type "$\infty-\infty$" by l'Hospitals's Theorem. Namely, for f,g with $\lim_{x\to x_0}f(x)=0$, $\lim_{x\to x_0}g(x)=0$, we get that

$$\lim_{x\to x_0}\left(\frac{1}{f(x)}-\frac{1}{g(x)}\right)=\lim_{x\to x_0}\frac{g(x)-f(x)}{f(x)\cdot g(x)}=\lim_{x\to x_0}\frac{g'(x)-f'(x)}{f'(x)\cdot g(x)+f(x)\cdot g'(x)}.$$

Also, expressions of type "$0\cdot\infty$" can be treated by a special trick. Namely, for f,g with $\lim_{x\to x_0}f(x)=0$, $\lim_{x\to x_0}g(x)=\infty$, we get that

$$\lim_{x\to x_0}f(x)\cdot g(x)=\lim_{x\to x_0}\frac{f(x)}{\frac{1}{g(x)}}=\lim_{x\to x_0}\frac{f'(x)}{-\frac{g'(x)}{(g(x))^2}}=-\lim_{x\to x_0}\frac{f'(x)(g(x))^2}{g'(x)}.$$

You do not have to keep the above two formulas in mind. These can always be derived in concrete examples.

Example 5.21. a) Let $n\in\mathbb{N}$ and consider

$$\lim_{x\to\frac{\pi}{2},x<\frac{\pi}{2}}\left(x-\frac{\pi}{2}\right)^n\cdot\tan(x).$$

This is an expression of type "$0 \cdot \infty$" and we can make use of

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}, x<\frac{\pi}{2}} \left(x-\frac{\pi}{2}\right)^n \cdot \tan(x) &= \lim_{x\to\frac{\pi}{2}, x<\frac{\pi}{2}} \frac{\left(x-\frac{\pi}{2}\right)^n}{\frac{1}{\tan(x)}} \\
&= \lim_{x\to\frac{\pi}{2}, x<\frac{\pi}{2}} \frac{n\cdot\left(x-\frac{\pi}{2}\right)^{n-1}}{-\frac{1}{\tan^2(x)}\frac{1}{\cos^2(x)}} \\
&= \lim_{x\to\frac{\pi}{2}, x<\frac{\pi}{2}} \frac{n\cdot\left(x-\frac{\pi}{2}\right)^{n-1}}{-\frac{1}{\sin^2(x)}} \\
&= -\lim_{x\to\frac{\pi}{2}, x<\frac{\pi}{2}} n\cdot\left(x-\frac{\pi}{2}\right)^{n-1}\sin^2(x) = \begin{cases} -1 & \text{if } n=1, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

b)

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin(x)}-\frac{1}{x}\right) &= \lim_{x\to 0}\frac{x-\sin(x)}{x\sin(x)} \\
&= \lim_{x\to 0}\frac{1-\cos(x)}{\sin(x)+x\cos(x)} = \lim_{x\to 0}\frac{\sin(x)}{\cos(x)-x\sin(x)+\cos(x)} = 0.
\end{aligned}$$

5.3 Higher Derivatives, Curve Sketching

Here we consider derivatives of derivatives (of derivatives ...) and discuss consequences for the search for local extrema of functions.

Definition 5.22.

If the derivative of a function is differentiable, we call $(f')'$ the second derivative. The n-th derivative of a function is inductively defined as the derivative of the $n-1$-th derivative.
For the second derivative at x_0, we write $f''(x_0)$ or $\frac{d^2}{dx^2}f(x_0)$. The n-th derivative is denoted by $f^{(n)}(x_0)$ or $\frac{d^n}{dx^n}f(x_0)$.
We call a function $f : I \to \mathbb{R}$ n-times differentiable if $f^{(n)}(x)$ exists for all $x \in I$.
Furthermore we call a function $f : I \to \mathbb{R}$ n-times continuously differentiable if $f^{(n)} : I \to \mathbb{R}$ exists and is continuous.

Theorem 5.23.

Let $I := [a, b]$ and $f : I \to \mathbb{R}$ be differentiable. Furthermore let $x_0 \in I$ such that $f'(x_0) = 0$ and f' is differentiable in x_0. Then

1. *if $f''(x_0) > 0$, then f has a local minimum in x_0;*
2. *if $f''(x_0) < 0$, then f has a local maximum in x_0.*

Proof: We only show the case $f''(x_0) > 0$ (the opposite case is analogous). By definition, we have

$$f''(x_0) = \lim_{x\to x_0}\frac{f'(x)-f'(x_0)}{x-x_0} > 0.$$

Since f' is continuous in x_0, we have that there exists some $\varepsilon > 0$ such that for all $x \in I \backslash \{x_0\}$ with $|x - x_0| < \varepsilon$ holds

$$\frac{f'(x) - f'(x_0)}{x - x_0} > 0.$$

Since $f'(x_0) = 0$, we have that

$$\begin{aligned} f'(x) < 0 &\quad \text{for all } x \in (x_0 - \varepsilon, x_0), \\ f'(x) > 0 &\quad \text{for all } x \in (x_0, x_0 + \varepsilon). \end{aligned}$$

Therefore, f is monotonically decreasing in $(x_0 - \varepsilon, x_0)$ and monotonically increasing in $(x_0, x_0 + \varepsilon)$. Therefore, f has a local minimum in x_0. □

Remark:

Note that in the case $f''(x_0) = 0$, we cannot make a decision whether f has a local extremum there. For instance, consider the three functions $f_1(x) = x^3$, $f_2(x) = x^4$ and $f_3(x) = -x^4$. We have $f_1'(0) = f_2'(0) = f_3'(0) = 0$ and, furthermore, $f_1''(0) = f_2''(0) = f_3''(0) = 0$. However, f_1 has no local extremum in 0, f_2 has a local minimum in 0 and f_3 has a local maximum in 0.

Example 5.24.

1. Consider the rational function

$$f(x) = \frac{x(x+5)}{x-4} = x + 9 + \frac{36}{x-4}.$$

   It can be easily seen that f has a first order pole at $x_0 = 4$.
   The first two derivatives of f are given by

$$f'(x) = \frac{x^2 - 8x - 20}{(x-4)^2} = \frac{(x+2)(x-10)}{(x-4)^2}, \qquad f''(x) = \frac{72}{(x-4)^3}.$$

   The zeros of f are given by $x_1 = 0$ and $x_2 = -5$.
   Now we determine the set of local extrema: We have that $f'(x) = 0$ is only fulfilled for $x_3 = -2$ and $x_4 = 10$. In this case, we have $f''(x_3) = -\frac{1}{3} < 0$ and $f''(x_4) = \frac{1}{3} > 0$. As a consequence, f has a local maximum in $x_3 = -2$ and a local minimum in $x_4 = 10$. We further have

   a) $x \in (-\infty, -2) \;\Rightarrow f'(x) > 0$, i.e., f is strictly monotonically increasing in $(-\infty, -2)$;
   b) $x \in (-2, 4) \;\Rightarrow f'(x) < 0$, i.e., f is strictly monotonically decreasing in $(-2, 4)$;
   c) $x \in (4, 10) \;\Rightarrow f'(x) < 0$, i.e., f is strictly monotonically decreasing in $(4, 10)$;
   d) $x \in (10, \infty) \;\Rightarrow f'(x) > 0$, i.e., f is strictly monotonically increasing in $(10, \infty)$.

2. $f(x) = \sin(x)$. The zeros are given by

$$\{0, \pi, -\pi, 2\pi, -2\pi, 3\pi, -3\pi, \ldots\} = \{n\pi \mid n \in \mathbb{Z}\}.$$

   The first two derivatives are given by $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$. The zeros of f' are given by

$$\left\{\frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \frac{5\pi}{2}, \frac{5\pi}{2}, \ldots\right\} = \left\{\frac{2n+1}{2}\pi \mid n \in \mathbb{Z}\right\}.$$

For $x_n = \frac{2n+1}{2}\pi$, we have $f''(x_n) = -\sin(\frac{2n+1}{2}\pi) = (-1)^{n+1}$. As a consequence, sin has a local maximum in $x_n = \frac{2n+1}{2}\pi$ if n is even and a local minimum in $x_n = \frac{2n+1}{2}\pi$ if n is odd.

Definition 5.25. Convexity/Concavity

A function $f : [a, b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$, is called convex, if for all $x_1 < x < x_2$ in $[a, b]$ holds

$$f(x) \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} \cdot (x - x_1) + f(x_1) \ . \tag{5.1}$$

It is called concave, if for all $x_1 < x < x_2$ in $[a, b]$ holds

$$f(x) \ge \frac{f(x_2) - f(x_1)}{x_2 - x_1} \cdot (x - x_1) + f(x_1) \ . \tag{5.2}$$

If the inequalities in (5.1) or (5.2) are strict then f is called strictly convex/concave. Geometrically this means that the graph of a convex (concave) function $f : [a, b] \to \mathbb{R}$ restricted to any subinterval $[x_1, x_2]$ of $[a, b]$ lies below (above) the secant

$$s(x) := \frac{f(x_2) - f(x_1)}{x_2 - x_1} \cdot (x - x_1) + f(x_1) \ .$$

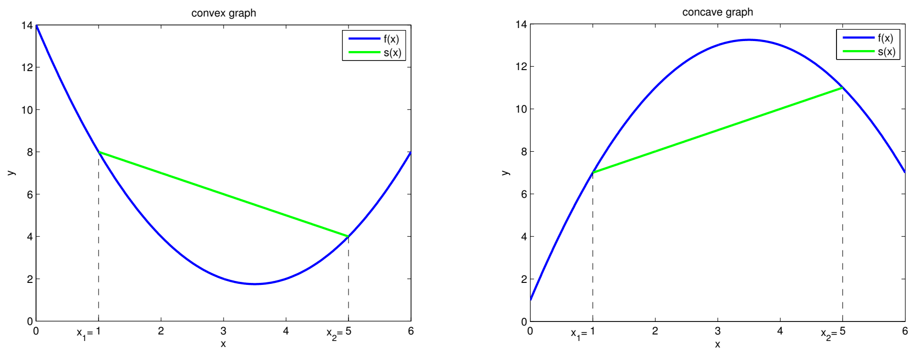


Figure 5.5: convex and concave graphs

Theorem 5.26.

Let $f : [a, b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$, be 2-times differentiable.

a) If $f''(x) \ge 0$ for all $x \in (a, b)$, then f is convex.

b) If $f''(x) > 0$ for all $x \in (a, b)$, then f is strictly convex.

c) If $f''(x) \le 0$ for all $x \in (a, b)$, then f is concave.

d) If $f''(x) < 0$ for all $x \in (a, b)$, then f is strictly concave.

Proof: We only prove a). The other results follow analogously. Let $x_1 < x < x_2$ in $[a, b]$. Since $f'' \ge 0$ we know that f' is monotonically increasing. By the intermediate value theorem there are $\xi_1 \in (x_1, x)$ and $\xi_2 \in (x, x_2)$ such that

$$\frac{f(x) - f(x_1)}{x - x_1} = f'(\xi_1) \leq f'(\xi_2) = \frac{f(x_2) - f(x)}{x_2 - x} .$$

This implies

$$(f(x) - f(x_1))(x_2 - x) \leq (f(x_2) - f(x))(x - x_1)$$

$$\Leftrightarrow \quad f(x)(x_2 - x_1) \leq (f(x_2) - f(x_1))(x - x_1) + f(x_1)(x_2 - x_1)$$
$$\Leftrightarrow \quad f(x) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1) .$$

□

Definition 5.27. Inflection point

Let $f : [a, b] \to \mathbb{R}$ be a function. We say that $x_0 \in (a, b)$ is an inflection point of f if there is an $\varepsilon > 0$ with $[x_0 - \varepsilon, x_0 + \varepsilon] \subset [a, b]$ such that one of the following two statements holds true:

a) f is convex on $[x_0 - \varepsilon, x_0]$ and concave on $[x_0, x_0 + \varepsilon]$.

b) f is concave on $[x_0 - \varepsilon, x_0]$ and convex on $[x_0, x_0 + \varepsilon]$.

Theorem 5.28.

Let $f : [a, b] \to \mathbb{R}$ be 3-times continuously differentiable and $x_0 \in (a, b)$.

a) If x_0 is an inflection point, then $f''(x_0) = 0$.

b) If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then x_0 is an inflection point.

Proof: a) follows from Theorem 5.26 and continuity of f''.

b) If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then by continuity of f''' there is an $\varepsilon > 0$ with $[x_0 - \varepsilon, x_0 + \varepsilon] \subset [a, b]$ such that f''' does not have a zero on $[x_0 - \varepsilon, x_0 + \varepsilon]$. This implies that f'' is strictly monotonic on $[x_0 - \varepsilon, x_0 + \varepsilon]$. Thus either

i) $f'' > 0$ on $[x_0 - \varepsilon, x_0]$ and $f'' < 0$ on $[x_0, x_0 + \varepsilon]$ or

ii) $f'' < 0$ on $[x_0 - \varepsilon, x_0]$ and $f'' > 0$ on $[x_0, x_0 + \varepsilon]$

holds true. In case i), f is convex on $[x_0 - \varepsilon, x_0]$ and concave on $[x_0, x_0 + \varepsilon]$ and in case ii) the reversed behavior is given. Therefore x_0 is an inflection point. □

The aim of a so-called *curve discussion* of a function $f : D \to \mathbb{R}$, $D \subset \mathbb{R}$, is to determine its qualitative and quantitative behaviour. We give a short list of things that have to be investigated/determined:

1. Domain of definition D
2. Symmetries
   a) f is symmetrical with respect to the y-axis if $f(x) = f(-x)$ for all x in the domain of definition. In this case f is called an even function.

   b) f is point-symmetrical with respect to the origin if $f(-x) = -f(x)$ for all x in the domain of definition. In this case f is called an odd function.

3. Poles
4. Behaviour for $x \longrightarrow \pm\infty$, asymptotes

   A straight line $g(x) = ax + b$, $a, b \in \mathbb{R}$, is called an asymptote of f for $x \to \pm\infty$ if $\lim_{x\to\pm\infty}(f(x) - (ax + b)) = 0$. In this case the coefficients a, b can be successively determined by

$$\begin{aligned} a &= \lim_{x\to\pm\infty} \frac{f(x)}{x} , \\ b &= \lim_{x\to\pm\infty} (f(x) - ax) . \end{aligned}$$

5. Zeros
6. Extrema, monotonicity behaviour
7. Inflection points, convexity/concavity behaviour
8. Function graph

Example 5.29. We want to give a complete curve discussion for the rational function

$$f(x) = \frac{2x^2 + 3x - 4}{x^2} .$$

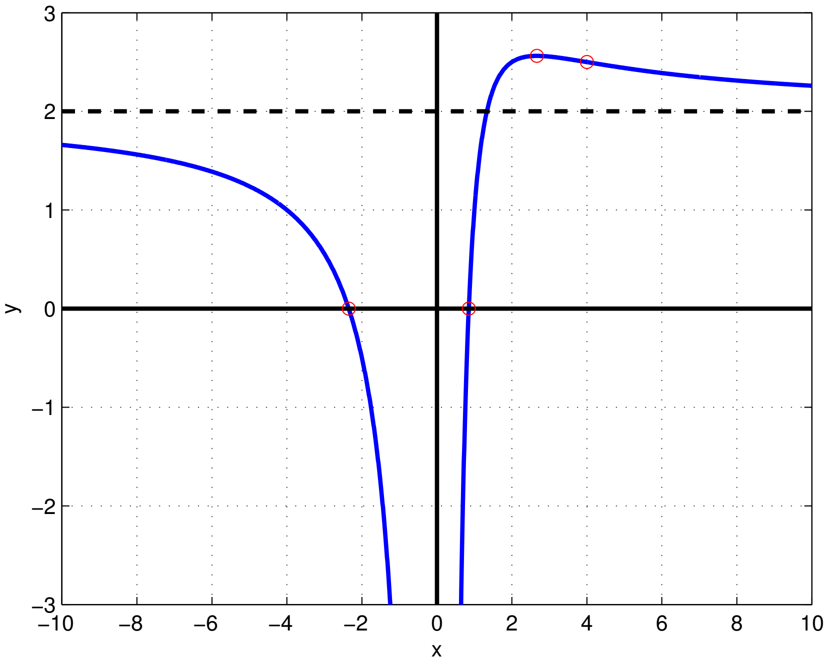
1. Domain of definition: $D = \mathbb{R}\backslash\{0\}$
2. Symmetries: f is neither an even nor an odd function.
3. Poles: $x_0 = 0$ is a pole of order 2, $\lim_{x\nearrow 0} f(x) = -\infty = \lim_{x\searrow 0} f(x)$
4. Behaviour for $x \longrightarrow \pm\infty$, asymptotes: $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$, $\lim_{x\to\pm\infty} f(x) = 2$. Thus the horizontal line at $y = 2$ is an asymptote of f for $x \longrightarrow \infty$ and also for $x \longrightarrow -\infty$.
5. Zeros: $f(x) = 0 \;\Leftrightarrow\; 2x^2 + 3x - 4 = 0 \;\Leftrightarrow\; x = x_{1,2} = \frac{1}{4}(-3 \pm \sqrt{41})$
   $x_1 \approx -2.35$, $x_2 \approx 0.85$
6. Extrema, monotonicity behaviour:
   $f'(x) = \frac{-3x+8}{x^3} = 0 \;\Leftrightarrow\; x = x_3 = \frac{8}{3}$, $y_3 := f(x_3) \approx 2.56$
   $f''(x) = \frac{6x-24}{x^4}$
   $f''(x_3) < 0 \;\Rightarrow\; f$ has a local maximum at x_3.

$$f'(x) \quad \begin{cases} < 0 & , \frac{8}{3} < x < \infty & \text{, strictly monotonically decreasing} \\ > 0 & , 0 < x < \frac{8}{3} & \text{, strictly monotonically increasing} \\ < 0 & , -\infty < x < 0 & \text{, strictly monotonically decreasing} \end{cases}$$

7. Inflection points, convexity/concavity behaviour:
   $f''(x) = 0 \;\Leftrightarrow\; x = x_4 = 4$, $y_4 = f(x_4) = \frac{5}{2}$
   $f''' = \frac{96-18x}{x^5}$
   $f'''(x_4) > 0 \;\Rightarrow\; x_4$ is an inflection point.

$$f''(x) \begin{cases} > 0 & , 4 < x < \infty & \text{, strictly convex} \\ < 0 & , 0 < x < 4 & \text{, strictly concave} \\ < 0 & , -\infty < x < 0 & \text{, strictly concave} \end{cases}$$

8. Function graph



5.4 Taylor's Formula

The aim of this part is to approximate a sufficiently smooth (that means sufficiently often differentiable) function by a polynomial. More precisely, we will perform the approximation by

$$f(x_0+h) \approx \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}h^k = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{6}h^3 + \ldots + \frac{f^{(n)}(x_0)}{n!}h^n.$$

We will also estimate the approximation error.

Theorem 5.30. Taylor's formula

Let I be an interval and assume that $f : I \to \mathbb{R}$ is $n+1$-times differentiable. Let $x_0 \in I$ and $h \in \mathbb{R}$ such that $x_0 + h \in I$. Then there exists some $\theta \in (0,1)$ such that

$$f(x_0+h) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}h^k}_{\text{“Taylor polynomial”}} + \underbrace{\frac{f^{(n+1)}(x_0+\theta h)}{(n+1)!}h^{n+1}}_{\text{“remainder term”}}.$$

The number x_0 is called expansion point.

Proof: Remember the generalised mean value theorem

$$\frac{F(x_1)-F(x_0)}{g(x_1)-g(x_0)}=\frac{F'(x_0+\theta(x_1-x_0))}{g'(x_0+\theta(x_1-x_0))}$$

for some $\theta \in (0,1)$, which can be reformulated as (with $h = x_1 - x_0$)

$$F(x_0+h)-F(x_0)=\frac{g(x_0+h)-g(x_0)}{g'(x_0+\theta h)}\cdot F'(x_0+\theta h). \tag{5.3}$$

Now consider the functions

$$F(x):=\sum_{k=0}^{n}\frac{f^{(k)}(x)}{k!}(x_0+h-x)^k, \qquad g(x):=(x_0+h-x)^{n+1}.$$

Then we have

$$F'(x)=\sum_{k=0}^{n}\frac{f^{(k+1)}(x)}{k!}(x_0+h-x)^k-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{(k-1)!}(x_0+h-x)^{k-1}=\frac{f^{(n+1)}(x)}{n!}(x_0+h-x)^n.$$

and

$$g'(x)=-(n+1)(x_0+h-x)^n.$$

Moreover, we have

$$F(x_0)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}h^k, \quad F(x_0+h)=f(x_0+h), \quad g(x_0)=h^{n+1}, \quad g(x_0+h)=0.$$

Then using (5.3), we obtain

$$\begin{aligned}
&f(x_0+h)-\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}h^k\\
=&F(x_0+h)-F(x_0)\\
=&\frac{g(x_0+h)-g(x_0)}{g'(x_0+\theta h)}\cdot F'(x_0+\theta h)\\
=&\frac{-h^{n+1}}{-(n+1)((1-\theta)h)^n}\cdot\frac{f^{(n+1)}(x_0+\theta h)}{n!}((1-\theta)h)^n\\
=&\frac{f^{(n+1)}(x_0+\theta h)}{(n+1)!}h^{n+1}.
\end{aligned}$$

This implies Taylor's formula. □

Note that, by the substitution $h := x - x_0$, Taylor's formula can also be written as follows:

Theorem 5.31. Taylor's formula, alternative version

Let I be an interval and assume that $f : I \to \mathbb{R}$ is $n+1$-times differentiable. Let

$x_0, x \in I$. Then there exists some $\hat{x}$ between x_0 and x such that

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{=:T_n(x,x_0)} + \underbrace{\frac{f^{(n+1)}(\hat{x})}{(n+1)!}(x-x_0)^{n+1}}_{=:R_n(x,x_0)}.$$

The application of Taylor's formula is twofold: First, it gives a polynomial that approximates a given function quite fine in some neighbourhood. The second application is the computation of values of "complicated functions". We will present examples for both kinds of application.

Example 5.32. Consider the function $f(x) = \sin(x)$. We want to determine the Taylor polynomial of degree 6 with expansion point $x_0 = \frac{\pi}{2}$. Since we have

$$\begin{aligned} \sin'(x) &= \cos(x), \quad \sin''(x) = -\sin(x), \quad \sin^{(3)}(x) = -\cos(x), \\ \sin^{(4)}(x) &= \sin(x), \quad \sin^{(5)}(x) = \cos(x), \quad \sin^{(6)}(x) = -\sin(x) \end{aligned}$$

and

$$\sin(x_0) = \sin\left(\frac{\pi}{2}\right) = 1, \quad \cos(x_0) = \cos\left(\frac{\pi}{2}\right) = 0,$$

the Taylor polynomial of degree 6 is given by

$$T_6(x) = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x - \frac{\pi}{2}\right)^6.$$

The remainder term reads

$$R_6(x, x_0) = \frac{\sin^{(7)}(\hat{x})}{7!}\left(x - \frac{\pi}{2}\right)^7 = \frac{-\cos(\hat{x})}{5040}\left(x - \frac{\pi}{2}\right)^7.$$

Taking into account that $|\cos(x)| \leq 1$ for all $x \in \mathbb{R}$, we have that

$$|R_6(x, x_0)| \leq \frac{\left|x - \frac{\pi}{2}\right|^7}{5040}.$$

This leads to the estimate

$$|\sin(x) - T_6(x)| = |R_6(x, x_0)| \leq \frac{\left|x - \frac{\pi}{2}\right|^7}{5040}.$$

Example 5.33. We want to compute $\log(1.2)$ up to 3 digit precision. A nice expansion point is $x_0 = 1$ since we know the precise values of $\log^{(k)}(1)$. Consider

$$\begin{aligned} \log'(x) &= \frac{1}{x}, \quad \log''(x) = -\frac{1!}{x^2}, \\ \log^{(3)}(x) &= \frac{2!}{x^3}, \quad \log^{(4)}(x) = -\frac{3!}{x^4} \end{aligned}$$

and therefore, the Taylor polynomial of degree 3 is given by

$$T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3.$$

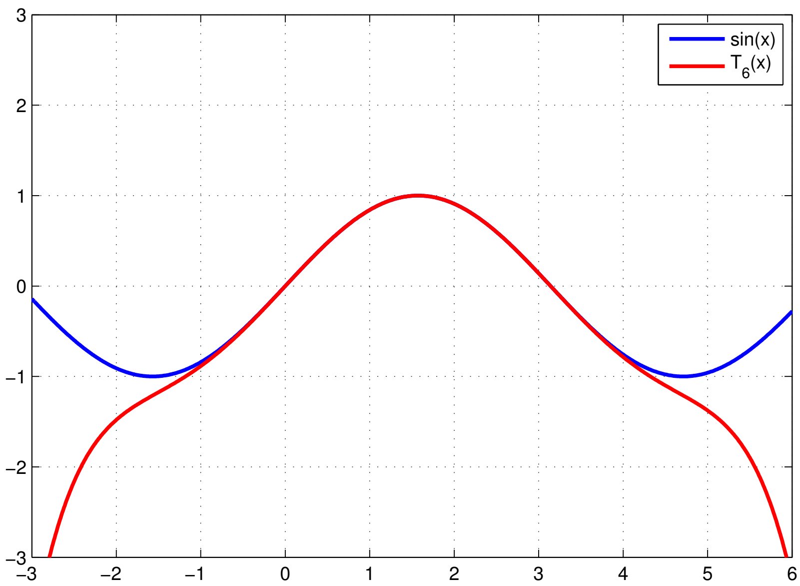


Figure 5.6: Taylor approximation of sin

In particular, we have

$$T_3(1.2) = 0.2 - \frac{1}{2} \cdot 0.2^2 + \frac{1}{3} \cdot 0.2^3 = \frac{137}{750}.$$

Now we estimate $|\log(1.2) - \frac{137}{750}|$: The remainder term is given by

$$R_3(x, x_0) = -\frac{3!}{\hat{x}^4} \frac{(x - x_0)^4}{4!} = -\frac{(x - x_0)^4}{4\hat{x}^4}$$

for some $\hat{x}$ between x and x_0. For $x = 1.2$, $x_0 = 1$ we have $1 < \hat{x} < 1.2$ and therefore

$$|R_3(1.2, 1)| = \frac{(0.2)^4}{4\hat{x}^4} = 4 \cdot 10^{-4} \frac{1}{\hat{x}^4} \leq 4 \cdot 10^{-4}.$$

This leads to

$$|\log(1.2) - 0.182\overline{6}| = |R_3(1.2, 1)| \leq 4 \cdot 10^{-4},$$

so we have determined $\log(1.2)$ up to three digits.

Theorem 5.34.

Let $I \subset \mathbb{R}$ be an open interval, $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ an n-times continuously differentiable function. Suppose that for $a \in I$ holds

$$f'(a) = f''(a) = \ldots = f^{(n-1)}(a) = 0 \quad \text{and} \quad f^{(n)}(a) \neq 0 \ .$$

If n is odd, then a is not a local extremum. If n is even and $f^{(n)}(a) > 0$, then a is a local minimum. If n is even and $f^{(n)}(a) < 0$, then a is a local maximum.

Proof: By assumption the Taylor expansion of f of degree $n-1$ in the expansion point a reads:

$$f(x) = f(a) + \frac{f^n(z)}{n!}(x-a)^n \tag{5.4}$$

where $z = z(x)$ lies between x and a. Since $f^{(n)}$ is continuous and $f^{(n)}(a) \neq 0$, there is a neighbourhood $U := (a-\varepsilon, a+\varepsilon) \subset I$, $\varepsilon > 0$, of a such that $f^{(n)}(x) \neq 0$ for all $x \in U$. This means that for all $x \in U$, $f^{(n)}(x)$ and $f^{(n)}(a)$ have the same sign. Then for any $x_l \in (a-\varepsilon, a)$ and any $x_r \in (a, a+\varepsilon)$ Equation (5.4) implies that for $z_l := z(x_l) \in (x_l, a)$ and $z_r := z(x_r) \in (a, x_r)$ holds

$$\begin{aligned} f(x_l) &= f(a) + \frac{f^n(z_l)}{n!}(x_l - a)^n \ , \\ f(x_r) &= f(a) + \frac{f^n(z_r)}{n!}(x_r - a)^n \ , \end{aligned}$$

and $0 \neq f^n(a), f^n(z_l), f^n(z_r)$ have the same sign. If n is odd, then $(x_l-a)^n < 0 < (x_r-a)^n$ and therefore either $f(x_l) < f(a) < f(x_r)$ or $f(x_l) > f(a) > f(x_r)$ so that a is not a local extremum. If n is even and $f^{(n)}(a) > 0$, then $(x_l-a)^n, (x_r-a)^n > 0$ and $f(x_l), f(x_r) > f(a)$ so that a is a local minimum. Finally, if n is even and $f^{(n)}(a) < 0$, then $(x_l-a)^n, (x_r-a)^n > 0$ and $f(x_l), f(x_r) < f(a)$ so that a is a local maximum. □

5.5 Simple methods for the numerical solution of equations

One of the most basic concerns of mathematics is solving equations. For example, if $f : \mathbb{R} \to \mathbb{R}$ is a given function and $y \in \mathbb{R}$ is a given number, then the question arises whether or not the equation

$$f(x) = y \tag{5.5}$$

can be solved for x. In practice the function f can be very complicated and a solution of (5.5) can in general not be found by simple arithmetical transformations. In such cases numerical procedures/algorithms might still be applicable to find approximate solutions $\hat{x}$ that fulfill $f(\hat{x}) \approx y$. In practical applications such approximate solutions are sufficient if $|f(\hat{x}) - y| \leq \varepsilon$, where $\varepsilon > 0$ is some "acceptable" error bound. In this section we will present two common basic numerical methods to determine such approximate solutions.

First of all note that the original Problem 5.5 can be transformed to

$$\tilde{f}(x) := f(x) - y + x = x \tag{5.6}$$

or to

$$\tilde{f}(x) := f(x) - y = 0 \ . \tag{5.7}$$

Thus solving Equation (5.6) means finding a fixed-point of $\tilde{f}$ and (5.7) means finding a zero of $\tilde{f}$. These are in some kind the most common normalized formulations for "solving" an equation. First we will present a simple numerical method for finding a fixed point of a given function based on Banach's fixed-point theorem. Afterwards we will state Newton's method for finding a zero of a given differentiable function.

Theorem 5.35. Weissinger fixed-point theorem

Let (X,d) be a complete metric space and $f : X \to X$ be a function on X. Assume that there is a convergent series $\sum_{n=0}^{\infty} a_n$ with $a_n \geq 0$ such that

$$d(x_n, y_n) \leq a_n \cdot d(x_0, y_0)$$

for arbitrary $x_0, y_0 \in X$ and recursively defined sequences

$$x_{n+1} := f(x_n)\ , \quad y_{n+1} := f(y_n)\ , \qquad n \in \mathbb{N}_0.$$

Then f possesses exactly one fixed point $z \in X$ and for an arbitrary starting point $z_0 \in X$ the recursively defined sequence $z_{n+1} := f(z_n)$, $n \in \mathbb{N}_0$, converges to z. Moreover, the following error estimates hold:

$$d(z, z_k) \leq \sum_{n=k}^{\infty} a_n \cdot d(z_0, z_1) \qquad (\textit{a priori estimate}) \tag{5.8}$$

$$d(z, z_k) \leq \sum_{n=1}^{\infty} a_n \cdot d(z_{k-1}, z_k) \qquad (\textit{a posteriori estimate}) \tag{5.9}$$

for all $k \in \mathbb{N}$.

Proof: First of all we show that f is continuous. Let $x \in X$, $\varepsilon > 0$ and set $\delta := \frac{\varepsilon}{a_1+1} > 0$. Then for $y \in X$ with $d(x,y) < \delta$ holds by assumption

$$d(f(x), f(y)) \leq a_1 \cdot d(x,y) \leq a_1 \cdot \frac{\varepsilon}{a_1 + 1} < \varepsilon\ .$$

From the ε-δ-criterion it follows that f is continuous in x.

Now let $x_0 \in X$ be arbitrary and $x_{n+1} := f(x_n)$, $n \in \mathbb{N}_0$. Then for $y_0 := x_1$ and $y_{n+1} := f(y_n)$, $n \in \mathbb{N}_0$, we have $y_n = x_{n+1}$ for all $n \in \mathbb{N}_0$ and by assumption

$$d(x_n, x_{n+1}) = d(x_n, y_n) \leq a_n \cdot d(x_0, y_0) = a_n \cdot d(x_0, x_1),$$

for all $n \in \mathbb{N}$. We will show now that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $\sum_{n=0}^{\infty} a_n$ converges, there is an $N \in \mathbb{N}$ such that for all $k, m > N$ with $k \geq m$ holds

$$\sum_{n=k}^{m} a_n < \frac{\varepsilon}{d(x_0, x_1) + 1}\ .$$

Using the triangle inequality, for such k, m holds:

$$d(x_k, x_m) \underset{\text{triangle ineq.}}{\leq} \sum_{n=k}^{m-1} d(x_n, x_{n+1}) = \sum_{n=k}^{m-1} d(x_n, y_n) \leq \sum_{n=k}^{m-1} a_n \cdot d(x_0, x_1) < \varepsilon\ .$$

Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X,d) is a complete, $(x_n)_{n\in\mathbb{N}}$ converges to some limit x and since f is continuous we conclude

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x)\ ,$$

i.e. x is a fixed point of f. For any other starting point $z_0 \in X$ and recursively defined $z_{n+1} := f(z_n)$, $n \in \mathbb{N}_0$, we obtain by the previous that $z := \lim_{n\to\infty} z_n$ is also a fixed point of f. But the assumption applied to $u_0 := x$, $v_0 := z$, $u_{n+1} := f(u_n) = x$, $v_{n+1} := f(v_n) = z$ for $n \in \mathbb{N}_0$ yields

$$0 \le d(x,z) = d(u_n, v_n) \le a_n \cdot d(u_0, v_0) = a_n \cdot d(x,z).$$

Since $(a_n)_{n\in\mathbb{N}_0}$ converges to zero, the right-hand side converges to zero which implies $d(x,z) = 0$, i.e. $x = z$. Thus f has a unique fixed point $z \in X$.
Next we derive (5.8). Let $k \in \mathbb{N}$. For arbitrary $m \ge k$ holds

$$\begin{aligned} d(z, z_k) &\le d(z, z_m) + d(z_m, z_k) \le d(z, z_m) + \sum_{n=k}^{m-1} d(z_n, z_{n+1}) \\ &\le \underbrace{d(z, z_m)}_{\xrightarrow[m\to\infty]{} 0} + \underbrace{\sum_{n=k}^{m-1} a_n}_{\xrightarrow[m\to\infty]{} \sum_{n=k}^{\infty} a_n} \cdot d(z_0, z_1) \xrightarrow{m\to\infty} \sum_{n=k}^{\infty} a_n \cdot d(z_0, z_1) \ . \end{aligned}$$

Finally, if we define $\tilde{z}_0 := z_{k-1}$ and $\tilde{z}_{j+1} := f(\tilde{z}_j) = z_{k+j}$, $j \in \mathbb{N}_0$, (5.9) follows from (5.8) applied to the sequence $(\tilde{z}_j)_{j\in\mathbb{N}_0}$ for $j := 1$, namely:

$$d(z, z_k) = d(z, \tilde{z}_1) \le \sum_{n=1}^{\infty} a_n \cdot d(\tilde{z}_0, \tilde{z}_1) = \sum_{n=1}^{\infty} a_n \cdot d(z_{k-1}, z_k) \ .$$

□

The so-called Banach fixed-point theorem is a special case of Weissinger's fixed point theorem.

Theorem 5.36. Banach fixed-point theorem

Let (X, d) be a complete metric space and $f : X \to X$ be a function on X such that $d(f(x), f(y)) \le q \cdot d(x,y)$ for all $x, y \in X$, where $q \in [0,1)$ is a fixed nonnegative constant less then one [a]. Then f has exactly one fixed point $z \in X$ and for an arbitrary $z_0 \in X$ the recursively defined sequence $z_{n+1} := f(z_n)$, $n \in \mathbb{N}_0$, converges to z. Moreover the following error estimates hold for $k \in \mathbb{N}$:

$$d(z, z_k) \le \frac{q^k}{1-q} \cdot d(z_0, z_1) \qquad (\textit{a priori estimate}) \tag{5.10}$$

$$d(z, z_k) \le \frac{q}{1-q} \cdot d(z_{k-1}, z_k) \qquad (\textit{a posteriori estimate}) \tag{5.11}$$

[a] Functions with this property are called *contractions* and q is called a *contraction constant* for f.

Proof: With $a_n := q^n$ for $n \in \mathbb{N}_0$ the asumptions of Weissinger's fixed point theorem are fulfilled. The estimates (5.10) and (5.11) follow directly from (5.8) and (5.9) respectively.
□

We want to reformulate Banach's fixed-point theorem for the special case where X is a closed subset of $\mathbb{R}$ and d is the Euclidean metric on X, i.e. $d(x,y) := |x-y|$ for $x, y \in X$. Recall that in this case (X, d) is complete.

Theorem 5.37.

Let $X \subset \mathbb{R}$ be closed and $f : X \to X$ be a function such that $|f(x)-f(y)| \leq q\cdot|x-y|$ for all $x, y \in X$, where $q \in [0,1)$ is a fixed nonnegative constant less then one, i.e. f is a contraction on X with contraction constant q. Then f has exactly one fixed point $z \in X$ and for an arbitrary $z_0 \in X$ the recursively defined sequence $z_{n+1} := f(z_n)$, $n \in \mathbb{N}_0$, converges to z. Moreover the following error estimates hold for $k \in \mathbb{N}$:

$$|z - z_k| \leq \frac{q^k}{1-q}|z_1 - z_0| \qquad (\textit{a priori estimate}) \tag{5.12}$$

$$|z - z_k| \leq \frac{q}{1-q}|z_{k-1} - z_k| \qquad (\textit{a posteriori estimate}) \tag{5.13}$$

In practical applications the function f is given but in order to apply Theorem 5.37 an appropriate domain X with $f(X) \subset X$ and a contraction constant $q \in [0,1)$ must be determined. In practise a closed area X is guessed where a fixed-point of a given function f might be located. Then $f(X) \subset X$ must be verified and a contraction constant must be found. If this is not possible, the guessed domain X must be changed. The following Theorem states a standard procedure for finding an contraction constant q on a given closed interval domain X by using an upper bound of the first derivative of f, which requires that f is continuously differentiable on X.

Theorem 5.38.

Let $X \subset \mathbb{R}$ be a closed interval and $f : X \to X$ be continuously differentiable on X. If $q := ||f'||_\infty = \sup\{|f'(x)| \mid x \in X\} < 1$, then f is a contraction on X with contraction constant q. In particular, by Banach's fixed-point theorem, f has exactly one fixed point in X and for an arbitrary $z_0 \in X$ the recursively defined sequence $z_{n+1} := f(z_n)$, $n \in \mathbb{N}_0$, converges to z and the error estimates (5.12) and (5.13) hold.

Proof: Let $x, y \in X$ with $x < y$. By the mean value theorem there is a $\xi \in (x, y)$ such that

$$|f(x) - f(y)| = |f'(\xi) \cdot (y - x)| = |f'(\xi)| \cdot |(y - x)| \leq q \cdot |x - y|.$$

This implies that f is a contraction on X with contraction constant q and the conclusion follows from Theorem 5.37 . □

Now we will state Newton's method for finding zeros of differentiable functions f. Newton's method is based on the following simple iteration principle: If x_0 is an approximate solution of $f(x) = 0$, then f is replaced in a vicinity of x_0 by the tangent of f in x_0, namely by

$$t(x) := f'(x_0)(x - x_0) + f(x_0) \ .$$

Then, if $f'(x_0) \neq 0$, the solution x_1 of $t(x) = 0$, which is

$$x_1 := x_0 - \frac{f(x_0)}{f'(x_0)} \ ,$$

is taken as an improved approximate solution of $f(x) = 0$.

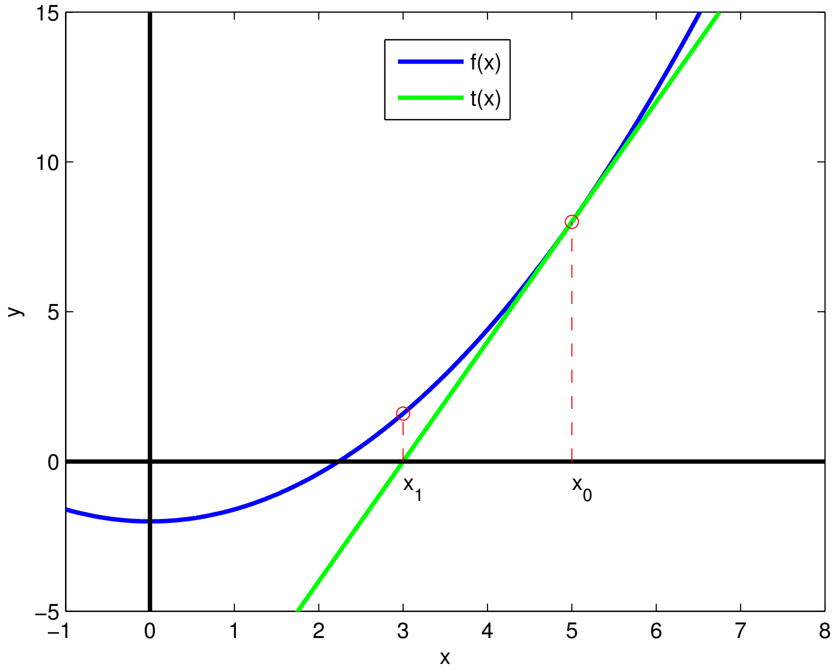


Figure 5.7: Newton's method

This procedure generates a sequence

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}\ , \qquad n \in \mathbb{N}_0$$

a so-called *Newton iteration*, which for a general differentiable function f and an arbitrary starting point x_0 in the domain of f might not be well-defined or might not converge to a root of f.

The following Theorem gives sufficient conditions that assure that the Newton iteration converges to a root of f.

Theorem 5.39.

Suppose that $f : [a,b] \to \mathbb{R}$ is two times differentiable and convex with $f(a) < 0$ and $f(b) > 0$. Then the following holds:

a) $f(\xi) = 0$ for exactly one $\xi \in (a,b)$.

b) For arbitrary $x_0 \in [a,b]$ with $f(x_0) \geq 0$ the Newton iteration

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}\ , \qquad n \in \mathbb{N}_0$$

is well-defined, monotonically decreasing and converges to ξ.

c) If there are nonnegative constants C, K such that $f'(\xi) \geq C > 0$ and $f''(x) \leq K$ for all $x \in (\xi, b)$, then the following error estimate holds for $n \in \mathbb{N}$:

$$|x_{n+1} - x_n| \leq |\xi - x_n| \leq \frac{K}{2C} \cdot |x_n - x_{n-1}|^2\ .$$

Proof: a) By the intermediate value Theorem f has a root $\xi \in (a,b)$. In order to prove uniqueness, we assume that f has two distinct roots $\xi, \eta \in (a,b)$ with $\xi < \eta$. Since f is convex, for $x_1 := a < x := \xi < x_2 := \eta$ holds

$$\begin{aligned} 0 &= f(\xi) = f(x) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1) = \frac{f(\eta) - f(a)}{\eta - a}(\xi - a) + f(a) \\ &= \frac{-f(a)}{\eta - a}(\xi - a) + f(a) = (1 - \underbrace{\frac{\xi - a}{\eta - a}}_{<1}) \cdot f(a) < 0 \ , \end{aligned}$$

a contradiction. Therefore, f has a unique root ξ in (a,b) .

b) By the mean value theorem there exists an $\eta \in (a,\xi)$ such that

$$f'(\eta) = \frac{f(\xi) - f(a)}{\xi - a} = \frac{-f(a)}{\xi - a} > 0 \ .$$

Since f is convex, f' is monotonically increasing so that for all $x \in [\xi, b]$ holds $f'(x) \geq f'(\eta) > 0$, i.e. f' is positive on $[\xi, b]$. In particular, f is strictly monotonically increasing on $[\xi, b]$ so that $f(x) > f(\xi) = 0$ for all $x \in (\xi, b]$.

Now let $x_0 \in [a,b]$ with $f(x_0) \geq 0$. Then necessarily $x_0 \geq \xi$. (Otherwise, if $x_0 < \xi$, then $f(a) < 0 \leq f(x_0)$ would imply that f has another root in $[a, x_0]$ which is not possible due to a).) By the previous $f'(x_0) > 0$ and therefore

$$x_1 := x_0 - \frac{f(x_0)}{f'(x_0)}$$

is well-defined. We know that x_1 is the root of the tangent

$$t(x) := f'(x_0)(x - x_0) + f(x_0) \ .$$

Since $t(x_0) = f(x_0) \geq 0$ and $t'(x_0) = f'(x_0) > 0$ we immediately see that $x_{n+1} \leq x_0$. Moreover, $(f - t)(x_0) = 0$ and $(f - t)'(x) = f'(x) - f'(x_0) \leq 0$ for $x \leq x_0$ imply $f(x) \geq t(x)$ for $x \leq x_0$. In particular $t(\xi) \leq f(\xi) = 0 \leq f(x_0) = t(x_0)$ and therefore $\xi \leq x_1 \leq x_0$ as $t(x)$ is a straight line. Replacing x_0 by x_1 we conclude inductively that the sequence

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)} \ , \quad n \in \mathbb{N}_0,$$

is well-defined, monotonically decreasing and bounded from below by ξ. Thus

$$\eta := \lim_{n \to \infty} x_n$$

exists and fulfils $\xi \leq \eta$. Since f' is bounded from below on $[\xi, b]$ by some positive constant, we also have

$$\eta = \eta - \frac{f(\eta)}{f'(\eta)}$$

which implies $f(\eta) = 0$. By a) this proves $\eta = \xi$.

c) Since f' is monotonically increasing we have $f'(x) \geq f'(\xi) \geq C > 0$ for all $x \in [\xi, b]$. Therefore, $f(x) \geq C(x-\xi)$ for all $x \in [\xi, b]$. In particular, this implies

$$|x_n - \xi| = x_n - \xi \leq \frac{f(x_n)}{C} \quad \text{for all } n \in \mathbb{N}.$$

In order to estimate $f(x_n)$ the following function $g(x)$ is considered:

$$\begin{aligned} g(x) &:= f(x) - f(x_{n-1}) - f'(x_{n-1})(x - x_{n-1}) - \frac{K}{2}(x - x_{n-1})^2 \\ g'(x) &= f'(x) - f'(x_{n-1}) - K(x - x_{n-1}) \\ g''(x) &= f''(x) - K \leq 0 \quad \text{for } x \in \ (\xi, b). \end{aligned}$$

Thus g' is monotonically decreasing on $[\xi, b]$. Since $g'(x_{n-1}) = 0$, this implies $g'(x) \geq 0$ for $x \in [\xi, x_{n-1}]$. Since $g(x_{n-1}) = 0$, this implies $g(x) \leq 0$ for $x \in [\xi, x_{n-1}]$. In particular,

$$\begin{aligned} 0 \ &\geq \ g(x_n) = f(x_n) - f(x_{n-1}) - f'(x_{n-1})(x_n - x_{n-1}) - \frac{K}{2}(x_n - x_{n-1})^2 \\ &= \ f(x_n) - f(x_{n-1}) + f(x_{n-1}) - \frac{K}{2}(x_n - x_{n-1})^2 \\ &= \ f(x_n) - \frac{K}{2}(x_n - x_{n-1})^2 \ , \end{aligned}$$

that is

$$f(x_n) \ \leq \ \frac{K}{2}(x_n - x_{n-1})^2$$

and we conclude

$$|x_n - \xi| \leq \frac{f(x_n)}{C} \leq \frac{K}{2C}|x_n - x_{n-1}|^2.$$

Finally, by b) $(x_n)_{n\in\mathbb{N}_0}$ decreases monotonically with limit ξ which directly yields $|x_{n+1} - x_n| \leq |\xi - x_n|$.

□

Remark:

a) The error estimate given in Theorem 5.39 c) says that Newton's method (locally) converges quadratically. If f' is bounded from below by some $c_1 > 0$ and if $|f''|$ is bounded from above by some c_2, then we may choose $C := c_1$ and $K := c_2$.

b) Analog formulations of Theorem 5.39 hold if the function f is concave or if $f(a) > 0$ and $f(b) < 0$.

6 The Riemann Integral

Prince George: *Someone said I had the wit and intellect of a donkey.*

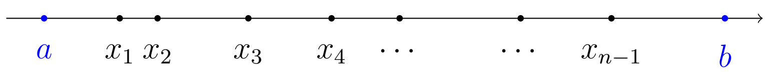
Blackadder: *Oh, an absurd suggestion sir. Unless it was a particularly stupid donkey.*

Rowan Atkinson, Blackadder

Definition 6.1.

Let $[a, b] \subset \mathbb{R}$. A set $\{x_0, x_1, \dots, x_n\}$ is called a decomposition or partition of $[a, b]$ if

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

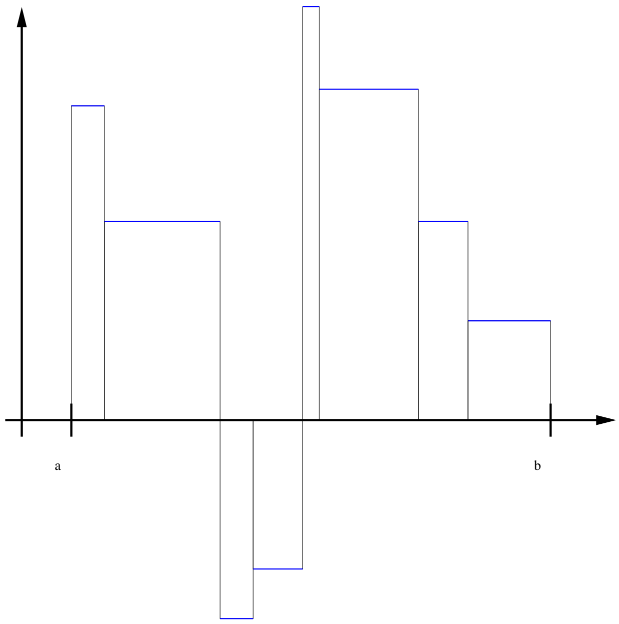


Definition 6.2. Step function

$f : [a, b] \to \mathbb{R}$ is called a step function if it is piecewisely constant, i.e. there exists a decomposition $\{x_0, \dots, x_n\}$ of $[a, b]$ and some $c_1, \dots, c_n \in \mathbb{R}$ such that for all $i = 1, \dots, n$ holds

$$f(x) = c_i \text{ for all } x \in (x_{i-1}, x_i).$$

The set of step functions on $[a, b]$ is denoted by $\mathcal{T}([a, b])$.



It can be readily verified that for two step functions $f_1, f_2 \in \mathcal{T}([a,b])$ holds $f_1 + f_2 \in \mathcal{T}([a,b])$. As well, we have $\lambda f_1 \in \mathcal{T}([a,b])$. Hence, $\mathcal{T}([a,b])$ is a vector space. Furthermore, since step functions only attain finitely many values, they are bounded.

Definition 6.3. Integral of step functions

Let $\phi \in \mathcal{T}([a,b])$ and a decomposition $\{x_0, \dots, x_n\}$ of $[a,b]$ be given such that for all $i = 1, \dots, n$

$$\phi(x) = c_i \text{ for all } x \in (x_{i-1}, x_i).$$

Then the integral of ϕ is defined as

$$\int_a^b \phi(x)\,dx := \sum_{j=1}^{n} c_j(x_j - x_{j-1}).$$

However, to be sure that the integral is well-defined we need additionally that it is independent of the special choice of a decomposition.

Lemma 6.4.

$\int_a^b \phi(x)\,dx$ is independent of the choice of the decomposition, i.e., $\int_a^b \phi(x)\,dx$ is well-defined for all $\phi \in \mathcal{T}([a,b])$.

Proof: Let

$$\begin{aligned} Z_1 :\quad & a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b, \\ Z_2 :\quad & a = y_0 < y_1 < y_2 < \ldots < y_{m-1} < y_m = b \end{aligned}$$

be two decompositions of $[a,b]$ such that ϕ is constant on (x_{i-1}, x_i) and ϕ is constant on (y_{j-1}, y_j) for all $i = 1, \dots, n$, $j = 1, \dots, m$. We distinguish between two cases: **1st Case:** Z_2 is a refinement of Z_1. This means that for all $i \in \{1, \dots, n\}$ there exists some $j(i) \in \{1, \dots, m\}$ with $x_i = y_{j(i)}$. Then for i holds

$$x_i = y_{j(i)} < y_{j(i)+1} < \ldots < y_{j(i+1)} = x_{i+1}$$

and $\phi(x) = d_k = c_i$ for $x \in (y_{k-1}, y_k)$, $k \in \{j(i-1)+1, \ldots, j(i)\}$. This implies

$$\begin{aligned}\sum_{j=1}^{m} d_j(y_j - y_{j-1}) &= \sum_{i=1}^{n} \sum_{k=j(i-1)+1}^{j(i)} c_i(y_k - y_{k-1}) \\ &= \sum_{i=1}^{n} c_i \sum_{k=j(i-1)+1}^{j(i)} (y_k - y_{k-1}) \\ &= \sum_{i=1}^{n} c_i(y_{j(i)} - y_{j(i-1)}) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).\end{aligned}$$

2nd Case: There exists a common refinement Z_3 of Z_1 and Z_2, i.e., $Z_3 = Z_1 \cup Z_2$. Then we have (by using the 1st case)

$$\sum_{Z_1} \cdots = \sum_{Z_3} \ldots = \sum_{Z_2} \ldots .$$

□

The integral can be seen as a mapping from the space $\mathcal{T}([a,b])$ to $\mathbb{R}$. In the literature, mappings from vector spaces to a field (in this case $\mathbb{R}$) are called *functionals*. The following result shows that the integral is a linear and monotone functional.

Theorem 6.5.

$\int_a^b : \mathcal{T}([a,b]) \to \mathbb{R}$ is linear and monotonic, that is, for all $\phi, \psi \in \mathcal{T}([a,b])$ and all $\lambda, \mu \in \mathbb{R}$ holds

(i)

$$\int_a^b \Big(\lambda\phi(x) + \mu\psi(x)\Big)\, dx = \lambda \int_a^b \phi(x)\, dx + \mu \int_a^b \psi(x)\, dx$$

(ii) if $\phi(x) \le \psi(x)$ for all $x \in [a,b]$ $(\phi \le \psi)$, then

$$\int_a^b \phi(x)\, dx \le \int_a^b \psi(x)\, dx.$$

Proof: This directly follows by the definition of the integral. □

Now we will define the integral for bounded functions:

Definition 6.6.

Let $f : [a,b] \to \mathbb{R}$ be bounded. Then we define the Riemann upper integral

$$\overline{\int_a^b} f(x)\, dx := \inf \left\{ \int_a^b \phi(x)\, dx \;:\; \phi \in \mathcal{T}([a,b]) \text{ with } \phi \ge f \right\}$$

and the Riemann lower integral by

$$\underline{\int_a^b} f(x)\, dx := \sup \left\{ \int_a^b \phi(x)\, dx \;:\; \phi \in \mathcal{T}([a,b]) \text{ with } \phi \le f \right\}.$$

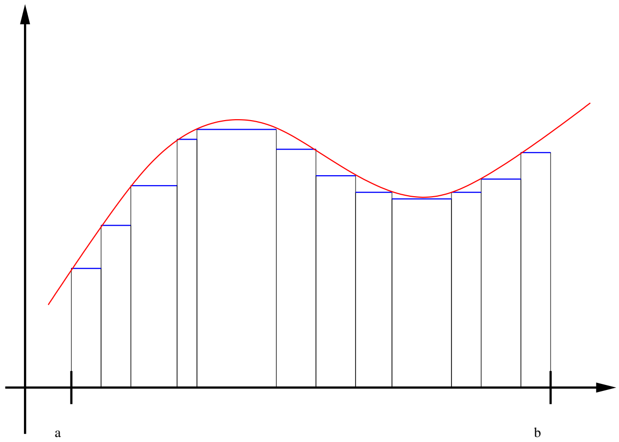


Figure 6.1: approximation for the lower integral

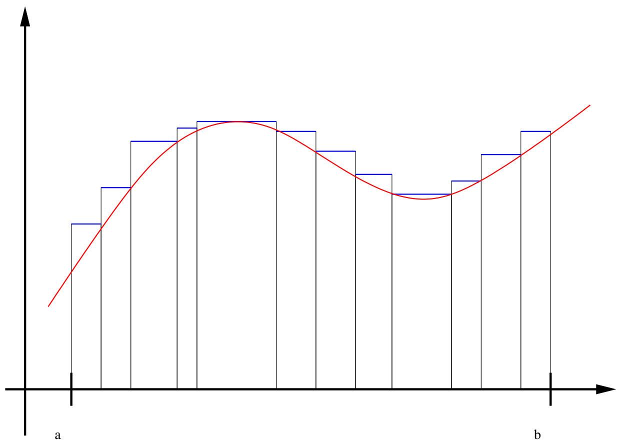


Figure 6.2: approximation for the upper integral

It can be seen from Fig. 6.1 and Fig. 6.2 that the integral can be seen as the "signed area" between the function graph and the x-axis. "Signed" means that the area of the negative parts of the function has to be counted negative.

By the above definition, we can directly deduce that for $\phi \in \mathcal{T}([a,b])$ holds

$$\overline{\int_a^b} \phi(x)\,dx = \underline{\int_a^b} \phi(x)\,dx = \int_a^b \phi(x)\,dx.$$

For general bounded functions $f : [a,b] \to \mathbb{R}$ holds

$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx.$$

Note that in general the upper integral does not coincide with the lower integral. For instance, consider the function $f : [0,1] \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1 & : \ x \in \mathbb{Q} \\ 0 & : \ x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

Then we have
$$\overline{\int_0^1} f(x)\,dx = 1 > 0 = \underline{\int_0^1} f(x)\,dx.$$

Theorem 6.7.

Let $f, g : [a,b] \to \mathbb{R}$ be bounded. Then

(i)
$$\overline{\int_a^b} f(x) + g(x)\,dx \le \overline{\int_a^b} f(x)\,dx + \overline{\int_a^b} g(x)\,dx.$$

(ii)
$$\underline{\int_a^b} f(x) + g(x)\,dx \ge \underline{\int_a^b} f(x)\,dx + \underline{\int_a^b} g(x)\,dx.$$

(iii) For $\lambda \ge 0$ holds
$$\underline{\int_a^b} \lambda f(x)\,dx = \lambda \underline{\int_a^b} f(x)\,dx, \quad \overline{\int_a^b} \lambda f(x)\,dx = \lambda \overline{\int_a^b} f(x)\,dx.$$

(iv) For $\lambda \le 0$ holds
$$\underline{\int_a^b} \lambda f(x)\,dx = \lambda \overline{\int_a^b} f(x)\,dx, \quad \overline{\int_a^b} \lambda f(x)\,dx = \lambda \underline{\int_a^b} f(x)\,dx.$$

Proof:
(i) Let $\varepsilon > 0$. Then there exist $\phi, \psi \in \mathcal{T}([a,b])$ with $\phi \ge f$, $\psi \ge g$ and
$$\overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2} \ge \int_a^b \phi(x)\,dx, \qquad \overline{\int_a^b} g(x)\,dx + \frac{\varepsilon}{2} \ge \int_a^b \psi(x)\,dx.$$

Then $f + g \le \phi + \psi$ and
$$\begin{aligned}
\overline{\int_a^b} f(x) + g(x)\,dx &= \inf\left\{ \int_a^b \zeta(x)\,dx \ : \ \zeta \in \mathcal{T}([a,b]) \text{ with } \zeta \ge f + g \right\} \\
&\le \int_a^b \phi(x) + \psi(x)\,dx \\
&\le \left(\overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2} \right) + \left(\overline{\int_a^b} g(x)\,dx + \frac{\varepsilon}{2} \right) \\
&= \overline{\int_a^b} f(x)\,dx + \overline{\int_a^b} g(x)\,dx + \varepsilon.
\end{aligned}$$

Since this holds true for all $\varepsilon > 0$, we conclude
$$\overline{\int_a^b} f(x) + g(x)\,dx \le \overline{\int_a^b} f(x)\,dx + \overline{\int_a^b} g(x)\,dx.$$

(ii): Analogous to (i).
(iii): We only show the statement for the upper integral. The other result can be shown analogously.
Let $\varepsilon > 0$. Then there exists some $\phi \in \mathcal{T}([a,b])$ with $\phi \geq f$ and

$$\overline{\int_a^b} f(x)\,dx + \varepsilon \geq \int_a^b \phi(x)\,dx.$$

Then $\lambda\phi \geq \lambda f$ and

$$\overline{\int_a^b} \lambda f(x)\,dx \leq \int_a^b \lambda\phi(x)\,dx = \lambda \int_a^b \phi(x)\,dx \leq \lambda \overline{\int_a^b} f(x)\,dx + \lambda\varepsilon.$$

Since this holds true for all $\varepsilon > 0$, we conclude

$$\overline{\int_a^b} \lambda f(x)\,dx \leq \lambda \overline{\int_a^b} f(x)\,dx.$$

The opposite inequality follows from the previous one applied to $g(x) := \lambda f(x)$ and $\mu := \frac{1}{\lambda} > 0$:

$$\overline{\int_a^b} f(x)\,dx = \overline{\int_a^b} \mu g(x)\,dx \leq \mu \overline{\int_a^b} g(x)\,dx = \frac{1}{\lambda} \overline{\int_a^b} \lambda f(x)\,dx.$$

Multiplying this inequality by $\lambda > 0$ yields

$$\lambda \overline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} \lambda f(x)\,dx.$$

(iv): This result can be shown by using (iii) and the fact

$$\overline{\int_a^b} - f(x)\,dx = - \underline{\int_a^b} f(x)\,dx.$$

□

Definition 6.8.

A bounded function $f : [a,b] \to \mathbb{R}$ is called Riemann-integrable if

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

In this case, we write

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

The set of Riemann-integrable functions is denoted by $\mathcal{R}([a,b])$.

We obviously have that $\mathcal{T}([a,b]) \subset \mathcal{R}([a,b])$. In the following we state that monotonic functions as well as continuous functions are Riemann-integrable. The proof is not presented here.

Remark:

As it holds true for summation, the integration variable can be renamed without changing the integral. That is, for $f \in \mathcal{R}([a,b])$ holds

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$$

Theorem 6.9.

Let $f : [a,b] \to \mathbb{R}$ be continuous or monotonic. Then f is Riemann-integrable.

The following result is straightforward:

Theorem 6.10.

$\mathcal{R}([a,b])$ is a real vector space and the mapping

$$\int_a^b \; : \; \mathcal{R}([a,b]) \to \mathbb{R}$$

is linear and monotonic.

So far we did not compute any integrals. We will now give an example of an integral that will be computed according to the definition. This will turn out to be really exhausting even for this quite simple example.

Example 6.11. Consider the function $f : [0,1] \to \mathbb{R}$ with $f(x) = x$. Determine $\int_0^1 f(x)\,dx = \int_0^1 x\,dx$. First we consider two sequences of step functions $(\phi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\in\mathbb{N}}$ with

$$\begin{aligned}
\phi_n(x) &= \frac{k-1}{n} \text{ for } x \in \left[\frac{k-1}{n}, \frac{k}{n}\right), \qquad k \in \{1,\dots,n\},\\
\psi_n(x) &= \frac{k}{n} \text{ for } x \in \left[\frac{k-1}{n}, \frac{k}{n}\right), \qquad k \in \{1,\dots,n\}.
\end{aligned}$$

Then for all $n \in \mathbb{N}$ holds $\phi_n \le f \le \psi_n$. Now we calculate

$$\begin{aligned}
\int_0^1 \phi_n(x)\,dx &= \sum_{k=1}^n \frac{k-1}{n} \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right)\\
&= \sum_{k=1}^n \frac{k-1}{n} \cdot \frac{1}{n}\\
&= \frac{1}{n^2} \sum_{k=1}^n (k-1)\\
&= \frac{1}{n^2} \cdot \frac{n(n-1)}{2} = \frac{1}{2} - \frac{1}{2n}
\end{aligned}$$

and

$$\int_0^1 \psi_n(x)\,dx = \sum_{k=1}^{n} \frac{k}{n} \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right)$$
$$= \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n}$$
$$= \frac{1}{n^2} \sum_{k=1}^{n} k$$
$$= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{1}{2} + \frac{1}{2n}.$$

In particular, we have for all $n \in \mathbb{N}$ that

$$\frac{1}{2} - \frac{1}{2n} = \int_0^1 \phi_n(x)\,dx \leq \int_0^1 x\,dx \leq \int_0^1 \psi_n(x)\,dx = \frac{1}{2} + \frac{1}{2n}$$

and thus

$$\int_0^1 x\,dx = \frac{1}{2}.$$

By the definition of the integral, it is not difficult to obtain that for $f \in \mathcal{R}([a,b])$ and $c \in (a,b)$ holds

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

To make this formula also valid for $c \geq b$ or $c \leq a$, we define for $a \geq b$ that

$$\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx.$$

Now we present the mean value theorem of integration and its manifold consequences.

Theorem 6.12. Mean Value Theorem of Integration

Let $f, g : [a,b] \to \mathbb{R}$ be continuous and let $g(x) \geq 0$ for all $x \in [a,b]$ (i.e., $g \geq 0$). Then there exists some $\hat{x} \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(\hat{x}) \cdot \int_a^b g(x)\,dx.$$

Proof: Let

$$m = \min\{f(x) \ : \ x \in [a,b]\}, \qquad M = \max\{f(x) \ : \ x \in [a,b]\}.$$

Since $g \geq 0$, for all $x \in [a,b]$ holds $mg(x) \leq f(x)g(x) \leq Mg(x)$ and, by the monotonicity of the integral holds

$$m \int_a^b g(x)\,dx = \int_a^b mg(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \int_a^b Mg(x)\,dx \leq M \int_a^b g(x)\,dx.$$

Then there exists some $m \leq \mu \leq M$ such that

$$\mu \int_a^b g(x)\,dx = \int_a^b f(x)g(x)\,dx$$

Since
$$\min\{f(x) : x \in [a,b]\} \le \mu \le \max\{f(x) : x \in [a,b]\},$$
the intermediate value theorem implies that there exists some $\hat{x} \in [a,b]$ with $\mu = f(\hat{x})$ and thus
$$f(\hat{x}) \int_a^b g(x)\,dx = \int_a^b f(x)g(x)\,dx.$$
□

Corollary 6.13.

Let $f : [a,b] \to \mathbb{R}$ be continuous. Then there exists some $\hat{x} \in [a,b]$ such that
$$\int_a^b f(x)\,dx = f(\hat{x}) \cdot (b-a).$$

Proof: Apply the mean value theorem to $g = 1$. Observing that
$$\int_a^b g(x)\,dx = \int_a^b 1\,dx = b-a,$$
the result then follows immediately. □

6.1 Differentiation and Integration

Definition 6.14.

Let I be an interval and $f : I \to \mathbb{R}$ be continuous. Then a differentiable function $F : I \to \mathbb{R}$ is called an antiderivative of f if $F' = f$.

Theorem 6.15.

Let I be an interval, $f : I \to \mathbb{R}$ be continuous and $a \in I$. For $x \in I$ define
$$F(x) = \int_a^x f(\xi)d\xi.$$
Then F is differentiable and an antiderivative of f.

Proof: Let $x \in I$ and $h \neq 0$ such that $x + h \in I$. Then, by using the mean value theorem of integration we obtain
$$\begin{aligned}\frac{1}{h}(F(x+h) - F(x)) &= \frac{1}{h}\left(\int_a^{x+h} f(\xi)d\xi - \int_a^x f(\xi)d\xi\right)\\ &= \frac{1}{h}\int_x^{x+h} f(\xi)d\xi = \frac{1}{h}\cdot hf(\hat{x}) = f(\hat{x})\end{aligned}$$
for some $\hat{x}$ between x and $x+h$. If h tends to 0 then $\hat{x} \to x$ and thus
$$\lim_{h\to 0} \frac{1}{h}(F(x+h) - F(x)) = f(x).$$
This shows the desired result. □

Now we consider how two antiderivatives of a given continuous $f : I \to \mathbb{R}$ differ.

Theorem 6.16.

Let I be an interval, $f : I \to \mathbb{R}$ be given and let $F : I \to \mathbb{R}$ be an antiderivative of f, i.e., $F' = f$. Then $G : I \to \mathbb{R}$ is an antiderivative of f if and only if $F - G$ is constant.

Proof: "$\Rightarrow$": Let $G : I \to \mathbb{R}$ be an antiderivative of f. Then

$$(F - G)' = F' - G' = f - f = 0$$

and hence, $F - G$ is constant due to the mean value theorem of differentiation.
"$\Leftarrow$": If $F - G$ is constant, i.e., $F(x) - G(x) = c$ for some $c \in \mathbb{R}$ and all $x \in I$, then $0 = (F - G)' = F' - G'$ and thus $f = F' = G'$. $\square$

Remark:

It is very important to note that the statement of Theorem 6.16 is only valid for functions defined on intervals. For instance, consider the function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ with $f(x) = \frac{1}{x}$. An antiderivative is given by $F : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ with

$$F(x) = \begin{cases} \log(x) & : x > 0, \\ \log(-x) & : x < 0 \end{cases} = \log(|x|).$$

Another antiderivative is given by

$$G(x) = \begin{cases} \log(x) & : x > 0, \\ \log(-x) + 1 & : x < 0 \end{cases}.$$

The difference between G and F is given by

$$G(x) - F(x) = \begin{cases} 0 & : x > 0, \\ 1 & : x < 0 \end{cases}$$

and therefore not constant.

The next result now states that integrals can be determined by inversion of differentiation.

Theorem 6.17. Fundamental theorem of differentiation and integration

Let I be an interval and a continuous $f : I \to \mathbb{R}$ be given. Let $F : I \to \mathbb{R}$ be an antiderivative of f. Then for all $a, b \in I$ holds

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

We write

$$\int_a^b f(x)\, dx = F(x)|_{x=a}^{x=b}.$$

Proof: Consider the function $F_0 : I \to \mathbb{R}$ defined by

$$F_0(x) = \int_a^x f(\xi) d\xi.$$

By Theorem 6.15, we know that F_0 is an antiderivative of f. In particular, we have that $F_0(a) = \int_a^a f(\xi)d\xi = 0$ and thus $F_0(b) - F_0(a) = F_0(b) = \int_a^b f(\xi)d\xi = \int_a^b f(x)\,dx$. Let $FI :\to \mathbb{R}$ be an antiderivative of f. Theorem 6.16 now implies that there exists some $c \in \mathbb{R}$ with $F(x) = F_0(x) + c$ for all $x \in I$. Therefore

$$F(b) - F(a) = (F_0(b) + c) - (F_0(a) + c) = F_0(b) - F_0(a) = \int_a^b f(x)\,dx.$$

□

The above result gives rise to the following notation for an antiderivative:

$$\int f(x)\,dx := F(x).$$

Based on our knowledge about differentiation, we now collect some antiderivatives of important functions in Table 6.1.

$f(x)$	$\int f(x)\,dx$
$x^n, \quad n \in \mathbb{N}$	$\frac{1}{n+1}x^{n+1}$
$x^{-1}, \quad x \neq 0$	$\log(\|x\|)$
$x^{-n}, \quad x \neq 0, n \in \mathbb{N}, n \neq 1$	$\frac{1}{1-n}x^{1-n}$
$\exp(x)$	$\exp(x)$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\frac{1}{\sqrt{1+x^2}}$	$\operatorname{arsinh}(x)$
$\frac{1}{\sqrt{x^2-1}}, \quad x > 1$	$\operatorname{arcosh}(x)$
$\frac{1}{1-x^2}, \quad \|x\| < 1$	$\operatorname{artanh}(x)$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$	$\tan(x)$
$\frac{1}{\sqrt{1-x^2}}, \quad \|x\| < 1$	$\arcsin(x)$
$-\frac{1}{\sqrt{1-x^2}}, \quad \|x\| < 1$	$\arccos(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$

Table 6.1: Some antiderivatives

6.2 Integration Rules

We now collect some rules for the integration of more complicated functions. Unfortunately, integration is not as straightforward as differentiation and one often has to have an "inspired guess" to find out the antiderivative.

6.2.1 Integration by Substitution

Theorem 6.18. Integration by Substitution

Let I be an Interval, $f : I \to \mathbb{R}$ be continuous and $\phi : [a,b] \to I$ be continuously differentiable. Then

$$\int_a^b f(\phi(t))\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} f(x)\,dx.$$

Proof: Let $F : I \to \mathbb{R}$ be an antiderivative of f. Then, according to the chain rule, the function $F \circ \phi : [a,b] \to \mathbb{R}$ is differentiable with

$$(F \circ \phi)'(t) = F'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t).$$

Therefore,

$$\int_a^b f(\phi(t))\phi'(t)\,dt = \int_a^b (F \circ \phi)'(t)dt = (F \circ \phi)(t)\big|_{t=a}^{t=b} = F(x)\big|_{x=\phi(a)}^{x=\phi(b)} = \int_{\phi(a)}^{\phi(b)} f(x)\,dx.$$

□

As a direct conclusion of this results, we can formulate the following:

Theorem 6.19. Integration by Substitution II

Let I be an interval, $g : I \to \mathbb{R}$ be continuously differentiable and injective with inverse function $g^{-1} : g(I) \to \mathbb{R}$. Let $f : J \to \mathbb{R}$ with $J \subset g(I)$. Then

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))g'(t)\,dt.$$

Example 6.20. We can use the substitution rule to determine the area of an ellipse. The equation of an ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This leads to

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}, \qquad x \in [-a,a].$$

As a consequence, the area of an ellipse is given by

$$A = 2\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\,dx = 2b\int_{-a}^{a} \sqrt{1 - \frac{x^2}{a^2}}\,dx$$

Now we set $g(t) = a\sin(t)$ and $f(x) = \sqrt{1 - \frac{x^2}{a^2}}$. According to the substitution rule, we now have

$$\begin{aligned}\int_{-a}^{a} \sqrt{1 - \frac{x^2}{a^2}}\, dx &= \int_{\arcsin\left(-\frac{a}{a}\right)}^{\arcsin\left(\frac{a}{a}\right)} \sqrt{1 - \frac{a^2\sin^2(t)}{a^2}} (a\sin)'(t)\, dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2(t)} a\cos(t)\, dt = a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(t)\, dt.\end{aligned}$$

With

$$\cos^2(t) = \frac{1}{4}(\exp(it) + \exp(-it))^2 = \frac{1}{4}(\exp(2it) + 2 + \exp(-2it)) = \frac{1}{2}\cos(2t) + \frac{1}{2},$$

we obtain

$$\begin{aligned}A =& 2\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\, dx = 2ab \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(t) dt \\ =& 2ab \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos(2t) + \frac{1}{2} dt = ab \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2t) + 1 dt \\ =& ab \cdot \left(\frac{1}{2}\sin(2t) + t \Big|_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} \right) = ab \cdot \left(\frac{1}{2}\sin(\pi) - \frac{1}{2}\sin(-\pi) + \frac{\pi}{2} + \frac{\pi}{2} \right) = \pi ab.\end{aligned}$$

Note that integration by substitution can also be applied by using the following formalism for determining $\int_a^b f(x)\, dx$: Consider the substitution $x = g(t) \Rightarrow g'(t) = \frac{d}{dt}g(t) = \frac{dx}{dt}$ and "a formal multiplication with dt yields $dx = g'(t)dt$. For a formal determination of the integration bounds, we consider the equations $a = g(t_l)$, $b = g(t_u)$ and thus $t_l = g^{-1}(a)$, $t_u = g^{-1}(b)$. Integration by substitution can then be formally done by

$$\underbrace{\int_a^b}_{\int_{g^{-1}(a)}^{g^{-1}(b)}} f(\underbrace{x}_{=g(t)}) \underbrace{dx}_{=g'(t)dt} = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))g'(t)dt.$$

Example 6.21. a) For $a, b \in \mathbb{R}$, determine

$$\int_a^b x^2 \sin(x^3) dx.$$

Consider the "new variable" $t = x^3$. Then $x = \sqrt[3]{t} = t^{1/3}$ and $\frac{dx}{dt} = \frac{1}{3}t^{-2/3}$ and thus $dx = \frac{1}{3}t^{-2/3}dt$. The integration bounds are given by $t_l = a^3$ and $t_u = b^3$ and thus

$$\int_{a^3}^{b^3} t^{2/3}\sin(t)\frac{1}{3}t^{-2/3}dt = \frac{1}{3}\int_{a^3}^{b^3} \sin(t)dt = -\frac{1}{3}\cos(t)\Big|_{t=a^3}^{t=b^3} = -\frac{1}{3}\cos(x^3)\Big|_{x=a}^{x=b}.$$

b) For $a \geq 0$, $b \geq 0$, determine

$$\int_a^b \exp(\sqrt{x}) dx.$$

Consider the substitution $x = t^2$. Then $\frac{dx}{dt} = 2t$ and thus $dx = 2tdt$. For the integration bounds, consider $a = t_l^2$ and $b = t_u^2$ which yields $t_l = \sqrt{a}$, $t_u = \sqrt{b}$. We now get

$$\begin{aligned}\int_a^b \exp(\sqrt{x})dx = &\int_{\sqrt{a}}^{\sqrt{b}} \exp(t)2tdt\\ =&2\int_{\sqrt{a}}^{\sqrt{b}} t\exp(t)dt\\ &= 2\exp(t)(t-1)|_{t=\sqrt{a}}^{t=\sqrt{b}} = 2\exp(\sqrt{x})(\sqrt{x}-1)|_{x=a}^{x=b}.\end{aligned}$$

c) For $a, b \in [-1, \infty)$, determine

$$\int_a^b \frac{x^2+1}{\sqrt{x+1}}dx.$$

Consider the "new variable" $t = \sqrt{x+1}$. Then $x = t^2 - 1$ and $\frac{dx}{dt} = 2t \Rightarrow dx = 2tdt$ and

$$\begin{aligned}\int_a^b \frac{x^2+1}{\sqrt{x+1}}dx &= \int_{\sqrt{a+1}}^{\sqrt{b+1}} \frac{(t^2-1)^2+1}{t}2tdt\\ &= \int_{\sqrt{a+1}}^{\sqrt{b+1}} 2t^4 - 4t^2 + 4dt\\ &= \frac{2}{5}t^5 - \frac{4}{3}t^3 + 4t\Big|_{t=\sqrt{a+1}}^{t=\sqrt{b+1}}\\ &= \frac{2}{5}\sqrt{x+1}^5 - \frac{4}{3}\sqrt{x+1}^3 + 4\sqrt{x+1}\Big|_{x=a}^{x=b}.\end{aligned}$$

By using the substitution rule, we can also integrate expressions of type $\frac{g'(x)}{g(x)}$.

Corollary 6.22.

For a differentiable function $g : [a, b] \to \mathbb{R}$ with $g(x) \neq 0$ for all $x \in [a, b]$ holds

$$\int_a^b \frac{g'(x)}{g(x)}dx = \log(|g(x)|)|_{x=a}^{x=b}.$$

Proof: For $f(y) = \frac{1}{y}$, the above integral is of type

$$\int_a^b f(g(x))g'(x)dx$$

and thus, the result follows by the substitution rule. □

Example 6.23. a) For $a, b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ holds

$$\begin{aligned}\int_a^b \tan(x)dx = \int_a^b \frac{\sin(x)}{\cos(x)}dx = \int_a^b \frac{-\cos'(x)}{\cos(x)}dx\\ = -\log(|\cos(x)|)|_{x=a}^{x=b}.\end{aligned}$$

b) For $a, b \in \mathbb{R}$ holds

$$\begin{aligned}\int_a^b \frac{x}{x^2+1}dx =& \frac{1}{2}\int_a^b \frac{2x}{x^2+1}dx\\ =&\frac{1}{2}\int_a^b \frac{(x^2+1)'}{x^2+1}dx\\ =&\frac{1}{2}\log(|x^2+1|)\Big|_{x=a}^{x=b}.\end{aligned}$$

6.2.2 Integration by Parts

In this part we integrate products of functions.

Theorem 6.24. Integration by Parts

Let I be an Interval and $f, g : I \to \mathbb{R}$ be differentiable. Then for $a, b \in I$ holds

$$\int_a^b f'(x)g(x)dx = f(x)g(x)|_{x=a}^{x=b} - \int_a^b f(x)g'(x)dx.$$

Proof: The product rule of differentiation implies

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

and thus

$$\begin{aligned}f(x)g(x)|_{x=a}^{x=b} &= \int_a^b (f(x)g(x))'dx\\ =\int_a^b f'(x)g(x) + f(x)g'(x)dx &= \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx.\end{aligned}$$

Solving this equation for $\int_a^b f'(x)g(x)dx$, we get the desired equation. □

Example 6.25. a) For $a, b \in \mathbb{R}$, determine

$$\int_a^b x\exp(x)dx.$$

We use integration by parts with $f'(x) = f(x) = \exp(x)$ and $g(x) = x$. Then

$$\begin{aligned}\int_a^b \exp(x)x dx &= x\exp(x)|_{x=a}^{x=b} - \int_a^b \exp(x)dx\\ &= x\exp(x)|_{x=a}^{x=b} - \exp(x)|_{x=a}^{x=b}\\ &= (x-1)\exp(x)|_{x=a}^{x=b}.\end{aligned}$$

It is very important to note that an unlucky choice of f and g may be misleading. For instance, if we choose $f'(x) = x$ and $g(x) = \exp(x)$. Then integration by parts gives

$$\int_a^b x\exp(x)dx = \frac{1}{2}x^2\exp(x)\Big|_{x=a}^{x=b} - \int_a^b \frac{1}{2}x^2\exp(x)dx.$$

This formula is mathematically correct, but it does not lead to the explicit determination of the integral.

b) Integrate $\sin(x)\cos(x)$:

$$\begin{aligned}
&\int_a^b \underbrace{\sin(x)}_{f'(x)=-\cos'(x)} \underbrace{\cos(x)}_{=g(x)}\,dx \\
&= -\cos^2(x)\big|_{x=a}^{x=b} - \int_a^b (-\cos(x))(-\sin(x))dx \\
&= -\cos^2(x)\big|_{x=a}^{x=b} - \int_a^b \cos(x)\sin(x)dx.
\end{aligned}$$

Solving this equation for $\int_a^b \sin(x)\cos(x)dx$, we obtain

$$\int_a^b \sin(x)\cos(x)dx = -\frac{1}{2}\cos^2(x)\bigg|_{x=a}^{x=b}.$$

c) For $a, b \in (0,\infty)$ and $\lambda \in \mathbb{R}\backslash\{-1\}$, determine the integral

$$\int_a^b x^\lambda \log(x)dx.$$

Defining $f(x) = \frac{1}{\lambda+1}x^{\lambda+1}$, $g(x) = \log(x)$, integration by parts leads to

$$\begin{aligned}
&\int_a^b x^\lambda \log(x)dx \\
&= \frac{1}{\lambda+1}x^{\lambda+1}\log(x)\bigg|_{x=a}^{x=b} - \int_a^b \frac{1}{\lambda+1}x^{\lambda+1}\frac{1}{x}dx \\
&= \frac{1}{\lambda+1}x^{\lambda+1}\log(x)\bigg|_{x=a}^{x=b} - \int_a^b \frac{1}{\lambda+1}x^{\lambda}dx \\
&= \frac{1}{\lambda+1}x^{\lambda+1}\log(x)\bigg|_{x=a}^{x=b} - \frac{1}{(\lambda+1)^2}x^{\lambda+1}\bigg|_{x=a}^{x=b} \\
&= \frac{1}{\lambda+1}x^{\lambda+1}\left(\log(x) - \frac{1}{\lambda+1}\right)\bigg|_{x=a}^{x=b}.
\end{aligned}$$

d) For $a, b \in (0,\infty)$, determine the integral

$$\int_a^b \frac{\log(x)}{x}dx.$$

Defining $f(x) = g(x) = \log(x)$, integration by parts leads to

$$\int_a^b \frac{\log(x)}{x}dx = \log^2(x)\big|_{x=a}^{x=b} - \int_a^b \frac{\log(x)}{x}dx.$$

Solving this for $\int_a^b \frac{\log(x)}{x}dx$, we obtain

$$\int_a^b \frac{\log(x)}{x}dx = \frac{1}{2}\log^2(x)\big|_{x=a}^{x=b}.$$

6.3 Integration of Rational Functions

We know by Theorem 4.29 that every rational function $f(x)$ possesses a representation as the sum of some polynomial r and a strict proper rational function $\frac{p(x)}{q(x)}$, i.e

$$f(x) = r(x) + \frac{p(x)}{q(x)}, \quad \deg(p) < \deg(q).$$

By using linearity of the integral, we can integrate each summand separately. Thus we restrict our discussion here to strict proper rational functions. Furthermore by Theorem 4.31 we can write every strict proper rational function as a sum of partial fractions $\frac{A}{(x-x_0)^s}$, with x_0 being a root of q and A being some constant and $s \geq 1$.

Altogether we have that it is enough to know the antiderivative of the partial fractions in order to integrate arbitrary rational functions.
Let us first restrict to the case $x_0 \in \mathbb{R}$ and $s = 1$. Then we have

$$\int \frac{A}{x - x_0}\, dx = A \log|x - x_0| + const., \qquad x_0 \in \mathbb{R}.$$

It is clear that this equality is only valid in the case $x_0 \in \mathbb{R}$ since the antiderivative is a real valued function.
Now consider $x_0 \in \mathbb{C} \setminus \mathbb{R}$. Because of q having only real coefficients we know that also $\overline{x_0}$ is a root of q thus the partial fraction decomposition at least has the terms $\frac{A}{x-x_0}$ and $\frac{B}{x-\overline{x_0}}$ included in the sum. Then setting $x_0 := \alpha + i\beta$

$$\int \frac{A}{x - x_0} + \frac{B}{x - \overline{x_0}}\, dx = \int \frac{\overbrace{(A+B)}^{=:a}(x-\alpha) + i\,\overbrace{(A-B)}^{=:b}\beta}{\beta^2 + (x-\alpha)^2}\, dx.$$

Two integrals arise in this case. For the first one the antiderivative can be calculated using the substitution $t = \frac{x-\alpha}{\beta}$, $dt = \frac{dx}{\beta}$

$$\begin{aligned}\int \frac{a(x-\alpha)}{\beta^2 + (x-\alpha)^2}\, dx &= a \int \frac{\frac{x-\alpha}{\beta}}{1 + (\frac{x-\alpha}{\beta})^2}\, \frac{dx}{\beta} = a \int \frac{t}{1+t^2}\, dt \\ &= \frac{a}{2} \log(1+t^2) + const. = \frac{a}{2} \log\left(1 + \left(\frac{x-\alpha}{\beta}\right)^2\right) + const.\end{aligned}$$

For the second term we have again using the same substitution as above

$$\begin{aligned}\int \frac{b\beta}{\beta^2 + (x-\alpha)^2}\, dx &= b \int \frac{1}{1 + (\frac{x-\alpha}{\beta})^2}\, \frac{dx}{\beta} = b \int \frac{1}{1+t^2}\, dt \\ &= b \arctan(t) + const = b \arctan\left(\frac{x-\alpha}{\beta}\right) + const.\end{aligned}$$

Altogether we have for $x_0 \in \mathbb{C} \setminus \mathbb{R}$ and $s = 1$

$$\int \frac{A}{x - x_0} + \frac{B}{x - \overline{x_0}}\, dx = \frac{A+B}{2} \log\left(1 + \left(\frac{x-\alpha}{\beta}\right)^2\right) + i(A-B) \arctan\left(\frac{x-\alpha}{\beta}\right) + const.$$

Now we want to investigate the case $s > 1$. Here it is not relevant if x_0 is real or complex. We have

$$\int \frac{A}{(x-x_0)^s}\, dx = \frac{A}{1-s}(x-x_0)^{1-s} + const.$$

Example 6.26. We want to integrate $f(x) = \frac{x^5+2x^3+4x^2-3x}{(x^2+1)^2}$. First of all we decompose f into a polynomial part and a strict proper function using polynomial division

$$\frac{x^5 + 2x^3 + 4x^2 - 3x}{(x^2+1)^2} = x + \frac{4x^2 - 4x}{(x^2+1)^2} = x + \frac{4x^2 - 4x}{(x+i)^2(x-i)^2}.$$

The polynomial part has the antiderivative

$$\int x\,dx = \frac{x^2}{2} + const.$$

The strict proper part has a partial fraction decomposition

$$\frac{4x^2 - 4x}{(x+i)^2(x-i)^2} = \frac{A}{x+i} + \frac{B}{(x+i)^2} + \frac{\overline{A}}{x-i} + \frac{\overline{B}}{(x-i)^2},$$

with $A = i$ and $B = 1 - i$. Thus we have

$$\begin{aligned}\int \frac{i}{x+i} - \frac{i}{x-i}\,dx &= 2\arctan(x) + const.\\ \int \frac{1-i}{(x+i)^2}\,dx &= -\frac{1-i}{x+i} + const.\\ \int \frac{1+i}{(x-i)^2}\,dx &= -\frac{1+i}{x-i} + const.\end{aligned}$$

Altogether the antiderivative is

$$\begin{aligned}\int f(x)\,dx &= \frac{1}{2}x^2 + 2\arctan(x) - \frac{1-i}{x+i} - \frac{1+i}{x-i} + const.\\ &= \frac{1}{2}x^2 + 2\arctan(x) - \frac{2x-2}{x^2+1} + const.\end{aligned}$$

6.4 Integration on Unbounded Domains and Integration of Unbounded Functions

So far, we have integrated bounded functions on compact intervals $[a,b]$. In this part we skip these two assumptions by extending the integral notion to functions that may have a pole and/or are defined on unbounded intervals.

6.4.1 Unbounded Interval

Definition 6.27. Integration on unbounded intervals

Let $f : [a,\infty) \to \mathbb{R}$ be a function with the property that for all $b > a$ the restriction of f to $[a,b]$ belongs to $\mathcal{R}([a,b])$. If

$$\lim_{b\to\infty} \int_a^b f(x)dx$$

exists, then we say that

$$\int_a^\infty f(x)dx$$

is convergent. Otherwise, we speak of divergence.

Example 6.28. a) For integrating the function $\exp(-x)$ on the interval $[0,\infty)$, we compute

$$\begin{aligned}\int_0^\infty \exp(-x)dx &= \lim_{b\to\infty}\int_0^b \exp(-x)dx\\ &= \lim_{b\to\infty} -\exp(-x)|_{x=0}^{x=b}\\ &=1-\lim_{b\to\infty}\exp(-b)=1.\end{aligned}$$

b) For $\alpha>0$, consider

$$\int_1^\infty \frac{1}{x^\alpha}dx.$$

We know that for $b>1$ holds

$$\int_1^b \frac{1}{x^\alpha}dx = \begin{cases}\frac{1}{1-\alpha}\frac{1}{x^{\alpha-1}}\Big|_{x=1}^{x=b} & :\alpha\neq 1,\\ \log(x)|_{x=1}^{x=b} & :\alpha=1.\end{cases}$$

Since $\lim_{b\to\infty}\log(b)=\infty$ and

$$\lim_{b\to\infty}\frac{1}{x^{\alpha-1}} = \begin{cases}0 & :\alpha>1,\\ \infty & :\alpha<1,\end{cases}$$

we have

$$\int_1^\infty \frac{1}{x^\alpha}dx = \begin{cases}\frac{1}{\alpha-1} & :\alpha>1,\\ \infty & :\alpha\leq 1.\end{cases}$$

It is straightforward to define the integral of a function defined on some interval unbounded from below by

$$\int_{-\infty}^a f(x)dx = \lim_{b\to-\infty}\int_b^a f(x)dx.$$

We now define the integral of functions defined on the whole real axis.

Definition 6.29.

Let $f:\mathbb{R}\to\mathbb{R}$ be a function with the property that for all $a,b\in\mathbb{R}$ the restriction of f to $[a,b]$ belongs to $\mathcal{R}([a,b])$. If there exists some $c\in\mathbb{R}$ such that both integrals

$$\int_c^\infty f(x)dx, \qquad \int_{-\infty}^c f(x)dx$$

exist, then we say that

$$\int_{-\infty}^\infty f(x)dx$$

is convergent. Otherwise, we speak of divergence. In case of convergence we set

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx.$$

We remark without proof that the above definition is independent of c.

Example 6.30. a) Consider

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx &= \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_{0}^{\infty} \frac{1}{1+x^2}dx \\ &= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}dx \\ &= \lim_{a\to-\infty} \arctan(x)|_{x=a}^{x=0} + \lim_{b\to\infty} \arctan(x)|_{x=0}^{x=b} \\ &= - \lim_{a\to-\infty} \arctan(a) + \lim_{b\to\infty} \arctan(b) \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi.\end{aligned}$$

b) The integral

$$\int_{-\infty}^{\infty} xdx$$

diverges since both integrals

$$\begin{aligned}\int_{-\infty}^{0} xdx &= \lim_{a\to-\infty} \int_{a}^{0} xdx = \lim_{a\to-\infty} \frac{1}{2}x^2\Big|_{x=a}^{x=0} = -\infty \\ \int_{0}^{\infty} xdx &= \lim_{b\to\infty} \int_{0}^{b} xdx = \lim_{b\to\infty} \frac{1}{2}x^2\Big|_{x=0}^{x=b} = \infty\end{aligned}$$

diverge. This example shows that the convergence of $\int_{-\infty}^{\infty} f(x)dx$ is not equivalent to the existence of the limit

$$\lim_{a\to\infty} \int_{-a}^{a} f(x)dx.$$

However, in case of convergence, the integral $\int_{-\infty}^{\infty} f(x)dx$ coincides with the above limit.

The majorant criterion for series says that if the absolut values of the addends of a given series can be bounded from above by the addends of a convergent series, then (absolut) convergence of the given series can be concluded.

Conversely, the minorant criterion for series says that if the addends of a given series can be bounded from below by the addends of a series that diverges to $+\infty$, then also the given series diverges to $+\infty$.

Analogue criteria hold true for integrals. We skip the proofs since they are totally analogous to those of the majorant and minorant criteria.

Theorem 6.31.

Let $f, g : [a, \infty) \to \mathbb{R}$ such that for all $b \in [a, \infty)$, the restrictions of f and g to $[a, b]$ are Riemann-integrable.

(i) If $|f(x)| \leq g(x)$ for all $x \in [a, \infty)$ and $\int_a^\infty g(x)dx$ converges, then also $\int_a^\infty f(x)dx$ converges and it holds that

$$\left|\int_a^\infty f(x)dx\right| \leq \int_a^\infty |f(x)|dx \leq \int_a^\infty g(x)dx.$$

(ii) If $g(x) \leq f(x)$ for all $x \in [a, \infty)$ and $\int_a^\infty g(x)dx = +\infty$, then $\int_a^\infty f(x)dx = +\infty$.

Example 6.32. a) Consider

$$\int_1^\infty \frac{x}{x^2+1}dx.$$

We have

$$\lim_{x\to\infty} x \cdot \frac{x}{x^2+1} = 1.$$

For large enough $x \in \mathbb{R}$, we therefore have

$$\left|x \cdot \frac{x}{x^2+1}\right| = \frac{x^2}{x^2+1} \geq \frac{1}{2}$$

and thus

$$\frac{x}{x^2+1} \geq \frac{1}{2x}.$$

Since

$$\int_1^\infty \frac{1}{2x}dx$$

is divergent,

$$\int_1^\infty \frac{x}{x^2+1}dx$$

is divergent, too.

b) Consider

$$\int_1^\infty \frac{\sqrt{x}}{x^2+1}dx.$$

We have

$$\lim_{x\to\infty} x^{3/2} \cdot \frac{\sqrt{x}}{x^2+1} = 1.$$

For large enough $x \in \mathbb{R}$, we therefore have

$$\left|x^{3/2} \cdot \frac{\sqrt{x}}{x^2+1}\right| = \frac{x^2}{x^2+1} \leq 2$$

and thus

$$\frac{\sqrt{x}}{x^2+1} \leq \frac{2}{x^{3/2}}.$$

Since
$$\int_1^\infty \frac{2}{x^{3/2}} dx$$
is convergent, so
$$\int_1^\infty \frac{\sqrt{x}}{x^2+1} dx$$
is convergent, too.

Now we use integrals on unbounded domains to check whether series are convergent or not.

Theorem 6.33. Integral Criterion for Series

Let $f : [0, \infty) \to \mathbb{R}$ be monotonically decreasing and non-negative. Then the series
$$\sum_{k=0}^{\infty} f(k)$$
is convergent if and only if the integral
$$\int_0^\infty f(x) dx$$
converges. In the case of convergence, the following estimate holds true:
$$0 \le \sum_{k=0}^{\infty} f(k) - \int_0^\infty f(x)\ dx \le f(0) \tag{6.1}$$

Proof: Monotonicity of f implies that for $k-1 \le x \le k$ holds $f(k) \le f(x) \le f(k-1)$. Monotonicity of the integral therefore leads to the inequality
$$f(k) = \int_{k-1}^{k} f(k) dx \le \int_{k-1}^{k} f(x) dx \le \int_{k-1}^{k} f(k-1) dx = f(k-1).$$
Therefore
$$\int_1^{n+2} f(x) dx \le \sum_{k=1}^{n+1} f(k) \le \int_0^{n+1} f(x) dx \le \sum_{k=0}^{n} f(k).$$
Using this inequality, we can directly conclude that, if one of the limits (integral or sum) as $n \to \infty$ exists, then the other limit (sum or integral) also exists.

In case of convergence necessarily $(f(k))_{k \in \mathbb{N}_0}$ is a zero sequence. Therefore
$$0 \le \sum_{k=0}^{n} f(k) - \int_0^{n+1} f(x)\ dx \le \sum_{k=0}^{n} f(k) - \sum_{k=1}^{n+1} f(k) = f(0) - f(n+1)$$
implies (6.1) for $n \to \infty$.

□

Remark:

Since the convergence of $\int_a^\infty f(x)dx$ does not depend on $a \in \mathbb{R}$, the above result can also be slightly generalised in a way that the convergence of the series $\sum_{k=a}^\infty f(k)$ for $a \in \mathbb{N}$ is equivalent to that of the integral $\int_a^\infty f(x)dx$.

Example 6.34. a) Using the results of Example 6.28 b), we see that

$$\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$$

converges for $\alpha > 1$ and is divergent for $\alpha \leq 1$.

b) For $\alpha \in \mathbb{R}$ consider

$$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^\alpha}.$$

We have

$$\begin{aligned}\int_2^\infty \frac{1}{x(\log(x))^\alpha}\,dx &= \lim_{n\to\infty} \begin{cases} \frac{(\log(x))^{1-\alpha}}{1-\alpha}\Big|_{x=2}^{x=n} & : \alpha \neq 1, \\ \log(\log(x))|_{x=2}^{x=n} & : \alpha = 1 \end{cases} \\ &= \begin{cases} -\frac{(\log(2))^{1-\alpha}}{1-\alpha} & : \alpha > 1, \\ \infty & : \alpha \leq 1. \end{cases}\end{aligned}$$

Therefore, the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^\alpha}.$$

is convergent if and only if $\alpha > 1$.

6.4.2 Unbounded Integrand

Here we consider integrals of functions that may have a pole in the domain of integration.

Definition 6.35. Integration of unbounded functions

Let $f : (a, b] \to \mathbb{R}$ be given with the property that for all $\varepsilon > 0$, the restriction of f to the interval $[a + \varepsilon, b]$ is Riemann-integrable. If

$$\lim_{\varepsilon \searrow 0} \int_{a+\varepsilon}^{b} f(x)dx$$

exists, then we say that

$$\int_a^b f(x)dx$$

is convergent. Otherwise, we speak of divergence.

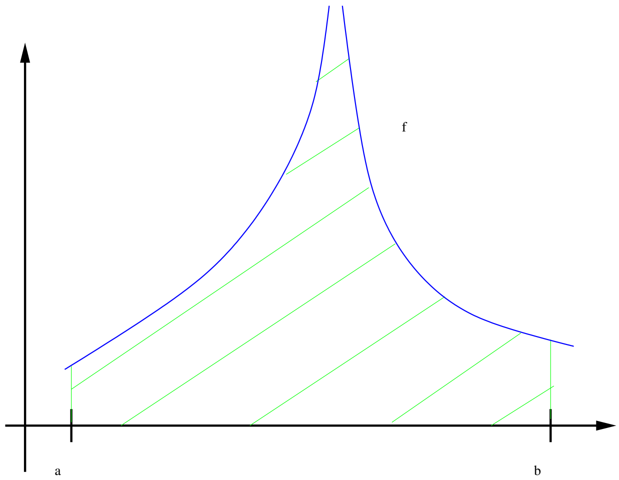


Figure 6.3: Integral of a function with a pole

Example 6.36. a) For integrating the function $\log(x)$ from 0 to 1, we first compute

$$\int_\varepsilon^1 \log(x)dx = \int_\varepsilon^1 1 \cdot \log(x)dx = x\log(x)\big|_{x=\varepsilon}^{x=1} - \int_\varepsilon^1 x \cdot \frac{1}{x}dx$$
$$= x(\log(x)-1)\big|_{x=\varepsilon}^{x=1} \ .$$

Therefore

$$\begin{aligned}\int_0^1 \log(x)dx &= \lim_{\varepsilon\searrow 0}\int_\varepsilon^1 \log(x)dx \\ &= \log(1) - 1 - \lim_{\varepsilon\searrow 0}\varepsilon(\log(\varepsilon)-1) \\ &= -1 - \lim_{\varepsilon\searrow 0}\varepsilon\log(\varepsilon) \\ &= -1 - \lim_{\varepsilon\searrow 0}\frac{\log(\varepsilon)}{\frac{1}{\varepsilon}} \\ &= -1 - \lim_{\varepsilon\searrow 0}\frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = -1 \ .\end{aligned}$$

b) For $\alpha > 0$, consider

$$\int_0^1 \frac{1}{x^\alpha}dx = \lim_{\varepsilon\searrow 0}\int_\varepsilon^1 \frac{1}{x^\alpha}dx = \lim_{\varepsilon\searrow 0}\begin{cases}\frac{1}{1-\alpha}\frac{1}{x^{\alpha-1}}\Big|_{x=\varepsilon}^{x=1} & : \alpha \neq 1, \\ \log(x)|_{x=\varepsilon}^{x=1} & : \alpha = 1.\end{cases} = \begin{cases}\frac{1}{1-\alpha} & : \alpha < 1, \\ \infty & : \alpha \geq 1.\end{cases}$$

Again we can formulate a majorant and a minorant criterion for the convergence of integrals of unbounded functions.

Theorem 6.37.

Let $f, g : (a, b] \to \mathbb{R}$ such that for all $\varepsilon > 0$, the restrictions of f and g to $[a+\varepsilon, b]$ are Riemann-integrable.

(i) If $|f(x)| \leq g(x)$ for all $x \in (a, b]$ and $\int_a^b g(x)dx$ converges, then also $\int_a^b f(x)dx$

converges and it holds that

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx \le \int_a^b g(x)dx.$$

(ii) If $g(x) \le f(x)$ for all $x \in (a,b]$ and $\int_a^b g(x)dx = +\infty$, then $\int_a^b f(x)dx = +\infty$.

Example 6.38. a) Consider

$$\int_0^1 \frac{\cos^2(x) + 2\sin(x)}{\sqrt[3]{x^2}}dx.$$

This integral is convergent due to

$$\left|\frac{\cos^2(x) + 2\sin(x)}{\sqrt[3]{x^2}}\right| \le \frac{3}{x^{2/3}} \qquad \text{for } x \in (0,1].$$

b) For integrating the function $\frac{\cos(x)}{\sqrt{x}}$ over the entire positive half-axis, we split

$$\int_0^\infty \frac{\cos(x)}{\sqrt{x}}dx = \int_0^1 \frac{\cos(x)}{\sqrt{x}}dx + \int_1^\infty \frac{\cos(x)}{\sqrt{x}}dx.$$

The first addend is convergent due to

$$\left|\frac{\cos(x)}{\sqrt{x}}\right| \le \frac{1}{\sqrt{x}} \quad \text{for } x \in \ (0,1], \qquad \int_0^1 \frac{1}{\sqrt{x}}\,dx = 2 < \infty.$$

For the second addend, we make use of integration by parts to determine

$$\int_1^b \frac{\cos(x)}{\sqrt{x}}dx = \left.\frac{\sin(x)}{\sqrt{x}}\right|_{x=1}^{x=b} + \frac{1}{2}\int_1^b \frac{\sin(x)}{x^{3/2}}.$$

The integral $\int_1^\infty \frac{\sin(x)}{x^{3/2}}$ converges due to $\left|\frac{\sin(x)}{x^{3/2}}\right| \le \frac{1}{x^{3/2}}$ for all $x \in [1,\infty)$. Furthermore,

$$\lim_{b\to\infty} \left.\frac{\sin(x)}{\sqrt{x}}\right|_{x=1}^{x=b} = \lim_{b\to\infty} \frac{\sin(b)}{\sqrt{b}} - \sin(1) = -\sin(1).$$

Therefore, the integral $\int_0^\infty \frac{\cos(x)}{\sqrt{x}}dx$ converges.

For functions $f : [a,b]\backslash\{c\} \to \mathbb{R}$ it is straightforward to define

$$\int_a^b f(x)dx = \lim_{\varepsilon\nearrow c}\int_a^\varepsilon f(x)dx + \lim_{\varepsilon\searrow c}\int_\varepsilon^b f(x)dx.$$

Example 6.39.

$$\begin{aligned}\int_{-1}^1 \frac{1}{\sqrt{|x|}}dx &= \lim_{\varepsilon\nearrow 0}\int_{-1}^\varepsilon \frac{1}{\sqrt{|x|}}dx + \lim_{\varepsilon\searrow 0}\int_\varepsilon^1 \frac{1}{\sqrt{|x|}}dx \\ &= \lim_{\varepsilon\nearrow 0} -2\sqrt{-x}\Big|_{x=-1}^{x=\varepsilon} + \lim_{\varepsilon\searrow 0} 2\sqrt{x}\Big|_{x=\varepsilon}^{x=1} = 4\end{aligned}$$

The remaining parts of this Chapter on parameter-dependent integrals, solids of revolution, path integrals and Fourier series follow [?].

6.5 Parameter-dependent integrals

In this section we consider integrals of the form

$$F(x) \quad := \quad \int_a^b f(x,y)\; dy \tag{6.2}$$

where the integrand f additionaly depends on some free parameter $x \in I$, where I is some real interval. Precisely, the function $f : I \times [a,b] \to \mathbb{R}$ has the property that for each fixed $x \in I$, the function $f(x,\cdot) : [a,b] \to \mathbb{R}, y \mapsto f(x,y)$ is Riemann integrable.

We will investigate continuity and differentiability of the integral function

$$F : I \to \mathbb{R},\ x \mapsto \int_a^b f(x,y)\; dy\ .$$

For preparation we need the following definition.

Definition 6.40.

A function $f : D \to \mathbb{R}^m$, $D \subset \mathbb{R}^n$, $m,n \in \mathbb{N}$, is called uniformly continuous, if for each $\varepsilon > 0$ there exits a $\delta > 0$ such that for all $x, x_0 \in D$ with $||x - x_0|| < \delta$ holds $||f(x) - f(x_0)|| < \varepsilon$. Here $||\cdot||$ denotes some norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, for example the Euclidean norm.

Note that a uniformly continuous function is continuous. In general the converse is not true, but if the domain of a continuous function is compact it is already uniformly continuous.

Theorem 6.41.

Let $D \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be compact. A continuous function $f : D \to \mathbb{R}^m$, $m \in \mathbb{N}$, is uniformly continuous.

Proof: Suppose that f is not uniformly continuous. Then there exists an $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there are $x_n, y_n \in D$ with $||x_n - y_n|| < \frac{1}{n}$ and $||f(x_n) - f(y_n)|| > \varepsilon$. Since D is compact, there are convergent subsequences $(x_{n_k})_{k\in\mathbb{N}}$ and $(y_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ which necessarily converge to the same limit. Set

$$z := \lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k}.$$

Since f is continuous, we conclude

$$\lim_{k\to\infty} (f(x_{n_k}) - f(y_{n_k})) = \lim_{k\to\infty} f(x_{n_k}) - \lim_{k\to\infty} (y_{n_k}) = f(z) - f(z) = 0,$$

a contradiction to $||f(x_{n_k}) - f(y_{n_k})|| > \varepsilon$ for all $k \in \mathbb{N}$. □

Now we consider continuity of the integral function (6.2).

Theorem 6.42.

Let $a, b \in \mathbb{R}$, $a < b$, and $I \subset \mathbb{R}$ be an interval. If $f : I \times [a,b] \to \mathbb{R}$ is continuous, then $F : I \to \mathbb{R},\ x \mapsto \int_a^b f(x,y)\; dy$ is well-defined and continuous on I.

Proof:

Since f is continuous, for each fixed $x \in I$, the function $f(x,\cdot) : [a,b] \to \mathbb{R}, y \mapsto f(x,y)$ is continuous too and therefore integrable on $[a,b]$. Hence F is well-defined. Now let $x_0 \in I$ and $I_0 \subset I$ be a compact interval that contains x_0. If x_0 is an inner point of I then we can also choose I_0 such that x_0 is an inner point of I_0. Now $I_0 \times [a,b]$ is compact in $\mathbb{R}^2$ and hence f is uniformly continuous on $I_0 \times [a,b]$ by Theorem 6.41. This means that for given $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in I_0$ with $|x - x_0| < \delta$ and all $y \in [a,b]$ holds $|f(x,y) - f(x_0,y)| < \varepsilon$.

Thus for $x, x_0 \in I_0$ with $|x - x_0| < \delta$ and $y \in [a,b]$ we have

$$|F(x) - F(x_0)| = \left|\int_a^b f(x,y) - f(x_0,y)\, dy\right| \le \int_a^b |f(x,y) - f(x_0,y)|\, dy \le \varepsilon(b-a) .$$

Therefore F is continuous in x_0. □

Next we consider differentiability of the integral function (6.2).

Theorem 6.43.

Let $a, b \in \mathbb{R}$, $a < b$, and $I \subset \mathbb{R}$ be an interval. If $f : I \times [a,b] \to \mathbb{R}$ is continuous and if for each fixed $y \in [a,b]$ the function $f(\cdot,y) : I \to \mathbb{R},\ x \mapsto f(x,y)$ is continuously differentiable, then also $F : I \to \mathbb{R},\ x \mapsto \int_a^b f(x,y)\, dy$ is continuously differentiable on I with

$$F'(x) = \int_a^b \frac{\partial f}{\partial x}(x,y)\, dy .$$

Proof:

Let $x_0 \in I$. Then for $x \in I\backslash\{x_0\}$ and each $y \in [a,b]$ the mean value theorem applied to $f(\cdot,y)$ supplies a $\xi_{x,y}$ between x and x_0 such that

$$\frac{F(x) - F(x_0)}{x - x_0} = \int_a^b \frac{f(x,y) - f(x_0,y)}{x - x_0}\, dy = \int_a^b \frac{\partial f}{\partial x}(\xi_{x,y}, y)\, dy .$$

By assumption and Theorem 6.42 the function

$$G : I \to \mathbb{R},\ x \mapsto \int_a^b \frac{\partial f}{\partial x}(x,y)\, dy$$

is well-defined and continuous on I. Since $\xi_{x,y} \to x_0$ for $x \to x_0$ this implies

$$F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = \int_a^b \frac{\partial f}{\partial x}(x_0,y)\, dy .$$

□

Example 6.44.

$$F(x) := \int_1^\pi \frac{\sin(tx)}{t}\, dt$$

$$F'(x) := \int_1^{\pi} \cos(tx)\ dt$$
$$F''(x) := -\int_1^{\pi} t\sin(tx)\ dt$$

For parameter-dependent improper intergrals analog statements for continuity and differentiability hold true which we will state without proof.

Definition 6.45.

Let $a \in \mathbb{R}$, $I \subset \mathbb{R}$ be an interval and $f : I \times [a, \infty) \to \mathbb{R}$. Suppose that for each fixed $x \in I$ the improper integral $\int_a^\infty f(x, y)\ dy$ exists. Then the integral $\int_a^\infty f(x, y)\ dy$ is called uniformly convergent if for each $\varepsilon > 0$ there is a constant $K > a$ such that for all $x \in I$ and all $b_1, b_2 \geq K$ holds

$$\left| \int_{b_1}^{b_2} f(x, y)\ dy \right| < \varepsilon$$

Theorem 6.46.

Let $a \in \mathbb{R}$ and $I \subset \mathbb{R}$ be an interval. If $f : I \times [a, \infty) \to \mathbb{R}$ is continuous and continuously differentiable with respect to the first variable x and if the integrals

$$\int_a^\infty f(x, y)\ dy \quad \text{and} \quad \int_a^\infty \frac{\partial f}{\partial x}(x, y)\ dy$$

converge on each compact subset of I uniformly, then

$$F : I \to \mathbb{R}, x \mapsto \int_a^\infty f(x, y)\ dy$$

is continuously differentiable and its derivative is given by

$$F'(x) = \int_a^\infty \frac{\partial f}{\partial x}(x, y)\ dy\ .$$

Example 6.47 (Gamma function)**.**

$$\Gamma \ : \ (0, \infty) \to \mathbb{R},\ x \mapsto \int_0^\infty e^{-y} y^{x-1}\ dy$$
$$\Gamma'(x) \ = \ \int_0^\infty e^{-y} y^{x-1} \log(y)\ dy$$

6.6 Solids of revolution, path integrals

In this section we give some simple geometrical applications of one-dimensional integration theory, for example computing volumes of certain bodies and length's of curves in the three-dimensional euclidean space.

A solid of revolution is a volume which is obtained by rotating a plane curve around an axis. For example if $f : [a, b] \to [0, \infty)$ is a nonnegative continuous function and if the graph of f is rotated around the x-axis, then the enclosed volume V_{rot} can easily be computed by

$$V_{\text{rot}} = \int_a^b \pi f(x)^2 \, dx = \pi \int_a^b f(x)^2 \, dx \ .$$

Example 6.48 (ellipsoid)**.** Let us consider an ellipse given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with semi-axes $a, b > 0$.

In this case $f : [-a, a] \to \mathbb{R}, x \mapsto b\sqrt{1 - \left(\frac{x}{a}\right)^2}$. Hence

$$\begin{aligned} V_{\text{rot}} &= \pi \int_{-a}^{a} f(x)^2 \, dx = \pi \int_{-a}^{a} b^2 \left(1 - \left(\frac{x}{a}\right)^2\right) \, dx \\ &= \pi b^2 (x - \frac{1}{3a^2} x^3)|_{x=-a}^{x=a} = \frac{4}{3}\pi a b^2. \end{aligned}$$

If $a = b =: r$, then the ellipsoid is a ball with radius r and volume

$$V_{\text{ball}} = \frac{4}{3}\pi r^3.$$

Now we will derive a formula for computing the lateral surface M_{rot} of solids of revolution. First recall that the lateral surface M_C of a cone with circular ground face of radius r and lateral height l is given by

$$M_C = \pi r l \ .$$

Thus the lateral surface M_{tC} of a truncated cone with circular ground face of radius r_1, circular top face of radius $r_2 < r_1$ and lateral height l is given by

$$\begin{aligned} M_{tC} &= M_{C_1} - M_{C_2} = \pi r_1 l_1 - \pi r_2 l_2 = \pi(r_1 l_1 + r_2 l_1 - r_1 l_2 - r_2 l_2) \\ &= \pi(r_1 + r_2)(l_1 - l_2) = \pi(r_1 + r_2) l \ , \end{aligned}$$

where l_1 is the lateral height of the complete cone and l_2 that of its truncated top. (Recall that $\frac{r_1}{l_1} = \frac{r_2}{l_2}$.)

Now let $f : [a, b] \to \mathbb{R}_{\geq 0}$ be a non-negative function which is continuously differentiable on (a, b). We want to approximate its lateral surface area by summing up lateral surfaces of certain truncated cones. Precisely, consider a decomposition Z: $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of $[a, b]$ and define $y_i := f(x_i)$, $\Delta x_i = x_{i+1} - x_i$ and $\Delta y_i = y_{i+1} - y_i$. Then the sum $M(Z)$ of all lateral surfaces of the n truncated cones with circular faces of radii y_i, y_{i+1} and lateral heights $l_i := \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$, $i = 0, ..., n-1$, is given by

$$M(Z) = \sum_{i=0}^{n-1} \pi(y_i + y_{i+1})\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= \quad 2\pi \sum_{i=0}^{n-1} \frac{y_i + y_{i+1}}{2} \cdot \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

Now, roughly speaking, for $\Delta x_i \to 0$ the right-hand side converges to the integral

$$M_{\text{rot}} \quad = \quad 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx \ .$$

Example 6.49 (Surface of a ball)**.** Let us consider a ball of radius $r > 0$

$$x^2 + y^2 = r^2,$$

The corresponding function $f : [-r, r] \to \mathbb{R}, x \mapsto \sqrt{r^2 - x^2}$ is continuously differentiable on $(-r, r)$ with derivative

$$f'(x) = \frac{-x}{\sqrt{r^2 - x^2}}.$$

Thus the surface of the ball is

$$\begin{aligned} M_{\text{rot}} \quad &= \quad 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\ &= \quad 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2 - x^2}}\, dx = 2\pi r \int_{-r}^{r} dx = 4\pi r^2 \ . \end{aligned}$$

In the following we will consider curves.

Definition 6.50.

a) A continuous function $c : [a, b] \to \mathbb{R}^n$, $x \mapsto (c_1(x), ..., c_n(x))^T$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$, $a < b$, is called a curve in $\mathbb{R}^n$. The vectors $c(a), c(b) \in \mathbb{R}^n$ are called starting point and end point of the curve. The curve is called closed if $c(a) = c(b)$.

b) If $c : [a, b] \to \mathbb{R}^n$ is continuously differentiable, i.e. $c \in C^1([a, b], \mathbb{R}^n)$, which is the case if and only if each of the coordinate functions c_i, $i = 1, ..., n$, is continuously differentiable, then c is called a C^1-curve. Moreover c is called a piecewise C^1-curve if there is a decomposition $a = t_0 < t_1 < ... < t_m = b$, $m \in \mathbb{N}$, such that c is a C^1-curve on each subinterval $[t_i, t_{i+1}]$, $i = 0, ..., m-1$.

c) A C^1-curve $c \in C^1([a, b], \mathbb{R}^n)$ is called smooth or regular if for all $t \in [a, b]$. holds

$$c'(t) = (c_1'(t), ..., c_n'(t)) \neq 0 \ .$$

Example 6.51. a) The curve $c : [0, 2\pi] \to \mathbb{R}^2$, $t \mapsto (r\cos(t), r\sin(t))^T$, $r > 0$, describes a circle in $\mathbb{R}^2$ of radius r. It is a closed C^1-curve with derivative $c'(t) = (-r\sin(t), r\cos(t))^T$. Since cosine and sine do not have common roots, $c'(t) \neq 0$ for all t so that c is also smooth.

b) The curve $c : [0, T] \to \mathbb{R}^2$, $t \mapsto (rt - a\sin(t), r - a\cos(t))^T$, $T, a, r > 0$ is called a cycloid. It is a C^1-curve with with derivative $c'(t) = (r - a\cos(t), a\sin(t))^T$. If $a = r$, then then $c'(2\pi k) = 0$ for all $k \in \mathbb{N}$. Hence c is not smooth in this case.

c) The curve $c : [0, T] \to \mathbb{R}^3,\ t \mapsto (r\cos(t), r\sin(t), ht)^T,\ T, r, h > 0$, describes a helix in $\mathbb{R}^3$. It is a smooth C^1-curve with derivative

$$c'(t) = (-r\sin(t), r\cos(t), h)^T \neq 0.$$

d) Each continuous function $f : [a, b] \to \mathbb{R}$ can be interpreted as a curve in $\mathbb{R}^2$ via $c : [a, b] \to \mathbb{R}^2,\ t \mapsto (t, f(t))^T$. The associated curve c is C^1 if and only if f is continuously differentiable. In this case holds $c'(t) = (1, f'(t))^T \neq 0$ and therefore c is smooth.

If $c : [a, b] \to \mathbb{R}^n$ is a curve and if $h : [\alpha, \beta] \to [a, b]$ is a continuous bijective and monotonically increasing function, then the "new" curve

$$\tilde{c} : [\alpha, \beta] \to \mathbb{R}^n,\ \tau \mapsto c(h(\tau))$$

has the same shape and the same oriented direction. In this case the function h is called a reparametrisation. In case of C^1-curves also only C^1-reparametrisations h with $h' > 0$ are permitted.

In general curves c_1 and c_2 which distinguish themselves only through a reparametrisation are considered as "equal".

Now we want to compute the length of a C^1-curve $c : [a, b] \to \mathbb{R}^n$. This is done by approximation by polygonal paths. For a given decomposition $Z := \{a = t_0 < t_1 < .... < t_m = b\}$, $m \in \mathbb{N}$, of the interval $[a, b]$ the length $L(Z)$ of the polygonal path with corners $c(t_i)$ is given by

$$L(Z) = \sum_{i=0}^{m-1} ||c(t_{i+1}) - c(t_i)|| = \sum_{i=0}^{m-1} ||\frac{c(t_{i+1}) - c(t_i)}{t_{i+1} - t_i}|| \cdot (t_{i+1} - t_i).$$

If the right-hand side converges for $t_{i+1} - t_i \to 0$ then the limit is the the curve length $L(c)$ which is given by

$$L(c) = \int_a^b ||c'(t)||\, dt.$$

Definition 6.52.

If the set $\{L(Z) \mid Z \text{ is a decomposition of } [a, b]\}$ is bounded from above, then the curve $c : [a, b] \to \mathbb{R}$ is called rectifiable and

$$L(c) := \sup\{L(Z) \mid Z \text{ is a decomposition of } [a, b]\} = \lim_{||Z|| \to 0} L(Z)$$

is called the length of the curve c where $||Z|| := \min\{|t_{j+1} - t_j| \mid j = 0, ..., m-1\}$ for a decomposition $Z = \{a = t_0 < t_1 < .... < t_m = b\}$, $m \in \mathbb{N}$, of the interval $[a, b]$.

Theorem 6.53.

Each C^1-curve c is rectifiable and

$$L(c) = \int_a^b ||c'(t)||\, dt.$$

Proof: Let $Z = \{a = t_0 < t_1 < .... < t_m = b\}$, $m \in \mathbb{N}$, be a decomposition of the interval $[a, b]$. Using the mean value theorem for c, we have

$$L(Z) = \sum_{j=0}^{m-1} \sqrt{\sum_{k=1}^{n} (c_k(t_{j+1}) - c_k(t_j))^2} = \sum_{j=0}^{m-1} \sqrt{\sum_{k=1}^{n} (c_k'(\tau_{k_j}))^2}\ (t_{j+1} - t_j)$$

with $t_j \le \tau_{k_j} \le t_{j+1}$. Set

$$R(Z) := \sum_{j=0}^{m-1} \sqrt{\sum_{k=1}^{n} (c_k'(t_j))^2}\ (t_{j+1} - t_j).$$

We will estimate $|L(Z) - R(Z)|$. Let $\varepsilon > 0$. Since c_k' is uniformly continuous on $[a, b]$, there is a $\delta > 0$ such that for all $t, \tilde{t} \in [a, b]$ with $|t - \tilde{t}| < \delta$ holds $|c_k'(\tilde{t}) - c_k'(t)| < \varepsilon$ for all $k = 1, ..., n$. Thus if Z fulfils $||Z|| < \delta$, then

$$\begin{aligned}
|L(Z) - R(Z)| &= \left| \sum_{j=0}^{m-1} (||c'(\tau_j)|| - ||c'(t_j)||)\ (t_{j+1} - t_j) \right| \\
&\le \sum_{j=0}^{m-1} |\ ||c'(\tau_j)|| - ||c'(t_j)||\ |\ (t_{j+1} - t_j) \\
&\le \sum_{j=0}^{m-1} ||c'(\tau_j) - c'(t_j)||\ (t_{j+1} - t_j) \\
&\le \sqrt{n}\varepsilon(b - a) \underset{\varepsilon \searrow 0}{\longrightarrow} 0\ ,
\end{aligned}$$

where $\tau_j := (\tau_{1_j}, ..., \tau_{n_j})$ and $c'(\tau_j) := (c_1'(\tau_{1_j}), ..., c_n'(\tau_{n_j}))$.

Since $R(Z) \to \int_a^b ||c'(t)||\ dt$ for $||Z|| \to 0$, this also holds for $L(Z)$.

Example 6.54. The length of a cycloid $c(t) = (r(t - \sin(t)), r(1 - \cos(t)))^T$, $0 \le t \le 2\pi$, can be calculated as follows:

$$\begin{aligned}
c'(t) &= (r(1 - \cos(t)), r(\sin(t)))^T \\
||c'(t)|| &= r\sqrt{(1 - \cos(t))^2 + \sin(t)^2} = r\sqrt{2(1 - \cos(t))} \\
&= r\sqrt{2(1 - (1 - 2\sin^2(\frac{t}{2})))} = 2r\sin(\frac{t}{2}) \\
L(c) &= 2r\int_0^{2\pi} \sin(\frac{t}{2})\ dt = -4r\cos(\frac{t}{2})|_{t=0}^{t=2\pi} = 8r\ .
\end{aligned}$$

The length of a curve is independent with respect to reparametrisations, because if $h : [\alpha, \beta] \to [a, b]$ is a C^1-reparametrisation, then $h' > 0$ and the substitution rule immediately yields

$$\begin{aligned}
L(c \circ h) &= \int_\alpha^\beta ||(c \circ h)'(\tau)||\ d\tau = \int_\alpha^\beta ||c'(h(\tau))h'(\tau)||\ d\tau \\
&= \int_\alpha^\beta ||c'(h(\tau))|| \cdot h'(\tau)\ d\tau = \int_a^b ||c'(t)|| \cdot\ dt\ .
\end{aligned}$$

Definition 6.55.

Let $c : [a, b] \to \mathbb{R}$ be a C^1-curve. The function

$$S : [a, b] \to \mathbb{R}, \quad t \mapsto \int_a^t ||c'(\tau)|| \, d\tau$$

is called the arc length function of the curve.

If c is a smooth C^1-curve, then the arc length function $S : [a, b] \to [0, L(c)]$ is a C^1-function. In particular, the inverse function $S^{-1} : [0, L(c)] \to [a, b]$ exists and is a C^1-reparametrisation. The parametrisation $\tilde{c} := c \circ S^{-1}$ is called *parametrisation with respect to the arc length*.

Its derivative is given by

$$\tilde{c}'(s) \;=\; c'(S^{-1}(s)) \cdot \frac{1}{||c'(S^{-1}(s))||}$$

Obviously, this is a vector in $\mathbb{R}^n$ of length one. This means that the "speed" of the curve is always constant one and that $\tilde{c}'(s)$ is the *unit tangent vector* at the curve in the point $t = S^{-1}(s)$.

Differentiation of $1 = ||\tilde{c}'(s)||^2 = \langle \tilde{c}'(s), \tilde{c}'(s)\rangle$ yields

$$\begin{aligned} 0 &= (\langle \tilde{c}'(s), \tilde{c}'(s)\rangle)' = \left(\sum_{k=1}^{n} (\tilde{c_k}'(s))^2\right)' \\ &= \sum_{k=1}^{n} 2\tilde{c_k}'(s)\tilde{c_k}''(s) = 2\langle \tilde{c}''(s), \tilde{c}'(s)\rangle \; . \end{aligned}$$

This means that the acceleration vector $\tilde{c}''(s)$ is perpendicular to the velocity vector $\tilde{c}'(s)$. The vector

$$n(s) := \frac{\tilde{c}''(s)}{||\tilde{c}''(s)||}$$

is called the *unit normal vector* and $||\tilde{c}''(s)||$ is called the *curvature* of $c(t)$ in the point $t = S^{-1}(s)$.

Finally, the plane spanned by the tangent vector $\tilde{c}'(t)$ and the normal vector $\tilde{c}''(t)$ is called *osculating plane* of $c(t)$ in the point $t = S^{-1}(s)$.

Example 6.56. For a curve $c : [a, b] \to \mathbb{R},\; x \mapsto (t, y(x))^T$ corresponding to the graph of a C^2-function $y : [a, b] \to \mathbb{R}$ holds

$$\begin{aligned} c'(x) &= (1, y'(x))^T \\ ||c'(x)|| &= \sqrt{1 + (y'(x))^2} \\ S(t) &= \int_a^t \sqrt{1 + (y'(x))^2} \, dx \\ \kappa(x) &= \frac{|y''(x)|}{(\sqrt{1 + (y'(x))^2})^3} \; . \end{aligned}$$

Now, for a given C^1-curve $c : [a, b] \to \mathbb{R}^2$, we want to compute the signed area $A(c)$ consisting of all points "between" the curve and the origin. These are all points $P = \lambda c(t)$, $t \in [a, b]$, $\lambda \in [0, 1]$. The area $A(c)$ is called the area *enclosed* by the curve c.

If $Z = \{a = t_0, t_1, ..., t_{m-1}, t_m = b\}$, $m \in \mathbb{N}$, is a decomposition of $[a, b]$. Then $A(c)$ can be approximated by the sum of the signed areas A_i of all triangles with corners $c(t_i), c(t_{i+1}), 0_{\mathbb{R}^2}$, $i = 0, ..., m-1$. These triangle areas can easily be calculated using the cross product:

$$\begin{aligned} |A_i| &= \frac{1}{2}||(c_1(t_i), c_2(t_i), 0)^T \times (c_1(t_{i+1}), c_2(t_{i+1}), 0)^T|| = \frac{1}{2}|c_1(t_i)c_2(t_{i+1}) - c_1(t_{i+1})c_2(t_i)| \\ A_i &= \frac{1}{2}(c_1(t_i)c_2(t_{i+1}) - c_1(t_{i+1})c_2(t_i)) \ . \end{aligned}$$

Setting $\Delta t_i := t_{i+1} - t_i$, $\Delta c_{j,i} := c_j(t_{i+1}) - c_j(t_i)$, $j = 1, 2$, $i = 0, ..., m-1$, the sum $A(Z)$ of all signed triangle areas is

$$\begin{aligned} A(Z) &:= \frac{1}{2}\sum_{i=0}^{m-1}(c_1(t_i)c_2(t_{i+1}) - c_1(t_{i+1})c_2(t_i)) \\ &= \frac{1}{2}\sum_{i=0}^{m-1}\frac{c_1(t_i)c_2(t_{i+1}) - c_1(t_{i+1})c_2(t_i)}{\Delta t_i} \cdot \Delta t_i \\ &= \frac{1}{2}\sum_{i=0}^{m-1}\left(c_1(t_i)\frac{\Delta c_{2,i}}{\Delta t_i} - c_2(t_i)\frac{\Delta c_{1,i}}{\Delta t_i}\right) \cdot \Delta t_i. \end{aligned}$$

For $||Z|| \to 0$, $A(Z)$ converges to

$$A(c) = \frac{1}{2}\int_a^b (c_1(t)c_2'(t) - c_2(t)c_1'(t)) \ dt \ . \tag{6.3}$$

Example 6.57. For given $a, b > 0$, $t_1, t_2 \in [0, 2\pi]$, $t_1 < t_2$ we want to compute the area A_{t_1,t_2} of the sector of the ellipse

$$c : [0, 2\pi] \to \mathbb{R}, \ t \mapsto (a\cos(t), b\sin(t))^T$$

given by $t_1 \leq t \leq t_2$. This computes as

$$A_{t_1,t_2} = \frac{1}{2}\int_{t_1}^{t_2} ab(\cos^2(t) + \sin^2(t)) \ dt = \frac{ab}{2}\int_{t_1}^{t_2} 1 \ dt = \frac{ab(t_2 - t_1)}{2}.$$

In particular, if $a = b =: r$, then the sector of the circle has the area $\frac{t_2-t_1}{2}r^2$.

The final topic of this section are so-called *curve integrals*. Consider the following problem: Given a curved wire with inhomogeneous density. By integration we want to determine its total mass. Assume that the position of the wire is parameterised by a C^1-curve $c : [a, b] \to \mathbb{R}^n$, $n := 3$. The density $\rho(c(t))$ of the wire in the point $c(t)$ is defined as

$$\rho(c(t)) := \frac{\text{mass}}{\text{length unit}} \ .$$

Now, in order to compute the total mass of the wire, we consider a decomposition $Z = \{a = t_0, t_1, ..., t_{m-1}, t_m = b\}$ of the interval $[a,b]$ and approximate the density in the interval $[t_i, t_{i+1}]$ by the constant value $\rho(c(t_i))$ [= density in the left boundary point $c(t_i)$]. Furthermore, also the length of the wire in the interval $[t_i, t_{i+1}]$ is approximated by the length of the straight line between $c(t_i)$ and $c(t_{i+1})$ which, by the mean value theorem, is

$$||c(t_{i+1}) - c(t_i)|| = \sqrt{\sum_{k=1}^{n}(c_k(t_{i+1}) - c_k(t_i))^2} = \sqrt{\sum_{k=1}^{n} c_k'(\tau_{k,i})^2}\ (t_{i+1} - t_i)$$

for suitable $\tau_{k,i} \in [t_i, t_{i+1}]$. Thus the total mass of the wire is approximated by

$$\sum_{i=0}^{m-1} \rho(c(t_i))||c(t_{i+1}) - c(t_i)|| = \sum_{i=0}^{m-1} \rho(c(t_i))\sqrt{\sum_{k=1}^{n} c_k'(\tau_{k,i})^2}\ (t_{i+1} - t_i).$$

For $||Z|| \to 0$ the right-hand side converges to

$$\int_a^b \rho(c(t))||c'(t)||\ dt\ .$$

According to this result the following notation is introduced.

Definition 6.58.

Let $D \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $f : D \to \mathbb{R}$ continuous and $c : [a,b] \to D$ a piecewise C^1-curve, $a, b \in \mathbb{R}$, $a < b$. Then the path integral (or line integral, contour integral, curve integral) of the first kind of f with respect to c is defined as

$$\int_c f(x)\ ds := \int_a^b f(c(t))||c'(t)||\ dt\ .$$

If c is a closed curve then also the symbol

$$\oint_c f(x)\ ds$$

is used instead.

The path integral of the first kind is invariant with respect to reparametrisations, since for a C^1-reparametrisation $h : [\alpha, \beta] \to [a,b]$ with $h' > 0$ holds

$$\begin{aligned}
\int_{c\circ h} f(x)\ ds &= \int_\alpha^\beta f((c\circ h)(\tau))||(c\circ h)'(\tau)||\ d\tau \\
&= \int_\alpha^\beta f(c(h(\tau)))||c'(h(\tau))h'(\tau)||\ d\tau \\
&\underset{h'>0}{=} \int_\alpha^\beta f(c(h(\tau)))||c'(h(\tau))||h'(\tau)\ d\tau \\
&= \int_a^b f(c(t))||c'(t)||\ dt \qquad \text{(substitution } t := h(\tau)) \\
&= \int_c f(x)\ ds\ .
\end{aligned}$$

Example 6.59 (center of gravity)**.** For a system of N mass points with point masses m_i at positions $x_i \in \mathbb{R}^n$ the center of gravity $x_s \in \mathbb{R}^n$ is given by

$$x_s = \frac{\sum_{i=1}^N m_i x_i}{\sum_{i=1}^N m_i} \ .$$

For computing the center of gravity x_s of a wire, the total mass of the piece of wire between two points $c(t_i)$ and $c(t_{i+1})$ is approximated by $\rho(c(t_i))||c(t_{i+1}) - c(t_i)||$. Thus the approximation of x_S reads

$$x_S \approx \frac{\sum_{i=0}^{m-1} \rho(c(t_i))||\frac{c(t_{i+1})-c(t_i)}{\Delta t_i}||c(t_i)\Delta t_i}{\sum_{i=0}^{m-1} \rho(c(t_i))||\frac{c(t_{i+1})-c(t_i)}{\Delta t_i}||\Delta t_i} \ .$$

For $||Z|| \to 0$ this becomes

$$x_s = \frac{\int_a^b \rho(c(t))||c'(t)||c(t)\ dt}{\int_a^b \rho(c(t))||c'(t)||\ dt} = \frac{\int_c \rho(x)x\ ds}{\int_c \rho(x)\ ds} \ .$$

The curve integral

$$\int_c \rho(x)x\ ds = \begin{pmatrix} \int_c \rho(x)x_1\ ds \\ \vdots \\ \int_c \rho(x)x_n\ ds \end{pmatrix} = \begin{pmatrix} \int_a^b \rho(c(t))||c'(t)||c_1(t)\ dt \\ \vdots \\ \int_a^b \rho(c(t))||c'(t)||c_n(t)\ dt \end{pmatrix}$$

in the numerator has to be computed componentwise.

Example 6.60 (moment of inertia)**.** If a mass point with mass m rotates around an axis with distance r and angular velocity ω, then its kinetic energy is

$$E_{\text{kin}} = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}\theta\omega^2 \ .$$

The term $\theta := mr^2$ is called *moment of inertia* of the masspoint with respect to the given axis of rotation. For a system of N mass points with masses m_i and distances r_i to the axis of rotation, the single moments of inertia $\theta_i = m_i r_i^2$ simply add up to a total moment of inertia

$$\theta = \sum_{i=1}^N m_i r_i^2.$$

For a wire as considered above the moment of inertia θ is approximated by

$$\theta \approx \sum_{i=0}^{m-1} \rho(c(t_i))||c(t_{i+1}) - c(t_i)||r(c(t_i))^2 = \sum_{i=0}^{m-1} \rho(c(t_i)) \left\|\frac{c(t_{i+1}) - c(t_i)}{\Delta t_i}\right\| r(c(t_i))^2\ \Delta t_i \ .$$

Here $r(c(t))$ denotes the orthogonal distance of $c(t)$ to the axis of rotation. For $||Z|| \to 0$ the right-hand side converges to

$$\theta = \int_a^b \rho(c(t))||c'(t)||r^2(c(t))\ dt = \int_c \rho(x)r^2(x)\ ds \ .$$

For example if the wire has constant density ρ and is placed along a straight line of length $l > 0$ in the (x, z)-plane that encloses an angle α with the x-axis, then

$$c : [0, l] \to \mathbb{R},\ t \mapsto (t\cos(\alpha), 0, t\sin(\alpha))^T$$

and rotation around the x-axis yields

$$\begin{aligned} r(c(t)) &= t\sin(\alpha) \\ \theta_{x\text{-axis}} &= \int_0^l \rho \cdot 1 \cdot (t\sin(\alpha))^2\, dt = \frac{1}{3} l^3 \rho \sin^2(\alpha)\ . \end{aligned}$$

6.7 Fourier series

Definition 6.61.

A function $f : \mathbb{R} \to \mathbb{C}$ is called periodic with period $T \in \mathbb{R}$ or shortly T-periodic if $f(t + T) = f(t)$ for all $t \in \mathbb{R}$. In this case, if $T \neq 0$, $\nu := \frac{1}{T}$ is called frequency of f.

For example $\sin(t)$, $\cos(t)$ and $\exp(it) = \cos(t) + i\sin(t)$ are 2π-periodic functions. If a function f is T-periodic, then it is also kT-periodic for each $k \in \mathbb{Z}$. Without proof we mention that if f is a nonconstant continuous periodic function, then there exists always a smallest positive period $T > 0$ of f.

Definition 6.62.

A series of the form

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t)$$

with $a_k, b_k \in \mathbb{C}$ and $\omega > 0$ is called Fourier series or trigonometric series. The corresponding partial sums

$$f_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(k\omega t) + b_k \sin(k\omega t))$$

are called trigonometric polynomials.

The trigonometric polynomials $f_n(t)$ can be transformed as follows:

$$\begin{aligned} f_n(t) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(k\omega t) + b_k \sin(k\omega t) \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} \frac{a_k}{2}(e^{ik\omega t} + e^{-ik\omega t}) + \frac{b_k}{2i}(e^{ik\omega t} - e^{-ik\omega t}) \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} \frac{a_k - ib_k}{2} e^{ik\omega t} + \frac{a_k + ib_k}{2} e^{-ik\omega t} \end{aligned}$$

$$= \sum_{k=-n}^{n} \gamma_k e^{ik\omega t}$$

with

$$\gamma_0 := \frac{a_0}{2} \tag{6.4}$$

$$\gamma_k := \frac{a_k - ib_k}{2} \tag{6.5}$$

$$\gamma_{-k} := \frac{a_k + ib_k}{2} \tag{6.6}$$

for $k \in \mathbb{N}$. Conversely, if coefficients $\gamma_k \in \mathbb{C}$, $k = -n, -n+1, ..., n$, are given, then corresponding coefficients $a_k \in \mathbb{C}$, $k = 0, ..., n$, $b_k \in \mathbb{C}$, $k = 1, ..., n$, compute as

$$a_0 := 2\gamma_0 \tag{6.7}$$

$$a_k := \gamma_k + \gamma_{-k} \tag{6.8}$$

$$b_k := i(\gamma_k - \gamma_{-k}) \ . \tag{6.9}$$

For Fourier series we therefore have

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t) \\ &= \lim_{n\to\infty} \sum_{k=-n}^{n} \gamma_k e^{ik\omega t} =: \sum_{k=-\infty}^{\infty} \gamma_k e^{ik\omega t} \ . \end{aligned}$$

If these series converge for each $t \in \mathbb{R}$, then clearly the function $f(t)$ is well-defined and periodic with period $T := \frac{2\pi}{\omega}$.

Without proof we state that for two complex valued Riemann integrable functions $f, g \in \mathcal{R}([a,b],\mathbb{C})$, $a, b \in \mathbb{R}$, $a < b$, with integrals

$$\begin{aligned} \int_a^b f(x)\, dx &:= \int_a^b \operatorname{Re} f(x)\, dx + i \int_a^b \operatorname{Im} f(x)\, dx \\ \int_a^b g(x)\, dx &:= \int_a^b \operatorname{Re} g(x)\, dx + i \int_a^b \operatorname{Im} g(x)\, dx \end{aligned}$$

also their product $fg : [a,b] \to \mathbb{R}$, $x \mapsto f(x)g(x)$ is Riemann integrable with integral

$$\begin{aligned} \int_a^b (fg)(x)\, dx &= \int_a^b \big(\operatorname{Re} f(x) \operatorname{Re} g(x) - \operatorname{Im} f(x) \operatorname{Im} g(x)\big)\, dx + \\ & \quad i \int_a^b \big(\operatorname{Re} f(x) \operatorname{Im} g(x) + \operatorname{Im} f(x) \operatorname{Re} g(x)\big)\, dx \ . \end{aligned}$$

Definition 6.63.

Let $c > 0$. The mapping

$$\langle \cdot, \cdot \rangle \ : \ \mathcal{R}([a,b],\mathbb{C}) \times \mathcal{R}([a,b],\mathbb{C}) \to \mathbb{C},$$

$$\langle f, g\rangle = c\int_a^b \overline{f(x)}g(x)\ dx = c\int_a^b \operatorname{Re} f(x)\operatorname{Re} g(x) + \operatorname{Im} f(x)\operatorname{Im} g(x)\ dx +$$
$$ic\int_a^b \operatorname{Re} f(x)\operatorname{Im} g(x) - \operatorname{Im} f(x)\operatorname{Re} g(x)\ dx$$

is a so-called well-defined positive-semidefinite hermitian form on the $\mathbb{C}$-vector space $\mathcal{R}([a,b],\mathbb{C})$, which means that for all $f, g, h \in \mathcal{R}([a,b],\mathbb{C})$ and all $\lambda, \mu \in \mathbb{C}$ holds

(i) $\langle f, \lambda g + \mu h\rangle = \lambda\langle f, g\rangle + \mu\langle f, h\rangle$

(ii) $\langle f, g\rangle = \overline{\langle g, f\rangle}$

(iii) $\langle f, f\rangle = c\int_a^b |f(x)|^2\ dx \geq 0$.

If $f \in C([a,b],\mathbb{C})\setminus\{0\}$ is a non-zero complex valued continuous function on $[a,b]$, then there exists a $\xi \in (a,b)$ such that $|f(\xi)| > 0$ and by continuity there is an $\varepsilon > 0$ such that $[\xi-\varepsilon, \xi+\varepsilon] \subset [a,b]$ and such that $|f(x)| \geq \frac{1}{2}|f(\xi)|$ for all $x \in [\xi-\varepsilon, \xi+\varepsilon]$. Thus

$$\langle f, f\rangle = c\int_a^b |f(x)|^2\ dx \geq c\int_{\xi-\varepsilon}^{\xi+\varepsilon} |f(x)|^2\ dx \geq \frac{1}{2}c\varepsilon|f(\xi)|^2 > 0\ .$$

This shows that $\langle\cdot,\cdot\cdot\rangle$ restricted to $C([a,b],\mathbb{C}) \subset \mathcal{R}([a,b],\mathbb{C})$ is positive definite, i.e. for $f \in C([a,b],\mathbb{C})$ holds $\langle f, f\rangle = 0$ if and only if $f = 0$.

In other words $\langle\cdot,\cdot\cdot\rangle$ is an inner product on the complex vector space $C([a,b],\mathbb{C})$ which, equipped with this inner product, therefore becomes a complex inner product space.

Restricted to the $\mathbb{R}$-subspace $C([a,b],\mathbb{R})$ consisting of all real valued continuous functions on $[a,b]$ the inner product becomes $\langle f, g\rangle = c\int_a^b f(x)g(x)\ dx$ and equipped with this inner product $C([a,b],\mathbb{R})$ is a real inner product space.

The constant c is simply a scaling factor which for example can be set to the reciprocal of the length of the integration interval, that is $c := \frac{1}{b-a}$.

Similar arguments as given above show that $\langle\cdot,\cdot\cdot\rangle$ restricted to the $\mathbb{C}/\mathbb{R}$-vector space of piecewice complex-/real-valued continuous functions on $[a,b]$, denoted by $C_p([a,b],\mathbb{C})/C_p([a,b],\mathbb{R})$, makes this space to a unitary/Euclidean one.

Theorem 6.64.

Let $T > 0$ and $\omega := \frac{2\pi}{T}$.

a) The functions $e^{ik\omega t} \in C([0,T],\mathbb{C})$, $k \in \mathbb{Z}$, form an orthonormal system with respect to the inner product $\langle f, g\rangle := \frac{1}{T}\int_0^T \overline{f(t)}g(t)\ dt$, $f, g \in C([0,T],\mathbb{C})$. (Here, the inner product has the complex conjugation in the first component!)

b) The functions $\frac{1}{\sqrt{2}}, \cos(k\omega t), \sin(k\omega t) \in C([0,T],\mathbb{R}) \subset C([0,T],\mathbb{C})$, $k \in \mathbb{N}$, form an orthonormal system w.r.t. the inner product $2\langle f, g\rangle = \frac{2}{T}\int_0^T \overline{f(t)}g(t)\ dt$, $f, g \in C([0,T],\mathbb{C})$.

Proof: a) Let $k, l \in \mathbb{Z}$. If $k = l$, then

$$\langle e^{ik\omega t}, e^{il\omega t} \rangle = \frac{1}{T}\int_0^T e^{-ik\omega t} e^{ik\omega t}\ dt = \frac{1}{T}\int_0^T 1\ dt = 1\ .$$

If $k \neq l$, then

$$\langle e^{ik\omega t}, e^{il\omega t} \rangle \quad = \quad \frac{1}{T}\int_0^T e^{i(l-k)\omega t}\ dt = \frac{1}{i(l-k)\omega T} \cdot e^{i(l-k)\omega t}\Big|_{t=0}^{t=T} = 0\ .$$

b) Let $k, l \in \mathbb{N}$ with $k \neq l$. Using

$$\begin{aligned}
\int \cos^2(t)\ dt &= \frac{t + \sin(t)\cos(t)}{2} \\
\int \sin^2(t)\ dt &= \frac{t - \sin(t)\cos(t)}{2} \\
\int \sin(t)\cos(t)\ dt &= \frac{\sin^2(t)}{2}
\end{aligned}$$

immediately gives

$$\begin{aligned}
2\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle &= \frac{2}{T}\int_0^T \left(\frac{1}{\sqrt{2}}\right)^2\ dt = 1 \\
2\langle \frac{1}{\sqrt{2}}, \cos(k\omega t) \rangle &= \frac{\sqrt{2}}{T}\int_0^T \cos(k\omega t)\ dt = \frac{\sqrt{2}}{k\omega T}\sin(k\omega t)\Big|_{t=0}^{t=T} = 0 \\
2\langle \frac{1}{\sqrt{2}}, \sin(k\omega t) \rangle &= \frac{\sqrt{2}}{T}\int_0^T \sin(k\omega t)\ dt = \frac{-\sqrt{2}}{k\omega T}\cos(k\omega t)\Big|_{t=0}^{t=T} = 0 \\
2\langle \cos(k\omega t), \cos(k\omega t) \rangle &= \frac{2}{T}\int_0^T \cos^2(k\omega t)\ dt = \frac{k\omega t + \sin(k\omega t)\cos(k\omega t)}{k\omega T}\Big|_{t=0}^{t=T} = 1 \\
2\langle \sin(k\omega t), \sin(k\omega t) \rangle &= \frac{2}{T}\int_0^T \sin^2(k\omega t)\ dt = \frac{k\omega t - \sin(k\omega t)\cos(k\omega t)}{k\omega T}\Big|_{t=0}^{t=T} = 1 \\
2\langle \sin(k\omega t), \cos(k\omega t) \rangle &= \frac{2}{T}\int_0^T \sin(k\omega t)\cos(k\omega t)\ dt = \frac{\sin^2(k\omega t)}{k\omega T}\Big|_{t=0}^{t=T} = 0
\end{aligned}$$

□

In the sequel let $T > 0$ and $\omega := \frac{2\pi}{T}$ be fixed.

Theorem 6.65.

If the Fourier series $\sum_{k=-\infty}^{\infty} \gamma_k e^{ik\omega t} = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t)$, $\gamma_k \in \mathbb{C}$, $a_k, b_k \in \mathbb{C}$, converges uniformly on $[0, T]$ to a function $f(t)$, then $f(t)$ is continuous and

$$\gamma_k = \langle e^{ik\omega t}, f(t) \rangle = \frac{1}{T}\int_0^T e^{-ik\omega t} f(t)\ dt \quad , \quad k \in \mathbb{Z} \tag{6.10}$$

$$a_k = 2\langle \cos(k\omega t), f(t) \rangle = \frac{2}{T}\int_0^T \cos(k\omega t) f(t)\ dt \quad , \quad k \in \mathbb{N}_0 \tag{6.11}$$

$$b_k \quad = \quad 2\langle \sin(k\omega t), f(t)\rangle = \frac{2}{T}\int_0^T \sin(k\omega t) f(t)\ dt \quad , \quad k \in \mathbb{N}. \qquad (6.12)$$

Proof: For $n \in \mathbb{N}$ set $f_n := \sum_{k=-n}^{n} \gamma_k e^{ik\omega t} \in C([0,T],\mathbb{C})$. By assumption, the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. By Theorem 3.16 the function f must be continuous. Furthermore, for fixed $k \in \mathbb{N}_0$, $e^{-ik\omega t} f_n$ converges uniformly to $e^{-ik\omega t} f$ as

$$||e^{-ik\omega t} f_n - e^{-ik\omega t} f||_\infty = ||e^{-ik\omega t}(f_n - f)||_\infty \underset{|e^{-ik\omega t}|=1}{=} ||f_n - f||_\infty \underset{n\to\infty}{\longrightarrow} 0\ .$$

Thus using Theorem 6.64 a) we obtain

$$\begin{aligned}
\frac{1}{T}\int_0^T e^{-ik\omega t} f(t)\ dt &= \frac{1}{T}\int_0^T \lim_{n\to\infty} (e^{-ik\omega t} f_n(t))\ dt = \lim_{n\to\infty} \frac{1}{T}\int_0^T e^{-ik\omega t} f_n(t)\ dt \\
&= \lim_{n\to\infty} \underbrace{\langle e^{ik\omega t}, f_n(t)\rangle}_{=\gamma_k \text{ for } n \geq k} = \gamma_k\ .
\end{aligned}$$

Analogously using Theorem 6.64 b) we obtain the stated formulas for a_k, $k \geq 0$, and b_k, $k > 0$.

□

Note that for arbitrary $f \in \mathcal{R}([0,T],\mathbb{C})$ the coefficients γ_k, a_k, b_k defined in (6.10) to (6.12) fulfil the relations (6.4) to (6.9). We will mainly restrict to piecewise continuous functions.

Definition 6.66. Fourier series

For a piecewise continuous function $f : [0,T] \to \mathbb{C}$ the Fourier coefficients $(\gamma_k)_{k\in\mathbb{Z}}$, $(a_k)_{k\in\mathbb{N}_0}$, $(b_k)_{k\in\mathbb{N}}$ of f are defined by (6.10) to (6.12) and the Fourier sereies of f is defined by

$$F(f)(t) \quad := \quad \sum_{k=-\infty}^{\infty} \gamma_k e^{ik\omega t} = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t)\ .$$

Theorem 6.67.

Let $f : \mathbb{R} \to \mathbb{C}$ be a piecewise continuous T-periodic function.

a) If f is even, that is $f(t) = f(-t)$ for all $t \in \mathbb{R}$, then

$$\begin{aligned}
a_k &:= \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos(k\omega t)\ dt \\
b_k &= 0\ .
\end{aligned}$$

b) If f is odd, that is $f(t) = -f(-t)$ for all $t \in \mathbb{R}$, then

$$\begin{aligned}
a_k &= 0 \\
b_k &:= \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin(k\omega t)\ dt\ .
\end{aligned}$$

Proof: We only prove a) since b) follows analogously.

$$\begin{aligned}
b_k &= \frac{2}{T}\int_0^T f(t)\sin(k\omega t)\;dt \underset{\tau:=-t}{=} \frac{2}{T}\int_0^{-T} f(-\tau)\sin(k\omega\tau)\;d\tau \\
&= -\frac{2}{T}\int_{-T}^{0} f(\tau)\sin(k\omega\tau)\;d\tau = -\frac{2}{T}\int_0^{T} f(\tau)\sin(k\omega\tau)\;d\tau = -b_k
\end{aligned}$$

This implies $b_k = 0$. Furthermore, we compute

$$\begin{aligned}
a_k &= \frac{2}{T}\int_0^T f(t)\cos(k\omega t)\;dt = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(k\omega t)\;dt \\
&= \frac{2}{T}\left(\int_{-\frac{T}{2}}^{0} f(t)\cos(k\omega t)\;dt + \int_0^{\frac{T}{2}} f(t)\cos(k\omega t)\;dt\right) \\
&\underset{\tau:=-t}{=} \frac{2}{T}\left(\int_0^{\frac{T}{2}} f(-\tau)\cos(k\omega\tau)\;d\tau + \int_0^{\frac{T}{2}} f(t)\cos(k\omega t)\;dt\right) \\
&= \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos(k\omega t)\;dt\;.
\end{aligned}$$

In the following we list some calculation rules for Fourier series which are easily deduced and can be proved as an exercise.

Lemma 6.68.

Suppose that $f, g : \mathbb{R} \to \mathbb{C}$ are T-periodic piecewise continuous functions with

$$\begin{aligned}
F(f) = F(f)(t) &= \sum_{k=-\infty}^{\infty} \gamma_k e^{ik\omega t} = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(k\omega t) + b_k\sin(k\omega t) \\
F(g) = F(g)(t) &= \sum_{k=-\infty}^{\infty} \delta_k e^{ik\omega t}
\end{aligned}$$

a) ***linearity:***

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g) = \sum_{k=-\infty}^{\infty} (\alpha\gamma_k + \beta\delta_k)e^{ik\omega t}$$

b) ***conjugation:***

$$\overline{F(f)} = \sum_{k=-\infty}^{\infty} \overline{\gamma_{-k}}e^{ik\omega t}$$

c) ***time reversal:***

$$F(f(-t)) = \sum_{k=-\infty}^{\infty} \gamma_{-k}e^{ik\omega t}$$

d) ***stretch:***

$$F(f(ct)) = \sum_{k=-\infty}^{\infty} \gamma_k e^{ik(c\omega)t}, \quad c > 0$$

e) ***shift:***

$$\begin{aligned} F(f(t+a)) &= \sum_{k=-\infty}^{\infty} (\gamma_k e^{ik\omega a})\gamma_k e^{ik\omega t}, \quad a \in \mathbb{R} \\ F(e^{in\omega t} f(t)) &= \sum_{k=-\infty}^{\infty} \gamma_{k-n} e^{ik\omega t}, \quad n \in \mathbb{Z} \end{aligned}$$

f) ***derivation:*** *If f is continuous and piecewise continuously differentiable, then*

$$\begin{aligned} F(f') &= \sum_{k=-\infty}^{\infty} (ik\omega\gamma_k) e^{ik\omega t} \\ & \sum_{k=1}^{\infty} (k\omega)(b_k \cos(k\omega t) - a_k \sin(k\omega t)) \ . \end{aligned}$$

g) ***integration:*** *If $0 = \gamma_0 = \frac{1}{T}\int_0^T f(t)\ dt$, then*

$$F\left(\int_0^t f(\tau)\ d\tau\right) = -\frac{1}{T}\int_0^T t f(t)\ dt - \sum_{k=1}^{\infty} \frac{b_k}{k\omega}\cos(k\omega t) - \frac{a_k}{k\omega}\sin(k\omega t) \ .$$

Without proof we state the following theorem on convergence of Fourier series.

Theorem 6.69. convergence

Let $f : \mathbb{R} \to \mathbb{C}$ be T-perodic and piecewise continuously differentiable with Fourier series

$$F(f)(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t).$$

a) *$F(f)(t)$ converges pointwisely and for all $t \in \mathbb{R}$ holds:*

$$F(f)(t) = \frac{1}{2}(\underbrace{\lim_{s \searrow t} f(t)}_{f(t^+):=} + \underbrace{\lim_{s \nearrow t} f(t)}_{f(t^-):=}) \ .$$

b) *On each compact interval on which f is continuous, the Fourier series converges uniformly.*

c) *In each point t of discontinuity the so-called Gibbs phenomenon can be observed, which says that for $\delta := |f(t^+) - f(t^-)| > 0$ and*

$$\rho := \frac{1}{\pi}\int_0^{\pi} \frac{\sin(t)}{t}\ dt - \frac{1}{2} \approx 0.0895 \approx 9\%$$

holds

$$|F(f)(t) - f(t^+)| \approx \rho \cdot \delta \approx |F(f)(t) - f(t^-)|$$

i.e. the "error" between f and its Fourier series is approximately 9% of the step size δ of the discontinuity.

The following theorem clarifies the approximation quality of Fourier series.

Theorem 6.70.

Let $f : \mathbb{R} \to \mathbb{C}$ be T-periodic and piecewise continuous with Fourier series

$$F(f)(t) \quad = \quad \sum_{k=-\infty}^{\infty} \gamma_k e^{ik\omega t} = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t)$$

and let

$$S_n(t) \quad := \quad \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(k\omega t) + b_k \sin(k\omega t)).$$

denote the corresponding partial sums.

a) *$S_n(t)$ is the orthogonal projection of f onto the subspace*

$$T_n := \operatorname{span}\{\frac{1}{\sqrt{2}}, \cos(\omega t), ..., \cos(n\omega t), \sin(\omega t), ..., \cos(n \sin t)\} \subset C([0,T], \mathbb{C})$$

for $n \in \mathbb{N}_0$ with respect to the inner product

$$2\langle u, v\rangle = \frac{2}{T} \int \overline{u(t)} v(t) \ dt \ .$$

Setting $||u|| := \sqrt{2\langle u, u\rangle}$ this means that for all $g \in T_n$ holds

$$||f - S_n|| \leq ||f - g|| \ .$$

b) *The so-called Bessel inequality holds:*

$$\frac{|a_0|^2}{2} + \sum_{k=1}^{n} (|a_k|^2 + |b_k|^2) \leq \frac{2}{T} \int_0^T |f(t)|^2 \ dt \ .$$

This especially implies the convergence of the series

$$\sum_{k=0}^{\infty} |a_k|^2 \quad , \quad \sum_{k=0}^{\infty} |b_k|^2$$

and therefore also $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0$

c) *If f is $(m-1)$-times continuously differentiable and $(m+1)$-times piecewise continuously differentiable, then there is a constant $C > 0$ such that*

$$|\gamma_k| \leq \frac{C}{|k|^{m+1}}$$

for all $k \in \mathbb{Z} \backslash \{0\}$.

Proof: a) This is clear by Theorem 6.64.

b)

$$\begin{aligned} 0 &\leq ||f-S_n||^2 = 2\langle f-S_n, f-S_n\rangle \\ &= ||f||^2 - 4\,\mathrm{Re}\langle f, \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos(k\omega t) + b_k\sin(k\omega t))\rangle + ||S_n||^2 \\ &= ||f||^2 - \frac{|a_0|^2}{2} - \sum_{k=1}^{n}|a_k|^2 + |b_k|^2. \end{aligned}$$

c) By using the rule for differentiation of Lemma 6.68 it is sufficient to show the assertion for $m = 0$. Therefore let f be piecewise continuously differentiable. Choose a decomposition $0 = t_0 < t_1 < ... < t_m = T$ such that $f|_{[t_i,t_{i+1}]}$, $i = 0, ..., m-1$, is continuously differentiable. Then integration by parts yields

$$\begin{aligned} \gamma_k &= \int_0^T f(t)e^{ik\omega t}\,dt \\ &= -\frac{1}{ik\omega}\sum_{j=0}^{m-1}\left(f(t)e^{-ik\omega t}\Big|_{t=t_j}^{t=t_{j+1}} - \int_{t_j}^{t_{j+1}} f'(t)e^{ik\omega t}\,dt \right) \end{aligned}$$

and therefore

$$|\gamma_k| \leq \frac{1}{k}\underbrace{\left(\frac{1}{\omega}\sum_{j=0}^{m-1}|f(t_{j+1}^-)| + |f(t_j^+)| + \frac{1}{\omega}\int_0^T |f'(t)|\,dt\right)}_{C:=}.$$

□

Finally, without proof, we state the following theorem on uniqueness of Fourier series:

Theorem 6.71.

If two T-periodic, piecewise continuous functions $f, g : \mathbb{R} \to \mathbb{C}$ have the same Fourier series and if they fulfill

$$\begin{aligned} f(t) &= \frac{1}{2}(f(t^-) + f(t^+)) \\ g(t) &= \frac{1}{2}(g(t^-) + g(t^+)) \end{aligned}$$

for all $t \in \mathbb{R}$, then $f = g$.

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