A FINAL YEAR PROJECT
ON

Applications of Parabolic and Hyperbolic Partial Differential Equations in Science and Engineering

THE DEPARTMENT OF MATHEMATICS,

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DIFFERENTIAL EQUATIONS

Introduction

In a Differential Equation, there are one or more functions with their derivatives and the derivatives defines the rate of change of functions at a point. The rate of change of functions is basically the derivatives of the dependent variable with respect to the independent variable.

$$\frac{dy}{dx} = f(x)$$

where 'x' is an independent variable and 'y' is a dependent variable.
The primary purpose of the differential equation is the study of solutions that satisfy the equations and the properties of the solutions.

Types of Differential Equations

Differential equations can be divided into several types namely

Ordinary Differential Equations

Partial Differential Equations

Linear Differential Equations

Non-linear differential equations

Homogeneous Differential Equations

Non-homogenous Differential Equations

History of Differential Equations

With the invention of Calculus by Isaac Newton and Gottfried Leibnitz, Differential Equation came into existence.
As early as 1671, Newton, in rough, unpublished notes listed 3 types of Differential Equations,

$$\frac{dy}{dx} = f(x)$$

$$\frac{dy}{dx} = f(x, y)$$

$$x_1 \frac{\partial y}{\partial x_1} + x_2 \frac{\partial y}{\partial x_2} = y$$

where 'y' is a function of 'x' and f is a given function.
When Newton finally published these equations (circa 1736), he originally dubbed them "fluxions".

Around 1675, German mathematician Gottfried Leibniz, also in unpublished notes, introduced two key ideas: his own differential and the very first recorded instance of the integral symbol:

$$\int x \, dx = \frac{1}{2} x^2$$

Applications in Science and Engineering

The properties of Differential Equation of various types is used in many disciplines like in Pure and Applied Mathematics, Physics and Engineering, biology and so on. Differential Equations are very useful in modelling virtually every physical, technical or biological processes.

The propagation of light and sound in the atmosphere, and of waves on the surface of a pond, they all can be described by second order Partial Differential Equation, the Wave Equation. It allows us to think of light and sound as forms of waves. The theory of Conduction of Heat is governed by another second order Partial Differential Equation, the Heat Equation which was developed by Joseph Fourier. In finance, the Black-Scholes Equation is related to the heat Equation.

Definitions :

1.Ordinary Differential Equation

A differential equation involving derivatives with respect to a single independent variable is called an ordinary differential equation.

Example

1. $dy = (x + \sin x)dx$
2. $\dfrac{d^4x}{dt^4} + \dfrac{d^2x}{dt^2} + \left(\dfrac{dx}{dt}\right)^5 = e^t$
3. $= \sqrt{x}\dfrac{dy}{dx} + \dfrac{k}{\frac{dy}{dx}}$
4. Newton's Law of Cooling

The rate of change of temperature is proportional to the temperature difference between it and that of its surroundings.

$$\frac{dT}{dt} = k(T - T_\circ)$$

Here the unknown function 'T' depends on one variable 't' and the relation involves first order derivative '$\frac{dT}{dt}$'

2.Order and Degree of Differential Equations

Order of a differential equation is the order of the highest derivative (also known as differential coefficient) present in the equation.

Example (i):$\dfrac{d^3y}{dx^3} + 3x\dfrac{dy}{dx} = e^y$

In this equation, the order of the highest derivative is 3 hence, this is a third order differential equation.

Example (ii) : $\left(\dfrac{d^2y}{dx^2}\right)^5 + \dfrac{dy}{dx} = 3$

This equation represents a second order differential equation.

The **degree** of the differential equation is represented by the power of the highest order derivative in the given differential equation.

The differential equation must be a polynomial equation in derivatives for the degree to be defined.

Example 1:- $\dfrac{d^3y}{dx^3} + (\dfrac{d^2y}{dx^2})^5 + \dfrac{dy}{dx} = 9$

Here, the exponent of the highest order derivative is one and the given differential equation is a polynomial equation in derivatives. Hence, the degree of this equation is 1.

Example 2: $\left[\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]^5=k^2\left(\frac{d^3y}{dx^3}\right)^2$

The order of this equation is 3 and the degree is 2 as the highest derivative is of order 3 and the exponent raised to the highest derivative is 2.

3.Partial Differential Equations

An equation consisting of one dependent variable,one or more independent variable and partial derivatives of dependent variables with respect to independent variable is called partial differential equation.

General form:

Let u is dependent and x and y are independent variables in some domain D$\subseteq R^2$

$$f(u,x,y,u_x,u_y,u_{yy},u_{xy}.....)=0$$

where u_x is the partial derivative of u with respect to x,u_y is the partial derivative of u with respect to y, u_{yy} is the second order partial derivative of u with respect to y and u_{xy} is the partial derivative of u with respect to x and then y.

Examples:

1. $u_x+u_y=2$
2. $uu_x+u_y=\sin x$

3. $uu_{xx} + u_x = y$

Classification of partial differential equations:

1. Linear partial differential equation

A partial differential equation is said to be linear if it is linear in the unknown function and all its derivatives with coefficients depending only on the independent variables.

Example: $yu_{xx} + 2xu_{yy} + u = 1$

2. Quasi Linear partial differential equation

A partial differential equation is said to be quasi linear if it is linear in the highest order derivative of the unknown function.

Example: $u_x u_{xx} + 2xuu_y = \sin y$

3. Non Linear partial differential equation

A differential equation of order n is called nonlinear if it is not linear in the derivatives of order n.

Example: $yu_{xx}^2 + 2xyu_{yy} + u = 1$

4. Semi Linear differential equation

A quasi-linear differential equation of order n, where the coefficients of derivatives of order n are functions of the independent variables $x_1, ..., x_m$ alone is called a semi-linear differential equation

Example: $xuu_x + yuu_y = x^2 + y^2, \quad x > 0, y > 0$

4. Solution of Differential Equations

A solution of differential equation is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.It is a relation between the dependent and independent variables.

Example : $y = \dfrac{x^3}{3} + c$ is the solution of the differential equation $\dfrac{dy}{dx} = x^2$, as it satisfies the equation.

1.General solution of Differential Equation

The solution of a differential equation is said to be a general solution of the differential equation if the number of arbitrary constants in the solution is exactly equal to the order of the differential equation.

Example: $y = \dfrac{x^3}{3} - 3x + k$ is the general solution of the differential equation $\dfrac{dy}{dx} = x^2 - 3$, as it contains one arbitrary constants k whose number is equal to the order of the equation.

2.Particular Solution of a Differential Equation

An equation obtained by giving some particular values to the arbitrary constants in its general solution is called a particular solution of the equation.

Example: $y = \dfrac{-ln(1-2x^2)}{4}$, is the particular solution of the differential equation.
$\dfrac{dy}{dx} = e^{4y+lnx}$ as it is obtained by assigning particular value $\dfrac{-1}{4}$ to the arbitrary constant c_1 respectively in the general solution $\dfrac{e^{-4y}}{-4} = \dfrac{x^2}{2} + c$ of the equation.

5.Classification of second order Partial Differential Equations

The general second order linear Partial Differential Equation has the following form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + E_{uy} + Fu = G \text{ ....(1)}$$

where the coefficients A, B, C, D, F and the free term G are in general functions of the independent variables x, y but do not depend on the unknown function u
by the transformation $x = x(x^*, y^*), y = y(x^*, y^*)$
turning equation (1) into canonical form

$$a(x, y)u_{xx}+c(x, y)u_{yy}+d(x, y)u_x+e(x, y)u_y+f(x, y)u = 0 \text{ ....(2)}$$

then
(i) $a = -c$ is hyperbolic,

(ii) $a = 0$ or $c = 0$ is parabolic,
(iii) $a = c$ is elliptic.
Assuming the separable solution of (2) in the form

$$u(x, y) = X(x)Y(y) \neq 0 \text{ ....(3)}$$

where X and Y are, respectively, functions of x and of y alone, and are differentiable twice continuously.
On substituting equation (3) into equation (2), we get

$$aX''Y + cXY'' + dX'Y + eXY' + fXY = 0 \text{ ....(4)}$$

Then we assume there exists a function $q(x, y)$, such that, if we divide equation (4) by $q(x, y)$, we obtain

$$m(x)X''Y+n(y)XY''+m_1(x)X'Y+n_1(y)XY'+[m_2(x)+n_2(y)]XY = 0 \text{ ....(5)}$$

On dividing equation (5) again by XY , we get

$$\left[m\frac{X''}{X} + m_1\frac{X'}{X} + m_2\right] = -\left[n\frac{Y''}{Y} + n_1\frac{Y'}{Y} + n_2\right] \text{ ....(6)}$$

The LHS of equation (6) is a function of x only. The RHS of equation (6) depends only upon y.Hence, we can differentiate equation (6) with respect to x to get

$$\frac{d}{dx}\left[m\frac{X''}{X} + m_1\frac{X'}{X} + m_2\right] = 0 \text{ ....(7)}$$

Integration of equation (7) gives us

$$\left[m\frac{X''}{X} + m_1\frac{X'}{X} + m_2\right] = \lambda \text{ ....(8)}$$

where λ is a separation constant. From equations (6) and (8), we have

$$n\frac{Y''}{Y} + n_1\frac{Y'}{Y} + n_2 = -\lambda \text{ ....(9)}$$

We can rewrite equations (8) and (9) in the form

$$mX'' + m_1X' + (m_2\lambda)X = 0 \text{ ....(10)}$$

and

$$nY'' + n_1Y' + (n_2 + \lambda)Y = 0 \text{ ....(11)}$$

Thus, $u(x, y)$ is the solution of equation (2) if $X(x)$ and $Y(y)$ are the solutions of the ordinary differential equations (10) and (11) respectively. If the coefficients in equation (1) are constant, then the reduction of equation (1) to canonical form is not necessary. To show this, we consider the second-order equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + E_{uy} + Fu\ = 0 \text{ ....(12)}$$

where A, B, C, D, E, and F are constants which are not all zero.

As before, we assume a separable solution in the form

$$u(x,y) = X(x)Y(y) \neq 0.$$

Substituting this in equation (12), get

$$AX''Y+BX'Y'+CXY''+DX'Y+EXY'+FXY = 0. \text{ ....(13)}$$

Division of this equation by AXY and differentiating this equation with respect to x we get

$$\left(\frac{X''}{X}\right)' + \frac{B}{A}\left(\frac{X'}{X}\right)'\frac{Y'}{Y} + \frac{D}{A}\left(\frac{X'}{X}\right)' = 0 \text{ ....(14)}$$

Thus, we have

$$\frac{\left(\frac{X''}{X}\right)'}{\frac{B}{A}\left(\frac{X'}{X}\right)'} + \frac{D}{B} = -\frac{Y'}{Y} \text{ ....(15)}$$

This equation is obviously separable, so that both sides must be equal to a constant Î». Therefore, we obtain
$Y' + \lambda Y = 0$ ....(16)

$$\left(\frac{X''}{X}\right)' + \frac{B}{A}\left(\frac{X'}{X}\right)'\left(\frac{D}{B} - \lambda\right) = 0 \text{ ....(17)}$$

Integrating equation (17) with respect to x, we get

$$\left(\frac{X''}{X}\right) + \frac{B}{A}\left(\frac{X'}{X}\right)\left(\frac{D}{B} - \lambda\right) = -\beta \text{ ....(18)}$$

where β is a constant to be determined.Substituting equation

(16) into the original equation, we get

$$X'' + \frac{B}{A}X'\left(\frac{D}{B} - \lambda\right) + \left(\lambda^2 - \frac{E}{C}\lambda + \frac{F}{C}\right)\frac{C}{A}X = 0 \text{ ....(19)}$$

Comparing equations (18) and (19), we clearly find

$$\beta = \left(\lambda^2 - \frac{E}{C}\lambda + \frac{F}{C}\right)\frac{C}{A} \text{ ....(19)}$$

Therefore, $u(x, y)$ is a solution of equations (12) if $X(x)$ and $Y(y)$ satisfy the ordinary differential equations (19) and (16) respectively.

Methodology - FOURIER SERIES

1. Prerequisites

A function $f(x)$ is called a periodic function if $f(x)$ is defined $\forall$ real x and if $\exists$ some positive number p, called a period of $f(x)$ such that $f(x + p) = f(x) \qquad \forall x$

$f(x + np) = f(x) \qquad \forall x$

Dealing with various functions $f(x)$ of period 2π
$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\ldots\ldots, \cos nx, \sin nx, \ldots\ldots\ldots$

The series obtained is a trigonometric series of the form
$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots\ldots\ldots\ldots\ldots\ldots$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad ....(1)$$

$a_0, a_1, b_1, a_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ are coefficients of the series

Then, (1) is the fourier series of $f(x)$ given by Eular Formulas

(a) $a_0 = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\, dx$

(b) $a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad n = 1, 2, \ldots\ldots\ldots\ldots\ldots$

(c) $b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

Find Fourier Series of periodic function f(x)

$$f(x) = \begin{cases} -\mathbb{R}, & \text{if } -\pi < x < 0 \\ \mathbb{R}, & \text{if } 0 < x < \pi \\ \text{and } f(x + 2\pi) = f(x) \end{cases}$$

a_0, Since the area under the curve of $f(x)$ at a single point between $-\pi$ and π is zero.

We obtain coefficients of cosine term $(a_1, a_2, \ldots\ldots\ldots\ldots\ldots\ldots)$

Split in two integrals

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} (-k) \cos hx\, dx + \int_{0}^{\pi} k \cos hx\, dx \right]$$

$$= \frac{1}{\pi} \left[-k\, \frac{\sin nx}{n} \bigg|_{-\pi}^{0} + k\, \frac{\sin nx}{n} \bigg|_{0}^{\pi} = 0 \right.$$

$\therefore$ Series has no cosine terms, it is a fourier sine series with coefficients $b_1, b_2, \ldots\ldots\ldots\ldots\ldots$ obtained

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx + \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[k\,\frac{\cos nx}{n}\bigg|_{-\pi}^{0} - k\,\frac{\cos nx}{n}\bigg|_{0}^{\pi}\right.$$

$\because\ \cos(-\alpha) = \cos\alpha$

$$b_n = \frac{2k}{n\pi}(1-\cos n\pi)$$

In general,

$$\cos n\pi = \begin{cases} -1\,, & \text{odd } n \\ 1\,, & \text{even } n \end{cases} \quad ,$$

$$\text{1-}\cos n\pi = \begin{cases} 2\,, & \text{odd } n \\ 0\,, & \text{even } n \end{cases}$$

2. Fourier cosine and sine transformations

Fourier cosine series concerns even functions $f(x)$

$f(x) = \int_0^\infty A(w)\cos wx\,dw$, where

$$A(w) = \frac{2}{\pi} \int_0^\infty f(v) wv \, dv$$

We namely set $\quad A(w) = \sqrt{\frac{2}{\pi}} f_c(W)$

i.e., $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(w) \cos wx \, dw$$

Fourier sine transformation of $f(x)$ is given by

$$f_c(w) = \frac{2}{\pi} \int_0^\infty f(x) \sin wx \, dx$$

3. Fourier Series on arbitrary interval

If $f(x)$ is a periodic function defined on interval $[-l, l]$ of period $2l$, then it's fourier expansion is

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{k\pi x}{l}\right) + \text{b}_k \sin\left(\tfrac{k\pi x}{l}\right) \right.$$

where ,

$$a_k = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{k\pi x}{l}\right) \text{dx}$$

$$b_k = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{k\pi x}{l}\right) \text{dx} \qquad \text{k} = 1, 2, 3, \ldots\ldots\ldots\ldots$$

If $f(x)$ is an even function of period $2l$, then
$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{l}\right)$

where , $\qquad a_k = \frac{2}{l}\int_0^2 f(x)\cos\left(\frac{k\pi x}{l}\right)$ dx $\qquad$ k = 0, 1, 2, 3, ...

If function is an odd $f(x)$ on $2l$, then
$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{l}\right)$
where, $\qquad$ $\mathrm{b}_k = \frac{2}{l}\int_0^2 f(x)\sin\left(\frac{k\pi x}{l}\right)$ dx .

APPLICATION

Solution of Heat Equation

The Heat Equation is a certain Partial differential Equation. It's an example of parabolic Partial Differential Equation. The theory of heat equation was first developed by Joseph Fourier in 1822 for the purpose of modelling how a quantity such as heat diffuses through a given region.

Method of Separation of Variables

To find the solution of the form

$$u(x,t) = X(x)T(t),$$

where $X(x)$ is a function of x and $T(t)$ is a function of t.
Consider the following Initial Boundary Value Problem:

Partial Differential Equation: $u_t = \alpha^2 u_{xx}\,,\ 0 \le x \le L\,,\ 0 < t < \infty,$ (1)
Boundary Conditions : $u(0,t) = 0 u(L,t) = 0\,,\ 0 < t < \infty\,,$ (2)
Initial Conditions : $u(x,0) = f(x)\,,\ 0 \le x \le L\,.$ (3)

Step 1: (Reducing to the Ordinary Differential Equations) Assuming equation (1) has solutions of the form

$$u(x,t) = X(x)T(t),$$

where X is a function of x alone and T is a function of t alone. Now

$u_t = X(x)T'(t)$ *and* $u_{xx} = X''(x)T(t).$

Now, substituting these expression into $u_t = \alpha^2 u_{xx}$ and separating variables, we obtain

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$
$$\implies \qquad \frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}.$$

Since a function of t can equal a function of x only when both functions are constant.
Thus,

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = c$$

for some constant c. This leads to the following two ODEs:

$$T'(t) - \alpha^2 c T(t) = 0\,, \tag{4}$$

$$X''(x) - cX(x) = 0\,. \tag{5}$$

Thus, the problem of solving the PDE (1) is now reduced to solving the two ODEs.

Step 2: (Applying Boundary conditions)
Since the product solutions $u(x,t) = X(x)T(t)$ are to satisfy the Boundary Conditions (2), we have

$$u(0,t) = X(0)T(t) = 0 \qquad \text{and} \qquad X(L)T(t) = 0\,,\ t > 0.$$

Thus, either $T(t) = 0 \quad \forall \quad t > 0$,
$\implies u(x,t) = 0$, or $X(0) = X(L) = 0$.

Ignoring the trivial solution $u(x,t) = 0$, we combine the boundary conditions $X(0) = X(L) = 0$ with the differential equation for X in (5) to obtain the Boundary Value Problem:

$$X''(x) - cX(x) = 0, X(0) = X(L) = 0. \tag{6}$$

There are three cases: $c < 0\,,\ c > 0\,,\ c = 0$. It is convenient to set $c = \lambda^2$ when $c < 0$ and $c = \lambda^2$ when $c > 0\,,$ for some constant $\lambda > 0$.

<u>*Case 1.*</u> ($c = \lambda^2 > 0$ for some $\lambda > 0$). In this case, a general solution to the differential equation (5) is

$$X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x},$$

where C_2 and C_2 are arbitrary constants. To determine C_1 and C_2, we use the Boundary Condition $X(0) = 0\,,\ X(L) = 0$ to have

$$X(0) = C_1 + C_2 = 0, \tag{7}$$

$$X(L) = C_1 e^{\lambda L} + C_2^{-\lambda L} = 0. \tag{8}$$

From the first equation, it follows that $C_2 = -C_1$. The second equation leads to

$$C_1(e^{\lambda L} - e^{-\lambda L}) = 0,$$

$$\implies C_1(e^{2\lambda L}) = 0,$$

$$\implies C_1 = 0.$$

Since $(e^{2\lambda L} \ - 1) > 0$. Therefore, $C_1 = 0$ and hence $C_2 = 0$. Consequently $X(x) = 0$ and this implies $u(x,t) = 0$ i.e., there is no nontrivial solution to (5) for the case $c > 0$.

Case 2. . (when $c = 0$)

The general solution solution to (5) is given by

$$X(x) = C_3 + C_4 x.$$

Applying Boundary Conditons yields $C_3 = C_4 = 0$ and hence $X(x) = 0$. Again, $u(x,t) = X(x)T(t) = 0$.

Thus, there is no nontrivial solution to (5) for $c = 0$.

Case 3. (When $c = -\lambda^2 < 0$ for some $\lambda > 0$)

The general solution to (5) is

$$X(x) = C_5 \cos(\lambda x) + C_6 \sin(\lambda x).$$

This time the Boundary Condition $X(0) = 0$, $X(L) = 0$ gives the system

$$C_5 = 0,$$
$$C_5 \cos(\lambda L) + C_6 \sin(\lambda L) = 0.$$

As $C_5 = 0$, the system reduces to solving $C_6 \sin(\lambda L) = 0$. Hence, either $\sin(\lambda L) = 0$ or $C_6 = 0$. Now

$$\sin(\lambda L) = 0 \qquad \Longrightarrow \qquad \lambda L = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots .$$

Therefore, (5) has a nontrivial solution ($C6 \neq 0$) when

$$\lambda L = n\pi \quad \text{or} \quad \lambda = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots .$$

Here, excluding $n = 0$, since it makes $c = 0$. Therefore, the nontrivial solutions (eigen-functions) X_n corresponding to the eigenvalue $c = -\lambda^2$ are given by

$$X_n(x) = a_n \sin\left(\frac{n\pi x}{L}\right), \tag{9}$$

where a_n's are arbitrary constants.

Step 3: (Applying Initial Conditions)

Solving equation (4). The general solution to (4) with $c = -\lambda^2 = \left(\frac{n\pi}{L}\right)^2$ is

$$T_n(t) = b_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}.$$

Combing this with (9), the product solution $u(x,t) = X(x)T(t)$ becomes

$$
\begin{aligned}
u_n(x,t) &:= X_n(x)T_n(t) = a_n \sin\left(\frac{n\pi x}{L}\right) b_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \\
&= c_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots .
\end{aligned}
$$

where c_n is an arbitrary constant.

Since the problem (9) is linear and homogeneous, an application of superposition principle gives

$$
\underline{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right),} \tag{10}
$$

which will be a solution to (1)-(3), provided the infinite series has the proper convergence behaviour.

Since the solution (10) is to satisfy Initial Conditions (3), we must have

$$
u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x), \quad 0 < x < L.
$$

Thus, if $f(x)$ has an expansion of the form

$$
f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right), \tag{11}
$$

which is called a Fourier sine series (FSS) with c_n's are given by the formula

$$
c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \tag{12}
$$

Then the infinite series (10) with the coefficients c_n given by (12) is a solution to the problem (1)-(3).

Example :

Find the solution to the Initial Boundary Value Problem

$$4\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2]\,,$$

$$u(0, x) = 3\sin\left(\frac{\pi x}{2}\right), \quad u(t, 0) = 0, \quad u(t, 2) = 0.$$

Solution: Let $u_n(t, x) = v_n(t)w_n(x)$. Then

$$4w_n(x)\frac{dv}{dt}(t) = v_n(t)\frac{d^2w}{dx^2}(x) \qquad \Longrightarrow \qquad \frac{4v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$v_n'(t) + \frac{\lambda_n}{4}v_n(t) = 0, \qquad w_n''(x) + \lambda_n w_n(x) = 0.$$

We solve for v_n with the initial condition $v_n(0) = 1$.

$$e^{\frac{\lambda n}{4}t}v_n'(t) + \frac{\lambda_n}{4}e^{\frac{\lambda n}{4}t}v_n(t) \qquad \Longrightarrow \qquad \left[e^{\frac{\lambda n}{4}t}v_n(t)\right]' = 0$$

Therefore,

$$v_n(t) = ce^{\frac{\lambda n}{4}t}, \quad 1 = v_n(0) = c \quad \Longrightarrow \quad v_n(t) = e^{-\frac{\lambda n}{4}t}.$$

Next the Boundary Value Problem: $w_n''(x) + \lambda_n w_n(x) = 0$, with $w_n(0) = w_n(L) = 0$.

Since $\lambda_n > 0$, introduce $\lambda_n = \mu_n^2$. The characteristic polynomial is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Longrightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution, $w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)$.
The boundary conditions imply

$$0 = w_n(0) = c_1, \quad \Longrightarrow w_n(x) = c_2 \sin(\mu_n x).$$

$$0 = w_n(2) = c_2 \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Longrightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \left(\frac{n\pi}{2}\right)^2, \quad w_n(x) = \sin\left(\frac{n\pi x}{2}\right).$$

$$u(t,x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{4}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right).$$

The initial condition is $3\sin\left(\dfrac{\pi x}{2}\right) = \displaystyle\sum_{n=1}^{\infty} c_n \sin\left(\dfrac{n\pi x}{2}\right)$.

The orthogonality of the sine functions implies

$$3\int_0^2 \sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{m\pi x}{2}\right)dx = \sum_{n=1}^{\infty}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi x}{2}\right)dx.$$

If $m \neq 1$, then $0 = c_m \frac{2}{2}$, that is, $c_m = 0$ for $m \neq 1$. Therefore,

$$3\sin\left(\frac{\pi x}{2}\right) = c_1 \sin\left(\frac{\pi x}{2}\right) \quad \Longrightarrow \quad c_1 = 3.$$

We conclude that

$$u(t,x) = 3e^{-\left(\frac{\pi}{4}\right)^2 t} \sin\left(\frac{\pi x}{2}\right).$$

Solution of Wave Equation

A wave is a physical process in which energy is transported through space by a propagating disturbance of a physical field from that field's equilibrium state. The function $U(x, y, z, t)$ describing the wave is therefore a description of that field's state at each point in space and each moment of time.

The Problem :

Let $u(x,t)$ denote the vertical displacement of a string from the x axis at position x and time t. The string has length l. The left and right ends are held fixed at height zero and its initial configuration and speed are given.We choose a coordinate system so that the left hand end of the string is at $x = 0$ and the right hand end of the string is at $x = l$.

For the string undergoing small amplitude transverse vibrations $u(x,t)$ follows the wave equation.

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \textbf{(1)}$$

for all $0 < x < l$ and $t > 0$

This wave equation is a type of second-order partial differential equation involving two variables x and t.

conditions at the ends at height zero are the "boundary conditions"

$u(0,t) = 0$ for all $t > 0$ **(2)**

$u(l,t) = 0$ for all $t > 0$ **(3)**

we are given the position and speed of the string at time 0, there are functions $f(x)$ and $g(x)$ such that the "initial conditions"

$u(x,0) = f(x)$ for all $0 < x < l$ **(4)**

$u_t(x,0) = g(x) \qquad\qquad \text{for all } 0 < x < l \qquad\qquad \mathbf{(5)}$

We have to determine $u(x,t)$ for all x and t.

The general application of the Method of Separation of Variables for a wave equation involves **three** steps:

1.We find all solutions of the wave equation with the general form

$$u(x,t) = X(x)T(t)$$

for some function $X(x)$ that depends on x only and some function $T(t)$ that depends only on t. If we find a set of solutions $X_i(x)T_i(t)$ since the wave equation is a linear equation,

$$u(x,t) = \sum_i c_i X_i(x)T_i(t)$$

is also a solution for any choice of the constants c_i .

2. We try to impose the boundary conditions (2) and (3).

3. We try to impose the initial conditions (4) and (5).

Solution :

The First Step

The factorized function $u(x,t) = X(x)T(t)$ is a solution to the wave equation (1) iff

$$X(x)T''(t) = c^2 X''(x)T(t) \Longleftrightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

The LHS is independent of t. So the RHS, that is equal to the LHS, must be independent of t too. The RHS is independent of x.Therefore the LHS must be independent of x too. So both sides must be independent of both x and t. So both sides are constant. Let them be a constant λ. So we get

$$\frac{X''(x)}{X(x)} = \lambda \qquad \frac{1}{c^2}\frac{T''(t)}{T(t)} = \lambda \qquad \textbf{(6)}$$

$\Longleftrightarrow$

$$X''(x)\lambda X(x) = 0 \qquad T''(t)c^2\lambda T(t) = 0$$

We now have two constant coefficient ordinary differential equations, which is solved in the usual way. We put $X(x) = e^{mx}$ and $T(t) = e^{nt}$ for some constants m and n to be found. These are solutions iff

$$\frac{d^2}{dx^2}e^{mx} - \lambda e^{mx} = 0 \qquad \frac{d^2}{dx^2}e^{nt} - c^2\lambda e^{nt} = 0$$

$\Longleftrightarrow$

$$(m^2 - \lambda)e^{mx} = 0 \qquad (n^2 - c^2\lambda)e^{nt} = 0$$

$\Longleftrightarrow$

$$m^2 - \lambda = 0 \qquad n^2 - c^2\lambda = 0$$

$\Longleftrightarrow$

$$m \pm \sqrt{\lambda} \qquad n = \pm c\sqrt{\lambda}$$

If $\lambda \neq 0$, we now have two independent solutions, namely $e^{\sqrt{\lambda}x}$ and $e^{-\sqrt{\lambda}x}$, for $X(x)$ and two independent solutions, namely $e^{c\sqrt{\lambda}t}$ and $e^{-c\sqrt{\lambda}t}$, for $T(t)$. For $\lambda \neq 0$ and $\lambda = 0$, the general solution to (6) is

$$u(x,t) = (d_1 e^{\sqrt{\lambda}x} + d_2 e^{-\sqrt{\lambda}x})(d_3 e^{c\sqrt{\lambda}t} + d_4 e^{-c\sqrt{\lambda}t})$$

for arbitrary $\lambda \neq 0$ and arbitrary d_1, d_2, d_3, d_4

$$u(x,t) = (d_1 + d_2 x)(d_3 + d_4 t)$$

for arbitrary $\lambda = 0$ and arbitrary d_1, d_2, d_3, d_4

Second Step

If $X_i(x)T_i(t)$, $i = 1, 2, 3, \ldots.$ then $\sum_i c_i X_i(x) T_i(t)$ is also a solution for any choice of the constants c_i. This solution satisfies the boundary condition iff

$$\sum_i c_i X_i(x) T_i(t) = 0$$

for all $t > 0$

Going through the solutions that we found in Step 1 and discarding all of those that fail to satisfy $X(0) = X(l) = 0$.

Consider $\lambda = 0$ the conditions $X(0) = X(l) = 0$ are both satisfied only if $d_1 = d_2 = 0$, in which case $X(x)$ is identically zero. There is nothing to be gained by keeping an identically zero $X(x)$, so we discard $\lambda = 0$.

Consider $\lambda \neq 0$ so that $d_1 e^{\sqrt{\lambda}x} + d_2 e^{-\sqrt{\lambda}x}$. The condition $X(0) = 0$ is satisfied iff $d_1 + d_2 = 0$. So we require that $d_2 = -d_1$. The condition $X(l) = 0$ is satisfied iff

$$0 = d_1 e^{\sqrt{\lambda}l} + d_2 e^{-\sqrt{\lambda}l} = d_1(e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l})$$

If d_1 is zero, then $X(x)$ would again be identically zero. So instead, we discard any λ that does not obey

$$e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l} = 0 \iff e^{\sqrt{\lambda}l} = e^{-\sqrt{\lambda}l} \iff e^{2\sqrt{\lambda}l} = 1$$

this is true iff there is an integer r such that

$$2\sqrt{\lambda}l = 2r\pi \iff \sqrt{\lambda} = r\frac{\pi}{l} \iff \lambda = r^2\frac{\pi^2}{l^2}$$

$$X(x)T(t) = (d_1 e^{r\frac{\pi}{l}x} + d_2 e^{-r\frac{\pi}{l}x})(d_3 e^{cr\frac{\pi}{l}t} + d_4 e^{-cr\frac{\pi}{l}t})$$

$$= \sin(r\frac{\pi}{l}x)[\alpha_r \cos(cr\frac{\pi}{l}t) + \beta_r \sin(cr\frac{\pi}{l}t)]$$

where $\alpha_r = 2d_1(d_3 + d_4)$ and $\beta_r = 2d_1(d_3 - d_4)$

Third Step

We know that

$$u(x,t) = \sum_{r=1}^{\infty} \sin(r\frac{\pi}{l}x)[\alpha_r \cos(cr\frac{\pi}{l}t) + \beta_r \sin(cr\frac{\pi}{l}t)]$$

obeys the wave equation (1) and the boundary conditions (2) and (3)

$$f(x) = u(x,0) = \sum_{r=1}^{\infty} \alpha_r \sin(r\frac{\pi}{l}x)$$

$$g(x) = u_t(x,0) = \sum_{r=1}^{\infty} \beta_r \frac{ck\pi}{l} \sin(\frac{r\pi}{l}x)$$

But any function, $h(x)$, defined on the interval $0 < x < l$, has a unique presentation

$$h(x) = \sum_{r=1}^{\infty} b_r \sin(\frac{r\pi x}{l})$$

$$b_k = \frac{2}{l} \int_0^l h(x) \sin(\frac{r\pi x}{l}) dx$$

by choosing $f(x) = h(x)$ and $g(x) = h(x)$ we get the solution

$$u(x,t) = \sum_{r=1}^{\infty} \sin(\frac{r\pi x}{l})[\alpha_r \cos(\frac{cr\pi t}{l}) + \beta_r \sin(\frac{cr\pi t}{l})]$$

where

$$\alpha_k = \frac{2}{l} \int_0^l h(x) \sin(\frac{r\pi x}{l}) dx \qquad \beta_k = \frac{2}{cr\pi} \int_0^l g(x) \sin(\frac{r\pi x}{l}) dx$$

Example 1 : Find the deflection $u(x,t)$ of the vibrating string (length $l = \pi$, ends fixed and $c^2 = 1$) corresponding to zero initial velocity and initial deflection

$$f(x) = k(\sin x - \sin 2x)$$

Vibrations in the string can be calculated using the wave equation that is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(since } c^2 = 1 \text{ and } l = \pi\text{)}$$

Here the initial and boundary conditions are:

Boundary conditions:

$$\left.\begin{array}{rcl} u(0,t) & = & 0 \\ u(\pi,t) & = & 0 \end{array}\right\}$$

Initial conditions:

$u(x,0) = f(x) = k(\sin x - \sin 2x)$

and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$

The solution of wave equation is of the form:

$$u(x,t) = (A_1 \cos rx + A_2 \sin rx)(A_3 \cos rt + A_4 \sin rt)$$

Since, $u(t,0) = 0 \Rightarrow A_1(A_3 \cos rt + A_4 \sin rt) \Rightarrow A_1 = 0$

$A_3 \cos rt + A_4 \sin rt \neq 0$ as then $u(x,t) = 0$ is the trivial solution

Therefore,

$$u(x,t) = A_2 \sin rx (A_3 \cos rt + A_4 \sin rt) = \sin rx (A_2 A_3 \cos rt + A_2 A_4 \sin rt)$$

Also,

$u(\pi, t) = 0 \Rightarrow \sin r\pi (A_2 A_3 \cos rt + A_2 A_4 \sin rt) = 0$

here $\sin r\pi = 0$ otherwise the solution of the problem will be trivial

$\Rightarrow p = n$ where n is any integer

Therefore the solution of the wave equation satisfying the boundary conditions are:

$u_n(x,t) = (B_n \cos nt + C_n \sin nt) \sin nx$

For all the integral values of n, here $A_2 A_3 = B_n, A_2 A_4 = C_n$

Now to satisfy the initial boundary conditions take $n = 1, 3, ...$ then the general solution of the wave equation is :

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos(nt) + C_n \sin(nt) \sin nx$$

This also satisfies the boundary condition

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}(-nB_n \sin(nt) + nC_n \cos(nt)) \sin nx$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \Rightarrow nC_n \sin nx = 0 \Rightarrow C_n = 0$$

since other case is not possible

Hence we get,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos(nt) \sin nx$$

$$u(x,0) = f(x) \Rightarrow \sum_{n=1}^{\infty} B_n \sin nx = f(x) = k(\sin x - \sin 2x)$$

Equating coefficients of $\sin nx$ for $n = 1, 2, ...$ we get

$$A_1 = k, A_2 = -k \text{ and } A_3 = 0 = A_4 = ...$$

Hence,

$$\begin{aligned} u(x,t) &= A_1 \cos t \sin x + A_2 \cos 2t \sin 2x \\ &= k(\cos t \sin x - \cos 2t \sin 2x) \end{aligned}$$

is the required solution.

Example 2 : As a concrete example,consider that

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t) && \text{for all } 0<x<1 \text{ and } t>0\\
&u(0,t)=u(1,t)=0 && \text{for all } t>0\\
&u(x,0)=x(1-x) && \text{for all } 0<x<1\\
&u_t(x,0)=0 && \text{for all } 0<x<1
\end{aligned}
$$

This is a special case of equations (1-5) with $l=1$, $f(x)=x(1x)$ and $g(x)=0$

$$u(x,y)=\sum_{n=1}^{\infty}\sin(n\pi x)[\alpha_n\cos(cn\pi t)+\beta_n\sin(cn\pi t)]$$

with

$$\alpha_n = 2\int_0^1 x(1-x)\sin(n\pi x)dx \qquad \beta_n = 2\int_0^1 0\sin(n\pi x)dx = 0$$

$$\int_0^1 x\sin(n\pi x)dx = \int_0^1 \frac{-1}{\pi}\frac{d}{dk}\cos(n\pi x)dx = \frac{-1}{\pi}\cos(n\pi)$$

$$\int_0^1 x^2\sin(n\pi x)dx = \int_0^1 \frac{-1}{\pi^2}\frac{d^2}{dk^2}\sin(n\pi x)dx = \cos(n\pi)\frac{2-n^2\pi^2}{n^3\pi^3} - \frac{2}{n^3\pi^3}$$

we have

$$
\begin{aligned}
\alpha_n = 2\int_0^1 x(1-x)\sin(n\pi x)dx &= 2\left[\frac{-1}{\pi}\cos(n\pi)-\cos(n\pi)\frac{2-n^2\pi^2}{n^3\pi^3}+\frac{2}{n^3\pi^3}\right]\\
&= \frac{4}{n^3\pi^3}[1-\cos(n\pi)]
\end{aligned}
$$

$$= \begin{cases} \dfrac{8}{n^3\pi^3} & \text{for} \quad \text{n is odd} \\ 0 & \text{for} \quad \text{n is even} \end{cases}$$

and

$$u(x,y) = \sum_{n=1}^{\infty} \frac{8}{n^3\pi^3} \sin(n\pi x)\cos(cn\pi t) \qquad \text{when n is odd}$$

Conclusions

From above discussion we can say a differential equation is defined as an equation containing the derivative or derivatives of the dependent variable with respect to the independent variable. We have discussed differential equations formulas and different methods to solve differential equations by using basic differential equations formulas.
Wave equation and Heat equation are solved using method of separation of variables. Learning of wave equation will help us with not only the movement of strings and wires, but also the movement of fluid surfaces,such as water waves. Similarly heat equation can be used in simple engineering problems assuming there is equilibrium of the temperature fields and heat transport, with time.

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