

# Uniform Convergence

**Definition :** A sequence of functions  $f_n : X \rightarrow Y$  converges uniformly to  $f(x)$  if for every  $\epsilon > 0$ , there is  $N \in \mathbb{N}$  such that for all  $n \geq N$  and all  $x \in X$  we have  $|f_n(x) - f(x)| < \epsilon$

Discuss the uniform convergence of the sequence of functions  $f_n(x) = x^n$ ,  $0 \leq x \leq 0.5$

Here  $f(x) = 0$

So,  $|f_n(x) - f(x)| = |x^n - 0|$

```
In[5]:= b[n_] = Maximize[{Abs[x^n - 0], 0 <= x <= 0.5}, x]
```

NMaximize::nnum : The function value  $-e^{-3.77811 \text{Re}[n]}$  is not a number at  $\{x\} = \{0.0228659\}$ . >>

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```
Out[5]= Maximize[{Abs[x^n], 0 <= x <= 0.5}, x]
```

**b[1]**

$\{0.5, \{x \rightarrow 0.5\}\}$

**b[1][[1]]**

0.5

**N[b[1][[1]]]**

**IntegerPart[0.5]**

0

**NumberForm[x]** command Display the first 12 digits of a numeric approximation to x

```
In[7]:= NumberForm[N[b[2][[1]]], 12]
```

```
Out[7]/NumberForm=
0.25
```

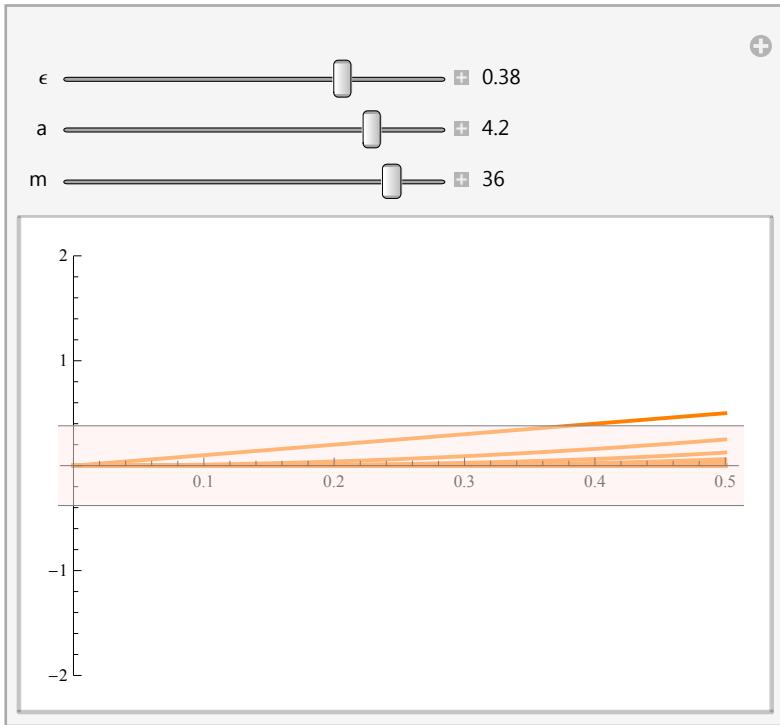
```
Maximize[{Abs[x^n], 0 <= x <= 0.5`}, x]
```

```
b[n_] := Maximize[{Abs[x^n - 0], 0 <= x <= 0.5}, x]
TableForm[Table[{n, N[b[n][[1]]]}, {n, 1, 20}],
TableHeadings -> {{}, {"n", "|f_n(x)-f(x)|"}}]
```

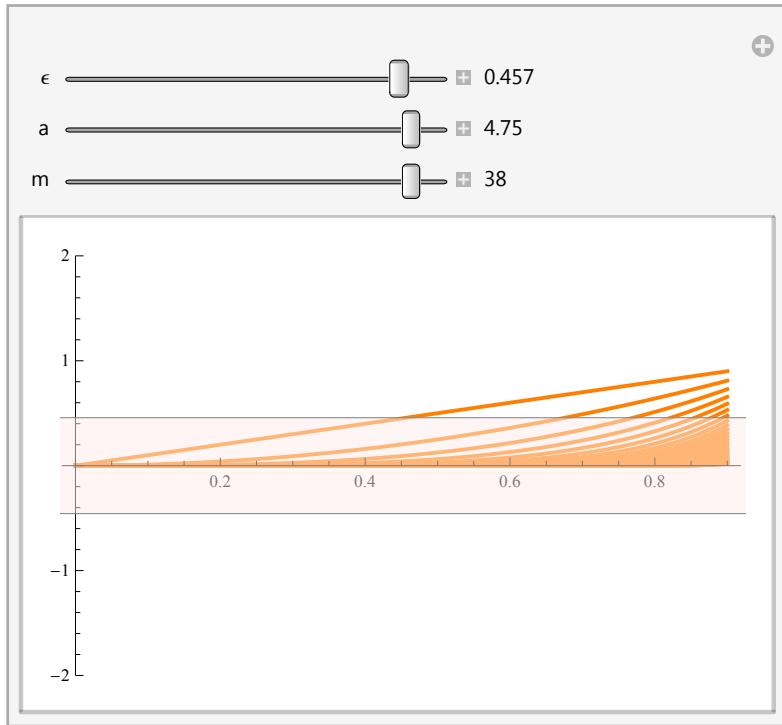
n	f <sub>n</sub> (x) - f(x)
1	0.5
2	0.25
3	0.125
4	0.0625
5	0.03125
6	0.015625
7	0.0078125
8	0.00390625
9	0.00195313
10	0.000976563
11	0.000488281
12	0.000244141
13	0.00012207
14	0.0000610352
15	0.0000305176
16	0.0000152588
17	7.62939 × 10 <sup>-6</sup>
18	3.8147 × 10 <sup>-6</sup>
19	1.90735 × 10 <sup>-6</sup>
20	9.53674 × 10 <sup>-7</sup>

```
Table[f[n, x], {n, 10}]
{x, x2, x3, x4, x5, x6, x7, x8, x9, x10}
```

```
In[38]:= f[n_, x_] := x^n
Manipulate[Plot[Table[f[n, x], {n, m}], {x, 0, 0.5},
  PlotRange -> {-2, 2}, PlotStyle -> {Orange, Thick}, Epilog ->
  {Opacity[.5], LightPink, EdgeForm[GrayLevel[.5]], Rectangle[{-a, -ε}, {a, ε}]}],
{ε, 0.01, 0.5, 0.001, Appearance -> "Labeled"}, {a, 0, 5, 0.01, Appearance -> "Labeled"}, {m, 1, 40, 1, Appearance -> "Labeled"}]
```



```
In[40]:= f[n_, x_] := x^n
Manipulate[Plot[Table[f[n, x], {n, m}], {x, 0, 0.9},
  PlotRange -> {-2, 2}, PlotStyle -> {Orange, Thick}, Epilog ->
  {Opacity[.5], LightPink, EdgeForm[GrayLevel[.5]], Rectangle[{-a, -ε}, {a, ε}]}],
{ε, 0.01, 0.5, 0.001, Appearance -> "Labeled"}, {a, 0, 5, 0.01, Appearance -> "Labeled"}, {m, 1, 40, 1, Appearance -> "Labeled"}]
```



```
In[42]:= f[n_, x_] := x^n
Manipulate[Plot[Table[f[n, x], {n, m}], {x, 0, 1},
  PlotRange -> {-2, 2}, PlotStyle -> {Orange, Thick}, Epilog ->
  {Opacity[.5], LightPink, EdgeForm[GrayLevel[.5]], Rectangle[{-a, -ε}, {a, ε}]}],
{ε, 0.01, 0.5, 0.001, Appearance -> "Labeled"}, {a, 0, 5, 0.01, Appearance -> "Labeled"}, {m, 1, 20, 0.1, Appearance -> "Labeled"}]
```

