ONE DIMENSIONAL ISING MODEL

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Introduction

In this paper using the transfer matrix method we briefly review the solution of the one-dimensional Ising model. It is an exactly solvable model of ferromagnetism. The aim is to find the thermodynamic limit of the partition function under the one dimensional lattice with finite number of interacting neighbors.The partition function Z then helps us to calculate the free energy F and the magnetization M.

The Ising Model with the Transfer Matrix Method

We start by considering a one dimensional lattice of N sites. Each site is labeled by an integer $\boldsymbol{i}$.

The assumption on boundary conditions is such that the setup consists of n spins where $\sigma_i = \pm 1$ denote the spin at site i.The conditions ensure that $\sigma_{N+1} = \sigma_1$. This is done so that the nth spin interacts with $(n-1)th$ spin and the first spin.

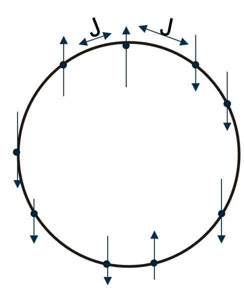

Figure shows the N lattice sites with either spin up or spin down. J being the interaction between the neighboring spins.

Furthermore, we assume that the nearest interaction of spins with the neighboring spins is given by

$$E_1 = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1}$$

where σ_i is the expectation value that is the value of the ith spin at the thermodynamic equilibrium.

The system is also subjected to an external field then the interaction between spins and applied external field is given by

$$E_2 = -h\sum_{i=1}^{N}\sigma_i$$

Defining the energy of each spin configuration to be given by the Hamiltonian function $\mathcal{H}$

$$\boxed{\mathcal{H} = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - h\sum_{i=1}^{N}\sigma_i}$$

It is known that the partition function is defined as

$$Z = \sum_{S} e^{-\beta E(S)}$$

here S is the configuration of the system, E(S) is the energy of the system at state S and β is the inverse of temperature
For the one dimensional Ising model the partition function is given as

$$Z_n = \sum_{\sigma} e^{\beta[J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} + h\sum_{i=1}^{N}\sigma_i]}$$

which gives the summation over all spin configurations of the system
Now we try to drop the summations from the partition function to be able

to calculate free energy . Expanding this summation in Boltzmann weight $e^{-\beta h}$ as

$$Z_n = \sum_{\sigma} e^{\beta[J(\sigma_1\sigma_2+\sigma_2\sigma_3)+.......+h(\sigma_1+\sigma_2+.....)}]$$

For this expression to be associated to only the near neighboring interactions we split the $h-terms$ such that

$$= \sum_{\sigma} e^{\beta[J\sigma_1\sigma_2+J\sigma_2\sigma_3+.....+\frac{h}{2}\sigma_1+\frac{h}{2}\sigma_2+\frac{h}{2}\sigma_1+\frac{h}{2}\sigma_2+.....]}$$

$$= \sum_{\sigma} e^{\beta[J\sigma_1\sigma_2+\frac{h}{2}(\sigma_1+\sigma_2)]}.....e^{[J\sigma_N\sigma_1+\frac{h}{2}(\sigma_N+\sigma_1)]}$$

Now we try to simplify the above expression to get the partition function.Define a function $'t'$ such that if

$$t_{\sigma_1\sigma_2} = e^{\beta[J\sigma_1\sigma_2+\frac{h}{2}(\sigma_1+\sigma_2)]}$$

then in general

$$t(\sigma,\sigma^{'}) = e^{\beta[J\sigma\sigma^{'}+\frac{h}{2}(\sigma+\sigma^{'})]}$$

Putting this $t(\sigma,\sigma^{'})$ in the partition function then we get

$$Z_n = \sum_{\sigma_1}\sum_{\sigma_2}......\sum_{\sigma_N} t_{\sigma_1\sigma_2}t_{\sigma_2\sigma_3}.......t_{\sigma_N\sigma_1}$$

This is the summation over each spin state

It is known that if there is a field between σ and $\sigma^{'}$ then the values possible for the first spin are $+1$ and -1 and similarly for the second spin the possible values are $+1$ and -1. Using this t can be written as a $2X2$ matrix with elements as all possible spin configurations such that

$$t = \begin{pmatrix} t_{+1,+1} & t_{+1,-1} \\ t_{-1,+1} & t_{-1,-1} \end{pmatrix} = \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix}$$

It is seen that if each $t_{\sigma_1\sigma_2}t_{\sigma_2\sigma_3}$ is summed over its respective spin variable then a matrix product defined is by $(AB)_{ik} = \sum_j A_{ij}B_{jk}$

For example

$$\sum_{\sigma_2} t_{\sigma_1\sigma_2}t_{\sigma_2\sigma_3} = (t.t)_{\sigma_1\sigma_2}$$

Then the partition function for n identical matrices becomes

$$Z_n = \sum_{\sigma_1} \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix} \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix} \cdots\cdots\cdots \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix}$$

$$Z_n = \sum_{\sigma_1} \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix}^N$$

Sum over σ_1 is the sum over the diagonal elements of the matrix
We know that the summation of a matrix is given by the trace of the power of the matrix

$$Z_N = Tr(t^n)$$

t is seen as a real and symmetric matrix which can be diagonalised as

$$t = QDQ^{-1}$$

Q is the matrix of left eigenvectors,Q^{-1} is the matrix of right eigenvectors and D is the diagonal matrix with eigenvalues λ_1 and λ_2

$$\Rightarrow t^n = QDQ^{-1}QDQ^{-1}....QDQ^{-1} = QD^nQ^{-1}$$

$$\Rightarrow tr(QDQ^{-1}) = tr(D^N)$$

$$\Rightarrow D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow D^2 = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \Rightarrow D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \Rightarrow tr(D^n) = \lambda_1^n + \lambda_2^n$$

$$\Rightarrow Z^N = tr(D^N) = \lambda_1^N + \lambda_2^N$$

To obtain the eigenvalues and hence the partition function we solve the characteristic equation $Av = \lambda v \Rightarrow (A - \lambda I)v = 0$ which is true only if $\det(A - \lambda I) = 0$ then

$$det[\begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}] = 0$$

$$det[\begin{pmatrix} e^{\beta(J+h)} - \lambda & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} - \lambda \end{pmatrix}] = 0$$

$$\Rightarrow (e^{\beta(J+h)} - \lambda)(e^{\beta(J-h)} - \lambda) - e^{-2\beta J} = 0$$

$$\Rightarrow \lambda^2 - \lambda e^{\beta J}(e^{\beta h} + e^{-\beta h}) + (e^{2\beta J} - e^{-2\beta J}) = 0$$

We know that $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$ then our quadratic equation becomes

$$2\sinh(2\beta J) - \lambda.2e^{\beta J}\cosh(\beta h) + \lambda^2 = 0$$

using the quadratic formula $\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{A^2}$ for $A = 1, B = -2e^{\beta J}\cosh(\beta h), C = 2\sinh(2\beta J)$ we get

$$\lambda = e^{\beta J}\cosh(\beta h) \pm \sqrt{e^{2\beta J}\sinh^2(\beta h) + e^{-2\beta J}}$$

Hence the partition function is given by

$$\boxed{Z = e^{N\beta J}[\left(\cosh(\beta h) + \sqrt{e^{-4\beta J} + \sinh^2(\beta h)}\right)^N + \left(\cosh(\beta h) - \sqrt{e^{-4\beta J} + \sinh^2(\beta h)}\right)^N]}$$

Free energy and Magnetization

Now we calculate the **Helmholtz free energy** that is given by

$$F = -k_B T ln(Z)$$

For the one dimensional Ising model

$$-(k_B T)^{-1} F = ln(Z_N) = ln(\lambda_1^N + \lambda_2^N) = ln[\lambda_1^N(1 + \frac{\lambda_2^N}{\lambda_1^N})]$$

In the thermodynamic limit where $N \to \infty$, free energy per spin is given by:

$$f = -k_B T \lim_{N\to\infty} \frac{1}{N} ln(Z_N = -k_B T ln(\lambda_1^N)$$

since

$$\lim_{N\to\infty} N^{-1} ln[\lambda_1^N(1+\frac{\lambda_2^N}{\lambda_1^N})] = 0$$

as $\frac{\lambda_2^N}{\lambda_1^N}$ is less than 1, so when we raise it to N, it tends to 0.

Free energy per lattice site as a functions of the the temperature T and h is given as

$$\boxed{\Rightarrow f(h,T) = -k_B T[e^{\beta J}\cosh(\beta h) + \sqrt{e^{2\beta J}\sinh^2(\beta h) + e^{-2\beta J}}]}$$

Magnetization M for the system of N particles is given by

$$M = \left(\frac{\partial f(h,T)}{\partial h}\right) = \left(\frac{\partial ln\lambda_1}{\partial H}\right)$$

$$\boxed{\Rightarrow M(h,T) = \frac{e^{\beta J}\sinh(\beta h)}{\sqrt{e^{2\beta J}\sinh^2(\beta h) + e^{-2\beta J}}}}$$

From the proof above the inference drawn is that $M(h,T)$ is an analytical function for real h and positive T. It is seen that there is no magnetization at any finite temperature in one dimension, hence no nontrivial critical point. Therefore, the model admits **no phase transition** at any positive temperature.

References

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