Fortnightly Problems

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Introduction

Note some of these answers may take up more space than a post-it note; this is to ensure comprehension on the reader's behalf. (e.g the first solution to challenge 2 can be written more concisely in terms of equivalences classes in which combinations of x_1 and x_2 are included)

Challenge 1 (14/01/2024)

> A knight move in chess is a translation 2 units North, East, South or West in one direction and 1 unit in a perpendicular direction. Given a knight starts at (0,0) in $\mathbb{Z}^2$, what is the minimum number of moves required for it to reach (n, n) for all $n \in \mathbb{N}$?

Consider the desired function is $f(n)$. The knight moves 3 unit translations a turn, and trivially it takes $2n$ unit translations to reach (n, n), therefore $f(n) \geq \frac{2n}{3}$ Hence $f(3n) \geq 2n$. We also know $f(n)$ is even because if we apply a checkerboard, all (n, n) squares are one colour and a knight alternates, hence $f(3n+1), f(3n+2) \geq 2n+1/2$ which can be rounded up to $2n+2$. By using $\binom{3}{3} = \binom{1}{2} + \binom{2}{1}$, $\binom{1}{1} = \binom{-1}{2} + \binom{2}{-1}$ and $\binom{5}{5} = 3\binom{1}{2} + \binom{2}{-1}$ we can attain these bounds for $n \neq 2$ hence $f(n) = \frac{2}{3}n$ if $n|3$ and $2\lfloor \frac{1}{3}n \rfloor + 2$ otherwise with the exception of $f(2) = 4$ ($\lfloor x \rfloor$ denotes the largest integer less than x).

Challenge 2 (22/01/2024)

Let $\langle x_n \rangle = 5n$ for values n from 1 to $4N$ inclusive. How many subsequences sum to a multiple of 4?

Method 1

Note that the sequence modulo 4 is $x_n = n$, simplifying the question. There are $2^{(4N-2)!}$ possibilities of subsequences for $n > 2$. However, note that, regardless of the sum of those subsequences mod 4, only 1 possibility of the first two terms can be included to get 0 mod 4. Explicitly, let the sum of the terms $n > 2$ be k. If:

- $k \equiv 0 \mod 4$, only by excluding $n = 1$ and $n = 2$ do we make the sum $0 \mod 4$
- $k \equiv 1 \mod 4$, only by including $n = 1$ and $n = 2$ do we make the sum 0 mod 4
- $k \equiv 2 \mod 4$, only by excluding $n = 1$ and including $n = 2$ do we make the sum $0 \mod 4$
- $k \equiv 3 \mod 4$, only by excluding $n - 1$ and including $n = 2$ do we make the sum $0 \mod 4$

This give the result of 2^{4N-2} subsequences.

Method 2

Consider the product $f(x) = (1 + x^5)(1 + x^{10})(1 + x^{15})...(1 + x^{20N})$. By choosing from each factor which term to multiply, we find that the coefficient of x^n gives the total number of ways we can add to get n from the terms in the sequence given. Let's rewrite f as follows:

$$f(x) = 1 + a_1x^5 + a_2x^{10} + a_3x^{15} + \dots$$

We can evaluate f for specific values as follows:

$$
\begin{aligned}
f(i) = 0 &= (1 - a_2 + a_4 - a_6 + \dots) + i(a_1 - a_3 + a_5 - a_7 \dots) \\
f(1) = 2^{4N} &= 1 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \dots \\
f(-1) = 2^{4N} &= 1 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 \dots \\
\frac{\mathrm{Re}(f(i)) + \frac{f(1)+f(-1)}{2}}{2} = 2^{4N-2} &= 1 + a_4 + a_8 + a_{12} \dots
\end{aligned}
$$

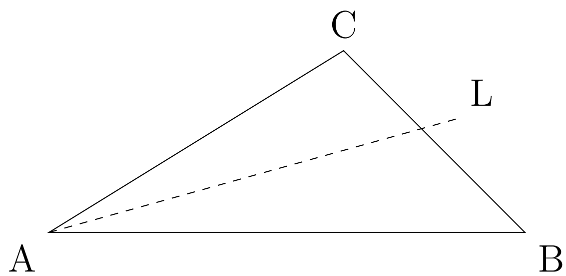
Since those are the coefficients we care about, the solution is 2^{4N-2}

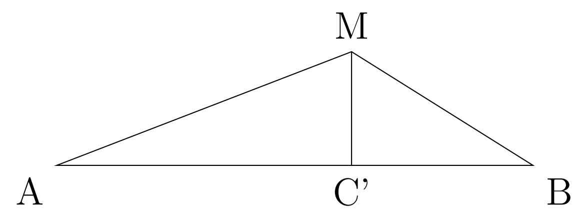
Challenge 3 (19/02/2024)

Consider the right angled triangle with sides $a < b < c$ (and angles A and B opposite a and b respectively). Prove:

$$a = b \tan\left(\frac{A}{2}\right) + \frac{c - b}{\cos(B)}$$

For shorthand I will refer to A, B and C interchangeably as points and angles at said points.





Fold (or reflect) the triangle as per the above diagram so that the line b overlaps c, bisecting A. Observe that $a = BM + MC'$. Noting a right angle at C' and that $BC' = c - b$ (as $C'A = b$), we get that $BM = \dfrac{c - b}{\cos(B)}$. We get that $MC' = b \tan\left(\dfrac{A}{2}\right)$. Putting these together in our equation for a gives the required result.

Challenge 4 (Added Later)

A projectile is projected at an angle of θ and a speed of U. Prove that the following quadratic is tangential for all projectile paths for fixed U:

$$y = \frac{U^2}{2g} - \frac{gx^2}{2U^2}$$

The parametric equation for motion is $\mathbf{r} = \begin{pmatrix} Ut\cos(\theta) \\ Ut\sin(\theta) - \frac{1}{2}gt^2 \end{pmatrix}$

First convert into parametric; $t = \dfrac{x}{U\cos(\theta)}$ and then substituting:

$$y = x\tan(\theta) - \frac{x^2}{2U^2}g\sec^2(\theta)$$

Rewrite in terms of $\tan\theta$:

$$0 = \frac{gx^2}{2U^2}\tan^2(\theta) - x\tan(\theta) + y + \frac{gx^2}{2U^2}$$

Note that, since this is tangential, each point the curve touches the given quadratic must have only one solution for θ, which means the discriminant in $\tan(\theta)$ must be 0:

$$x^2 - 4\left(\frac{gx^2}{2U^2}\right)\left(y + \frac{gx^2}{2U^2}\right) = 0$$

Rearranging:

$$-\left(y + \frac{gx^2}{2U^2}\right) = -\frac{U^2x^2}{2gx^2}$$

Rearranging further gives the final answer.

Challenge 5 (Added Later)

Let a 1D annulus be given by $D(p, r_1, r_2) := \{x \in \mathbb{R} : r_1 < |p - x| < r_2\}$. Prove that any set of $n-1$ points on the real line can always be covered by a collection of non-intersecting 1D annuli of the form $D\left(p, \frac{1}{2n}, \frac{1}{2}\right)$ (i.e an interval of unit length with a hole of size n^{-1})

Place the annuli such that $p \in \mathbb{Z}$; none of these overlap as they are 1 wide overall. If we translate all annuli by a value of $x \in [0, 1]$ the chance that an arbitrary point falls in the annuli is the proportion the annuli covers on a unit interval, $1 - \frac{n^{-1}}{1} = 1 - n^{-1}$. Let there be a point of 1 for each point covered and further $\mathbb{E}(X)$ be the expectation that X points are covered by the annuli. For $n-1$ points, enumerate them as $X_i, i = 1, 2, \ldots, n-1$. By the linearity of expectation, $\mathbb{E}(n-1) = \sum_{k=1}^{n-1} \mathbb{E}(X_k)$ which we know is just $\sum_{k=1}^{n-1}(1 - n^{-1}) = (1 - n^{-1})(n-1) = -1 - 1 + n + n^{-1} < n - 2$. Hence there must some choice of x that covers more than $n-2$ points, as required.