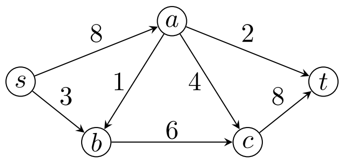
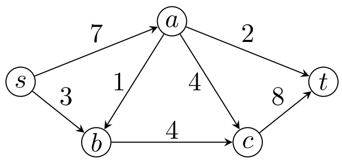
Problem 1 (20 points). For the following network $G = (V, E)$ with source s and sink t:

1. Give one maximum flow f^* of G (i.e., give $f^*(e)$ for every $e \in E$).
2. Draw the residual graph $G(f^*)$.
3. Give the s-t cut (S, T), where $S := \{v \in V \mid s \text{ can reach } v \text{ in } G(f^*)\}$, and $T := V \setminus S$, and give the capacity of this cut.
4. Give the s-t cut (S', T'), where $T' := \{v \in V \mid v \text{ can reach } t \text{ in } G(f^*)\}$, and $S' := V \setminus T'$, and give the capacity of this cut.

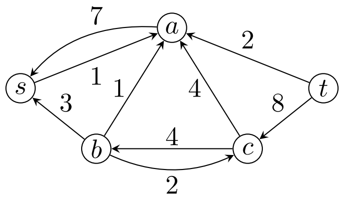


Solution.

1. One maximum-flow $f^*(e)$ is given below:



2. The corresponding residual graph $G(f^*)$ is given below:



3. $S = \{s, a\}$, $T = V \setminus S = \{b, c, t\}$. The capacity of this cut $c(S, T) = 2 + 4 + 1 + 3 = 10$.
4. $T' = \{t\}$, $S' = V \setminus T' = \{s, a, b, c\}$. The capacity of this cut $c(S', T') = 2 + 8 = 10$.

Problem 2 (20 points). Let f^* be one maximum flow of network $G = (V, E)$ with source $s \in V$ and sink $t \in V$. Let $T := \{v \in V \mid v \text{ can reach } t \text{ in } G(f^*)\}$. Let $S := V \setminus T$.

1. Prove that $s \in S$.
2. Prove that (S, T) is a minimum s-t cut of G.

Solution. The proof is analogy to the proof for the correctness of Ford-Fulkerson's algorithm.

1. Suppose coversely that $s \notin S$. Then we have $s \in T$ and therefore there exists an s-t path p in the residual graph $G(f^*)$. Hence we can use the *augment* (p, f^*) procedure to obtain a new flow f' with $|f'| > |f^*|$. This is contradicting to the fact that f^* is a maximum flow.

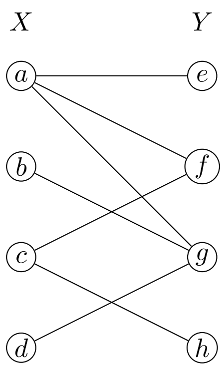
2. Consider these edges from S to T and those edges from T to S. Formally, define $E(S,T) = \{(u,v) \in E \mid u \in S, v \in T\}$, and $E(T,S) = \{(u,v) \in E \mid u \in T, v \in S\}$. For any edge $e = (u,v) \in E(S,T)$, we must have that e is saturated in f^*, i.e., $f^*(e) = c(e)$. This is because otherwise (i.e., $f^*(e) < c(e)$), there will be an edge from u to v in the residual graph $G(f^*)$, and therefore u can reach t, which contradicts to the definition of $E(S,T)$. Also, for any edge $e = (u,v) \in E(T,S)$, we must have that $f^*(e) = 0$. This is because otherwise (i.e., $f^*(e) > 0$), there will be an edge from v to u in the residual graph $G(f^*)$, and therefore v can reach t, which contradicts to the definition of $E(T,S)$. Then we compute the value of flow f^*: we proved that $|f^*| = \sum_{e \in E(S,T)} f^*(e) - \sum_{e \in E(T,S)} f^*(e)$. By using the above results, we have $|f^*| = \sum_{e \in E(S,T)} c(e) - 0 = \sum_{e \in E(S,T)} c(e)$, which is exactly the capacity of cut (S,T). As we know $c(S,T) \geq |f|$ for any cut (S,T) and any flow f, such an equation of $|f^*| = c(S,T)$ implies that (S,T) is a minimum-cut.

Problem 3 (20 points). Give a counter-example or prove this statement: Let f^* be one maximum flow of network $G = (V,E)$ with source $s \in V$ and sink $t \in V$. Let $S := \{v \in V \mid s \text{ can reach } v \text{ in } G(f^*)\}$. Let $T := \{v \in V \mid v \text{ can reach } t \text{ in } G(f^*)\}$. Then G has a *unique* minimum s-t cut if and only if $S \cup T = V$.

Solution. This claim is true. We first prove that if $S \cup T \neq V$, then the minimum s-t cut if not unique. As we have proved in Problem 2, both $(S, V \setminus S)$ and $(V \setminus T, T)$ are minimum s-t cut. The fact that $S \cup T \neq V$ implies that $S \neq V \setminus T$. Therefore these are two distinct minimum s-t cuts.

We now prove that if $S \cup T = V$, then G has unique minimum s-t cut. Suppose conversely that G have two distinct minimum s-t cuts, $(S_1, T_1 = V \setminus S_1)$, and $(S_2, T_2 = V \setminus S_2)$, and $S_1 \neq S_2$. Consider the residual graph $G(f^*)$ w.r.t. the maximum-flow f^*. We now show that $S \subset S_1$ and $T \subset T_1$. Define $E(S_1,T_1) = \{(u,v) \in E \mid u \in S_1, v \in T_1\}$, and $E(T_1,S_1) = \{(u,v) \in E \mid u \in T_1, v \in S_1\}$. Since (S_1,T_1) is a minimum s-t cut, we know that $c(S_1,T_1) = |f^*|$. Combining the facts that $c(S_1,T_1) = \sum_{e \in E(S_1,T_1)} c(e)$, $|f^*| = \sum_{e \in E(S_1,T_1)} f^*(e) - \sum_{e \in E(T_1,S_1)} f^*(e)$, and $0 \leq f^*(e) \leq c(e)$, we can conclude that $f^*(e) = c(e)$ for any edge $e \in E(S_1,T_1)$, $f^*(e) = 0$ for any edge $e \in E(T_1,S_1)$. Therefore, in the residual graph $G(f^*)$, s cannot reach any vertex in T_1, and no vertex in S_1 can reach t. Consequently, we have $S \subset S_1$ and $T \subset T_1$. Similarly, we can also prove that $S \subset S_2$ and $T \subset T_2$. Hence, we have $S \subset S_1 \cap S_2$ and $T \subset T_1 \cap T_2$. Since $S_1 \neq S_2$, we must have $S \cup T \neq V$.

Problem 4 (20 points). Find a maximum matching and minimum vertex cover of the following bipartite graph $B = (X \cup Y, E)$ by reducing to the maximum flow problem:



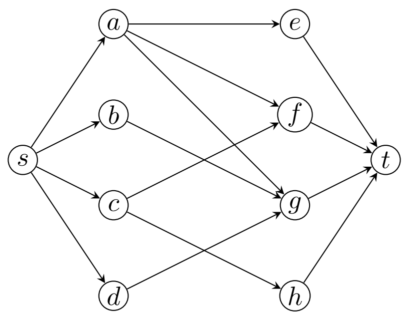
1. Draw the corresponding network $G = (V, E')$ together with capacity $c(e)$ for any $e \in E'$.

2. Give a maximum integral flow f^* of G.

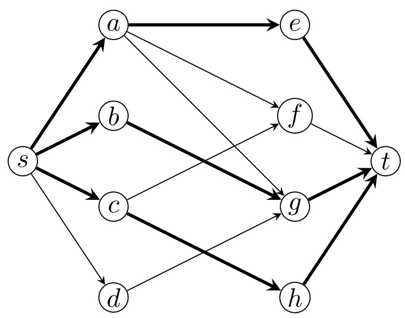
3. Give the maximum matching of B constructed from f^*, i.e., $\{e \in E \mid f^*(e) = 1\}$.
4. Draw the residual graph $G(f^*)$.
5. Give the s-t cut (S,T) of G, where $S := \{v \in V \mid s \text{ can reach } v \text{ in } G(f^*)\}$, and $T := V \setminus S$.
6. Give the minimum vertex cover of B constructed from $G(f^*)$, i.e., $(X \cap T) \cup (Y \cap S)$.

Solution.

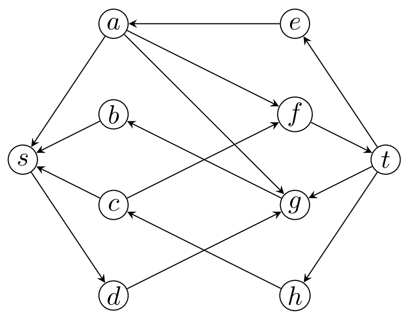
1. Network $G = (V, E')$ is given below, with capacity $c(e) = 1$ for any $e \in E'$:



2. A maximum flow f^* is represented below, with thick edges representing edges with $f^*(\cdot) = 1$ and thin edges representing edges with $f^*(\cdot) = 0$:



3. The maximum-matching for B is $\{(a,e), (b,g), (c,h)\}$.
4. The residual graph $G(f^*)$ is given below, with capacity $c(e) = 1$ for all edges:



5. $S = \{s, b, d, g\}$, $T = V \setminus S = \{a, c, e, f, h, t\}$.

6. $X \cap T = \{a, c\}$, and $Y \cap S = \{g\}$. The minimum vertex cover for B is $\{a, c, g\}$.

Problem 5 (20 points). Let $G = (V, E)$ be an directed graph with $s, t \in V$. Design an algorithm (by reducing to the max-flow problem) to find the maximum number of mutually edge-disjoint s-t paths in G. We define a set of s-t paths are mutually edge-disjoint if any two of them do not share any edge (they may share vertices).

Solution. We construct a max-flow instance from the given directed graph $G = (V, E)$. We first remove all in-edges of s and out-edges of t, as they will not appear in any s-t paths; denote the new graph as $G' = (V, E')$. The network will be exactly G' and the source and sink vertices are the given s and t respectively. We set $c(e) = 1$, for any edge $e \in E'$.

We then run any max-flow algorithm to find an integral maximum flow f^* of G'. Then we can construct $|f^*|$ mutually edge-disjoint s-t paths in G by using the edges in G' with $f^*(e) = 1$.

This algorithm is optimal. In fact, G contains k mutually edge-disjoint s-t paths if and only if the value of maximum-flow in G' equals k. Above we proved one side, i.e., if the value of maximum-flow of G' equals k, then we can construct k mutually edge-disjoint s-t paths in G. On the other side, if there are k mutually edge-disjoint s-t in G, then we can construct a flow f with $|f| = k$ by setting $f(e) = 1$ for each edge in these k paths.