Problem 1 (10 points). Let $S[1 \cdots n]$ be an *sorted* array with n *distinct* integers in ascending order. We select an integer k uniformly at random from $\{1, 2, \cdots, n\}$.

1. What is the probability that $S[k]$ is no less than the i-th smallest number in S *and* no larger than the j-th smallest number in S (assume that $i < j$)?

2. What is the probability that $S[k]$ is no larger than the i-th smallest number in S *or* no less than the j-th smallest number in S (assume that $i < j$)?

Problem 2 (10 points). Let $S[1 \cdots n]$ be an array with n *distinct* integers. We select an integer k uniformly at random from $\{1, 2, \cdots, n\}$.

1. What is the probability that $S[k]$ is no less than the i-th smallest number in S *and* no larger than the j-th smallest number in S (assume that $i < j$)?

2. What is the probability that $S[k]$ is no larger than the i-th smallest number in S *or* no less than the j-th smallest number in S (assume that $i < j$)?

Problem 3 (20 points). You are given two lists A and B, each of which is sorted in ascending order. It is guaranteed that all numbers in A and B are distinct. Given an integer k with $1 \leq k \leq |A| + |B|$, design an $O(\log |A| + \log |B|)$ time algorithm for computing the k-th smallest element in the union of A and B.

Problem 4 (20 points). Let $S[1 \cdots n]$ be an array with n *distinct* integers. Given an integer k with $1 \leq k \leq n$, design an algorithm to partition S into S_L (integers in S that are smaller than $S[k]$), $S[k]$, and S_R (integers in S that are larger than $S[k]$) using at most constant amount of extra memory.

Problem 5 (20 points). Let $S[1 \cdots n]$ be an array with n *distinct* integers. We say two indices (i, j) form an inversion if we have $i < j$ and $S[i] > S[j]$. Design an divide-and-conquer algorithm that counts the number of inversions in S. Your algorithm should run in $O(n \cdot \log n)$ time. For example, if you are given $S = (3, 8, 5, 2, 9)$, then your algorithm should return 4. The 4 inversions are $(3, 2), (8, 5), (8, 2), (5, 2)$.