Qualifying Exam—Theory and Algorithms, Fall 2019

Name (CAPITAL LETTERS): ____________________
Student email: ________________________________

P1	/25
P2	/25
P3	/25
P4	/25
P5	/25
P6	/25
Total	/150

Instructions: *Justify your answers.* Answer in the space allotted and use the back of its page if needed (in this case, clearly mark "continue on back").

1. **Logic, Quantifiers, Proofs**

   (a) Let $P(x, y)$ be the proposition, for integers x and y, that "$x + y = x - y$". Which of the following statements are true? Explain each answer in one sentence.

   i. $\forall x\ \exists y\ P(x, y)$

   ii. $\exists y\ \forall x\ P(x, y)$

   iii. $\forall y\ \exists x\ P(x, y)$

   (b) Write the negation of the proposition in 1(a)iii above (that is, $\neg\ \forall y\ \exists x\ P(x, y)$) in terms of the proposition $Q(x, y)$, which states that $x + y \neq x - y$.

   (c) Prove that if a and b are rational, then ab is rational.

   (d) Prove that if x is irrational, $x^{1/3}$ is irrational.

   (e) State the converse of 1d. State whether the converse is true or false and prove your assertion.

2. **Induction**

Prove by induction that $n^3 - 7n + 3$ is divisible by 3, for each integer $n \geq 0$.

Proof by induction on n:

Base Case:

Induction Hypothesis:

Induction Step:

3. **Dynamic Programming**

You are given the task of painting all n houses indexed $0, \ldots, n-1$ in your block. You can paint each house with one of the colors $c_1, c_2, \ldots, c_k$, but two neighboring houses cannot be painted with the same color. (The neighbors of house i are houses $i-1$ and $i+1$; the first and last houses have only one neighbor.) You know that the cost of coloring house i with color c_j is $p_{i,j}$. Your task is to find the minimum total cost required to paint all the houses.

4. **Modular Arithmetic**

(a) Compute 5^{17} mod 7.

(b) Solve for x and y; show all your work.

$$\begin{aligned} 2x + 3y &\equiv 2 \bmod 13 \\ x + 5y &\equiv 3 \bmod 13 \end{aligned}$$

5. **Greedy Algorithm**

Given n positive integers $a_1, a_2, \cdots, a_n$, design an polynomial time algorithm to determine whether there exists an undirected simple graph (self loops and multi-edges are not allowed) with n nodes such that the degrees of the n nodes are exactly the given n integers. Prove the correctness of your algorithm and analyze the running time of your algorithm.

6. **Shortest Path**

Given a directed graph $G = (V, E)$ with edge weight $w(e) \geq 0$ for any $e \in E$, two vertices $s \neq t \in V$, design an algorithm for each of the following problems. Your algorithm should run in $O((|V| + |E|) \cdot \log |V|)$ time. Prove the correctness of your algorithm and explain they run in $O((|V| + |E|) \cdot \log |V|)$ time.

(a) Find a path p from s to t such that $\min_{e \in p} w(e)$ is maximized.

(b) Find a path p from s to t such that $\max_{e \in p} w(e)$ is minimized.

Scratch paper. DO NOT put your answer here

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