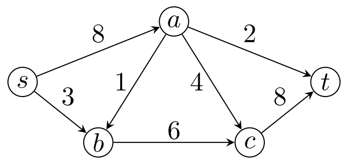
Problem 1 (20 points). For the following network $G = (V,E)$ with source s and sink t:

1. Give one maximum flow f^* of G (i.e., give $f^*(e)$ for every $e \in E$).
2. Draw the residual graph $G(f^*)$.
3. Give the s-t cut (S,T), where $S := \{v \in V \mid s \text{ can reach } v \text{ in } G(f^*)\}$, and $T := V \setminus S$, and give the capacity of this cut.
4. Give the s-t cut (S',T'), where $T' := \{v \in V \mid v \text{ can reach } t \text{ in } G(f^*)\}$, and $S' := V \setminus T'$, and give the capacity of this cut.

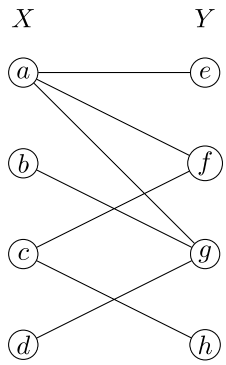


Problem 2 (20 points). Let f^* be one maximum flow of network $G = (V,E)$ with source $s \in V$ and sink $t \in V$. Let $T := \{v \in V \mid v \text{ can reach } t \text{ in } G(f^*)\}$. Let $S := V \setminus T$.

1. Prove that $s \in S$.
2. Prove that (S,T) is a minimum s-t cut of G.

Problem 3 (20 points). Give a counter-example or prove this statement: Let f^* be one maximum flow of network $G = (V,E)$ with source $s \in V$ and sink $t \in V$. Let $S := \{v \in V \mid s \text{ can reach } v \text{ in } G(f^*)\}$. Let $T := \{v \in V \mid v \text{ can reach } t \text{ in } G(f^*)\}$. Then G has a *unique* minimum s-t cut if and only if $S \cup T = V$.

Problem 4 (20 points). Find a maximum matching and minimum vertex cover of the following bipartite graph $B = (X \cup Y, E)$ by reducing to the maximum flow problem:



1. Draw the corresponding network $G = (V,E')$ together with capacity $c(e)$ for any $e \in E'$.
2. Give a maximum integral flow f^* of G.
3. Give the maximum matching of B constructed from f^*, i.e., $\{e \in E \mid f^*(e) = 1\}$.
4. Draw the residual graph $G(f^*)$.
5. Give the s-t cut (S,T) of G, where $S := \{v \in V \mid s \text{ can reach } v \text{ in } G(f^*)\}$, and $T := V \setminus S$.
6. Give the minimum vertex cover of B constructed from $G(f^*)$, i.e., $(X \cap T) \cup (Y \cap S)$.

Problem 5 (20 points). Let $G = (V, E)$ be an directed graph with $s, t \in V$. Design an algorithm (by inducing to the max-flow problem) to find the maximum number of mutually edge-disjoint *s*-*t* paths in G. We define a set of *s*-*t* paths are mutually edge-disjoint if any two of them do not share any edge (they may share vertices).