Fall 2018, CMPSC 465: Exam 0 (practice).
Closed book and closed notes, no 'cheat sheet', no calculators allowed.
Please don't use cell phones during the exam.
Answer questions in the space provided.
The exam is for 40 points (4 problems, 10 points each).

NAME:

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1. Solve the recurrence $T(n) = 2T(n/2) + n^2$ using the recursion tree method. Express your answer in the Θ-notation. In the recursion tree method, you
   - draw a recursion tree,
   - determine its height,
   - estimate the work associated with nodes at each level, and
   - simplify the summation to come up with a closed-form expression for the running time.

   Assume that $T(1) = \Theta(1)$ and that n is a power of 2.
   Solution:

Recursion tree :

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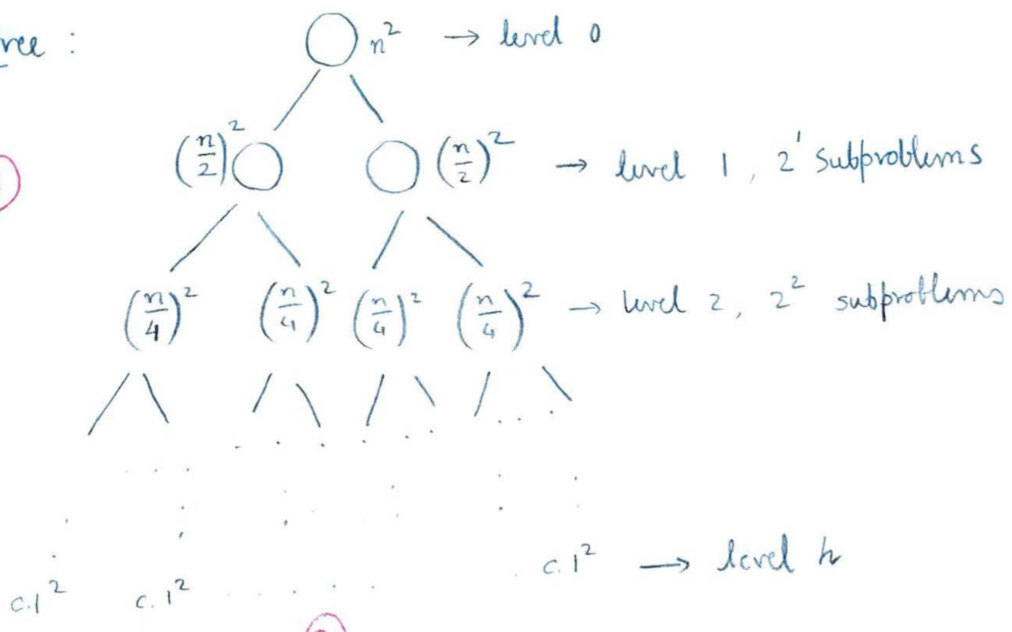

We assume $T(1) = c$ for $c > 0$.

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$$\frac{n}{2^h} = 1 \Rightarrow \boxed{h = \log n}.$$

number of subproblems

2

$$T(n) = \sum_{i=0}^{h} 2^i \cdot \left(\frac{n}{2^i}\right)^2$$

work associated with each subproblem

$$= n^2 \sum_{i=0}^{h} 2^{-i} = \frac{n^2\left(1 - \left(\frac{1}{2}\right)^{h+1}\right)}{1 - \frac{1}{2}} = 2n^2\left(1 - \frac{1}{2}\left(\frac{1}{2}\right)^{\log_2 n}\right)$$

$$= 2n^2\left(1 - \frac{1}{2} n^{\log_2 1/2}\right)$$

$$= 2n^2 - n$$

$$= \boxed{\Theta(n^2)}$$

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2. Solve the recurrence $T(n) = T(n^{\frac{1}{3}}) + \Theta(1)$ using the iterative substitution method.
Solution:

$\Theta(1) \le c_1$ for some $c_1 > 0$ and $\Theta(1) \geq c_2$ for some $c_2 > 0$.

Let us assume $\Theta(1) = c$ for some $c > 0$.

$$T(n) = T(n^{1/3}) + c$$
$$= \left(T\left((n^{1/3})^{1/3}\right) + c\right) + c$$
$$= T\left(n^{\frac{1}{3^2}}\right) + 2c$$
$$= \left(T\left((n^{1/3^2})^{1/3}\right) + c\right) + 2c$$
$$= T\left(n^{\frac{1}{3^3}}\right) + 3c$$
$$\vdots$$
$$= T\left(n^{\frac{1}{3^k}}\right) + kc \quad \text{(general pattern)}$$

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Assume $n^{\frac{1}{3^k}} = a$, where $a > 1$. Then $\frac{1}{3^k} \log n = \log a \Rightarrow 3^k = \dfrac{\log n}{\log a}$

$\Rightarrow k = \log_3 \log_a n$

$\therefore T(n) = T(a) + \log_3 \log_a n \cdot c$

If we assume $T(a) = k = \Theta(1)$,

$T(n) = \Theta(\log \log n)$. 4

3. Using the Big-Omega definition, show that $n^3 \log n = \Omega(n^3)$.
Solution:

$n^3 \log n = \Omega(n^3)$

$\Rightarrow$ there is a $c > 0$ and an $n_0 \geq 1$ such that $n^3 \log n \geq c n^3$ for all $n \geq n_0$. We need to find such a c and n_0.

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$n^3 \log n \geq c n^3 \Rightarrow \log n \geq c$ or $c \le \log n$

Consider $c = 2$ and $n_0 = 4$.

For $n = 4$, $2 \le \log_2 4 = 2$.

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For $n > 4$, $\log n > 2$.

Hence, $n^3 \log n = \Omega(n^3)$

4. Prove using induction that the sum of squares of the first n positive integers is $\frac{n(n+1)(2n+1)}{6}$.

Solution:

Base case: $n = 1$. $\frac{1(2)(3)}{6} = 1^2$. 2

Inductive step. Assume that the result holds for $n' = n-1$.

That means $\sum_{k=1}^{n-1} k^2 = \frac{(n-1)n(2n-1)}{6}$. 2

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2 = \frac{(n-1)n(2n-1)}{6} + n^2$$

$$= \frac{(n^2-n)(2n-1) + 6n^2}{6}$$

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$$= \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2+3n+1)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6}.$$