Problem 1 (20 points).

1. Prove by induction on $n \geq 1$ that for all $m \in \mathbb{Z}^+$, $\lceil \frac{n}{m} \rceil = \lfloor \frac{n+m-1}{m} \rfloor$.
2. For a positive integer N, we say that a is a quadratic residue modulo N if there exists x such that $a \equiv x^2 \mod N$. Let N be an odd prime and a be a non-zero quadratic residue modulo N. Show that there are exactly two values in $\{0, 1, \ldots, N\}$ satisfying $x^2 \equiv a \mod N$.

Problem 2 (20 points).

1. There are n balls and n bins. Each ball is placed into a bin uniformly at random. Prove that after all n balls are placed the expected number of empty bins is $\Theta(n)$.
2. Order the following functions in non-decreasing order by their asymptotic growth rate. Use $<$ and $=$, representing "small-o" and "Big-Theta", in your order (for example: $f_1 < f_2 = f_3 < f_4$).
   (a) $f_1 = \log(n!)$
   (b) $f_2 = (\log n)^{\log\log n}$
   (c) $f_3 = (\sum_{k=1}^{n} 1/k^2)^{\log n}$
   (d) f_4 is the solution of recursion $f_4(n) = f_4(n/2) + \log n$ with base $f_4(1) = 1$.

Problem 3 (15 points).
You are a licensed psychologist offering free Zoom therapy sessions during this pandemic. Every Sunday, you receive a set of meeting requests for Monday to Friday, each with a start and an end time. The duration of each request can vary from 15 minutes to several hours. Your objective is to maximize the number of therapy sessions you offer in a week, with the constraint that you can only participate in one Zoom session at any time. Design a greedy algorithm for this problem and show that it is correct. What is the running time of the algorithm (in terms of n, the number of requests received)?

Problem 4 (15 points).
You are given a directed graph $G = (V, E)$ and a subset of vertices $V_1 \subset V$. Design an algorithm to decide if for *every* vertex $v \in V \setminus V_1$ there exists a vertex $u \in V_1$ such that u can reach v in G and v can reach u in G. Your algorithm should run in $O(|V| + |E|)$ time. Explain the correctness and running time of your algorithm.

Problem 5 (15 points).
A country is running presidential elections via mail-in voting. Many candidates are running, and each is assigned a unique symbol. A registered voter can vote by drawing the symbol on a piece of paper and mailing the ballot to a central location. The country will have a president only if a candidate receives more than half of the total ballots received. Assume that n valid ballots are received and no voter votes more than once. Your objective as the chief election officers is to determine if there is a winner in this election. One constraint is that you cannot order the symbols assigned to candidates or the drawings, and so you cannot use a comparison-based sorting algorithm. However, you have the ability to inspect two ballots and answer in constant time if they are the same symbol. Can you design a linear-time divide-and-conquer algorithm for this problem?

Problem 6 (15 points).
You are given a binary square matrix M of size $n \times n$. We define a (p, q, l)-triangle of M, where $p \geq 1$, $q \geq 1$, $l \geq 1$, $p + l \leq n + 1$, and $q + l \leq n + 1$, as the set of elements $M[i, j]$ satisfying $p \leq i < p + l$, $q \leq j < q + l$, and $i + j < p + q + l$. Design an algorithm to find the largest (p, q, l)-triangle of M ("largest" means l is maximized) such that all its elements are "1". Your algorithm should run in $O(n^2)$ time. Explain the correctness and running time of your algorithm.