Problem 1 (15 points).

You are given an array of n elements, and you notice that some of the elements are duplicates; that is, they appear more than once in the array. Show how to remove all duplicates from the array in time $O(n \cdot \log n)$.

Problem 2 (15 points).

Assume you have functions f and g such that $f(n)$ is $O(g(n))$. For each of the following statements, decide whether you think it is true or false and give a proof or counterexample.

1. $\log_2 f(n) = O(\log_2 g(n))$.
2. $2^{f(n)} = O(2^{g(n)})$.
3. $f(n)^2 = O(g(n)^2)$.

Problem 3 (20 points).

For each pairs of functions below, indicate one of the three: $f = O(g),\ f = \Omega(g),\ f = \Theta(g)$.

1. $f(n) = 100 \cdot n;\ g(n) = 0.01 \cdot n$
2. $f(n) = n^{1.01};\ g(n) = n^{0.99}$
3. $f(n) = n;\ g(n) = n^{0.99} \cdot (\log n)^2$
4. $f(n) = 100 \cdot \log n;\ g(n) = n^{0.01}$
5. $f(n) = \log n^2;\ g(n) = \log n^3$
6. $f(n) = (\log n)^{\log n};\ g(n) = n$
7. $f(n) = 2^{\log n};\ g(n) = n^2$
8. $f(n) = 2^{n \cdot \log n};\ g(n) = 3^n$
9. $f(n) = n^n;\ g(n) = n!$
10. $f(n) = \log(n^n);\ g(n) = \log(n!)$

Problem 4 (20 points).

Solve each of the following recursions using master theorem.

1. $T(n) = 8 \cdot T(n/2) + 1000 \cdot n^2$
2. $T(n) = 2 \cdot T(n/2) + 10 \cdot n$
3. $T(n) = 2 \cdot T(n/2) + n^{0.5}$
4. $T(n) = 2 \cdot T(n/2) + n^{1.5}$
5. $T(n) = 4 \cdot T(n/2) + n \cdot \log(n)$

Problem 5 (15 points).

The Fibonacci sequence is a set of numbers that starts with a one, followed by a one, and proceeds based on the rule that each number is equal to the sum of the preceding two numbers.

$$F(n) = F(n-1) + F(n-2) \tag{1}$$

Can you design a divide-and-conquer algorithm to find $F(n)$ such that each sub-problem is around half the size? (You only need to give the recursive relationship and a brief explanation.)

Problem 6 (15 points).

Prove that, at iteration k in the Graham-Scan algorithm, when we find $p_b \to p_a \to p_k$ is turning "left", where p_a and p_b are the top and 2nd top elements in stack S, respectively, then p_k together with all points in S form the convex hull of $\{p_1, p_2, \cdots, p_k\}$.