Theory and Algorithms Qualifying Exam for Fall 2022

November 4th, 2022: 4pm - 7pm

NAME:

PSU EMAIL ID: @psu.edu

This exam is closed note. No additional devices or aids are allowed. Scratch paper is included at the back of the exam; scratch space will not be graded. For ease of gradescope scanning, please answer all questions in the space provided. Please clearly write your name and your PSU access ID (i.e., xyz1234) in the box on top of every page.

Choose 5 of the below problems to solve; indicate which problems are being attempted here - only 5 will be graded; all are worth 20 points.

	1	2	3	4	5	6
Indicate Problems Attempted						

1 Induction

Prove by induction that $n^3 - 7n + 3$ is divisible by 3, for all integers $n \geq 0$.

Base Case:

Induction Hypothesis:

Induction Step:

2 Recurrence, Growth, and Counting

a) Suppose a staircase can be climbed by a combination of single steps (i.e. advancing one step at a time) and double steps (i.e. skipping over a step by striding up two steps at a time). Let c_n be the number of different ways, **where order matters**, to climb a staircase of $n \geq 1$ stairs.

i. Find *and explain* a recurrence relation for c_n, $n \geq 1$

ii. Specify the concrete, numerical values of c_n, for all $n \leq 5$

b) Count the total number of possible n-bit to M-bit Boolean functions $f : \{0,1\}^n \rightarrow \{0,1\}^M$

3 Divide and Conquer

Given two sorted lists of size m and n, we want to find the kth smallest element in the concatenation of the two lists. Provide an algorithm (pseudocode) whose running time is $O(\lg m + \lg n)$. Assume $k < m$ and $k < n$. Justify your algorithm's correctness and runtime.

4 Greedy Algorithms

Aerith has challenged her friend Ciri to what she believes is a card game of chance: Each player starts with a set of n cards, each of which maps to a numerical value(e.g. ace=1, Shivan Dragon = 500, etc), in some shuffled order. In each round, both players simultaneously reveal their top card. The card with the larger value wins that round and "captures" the smaller card, with ties resolving in favor of the challenger; scoring is based on the total value of all cards lost by each player, with the lowest score winning. However, Ciri finds the tie resolution rule unfair, and has decided to cheat by distracting Aerith, memorizing Aerith's card order, and reshuffling her own cards to optimize her score.

a) Treating Aerith and Ciri's cards as two arrays of positive integers, A and C, each of size n, that specify the values of each card in the deck, design an algorithm that computes the minimum value of Ciri's cards that are captured (note: for this general formulation, unlike **b** below, you may **NOT** assume that A and C are permutations of the same set).

b) Assuming that both Aerith and Ciri start with a standard 52-card card deck (4 suits of 13 cards each), with values ace=1, ..., king=13, what is Ciri's final score?

5 Graph Algorithms

Consider a set of colonized planets, P, and the set of charted routes through space among them, R, in the form of an undirected graph $G = (P, R)$. Each space route $r \in R$ connects exactly one colonized planet to one other, and you know its length in parsecs, l_r. You want to go from planet s to planet t, and your starship can only hold enough fuel mass to cover L parsecs at constant burn (and you're too impatient for any slower transits). There are refuelling stations around every inhabited planet, but none in the vast, uncaring emptiness of space between them. Therefore, you only consider a route viable if every one of its edges has length $l_r \leq L$.

a) Given the limitation on how fuel mass limits your range, provide an algorithm that can determine, in time linear in $O(|P| + |R|)$, whether there is a viable route from s to t. Justify the algorithm's correctness and running time.

Graph Algorithms, continued

b) Your starship is wearing out and you need to buy a new one, but inflation is making you cost-averse and you're hoping to save some money by buying a starship with the minimum fuel capacity to meet your travel needs. Provide an $O((|P| + |R|) \log |P|)$ algorithm to determine this minimum fuel capacity. Justify the algorithm's correctness and runtime.

6 Dynamic Programming

Consider the following 3-partition problem. Given integers $a_1, \ldots, a_n$, we want to determine whether it is possible to find a partition of $\{1, \ldots, n\}$ into three disjoint subsets $I, J, K \subseteq \{1, \ldots, n\}$ such that

$$\sum_{i \in I} a_i = \sum_{j \in J} a_j = \sum_{k \in K} a_k = \frac{1}{3} \sum_{i=1}^{n} a_i$$

For example, $(2, 2, 3, 4, 4, 5, 7)$ is a YES-instance, because there is the partition $(2, 7), (4, 5), (2, 3, 4)$; while $(2, 2, 3, 5)$ is a NO-instance.

a) Let $A = (1/3) \sum_i a_i$. Define a true/false matrix $M[\cdot, \cdot, \cdot]$ of size $A \times A \times (n + 1)$ with the meaning that $M[x, y, k]$ is true if and only if there are two disjoint subsets $I, J \subseteq \{1, \ldots, k\}$ such that $\sum_{i \in I} a_i = x$ and $\sum_{j \in J} a_j = y$. Which entry represents the answer to the 3-partition problem?

b) Write a recurrence relation to construct M.

Dynamic Programming, continued

c) Give a dynamic programming algorithm for 3-partition that runs in polynomial in n and in $\sum_i a_i$. Prove the correctness and the running time.

Scratch Space 1 - not graded