Selected Problems Chapter 3
Linear Algebra Done Wrong, Sergei Treil, 1st Edition

Mustaf Ahmed

December 9, 2021

Problem Uniqueness of Determinant. Let $C \in \mathbb{R}^n$ be a column vector, i.e. $C = (c_i)_{i=1,\dots,n}$.

Show that if $D : (\mathbb{R}^n)^n \to \mathbb{R}$ satisfies

multi-linearity. linearity in each argument

anti-symmetry. switching arguments induces a sign change

normalization. $D(e_1, \dots, e_n) = 1$

then

$$D(C_1, \dots, C_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} c_{\sigma(k)k}$$

Proof. Let $D : (\mathbb{R}^n)^n \to \mathbb{R}$ be a function satisfying the three conditions. For each index j, we have $C_j = \sum_i^n c_{ij} e_i$. Repeatedly applying the multi-linear property we have

$$\begin{aligned} D(C_1, \dots, C_n) &= D(\sum_{i_1}^{n} c_{i_1 1} e_{i_1}, \dots, \sum_{i_n}^{n} c_{i_n n} e_{i_n}) \\ &= \sum_{i_1}^{n} c_{i_1 1} D(e_{i_1}, \dots, \sum_{i_n} c_{i_n n} e_{i_n}) \\ &= \dots \\ &= \sum_{i_1}^{n} \cdots \sum_{i_n}^{n} \prod_{k=1}^{n} c_{i_k k} D(e_{i_1}, \dots, e_{i_n}) \end{aligned}$$

Simplifying the iterated sum, we have

$$= \sum_{i_1, \dots, i_n} \prod_{k=1}^{n} c_{i_k k} D(e_{i_1}, \dots, e_{i_n}).$$

By proposition 3.1, $D(e_{i_1}, \ldots, e_{i_n}) = 0$ whenever any two of its arguments are the same. Thus, all products in the sum contain a determinant that permutes the standard basis. By anti-symmetry and normalization, we must multiply by the sign of the permutation. We have

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} c_{\sigma(k)k}.$$

$\square$

Problem Determinant of diagonal matrix. Let A be the diagonal matrix $diag(a_{11}, \ldots, a_{nn})$. Show that $det(A) = \prod_{k=1}^{n} a_{kk}$.

Proof. The jth column of A is written as $A_j = a_j e_j$. We have

$$
\begin{aligned}
det(A) &= det(A_1, \ldots, A_n) \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} a_{\sigma(k)k}
\end{aligned}
$$

$\square$

Assume that $\sigma \in S_n$ and $\sigma(k) \neq k$ for some k. Then $\prod_{k=1}^{n} a_{\sigma(k)k} = 0$ because one of its products will be 0, since it is off the diagonal of A. Thus the only valid permutation is the identity, which has a sign of 1. We have

$$
= \prod_{k=1}^{n} a_{kk},
$$

as desired.

Problem Determinant of Row Multiplication Elementary Matrix. Let $M_i(c) : \mathbb{R}^n \to \mathbb{R}^n$ be a row multiplication elementary matrix for row i defined by

$$M_i(c) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}.$$

Show that $det(M_i(c)) = c$.

Proof.

$$\begin{aligned} det(M_i(c)) &= det(e_1, \ldots, c * e_i, \ldots, e_n) \\ &= c * det(e_1, \ldots, e_n) \\ &= c * 1 \\ &= c \end{aligned}$$

, as desired. $\square$

Problem Determinant of Row Swap Elementary Matrix. Let $S_{i,j} : \mathbb{R}^n \to \mathbb{R}^n$ be the row swap elementary matrix for rows (i,j). We define $S_{i,j}$ by its columns. For each k,

k $\neq$ i and k $\neq$ j. $C_k = e_k$.

k = i. $C_k = e_j$.

k = j. $C_k = e_i$.

Prove that $det(S_{i,j}) = -1$.

Proof. Since each column of $S_{i,j}$ is a distinct standard basis vector, each argument in $det(S_{i,j})$ is a distinct standard basis vector. The arguments of $det(S_{i,j})$ are a permutation of $(e_1, \ldots, e_n)$ where the ith standard basis vector is switched with the jth standard basis vector. By anti-symmetry of the determinant,

$$\begin{aligned} det(S_{i,j}) &= -det(e_1, \ldots, e_n) \\ &= -1. \end{aligned}$$

$\square$

Problem Determinant of Row Addition Elementary Matrix. Let $A_{i,j}(c) : \mathbb{R}^n \to \mathbb{R}^n$ be the row addition elementary matrix that adds c times row i to row j. We define $A_{i,j}(c)$ by an identity matrix with c in the (j, i) position.

Prove that $det(A_{i,j}(c)) = 1$.

Proof. Taking the determinant, we have

$$\begin{aligned} det(A_{i,j}(c)) &= det(e_1, \ldots, e_i + ce_j, \ldots, e_n) \\ &= c * 0 + det(e_1, \ldots, e_j) \\ &= 1 \end{aligned}$$

. The first term of the second equality holds because two arguments are e_j, which allows us to use proposition 3.1.

$\square$

Problem Invertible Matrices and RREF. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix. Prove that A is Invertible if and only if the RREF of A is I.

Proof. For the forward direction, assume that A is invertible and the RREF is R. We have $E_n \dots E_1 A = R$. We want R to be invertible. For a contradiction, assume that R is not invertible. Since R is the multiplication of invertible matrices, from the first equation, R is invertible. This is a contradiciton.

We've shown that R is invertible. We want that $R = I$. If R has a row with all zeros, R can't be invertible because $dim(rank(R)) < n$. Thus, R has no rows with all zeros. Since R is in RREF, $R = I$.

For the backward direction, assume that R is in RREF and $R = I$. We have $E_n \dots E_1 A = I$, so by multiplying by inverses, $A = E_1^{-1} \dots E_n^{-1}$. We've shown that A is the product of invertible matrices, so A must be invertible. $\square$

Problem Invertible Matrices and Elementary Matrices Representation. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix. Prove that if A is invertible then A can be written as a product of elementary matrices.

Proof. By the previous theorem, we can write $E_n \dots E_1 A = I$. By applying the inverse of each elementary matrix on the right side, we have $A = E_1^{-1} \dots E_n^{-1}$, as desired. $\square$

Problem Lemma 3.6. For a square matrix A and an elementary matrix E, (of the same size) $det(AE) = det(A)det(E)$.

Proof. **Case 1.** Assume that E is elementary matrix that scale row i by c. By a previous result, $det(E) = c$. We want to show $det(AE) = cdet(A)$. The result AE is A with the ith column scaled. The result holds by multi-linearity.

Case 2. Assume that E is elementary matrix that swaps row i with row j. We want to show $det(AE) = -det(A)$. The result AE is A with columns i and j swapped. By anti-symmetry, $det(AE) = -det(A)$, as desired.

Case 3. Assume that E is elementary matrix that adds c times row i to row j. We want to show $det(AE) = det(A)$. The result AE is A with column i being the sum of column i and c times column j. By multi-linearity and proposition 3.1, $det(AE) = det(A)$, as desired. $\square$

Problem Corollary 3.7. For any matrix A and any sequence of elementary matrcies $E_1, \ldots, E_n$ (all matrices are n by n).

$$det(AE_1 \ldots E_n) = det(A)det(E_1) \ldots det(E_n)$$

. Using induction with lemma 3.6, it is straightforward to show this result.

Problem Theorem 3.5 (Determinant of Product). For n by n matrices A and B,

$$det(AB) = det(A)det(B).$$

Proof. **Case 1.** Assume that B is invertible. We can write $B = E_1, \ldots, E_n$, a product of elementary matrices. By corollary 3.7, we have

$$\begin{aligned} det(AB) &= det(AE_1, \ldots, E_n) \\ &= det(A)(det(E_1) \ldots det(E_n)) \\ &= det(A)det(B). \end{aligned}$$

Case 2. Assume that B is not invertible. By proposition 3.3, $det(B) = 0$. We want that $det(AB) = 0$. Since B is not invertible, AB is not invertible. By proposition 3.3, $det(AB) = 0$.

□