Selected Problems Chapter 5
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem Example 5.8. Suppose $T \in \mathcal{L}(F^2)$ is defined by $T(w, z) = (-z, w)$. Find the eigenvectors and eigenvalues of T if $F = \mathbb{R}$. Find the eigenvectors and eigenvalues of T if $F = \mathbb{C}$

Proof. **Part(a).**

Assume T has eigenvectors and eigenvalues with $F = \mathbb{R}$. The equation $\lambda(w, z) = (-z, w)$ holds and leads to the following system of equations:

$$\begin{aligned} \lambda w &= -z \\ \lambda z &= w. \end{aligned}$$

Solving for λ, we have $\lambda^2 = -1$, which only has solutions in $\mathbb{C}$. This contradiction means T has no eigenvectors and eigenvalues.

Part(b).

In part(a), we showed that the eigenvalues of T must be in the complex numbers. The equation from part(a) $\lambda^2 = -1$ has the solutions $\lambda = i$ and $\lambda = -i$. The eigenvectors corresponding to $\lambda = i$ are of the form $(w, -iw)$ for any $w \in \mathbb{C}$; the eigenvectors corresponding to $\lambda = -i$ are of the form (w, iw).

□

Problem Theorem 5.10 Linearly Independent Eigenvectors. Let $T \in \mathcal{L}(V)$. Suppose $(\lambda_1, \dots, \lambda_n)$ are distinct eigenvalues of T, and $(v_1, \dots, v_n)$ are corresponding eigenvectors. Then $(v_1, \dots, v_n)$ is a linearly independent list.

Proof. For a contradiction, suppose $(v_1, \dots, v_n)$ is a linearly dependent list. Then choose $a_1, \dots, a_n \in F$ where not all are zero such that $0 = a_1 v_1 + \dots a_n v_n$. By the linear dependence lemma, choose the smallest j such that $v_j = \frac{a_1}{a_j} v_1 + \cdots + \frac{a_{j-1}}{a_j} v_{j-1}$. Applying T, we have

$$\lambda_j v_j = \frac{a_1 \lambda_1}{a_j} v_1 + \cdots + \frac{a_{j-1}\lambda_{j-1}}{a_j} v_{j-1}.$$

Subtracting the left side, we have

$$0 = \frac{a_1(\lambda_1 - \lambda_j)}{a_j} v_1 + \cdots + \frac{a_{j-1}(\lambda_{j-1} - \lambda_j)}{a_j} v_{j-1}.$$

Each $(\lambda_k - \lambda_j) \neq 0$ because the eigenvalues are distinct. Since we chose j to be the smallest such that the v_j is in the span of the preceding vectors, $a_k = 0$ for $k = 1, \dots, j-1$. Thus, $v_j = 0$, but that is a contradiction because v_j is an eigenvector.

$\square$

Problem Theorem 5.13 Number of Eigenvalues. Suppose V is finite-dimensional. Then each operator on V has at most $dim(V)$ distinct eigenvalues.

Proof. Let $T \in \mathcal{L}(V)$. Let $(\lambda_1, \ldots \lambda_n)$ be a list of distinct eigenvalues in F, and let $(u_1, \ldots, u_m)$ be a corresponding list of eigenvectors. By Theorem 5.10, $(v_1, \ldots, v_n)$ is a linearly independent list. Choose a basis $(v_1, \ldots, v_n)$ of V. By Theorem 2.23, $m \leq n = dim(V)$. $\square$

Problem 5.A.12. Define $T \in \mathcal{L}(P_4(\mathbb{R}))$ by

$$T(p(x)) = xp^{'}(x)$$

for all $x \in \mathbb{R}$. Find all the eigenvalues and eigenvectors.

Assume T has an eigenvalue λ. Choose a non-zero $p(x) = a_0 + a_1x + \cdots + a_4x^4 \in P_4(\mathbb{R})$ such that $T(p(x)) = \lambda p(x)$. This is equivalent to the system of equations

$$\begin{aligned}
\lambda a_0 &= 0 \\
\lambda a_1 &= a_1 \\
\lambda a_2 &= 2a_2 \\
\lambda a_3 &= 3a_3 \\
\lambda a_4 &= 4a_4
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
\lambda a_0 &= 0 \\
a_1(\lambda - 1) &= 0 \\
a_2(\lambda - 2) &= 0 \\
a_3(\lambda - 3) &= 0 \\
a_4(\lambda - 4) &= 0 \\
&.
\end{aligned}$$

Let $\lambda \in \{0, 1, 2, 3, 4\}$. Then for each $j \neq \lambda$

$$\begin{aligned}
a_j(\lambda - j) &= 0 \\
a_j &= 0
\end{aligned}$$

by diving by $(\lambda - j)$. The corresponding eigenvector for λ is then $p(x) = a_\lambda x^\lambda$. Let $\lambda \notin \{0, 1, 2, 3, 4\}$. By diving the coefficient $(\lambda - j)$ for $j = 0, 1, 2, 3, 4$, we have $a_j = 0$ for each j. Thus, $p(x) = 0$, which is not an eigenvector.

We've shown that the eigenvalues are $\lambda = 0, 1, 2, 3, 4$ with the corresponding eigenvectors $c, cx, cx^2 cx^3, cx^4$ for $c \in \mathbb{R}$.

Problem 5.A.15. Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

Proof. **Part(a)**

We must show that λ is an eigenvalue of T if and only if λ is an eigenvalue of $S^{-1}TS$. For the forward direction, assume λ is an eigenvalue of T. Choose $v \in V$ such that $Tv = \lambda v$. Since S is invertible, S is also injective and surjective. We can choose $u \in V$ such that $Su = v$. We have

$$\begin{aligned}(S^{-1}TS)u &= (S^{-1}T)(Su) \\ &= (S^{-1}T)(v) \\ &= (S^{-1})(\lambda v) \\ &= \lambda u.\end{aligned}$$

For the backward direction, assume λ is an eigenvalue of $S^{-1}TS$. Choose $u \in V$ such that $(S^{-1}TS)u = \lambda u$. Applying S to $(S^{-1}TS)u$, we have

$$\begin{aligned}(TS)u &= T(Su) \\ &= \lambda(Su)\end{aligned}$$
,

as desired.

Part(b)

If v is an eigenvector of T, then $S^{-1}v$ is an eigenvector of $S^{-1}TS$. if v is an eigenvector of $S^{-1}TS$, Sv is an eigenvector of T.

□

Problem 5.A.18. Show that the operator $T \in \mathcal{L}(C^\infty)$ defined by

$$T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$$

has no eigenvalues.

Proof. Assume that $T(z_1, z_2, \dots) = \lambda(z_1, z_2, \dots)$. This is equivalent to the following system of equations

$$\begin{aligned} \lambda z_1 &= 0 \\ \lambda z_2 &= z_1 \\ \lambda z_3 &= z_2 \\ &\dots \end{aligned}$$

The first equation implies $\lambda = 0$ or $z_1 = 0$. If $\lambda = 0$, then $(z_1, z_2, \dots)$ is the zero vector.If $z_1 = 0$ each $z_j = 0$,and thus $(z_1, z_2, \dots)$ is the zero vector. We have that $(z_1, z_2, \dots)$ is the zero vector, which can't be associated with any eigenvalue by definition. $\square$

Problem 5.A.20. Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(F^\infty)$ defined by

$$T(z_1, z_2, \dots) = (z_2, z_3, \dots)$$

Assume $T(z_1, z_2, \dots) = \lambda(z_1, z_2, \dots)$ with $(z_1, z_2, \dots)$ being non-zero. The previous relation is equivalent to the system of equations

$$\begin{aligned} \lambda z_1 &= z_2 \\ \lambda z_2 &= z_3 \\ \lambda z_3 &= z_4 \\ &\dots \end{aligned}$$

By substitution of variables, another equivalent form is

$$\begin{aligned} \lambda z_1 &= z_2 \\ \lambda^2 z_1 &= z_3 \\ \lambda^3 z_3 &= z_4 \\ &\dots \end{aligned}$$

Thus, all eigenvectors are of the form $(z_1, z_2, z_3, \dots) = (z_1, \lambda z_1, \lambda^2 z_1, \dots)$ with $z_1 \neq 0$. Each $\lambda \in F$ is an eigen vector.

Problem 5.A.22. Suppose $T \in \mathcal{L}(V)$ and there exists nonzero vectors v and w in V such that $T(v) = 3w$ and $T(w) = 3v$. Prove that 3 and -3 are eigenvalues of T.

Proof. We have $T(v + w) = 3(v + w)$ by linearity. Similarly, $= T(v - w) = 3(w - v) = -3(v - w)$.

We must show that $(v+w)$ and $(v-w)$ are not both the zero vector. For a contradiction suppose $(v + w) = (v - w) = 0$. Adding the two vectors, it is clear that $v = 0$.

$\square$

Problem 5.A.30. Suppose $T \in \mathcal{L}(\mathbb{R}^3)$, and $-4, 5$ and $\sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

Proof. We have $Tx - 9x = (T - 9I)x$. Thus, it is enough to show that $(T - 9I)$ is surjective. We can show that $(T - 9I)$ is surjective by showing that it is injective by Theorem 3.69.

For a contradiction, assume that $(T - 9I)$ is not injective. By Theorem 3.16, choose a non-zero vector x in $\mathbb{R}^3$ such that $(T - 9I)x = 0$. Simplifying the right side , we have

$$Tx - 9x = 0,$$

so

$$Tx = 9x.$$

Thus, 9 is an eigenvalue, so we have 4 eigenvalues in total. By Theorem 5.13, there can be at most 3 eigenvalues, which is a contradiction.

□

Problem 5.A.32. Suppose $(\lambda_1, \dots, \lambda_n)$ is a list of distinct real numbers. Prove that the list $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$.

Proof. Let the differentiation operator for real-valued function on $\mathbb{R}$ be $T(f(x)) = f'(x)$. By Theorem 5.10, it suffices to show that $(\lambda_1, \dots, \lambda_n)$ are eigenvalues with corresponding eigenvectors $(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$.

Applying the differentiation operator for any j, we have

$$T(e^{\lambda_j x}) = \lambda_j e^{\lambda_j x},$$

as desired. $\square$

Problem 5.B.2. Suppose $T \in \mathcal{L}(V)$ and $(T-2I)(T-3I)(T-4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Proof. Choose a non-zero vector $v \in V$ such that $Tv = \lambda v$. Applying $(T-2I)(T-3I)(T-4I)$ to v, we have

$$\begin{aligned}(T-2I)(T-3I)(T-4I)v &= (T-2I)(T-3I)((\lambda-4)v) \\ &= (T-2I)((\lambda^2-7\lambda+12)v) \\ &= (\lambda^3-9\lambda^2+26\lambda-24)v\end{aligned}$$

The polynomial $(\lambda^3-9\lambda^2+26\lambda-24)$ has the roots $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$, as desired. $\square$

Problem 5.B.18. Suppose V is finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Define a function $f : \mathbb{C} \to \mathbb{R}$ by

$$f(\lambda) = dim(range(T - \lambda I)).$$

Prove that f is not a continuous function.

Proof. By the rank-nullity theorem, we have

$$dim(range(T - \lambda I)) = dim(V) - dim(null(T - \lambda I)),$$

so we can write $f(\lambda) = dim(V) - dim(null(T - \lambda I))$. Further, If λ is not an eigenvalue of T, $dim(null(T - \lambda I)) = 0$; If λ is an eigenvalue of T, $dim(null(T - \lambda I)) > 0$, because $null(T - \lambda I)$ contains eigenvectors corresponding to λ. Thus, we can write

$$f(\lambda) = \begin{cases} dim(V) - dim(null(T - \lambda I)) & \text{for } \lambda \text{ an eigenvalue of } T \\ dim(V) & \text{for } \lambda \text{ not eigenvalue of } T \end{cases}$$

.

We want a discontinuity at an eigenvalue of T. The vector space V is finite-dimensional, complex and non-zero, so by Theorem 5.27 there is at least one eigenvalue for T. Choose an eigenvalue $\lambda_0 \in \mathbb{C}$.

We want to show a discontinuity at λ_0. Choose $\epsilon = \frac{dim(null(T-\lambda_0 I))}{2}$, which is greater than 0. Let $\delta > 0$. Choose $x \in \mathbb{C}$ such that $|x - \lambda_0| < \delta$ and x is not an eigenvalue; we can do this because there are at most $dim(V)$ eigenvalues. We have

$$\begin{aligned} |f(x) - f(\lambda_0)| &= |dim(V) - dim(V) - dim(null(T - \lambda_0 I)| \\ &= dim(null(T - \lambda_0 I)) \\ &> \frac{dim(null(T - \lambda_0 I))}{2} = \epsilon, \end{aligned}$$

as desired.

□

Problem 5.C.1. Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = null(T) \oplus range(T)$.

Proof. Since T is diagonalizable, we can write

$$M(T) = \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix}$$

with respect to a basis $(v_1, \ldots, v_n)$ of eigenvectors. Let $(v_{n1}, \ldots, v_{ni})$ be a sublist of eigenvectors associated zero eigenvalue. Let $(v_{r1}, \ldots, v_{rj})$ be a sublist of eigenvectors associated with non-zero eigenvalue.

$V = span(v_{n1}, \ldots, v_{nk}) \oplus span(v_{r1}, \ldots, v_{rj})$ because the each sublist is linearly independent and together form a basis for V. It suffices to show two things: $(v_{n1}, \ldots, v_{nk})$ is a basis for $null(T)$ and $(v_{r1}, \ldots, v_{rj})$ is a basis for $range(T)$.

Lets show that $R = (v_{r1}, \ldots, v_{rj})$ is a basis for $range(T)$. Let $v \in V$, which can be written as a linear combination of our two sublists. Applying T, we have $T(v) = b_1 c_{r1} v_{r1} + \cdots + b_j c_{rj} v_{rj}$ because the eigenvectors associated with zero eigenvalue disappear, so $range(T) \subseteq span(R)$. Let $u \in span(R)$, which can be written as $u = b_1 v_{r1} + \cdots + b_j v_{rj}$. Choose $v = \frac{b_1}{c_{r1}} v_{r_1} + \cdots + \frac{b_j}{c_{rj}} v_{rj}$. Clearly, $Tv = u$, so $span(R) \subseteq range(T)$.

Now, we must show that $N = (v_{n1}, \ldots, v_{nk})$ is a basis for $null(T)$. Let $v \in span(N)$, which can be written as $v = a_1 v_{n1} + \cdots + a_n v_{nk}$. Applying T to v results in the zero vectors because v is a linear combination of vectors associated with the zero eigenvalue, so $span(N) \subseteq null(T)$. Let $v \in null(T)$, which can be written as $v = \sum_{i=1}^{k} a_i v_{ni} + \sum_{i=1}^{j} b_i v_{ri}$. Applying T, we have $Tv = b_1 c_{r1} v_{r1} + \cdots + b_j c_{rj} v_{rj} = 0$. Since $(v_{r1}, \ldots, v_{rj}$ is linearly independent and each eigenvalue is non-zero, each $b_m = 0$. We have $v = a_1 v_{n1} + \cdots + a_k v_{nk}$, so $null(T) \subseteq span(N)$, as desired.

$\square$

Problem 5.C.2. Prove or disprove the converse of 5.C.1.

Proof. Define the rotation map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T = \begin{bmatrix} cos(90) & -sin(90) \\ sin(90) & cos(90) \end{bmatrix}.$$

Intuitively, there are no eigenvectors because each non-zero vector is rotated out of alignment, so T can't be diagonalized. Furthermore, $range(T) = \mathbb{R}^2$ and $null(T) = \{0\}$, so $range(T) \cap null(T) = \{0\}$. By Theorem 1.45, we have $\mathbb{R}^2 = null(T) \oplus range(T)$.

$\square$

Problem 5.C.8. Suppose $T \in \mathcal{L}(F^5)$ and $dim(E(8,T)) = 4$. Prove that $T - 2I$ or $T - 6I$ is invertible.

Proof. For a contradiction, assume that $T - 2I$ and $T - 6I$ are non-invertible. This means 2 and 6 are eigenvalues. We have

$$\begin{aligned} dim(E(8,T)) + dim(E(2,T)) + dim(E(6,T)) &\geq 4 + 1 + 1 \\ &\geq 6 \\ &> dim(F^5), \end{aligned}$$

which is not possible by Theorem 5.38.

□

Problem 5.C.14. Find $T \in \mathcal{L}(\mathbb{C}^3)$ such that 6 and 7 are eigenvalues of T and such that T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$.

Proof. $\square$