Selected Problems Chapter 1
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

Mustaf Ahmed

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Problem 1.A.2. Show that $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1 (meaning that its cube equals 1.)

Proof. We can use the definition of complex multiplication :

$$\begin{aligned}\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \left(\frac{-1+\sqrt{3}i}{2}\right)^2\left(\frac{-1+\sqrt{3}i}{2}\right)\\ &= \left(\frac{-1-\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right)\\ &= \frac{1}{4}+\frac{-\sqrt{3}i}{2}+\frac{\sqrt{3}i}{2}+\frac{3}{4}\\ &= 1\end{aligned}$$

□

Problem 1.A.3. Find two distinct roots of i.

Let $z = (a + bi)$ be some root of i. We have :

$$z^2 = (a+bi)^2 = a^2 - b^2 + 2abi = i$$

Since i has no real component, this means that $a^2 - b^2 = 0$. Also, since the coefficient of i is 1, $2ab = 1$, which also means that a, b must have the same sign. Thus, $a = b$, and

$$\begin{aligned}2ab = 2a^2 &= 1\\ a^2 &= \frac{1}{2}\\ a = b &= \pm\frac{1}{\sqrt{2}}\end{aligned}$$

,

so the two solutions are $z = (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})$ and $z = (-\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}})$.

Problem 1.B.1. Prove that $-(-v) = v$ for each $v \in V$.

Proof. Given $v \in V$, we have :

$$-(-v) = -1(-1v) = (-1^2)v = 1(v) = v.$$

□

Problem 1.B.1. Suppose $a \in F, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Proof. There are two cases : $a = 0$ or $a \neq 0$.

Case 1: $a = 0$. We are done.

Case 2: $a \neq 0$. Since F is a field and $a \neq 0$, the multiplicative inverse of a exists. We have that

$$v = (\frac{1}{a})av = (\frac{1}{a})0 = 0.$$

The first equality holds because $\frac{1}{a}$ is the multiplicative inverse of a, and the third equality holds because the vector 0 is invariant to scalar multiplication.

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Problem 1.C.4. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Proof. Define the set
$$C = \{f \in \mathbb{R}^{[0,1]} \mid \int_0^1 f = b\}.$$
For the forward direction, assume C is a subspace of the real-valued functions from the interval $[0,1]$ to $\mathbb{R}$. Since C is a subspace, $0 \in C$. We have $0 + 0 \in C$ because addition is an operation on C, and
$$b = \int_0^1 0 = \int_0^1 (0+0) = \int_0^1 0 + \int_0^1 0 = b + b = 2b.$$
Subtracting b from both side, we get $b = 0$.

Conversely, assume $b = 0$. We must show that C is a subspace. Define the identity $0(x) = 0, \forall x \in [0,1]$. Given $f \in C$, we have
$$\int_0^1 (f+0) = \int_0^1 f + \int_0^1 0 = 0,$$
so $f + 0 \in C$, meaning C contains an additive identity. The other two properties (for the minimum conditions to be a subspace) are satisfied because the integral is linear over functions and scalar multiplication on a zero value integral is still equal to zero.

□