Selected Problems Chapter 1
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem 1.A.2. Show that $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1 (meaning that its cube equals 1.)

Proof. We can use the definition of complex multiplication :

$$\begin{aligned}
\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \left(\frac{-1+\sqrt{3}i}{2}\right)^2\left(\frac{-1+\sqrt{3}i}{2}\right) \\
&= \left(\frac{-1-\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right) \\
&= \frac{1}{4} + \frac{-\sqrt{3}i}{2} + \frac{\sqrt{3}i}{2} + \frac{3}{4} \\
&= 1
\end{aligned}$$

□

Problem 1.A.3. Find two distinct roots of i.

Let $z = (a + bi)$ be some root of i. We have :

$$z^2 = (a+bi)^2 = a^2 - b^2 + 2abi = i$$

Since i has no real component, this means that $a^2 - b^2 = 0$. Also, since the coefficient of i is 1, $2ab = 1$, which also means that a, b must have the same sign. Thus, $a = b$, and

$$\begin{aligned}
2ab = 2a^2 &= 1 \\
a^2 &= \frac{1}{2} \\
a = b &= \pm\frac{1}{\sqrt{2}}
\end{aligned}$$

,

so the two solutions are $z = (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})$ and $z = (-\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}})$.

Problem 1.B.1. Prove that $-(-v) = v$ for each $v \in V$.

Proof. Given $v \in V$, we have :

$$-(-v) = -1(-1v) = (-1^2)v = 1(v) = v.$$

□

Problem 1.B.1. Suppose $a \in F, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Proof. There are two cases : $a = 0$ or $a \neq 0$.

Case 1: $a = 0$. We are done.

Case 2: $a \neq 0$. Since F is a field and $a \neq 0$, the multiplicative inverse of a exists. We have that

$$v = (\frac{1}{a})av = (\frac{1}{a})0 = 0.$$

The first equality holds because $\frac{1}{a}$ is the multiplicative inverse of a, and the third equality holds because the vector 0 is invariant to scalar multiplication.

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Problem 1.C.4. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Proof. Define the set

$$C = \{f \in \mathbb{R}^{[0,1]} \mid \int_0^1 f = b\}.$$

For the forward direction, assume C is a subspace of the real-valued functions from the interval $[0,1]$ to $\mathbb{R}$. Since C is a subspace, $0 \in C$, defined as $0(x) = 0$ for all $x \in [0,1]$. Thus, $0 + 0 \in C$ because addition is an operation on C, and

$$b = \int_0^1 0 = \int_0^1 (0+0) = \int_0^1 0 + \int_0^1 0 = b + b = 2b.$$

Subtracting b from both side, we get $b = 0$.

Conversely, assume $b = 0$. We must show that C is a subspace. We define the zero vector as above. Given $f \in C$, we have

$$\int_0^1 (f+0) = \int_0^1 f + \int_0^1 0 = 0,$$

so $f + 0 \in C$, meaning C contains an additive identity.

We must now show that addition and scalar multiplication are operations on C. Given $f, g \in C$, we have :

$$\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0 + 0 = 0,$$

so $f + g \in C$. Also, Given $\lambda \in F$ and $f \in C$, we have :

$$\int_0^1 \lambda(f) = \lambda \int_0^1 f = \lambda * 0 = 0,$$

so $\lambda(f) \in C$. Thus, The minimum properties for C to be a subspace of $\mathbb{R}^{[0,1]}$ are satisfied. $\square$

Problem 1.C.24. A function $f : \mathbb{R} \to \mathbb{R}$ is called even if

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called odd if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

Proof. $\square$

We will first show that U_e and U_o are subspaces of $\mathbb{R}^{\mathbb{R}}$. The zero element of $\mathbb{R}^{\mathbb{R}}$ is defined to be $z(x) = 0$, for all $x \in \mathbb{R}$. We have that

$$z(-x) = 0$$

because $z(x)$ is constant, so $z(x) \in U_e$. We also have that

$$z(-x) = -0 = 0$$

because $z(x)$ is constant, so $z(x) \in U_o$. Next, we must show that addition and scalar multiplication are valid operations on U_e and U_o. Given $f, g \in U_e$, we have :

$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x),$$

since f and g are even. Also, Given
