University of Rochester
Math 235 Midterm 1 Practice Fall 13

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Problem 1. Let $\{v_1, v_2, v_3\}$ be a subset of a vector space V. Suppose that $\{v_1, v_2, v_3\}$ is linearly dependent. Suppose that $v_1 \neq 0$. Prove that we must have $v_2 \in span(v_1)$ or $v_3 \in span(v_1, v_2)$.

Proof. Since $\{v_1, v_2, v_3\}$ is linearly dependent, we have $a_1v_1 + a_2v_2 + a_3v_3 = 0$ where not all the coefficients are equal to zero. Assume that $a_2 = a_3 = 0$. It is straightforward that $v_1 = 0$, which is not possible. Thus, there are two cases: $a_2 = 0$ and $a_3 \neq 0$ or $a_2 \neq 0$ and $a_3 = 0$.

Assume that $a_2 = 0$ and $a_3 \neq 0$. We have $a_1v_1 + 0v_2 + a_3v_3 = 0$, so $v_3 = \frac{-a_1}{a_3}v_1 + 0v_2$.

Assume that $a_2 \neq 0$ and $a_3 = 0$. We have $a_1v_1 + a_2v_2 = 0$, so $v_2 = \frac{-a_1}{a_2}v_1$.

$\square$

Problem 2(a). Let V be a vector space of dimension $n \geq 2$. Let W_1 and W_2 be subspaces of V such that $W_1 \neq V$, $W_2 \neq V$, and $W_1 \neq W_2$. Show that

$$dim(W_1 \cap W_2) \leq dim(V) - 2.$$

Proof. Since $W_1 \neq V$ and $W_2 \neq V$, we have $dim(W_1) < dim(V)$ and $dim(W_2) < dim(V)$. Since $W_1 \neq W_2$, we have $dim(W_1 \cap W_2) < dim(V)$.

We want that $dim(W_1 \cap W_2) \leq dim(V) - 2$. It sufficies to show that $dim(W_1 \cap W_2) \neq dim(V) - 1$. For a contradiction, assume that $dim(W_1 \cap W_2) = dim(V) - 1$. Since $W_1 \neq W_2$, we can non-trivially extend any basis of $W_1 \cap W_2$ to a basis for W_1 or a basis for W_2. We have $dim(W_1) = dim(V)$ or $dim(W_2) = dim(V)$, a contradiction.

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