Selected Problems Chapter 3
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem Integration. Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ by

$$Tp = \int_0^1 p(x)dx.$$

Show that T is a linear map.

Proof.

Additivity
Given $p, q \in \mathcal{P}(\mathbb{R})$, we want the additivity propriety to hold for T. Applying T to the sum of p and q, we have

$$\begin{aligned} T(p+q) &= \int_0^1 p(x) + q(x)dx \\ &= \int_0^1 p(x)dx + \int_0^1 q(x)dx \\ &= T(p) + T(q), \end{aligned}$$

since integration of a sum is equal to the sum of the integrated parts.
Homogeneity
Given $p \in \mathcal{P}(\mathbb{R})$ and $a \in F$, we want the homogeneity property to hold. Applying T to the scalar multiple of p, we have

$$\begin{aligned} T(a * p) &= \int_0^1 a * p(x)dx \\ &= a * \int_0^1 p(x)dx \\ &= a * T(p), \end{aligned}$$

since constants can be separated in integration.

□

Problem Theorem 3.5. Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m \in W$. Show that there exists a unique linear map $T : V \to W$ such that

$$T(v_j) = w_j$$

for each $j = 1, \dots, n$.

Proof. We must first show the existence of a linear map with the desired properties. Define $T : V \to W$ by

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_n w_n$$

, where $a_1, \dots, a_n$ are coefficients in F.

We must show that T is a linear map. Given $a_1 v_1, \dots, a_n v_n \in V$ and $b_1 v_1, \dots, bn_v n \in V$, we have

$$\begin{aligned} T'((a_1 v_1 + \cdots + a_n v_n) + ((b_1 v_1 + \cdots + b_n v_n))) &= T((a_1 + b_1)v_1 + \dots (a_n + b_n)v_n) \\ &= (a_1 + b_1)w_1 + \cdots + (a_n + b_n)w_n \\ &= (a_1 w_1 + \cdots + a_n w_n) + (b_1 w_1 + \cdots + b_n w_n) \\ &= T(a_1 v_1 + \cdots + a_n v_n) + T(b_1 v_1 + \cdots + b_n v_n). \end{aligned}$$

Similarly, given $a_1 v_1 + \cdots + a_n v_n \in V$ and $\lambda \in F$, we have

$$\begin{aligned} T(\lambda(a_1 v_1 + \cdots + a_n v_n)) &= T((\lambda * a_1)v_1 + \cdots + (\lambda * a_n)v_n \\ &= (\lambda * a_1)w_1 + \cdots + (\lambda * a_n)w_n \\ &= \lambda(a_1 w_1 + \cdots + a_n w_n) \\ &= \lambda * T(a_1 v_1 + \cdots + a_n v_n) \end{aligned}$$

.

Thus, we have shown T to be a linear map.

Assume the existence of another linear map $T' : V \to W$ with the property

$$T'(v_j) = w_j$$

for each $j = 1, \dots, n$. To show uniqueness, we want that $T(v) = T'(v)$ for all $v \in V$.

Given $v \in V$, we can write $v = a_1 v_1 + \cdots + a_n v_n$, since we have a basis for V. Then,

$$\begin{aligned} T(v) &= T(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 T(v_1) + \cdots + a_n T(v_n) \\ &= a_1 T'(v_1) + \cdots + a_n T'(v_n) \\ &= T'(a_1 v_1 + \cdots + a_n v_n) \\ &= T'(v), \end{aligned}$$

since $T(v_i) = w_i = T'(v_i)$ for each $j = 1, \dots, n$. □

Problem 3.A.11. Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Proof. Given a subspace U of V and a linear map $S \in \mathcal{L}(U, W)$, we want to extend S to be a linear map on V. Choose a basis of U to be $u_1, \dots, u_m$. We can extend our chosen basis for U to be a basis of V as the list $u_1, \dots, u_m, v_1, \dots, v_n$. Let

$$w_j = Su_j$$

for $j = 1, \dots, m$. Thus the linear map S can be explicitly written as

$$\begin{aligned} Su &= S(a_1u_1 + \cdots + a_mu_m) \\ &= a_1w_1 + \dots a_mw_m \end{aligned}$$

, for all $u \in U$.

We are now in the position to define the extension linear map T. We define T by

$$\begin{aligned} Tv &= T(a_1u_1 + \cdots + a_mu_m + a_{m+1}v_1 + \cdots + a_{m+n}v_n) \\ &= a_1w_1 + \dots a_mw_m \end{aligned}$$

, for all $v \in V$. It is clear that $Tu = Su$ for all $u \in U$ by the definition of T; T is also a linear map.

□

Problem 3.A.14. Suppose V is finite-dimensional with $dim\ V \geq 2$. Prove that there exists $S, T \in \mathcal{L}(V, V)$ such that $ST \neq TS$.

Proof. Assume that $n \geq 2$. Choose a basis $v_1, \ldots, v_n \in V$. We will define both linear maps as follows:

$$\begin{aligned} Tv &= T(a_1 v_1 + \cdots + a_n v_n) \\ &= a_n v_1 + a_1 v_2 + \cdots + a_{n-1} v_n \end{aligned}$$

, which shifts coefficients by 1 in a circle; Also,

$$\begin{aligned} Sv &= S(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 v_1. \end{aligned}$$

We will now show that these linear maps do not commute.Let $v = v_1 + 2v_2$. We have

$$\begin{aligned} ST(v) &= ST(v_1 + 2v_2) \\ &= S(2v_1 + v_2) \\ &= 2v_1, \end{aligned}$$

and

$$\begin{aligned} TS(v) &= TS(v_1 + 2v_2) \\ &= T(v_1) \\ &= v_2. \end{aligned}$$

Thus, $ST \neq TS$. $\square$

Problem Extra (not from LADR). If V is a vector space over the field F, the dual vector space V^* is the vector space $\mathcal{L}(V, F)$ of linear maps from V to F.

Assume that *dim* $V = n$, and that $v_1, \ldots, v_n$ is a basis for V. Find a basis for V^*. What is *dim* V^*?

Proof. Let $T_i : V \to F$ be defined as

$$T_i(v) = T_i(a_1 v_1 + \ldots a_n v_n) = a_i$$

for $i = 1, \ldots, n$ be our chosen list of vectors. □

We will first show that our list spans $\mathcal{L}(V, F)$. Choose $M \in \mathcal{L}(V, F)$ and choose $v \in V$. We have that

$$\begin{aligned} M(v) &= M(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 M(v_1) + \cdots + a_n M(v_n) \\ &= M(v_1) T_1(v) + \cdots + M(v_n) T_n(v) \end{aligned}$$

Next, we will show that our list is linearly independent. Assume that

$$c_1 T_1(v) + \cdots + c_n T_n(v) = 0$$