Selected Problems Chapter 1
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem 1.A.2. Show that $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1 (meaning that its cube equals 1.)

Proof. We can use the definition of complex multiplication :

$$\begin{aligned}\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \left(\frac{-1+\sqrt{3}i}{2}\right)^2\left(\frac{-1+\sqrt{3}i}{2}\right)\\ &= \left(\frac{-1-\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right)\\ &= \frac{1}{4}+\frac{-\sqrt{3}i}{2}+\frac{\sqrt{3}i}{2}+\frac{3}{4}\\ &= 1\end{aligned}$$

□

Problem 1.A.3. Find two distinct roots of i.

Let $z = (a + bi)$ be some root of i. We have :

$$z^2 = (a+bi)^2 = a^2 - b^2 + 2abi = i$$

Since i has no real component, this means that $a^2 - b^2 = 0$. Also, since the coefficient of i is 1, $2ab = 1$, which also means that a, b must have the same sign. Thus, $a = b$, and

$$\begin{aligned}2ab = 2a^2 &= 1\\ a^2 &= \frac{1}{2}\\ a = b &= \pm\frac{1}{\sqrt{2}}\end{aligned}$$

,

so the two solutions are $z = (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})$ and $z = (-\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}})$.

Problem 1.B.1. Prove that $-(-v) = v$ for each $v \in V$.

Proof. Given $v \in V$, we have :

$$-(-v) = -1(-1v) = (-1^2)v = 1(v) = v.$$

□

Problem 1.B.1. Suppose $a \in F, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Proof. There are two cases : $a = 0$ or $a \neq 0$.

Case 1: $a = 0$. We are done.

Case 2: $a \neq 0$. Since F is a field and $a \neq 0$, the multiplicative inverse of a exists. We have that

$$v = (\frac{1}{a})av = (\frac{1}{a})0 = 0.$$

The first equality holds because $\frac{1}{a}$ is the multiplicative inverse of a, and the third equality holds because the vector 0 is invariant to scalar multiplication.

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Problem 1.C.4. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Proof. Define the set

$$C = \{f \in \mathbb{R}^{[0,1]} \mid \int_0^1 f = b\}.$$

For the forward direction, assume C is a subspace of the real-valued functions from the interval $[0,1]$ to $\mathbb{R}$. Since C is a subspace, $0 \in C$, defined as $0(x) = 0$ for all $x \in [0,1]$. Thus, $0 + 0 \in C$ because addition is an operation on C, and

$$b = \int_0^1 0 = \int_0^1 (0+0) = \int_0^1 0 + \int_0^1 0 = b + b = 2b.$$

Subtracting b from both side, we get $b = 0$.

Conversely, assume $b = 0$. We must show that C is a subspace. We define the zero vector as above. Given $f \in C$, we have

$$\int_0^1 (f+0) = \int_0^1 f + \int_0^1 0 = 0,$$

so $f + 0 \in C$, meaning C contains an additive identity.

We must now show that addition and scalar multiplication are operations on C. Given $f, g \in C$, we have :

$$\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0 + 0 = 0,$$

so $f + g \in C$. Also, Given $\lambda \in F$ and $f \in C$, we have :

$$\int_0^1 \lambda(f) = \lambda \int_0^1 f = \lambda * 0 = 0,$$

so $\lambda(f) \in C$. Thus, The minimum properties for C to be a subspace of $\mathbb{R}^{[0,1]}$ are satisfied. □

Problem 1.C.24. A function $f : \mathbb{R} \to \mathbb{R}$ is called even if

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called odd if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

Proof. □

We will first show that U_e and U_o are subspaces of $\mathbb{R}^{\mathbb{R}}$. The zero element of $\mathbb{R}^{\mathbb{R}}$ is defined to be $z(x) = 0$, for all $x \in \mathbb{R}$. We have that

$$z(-x) = 0 = z(x)$$

because $z(x)$ is constant, so $z(x) \in U_e$. We also have that

$$z(-x) = -0 = -z(x)$$

because $z(x)$ is constant, so $z(x) \in U_o$.

Now, we must show that addition and scalar multiplication are valid operations on the two sets. Given $f, g \in U_e$, we have

$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x),$$

since f and g are even. Similarly, given $f, g \in U_o$, we have

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f(x) + g(x)) = -(f+g)(x),$$

since f and g are odd. Given $\lambda \in F$ and $f \in U_e$, we have

$$(\lambda f)(-x) = \lambda(f)(-x) = \lambda(f)(x),$$

since f is even. Similarly, given $g \in U_o$, we know that

$$(\lambda g)(-x) = \lambda(g)(-x) = -\lambda(g)(x),$$

because g is odd. Thus, U_e and U_o are subspaces of $\mathbb{R}^{\mathbb{R}}$.

Finally, we must demonstrate that $U_e \oplus U_o = \mathbb{R}^{\mathbb{R}}$, i.e. we must show that the sum of the two subspaces equals $\mathbb{R}^{\mathbb{R}}$, and it's a direct sum. Given $f \in U_e$ and $g \in U_e$ (both functions from the real numbers to the real numbers), $(f+g)(x)$ is a function from the real numbers to the real numbers; thus, $U_e + U_o \subseteq \mathbb{R}^{\mathbb{R}}$. To finish showing that $U_e + U_o = \mathbb{R}^{\mathbb{R}}$, we need to prove that every function from the real numbers to the real numbers can be expressed as the sum of an even and odd function. Given $f \in \mathbb{R}^{\mathbb{R}}$, we must show that there exists $g \in U_e$ and $h \in U_o$ such that

$$f(x) = g(x) + h(x).$$

Define $f_e(x) = \frac{f(x)+f(-x)}{2}$ and $f_o(x) = \frac{f(x)-f(-x)}{2}$. Then,

$$\begin{aligned} f_e(x) + f_o(x) &= \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} \\ &= \frac{f(x)+f(-x)+f(x)-f(-x)}{2} \\ &= \frac{2f(x)}{2} \\ &= f(x), \end{aligned}$$

so $\mathbb{R}^{\mathbb{R}} \subseteq U_e + U_o$.

The last requirement is that $U_e + U_o$ is a direct sum; we can prove this by showing $U_e \cap U_o = \{z(x)\}$. Assume $f \in U_e \cap U_o$. Thus,

$$0 = f(x) + -f(x) = f(-x) + f(-x) = 2f(-x) = 2f(x),$$

so dividing both sides by two leads to $f(x) = 0$ for all $x \in \mathbb{R}$.

Problem Theorem 1.44 Condition for a direct sum. Suppose $U_1, \ldots, U_m$ are subspace of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 in the sum is the sum $u_1 + \cdots + u_m$ where each u_j is 0.

Proof. For the forward direction, assume that $U_1 + \cdots + U_m$ is a direct sum. We know that 0 in $U_1 + \cdots + U_m$ is written the sum of the zero vector in each U_j. By the definition of direct sum, this representation is unique.

For the backward direction, assume that 0 is uniquely represented in $U_1 + \cdots + U_m$ as the sum of the zero vector in each U_j. Given $v = v_1 + \cdots + v_m \in U_1 + \cdots + U_m$, assume there exists $w = w_1 + \cdots + w_m \in U_1 + \cdots + U_m$ where $v = w$. We want $v_j = w_j$ for each j.

Subtracting the two representations, we have

$$\begin{aligned} 0 &= v - w \\ &= (v_1 - w_1) + \cdots + (v_m - w_m) \end{aligned}$$

.

Since 0 is represented uniquely as the sum of zero vectors from each subspace, $v_j = w_j$ for each j, as desired.

□

Problem Theorem 1.45 Direct sum of two subspaces. Suppose U and W are subspaces of V. The $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof. We will prove the forward directly. Given $v \in U \cap W$, we can identify v in $U + W$ as $v = 0 + v$ and $v = v + 0$. Subtracting these representations, we have

$$\begin{aligned} 0 &= (0 - v) + (v - 0) \\ &= -v + v, \end{aligned}$$

which is a representation of 0 in $U + W$. Since $U + W$ is a direct sum, we know 0 has a unique representation as the sum of zero-vectors. Thus, $v = 0$.

Now assume that $U \cap W = \{0\}$. We must show that each vector in $U + W$ is written uniquely. By Theorem 1.44, we can show that $0 \in U + W$ is written uniquely. Let $0 = u + w$ for some $u \in U$ and $w \in W$. Subtracting w from both sides, we have $-w = u$, but since the intersection of U and W only contains the zero vector, $u = w = 0$.

$\square$
