Selected Problems Chapter 3
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem Integration. Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ by

$$Tp = \int_0^1 p(x)dx.$$

Show that T is a linear map.

Proof.

Additivity

Given $p, q \in \mathcal{P}(\mathbb{R})$, we want the additivity propriety to hold for T. Applying T to the sum of p and q, we have

$$\begin{aligned} T(p+q) &= \int_0^1 p(x) + q(x)dx \\ &= \int_0^1 p(x)dx + \int_0^1 q(x)dx \\ &= T(p) + T(q), \end{aligned}$$

since integration of a sum is equal to the sum of the integrated parts.

Homogeneity

Given $p \in \mathcal{P}(\mathbb{R})$ and $a \in F$, we want the homogeneity property to hold. Applying T to the scalar multiple of p, we have

$$\begin{aligned} T(a * p) &= \int_0^1 a * p(x)dx \\ &= a * \int_0^1 p(x)dx \\ &= a * T(p), \end{aligned}$$

since constants can be separated in integration.

□

Problem Theorem 3.5. Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m \in W$. Show that there exists a unique linear map $T : V \to W$ such that

$$T(v_j) = w_j$$

for each $j = 1, \dots, n$.

Proof. We must first show the existence of a linear map with the desired properties. Define $T : V \to W$ by

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_n w_n$$

, where $a_1, \dots, a_n$ are coefficients in F.

We must show that T is a linear map. Given $a_1 v_1, \dots, a_n v_n \in V$ and $b_1 v_1, \dots, bn_v n \in V$, we have

$$\begin{aligned} T'((a_1 v_1 + \cdots + a_n v_n) + ((b_1 v_1 + \cdots + b_n v_n))) &= T((a_1 + b_1)v_1 + \dots (a_n + b_n)v_n) \\ &= (a_1 + b_1)w_1 + \cdots + (a_n + b_n)w_n \\ &= (a_1 w_1 + \cdots + a_n w_n) + (b_1 w_1 + \cdots + b_n w_n) \\ &= T(a_1 v_1 + \cdots + a_n v_n) + T(b_1 v_1 + \cdots + b_n v_n). \end{aligned}$$

Similarly, given $a_1 v_1 + \cdots + a_n v_n \in V$ and $\lambda \in F$, we have

$$\begin{aligned} T(\lambda(a_1 v_1 + \cdots + a_n v_n)) &= T((\lambda * a_1)v_1 + \cdots + (\lambda * a_n)v_n \\ &= (\lambda * a_1)w_1 + \cdots + (\lambda * a_n)w_n \\ &= \lambda(a_1 w_1 + \cdots + a_n w_n) \\ &= \lambda * T(a_1 v_1 + \cdots + a_n v_n) \end{aligned}$$

.

Thus, we have shown T to be a linear map.

Assume the existence of another linear map $T' : V \to W$ with the property

$$T'(v_j) = w_j$$

for each $j = 1, \dots, n$. To show uniqueness, we want that $T(v) = T'(v)$ for all $v \in V$.

Given $v \in V$, we can write $v = a_1 v_1 + \cdots + a_n v_n$, since we have a basis for V. Then,

$$\begin{aligned} T(v) &= T(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 T(v_1) + \cdots + a_n T(v_n) \\ &= a_1 T'(v_1) + \cdots + a_n T'(v_n) \\ &= T'(a_1 v_1 + \cdots + a_n v_n) \\ &= T'(v), \end{aligned}$$

since $T(v_i) = w_i = T'(v_i)$ for each $j = 1, \dots, n$. $\square$

Problem 3.A.11. Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Proof. Given a subspace U of V and a linear map $S \in \mathcal{L}(U, W)$, we want to extend S to be a linear map on V. Choose a basis of U to be $u_1, \ldots, u_m$. We can extend our chosen basis for U to be a basis of V as the list $u_1, \ldots, u_m, v_1, \ldots, v_n$. Let

$$w_j = Su_j$$

for $j = 1, \ldots, m$. Thus the linear map S can be explicitly written as

$$\begin{aligned} Su &= S(a_1u_1 + \cdots + a_mu_m) \\ &= a_1w_1 + \ldots a_mw_m \end{aligned}$$

, for all $u \in U$.

We are now in the position to define the extension linear map T. We define T by

$$\begin{aligned} Tv &= T(a_1u_1 + \cdots + a_mu_m + a_{m+1}v_1 + \cdots + a_{m+n}v_n) \\ &= a_1w_1 + \ldots a_mw_m \end{aligned}$$

, for all $v \in V$. It is clear that $Tu = Su$ for all $u \in U$ by the definition of T; T is also a linear map.

□

Problem 3.A.14. Suppose V is finite-dimensional with $dim\ V \geq 2$. Prove that there exists $S, T \in \mathcal{L}(V, V)$ such that $ST \neq TS$.

Proof. Assume that $n \geq 2$. Choose a basis $v_1, \ldots, v_n \in V$. We will define both linear maps as follows:

$$\begin{aligned} Tv &= T(a_1 v_1 + \cdots + a_n v_n) \\ &= a_n v_1 + a_1 v_2 + \cdots + a_{n-1} v_n \end{aligned}$$

, which shifts coefficients by 1 in a circle; Also,

$$\begin{aligned} Sv &= S(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 v_1. \end{aligned}$$

We will now show that these linear maps do not commute.Let $v = v_1 + 2v_2$. We have

$$\begin{aligned} ST(v) &= ST(v_1 + 2v_2) \\ &= S(2v_1 + v_2) \\ &= 2v_1, \end{aligned}$$

and

$$\begin{aligned} TS(v) &= TS(v_1 + 2v_2) \\ &= T(v_1) \\ &= v_2. \end{aligned}$$

Thus, $ST \neq TS$. $\square$

Problem 3.B.2. Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that $range(S) \subset null(T)$. Prove that $(ST)^2 = 0$.

Proof. Given $u \in V$, we must show that $((ST)(ST))u = 0$. By the definition of the product of linear maps, we have

$$((ST)(ST))u = (ST)(STu)$$

Let $w = (STu)$, which is in $range(S)$. Because $w \in range(S)$, w is also in $null(T)$ by our assumption. We have

$$\begin{aligned}((ST)(ST))u &= (ST)(STu) \\ &= (ST)w \\ &= (STw) \\ &= (S0) \\ &= 0,\end{aligned}$$

as desired.

$\square$

Problem 3.B.12. Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap null(T) = \{0\}$ and $range(T) = \{Tu \mid u \in U\}$.

Proof. Choose a subspace U of V such that $U \oplus null(T) = V$, which is guaranteed by Theorem 2.34. Since U and $null(T)$ form a direct sum, we have

$$U \cap null(T) = \{0\}.$$

The next part is to show that $range(T) = \{Tu \mid u \in U\}$. The right to left inclusion is clear by the definition of range; we must show the left to right inclusion to finish the proof. Given $w \in range(T)$, choose $v \in V$ such that $w = Tv$. Since $U \oplus null(T) = V$, choose $u \in U$ and $n \in null(T)$ such that $v = u + n$. We have

$$\begin{aligned} w &= Tv \\ &= T(u+n) \\ &= T(u) + T(n) \\ &= T(u) + 0 \\ &= T(u), \end{aligned}$$

as desired.

□

Problem Fundamental Theorem of Linear Maps. Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $range(T)$ is finite-dimensional and

$$dim(V) = dim(null(T)) + dim(range(T))$$

Proof. By Theorem 2.34, we can choose a subspace U of V such that $U \oplus null(T) = V$. By Problem 3.B.12, $range(T) = \{Tu \mid u \in U\}$. Let $B_1 = \{u_1, \ldots, u_m\}$ of U, and let $B_2 = \{n_1, \ldots, n_k\}$ of $null(T)$.

We want $B_1 \cup B_2$ to be a basis for V. Since $U \oplus null(T) = V$, $B_1 \cup B_2$ spans V. We need to show that $B_1 \cup B_2$ is also linearly independent. It suffices to show that the zero-vector can be represented uniquely. Assume that the zero-vector can be written as

$$0 = \sum_{i=1}^{m} \lambda_i u_i + \sum_{i=m+1}^{m+k} \lambda_i n_i$$

and

$$0 = \sum_{i=1}^{m} \alpha_i u_i + \sum_{i=m+1}^{m+k} \alpha_i n_i.$$

Since U and $null(T)$ form a direct sum, any vector in V can be written uniquely as a vector from U and a vector from $null(T)$; Thus, we have

$$\sum_{i=1}^{m} \lambda_i u_i = \sum_{i=1}^{m} \alpha_i u_i$$

and

$$\sum_{i=m+1}^{m+k} \lambda_i n_i = \sum_{i=m+1}^{m+k} \alpha_i n_i.$$

It follows from the fact that B_1 and B_2 are basis lists that $\lambda_i = \alpha_i$ for all i. Thus, $B_1 \cup B_2$ forms a basis for V.

We know that $dim(V) = m + k$ from $B_1 \cup B_2$ being a basis. We also know that $dim(null(T)) = k$ from B_2 being a basis of $null(T)$. It suffices to show that $dim(range(T)) = m$. We must show that $R = \{Tu_1, \ldots, Tu_m\}$ is a basis for $range(T)$. By Problem 3.B.12, R spans $range(T)$. For a contradiction, suppose that R is linearly dependent. We can choose Tu_j in $span(Tu_1, \ldots, Tu_{j-1})$ by the Linear Dependence Lemma. We have

$$T(u_j) = \lambda_1 T(u_1) + \cdots + \lambda_{j_1} T(u_{j-1}),$$

so

$$\begin{aligned} T(u_j) &= T(\lambda_1 u_1 + \cdots + \lambda_{j_1} u_{j-1}) \\ 0 &= T(\lambda_1 u_1 + \cdots + \lambda_{j_1} u_{j-1} - u_j) \end{aligned}$$

Thus, $\lambda_1 u_1 + \cdots + \lambda_{j_1} u_{j-1} - u_j \in null(T)$, with not all the constants being zero, but we know that $U \cap Null(T) = \{0\}$, which is a contradiction.

$\square$

Problem A map to a smaller dimensional space is not injective. Suppose V and W are finite-dimensional vector spaces and $dim(V) > dim(W)$. Then no linear map from V to W is injective.

The intuition is that v is being transferred into a smaller space, so there is no way that points in W receive a unique member in V, since there are simply too many elements in V.

Proof. We are given a linear map $T \in \mathcal{L}(V, W)$. Since $range(T) \subseteq W$, $dim(range(T)) \leq dim(W)$. By the Fundamental Theorem of Linear Maps, we have

$$\begin{aligned} dim(null(T)) &= dim(V) - dim(range(T)) \\ &\geq dim(V) - dim(W) \\ &\geq 1 \end{aligned}$$

.

Thus, for any basis of $null(T)$, there is at least one non-zero element which is mapped to the zero-vector in W. The zero-vector in V is also mapped to the zero-vector in W. T is not injective as desired.

□

Problem A map to a larger dimensional space is not surjective. Suppose V and W are finite-dimensional vector spaces and $dim(V) < dim(W)$. Then no linear map from V to W is surjective.

Proof. We are given a linear map $T \in \mathcal{L}(V, W)$. By the Fundamental Theorem of Linear Maps, we have

$$\begin{aligned} dim(range(T)) &= dim(V) - dim(null(T)) \\ &\leq dim(V) \\ &< dim(W). \end{aligned}$$

Thus, for any basis of $range(T)$, we can extend it to a basis of W, revealing a vector not in $range(T)$; thus, $T : V \to W$ is not surjective.

$\square$

Problem A map to a larger dimensional space is not surjective. Suppose V and W are finite-dimensional vector spaces and $dim(V) < dim(W)$. Then no linear map from V to W is surjective.

Proof. We are given a linear map $T \in \mathcal{L}(V, W)$. By the Fundamental Theorem of Linear Maps, we have

$$\begin{aligned} dim(range(T)) &= dim(V) - dim(null(T)) \\ &\leq dim(V) \\ &< dim(W). \end{aligned}$$

Thus, for any basis of $range(T)$, we can extend it to a basis of W, revealing a vector not in $range(T)$; thus, $T : V \to W$ is not surjective.

□

Problem 3.B.20. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

Proof. We will first prove the forward direction. Assume that T is injective. Define $S : range(T) \to V$ by

$$Sw = v$$

where v is the unique element in V such that $Tv = w$. We must show that ST is the identity map on V. Choose $v \in V$. Then

$$\begin{aligned} (ST)v &= S(Tv) \\ &= v, \end{aligned}$$

by the definition of S.

We will prove the backward direction. Choose $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V. Assume that $Tu = Tw$ for some $u, w \in V$. Using (ST), we have

$$\begin{aligned} u &= (ST)u \\ &= S(Tu) \\ &= S(Tw) \\ &= (ST)w \\ &= w, \end{aligned}$$

meaning T is injective.

□

Problem Edited 3.B.29 (V is assumed to be finite-dimensional). Suppose $T \in \mathcal{L}(V, F)$. Suppose $u \in V$ is not in $null(T)$. Prove that

$$V = null(T) \oplus \{au \mid a \in F\}$$

Proof. We will first construct a certain basis for V. Choose $u \in V$ such that u is not in $null(T)$. Let $\{n_1, \ldots, n_m\}$ be a basis for $null(T)$. We want

$$B = \{n_1, \ldots, n_m, u\}$$

to be a basis for V.

For a contradiction, suppose that B is not a basis for V. Thus, we can extend B to a basis of V:

$$B_e = \{n_1, \ldots, n_m, u, e_1, \ldots, e_n\}.$$

We consider the complex numbers to be a vector space over itself, so $dim(F) = 1$. We want a contradiction with $dim(range(T)) \leq dim(F)$. We will do this by showing that $R = \{Tu, Te_1, \ldots, Te_n\}$ is a basis for $range(T)$. Given $r \in range(T)$, choose $a_1, \ldots, a_{m+n+1} \in F$ such that

$$\begin{aligned} r &= T((a_1 n_1 + \cdots + a_m n_m) + a_{m+1} u + (a_{m+2} e_1 + \cdots + a_{m+n+1} e_n)) \\ &= a_{m+1} T(u_1) + a_{m+2} T(e_1) + \cdots + a_{n+1} T(e_n) \end{aligned}$$

, so R spans $range(T)$. Next, we need to show that R is linearly independent. Choose $a_1, \ldots, a_{n+1} \in F$ such that

$$\begin{aligned} 0 &= a_1 T(u_1) + a_2 T(e_1) + \cdots + a_{n+1} T(e_m) \\ &= T(a_1 u_1 + a_2 e_1 + \cdots + a_{n+1} e_m), \end{aligned}$$

so $a_1 u_1 + a_2 e_1 + \cdots + a_{n+1} e_m \in span(n_1, \ldots, n_m)$. For some $b_1, \ldots, b_m \in F$, we have

$$0 = (a_1 u_1 + a_2 e_1 + \cdots + a_{n+1} e_m) - (b_1 n_1 + \ldots b_m n_m)$$

,

which means that all the scalars must be 0 because the vectors form a basis for V. Thus, R is a basis for $range(T)$.

Now, for the contradiction.

$$\begin{aligned} dim(range(T)) &= 1 + n \\ &> 1 \\ &= dim(F), \end{aligned}$$

which is not possible because $dim(F) = 1$. So B is a basis for V.

We've established that B is a basis for V. All that is left is to show that $V = null(T) + \{au \mid a \in F\}$ and $null(T) \cap \{au \mid a \in F\} = \{0\}$. Because B span V, it is clear that $V = null(T) + \{au \mid a \in F\}$. We assumed that $u \notin null(T)$, so the $null(T) \cap \{au \mid a \in F\} = \{0\}$. $\square$

Problem Theorem 3.36 matrix of sum of linear maps. Suppose $S, T \in \mathcal{L}(V, W)$. Then $M(S+T) = M(S) + M(T)$.

Proof. Let $v_1, \ldots v_n$ be our chosen basis for V and $w_1 \ldots, w_m$ be our chosen basis for W. The sum of S and T applied to v_j is

$$\begin{aligned}
(S+T)v_j &= S(v_j) + T(v_j) \\
&= \sum_{i=1}^{m} a_{i,j} w_i + \sum_{i=1}^{m} b_{i,j} w_i \\
&= \sum_{i=1}^{m} (a_{i,j} + b_{i,j}) w_j
\end{aligned}$$

Thus, $M(S+T)$ is defined as

$$M(S+T) = \begin{pmatrix} a_{1,1} + b_{1,1} & \cdots & a_{1,n} + b_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} + b_{m,1} & \cdots & a_{m,n} + b_{m,n} \end{pmatrix}.$$

From our decomposition of $(S+T)v_j$, for each j, above, we have

$$M(S) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix},$$

, and

$$M(T) = \begin{pmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{m,1} & \cdots & b_{m,n} \end{pmatrix}.$$

It is clear that $M(S+T) = M(S) + M(T)$ follows from the definition of matrix addition. $\square$

Problem 3.C.3. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a basis of V and a basis of W such that with respect to these bases, all entries of $M(T)$ are 0 except that the entries in row j, column j, equal 1 for $1 \leq j \leq dim(range(T))$.

Proof. Let $n_1, \ldots, n_m$ be a basis for $null(T)$. We can extend this list to a basis for V:

$$B_v = \{v_1, \ldots, v_n, n_1, \ldots, n_m\}.$$

It is clear that $Tv_1, \ldots, T_{vn}$ is a basis for $Range(T)$. We can extend this to a basis of W:

$$B_w = \{Tv_1, \ldots, Tv_n, w_1, \ldots, w_k\}.$$

Now that we have a basis for V and W, we must show that $M(T)$ is in the desired form. The column associated with each v_j has a single 1 in $M(T)_{j,j}$ and 0 everywhere else. The column associated with each n_k is filled with only 0, since $n_k \in null(T)$. We are done.

$\square$

Problem 3.C.4. Suppose $v_1, \ldots, v_m$ is a basis for V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove there exists a basis $w_1, \ldots, w_n$ of W such that all the entries in the first column of $M(T)$ (with respect to the bases) are 0 except for possibly a 1 in the first row, first column.

Proof. **Case 1.** v_1 is in $null(T)$. Let $w_1, \ldots, w_n$ be any basis for W. We can write Tv_1 as

$$Tv_1 = 0w_1 + 0w_2 + \cdots + 0w_n,$$

meaning that the first column of $M(T)$ contains all zeros.

Case 2. v_1 is not in $null(T)$. Let $w_1 = Tv_1$. We can extend the list containing only w_1 to a basis of W: $w_1, \ldots, w_n$. Then, Tv_1 is written as

$$Tv_1 = 1w_1 + 0w_2 + \cdots + 0w_n,$$

,so the first column of $M(T)$ starts with a 1 and the rest of the entries are 0. We are done. $\square$

Problem Theorem Unique Inverse. An invertible linear map has a unique inverse.

Proof. Let $T \in \mathcal{L}(V, W)$, and let $R, S \in \mathcal{L}(W, V)$ be inverses of T. We have

$$\begin{aligned} R &= RI \\ &= R(TS) \\ &= (RT)S \\ &= IS \\ &= S \end{aligned}$$

,

as desired. □

Problem Theorem 3.56 Invertibility is equivalent to injectivity and surjectivity. A linear map is invertible if and only if it is injective and surjective.

Proof. Let $T : V \to W$ be a linear map. Assume that T is invertible. Choose $T^{-1} \in \mathcal{L}(W, V)$ to be the unique inverse for T. For injectivity, assume that $Tv = Tu$ for $u, v \in V$. Applying the inverse, we have

$$\begin{aligned} u &= Iu \\ &= (T^{-1}T)u \\ &= T^{-1}(Tu) \\ &= T^{-1}(Tv) \\ &= (T^{-1}T)v \\ &= Iv \\ &= v. \end{aligned}$$

Next is surjectivity. Choose $w \in W$. Then $T^{-1}w \in V$. Applying T to this vector, we have $T(T^{-1}w) = (TT^{-1})w = Iw = w$.

For the backwards direction, assume that T is injective and surjective. Given $w \in W$, we define the inverse $S : W \to V$ by

$$Sw = v$$

where v is the unique element in V such that $Tv = w$. We have

$$\begin{aligned} (ST)v &= S(Tv) \\ &= Sw \\ &= v \end{aligned}$$

,

and

$$\begin{aligned} (TS)w &= T(Sw) \\ &= Tv \\ &= w \end{aligned}$$

.

$\square$

Problem Theorem 3.59. Two finite-dimensional vector spaces over F are isomorphic if and only if they have the same dimension.

Proof. Let V and W be finite-dimensional vector spaces over F. First we will show that if two vector spaces over F are isomorphic then they have the same dimension. Assume that V and W are isomorphic. Then choose the invertible linear map $T : V \to W$. Since T is inverible, T is also surjective and injective. By the fundamental theorem of linear maps, we have

$$\begin{aligned} dim(V) &= dim(null(T)) + dim(range(T)) \\ &= dim(range(T)), \end{aligned}$$

since T is injective, resulting in $dim(null(T)) = 0$. Since T is surjective, $range(T) = W$ which means that $dim(V) = dim(W)$.

Next we will show that if the two vector spaces have the same dimension, then they are isomorphic. Assume that $dim(V) = dim(W)$. Choose $v_1, \ldots, v_n$ to be a basis for V and $w_1, \ldots, w_n$ to be a basis for W. We will define our isomorphism $T : V \to W$ by

$$\begin{aligned} T(v) &= T(a_1v_1 + \ldots a_nv_n) \\ &= a_1w_1 + \ldots a_nw_n \end{aligned}$$

We need to first show that T is a linear map. For additivity, we have

$$\begin{aligned} T(u + v) &= T((a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n) \\ &= (a_1 + b_1)w_1 + \ldots (a_n + b_n)w_n \\ &= (a_1w_1 + \cdots + a_nw_n) + (b_1w_1 + \cdots + b_nw_n) \\ &= T(u) + T(v). \end{aligned}$$

For homogeneity, we have

$$\begin{aligned} T(\lambda v) &= T((\lambda a_1)v_1 + \cdots + (\lambda a_n)v_n) \\ &= (\lambda a_1)w_1 + \ldots (\lambda a_n)w_n \\ &= \lambda(a_1w_1 + \cdots + a_nw_n) \\ &= \lambda T(v). \end{aligned}$$

The last part of this proof is to show that T is injective and surjective. Given $u, v \in V$, assume that $T(u) = T(v)$. We have

$$\begin{aligned} 0 &= T(v) - T(u) \\ &= (a_1 - b_1)w_1 + \ldots (a_n - b_n)w_n, \end{aligned}$$

so $a_i = b_i$ for each $i \in \{1, \ldots, n\}$ because we have a basis for W. Finally, we must show surjectivity. Given $w \in W$, we can write it as $w = a_1w_1 + \cdots + a_nw_n$. Thus,

$$T(a_1v_1 + \cdots + a_nv_n) = a_1w_1 + \ldots a_nw_n,$$

as desired. $\square$

Problem Theorem 3.60. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then M is an isomorphism between $\mathcal{L}(V, W)$ and $F^{m,n}$.

This theorem is essentially saying that the set of all linear maps from V to W is the same thing as the set of all linear map encodings, once we fix bases.

Proof. By Theorem 3.36 and Theorem 3.38, M is linear. All that is left to show is that M is injective and surjective.

We will first show injectivity. Assume that $M(T) = M(S)$. Then

$$T(v_j) = S(v_j) = a_{1,j}w_1 + \cdots + a_{m,j}w_m,$$

for $j = 1, \ldots, n$, which is given by how M is constructed. Since linear maps are uniquely determined by where they send a basis list (Theorem 3.5), $T = S$.

Next, we will show surjectivity. Let $M \in F^{m,n}$ be

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}.$$

We will define the linear map $T : V \to W$ by

$$T(v_j) = \sum_{i=1}^{m} a_{i,j}w_i,$$

for each $j = 1, \ldots n$. Then $M(T) = M$ by definition.

$\square$

Problem 3.D.7. Suppose V and W are finite-dimensional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) \mid Tv = 0\}.$$

(a) Show that E is a subspace of $\mathcal{L}(V, W)$.
(b) Supose $v \neq 0$. What is $dim(E)$?

Proof.

Part(a). Clearly, $E \subseteq \mathcal{L}(V, W)$. So we just need to show that E is closed under addition, closed under scalar multiplication, and contains the zero linear map.

We will begin with closure under addition and scalar multiplication. Let $T, S \in E$. We have

$$\begin{aligned}(T + S)v &= Tv + Sv \\ &= 0.\end{aligned}$$

Similarly for scalar multiplication, let $T \in E$ and $\lambda \in F$. We have

$$\begin{aligned}(\lambda T)v &= \lambda Tv \\ &= 0.\end{aligned}$$

Finally, we must show that the zero linear map is in E. This is clear from the fact that the linear map that sends every vector in V to the zero vector in W is in E.

Part(b). Let $F : \mathcal{L}(W, V) \to W$ be defined as

$$F(T) = Tv.$$

It is clear then that $E = null(F)$. By the fundamental theorem of linear algebra, we have

$$\begin{aligned}dim(Null(F)) &= dim(\mathcal{L}(W, V)) - dim(range(F)) \\ &= dim(W)dim(V) - dim(range(F))\end{aligned}$$

By Theorem 3.5, F is surjective, so $dim(range(F)) = dim(W)$. Thus,

$$dim(E) = dim(W)dim(V) - dim(W).$$

$\square$

Problem 3.D.9. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

Proof.

We will prove the forward direction first. Assume that ST is invertible. Choose the inverse $U \in \mathcal{L}(V)$. Composing U and ST we have

$$\begin{aligned} U(ST) &= (US)T \\ &= I. \end{aligned}$$

By problem 3.B.20, T must be injective, so by Theorem 3.69, T must be invertible. It suffices to show that S is surjective by Theorem 3.69. For a contradiction, suppose S is not surjective. Then choose $u \in V$ such that $u \notin range(S)$. Then for all $v \in V$, $S(Tv) \neq u$, but that means ST is not sujrective, which is a contradiction.

We will now prove the backward direction. Assume that S and T are invertible. Composing ST with $T^{-1}S^{-1}$, we have

$$\begin{aligned} (ST)(T^{-1}S^{-1}) &= (S(TT^{-1}))(S^{-1}) \\ &= SS^{-1} \\ &= I. \end{aligned}$$

Similarly, composing $T^{-1}S^{-1}$ with ST, we have

$$\begin{aligned} (T^{-1}S^{-1})(ST) &= (T(SS^{-1}))(T^{-1}) \\ &= TT^{-1} \\ &= I. \end{aligned}$$

We are done.

□

Problem 3.D.10. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that if $ST = I$ then $TS = I$.

For linear operators, the left or right inverse is the inverse.

Proof.

Assume that $ST = I$. By problem 3.B.20, T is injective. T is an operator, so by Theorem 3.69, T is invertible. Composing ST with T^{-1}, we have

$$\begin{aligned}(ST)(T^{-1}) &= S(TT^{-1}) \\ &= S.\end{aligned}$$

We also have

$$\begin{aligned}(ST)(T^{-1}) &= IT^{-1} \\ &= T^{-1}.\end{aligned}$$

So $T^{-1} = S$, which means that $TS = I$.

$\square$

Problem 3.D.16. Suppose that V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of I iff $ST = TS$ for every $S \in \mathcal{L}(V)$.

Proof.

Given $S \in \mathcal{L}(V)$, assume that T is a scalar multiple of I. We can write $T = \lambda I$ for some $\lambda \in F$. Composing S with T, we have

$$\begin{aligned}(ST)v &= S(\lambda I v)\\ &= S(\lambda v)\\ &= \lambda(Sv)\\ &= \lambda I(Sv)\\ &= (TS)v.\end{aligned}$$

Now, assume $ST = TS$ for every $S \in \mathcal{L}(V)$. Choose $v_1, \ldots, v_n$ to be a basis of V. Let $S_{ij} \in \mathcal{L}(V)$ be defined as

$$S_{ij}(a_1 v_1 + \cdots + a_n v_n) = a_i v_j,$$

which is a linear map. For each v_i, let

$$T(v_i) = c_{1i} v_1 + \cdots + c_{ni} v_n.$$

For each v_i, we have

$$\begin{aligned}S_{ii}T(v_i) &= S_{ii}(c_{1i}v_1 + \cdots + c_{ni}v_n)\\ &= c_{ii}v_i\\ &= TS_{ii}(v_i)\\ &= T(v_i)\\ &= c_{1i}v_1 + \cdots + c_{ni}v_n\end{aligned}$$

Subtracting $c_{ii}v_i$ from the final result and using the linear independence of our basis, $c_{nm} = 0$ for all pairs where $n \neq m$.

To finish the proof, we will show that there is a scalar $a = c_{ii}$ for each i. We can define a linear map by what it does to a basis, so define $U \in \mathcal{L}(V)$ by $U(v_i) = v_{i+1}$ for $i < n$ and $U(v_n) = v_1$. For each v_i, we have

$$
\begin{aligned}
UT(v_i) &= U(c_{ii}v_i) \\
&= c_{ii}v_{i+1} \\
&= TU(v_i) \\
&= T(v_{i+1}) \\
&= c_{i+1i+1}v_{i+1},
\end{aligned}
$$

,

so all the scalars are the same, call it a.

Finally, we will show that $T = aI$. We have

$$
\begin{aligned}
T(c_1v_1 + \cdots + c_nv_n) &= c_1T(v_1) + \cdots + c_nT(v_n) \\
&= (c_1a)v_1 + \cdots + (c_na)v_n \\
&= a(c_1v_1 + \cdots + c_nv_n),
\end{aligned}
$$

, as desired.

□