Conditional Distributions

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Conditional distributions are not as a straight forward when considering continuous random variables and mixtures of continuous and discrete random variables. We will go over some intuitive arguments for the definition of conditional distributions under these conditions.

Suppose X, Y are continuous random variables and $f_Y(y) > 0$. To derive the conditional density, we can first derive the conditional cumulative distribution $F_{X|Y=y}$. Through a limit, we can get finer approximation of $F_{X|Y=y}$:

$$\begin{aligned}
F_{X|Y=y} &= \lim_{\epsilon \to 0} P\Big(X \geq x \mid Y \in [y, y+\epsilon)\Big) \\
&= \lim_{\epsilon \to 0} \frac{P\Big(X \geq x \cap Y \in [y, y+\epsilon)\Big)}{P\Big(Y \in [y, y+\epsilon)\Big)} \\
&= \lim_{\epsilon \to 0} \frac{\dfrac{P\Big(X \geq x \cap Y \in [y, y+\epsilon)\Big)}{\epsilon}}{\dfrac{P\Big(Y \in [y, y+\epsilon)\Big)}{\epsilon}}
\end{aligned}$$

The denominator, $\frac{P\left(Y \in [y, y+\epsilon)\right)}{\epsilon}$, tends to $f_Y(y)$ as $\epsilon \to 0$, while the numerator, $\frac{P\left(X \geq x \cap Y \in [y, y+\epsilon)\right)}{\epsilon}$, tends to $\frac{\partial}{\partial y} F_{X,Y}(x, y)$. Hence, we obtain

$$F_{X|Y=y}(x) = \frac{\frac{\partial}{\partial y} F_{X,Y}(x, y)}{f_Y(y)}.$$

The conditional density of is obtained by differentiating $F_{X|Y=y}(x)$ with respect to x:

$$
\begin{aligned}
f_{X|Y=y}(x) &= \frac{\partial}{\partial x}\left(\frac{\frac{\partial}{\partial y}F_{X,Y}(x,y)}{f_Y(y)}\right) \\
&= \frac{1}{f_Y(y)}\,\frac{\partial^2}{\partial x\,\partial y}F_{X,Y}(x,y) \\
&= \frac{f_{X,Y}(x,y)}{f_Y(y)}
\end{aligned}
$$