Selected Problems Chapter 6
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem Inner Product Bilinearity. Let V be a vector space equipped with an inner product $\langle .,. \rangle : V \times V \to \mathbb{R}$. Show that the inner product is bilnear.

Proof. We'll first show additivity in the second slot. We have

$$\begin{aligned} \langle u, v + w \rangle &= \overline{\langle v + w, u \rangle} \\ &= \overline{\langle v, u \rangle + \langle w, u \rangle} \\ &= \langle v, u \rangle + \langle w, u \rangle \\ &= \overline{\langle u, v \rangle} + \overline{\langle u, w \rangle} \\ &= \langle u, v \rangle + \langle u, w \rangle. \end{aligned}$$

For homogenity in the second slot, we have

$$\begin{aligned} \langle u, \lambda v \rangle &= \overline{\langle \lambda v, u \rangle} \\ &= \overline{\lambda \langle v, u \rangle} \\ &= \lambda \langle v, u \rangle \\ &= \lambda \overline{\langle u, v \rangle} \\ &= \lambda \langle u, v \rangle. \end{aligned}$$

□

Problem Example 6.4(a). Show that the function $\langle ., . \rangle : F^n \times F^n \to \mathbb{C}$ define by

$$\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle = w_1\overline{z_1} + \cdots + w_n\overline{z_n}$$

is an inner product.

Proof. **Positivity.** Let $(w_1, \ldots, w_n) \in F^n$. We must show that $w_1\overline{w_1} + \cdots + w_n\overline{w_n}$ is real and non-negative. It suffices to show that $w_k\overline{w_k}$ is real and non-negative for each k. Given $k \in \{1, \ldots, n\}$, choose $a, b \in \mathbb{R}$ such that $w_k = a + bi$. We have $w_k\overline{w_k} = (a + bi)(a - bi) = a^2 + b^2 \geq 0$.

Definiteness. For the forward direction, assume that $\langle w, w \rangle = 0$. We'll show that each $w_k = a + bi = 0$. In the positivity proof, we showed that $w_k\overline{w_k} \geq 0$. Since $\langle w, w \rangle = 0$, each $w_k\overline{w_k} = 0$. We have

$$\begin{aligned} 0 &= w_k\overline{w_k} \\ &= a^2 + b^2 \end{aligned}$$

,so $a = 0$ and $b = 0$. The backward direction is straightforward.

Additivity in first slot. Let $u, v, w \in F^n$. Then,

$$\begin{aligned} \langle u + v, w \rangle &= (u_1 + v_1)\overline{w_n} + \cdots + (u_n + v_n)\overline{w_n} \\ &= (u_1\overline{w_1} + \cdots + u_n\overline{w_n}) + (v_1\overline{w_1} + \cdots + v_n\overline{w_n}) \\ &= \langle u, w \rangle + \langle v, w \rangle. \end{aligned}$$

Homogeneity in first slot. Let $\lambda \in F$ and let $u, v \in F^n$. Then,

$$\begin{aligned} \langle \lambda u, v \rangle &= \lambda u_1\overline{v_1} + \cdots + \lambda u_n\overline{v_n} \\ &= \lambda(u_1\overline{v_1} + \cdots + u_n\overline{v_n}) \\ &= \lambda\langle u, v \rangle. \end{aligned}$$

Conjugate symmetry. Let $u, v \in F^n$. We have

$$\begin{aligned} \overline{\langle v, u \rangle} &= \overline{v_1\overline{u_1} + \cdots + v_n\overline{u_n}} \\ &= \overline{v_1}u_1 + \cdots + \overline{v_n}u_n \\ &= u_1\overline{v_1} + \cdots + u_n\overline{v_n} \\ &= \langle u, v \rangle. \end{aligned}$$

$\square$

Problem Theorem 6.10. Let $v \in V$.

(a). $\|v\| = 0$ if and only if $v = 0$.

(b). $\|\lambda v\| = \lambda \|v\|$ for all $\lambda \in F$.

Proof. **Part (a).** Follows straightforwardly from the fact that $\langle v, v \rangle = 0$ if and only if $v = 0$.

Part (b). Let $\lambda \in F$. We have

$$\begin{aligned} \|\lambda v\|^2 &= \langle \lambda v, \lambda v \rangle \\ &= \lambda \overline{\lambda} \langle v, v \rangle \\ &= |\lambda|^2 \langle v, v \rangle. \end{aligned}$$

Taking the square root gives the desired result. $\square$