Selected Problems Chapter 2
Introduction to Probability for Data Science
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Problem 2.1. A space S and three of its subsets are given by $S = \{1, 3, 5, 7, 9, 11\}$, $A = \{1, 3, 5\}$, $B = \{7, 9, 11\}$, and $C = \{1, 3, 9, 11\}$. Find $A \cap B \cap C$, $A^c \cap B$, $A - C$, and $(A - B) \cup B$.

1. $A \cap B \cap C = \{\}$
2. $A^c \cap B = B$
3. $A - C = \{5\}$
4. $(A - B) \cup B = S$

Problem 2.2. Let $A = (-\infty, r]$ and $B = (-\infty, s]$ where $r \leq s$. Find an expression for $C = (r, s]$ in terms of A and B. Show that $B = A \cup C$, and $A \cap C = \emptyset$.

Problem 2.3. Simplify the following sets.

(a). $[1,4] \cap ([0,2] \cup [3,5])$
(b). $([0,1] \cup [2,3])^c$
(c). $\bigcap_{i=1}^{\infty} (\frac{-1}{n}, \frac{1}{n})$
(d). $\bigcup_{i=1}^{\infty} [5, 8 - \frac{1}{2^n}]$

(a). $[1,2] \cup [3,4]$
(b). $(-\infty, 0) \cup (1,2) \cup (3, \infty)$
(c). $\{0\}$
(d). $[5,8)$

Problem Theorem 2.5 Part 2. Prove that $(A \cup B)^c = A^c \cap B^c$.

Proof. We'll prove this with a series of equivalences:

$$\begin{aligned} x \in (A \cup B)^c &\iff x \notin A \cup B \\ &\iff x \notin A \text{ and } x \notin B \\ &\iff x \in A^c \text{ and } x \in B^c \\ &\iff x \in A^c \cap B^c \\ &\iff x \in A^c \cap B^c \end{aligned}$$

□

Problem Corollary 2.1 (a). Prove that $P(A^c) = 1 - P(A)$.

Proof.

$$\begin{aligned} 1 &= P(\Omega) \\ &= P(A^c \cup A) \\ &= P(A^c) + P(A). \end{aligned}$$

Subtracting $P(A)$ from both sides gives the desired result. □

Problem Corollary 2.1 (b). Prove that $P(A) \leq 1$.

Proof. By corollary 2.1.(a), we have $P(A) = 1 - P(A^c)$. Since $P(A^c) \geq 0$, it is clear that $P(A) \leq 1$.

□

Problem Corollary 2.1 (c). Prove that $P(\emptyset) = 0$.

Proof.

$$\begin{aligned} P(\emptyset) &= P(\emptyset \cup \emptyset) \\ &= P(\emptyset) + P(\emptyset). \end{aligned}$$

Subtracting $P(\emptyset)$ from both sides gives the desired result.

□

Problem Corollary 2.3 (a). Prove that $P(A \cup B) \leq P(A) + P(B)$.

Proof. By Corollary 2.2, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Since $P(A \cap B) \geq 0$, the desired result is clear.

$\square$

Problem Corollary 2.3 (b). Prove that if $A \subseteq B$, $P(A) \leq P(B)$.

Proof. Assume that $A \subseteq B$. We have

$$\begin{aligned} P(A) &\leq P(A) + P(B - A) \\ &= P(A \cup (B - A)) \\ &= P(B), \end{aligned}$$

as desired.

$\square$