

# Part III: Continuous Random Variables

## Introduction to Probability for Computing

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**Problem Theorem 9.3.** Let  $X \sim \mathcal{N}(\mu, \sigma^2)$ . Then  $\text{Var}(X) = \sigma^2$ .

*Proof.* Assume that  $X \sim \mathcal{N}(\mu, \sigma^2)$ . Using the definition of  $\text{Var}(X)$ , we have

$$\begin{aligned}\text{Var}(X) &= E[(X - \mu)^2] \\ &= \int_{\mathbb{R}} (x - \mu)^2 f(x) dx \\ &= \int_{-\infty}^{\infty} (x - \mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx\end{aligned}$$

Let  $z = (x - \mu)$  for a substitution. Thus,

$$\begin{aligned}&= \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} dz \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2\sigma^2}} dz \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2\sigma^2}} dz\end{aligned}$$

We can use symmetry of the integrand to change the bounds:

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-\frac{z^2}{2\sigma^2}} dz$$

Let  $y = \frac{z^2}{2\sigma^2}$ , so  $dz = \frac{\sigma^2}{z} dy$ . Hence,

$$\begin{aligned}
&= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty 2\sigma^2 y e^{-y} \frac{\sigma^2}{z} dy \\
&= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty \sqrt{y} \sqrt{2}\sigma^3 e^{-y} dy \\
&= \frac{2\sqrt{2}\sigma^3}{\sigma\sqrt{2\pi}} \int_0^\infty \sqrt{y} e^{-y} dy \\
&= \frac{2\sqrt{2}\sigma^3}{\sigma\sqrt{2\pi}} \int_0^\infty \sqrt{y} e^{-y} dy \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty \sqrt{y} e^{-y} dy
\end{aligned}$$

The integral  $\int_0^\infty y^{1/2} e^{-y} dy$  is a standard Gamma function, which simplifies to  $\Gamma\left(\frac{3}{2}\right)$ . Thus,

$$\begin{aligned}
&= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \\
&= \sigma^2,
\end{aligned}$$

as desired. □

**Problem Theorem 9.3.** Let  $X \sim \mathcal{N}(\mu, \sigma^2)$ . Then  $E(X) = \mu$ .

*Proof.* Assume that  $X \sim \mathcal{N}(\mu, \sigma^2)$ . Using the definition of  $E(X)$ , we have

$$\begin{aligned} E(X) &= \int_{\mathbb{R}} x f(x) dx \\ &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \end{aligned}$$

Let  $y = \frac{x-\mu}{\sigma}$ , so  $dx = \sigma dy$ . Substituting,

$$\begin{aligned} &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma y + \mu) e^{-\frac{y^2}{2}} \sigma dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma y + \mu) e^{-\frac{y^2}{2}} \sigma dy \\ &= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2}} dy + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \end{aligned}$$

The integral  $\frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2}} dy$  evaluates to 0 since  $y$  is an odd function. Hence,

$$\begin{aligned} &= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \\ &= \frac{\mu\sqrt{2\pi}}{\sqrt{2\pi}} \\ &= \mu \end{aligned}$$

□

**Lemma 1.** *The following lemma will be used in the proof of Theorem 9.5.*

Let  $X \sim \mathcal{N}(\mu, \sigma^2)$ , and let  $Y = aX + b$ , where  $a > 0$  and  $b \in \mathbb{R}$ . Let  $F_X(x)$  denote the cumulative distribution function of  $X$ . Then, the cumulative distribution function of  $Y$ , denoted  $F_Y(y)$ , satisfies:

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right).$$

*Proof.*

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(aX + b \leq y) \\ &= P(X \leq \frac{y-b}{a}) \\ &= F_X\left(\frac{y-b}{a}\right) \end{aligned}$$

□

**Problem Theorem 9.5 (Linear Transformation Property).** Let  $X \sim \mathcal{N}(\mu, \sigma^2)$ , and let  $Y = aX + b$ , where  $a > 0$  and  $b \in \mathbb{R}$ . Then,  $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ .

*Proof.* Using Lemma 1,  $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$ . By the fundamental theorem of calculus, differentiating  $F_Y(y)$  yields the probability density function for  $Y$ . Hence,

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) \\ &= \frac{d}{dy} \left(\frac{y-b}{a}\right) f_X\left(\frac{y-b}{a}\right) \\ &= \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \\ &= \frac{1}{a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-(a\mu+b))^2}{2a^2\sigma^2}} \\ &= \frac{1}{a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}} \\ &= \frac{1}{\sqrt{2\pi a^2\sigma^2}} e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}} \\ &= \frac{1}{\sqrt{2\pi a^2\sigma^2}} e^{-\frac{(y-b-a\mu)^2}{2\sigma^2}} \\ &= \frac{1}{\sqrt{2\pi a^2\sigma^2}} e^{-\frac{(y-(b+a\mu))^2}{2\sigma^2}} \end{aligned}$$

By the definition of the normal distribution,  $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ , as desired.

□