Selected Problems Chapter 3
Linear Algebra Done Wrong, Sergei Treil, 1st Edition

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Problem Uniqueness of Determinant. Let $C \in \mathbb{R}^n$ be a column vector, i.e. $C = (c_i)_{i=1,\dots,n}$.

Show that if $D : (\mathbb{R}^n)^n \to \mathbb{R}$ satisfies

multi-linearity. linearity in each argument

anti-symmetry. switching arguments induces a sign change

normalization. $D(e_1, \dots, e_n) = 1$

then

$$D(C_1, \dots, C_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} c_{\sigma(i),i}$$

Proof. Let $D : (\mathbb{R}^n)^n \to \mathbb{R}$ be a function satisfying the three conditions. For each index j, we have $C_j = \sum_i^n c_{ij} e_i$. Repeatedly applying the multi-linear property and normalization, we have

$$\begin{aligned}
D(C_1, \dots, C_n) &= D(\sum_{i_1}^{n} c_{i_1 1} e_i, \dots, \sum_{i_n}^{n} c_{i_n n} e_n) \\
&= \sum_{i_1}^{n} c_{i_1 1} D(e_1, \dots, \sum_{i_n} c_{i_n n} e_n) \\
&= \dots \\
&= \sum_{i_1}^{n} \cdots \sum_{i_n}^{n} \prod_{k=1}^{n} c_{i_k k} D(e_1, \dots, e_n) \\
&= \sum_{i_1}^{n} \cdots \sum_{i_n}^{n} \prod_{k=1}^{n} c_{i_k k}
\end{aligned}$$

The iterated sum goes through all possible values of $i_1, \dots, i_n$, so we can write

$$= \sum_{i_1,\ldots,i_n} \prod_{k=1}^{n} c_{i_k k}$$

$\square$