Selected Problems Chapter 3
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem Integration. Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ by

$$Tp = \int_0^1 p(x)dx.$$

Show that T is a linear map.

Proof.

Additivity

Given $p, q \in \mathcal{P}(\mathbb{R})$, we want the additivity propriety to hold for T. Applying T to the sum of p and q, we have

$$\begin{aligned} T(p+q) &= \int_0^1 p(x) + q(x)dx \\ &= \int_0^1 p(x)dx + \int_0^1 q(x)dx \\ &= T(p) + T(q), \end{aligned}$$

since integration of a sum is equal to the sum of the integrated parts.

Homogeneity

Given $p \in \mathcal{P}(\mathbb{R})$ and $a \in F$, we want the homogeneity property to hold. Applying T to the scalar multiple of p, we have

$$\begin{aligned} T(a * p) &= \int_0^1 a * p(x)dx \\ &= a * \int_0^1 p(x)dx \\ &= a * T(p), \end{aligned}$$

since constants can be separated in integration.

□

Problem Theorem 3.5. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m \in W$. Show that there exists a unique linear map $T : V \to W$ such that

$$T(v_j) = w_j$$

for each $j = 1, \ldots, n$.

Proof. We must first show the existence of a linear map with the desired properties. Define $T : V \to W$ by

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_n w_n$$

, where $a_1, \ldots, a_n$ are coefficients in F.

Assume the existence of another linear map $T' : V \to W$ with the property

$$T'(v_j) = w_j$$

for each $j = 1, \ldots, n$. To show uniqueness, we want that $T(v) = T'(v)$ for all $v \in V$.

Given $v \in V$, we can write $v = a_1 v_1 + \cdots + a_n v_n$, since we have a basis for V. Then,

$$\begin{aligned} T(v) &= T(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 T(v_1) + \cdots + a_n T(v_n) \\ &= a_1 T'(v_1) + \cdots + a_n T'(v_n) \\ &= T'(a_1 v_1 + \cdots + a_n v_n) \\ &= T'(v), \end{aligned}$$

since $T(v_i) = w_i = T'(v_i)$ for each $j = 1, \ldots, n$.

$\square$