

Selected Problems Chapter 2
Introduction to Probability for Data Science
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Problem 2.1. A space S and three of its subsets are given by $S = \{1, 3, 5, 7, 9, 11\}$, $A = \{1, 3, 5\}$, $B = \{7, 9, 11\}$, and $C = \{1, 3, 9, 11\}$. Find $A \cap B \cap C$, $A^c \cap B$, $A - C$, and $(A - B) \cup B$.

1. $A \cap B \cap C = \{\}$
2. $A^c \cap B = B$
3. $A - C = \{5\}$
4. $(A - B) \cup B = S$

Problem 2.2. Let $A = (-\infty, r]$ and $B = (-\infty, s]$ where $r \leq s$. Find an expression for $C = (r, s]$ in terms of A and B . Show that $B = A \cup C$, and $A \cap C = \emptyset$.

Problem 2.3. Simplify the following sets.

(a). $[1, 4] \cap ([0, 2] \cup [3, 5])$

(b). $([0, 1] \cup [2, 3])^c$

(c). $\bigcap_{i=1}^{\infty} (\frac{-1}{n}, \frac{1}{n})$

(d). $\bigcup_{i=1}^{\infty} [5, 8 - \frac{1}{2^n}]$

(a). $[1, 2] \cup [3, 4]$

(b). $(-\infty, 0) \cup (1, 2) \cup (3, \infty)$

(c). $\{0\}$

(d). $[5, 8)$

Problem Theorem 2.5 Part 2. Prove that $(A \cup B)^c = A^c \cap B^c$.

Proof. We'll prove this with a series of equivalences:

$$\begin{aligned}x \in (A \cup B)^c &\iff x \notin A \cup B \\&\iff x \notin A \text{ and } x \notin B \\&\iff x \in A^c \text{ and } x \in B^c \\&\iff x \in A^c \cap B^c \\&\iff x \in A^c \cap B^c\end{aligned}$$

□

Problem Corollary 2.1 (a). Prove that $P(A^c) = 1 - P(A)$.

Proof.

$$\begin{aligned}1 &= P(\Omega) \\&= P(A^c \cup A) \\&= P(A^c) + P(A).\end{aligned}$$

Subtracting $P(A)$ from both sides gives the desired result.

□

Problem Corollary 2.1 (b). Prove that $P(A) \leq 1$.

Proof. By corollary 2.1.(a), we have $P(A) = 1 - P(A^c)$. Since $P(A^c) \geq 0$, it is clear that $P(A) \leq 1$.

□

Problem Corollary 2.1 (c). Prove that $P(\emptyset) = 0$.

Proof.

$$\begin{aligned}P(\emptyset) &= P(\emptyset \cup \emptyset) \\&= P(\emptyset) + P(\emptyset).\end{aligned}$$

Subtracting $P(\emptyset)$ from both sides gives the desired result.

□

Problem Corollary 2.3 (a). Prove that $P(A \cup B) \leq P(A) + P(B)$.

Proof. By Corollary 2.2, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Since $P(A \cap B) \geq 0$, the desired result is clear. □

Problem Corollary 2.3 (b). Prove that if $A \subseteq B$, $P(A) \leq P(B)$.

Proof. Assume that $A \subseteq B$. We have

$$\begin{aligned} P(A) &\leq P(A) + P(B - A) \\ &= P(A \cup (B - A)) \\ &= P(B), \end{aligned}$$

as desired. □

Problem 2.8 . Consider an experiment consisting of rolling a die twice. The outcome of this experiment is an ordered pair whose first element is the first value rolled and whose second element is the second value rolled.

- (a) Find the sample space.
- (b) Find the set A representing the event that the value on the first roll is greater than or equal to the value on the second roll.
- (c) Find the set B corresponding to the event that the first roll is a six.
- (d) Let C correspond to the event that the first valued rolled and the second value rolled differ by two. Find $A \cap C$.

Part (a). $\Omega = \{(x, y) \mid x \in \{1, 2, 3, 4, 5, 6\} \text{ and } y \in \{1, 2, 3, 4, 5, 6\}\}$

Part (b). $A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (3, 2), (4, 2), (5, 2),$

Part (c). $B = \{(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

Part (d). $A \cap C = \{(3, 1), (4, 2), (5, 3), (6, 4)\}$

Problem 2.6 . A space S is defined as $S = \{1, 3, 5, 7, 9, 11\}$, and three subsets as $A = \{1, 3, 5\}$, $B = \{7, 9, 11\}$, and $C = \{1, 3, 9, 11\}$. Assume that each element has probability $1/6$. Find the following probabilities:

- (a) $P(A)$
- (b) $P(B)$
- (c) $P(C)$
- (d) $P(A \cup B)$
- (e) $P(A \cup C)$
- (f) $P((A \setminus B) \cup B)$

Part (a).

$$\begin{aligned} P(A) &= P(\{1\}) + P(\{3\}) + P(\{5\}) \\ &= \frac{3}{6} \end{aligned}$$

Part (b).

$$\begin{aligned} P(B) &= P(\{7\}) + P(\{9\}) + P(\{11\}) \\ &= \frac{3}{6} \end{aligned}$$

Part (c).

$$\begin{aligned} P(C) &= P(\{1\}) + P(\{3\}) + P(\{9\}) + P(\{11\}) \\ &= \frac{4}{6} \end{aligned}$$

Part (d).

$$\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= \frac{3}{6} + \frac{3}{6} - 0 \\
&= 1
\end{aligned}$$

Part (e).

$$\begin{aligned}
P(A \cup C) &= P(A) + P(C) - P(A \cap C) \\
&= \frac{3}{6} + \frac{4}{6} - \frac{2}{6} \\
&= \frac{5}{6}
\end{aligned}$$

Part (f).

Since $(A \setminus B) \cup B = A \cup B$, we have

$$\begin{aligned}
P((A \setminus B) \cup B) &= P(A \cup B) \\
&= 1
\end{aligned}$$

Problem 2.13 . Let the events A and B have $P(A) = x$, $P(B) = y$ and $P(A \cup B) = z$. Find the following probabilities: (a) $P(A \cap B)$, (b) $P(A^c \cap B^c)$, (c) $P(A^c \cup B^c)$, (d) $P(A \cap B^c)$ and (e) $P(A^c \cup B)$.

Part (a).

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= x + y - z \end{aligned}$$

Part (b).

$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) \\ &= 1 - z \end{aligned}$$

Part (c).

$$\begin{aligned} P(A^c \cup B^c) &= P((A \cap B)^c) \\ &= 1 - x - y + z \end{aligned}$$

Part (d). Since $A \cap B^c = A \setminus B$, we have

$$\begin{aligned} P(A \cap B^c) &= P(A \setminus B) \\ &= P(A) - P(A \cap B) \\ &= -y + z \end{aligned}$$

Part (e).

$$\begin{aligned} P(A^c \cup B) &= P((A \cap B^c)^c) \\ &= 1 - P(A \cap B^c) \\ &= 1 + y - z \end{aligned}$$

Problem 2.12. A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x \mid x < 0\}$, $B = \{x \mid |x - .5| < .5\}$ and $C = \{x \mid x > 0.75\}$. Find (a) $P(A \mid B)$, (b) $P(B \mid C)$, (c) $P(A \mid C^c)$ and (d) $P(B \mid C^c)$.

Assume that the measure P uses the weighting function $f(x) = \frac{1}{3}$.

Part (a). $A \cap B = \emptyset$, so $P(A \cap B) = 0$. By the definition of conditional probability, $P(A \mid B) = 0$.

Part (b). $B \cap C = (.75, 1]$. $P(B \cap C) = \int_{.75}^1 \frac{1}{3} dx = \frac{1}{12}$, and $P(C) = \int_{.75}^2 \frac{1}{3} dx = \frac{5}{12}$. Calculating the conditional probability, we have

$$\begin{aligned} P(B \mid C) &= \frac{P(B \cap C)}{P(C)} \\ &= \frac{1}{5} \end{aligned}$$

Part (c). $C^c = [-1, .75]$, so $A \cap C^c = A$. $P(C^c) = \int_{-1}^{.75} \frac{1}{3} dx = \frac{7}{12}$, and $P(A) = \int_{-1}^0 \frac{1}{3} dx = \frac{1}{3}$. Thus,

$$\begin{aligned} P(A \mid C^c) &= \frac{P(A \cap C^c)}{P(C^c)} \\ &= \frac{P(A)}{P(C^c)} \\ &= \frac{4}{7} \end{aligned}$$

Part (d). $B \cap C^c = [0, .75]$, so $P(B \cap C^c) = \int_0^{.75} \frac{1}{3} dx = \frac{1}{4}$. We calculated $P(C^c)$ in part (c). We have

$$\begin{aligned} P(B \mid C^c) &= \frac{P(B \cap C^c)}{P(C^c)} \\ &= \frac{3}{7} \end{aligned}$$