Part III: Continuous Random Variables
Introduction to Probability for Computing

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Problem Theorem 9.3. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then $Var(X) = \sigma^2$.

Proof. Assume that $X \sim \mathcal{N}(\mu, \sigma^2)$. Using the definition of $Var(X)$, we have

$$\begin{aligned}
Var(X) &= E[(X-\mu)^2] \\
&= \int_{\mathbb{R}} (x-\mu)^2 f(x)\, dx \\
&= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\ dx
\end{aligned}$$

Let $z = (x-\mu)$ for a substitution. Thus,

$$\begin{aligned}
&= \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}\ dz \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2\sigma^2}}\ dz \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2\sigma^2}}\ dz
\end{aligned}$$

We can use symmetry of the integrand to change the bounds:

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-\frac{z^2}{2\sigma^2}}\ dz$$

Let $y = \frac{z^2}{2\sigma^2}$, so $dz = \frac{\sigma^2}{z} dy$. Hence,

$$
\begin{aligned}
&= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty 2\sigma^2 y e^{-y} \frac{\sigma^2}{z} \; dy \\
&= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty \sqrt{y}\sqrt{2}\sigma^3 e^{-y} \; dy \\
&= \frac{2\sqrt{2}\sigma^3}{\sigma\sqrt{2\pi}} \int_0^\infty \sqrt{y} e^{-y} \; dy \\
&= \frac{2\sqrt{2}\sigma^3}{\sigma\sqrt{2\pi}} \int_0^\infty \sqrt{y} e^{-y} \; dy \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty \sqrt{y} e^{-y} \; dy
\end{aligned}
$$

The integral $\int_0^\infty y^{1/2} e^{-y} \, dy$ is a standard Gamma function, which simplifies to $\Gamma\left(\frac{3}{2}\right)$. Thus,

$$
\begin{aligned}
&= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma(\frac{3}{2}) \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \\
&= \sigma^2,
\end{aligned}
$$

as desired.

$\square$