Selected Problems Chapter 1
Real Mathematical Analysis, Pugh, Second Edition

Mustaf Ahmed

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Problem 8 Statement. Suppose that the natural number k is not a perfect n^{th}.

(a) Prove that its n^{th} root is irrational

(b) Infer that the n^{th} root of a natural number is either a natural number or it is irrational. It is never a fraction.

Problem 8 (a).

Proof. We will prove this by contradiction. Suppose the n^{th} root of k is rational. Choose $p, q \in \mathbb{Z}$, where $q \neq 0$, such that the n^{th} root $r = \frac{p}{q}$. Then $k = r^n = \frac{p^n}{q^n}$. Since k is an integer, q must divide p. This r is an integer, and therefore k is a perfect n^{th} root, a contradiction. $\square$

Problem 8 (b).

A natural number is either a perfect n^{th} root or it is not. If it is not a perfect n^{th} root, By (a), we know the n^{th} root must be irrational. If it is a perfect n^{th} root, by definition the n^{th} root must be a an integer.

Problem 12 Statement. Prove that there exists no smallest positive real number. Does there exists a smallest positive rational number? Given a real number x, does there exist a smallest real number $y > x$?

Proof. We the first part by contradiction. Suppose there did exist a smallest positive real number $x \in \mathbb{R}$. Consider $y = \frac{x}{2}$. Clearly $y < x$ .Similarly, given $x \in \mathbb{Q}$ we can define $y = x - 1$, another rational number, and it is clear that $y = x - 1 < x$.

We will prove the final part by contradiction. Suppose there did exist a smallest number $y > x$. We will construct an even smaller real number that satisfies this property. Let $z = x + \frac{|y-x|}{2}$. We have

$$\begin{aligned} z &= x + \frac{|y-x|}{2} \\ &< x + |y-x| \\ &= y \end{aligned}$$

$\square$

Problem 13 Statement. Let $b = \text{l.u.b } S$, where S is a bounded nonempty subset of $\mathbb{R}$.
(a) Given $\epsilon > 0$ show that there exists an $s \in S$ with

$$b - \epsilon \leq s \leq b.$$

(b) Can $s \in S$ always be found so that $b - \epsilon < s < b$.

Problem 13 (a).

Proof. Since $b = \text{Sup } S$, it must be the case that there exist some $s \in S$ satisfying this property, otherwise b would no longer be the Sup for S. $\square$

Problem 13 (b).

Proof. No. Consider $S = \{1, 3\}$ and let $\epsilon = 1$. There does not exist $s \in S$ where

$$3 - 1 < s < 3.$$

$\square$

Problem 20 Statement. Prove that limits are unique, i.e., if (a_n) is a sequence of real numbers that converges to a real number b and also converges to a real number $b^{'}$, then $b = b^{'}$.

Problem 20.

Proof. Assume that (a_n) converges to b and $b^{'}$ in $\mathbb{R}$. By the epsilon principle, we want :

$$\forall \epsilon > 0 \ , \ |b - b^{'}| < \epsilon.$$

Given $\epsilon > 0$ in $\mathbb{R}$, choose $N, N^{'} \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, \text{ if } n \geq N, \ |a_n - b| < \frac{\epsilon}{2},$$

and

$$\forall n \in \mathbb{N}^{'}, \text{ if } n \geq N^{'}, \ |a_n - b^{'}| < \frac{\epsilon}{2}.$$

Let $M = max(N, N^{'})$, and choose $n \in \mathbb{N}$ such that $n \geq M$. Since $n \geq N, N^{'}$, we have

$$|a_n - b^{'}| < \frac{\epsilon}{2},$$

and

$$|a_n - b| < \frac{\epsilon}{2}.$$

Thus, we have

$$-\epsilon < b - a_n + a_n - b^{'} < \epsilon,$$

which means that

$$|b - b^{'}| < \epsilon.$$

$\square$