Part III: Continuous Random Variables
Introduction to Probability for Computing

Mustaf Ahmed

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Problem Theorem 9.3. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then $Var(X) = \sigma^2$.

Proof. Assume that $X \sim \mathcal{N}(\mu, \sigma^2)$. Using the definition of $Var(X)$, we have

$$\begin{aligned} Var(X) &= E[(X-\mu)^2] \\ &= \int_{\mathbb{R}} (x-\mu)^2 f(x)\, dx \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\ dx \end{aligned}$$

Let $z = (x - \mu)$ for a substitution. Thus,

$$\begin{aligned} &= \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}\ dz \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2\sigma^2}}\ dz \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2\sigma^2}}\ dz \end{aligned}$$

We can use symmetry of the integrand to change the bounds:

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-\frac{z^2}{2\sigma^2}}\ dz$$

Let $y = \frac{z^2}{2\sigma^2}$, so $dz = \frac{\sigma^2}{z} dy$. Hence,

$$
\begin{aligned}
&= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty 2\sigma^2 y e^{-y} \frac{\sigma^2}{z} \; dy \\
&= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty \sqrt{y}\sqrt{2}\sigma^3 e^{-y} \; dy \\
&= \frac{2\sqrt{2}\sigma^3}{\sigma\sqrt{2\pi}} \int_0^\infty \sqrt{y} e^{-y} \; dy \\
&= \frac{2\sqrt{2}\sigma^3}{\sigma\sqrt{2\pi}} \int_0^\infty \sqrt{y} e^{-y} \; dy \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty \sqrt{y} e^{-y} \; dy
\end{aligned}
$$

The integral $\int_0^\infty y^{1/2} e^{-y}\, dy$ is a standard Gamma function, which simplifies to $\Gamma\left(\frac{3}{2}\right)$. Thus,

$$
\begin{aligned}
&= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma(\frac{3}{2}) \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \\
&= \sigma^2,
\end{aligned}
$$

as desired.

$\square$

Problem Theorem 9.3. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then $E(X) = \mu$.

Proof. Assume that $X \sim \mathcal{N}(\mu, \sigma^2)$. Using the definition of $E(X)$, we have

$$\begin{aligned} E(X) &= \int_{\mathbb{R}} x f(x)\, dx \\ &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\ dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\ dx \end{aligned}$$

Let $y = \frac{x-\mu}{\sigma}$, so $dx = \sigma dy$. Substituting,

$$\begin{aligned} &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma y + \mu) e^{-\frac{y^2}{2}} \sigma\ dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma y + \mu) e^{-\frac{y^2}{2}} \sigma\ dy \\ &= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2}}\ dy + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\ dy \end{aligned}$$

The integral $\frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2}}\ dy$ evaluates to 0 since y is an odd function. Hence,

$$\begin{aligned} &= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\ dy \\ &= \frac{\mu\sqrt{2\pi}}{\sqrt{2\pi}} \\ &= \mu \end{aligned}$$

$\square$