Selected Problems Chapter 2
Introduction to Probability for Data Science
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Problem 2.1. A space S and three of its subsets are given by $S = \{1, 3, 5, 7, 9, 11\}$, $A = \{1, 3, 5\}$, $B = \{7, 9, 11\}$, and $C = \{1, 3, 9, 11\}$. Find $A \cap B \cap C$, $A^c \cap B$, $A - C$, and $(A - B) \cup B$.

1. $A \cap B \cap C = \{\}$
2. $A^c \cap B = B$
3. $A - C = \{5\}$
4. $(A - B) \cup B = S$

Problem 2.2. Let $A = (-\infty, r]$ and $B = (-\infty, s]$ where $r \leq s$. Find an expression for $C = (r, s]$ in terms of A and B. Show that $B = A \cup C$, and $A \cap C = \emptyset$.

Problem 2.3. Simplify the following sets.

(a). $[1,4] \cap ([0,2] \cup [3,5])$
(b). $([0,1] \cup [2,3])^c$
(c). $\bigcap_{i=1}^{\infty} (\frac{-1}{n}, \frac{1}{n})$
(d). $\bigcup_{i=1}^{\infty} [5, 8 - \frac{1}{2^n}]$

(a). $[1,2] \cup [3,4]$
(b). $(-\infty, 0) \cup (1,2) \cup (3, \infty)$
(c).$\{0\}$
(d).$[5,8)$

Problem Theorem 2.5 Part 2. Prove that $(A \cup B)^c = A^c \cap B^c$.

Proof. We'll prove this with a series of equivalences:

$$\begin{aligned} x \in (A \cup B)^c &\iff x \notin A \cup B \\ &\iff x \notin A \text{ and } x \notin B \\ &\iff x \in A^c \text{ and } x \in B^c \\ &\iff x \in A^c \cap B^c \\ &\iff x \in A^c \cap B^c \end{aligned}$$

□

Problem Corollary 2.1 (a). Prove that $P(A^c) = 1 - P(A)$.

Proof.

$$\begin{aligned} 1 &= P(\Omega) \\ &= P(A^c \cup A) \\ &= P(A^c) + P(A). \end{aligned}$$

Subtracting $P(A)$ from both sides gives the desired result. □

Problem Corollary 2.1 (b). Prove that $P(A) \leq 1$.

Proof. By corollary 2.1.(a), we have $P(A) = 1 - P(A^c)$. Since $P(A^c) \geq 0$, it is clear that $P(A) \leq 1$.

□

Problem Corollary 2.1 (c). Prove that $P(\emptyset) = 0$.

Proof.

$$\begin{aligned} P(\emptyset) &= P(\emptyset \cup \emptyset) \\ &= P(\emptyset) + P(\emptyset). \end{aligned}$$

Subtracting $P(\emptyset)$ from both sides gives the desired result.

□

Problem Corollary 2.3 (a). Prove that $P(A \cup B) \leq P(A) + P(B)$.

Proof. By Corollary 2.2, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Since $P(A \cap B) \geq 0$, the desired result is clear.

$\square$

Problem Corollary 2.3 (b). Prove that if $A \subseteq B$, $P(A) \leq P(B)$.

Proof. Assume that $A \subseteq B$. We have

$$\begin{aligned} P(A) &\leq P(A) + P(B - A) \\ &= P(A \cup (B - A)) \\ &= P(B), \end{aligned}$$

as desired.

$\square$

Problem 2.8 . Consider an experiment consisting of rolling a die twice. The outcome of this experiment is an ordered pair whose first element is the first value rolled and whose second element is the second value rolled.

(a) Find the sample space.

(b) Find the set A representing the event that the value on the first roll is greater than or equal to the value on the second roll.

(c) Find the set B corresponding to the event that the first roll is a six.

(d) Let C correspond to the event that the first valued rolled and the second value rolled differ by two. Find $A \cap C$.

Part (a). $\Omega = \{(x, y) \mid x \in \{1, 2, 3, 4, 5, 6\}$ and $y \in \{1, 2, 3, 4, 5, 6\}\}$

Part (b). $A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (3, 2), (4, 2), (5, 2),$

Part (c). $B = \{(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

Part (d).$A \cap C = \{(3, 1), (4, 2), (5, 3), (6, 4)\}$

Problem 2.6 . A space S is defined as $S = \{1, 3, 5, 7, 9, 11\}$, and three subsets as $A = \{1, 3, 5\}$, $B = \{7, 9, 11\}$, and $C = \{1, 3, 9, 11\}$. Assume that each element has probability 1/6. Find the following probabilities:

(a) $P(A)$

(b) $P(B)$

(c) $P(C)$

(d) $P(A \cup B)$

(e) $P(A \cup C)$

(f) $P((A \setminus B) \cup B)$

Part (a).

$$\begin{aligned} P(A) &= P(\{1\}) + P(\{3\}) + P(\{5\}) \\ &= \frac{3}{6} \end{aligned}$$

Part (b).

$$\begin{aligned} P(B) &= P(\{7\}) + P(\{9\}) + P(\{11\}) \\ &= \frac{3}{6} \end{aligned}$$

Part (c).

$$\begin{aligned} P(C) &= P(\{1\}) + P(\{3\}) + P(\{9\}) + P(\{11\}) \\ &= \frac{4}{6} \end{aligned}$$

Part (d).

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= \frac{3}{6} + \frac{3}{6} - 0 \\
&= 1
\end{aligned}
$$

Part (e).

$$
\begin{aligned}
P(A \cup C) &= P(A) + P(C) - P(A \cap C) \\
&= \frac{3}{6} + \frac{4}{6} - \frac{2}{6} \\
&= \frac{5}{6}
\end{aligned}
$$

Part (f).
Since $(A \setminus B) \cup B = A \cup B$, we have

$$
\begin{aligned}
P((A \setminus B) \cup B) &= P(A \cup B) \\
&= 1
\end{aligned}
$$

Problem 2.13 . Let the events A and B have $P(A) = x$, $P(B) = y$ and $P(A \cup B) = z$. Find the following probabilities: (a)$P(A \cap B)$, (b)$P(A^c \cap B^c)$, (c) $P(A^c \cup B^c)$, (d) $P(A \cap B^c)$ and (e) $P(A^c \cup B)$.

Part (a).

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= x + y - z \end{aligned}$$

Part (b).

$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) \\ &= 1 - z \end{aligned}$$

Part (c).

$$\begin{aligned} P(A^c \cup B^c) &= P((A \cap B)^c) \\ &= 1 - x - y + z \end{aligned}$$

Part (d). Since $A \cap B^c = A \setminus B$, we have

$$\begin{aligned} P(A \cap B^c) &= P(A \setminus B) \\ &= P(A) - P(A \cap B) \\ &= -y + z \end{aligned}$$

Part (e).

$$\begin{aligned} P(A^c \cup B) &= P((A \cap B^c)^c) \\ &= 1 - P(A \cap B^c) \\ &= 1 + y - z \end{aligned}$$

Problem 2.12. A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x \mid x < 0\}$, $B = \{x \mid |x - .5| < .5\}$ and $C = \{x \mid x > 0.75\}$. Find (a) $P(A \mid B)$, (b) $P(B \mid C)$, (c) $P(A \mid C^c)$ and (d) $P(B \mid C^c)$.

Assume that the measure P uses the weighting function $f(x) = \frac{1}{3}$.

Part (a). $A \cap B = \emptyset$, so $P(A \cap B) = 0$. By the definition of conditional probability, $P(A \mid B) = 0$.

Part (b). $B \cap C = (.75, 1]$. $P(B \cap C) = \int_{.75}^{1} \frac{1}{3}\, dx = \frac{1}{12}$, and $P(C) = \int_{.75}^{2} \frac{1}{3}\, dx = \frac{5}{12}$. Calculating the conditional probability, we have

$$\begin{aligned} P(B \mid C) &= \frac{P(B \cap C)}{P(C)} \\ &= \frac{1}{5} \end{aligned}$$

Part (c). $C^c = [-1, .75]$, so $A \cap C^c = A$. $P(C^c) = \int_{-1}^{.75} \frac{1}{3}\, dx = \frac{7}{12}$, and $P(A) = \int_{-1}^{0} \frac{1}{3}\, dx = \frac{1}{3}$. Thus,

$$\begin{aligned} P(A \mid C^c) &= \frac{P(A \cap C^c)}{P(C^c)} \\ &= \frac{P(A)}{P(C^c)} \\ &= \frac{4}{7} \end{aligned}$$

Part (d). $B \cap C^c = [0, .75]$, so $P(B \cap C^c) = \int_{0}^{.75} \frac{1}{3}\, dx = \frac{1}{4}$. We calculated $P(C^c)$ in part (c). We have

$$\begin{aligned} P(B \mid C^c) &= \frac{P(B \cap C^c)}{P(C^c)} \\ &= \frac{3}{7} \end{aligned}$$

Problem 2.18. A block of information is transmitted repeated over a noisy channel until an error-free block is received. Let $M \geq 1$ be the number of blocks required for a transmission. Define the following sets.

(i) $A = \{\text{M is even}\}$

(ii)$B = \{\text{M is a multiple of 3}\}$

(iii)$C = \{\text{M is less than or equal to 6}\}$

Assume that the probability of requiring one additional block is half of the probability without the additional block. That is:

$$P(M = k) = \left(\frac{1}{2}\right)^k, k = 1, 2, 3, \ldots$$

Determine the following probabilities:

(a) $P(A)$, $P(B)$, $P(C)$, $P(C^c)$

(b) $P(A \cap B)$, $P(A \setminus B)$, $P(A \cap B \cap C)$

(c) $P(A \mid B)$, $P(B \mid A)$

(d)$P(A \mid B \cap C)$, $P(A \cap B \mid C)$

Part (a).

$$\begin{aligned} P(A) &= \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{2i} \\ &= \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^{i} \\ &= \frac{1}{4} \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^{i} \\ &= \left(\frac{1}{4}\right)\left(\frac{1}{1 - \frac{1}{4}}\right) \\ &= \frac{1}{3} \end{aligned}$$

$$
\begin{aligned}
P(B) &= \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{3i} \\
&= \sum_{i=1}^{\infty} \left(\frac{1}{8}\right)^{i} \\
&= \frac{1}{8} \sum_{i=0}^{\infty} \left(\frac{1}{8}\right)^{i} \\
&= \left(\frac{1}{8}\right)\left(\frac{1}{1-\frac{1}{8}}\right) \\
&= \frac{1}{7}
\end{aligned}
$$

$$
\begin{aligned}
P(C) &= P(M=1)+P(M=2)+P(M=3)+P(M=4)+P(M=5)+P(M=6) \\
&= \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64} \\
&= \frac{63}{64}
\end{aligned}
$$

$$
P(C^c) = 1 - P(C) = \frac{1}{64}
$$

Part (b).

$$
\begin{aligned}
P(A \cap B) &= P(\{\text{M is an even multiple of 3}\}) \\
&= P(\{M = 6i \mid i \in \mathbb{N}\}) \\
&= \sum_{i=1}^{\infty} \left(\frac{1}{2^6}\right)^{i} \\
&= \frac{1}{2^6} \sum_{i=0}^{\infty} \left(\frac{1}{2^6}\right)^{i} \\
&= \left(\frac{1}{2^6}\right)\left(\frac{1}{1-\frac{1}{2^6}}\right) \\
&= \frac{1}{63}
\end{aligned}
$$

We know that $P(A \cap B) + P(A \setminus B) = P(A)$. Thus, $P(A \setminus B) = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{63}$.

$$
\begin{aligned}
P(A \cap B \cap C) &= P(\{\mathrm{M} = 6\}) \\
&= \frac{1}{2^6}
\end{aligned}
$$

Part (c).

$$
\begin{aligned}
P(A \mid B) &= \frac{P(A \cap B)}{P(B)} \\
&= \frac{1}{9}
\end{aligned}
$$

$$
\begin{aligned}
P(B \mid A) &= \frac{P(B \cap A)}{P(A)} \\
&= \frac{1}{21}
\end{aligned}
$$

Part (d).

$$
P(A \mid B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)}
$$

We need to calculate the missing value $P(B \cap C)$ first.

$$
\begin{aligned}
P(B \cap C) &= \frac{1}{2^3} + \frac{1}{2^6} \\
&= \frac{9}{64}
\end{aligned}
$$

Thus, $P(A \mid B \cap C) = \frac{1}{9}$.

$$
\begin{aligned}
P(A \cap B \mid C) &= \frac{P(A \cap B \cap C)}{P(C)} \\
&= \frac{1}{63}
\end{aligned}
$$