Selected Problems Chapter 5
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem Example 5.8. Suppose $T \in \mathcal{L}(F^2)$ is defined by $T(w, z) = (-z, w)$. Find the eigenvectors and eigenvalues of T if $F = \mathbb{R}$. Find the eigenvectors and eigenvalues of T if $F = \mathbb{C}$

Proof. **Part(a).**

Assume T has eigenvectors and eigenvalues with $F = \mathbb{R}$. The equation $\lambda(w, z) = (-z, w)$ holds and leads to the following system of equations:

$$\lambda w = -z \tag{1}$$

$$\lambda z = w. \tag{2}$$

$$\tag{3}$$

Solving for λ, we have $\lambda^2 = -1$, which only has solutions in $\mathbb{C}$. This contradiction means T has no eigenvectors and eigenvalues.

Part(b).

In part(a), we showed that the eigenvalues of T must be in the complex numbers. The equation from part(a) $\lambda^2 = -1$ has the solutions $\lambda = i$ and $\lambda = -i$. The eigenvectors corresponding to $\lambda = i$ are of the form $(w, -iw)$ for any $w \in \mathbb{C}$; the eigenvectors corresponding to $\lambda = -i$ are of the form (w, iw).

□

Problem Theorem 5.10 Linearly Independent Eigenvectors. Let $T \in \mathcal{L}(V)$. Suppose $(\lambda_1, \ldots, \lambda_n)$ are distinct eigenvalues of T, and $(v_1, \ldots, v_n)$ are corresponding eigenvectors. Then $(v_1, \ldots, v_n)$ is a linearly independent list.

Proof. For a contradiction, suppose $(v_1, \ldots, v_n)$ is a linearly dependent list. Then choose $a_1, \ldots, a_n \in F$ where not all are zero such that $0 = a_1 v_1 + \ldots a_n v_n$. By the linear dependence lemma, choose the smallest j such that $v_j = \frac{a_1}{a_j} v_1 + \cdots + \frac{a_{j-1}}{a_j} v_{j-1}$. Applying T, we have

$$\lambda_j v_j = \frac{a_1 \lambda_1}{a_j} v_1 + \cdots + \frac{a_{j-1} \lambda_{j-1}}{a_j} v_{j-1}. \tag{4}$$

Subtracting the left side, we have

$$0 = \frac{a_1(\lambda_1 - \lambda_j)}{a_j} v_1 + \cdots + \frac{a_{j-1}(\lambda_{j-1} - \lambda_j)}{a_j} v_{j-1}. \tag{5}$$

Each $(\lambda_k - \lambda_j) > 0$ because the eigenvalues are distinct. Since we chose j to be the smallest such that the v_j is in the span of the preceding vectors, $a_k = 0$ for $k = 1, \ldots, j-1$. Thus, $v_j = 0$, but that is a contradiction because v_j is an eigenvector.

□