

Selected Problems Chapter 3

Linear Algebra Done Right, Sheldon Axler, 3rd Edition

Mustaf Ahmed

July 31, 2021

Problem Integration. Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ by

$$Tp = \int_0^1 p(x)dx.$$

Show that T is a linear map.

Proof.

Additivity

Given $p, q \in \mathcal{P}(\mathbb{R})$, we want the additivity propriety to hold for T . Applying T to the sum of p and q , we have

$$\begin{aligned} T(p + q) &= \int_0^1 p(x) + q(x)dx \\ &= \int_0^1 p(x)dx + \int_0^1 q(x)dx \\ &= T(p) + T(q), \end{aligned}$$

since integration of a sum is equal to the sum of the integrated parts.

Homogeneity

Given $p \in \mathcal{P}(\mathbb{R})$ and $a \in F$, we want the homogeneity property to hold. Applying T to the scalar multiple of p , we have

$$\begin{aligned} T(a * p) &= \int_0^1 a * p(x)dx \\ &= a * \int_0^1 p(x)dx \\ &= a * T(p), \end{aligned}$$

since constants can be separated in integration.

□

Problem Theorem 3.5. Suppose v_1, \dots, v_n is a basis of V and $w_1, \dots, w_m \in W$. Show that there exists a unique linear map $T : V \rightarrow W$ such that

$$T(v_j) = w_j$$

for each $j = 1, \dots, n$.

Proof. We must first show the existence of a linear map with the desired properties. Define $T : V \rightarrow W$ by

$$T(a_1v_1 + \dots + a_nv_n) = a_1w_1 + \dots + a_nw_n$$

, where a_1, \dots, a_n are coefficients in F .

We must show that T is a linear map. Given $a_1v_1, \dots, a_nv_n \in V$ and $b_1v_1, \dots, b_nv_n \in V$, we have

$$\begin{aligned} T'((a_1v_1 + \dots + a_nv_n) + (b_1v_1 + \dots + b_nv_n)) &= T'((a_1 + b_1)v_1 + \dots + (a_n + b_n)v_n) \\ &= (a_1 + b_1)w_1 + \dots + (a_n + b_n)w_n \\ &= (a_1w_1 + \dots + a_nw_n) + (b_1w_1 + \dots + b_nw_n) \\ &= T(a_1v_1 + \dots + a_nv_n) + T(b_1v_1 + \dots + b_nv_n). \end{aligned}$$

Similarly, given $a_1v_1 + \dots + a_nv_n \in V$ and $\lambda \in F$, we have

$$\begin{aligned} T(\lambda(a_1v_1 + \dots + a_nv_n)) &= T((\lambda * a_1)v_1 + \dots + (\lambda * a_n)v_n) \\ &= (\lambda * a_1)w_1 + \dots + (\lambda * a_n)w_n \\ &= \lambda(a_1w_1 + \dots + a_nw_n) \\ &= \lambda * T(a_1v_1 + \dots + a_nv_n) \end{aligned}$$

Thus, we have shown T to be a linear map.

Assume the existence of another linear map $T' : V \rightarrow W$ with the property

$$T'(v_j) = w_j$$

for each $j = 1, \dots, n$. To show uniqueness, we want that $T(v) = T'(v)$ for all $v \in V$.

Given $v \in V$, we can write $v = a_1v_1 + \dots + a_nv_n$, since we have a basis for V . Then,

$$\begin{aligned} T(v) &= T(a_1v_1 + \dots + a_nv_n) \\ &= a_1T(v_1) + \dots + a_nT(v_n) \\ &= a_1T'(v_1) + \dots + a_nT'(v_n) \\ &= T'(a_1v_1 + \dots + a_nv_n) \\ &= T'(v), \end{aligned}$$

since $T(v_i) = w_i = T'(v_i)$ for each $j = 1, \dots, n$. □

Problem 3.A.11. Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V . In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Proof. Given a subspace U of V and a linear map $S \in \mathcal{L}(U, W)$, we want to extend S to be a linear map on V . Choose a basis of U to be u_1, \dots, u_m . We can extend our chosen basis for U to be a basis of V as the list $u_1, \dots, u_m, v_1, \dots, v_n$. Let

$$w_j = Su_j$$

for $j = 1, \dots, m$. Thus the linear map S can be explicitly written as

$$\begin{aligned} Su &= S(a_1u_1 + \dots + a_mu_m) \\ &= a_1w_1 + \dots + a_mw_m \end{aligned}$$

, for all $u \in U$.

We are now in the position to define the extension linear map T . We define T by

$$\begin{aligned} Tv &= T(a_1u_1 + \dots + a_mu_m + a_{m+1}v_1 + \dots + a_{m+n}v_n) \\ &= a_1w_1 + \dots + a_mw_m \end{aligned}$$

, for all $v \in V$. It is clear that $Tu = Su$ for all $u \in U$ by the definition of T ; T is also a linear map.

□

Problem 3.A.14. Suppose V is finite-dimensional with $\dim V \geq 2$. Prove that there exists $S, T \in \mathcal{L}(V, V)$ such that $ST \neq TS$.

Proof. Assume that $n \geq 2$. Choose a basis $v_1, \dots, v_n \in V$. We will define both linear maps as follows:

$$\begin{aligned} Tv &= T(a_1v_1 + \dots + a_nv_n) \\ &= a_nv_1 + a_1v_2 + \dots + a_{n-1}v_n \end{aligned}$$

, which shifts coefficients by 1 in a circle; Also,

$$\begin{aligned} Sv &= S(a_1v_1 + \dots + a_nv_n) \\ &= a_1v_1. \end{aligned}$$

We will now show that these linear maps do not commute. Let $v = v_1 + 2v_2$. We have

$$\begin{aligned} ST(v) &= ST(v_1 + 2v_2) \\ &= S(2v_1 + v_2) \\ &= 2v_1, \end{aligned}$$

and

$$\begin{aligned} TS(v) &= TS(v_1 + 2v_2) \\ &= T(v_1) \\ &= v_2. \end{aligned}$$

Thus, $ST \neq TS$. □

Problem 3.B.2. Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that $\text{range}(S) \subset \text{null}(T)$. Prove that $(ST)^2 = 0$.

Proof. Given $u \in V$, we must show that $((ST)(ST))u = 0$. By the definition of the product of linear maps, we have

$$((ST)(ST))u = (ST)(STu)$$

Let $w = (STu)$, which is in $\text{range}(S)$. Because $w \in \text{range}(S)$, w is also in $\text{null}(T)$ by our assumption. We have

$$\begin{aligned} ((ST)(ST))u &= (ST)(STu) \\ &= (ST)w \\ &= (STw) \\ &= (S0) \\ &= 0, \end{aligned}$$

as desired.

□

Problem 3.B.12. Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \text{null}(T) = \{0\}$ and $\text{range}(T) = \{Tu \mid u \in U\}$.

Proof. Choose a subspace U of V such that $U \oplus \text{null}(T) = V$, which is guaranteed by Theorem 2.34. Since U and $\text{null}(T)$ form a direct sum, we have

$$U \cap \text{null}(T) = \{0\}.$$

The next part is to show that $\text{range}(T) = \{Tu \mid u \in U\}$. The right to left inclusion is clear by the definition of range; we must show the left to right inclusion to finish the proof. Given $w \in \text{range}(T)$, choose $v \in V$ such that $w = Tv$. Since $U \oplus \text{null}(T) = V$, choose $u \in U$ and $n \in \text{null}(T)$ such that $v = u + n$. We have

$$\begin{aligned} w &= Tv \\ &= T(u + n) \\ &= T(u) + T(n) \\ &= T(u) + 0 \\ &= T(u), \end{aligned}$$

as desired.

□

Problem Fundamental Theorem of Linear Maps. Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\text{range}(T)$ is finite-dimensional and

$$\dim(V) = \dim(\text{null}(T)) + \dim(\text{range}(T))$$

Proof. By Theorem 2.34, we can choose a subspace U of V such that $U \oplus \text{null}(T) = V$. By Problem 3.B.12, $\text{range}(T) = \{Tu \mid u \in U\}$. Let $B_1 = \{u_1, \dots, u_m\}$ of U , and let $B_2 = \{n_1, \dots, n_k\}$ of $\text{null}(T)$.

We want $B_1 \cup B_2$ to be a basis for V . Since $U \oplus \text{null}(T) = V$, $B_1 \cup B_2$ spans V . We need to show that $B_1 \cup B_2$ is also linearly independent. It suffices to show that the zero-vector can be represented uniquely. Assume that the zero-vector can be written as

$$0 = \sum_{i=1}^m \lambda_i u_i + \sum_{i=m+1}^{m+k} \lambda_i n_i$$

and

$$0 = \sum_{i=1}^m \alpha_i u_i + \sum_{i=m+1}^{m+k} \alpha_i n_i.$$

Since U and $\text{null}(T)$ form a direct sum, any vector in V can be written uniquely as a vector from U and a vector from $\text{null}(T)$; Thus, we have

$$\sum_{i=1}^m \lambda_i u_i = \sum_{i=1}^m \alpha_i u_i$$

and

$$\sum_{i=m+1}^{m+k} \lambda_i n_i = \sum_{i=m+1}^{m+k} \alpha_i n_i.$$

It follows from the fact that B_1 and B_2 are basis lists that $\lambda_i = \alpha_i$ for all i . Thus, $B_1 \cup B_2$ forms a basis for V .

We know that $\dim(V) = m + k$ from $B_1 \cup B_2$ being a basis. We also know that $\dim(\text{null}(T)) = k$ from B_2 being a basis of $\text{null}(T)$. It suffices to show that $\dim(\text{range}(T)) = m$. We must show that $R = \{Tu_1, \dots, Tu_m\}$ is a basis for $\text{range}(T)$. By Problem 3.B.12, R spans $\text{range}(T)$. For a contradiction, suppose that R is linearly dependent. We can choose Tu_j in $\text{span}(Tu_1, \dots, Tu_{j-1})$ by the Linear Dependence Lemma. We have

$$T(u_j) = \lambda_1 T(u_1) + \dots + \lambda_{j-1} T(u_{j-1}),$$

so

$$\begin{aligned}T(u_j) &= T(\lambda_1 u_1 + \cdots + \lambda_{j_1} u_{j-1}) \\0 &= T(\lambda_1 u_1 + \cdots + \lambda_{j_1} u_{j-1} - u_j)\end{aligned}$$

Thus, $\lambda_1 u_1 + \cdots + \lambda_{j_1} u_{j-1} - u_j \in \text{null}(T)$, with not all the constants being zero, but we know that $U \cap \text{Null}(T) = \{0\}$, which is a contradiction.

□

Problem A map to a smaller dimensional space is not injective. Suppose V and W are finite-dimensional vector spaces and $\dim(V) > \dim(W)$. Then no linear map from V to W is injective.

The intuition is that v is being transferred into a smaller space, so there is no way that points in W receive a unique member in V , since there are simply too many elements in V .

Proof. We are given a linear map $T \in \mathcal{L}(V, W)$. Since $\text{range}(T) \subseteq W$, $\dim(\text{range}(T)) \leq \dim(W)$. By the Fundamental Theorem of Linear Maps, we have

$$\begin{aligned} \dim(\text{null}(T)) &= \dim(V) - \dim(\text{range}(T)) \\ &\geq \dim(V) - \dim(W) \\ &\geq 1 \end{aligned}$$

Thus, for any basis of $\text{null}(T)$, there is at least one non-zero element which is mapped to the zero-vector in W . The zero-vector in V is also mapped to the zero-vector in W . T is not injective as desired.

□

Problem A map to a larger dimensional space is not surjective. Suppose V and W are finite-dimensional vector spaces and $\dim(V) < \dim(W)$. Then no linear map from V to W is surjective.

Proof. We are given a linear map $T \in \mathcal{L}(V, W)$. By the Fundamental Theorem of Linear Maps, we have

$$\begin{aligned}\dim(\text{range}(T)) &= \dim(V) - \dim(\text{null}(T)) \\ &\leq \dim(V) \\ &< \dim(W).\end{aligned}$$

Thus, for any basis of $\text{range}(T)$, we can extend it to a basis of W , revealing a vector not in $\text{range}(T)$; thus, $T : V \rightarrow W$ is not surjective.

□

Problem A map to a larger dimensional space is not surjective. Suppose V and W are finite-dimensional vector spaces and $\dim(V) < \dim(W)$. Then no linear map from V to W is surjective.

Proof. We are given a linear map $T \in \mathcal{L}(V, W)$. By the Fundamental Theorem of Linear Maps, we have

$$\begin{aligned}\dim(\text{range}(T)) &= \dim(V) - \dim(\text{null}(T)) \\ &\leq \dim(V) \\ &< \dim(W).\end{aligned}$$

Thus, for any basis of $\text{range}(T)$, we can extend it to a basis of W , revealing a vector not in $\text{range}(T)$; thus, $T : V \rightarrow W$ is not surjective.

□

Problem 3.B.20. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V .

Proof. We will first prove the forward direction. Assume that T is injective. Define $S : \text{range}(T) \rightarrow V$ by

$$Sw = v$$

where v is the unique element in V such that $Tv = w$. We must show that ST is the identity map on V . Choose $v \in V$. Then

$$\begin{aligned} (ST)v &= S(Tv) \\ &= v, \end{aligned}$$

by the definition of S .

We will prove the backward direction. Choose $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V . Assume that $Tu = Tw$ for some $u, w \in V$. Using (ST) , we have

$$\begin{aligned} u &= (ST)u \\ &= S(Tu) \\ &= S(Tw) \\ &= (ST)w \\ &= w, \end{aligned}$$

meaning T is injective.

□

Problem Edited 3.B.29 (V is assumed to be finite-dimensional). Suppose $T \in \mathcal{L}(V, F)$. Suppose $u \in V$ is not in $\text{null}(T)$. Prove that

$$V = \text{null}(T) \oplus \{au \mid a \in F\}$$

Proof. We will first construct a certain basis for V . Choose $u \in V$ such that u is not in $\text{null}(T)$. Let $\{n_1, \dots, n_m\}$ be a basis for $\text{null}(T)$. We want

$$B = \{n_1, \dots, n_m, u\}$$

to be a basis for V .

For a contradiction, suppose that B is not a basis for V . Thus, we can extend B to a basis of V :

$$B_e = \{n_1, \dots, n_m, u, e_1, \dots, e_n\}.$$

We consider the complex numbers to be a vector space over itself, so $\dim(F) = 1$. We want a contradiction with $\dim(\text{range}(T)) \leq \dim(F)$. We will do this by showing that $R = \{Tu, Te_1, \dots, Te_n\}$ is a basis for $\text{range}(T)$. Given $r \in \text{range}(T)$, choose $a_1, \dots, a_{m+n+1} \in F$ such that

$$\begin{aligned} r &= T((a_1n_1 + \dots + a_mn_m) + a_{m+1}u + (a_{m+2}e_1 + \dots + a_{m+n+1}e_n)) \\ &= a_{m+1}T(u) + a_{m+2}T(e_1) + \dots + a_{m+n+1}T(e_n) \end{aligned}$$

, so R spans $\text{range}(T)$. Next, we need to show that R is linearly independent. Choose $a_1, \dots, a_{n+1} \in F$ such that

$$\begin{aligned} 0 &= a_1T(u_1) + a_2T(e_1) + \dots + a_{n+1}T(e_m) \\ &= T(a_1u_1 + a_2e_1 + \dots + a_{n+1}e_m), \end{aligned}$$

so $a_1u_1 + a_2e_1 + \dots + a_{n+1}e_m \in \text{span}(n_1, \dots, n_m)$. For some $b_1, \dots, b_m \in F$, we have

$$\begin{aligned} 0 &= (a_1u_1 + a_2e_1 + \dots + a_{n+1}e_m) - (b_1n_1 + \dots + b_mn_m) \\ &, \end{aligned}$$

which means that all the scalars must be 0 because the vectors form a basis for V . Thus, R is a basis for $\text{range}(T)$.

Now, for the contradiction.

$$\begin{aligned} \dim(\text{range}(T)) &= 1 + n \\ &> 1 \\ &= \dim(F), \end{aligned}$$

which is not possible because $\dim(F) = 1$. So B is a basis for V .

We've established that B is a basis for V . All that is left is to show that $V = \text{null}(T) + \{au \mid a \in F\}$ and $\text{null}(T) \cap \{au \mid a \in F\} = \{0\}$. Because B span V , it is clear that $V = \text{null}(T) + \{au \mid a \in F\}$. We assumed that $u \notin \text{null}(T)$, so the $\text{null}(T) \cap \{au \mid a \in F\} = \{0\}$. \square

Problem Theorem 3.36 matrix of sum of linear maps. Suppose $S, T \in \mathcal{L}(V, W)$. Then $M(S + T) = M(S) + M(T)$.

Proof. Let v_1, \dots, v_n be our chosen basis for V and w_1, \dots, w_m be our chosen basis for W . The sum of S and T applied to v_j is

$$\begin{aligned}(S + T)v_j &= S(v_j) + T(v_j) \\ &= \sum_{i=1}^m a_{i,j}w_i + \sum_{i=1}^m b_{i,j}w_i \\ &= \sum_{i=1}^m (a_{i,j} + b_{i,j})w_i\end{aligned}$$

Thus, $M(S + T)$ is defined as

$$M(S + T) = \begin{pmatrix} a_{1,1} + b_{1,1} & \cdots & a_{1,n} + b_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} + b_{m,1} & \cdots & a_{m,n} + b_{m,n} \end{pmatrix}.$$

From our decomposition of $(S + T)v_j$, for each j , above, we have

$$M(S) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix},$$

, and

$$M(T) = \begin{pmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{m,1} & \cdots & b_{m,n} \end{pmatrix}.$$

It is clear that $M(S + T) = M(S) + M(T)$ follows from the definition of matrix addition. □

Problem 3.C.3. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a basis of V and a basis of W such that with respect to these bases, all entries of $M(T)$ are 0 except that the entries in row j , column j , equal 1 for $1 \leq j \leq \dim(\text{range}(T))$.

Proof. Let n_1, \dots, n_m be a basis for $\text{null}(T)$. We can extend this list to a basis for V :

$$B_v = \{v_1, \dots, v_n, n_1, \dots, n_m\}.$$

It is clear that Tv_1, \dots, Tv_n is a basis for $\text{Range}(T)$. We can extend this to a basis of W :

$$B_w = \{Tv_1, \dots, Tv_n, w_1, \dots, w_k\}.$$

Now that we have a basis for V and W , we must show that $M(T)$ is in the desired form. The column associated with each v_j has a single 1 in $M(T)_{j,j}$ and 0 everywhere else. The column associated with each n_k is filled with only 0, since $n_k \in \text{null}(T)$. We are done. \square

Problem 3.C.4. Suppose v_1, \dots, v_m is a basis for V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove there exists a basis w_1, \dots, w_n of W such that all the entries in the first column of $M(T)$ (with respect to the bases) are 0 except for possibly a 1 in the first row, first column.

Proof. **Case 1.** v_1 is in $\text{null}(T)$. Let w_1, \dots, w_n be any basis for W . We can write Tv_1 as

$$Tv_1 = 0w_1 + 0w_2 + \cdots + 0w_n,$$

meaning that the first column of $M(T)$ contains all zeros.

Case 2. v_1 is not in $\text{null}(T)$. Let $w_1 = Tv_1$. We can extend the list containing only w_1 to a basis of W : w_1, \dots, w_n . Then, Tv_1 is written as

$$Tv_1 = 1w_1 + 0w_2 + \cdots + 0w_n,$$

,so the first column of $M(T)$ starts with a 1 and the rest of the entries are 0. We are done. \square

Problem Theorem Unique Inverse. An invertible linear map has a unique inverse.

Proof. Let $T \in \mathcal{L}(V, W)$, and let $R, S \in \mathcal{L}(W, V)$ be inverses of T . We have

$$\begin{aligned} R &= RI \\ &= R(TS) \\ &= (RT)S \\ &= IS \\ &= S \end{aligned}$$

,

as desired.

□

Problem Theorem 3.56 Invertibility is equivalent to injectivity and surjectivity.
A linear map is invertible if and only if it is injective and surjective.

Proof. Let $T : V \rightarrow W$ be a linear map. Assume that T is invertible. Choose $T^{-1} \in \mathcal{L}(W, V)$ to be the unique inverse for T . For injectivity, assume that $Tv = Tu$ for $u, v \in V$. Applying the inverse, we have

$$\begin{aligned} u &= Iu \\ &= (T^{-1}T)u \\ &= T^{-1}(Tu) \\ &= T^{-1}(Tv) \\ &= (T^{-1}T)v \\ &= Iv \\ &= v. \end{aligned}$$

Next is surjectivity. Choose $w \in W$. Then $T^{-1}w \in V$. Applying T to this vector, we have $T(T^{-1}w) = (TT^{-1})w = Iw = w$.

For the backwards direction, assume that T is injective and surjective. Given $w \in W$, we define the inverse $S : W \rightarrow V$ by

$$Sw = v$$

where v is the unique element in V such that $Tv = w$. We have

$$\begin{aligned} (ST)v &= S(Tv) \\ &= Sw \\ &= v \end{aligned}$$

and

$$\begin{aligned} (TS)w &= T(Sw) \\ &= Tv \\ &= w \end{aligned}$$

□

Problem Theorem 3.59. Two finite-dimensional vector spaces over F are isomorphic if and only if they have the same dimension.

Proof. Let V and W be finite-dimensional vector spaces over F . First we will show that if two vector spaces over F are isomorphic then they have the same dimension. Assume that V and W are isomorphic. Then choose the invertible linear map $T : V \rightarrow W$. Since T is invertible, T is also surjective and injective. By the fundamental theorem of linear maps, we have

$$\begin{aligned} \dim(V) &= \dim(\text{null}(T)) + \dim(\text{range}(T)) \\ &= \dim(\text{range}(T)), \end{aligned}$$

since T is injective, resulting in $\dim(\text{null}(T)) = 0$. Since T is surjective, $\text{range}(T) = W$ which means that $\dim(V) = \dim(W)$.

Next we will show that if the two vector spaces have the same dimension, then they are isomorphic. Assume that $\dim(V) = \dim(W)$. Choose v_1, \dots, v_n to be a basis for V and w_1, \dots, w_n to be a basis for W . We will define our isomorphism $T : V \rightarrow W$ by

$$\begin{aligned} T(v) &= T(a_1v_1 + \dots + a_nv_n) \\ &= a_1w_1 + \dots + a_nw_n \end{aligned}$$

We need to first show that T is a linear map. For additivity, we have

$$\begin{aligned} T(u + v) &= T((a_1 + b_1)v_1 + \dots + (a_n + b_n)v_n) \\ &= (a_1 + b_1)w_1 + \dots + (a_n + b_n)w_n \\ &= (a_1w_1 + \dots + a_nw_n) + (b_1w_1 + \dots + b_nw_n) \\ &= T(u) + T(v). \end{aligned}$$

For homogeneity, we have

$$\begin{aligned} T(\lambda v) &= T((\lambda a_1)v_1 + \dots + (\lambda a_n)v_n) \\ &= (\lambda a_1)w_1 + \dots + (\lambda a_n)w_n \\ &= \lambda(a_1w_1 + \dots + a_nw_n) \\ &= \lambda T(v). \end{aligned}$$

The last part of this proof is to show that T is injective and surjective. Given $u, v \in V$, assume that $T(u) = T(v)$. We have

$$\begin{aligned} 0 &= T(v) - T(u) \\ &= (a_1 - b_1)w_1 + \dots + (a_n - b_n)w_n, \end{aligned}$$

so $a_i = b_i$ for each $i \in \{1, \dots, n\}$ because we have a basis for W . Finally, we must show surjectivity. Given $w \in W$, we can write it as $w = a_1w_1 + \dots + a_nw_n$. Thus,

$$T(a_1v_1 + \dots + a_nv_n) = a_1w_1 + \dots + a_nw_n,$$

as desired. □

Problem Theorem 3.60. Suppose v_1, \dots, v_n is a basis of V and w_1, \dots, w_m is a basis of W . Then M is an isomorphism between $\mathcal{L}(V, W)$ and $F^{m,n}$.

This theorem is essentially saying that the set of all linear maps from V to W is the same thing as the set of all linear map encodings, once we fix bases.

Proof. By Theorem 3.36 and Theorem 3.38, M is linear. All that is left to show is that M is injective and surjective.

We will first show injectivity. Assume that $M(T) = M(S)$. Then

$$T(v_j) = S(v_j) = a_{1,j}w_1 + \dots + a_{m,j}w_m,$$

for $j = 1, \dots, n$, which is given by how M is constructed. Since linear maps are uniquely determined by where they send a basis list (Theorem 3.5), $T = S$.

Next, we will show surjectivity. Let $M \in F^{m,n}$ be

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix}.$$

We will define the linear map $T : V \rightarrow W$ by

$$T(v_j) = \sum_{i=1}^m a_{i,j}w_i,$$

for each $j = 1, \dots, n$. Then $M(T) = M$ by definition.

□

Problem 3.D.7. Suppose V and W are finite-dimensional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) \mid Tv = 0\}.$$

- (a) Show that E is a subspace of $\mathcal{L}(V, W)$.
- (b) Suppose $v \neq 0$. What is $\dim(E)$?

Proof.

Part(a). Clearly, $E \subseteq \mathcal{L}(V, W)$. So we just need to show that E is closed under addition, closed under scalar multiplication, and contains the zero linear map.

We will begin with closure under addition and scalar multiplication. Let $T, S \in E$. We have

$$\begin{aligned}(T + S)v &= Tv + Sv \\ &= 0.\end{aligned}$$

Similarly for scalar multiplication, let $T \in E$ and $\lambda \in F$. We have

$$\begin{aligned}(\lambda T)v &= \lambda Tv \\ &= 0.\end{aligned}$$

Finally, we must show that the zero linear map is in E . This is clear from the fact that the linear map that sends every vector in V to the zero vector in W is in E .

Part(b). Let $F : \mathcal{L}(W, V) \rightarrow W$ be defined as

$$F(T) = Tv.$$

It is clear then that $E = \text{null}(F)$. By the fundamental theorem of linear algebra, we have

$$\begin{aligned}\dim(\text{Null}(F)) &= \dim(\mathcal{L}(W, V)) - \dim(\text{range}(F)) \\ &= \dim(W)\dim(V) - \dim(\text{range}(F))\end{aligned}$$

By Theorem 3.5, F is surjective, so $\dim(\text{range}(F)) = \dim(W)$. Thus,

$$\dim(E) = \dim(W)\dim(V) - \dim(W).$$

□

Problem 3.D.9. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

Proof.

We will prove the forward direction first. Assume that ST is invertible. Choose the inverse $U \in \mathcal{L}(V)$. Composing U and ST we have

$$\begin{aligned} U(ST) &= (US)T \\ &= I. \end{aligned}$$

By problem 3.B.20, T must be injective, so by Theorem 3.69, T must be invertible. It suffices to show that S is surjective by Theorem 3.69. For a contradiction, suppose S is not surjective. Then choose $u \in V$ such that $u \notin \text{range}(S)$. Then for all $v \in V$, $S(Tv) \neq u$, but that means ST is not surjective, which is a contradiction.

We will now prove the backward direction. Assume that S and T are invertible. Composing ST with $T^{-1}S^{-1}$, we have

$$\begin{aligned} (ST)(T^{-1}S^{-1}) &= (S(TT^{-1}))(S^{-1}) \\ &= SS^{-1} \\ &= I. \end{aligned}$$

Similarly, composing $T^{-1}S^{-1}$ with ST , we have

$$\begin{aligned} (T^{-1}S^{-1})(ST) &= (T(SS^{-1}))(T^{-1}) \\ &= TT^{-1} \\ &= I. \end{aligned}$$

We are done. □

Problem 3.D.10. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that if $ST = I$ then $TS = I$.

For linear operators, the left or right inverse is the inverse.

Proof.

Assume that $ST = I$. By problem 3.B.20, T is injective. T is an operator, so by Theorem 3.69, T is invertible. Composing ST with T^{-1} , we have

$$\begin{aligned}(ST)(T^{-1}) &= S(TT^{-1}) \\ &= S.\end{aligned}$$

We also have

$$\begin{aligned}(ST)(T^{-1}) &= IT^{-1} \\ &= T^{-1}.\end{aligned}$$

So $T^{-1} = S$, which means that $TS = I$.

□

Problem 3.D.16. Suppose that V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of I iff $ST = TS$ for every $S \in \mathcal{L}(V)$.

Proof.

Given $S \in \mathcal{L}(V)$, assume that T is a scalar multiple of I . We can write $T = \lambda I$ for some $\lambda \in F$. Composing S with T , we have

$$\begin{aligned}(ST)v &= S(\lambda Iv) \\ &= S(\lambda v) \\ &= \lambda(Sv) \\ &= \lambda I(Sv) \\ &= (TS)v.\end{aligned}$$

Now, assume $ST = TS$ for every $S \in \mathcal{L}(V)$. Choose v_1, \dots, v_n to be a basis of V . Let $S_{ij} \in \mathcal{L}(V)$ be defined as

$$S_{ij}(a_1v_1 + \dots + a_nv_n) = a_iv_j,$$

which is a linear map. For each v_i , let

$$T(v_i) = c_{1i}v_1 + \dots + c_{ni}v_n.$$

For each v_i , we have

$$\begin{aligned}S_{ii}T(v_i) &= S_{ii}(c_{1i}v_1 + \dots + c_{ni}v_n) \\ &= c_{ii}v_i \\ &= TS_{ii}(v_i) \\ &= T(v_i) \\ &= c_{1i}v_1 + \dots + c_{ni}v_n\end{aligned}$$

Subtracting $c_{ii}v_i$ from the final result and using the linear independence of our basis, $c_{nm} = 0$ for all pairs where $n \neq m$.

To finish the proof, we will show that there is a scalar $a = c_{ii}$ for each i . We can define a linear map by what it does to a basis, so define $U \in \mathcal{L}(V)$ by $U(v_i) = v_{i+1}$ for $i < n$ and $U(v_n) = v_1$. For each v_i , we have

$$\begin{aligned}
UT(v_i) &= U(c_{ii}v_i) \\
&= c_{ii}v_{i+1} \\
&= TU(v_i) \\
&= T(v_{i+1}) \\
&= c_{i+1i+1}v_{i+1},
\end{aligned}$$

,

so all the scalars are the same, call it a .

Finally, we will show that $T = aI$. We have

$$\begin{aligned}
T(c_1v_1 + \cdots + c_nv_n) &= c_1T(v_1) + \cdots + c_nT(v_n) \\
&= (c_1a)v_1 + \cdots + (c_na)v_n \\
&= aT(c_1v_1 + \cdots + c_nv_n),
\end{aligned}$$

, as desired.

□