Introduction to Modern Algebra I, Spring 2017, Columbia University

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Problem Theorem 1.25. Given an equivalence relation $\sim$ on a set X, the equivalence classes of X form a partition of X. Conversely, if $\mathcal{P} = \{X_i\}$ is a partition of a set X, then there is an equivalence relation on X with equivalence classes X_i.

Proof. For the forward direction, assume that $\sim$ is an equivalence relation on X. Let $x \in X$. The equivalence class $[x]$ is non empty because $x \sim x$. It follows that $\bigcup_{x \in X}[x] = X$. To finish this direction, we need to show that $[x] \cap [y] = \emptyset$ or $[x] = [y]$. Assume $[x] \cap [y] \neq \emptyset$. Choose $z \in [x] \cap [y]$. By symmetry and transitivity $x \sim y$. Let $w \in [y]$. By symmetry and transitivity, $w \sim x$, so $[y] \subseteq [x]$; A similar argument can be made for $[x] \subseteq [y]$.

For the backward direction, assume $\mathcal{P} = \{X_i\}$ is a partition of a set X. We'll define the relation $R = \{(x, y) \mid x, y \in X_i\}$.

Reflexivity. Let $x \in X$. Since $\mathcal{P}$ is a partition, x must be in some X_i. It is clear that x and itself are in the same partition, so R has the reflexive property.

Symmetry. Assume $(x, y) \in R$. Then $x, y \in X_i$ for some i. By the definition of R, $(y, x) \in R$ as well.

Transitivity. Assume $(x, y) \in R$ and $(y, z) \in R$. Then, $x, y \in X_i$ for some i, and $y, z \in X_j$ for some j. We want that $i = j$. Since the partition is formed from mutually disjoint sets, $i = j$. Thus, $x, z \in X_i$ for some i, so $x \sim z$ as desired.

□

Problem Corollary 1.26. Two equivalence classes of an equivalence relation are either disjoint or equal.

Proof. Shown in the forward direction of Theorem 1.25.

□

Problem 1. List all subsets of the 3-element set $A = \{1, 2, 3\}$. How many subsets does a set with n elements have? How many of these subsets have at most two elements?

Part (a). The sets are $\emptyset, \{1\}\{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

Part (b). Find how many subsets exist for a set with n elements amounts to summing all possible sizes for combinations of elements in the set:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Part (c). We need to exclude the combinations where $i <= 1$:

$$\begin{aligned}\sum_{i=0}^{n} \binom{n}{i} &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} \\ &= 1 + n + \frac{n(n-1)}{2}\end{aligned}$$

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Problem 2. Simplify descriptions of the following sets. These sets depend on subsets A, B of a universal set X, so that $A^{'} = X \backslash A$, and so on.

(a) $A^{'} \cup A$, $A^{'} \cap A$, $(A^{'} \cup A^{'}) \cup (A^{'} \cap A)$

(b) $(A \cap B) \cup (A \cup B)$, $(A \cup B^{'}) \cap (A^{'} \cap B)$, $(A \cup B) \backslash B$,$(A \cap B) \backslash B$, $(A \cap B) \cup (A \backslash B)$.

Part (a). $A^{'} \cup A = X$.

$A^{'} \cap A = \emptyset$ because the two sets are disjoint.

$(A^{'} \cup A^{'}) \cup (A^{'} \cap A)$ simplifies to $A^{'} \cup \emptyset$. This further simplifies to A'.

Part (b).