Selected Problems Chapter 1
Introduction to Probability for Data Science
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Problem 1.1. (a) Show that

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

for any $0 < r < 1$. Evaluate $\sum_{k=0}^{\infty} r^k$.

(b) Using the result of (a), evaluate

$$1 + 2r + 3r^2 + \dots.$$

(c) Evaluate the sums

$$\sum_{k=0}^{\infty} k \left(\frac{1}{3}\right)^{k+1}, \text{ and } \sum_{k=2}^{\infty} k \left(\frac{1}{4}\right)^{k-1}$$

Proof.

Part (a). The first part is true by Theorem 1.1. For the second part, we must evaluate $\sum_{k=0}^{\infty} r^k = \lim_{k\to\infty} \frac{1-r^{k+1}}{1-r}$. Since the limit of the denominator is not zero, we have

$$\begin{aligned}\sum_{k=0}^{\infty} r^k &= \lim_{k\to\infty} \frac{1-r^{k+1}}{1-r} \\ &= \frac{\lim_{k\to\infty} 1-r^{k+1}}{\lim_{k\to\infty} 1-r} \\ &= \frac{1}{1-r}.\end{aligned}$$

Part (b). By part (a), we know that $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. Taking the derivative of both sides gives us

$$1 + 2r + 3r^2 + \ldots = \frac{1}{(1-r)^2}.$$

Part (c).

$$\begin{aligned}
\sum_{k=0}^{\infty} k \left(\frac{1}{3}\right)^{k+1} &= \sum_{k=1}^{\infty} k \left(\frac{1}{3}\right)^{k+1} \\
&= \left(\frac{1}{3}\right)^2 \sum_{k=1}^{\infty} k \left(\frac{1}{3}\right)^{k-1} \\
&= \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
\sum_{k=2}^{\infty} k \left(\frac{1}{4}\right)^{k-1} &= -1 + \sum_{k=1}^{\infty} k \left(\frac{1}{4}\right)^{k-1} \\
&= \frac{7}{9}
\end{aligned}$$

$\square$