

# Selected Problems Chapter 5

## Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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**Problem Example 5.8.** Suppose  $T \in \mathcal{L}(F^2)$  is defined by  $T(w, z) = (-z, w)$ . Find the eigenvectors and eigenvalues of  $T$  if  $F = \mathbb{R}$ . Find the eigenvectors and eigenvalues of  $T$  if  $F = \mathbb{C}$

*Proof.* **Part(a).**

Assume  $T$  has eigenvectors and eigenvalues with  $F = \mathbb{R}$ . The equation  $\lambda(w, z) = (-z, w)$  holds and leads to the following system of equations:

$$\lambda w = -z$$

$$\lambda z = w.$$

Solving for  $\lambda$ , we have  $\lambda^2 = -1$ , which only has solutions in  $\mathbb{C}$ . This contradiction means  $T$  has no eigenvectors and eigenvalues.

**Part(b).**

In part(a), we showed that the eigenvalues of  $T$  must be in the complex numbers. The equation from part(a)  $\lambda^2 = -1$  has the solutions  $\lambda = i$  and  $\lambda = -i$ . The eigenvectors corresponding to  $\lambda = i$  are of the form  $(w, -iw)$  for any  $w \in \mathbb{C}$ ; the eigenvectors corresponding to  $\lambda = -i$  are of the form  $(w, iw)$ .

□

**Problem Theorem 5.10 Linearly Independent Eigenvectors.** Let  $T \in \mathcal{L}(V)$ . Suppose  $(\lambda_1, \dots, \lambda_n)$  are distinct eigenvalues of  $T$ , and  $(v_1, \dots, v_n)$  are corresponding eigenvectors. Then  $(v_1, \dots, v_n)$  is a linearly independent list.

*Proof.* For a contradiction, suppose  $(v_1, \dots, v_n)$  is a linearly dependent list. Then choose  $a_1, \dots, a_n \in F$  where not all are zero such that  $0 = a_1 v_1 + \dots + a_n v_n$ . By the linear dependence lemma, choose the smallest  $j$  such that  $v_j = \frac{a_1}{a_j} v_1 + \dots + \frac{a_{j-1}}{a_j} v_{j-1}$ . Applying  $T$ , we have

$$\lambda_j v_j = \frac{a_1 \lambda_1}{a_j} v_1 + \dots + \frac{a_{j-1} \lambda_{j-1}}{a_j} v_{j-1}.$$

Subtracting the left side, we have

$$0 = \frac{a_1(\lambda_1 - \lambda_j)}{a_j} v_1 + \dots + \frac{a_{j-1}(\lambda_{j-1} - \lambda_j)}{a_j} v_{j-1}.$$

Each  $(\lambda_k - \lambda_j) \neq 0$  because the eigenvalues are distinct. Since we chose  $j$  to be the smallest such that the  $v_j$  is in the span of the preceding vectors,  $a_k = 0$  for  $k = 1, \dots, j-1$ . Thus,  $v_j = 0$ , but that is a contradiction because  $v_j$  is an eigenvector. □

**Problem Theorem 5.13 Number of Eigenvalues.** Suppose  $V$  is finite-dimensional. Then each operator on  $V$  has at most  $\dim(V)$  distinct eigenvalues.

*Proof.* Let  $T \in \mathcal{L}(V)$ . Let  $(\lambda_1, \dots, \lambda_n)$  be a list of distinct eigenvalues in  $F$ , and let  $(u_1, \dots, u_m)$  be a corresponding list of eigenvectors. By Theorem 5.10,  $(v_1, \dots, v_n)$  is a linearly independent list. Choose a basis  $(v_1, \dots, v_n)$  of  $V$ . By Theorem 2.23,  $m \leq n = \dim(V)$ .  $\square$

**Problem 5.A.12.** Define  $T \in \mathcal{L}(P_4(\mathbb{R}))$  by

$$T(p(x)) = xp'(x)$$

for all  $x \in \mathbb{R}$ . Find all the eigenvalues and eigenvectors.

Let  $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ . Assume  $T(p(x)) = \lambda p(x)$  with  $\lambda \in \mathbb{R}$ . We have  $\lambda p(x) = xp'(x)$ , so

$$\lambda a_0 + (\lambda a_1 - a_1)x + \cdots + (\lambda a_4 - a_4)x^4 = 0.$$

The list  $(1, x, \dots, x^4)$  is linearly independent in  $P_4(\mathbb{R})$ , which leads to the following system of equations:

$$\begin{aligned}\lambda a_0 &= 0 \\ \lambda a_1 &= a_1 \\ \lambda a_2 &= 2a_2 \\ \lambda a_3 &= 3a_3 \\ \lambda a_4 &= 4a_4\end{aligned}$$

Let  $\lambda \neq 0, 1, 2, 3, 4$ . Then each  $a_j = 0$ , but this is not possible because eigenvectors are non-zero. Thus, the possibilities are  $\lambda = 0, 1, 2, 3, 4$ . Choose  $\lambda \in \{0, 1, 2, 3, 4\}$ . For each  $a_j$  where  $j \neq \lambda$ , we have  $a_j = 0$ . Thus, the corresponding eigenvectors are  $c, cx, cx^2, cx^3, cx^4$  for  $c \in \mathbb{R}$ .

**Problem 5.A.15.** Suppose  $T \in \mathcal{L}(V)$ . Suppose  $S \in \mathcal{L}(V)$  is invertible.

- (a) Prove that  $T$  and  $S^{-1}TS$  have the same eigenvalues
- (b) What is the relationship between the eigenvectors of  $T$  and the eigenvectors of  $S^{-1}TS$ ?

*Proof.*

□