# Selected Problems Chapter 5
# Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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**Problem Example 5.8.** Suppose $T \in \mathcal{L}(F^2)$ is defined by $T(w, z) = (-z, w)$. Find the eigenvectors and eigenvalues of $T$ if $F = \mathbb{R}$. Find the eigenvectors and eigenvalues of $T$ if $F = \mathbb{C}$

*Proof.* **Part(a).**

Assume $T$ has eigenvectors and eigenvalues with $F = \mathbb{R}$. The equation $\lambda(w, z) = (-z, w)$ holds and leads to the following system of equations:

$$\lambda w = -z$$
$$\lambda z = w.$$

Solving for $\lambda$, we have $\lambda^2 = -1$, which only has solutions in $\mathbb{C}$. This contradiction means $T$ has no eigenvectors and eigenvalues.

**Part(b).**

In part(a), we showed that the eigenvalues of $T$ must be in the complex numbers. The equation from part(a) $\lambda^2 = -1$ has the solutions $\lambda = i$ and $\lambda = -i$. The eigenvectors corresponding to $\lambda = i$ are of the form $(w, -iw)$ for any $w \in \mathbb{C}$; the eigenvectors corresponding to $\lambda = -i$ are of the form $(w, iw)$.

$\square$

**Problem Theorem 5.10 Linearly Independent Eigenvectors.** Let $T \in \mathcal{L}(V)$. Suppose $(\lambda_1, \dots, \lambda_n)$ are distinct eigenvalues of $T$, and $(v_1, \dots, v_n)$ are corresponding eigenvectors. Then $(v_1, \dots, v_n)$ is a linearly independent list.

*Proof.* For a contradiction, suppose $(v_1, \dots, v_n)$ is a linearly dependent list. Then choose $a_1, \dots, a_n \in F$ where not all are zero such that $0 = a_1 v_1 + \dots a_n v_n$. By the linear dependence lemma, choose the smallest $j$ such that $v_j = \frac{a_1}{a_j} v_1 + \cdots + \frac{a_{j-1}}{a_j} v_{j-1}$. Applying $T$, we have

$$\lambda_j v_j = \frac{a_1 \lambda_1}{a_j} v_1 + \cdots + \frac{a_{j-1} \lambda_{j-1}}{a_j} v_{j-1}.$$

Subtracting the left side, we have

$$0 = \frac{a_1(\lambda_1 - \lambda_j)}{a_j} v_1 + \cdots + \frac{a_{j-1}(\lambda_{j-1} - \lambda_j)}{a_j} v_{j-1}.$$

Each $(\lambda_k - \lambda_j) \neq 0$ because the eigenvalues are distinct. Since we chose $j$ to be the smallest such that the $v_j$ is in the span of the preceding vectors, $a_k = 0$ for $k = 1, \dots, j-1$. Thus, $v_j = 0$, but that is a contradiction because $v_j$ is an eigenvector.

$\square$

**Problem Theorem 5.13 Number of Eigenvalues.** Suppose $V$ is finite-dimensional. Then each operator on $V$ has at most $dim(V)$ distinct eigenvalues.

*Proof.* Let $T \in \mathcal{L}(V)$. Let $(\lambda_1, \ldots \lambda_n)$ be a list of distinct eigenvalues in $F$, and let $(u_1, \ldots, u_m)$ be a corresponding list of eigenvectors. By Theorem 5.10, $(v_1, \ldots, v_n)$ is a linearly independent list. Choose a basis $(v_1, \ldots, v_n)$ of $V$. By Theorem 2.23, $m \leq n = dim(V)$. $\square$

**Problem 5.A.12.** Define $T \in \mathcal{L}(P_4(\mathbb{R}))$ by

$$T(p(x)) = xp^{'}(x)$$

for all $x \in \mathbb{R}$. Find all the eigenvalues and eigenvectors.

**Eigenvalues.** Let $p(x) = a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4$. Assume $T(p(x)) = \lambda p(x)$ with $\lambda \in \mathbb{R}$. We have $\lambda p(x) = xp^{'}(x)$, so

$$\lambda a_1 + (\lambda a_2 - a_2)x + \cdots + (\lambda a_5 - a_5)x^4 = 0.$$

The list $(1, x, \ldots, x^4)$ is linearly independent in $P_4(\mathbb{R})$, which leads to $\lambda = 1$.

**Eigenvectors.** We have shown that all solution pairs must have $\lambda = 1$. Assume that $T(p(x)) = 1p(x)$. We have

$$a_1 + (a_2 - a_2)x + \cdots + (a_5 - a_5)x^4 = 0,$$

which means solutions are of the form $(0, a_2, \ldots, a_5)$ for $a_j \in \mathbb{R}$.