

# Selected Problems Chapter 3

## Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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**Problem Integration.** Define  $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$  by

$$Tp = \int_0^1 p(x)dx.$$

Show that  $T$  is a linear map.

*Proof.*

### **Additivity**

Given  $p, q \in \mathcal{P}(\mathbb{R})$ , we want the additivity propriety to hold for  $T$ . Applying  $T$  to the sum of  $p$  and  $q$ , we have

$$\begin{aligned} T(p + q) &= \int_0^1 p(x) + q(x)dx \\ &= \int_0^1 p(x)dx + \int_0^1 q(x)dx \\ &= T(p) + T(q), \end{aligned}$$

since integration of a sum is equal to the sum of the integrated parts.

### **Homogeneity**

Given  $p \in \mathcal{P}(\mathbb{R})$  and  $a \in F$ , we want the homogeneity property to hold. Applying  $T$  to the scalar multiple of  $p$ , we have

$$\begin{aligned} T(a * p) &= \int_0^1 a * p(x)dx \\ &= a * \int_0^1 p(x)dx \\ &= a * T(p), \end{aligned}$$

since constants can be separated in integration.

□

**Problem Theorem 3.5.** Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_m \in W$ . Show that there exists a unique linear map  $T : V \rightarrow W$  such that

$$T(v_j) = w_j$$

for each  $j = 1, \dots, n$ .

*Proof.* We must first show the existence of a linear map with the desired properties. Define  $T : V \rightarrow W$  by

$$T(a_1v_1 + \dots + a_nv_n) = a_1w_1 + \dots + a_nw_n$$

, where  $a_1, \dots, a_n$  are coefficients in  $F$ .

We must show that  $T$  is a linear map. Given  $a_1v_1, \dots, a_nv_n \in V$  and  $b_1v_1, \dots, b_nv_n \in V$ , we have

$$\begin{aligned} T'((a_1v_1 + \dots + a_nv_n) + (b_1v_1 + \dots + b_nv_n)) &= T'((a_1 + b_1)v_1 + \dots + (a_n + b_n)v_n) \\ &= (a_1 + b_1)w_1 + \dots + (a_n + b_n)w_n \\ &= (a_1w_1 + \dots + a_nw_n) + (b_1w_1 + \dots + b_nw_n) \\ &= T(a_1v_1 + \dots + a_nv_n) + T(b_1v_1 + \dots + b_nv_n). \end{aligned}$$

Similarly, given  $a_1v_1 + \dots + a_nv_n \in V$  and  $\lambda \in F$ , we have

$$\begin{aligned} T(\lambda(a_1v_1 + \dots + a_nv_n)) &= T((\lambda * a_1)v_1 + \dots + (\lambda * a_n)v_n) \\ &= (\lambda * a_1)w_1 + \dots + (\lambda * a_n)w_n \\ &= \lambda(a_1w_1 + \dots + a_nw_n) \\ &= \lambda * T(a_1v_1 + \dots + a_nv_n) \end{aligned}$$

Thus, we have shown  $T$  to be a linear map.

Assume the existence of another linear map  $T' : V \rightarrow W$  with the property

$$T'(v_j) = w_j$$

for each  $j = 1, \dots, n$ . To show uniqueness, we want that  $T(v) = T'(v)$  for all  $v \in V$ .

Given  $v \in V$ , we can write  $v = a_1v_1 + \dots + a_nv_n$ , since we have a basis for  $V$ . Then,

$$\begin{aligned} T(v) &= T(a_1v_1 + \dots + a_nv_n) \\ &= a_1T(v_1) + \dots + a_nT(v_n) \\ &= a_1T'(v_1) + \dots + a_nT'(v_n) \\ &= T'(a_1v_1 + \dots + a_nv_n) \\ &= T'(v), \end{aligned}$$

since  $T(v_i) = w_i = T'(v_i)$  for each  $j = 1, \dots, n$ . □

**Problem 3.A.11.** Suppose  $V$  is finite-dimensional. Prove that every linear map on a subspace of  $V$  can be extended to a linear map on  $V$ . In other words, show that if  $U$  is a subspace of  $V$  and  $S \in \mathcal{L}(U, W)$ , then there exists  $T \in \mathcal{L}(V, W)$  such that  $Tu = Su$  for all  $u \in U$ .

*Proof.* Given a subspace  $U$  of  $V$  and a linear map  $S \in \mathcal{L}(U, W)$ , we want to extend  $S$  to be a linear map on  $V$ . Choose a basis of  $U$  to be  $u_1, \dots, u_m$ . We can extend our chosen basis for  $U$  to be a basis of  $V$  as the list  $u_1, \dots, u_m, v_1, \dots, v_n$ . Let

$$w_j = Su_j$$

for  $j = 1, \dots, m$ . Thus the linear map  $S$  can be explicitly written as

$$\begin{aligned} Su &= S(a_1u_1 + \dots + a_mu_m) \\ &= a_1w_1 + \dots + a_mw_m \end{aligned}$$

, for all  $u \in U$ .

We are now in the position to define the extension linear map  $T$ . We define  $T$  by

$$\begin{aligned} Tv &= T(a_1u_1 + \dots + a_mu_m + a_{m+1}v_1 + \dots + a_{m+n}v_n) \\ &= a_1w_1 + \dots + a_mw_m \end{aligned}$$

, for all  $v \in V$ . It is clear that  $Tu = Su$  for all  $u \in U$  by the definition of  $T$ ;  $T$  is also a linear map.

□

**Problem 3.A.14.** Suppose  $V$  is finite-dimensional with  $\dim V \geq 2$ . Prove that there exists  $S, T \in \mathcal{L}(V, V)$  such that  $ST \neq TS$ .

*Proof.* Assume that  $n \geq 2$ . Choose a basis  $v_1, \dots, v_n \in V$ . We will define both linear maps as follows:

$$\begin{aligned} Tv &= T(a_1v_1 + \dots + a_nv_n) \\ &= a_nv_1 + a_1v_2 + \dots + a_{n-1}v_n \end{aligned}$$

, which shifts coefficients by 1 in a circle; Also,

$$\begin{aligned} Sv &= S(a_1v_1 + \dots + a_nv_n) \\ &= a_1v_1. \end{aligned}$$

We will now show that these linear maps do not commute. Let  $v = v_1 + 2v_2$ . We have

$$\begin{aligned} ST(v) &= ST(v_1 + 2v_2) \\ &= S(2v_1 + v_2) \\ &= 2v_1, \end{aligned}$$

and

$$\begin{aligned} TS(v) &= TS(v_1 + 2v_2) \\ &= T(v_1) \\ &= v_2. \end{aligned}$$

Thus,  $ST \neq TS$ . □

**Problem 3.B.2.** Suppose  $V$  is a vector space and  $S, T \in \mathcal{L}(V, V)$  are such that  $\text{range}(S) \subset \text{null}(T)$ . Prove that  $(ST)^2 = 0$ .

*Proof.* Given  $u \in V$ , we must show that  $((ST)(ST))u = 0$ . By the definition of the product of linear maps, we have

$$((ST)(ST))u = (ST)(STu)$$

Let  $w = (STu)$ , which is in  $\text{range}(S)$ . Because  $w \in \text{range}(S)$ ,  $w$  is also in  $\text{null}(T)$  by our assumption. We have

$$\begin{aligned} ((ST)(ST))u &= (ST)(STu) \\ &= (ST)w \\ &= (STw) \\ &= (S0) \\ &= 0, \end{aligned}$$

as desired.

□

**Problem 3.B.12.** Suppose that  $V$  is finite-dimensional and that  $T \in \mathcal{L}(V, W)$ . Prove that there exists a subspace  $U$  of  $V$  such that  $U \cap \text{null}(T) = \{0\}$  and  $\text{range}(T) = \{Tu \mid u \in U\}$ .

*Proof.* Choose a subspace  $U$  of  $V$  such that  $U \oplus \text{null}(T) = V$ , which is guaranteed by Theorem 2.34. Since  $U$  and  $\text{null}(T)$  form a direct sum, we have

$$U \cap \text{null}(T) = \{0\}.$$

The next part is to show that  $\text{range}(T) = \{Tu \mid u \in U\}$ . The right to left inclusion is clear by the definition of range; we must show the left to right inclusion to finish the proof. Given  $w \in \text{range}(T)$ , choose  $v \in V$  such that  $w = Tv$ . Since  $U \oplus \text{null}(T) = V$ , choose  $u \in U$  and  $n \in \text{null}(T)$  such that  $v = u + n$ . We have

$$\begin{aligned} w &= Tv \\ &= T(u + n) \\ &= T(u) + T(n) \\ &= T(u) + 0 \\ &= T(u), \end{aligned}$$

as desired.

□

**Problem Fundamental Theorem of Linear Maps.** Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then  $\text{range}(T)$  is finite-dimensional and

$$\dim(V) = \dim(\text{null}(T)) + \dim(\text{range}(T))$$

*Proof.* By Theorem 2.34, we can choose a subspace  $U$  of  $V$  such that  $U \oplus \text{null}(T) = V$ . By Problem 3.B.12,  $\text{range}(T) = \{Tu \mid u \in U\}$ . Let  $B_1 = \{u_1, \dots, u_m\}$  of  $U$ , and let  $B_2 = \{n_1, \dots, n_k\}$  of  $\text{null}(T)$ .

We want  $B_1 \cup B_2$  to be a basis for  $V$ . Since  $U \oplus \text{null}(T) = V$ ,  $B_1 \cup B_2$  spans  $V$ . We need to show that  $B_1 \cup B_2$  is also linearly independent. It suffices to show that the zero-vector can be represented uniquely. Assume that the zero-vector can be written as

$$0 = \sum_{i=1}^m \lambda_i u_i + \sum_{i=m+1}^{m+k} \lambda_i n_i$$

and

$$0 = \sum_{i=1}^m \alpha_i u_i + \sum_{i=m+1}^{m+k} \alpha_i n_i.$$

Since  $U$  and  $\text{null}(T)$  form a direct sum, any vector in  $V$  can be written uniquely as a vector from  $U$  and a vector from  $\text{null}(T)$ ; Thus, we have

$$\sum_{i=1}^m \lambda_i u_i = \sum_{i=1}^m \alpha_i u_i$$

and

$$\sum_{i=m+1}^{m+k} \lambda_i n_i = \sum_{i=m+1}^{m+k} \alpha_i n_i.$$

It follows from the fact that  $B_1$  and  $B_2$  are basis lists that  $\lambda_i = \alpha_i$  for all  $i$ . Thus,  $B_1 \cup B_2$  forms a basis for  $V$ .

We know that  $\dim(V) = m + k$  from  $B_1 \cup B_2$  being a basis. We also know that  $\dim(\text{null}(T)) = k$  from  $B_2$  being a basis of  $\text{null}(T)$ . It suffices to show that  $\dim(\text{range}(T)) = m$ . We must show that  $R = \{Tu_1, \dots, Tu_m\}$  is a basis for  $\text{range}(T)$ . By Problem 3.B.12,  $R$  spans  $\text{range}(T)$ . For a contradiction, suppose that  $R$  is linearly dependent. We can choose  $Tu_j$  in  $\text{span}(Tu_1, \dots, Tu_{j-1})$  by the Linear Dependence Lemma. We have

$$T(u_j) = \lambda_1 T(u_1) + \dots + \lambda_{j-1} T(u_{j-1}),$$

so

$$\begin{aligned}T(u_j) &= T(\lambda_1 u_1 + \cdots + \lambda_{j_1} u_{j-1}) \\0 &= T(\lambda_1 u_1 + \cdots + \lambda_{j_1} u_{j-1} - u_j)\end{aligned}$$

Thus,  $\lambda_1 u_1 + \cdots + \lambda_{j_1} u_{j-1} - u_j \in \text{null}(T)$ , with not all the constants being zero, but we know that  $U \cap \text{Null}(T) = \{0\}$ , which is a contradiction.

□



**Problem A map to a smaller dimensional space is not injective.** Suppose  $V$  and  $W$  are finite-dimensional vector spaces and  $\dim(V) > \dim(W)$ . Then no linear map from  $V$  to  $W$  is injective.

The intuition is that  $v$  is being transferred into a smaller space, so there is no way that points in  $W$  receive a unique member in  $V$ , since there are simply too many elements in  $V$ .

*Proof.* We are given a linear map  $T \in \mathcal{L}(V, W)$ . Since  $\text{range}(T) \subseteq W$ ,  $\dim(\text{range}(T)) \leq \dim(W)$ . By the Fundamental Theorem of Linear Maps, we have

$$\begin{aligned}\dim(\text{null}(T)) &= \dim(V) - \dim(\text{range}(T)) \\ &\geq \dim(V) - \dim(W) \\ &\geq 1\end{aligned}$$

Thus, for any basis of  $\text{null}(T)$ , there is at least one non-zero element which is mapped to the zero-vector in  $W$ . The zero-vector in  $V$  is also mapped to the zero-vector in  $W$ .  $T$  is not injective as desired.

□

**Problem A map to a larger dimensional space is not surjective.** Suppose  $V$  and  $W$  are finite-dimensional vector spaces and  $\dim(V) < \dim(W)$ . Then no linear map from  $V$  to  $W$  is surjective.

*Proof.* We are given a linear map  $T \in \mathcal{L}(V, W)$ . By the Fundamental Theorem of Linear Maps, we have

$$\begin{aligned}\dim(\text{range}(T)) &= \dim(V) - \dim(\text{null}(T)) \\ &\leq \dim(V) \\ &< \dim(W).\end{aligned}$$

Thus, for any basis of  $\text{range}(T)$ , we can extend it to a basis of  $W$ , revealing a vector not in  $\text{range}(T)$ ; thus,  $T : V \rightarrow W$  is not surjective.

□

**Problem A map to a larger dimensional space is not surjective.** Suppose  $V$  and  $W$  are finite-dimensional vector spaces and  $\dim(V) < \dim(W)$ . Then no linear map from  $V$  to  $W$  is surjective.

*Proof.* We are given a linear map  $T \in \mathcal{L}(V, W)$ . By the Fundamental Theorem of Linear Maps, we have

$$\begin{aligned}\dim(\text{range}(T)) &= \dim(V) - \dim(\text{null}(T)) \\ &\leq \dim(V) \\ &< \dim(W).\end{aligned}$$

Thus, for any basis of  $\text{range}(T)$ , we can extend it to a basis of  $W$ , revealing a vector not in  $\text{range}(T)$ ; thus,  $T : V \rightarrow W$  is not surjective.

□

**Problem 3.B.20.** Suppose  $W$  is finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Prove that  $T$  is injective if and only if there exists  $S \in \mathcal{L}(W, V)$  such that  $ST$  is the identity map on  $V$ .

*Proof.* We will first prove the forward direction. Assume that  $T$  is injective. Define  $S : \text{range}(T) \rightarrow V$  by

$$Sw = v$$

where  $v$  is the unique element in  $V$  such that  $Tv = w$ . We must show that  $ST$  is the identity map on  $V$ . Choose  $v \in V$ . Then

$$\begin{aligned} (ST)v &= S(Tv) \\ &= v, \end{aligned}$$

by the definition of  $S$ .

We will prove the backward direction. Choose  $S \in \mathcal{L}(W, V)$  such that  $ST$  is the identity map on  $V$ . Assume that  $Tu = Tw$  for some  $u, w \in V$ . Using  $(ST)$ , we have

$$\begin{aligned} u &= (ST)u \\ &= S(Tu) \\ &= S(Tw) \\ &= (ST)w \\ &= w, \end{aligned}$$

meaning  $T$  is injective.

□

**Problem Edited 3.B.29** ( $V$  is assumed to be finite-dimensional). Suppose  $T \in \mathcal{L}(V, F)$ . Suppose  $u \in V$  is not in  $\text{null}(T)$ . Prove that

$$V = \text{null}(T) \oplus \{au \mid a \in F\}$$

*Proof.* We will first construct a certain basis for  $V$ . Choose  $u \in V$  such that  $u$  is not in  $\text{null}(T)$ . Let  $\{n_1, \dots, n_m\}$  be a basis for  $\text{null}(T)$ . We want

$$B = \{n_1, \dots, n_m, u\}$$

to be a basis for  $V$ .

For a contradiction, suppose that  $B$  is not a basis for  $V$ . Thus, we can extend  $B$  to a basis of  $V$ :

$$B_e = \{n_1, \dots, n_m, u, e_1, \dots, e_n\}.$$

We consider the complex numbers to be a vector space over itself, so  $\dim(F) = 1$ . We want a contradiction with  $\dim(\text{range}(T)) \leq \dim(F)$ . We will do this by showing that  $R = \{Tu, Te_1, \dots, Te_n\}$  is a basis for  $\text{range}(T)$ . Given  $r \in \text{range}(T)$ , choose  $a_1, \dots, a_{m+n+1} \in F$  such that

$$\begin{aligned} r &= T((a_1n_1 + \dots + a_mn_m) + a_{m+1}u + (a_{m+2}e_1 + \dots + a_{m+n+1}e_n)) \\ &= a_{m+1}T(u) + a_{m+2}T(e_1) + \dots + a_{m+n+1}T(e_n) \end{aligned}$$

, so  $R$  spans  $\text{range}(T)$ . Next, we need to show that  $R$  is linearly independent. Choose  $a_1, \dots, a_{n+1} \in F$  such that

$$\begin{aligned} 0 &= a_1T(u_1) + a_2T(e_1) + \dots + a_{n+1}T(e_m) \\ &= T(a_1u_1 + a_2e_1 + \dots + a_{n+1}e_m), \end{aligned}$$

so  $a_1u_1 + a_2e_1 + \dots + a_{n+1}e_m \in \text{span}(n_1, \dots, n_m)$ . For some  $b_1, \dots, b_m \in F$ , we have

$$\begin{aligned} 0 &= (a_1u_1 + a_2e_1 + \dots + a_{n+1}e_m) - (b_1n_1 + \dots + b_mn_m) \\ &, \end{aligned}$$

which means that all the scalars must be 0 because the vectors form a basis for  $V$ . Thus,  $R$  is a basis for  $\text{range}(T)$ .

Now, for the contradiction.

$$\begin{aligned} \dim(\text{range}(T)) &= 1 + n \\ &> 1 \\ &= \dim(F), \end{aligned}$$

which is not possible because  $\dim(F) = 1$ . So  $B$  is a basis for  $V$ .

We've established that  $B$  is a basis for  $V$ . All that is left is to show that  $V = \text{null}(T) + \{au \mid a \in F\}$  and  $\text{null}(T) \cap \{au \mid a \in F\} = \{0\}$ . Because  $B$  span  $V$ , it is clear that  $V = \text{null}(T) + \{au \mid a \in F\}$ . We assumed that  $u \notin \text{null}(T)$ , so the  $\text{null}(T) \cap \{au \mid a \in F\} = \{0\}$ .  $\square$

**Problem Theorem 3.36 matrix of sum of linear maps.** Suppose  $S, T \in \mathcal{L}(V, W)$ . Then  $M(S + T) = M(S) + M(T)$ .

*Proof.* Let  $v_1, \dots, v_n$  be our chosen basis for  $V$  and  $w_1, \dots, w_m$  be our chosen basis for  $W$ . The sum of  $S$  and  $T$  applied to  $v_j$  is

$$\begin{aligned}(S + T)v_j &= S(v_j) + T(v_j) \\ &= \sum_{i=1}^m a_{i,j}w_i + \sum_{i=1}^m b_{i,j}w_i \\ &= \sum_{i=1}^m (a_{i,j} + b_{i,j})w_i\end{aligned}$$

Thus,  $M(S + T)$  is defined as

$$M(S + T) = \begin{pmatrix} a_{1,1} + b_{1,1} & \cdots & a_{1,n} + b_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} + b_{m,1} & \cdots & a_{m,n} + b_{m,n} \end{pmatrix}.$$

From our decomposition of  $(S + T)v_j$ , for each  $j$ , above, we have

$$M(S) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix},$$

, and

$$M(T) = \begin{pmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{m,1} & \cdots & b_{m,n} \end{pmatrix}.$$

It is clear that  $M(S + T) = M(S) + M(T)$  follows from the definition of matrix addition.  $\square$

**Problem 3.C.3.** Suppose  $v$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Prove that there exists a basis of  $v$  and a basis of  $W$  such that with respect to these bases, all entries of  $M(T)$  are 0 except that the entries in row  $j$ , column  $j$ , equal 1 for  $1 \leq j \leq \dim(\text{range}(T))$ .

*Proof.*

□