Selected Problems Chapter 2
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Problem 2.A.11. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \ldots v_m, w$ is linearly independent if and only if $w \notin span(v_1, \ldots, v_m)$.

Proof. It is straightforward to show the backward direction of the linear dependence lemma; we'll use this fact in this proof.

For the forward direction, assume that $v_1, \ldots, v_m, w$ is linearly independent. By the linear dependence lemma, there is not a vector in the span of the vectors previous to it. Thus, $w \notin span(v_1, \ldots, v_m)$.

For the backward direction, assume that $w \notin span(v_1, \ldots, v_m)$. By the linear dependence lemma, it is enough to show that there is not a vector in the span of the vectors previous to it. By the linear dependence lemma, there is not a vector in the span of the previous ones in the list $v_1, \ldots, v_m$ because it is linearly independent. All that is left to verify is that $w \notin span(v_1, \ldots, v_m)$; this is our assumption, so the result holds.

$\square$

Problem 2.29 Basis Criterion. A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in F$.

Proof. For the forward direction, assume $v_1, \ldots, v_n$ are vectors in V that form a basis for V. Given $v \in V$, we want v to be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in F$. Since the list forms a basis for V, we can choose $a_1, \ldots, a_n \in F$ such that

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Suppose there exists $b_1, \dots, b_n \in V$ such that

$$v = b_1 v_1 + \cdots + b_n v_n.$$

. Then,

$$0 = (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n,$$

. so by linear independence, each coefficient must be equal, meaning that v is uniquely determined.

Next, we must show the backwards direction. Assume that every $v \in V$ can be written uniquely as a linear combination of $v_1, \dots, v_n$. By definition, $v_1, \dots, v_n$ spans V. We must now show the list in linearly independent. The zero vector is in V, so $0 \in V$ can be written as a linear combination of $v_1, \dots, v_n$, namely

$$0 = 0v_1 + \cdots + 0v_n,$$

. which is unique by assumption. Thus, the list satisfies the conditions of linear independence. □

Problem 2.B.5. Prove or disprove : there exists a basis p_0, p_1, p_2, p_3 of $P_3(F)$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Proof. This is a true statement. Let $p_0 = 1, p_1 = x, p_2 = x^2 + x^3, p3 = x^3$. We must show that this list of vectors spans $P_3(F)$ and is linearly independent.

Given $g = a + bx + cx^2 + dx^3 \in P_3(F)$, we must show the existence of coefficients in F such that g is in the span of the list of vectors defined earlier. Choosing $c_0 = a, c_1 = b, c_2 = c, c_3 = d - c$, we have that

$$\begin{aligned}
(\sum_{i=0}^{3} c_i p_i) &= a + bx + c(x^2 + x^3) + (d - c)x^3 \\
&= a + bx + cx^2 + c(x^3 - x^3) + dx^3 \\
&= a + bx + cx^2 + dx^3 \\
&= g.
\end{aligned}$$

Next, we must show that our list of vectors in $P_3(F)$ is linearly independent. Given $g \in P_3(F)$, suppose there exists $a_0, a_1, a_2, a_3 \in F$ and $b_0, b_1, b_2, b_3 \in F$ such that

$$g = (\sum_{i=0}^{3} a_i p_i) = (\sum_{i=0}^{3} b_i p_i).$$

We want a unique representation g. Subtracting the two representations, we have that

$$
\begin{aligned}
0 = (\sum_{i=0}^{3} a_i p_i) - (\sum_{i=0}^{3} b_i p_i) = (\sum_{i=0}^{3}(a_i - b_i)p_i) \\
&= (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)(x^2 + x^3) + (a_3 - b_3)x^3 \\
&= (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + (a_2 - b_2)(x^3) + (a_3 - b_3)x^3 \\
&.
\end{aligned}
$$

This implies that $a_0 = b_0, a_1 = b_1, a_2 = b_2$. Since $a_2 = b_2$, the fourth term disappears and the final term must also have that $a_3 = b_3$. Thus, all vectors in $P_3(F)$ can be represented uniquely with our list of vectors.

□

Problem 2.B.7. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis for V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis for U.

Proof. This is a false statement. Let $V = \mathbb{R}^4$, with the basis being the standard basis. Consider the span following collection of vectors:

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}
$$

□

Problem 2.C.1. Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Proof. Let $m = \dim U$. We can choose some basis for U to be $u_1, \ldots, u_m \in U$. Since U is a subset of V, we can extend this basis to be a basis of V. However, since all basis for V have the same length, our basis for U can not be extended further, and thus, is already a basis for V.

□

Problem 2.C.6(a). Let $U = \{p \in P_4(F) : p(2) = p(5)\}$. Find a basis for U.

Proof. First, we will propose a basis, and then prove that our list is a basis.

The constraints on the set U lead to the following equation:

$$
a + 2b + 4c + 8d + 16e = a + 5b + 25c + 125d + 625e.
$$

Solving for b, we get

$$
b = -7c - 39d - 203e.
$$

Thus, U is spanned by the following vectors:

$$a\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+c\begin{pmatrix}0\\-7\\1\\0\\0\end{pmatrix}+d\begin{pmatrix}1\\-39\\0\\1\\0\end{pmatrix}+e\begin{pmatrix}1\\-203\\0\\0\\1\end{pmatrix}$$

. The vectors represent the polynomials $1, x^2-7x, x^3-39x, x^4-203x$, which are our proposed basis.

Since the list spans U, it is sufficient to show that the list is linearly independent. Each polynomial is not in the span of the previous polynomials in the list, so by the linear dependence lemma, the list is linearly independent.

□

Problem 2.C.6(b). Extend the basis in part (a) to a basis of $P_4(F)$.

Proof. The standard basis of $P_4(F)$ is $1, x, x^2, x^3, x^4$; this means that the dimension of $P_4(F)$ is 5. We know that if a linearly independent list has the same length as a basis, the linear independent list is a basis itself. Thus, it is sufficient to extend our basis for U by a single polynomial.

Adding the polynomial x to our list, we get a new list: $1, x, x^2-7x, x^3-39x, x^4-203x$. Each polynomial in this new list is not in the span of polynomials before it because each polynomial has a unique highest degree. By the linear dependence lemma, this new list is linearly independent.

□

Problem 2.C.6(c). Find a subspace W of $P_4(F)$ such that $P_4(F) = U \oplus W$.

Proof. We will first define the second vector space by

$$W = \{ax \mid a \in F\}.$$

Clearly, this is a subspace of $P_4(F)$.

We want that $U \cap W = \{0\}$ because this will imply that U and W form a direct sum. Suppose $e \in U \cap W$. Then we have the following equation

$$e = ax = b + c(x^2-7x) + d(x^3-39x) + e(x^4-203x)$$

, where all coefficients are in F. Since e is at most degree 1, c, d and e must be zero. Thus, we have that $e = ax = b$, which means that a is zero and e is the zero vector.

Next, we must show that $P_4(F) = U \oplus W$. A basis $U \oplus W$ can clearly be the basis we defined for $P_4(F)$, so the two sets are equal.

□

Problem 2.C.11. Suppose that U and W are subspaces of $\mathbb{R}^8$ such that dim $U = 3$, dim $W = 5$ and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$

Proof. We will prove this by contradiction. Suppose there exists a non-zero vector $v \in U \cap W$. Let $B = \{u_1, u_2, u_3, w_1, w_2, w_3, w_4, w_5\}$, where u_i form a basis for U and w_j form a basis for W. Since $U + W = \mathbb{R}^8$, B spans $\mathbb{R}^8$. Also, $|B| = 8$ and dim $\mathbb{R}^8 = 8$, so B is a basis for $\mathbb{R}^8$.

We want a contradiction with the fact that B is a linearly independent list. Choose $a_1, a_2, a_3 \in F$ and $b_1, b_2, b_3, b_4, b_5 \in F$ such that

$$v = a_1u_1 + a_2u_2 + a_3u_3,$$

and

$$v = b_1w_1 + b_2w_2 + b_3w_3 + b_4w_4 + b_5w_5.$$

Then, we have that

$$0 = a_1u_1 + a_2u_2 + a_3u_3 - (b_1w_1 + b_2w_2 + b_3w_3 + b_4w_4 + b_5w_5),$$

where not all the coefficients are zero. This means that B is not a linearly independent list, which is a contradiction.

□

Problem 2.C.12. Suppose U and W are both five-dimensional subspaces of $\mathbb{R}^9$ Prove that $U \cap W \neq \{0\}$.

Proof. Using theorem 2.43, we have that

$$\begin{aligned} dim(U + W) &= dimU + dimW - dim(U \cap W) \\ &= \text{`}10 - dim(U \cap W). \end{aligned}$$

Since $U + W$ is a subspace of $\mathbb{R}^9$, $dim(U + W) \leq dim(\mathbb{R}^9)$. Thus, we have

$$dim(\mathbb{R}^9) = 9 \geq 10 - dim(U \cap W)$$

.

By rearranging terms, $dim(U \cap W) \geq 1$, so $U \cap W$ is not the trivial subspace.

□

Problem 2.21 Linear Dependence Lemma. The Linear Dependence Lemma shows that linearly dependence is tied to the idea of redundancy. The redundancy is the fact that we can remove a vector from the list without changing the span, if a list is linearly dependent.

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such that the following hold: (a) $v_j \in span(v_1, \ldots, v_{j-1})$; (b) if the j^{th} term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $span(v_1, \ldots, v_m)$.

Proof. We will first show part (a). Since our list is linearly dependent, the zero vector is not written uniquely:

$$0 = a_1v_1 + \cdots + a_mv_m,$$

where not all a_j are zero. Choose j to be the largest index where $a_j v_j$ is non-zero. Then, we can rewrite zero as

$$0 = a_1 v_1 + \cdots + a_j v_j.$$

Subtracting $a_j v_j$, we have

$$-a_j v_j = a_1 v_1 + \cdots + a_{j-1} v_{j-1},$$

, and then solving for v_j we have

$$v_j = (-a_1/a_j)v_1 + \cdots + (-a_{j-1}/a_j)v_{j-1},$$

proving (a).

To prove (b), we must show that the span of the list with v_j is a subset of the span of the list without v_j. Given $v = b_1 v_1 + \cdots + b_m v_m \in span(v_1, \ldots, v_m)$, we can write v as

$$v = (\sum_{i=1}^{j-1} (b_i + \frac{-a_i * b_j}{a_j}) v_i) + (\sum_{i=j+1}^{m} b_i v_i),$$

by replacing v_j with its representation from (a) and simplifying. Thus, any v in $span(v_1, \ldots, v_m)$ can be written without v_J.

□

Problem Theorem 2.23 Length of linearly independent list ≤ spanning list. In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list.

The spanning list describes all of V, and the linearly independent list describes a subspace contained in V in the smallest way. Intuitively, you'd expect then that the smallest description of a subspace of V should be smaller than a full description of V.

Proof. Let $U = \{u_1, \ldots, u_n\}$ be a linearly independent list in V, and let $W = \{w_1, \ldots w_m\}$ be a spanning list in V.

We will first describe a finite sequence of sets. Take W and add u_1 to the front of it. Since the resulting list is linearly dependent, we can remove some w_j without changing the span by Theorem 2.21. Let S_1 be this resulting list.

We will now define the rest of the sequence. While there are still vectors from W in S_{n-1} and unused vectors in V, take S_{n-1} and add u_n after u_{n-1}. The resulting list is linearly dependent, so we can remove some w_j without changing the span; We can only remove some w_j because otherwise V would not be linearly independent. Let S_n be this resulting list.

For a contradiction, suppose that $|U| > |W|$. The sequence of sets described will terminate with $S_j = \{u_1, \ldots, u_j\}$ for some $j < min(m, n)$, replacing all the vectors from W. The list of vectors S_j spans V, so U must be linearly dependent, since U is an extension of S_j. This a contradiction because we assumed that U is linearly independent.

□

Problem Theorem 2.31 Reduction Theorem. Every spanning list in a vector space can be reduced to a basis of the vector space.

Proof. From a given spanning list, we want to reduce it to a list that is spanning and linearly independent. The Linear Dependence Lemma (Theorem 2.21) states that if a list is linearly dependent, there is some vector within the span of the vectors that preceded it. The contrapositive of this theorem allows us to conclude that a list is linearly independent if there is no vector within the list in the span of the ones that precede it. Thus, we will reduce our spanning list in this way.

Let V be our vector space. We are given a spanning list $B = \{b_1, \ldots, b_m\}$. We will state a process that defines a reduced set of B which is linearly dependent and maintains the span of B.

Step 1. If $v_1 = 0$, remove it from the list. Otherwise, leave the list as is.

Step j. If $v_j \in span(v_1, \ldots, v_{j-1})$, remove v_j from the list. Otherwise, leave the list as is.

This finite process results in a new set B' which has the same span at B and is linearly independent by Theorem 2.21.

□

Problem Theorem 2.33 Extension Theorem. Every linearly independent list in a finite-dimensional vector space can be extended to a basis.

Proof. Let V be a finite-dimensional vector space ,and let $u_1, \ldots, u_n$ be a linearly independent list in V. Let $w_1, \ldots, w_m$ be a basis for V. Adjoining the two lists we have $u_1, \ldots, u_n, w_1, \ldots, w_m$, which is a spanning list. Theorem 2.31 reduces this list to a basis; no u_j is removed because $u_1, \ldots, u_n$ is linearly independent, so vector is in the span of the ones that precede it.

□