Selected Problems Chapter 3
Linear Algebra Done Wrong, Sergei Treil, 1st Edition

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Problem Uniqueness of Determinant. Let $C \in \mathbb{R}^n$ be a column vector, i.e. $C = (c_i)_{i=1,\ldots,n}$.

Show that if $D : (\mathbb{R}^n)^n \to \mathbb{R}$ satisfies

multi-linearity. linearity in each argument

anti-symmetry. switching arguments induces a sign change

normalization. $D(e_1, \ldots, e_n) = 1$

then

$$D(C_1, \ldots, C_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{k=1}^{n} c_{\sigma(k)k}$$

Proof. Let $D : (\mathbb{R}^n)^n \to \mathbb{R}$ be a function satisfying the three conditions. For each index j, we have $C_j = \sum_i^n c_{ij} e_i$. Repeatedly applying the multi-linear property we have

$$D(C_1, \ldots, C_n) = D(\sum_{i_1}^{n} c_{i_1 1} e_{i_1}, \ldots, \sum_{i_n}^{n} c_{i_n n} e_{i_n})$$

$$= \sum_{i_1}^{n} c_{i_1 1} D(e_{i_1}, \ldots, \sum_{i_n}^{n} c_{i_n n} e_{i_n})$$

$$= \ldots$$

$$= \sum_{i_1}^{n} \cdots \sum_{i_n}^{n} \prod_{k=1}^{n} c_{i_k k} D(e_{i_1}, \ldots, e_{i_n})$$

Simplifying the iterated sum, we have

$$= \sum_{i_1, \ldots, i_n} \prod_{k=1}^{n} c_{i_k k} D(e_{i_1}, \ldots, e_{i_n}).$$

By proposition 3.1, $D(e_{i_1}, \ldots, e_{i_n}) = 0$ whenever any two of its arguments are the same. Thus, all products in the sum contain a determinant that permutes the standard basis. By anti-symmetry and normalization, we must multiply by the sign of the permutation. We have

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} c_{\sigma(k)k}.$$

□

Problem Determinant of diagonal matrix. Let A be the diagonal matrix $diag(a_{11}, \ldots, a_{nn})$. Show that $det(A) = \prod_{k=1}^{n} a_{kk}$.

Proof. The *jth* column of A is written as $A_j = a_j e_j$. We have

$$det(A) = det(A_1, \ldots, A_n)$$
$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{k=1}^{n} a_{\sigma(k)k}$$

$\square$

Assume that $\sigma \in S_n$ and $\sigma(k) \neq k$ for some k. Then $\prod_{k=1}^{n} a_{\sigma(k)k} = 0$ because one of its products will be 0, since it is off the diagonal of A. Thus the only valid permutation is the identity, which has a sign of 1. We have

$$= \prod_{k=1}^{n} a_{kk},$$

as desired.

Problem Determinant of Row Multiplication Elementary Matrix. Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be a row multiplication elementary matrix defined by

$$M = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & m & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}.$$

Show that $det(M) = m$.

Proof.

$$\begin{aligned} det(M) &= det(e_1, \ldots, e_m * k, \ldots, e_n) \\ &= m * det(e_1, \ldots, e_n) \\ &= m * 1 \\ &= m \end{aligned}$$

, as desired.

$\square$

Problem Determinant of Row Swap Elementary Matrix. Let $S_{i,j} : \mathbb{R}^n \to \mathbb{R}^n$ be the row swap elementary matrix for rows (i, j). We will define $S_{i,j}$ by its columns. For each k,

   k ≠ i and k ≠ j. $C_k = e_k$.
   k = i. $C_k = e_j$.
   k = j. $C_k = e_i$.
   Prove that $det(S_{i,j}) = -1$.

Proof. Since each column of $S_{i,j}$ is a distinct standard basis vector, each argument in $det(S_{i,j})$ is a distinct standard basis vector. The arguments of $det(S_{i,j})$ are a permutation of $(e_1, \ldots, e_n)$ where the *ith* standard basis vector is switched with the *jth* standard basis vector. By anti-symmetry of the determinant,

$$det(S_{i,j}) = -det(e_1, \ldots, e_n)$$
$$= -1.$$

□