Selected Problems Chapter 2
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem 2.A.11. Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \dots v_m, w$ is linearly independent if and only if $w \notin span(v_1, \dots, v_m)$.

Proof. For the forward direction, assume for a contradiction that the list $v_1, \dots v_m, w$ is linearly independent and $w \in span(v_1, \dots, v_m)$. We can then choose $a_1, \dots, a_m \in F$ such that

$$w = \sum_{i=1}^{m} a_i v_i,$$

so we have that

$$(\sum_{i=1}^{m} a_i v_i) - w = 0.$$

Since not all of the coefficients are equal to zero, the list $v_1, \dots, v_m, w$ is not linearly independent, which is a contradiction.

For the backwards direction, assume for a contradiction that $w \notin span(v_1, \dots, v_m)$ and $v_1, \dots, v_m, w$ is linearly dependent. We can choose $a_1, \dots, a_m, a_{m+1} \in F$, where not all of the coefficients are zero, such that

$$(\sum_{i=1}^{m} a_i v_i) + a_{m+1} w = 0.$$

Since $v_1, \dots, v_m$ are linearly independent, $a_m + 1$ can't be equal to zero, otherwise we'd reach a contradiction. Thus, a_{m+1} is a non-zero coefficient, so we have that

$$w = \sum_{i=1}^{m} \frac{-a_i}{a_{m+1}} v_i$$

, by subtracting $a_{m+1} w$ and dividing out by $-a_{m+1}$. Thus, we conclude that $w \in span(v_1, \dots, v_m)$, which is a contradiction.

$\square$

Problem 2.29 Basis Criterion. A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in F$.

Proof. For the forward direction, assume $v_1, \ldots, v_n$ are vectors in V that form a basis for V. Given $v \in V$, we want v to be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in F$. Since the list forms a basis for V, we can choose $a_1, \ldots, a_n \in F$ such that

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Suppose there exists $b_1, \ldots, b_n \in V$ such that

$$v = b_1 v_1 + \cdots + b_n v_n.$$

. Then,

$$0 = (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n,$$

. so by linear independence, each coefficient must be equal, meaning that v is uniquely determined.

Next, we must show the backwards direction. Assume that every $v \in V$ can be written uniquely as a linear combination of $v_1, \ldots, v_n$. By definition, $v_1, \ldots, v_n$ spans V. We must now show the list in linearly independent. The zero vector is in V, so $0 \in V$ can be written as a linear combination of $v_1, \ldots, v_n$, namely

$$0 = 0v_1 + \cdots + 0v_n,$$

. which is unique by assumption. Thus, the list satisfies the conditions of linear independence. $\square$

Problem 2.B.5. Prove or disprove : there exists a basis p_0, p_1, p_2, p_3 of $P_3(F)$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Proof. This is a true statement. Let $p_0 = 1, p_1 = x, p_2 = x^2 + x^3, p3 = x^3$. We must show that this list of vectors spans $P_3(F)$ and is linearly independent.

Given $g = a + bx + cx^2 + dx^3 \in P_3(F)$, we must show the existence of coefficients in F such that g is in the span of the list of vectors defined earlier. Choosing $c_0 = a, c_1 = b, c_2 = c, c_3 = d - c$, we have that

$$\begin{aligned}
(\sum_{i=0}^{3} c_i p_i) &= a + bx + c(x^2 + x^3) + (d - c)x^3 \\
&= a + bx + cx^2 + c(x^3 - x^3) + dx^3 \\
&= a + bx + cx^2 + dx^3 \\
&= p.
\end{aligned}$$

Next, we must show that our list of vectors in $P_3(F)$ is linearly independent. Given $g \in P_3(F)$, suppose there exists $a_0, a_1, a_2, a_3 \in F$ and $b_0, b_1, b_2, b_3 \in F$ such that

$$g = (\sum_{i=0}^{3} a_i p_i) = (\sum_{i=0}^{3} b_i p_i).$$

We want a unique representation g. Subtracting the two representations, we have that

$$\begin{aligned}
0 = (\sum_{i=0}^{3} a_i p_i) - (\sum_{i=0}^{3} b_i p_i) &= (\sum_{i=0}^{3} (a_i - b_i) p_i) \\
&= (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)(x^2 + x^3) + (a_3 - b_3)x^3 \\
&= (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + (a_2 - b_2)(x^3) + (a_3 - b_3)x^3 \\
&.
\end{aligned}$$

This implies that $a_0 = b_0, a_1 = b_1, a_2 = b_2$. Since $a_2 = b_2$, the fourth term disappears and the final term must also have that $a_3 = b_3$. Thus, all vectors in $P_3(F)$ can be represented uniquely with our list of vectors.

□
