Selected Problems Chapter 1
Introduction to Probability for Data Science
Stanley Chan

Mustaf Ahmed

March 28, 2022

Problem 1.1. (a) Show that

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

for any $0 < r < 1$. Evaluate $\sum_{k=0}^{\infty} r^k$.

(b) Using the result of (a), evaluate

$$1 + 2r + 3r^2 + \dots .$$

(c) Evaluate the sums

$$\sum_{k=0}^{\infty} k \left(\frac{1}{3}\right)^{k+1}, \text{ and } \sum_{k=2}^{\infty} k \left(\frac{1}{4}\right)^{k-1}$$

Proof.

Part (a). The first part is true by Theorem 1.1. For the second part, we must evaluate $\sum_{k=0}^{\infty} r^k = \lim_{k\to\infty} \frac{1-r^{k+1}}{1-r}$. Since the limit of the denominator is not zero, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} r^k &= \lim_{k\to\infty} \frac{1 - r^{k+1}}{1 - r} \\
&= \frac{\lim_{k\to\infty} 1 - r^{k+1}}{\lim_{k\to\infty} 1 - r} \\
&= \frac{1}{1 - r}.
\end{aligned}$$

Part (b). By part (a), we know that $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. Taking the derivative of both sides gives us

$$1 + 2r + 3r^2 + \ldots = \frac{1}{(1-r)^2}.$$

Part (c).

$$\begin{aligned}
\sum_{k=0}^{\infty} k \left(\frac{1}{3}\right)^{k+1} &= \sum_{k=1}^{\infty} k \left(\frac{1}{3}\right)^{k+1} \\
&= \left(\frac{1}{3}\right)^2 \sum_{k=1}^{\infty} k \left(\frac{1}{3}\right)^{k-1} \\
&= \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
\sum_{k=2}^{\infty} k \left(\frac{1}{4}\right)^{k-1} &= -1 + \sum_{k=1}^{\infty} k \left(\frac{1}{4}\right)^{k-1} \\
&= \frac{7}{9}
\end{aligned}$$

$\square$

Problem 1.2. Recall that

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}.$$

Evaluate

$$\sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!}, \text{ and } \sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!}.$$

(a).

$$\begin{aligned}
\sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} &= e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} \\
&= \lambda e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^{k-1}}{k!} \\
&= \lambda e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{k!} \\
&= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda e^{-\lambda} \sum_{k'=0}^{\infty} \frac{\lambda^{k'}}{k'!} \\
&= \lambda e^{-\lambda} e^{\lambda} \\
&= \lambda
\end{aligned}$$

(b).

$$
\begin{aligned}
\sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!} &= e^{-\lambda} \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=1}^{\infty} k^2 \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \lambda \sum_{k=1}^{\infty} k^2 \frac{\lambda^{k-1}}{k!} \\
&= e^{-\lambda} \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} \\
&= e^{-\lambda} \lambda \sum_{k^{'}=0}^{\infty} (k^{'}+1) \frac{\lambda^{k^{'}}}{k^{'}!} \\
&= e^{-\lambda} \lambda (\sum_{k^{'}=0}^{\infty} k^{'} \frac{\lambda^{k^{'}}}{k^{'}!} + \sum_{k^{'}=0}^{\infty} \frac{\lambda^{k^{'}}}{k^{'}!}) \\
&= e^{-\lambda} \lambda (e^{\lambda}\lambda + e^{\lambda}) \\
&= \lambda^2 + \lambda
\end{aligned}
$$

Problem 1.4. (a)

```
import numpy as np

A = np.array([[1,2,3],[4,5,6],[7,8,9]])
x = np.array([[1],[2],[3]])

b = A.dot(x) # answer [14,32,50]
```

(b)

```
import math
import matplotlib.pyplot as plt


inputs = []
start = -math.pi
change = ((2 * math.pi)/1000)

# add values in domain to inputs
for i in range(1000):
  inputs.append(start)
  start = start + change

outputs = [math.sin(inputs[i]) for i in range(len(inputs))]

plt.plot(inputs, outputs)
plt.show()
```

Problem 1.9. A collection of 26 English letters, a-z, is mixed in a jar. Two letters are drawn at random, one after the other.

(a) What is the probability of drawing a vowel (a,e,i,o,u) and a consonant in either order?

(b) Write a MATLAB / Python program to verify your answer in part (a). Randomly draw two letters without replacement and check whether one is a vowel and the other is a consonant. Compute the probability by repeating the experiment 10000 times.

(a). The size of the sample space is $26 * 25 = 650$. The total ways of getting a vowel first and a consonant second is $5 * 21 = 105$. The total ways of getting a vowel second and a consonant first is $5 * 21 = 105$. Thus, the probability is

$$P(A) = \frac{210}{650} \approx .32$$

(b).

```
import random

count = 0

for i in range(100000):
  num_alphabet = [n for n in range(1,27)]

  # get first alphabet letter
  first = random.choice(num_alphabet)

  # remove first choice
  num_alphabet.remove(first)

  # get second alphabet letter
  second = random.choice(num_alphabet)

  if (first <= 5 and second > 5) or (first > 5 and second <= 5):
    count = count + 1

probability = count/100000

print(probability) # approximately .32
```