Selected Problems Chapter 3
Linear Algebra Done Wrong, Sergei Treil, 1st Edition

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Problem Uniqueness of Determinant. Let $C \in \mathbb{R}^n$ be a column vector, i.e. $C = (c_i)_{i=1,\ldots,n}$.

Show that if $D : (\mathbb{R}^n)^n \to \mathbb{R}$ satisfies

multi-linearity. linearity in each argument

anti-symmetry. switching arguments induces a sign change

normalization. $D(e_1, \ldots, e_n) = 1$

then

$$D(C_1, \ldots, C_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} c_{\sigma(k)k}$$

Proof. Let $D : (\mathbb{R}^n)^n \to \mathbb{R}$ be a function satisfying the three conditions. For each index j, we have $C_j = \sum_i^n c_{ij} e_i$. Repeatedly applying the multi-linear property anwe have

$$\begin{aligned}
D(C_1, \ldots, C_n) &= D(\sum_{i_1}^{n} c_{i_1 1} e_{i_1}, \ldots, \sum_{i_n}^{n} c_{i_n n} e_{i_n}) \\
&= \sum_{i_1}^{n} c_{i_1 1} D(e_{i_1}, \ldots, \sum_{i_n} c_{i_n n} e_{i_n}) \\
&= \ldots \\
&= \sum_{i_1}^{n} \cdots \sum_{i_n}^{n} \prod_{k=1}^{n} c_{i_k k} D(e_{i_1}, \ldots, e_{i_n})
\end{aligned}$$

Simplifying the iterated sum, we have

$$= \sum_{i_1, \ldots, i_n} \prod_{k=1}^{n} c_{i_k k} D(e_{i_1}, \ldots, e_{i_n}).$$

By proposition 3.1, $D(e_{i_1}, \ldots, e_{i_n}) = 0$ whenever any two of its arguments are the same. Thus, all products in the sum contain a determinant that permutes the standard basis. By anti-symmetry and normalization, we must multiply by the sign of the permutation. We have

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} c_{\sigma(k)k}.$$

$\square$

Problem Determinant of diagonal matrix. Let A be the diagonal matrix $diag(a_{11}, \ldots, a_{nn})$. Show that $det(A) = \prod_{k=1}^{n} a_{kk}$.

Proof. The jth column of A is written as $A_j = a_j e_j$. We have

$$\begin{aligned} Det(A) &= Det(A_1, \ldots, A_n) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} a_{\sigma(k)k} \end{aligned}$$

□

Assume that $\sigma \in S_n$ and $\sigma(k) \neq k$ for some k. Then $\prod_{k=1}^{n} a_{\sigma(k)k} = 0$ because one of its products will be 0, since it is off the diagonal of A. Thus the only valid permutation is the identity, which has a sign of 1. We have

$$= \prod_{k=1}^{n} a_{kk},$$

as desired.