Convex Functions

Mustaf Ahmed

May 17, 2025

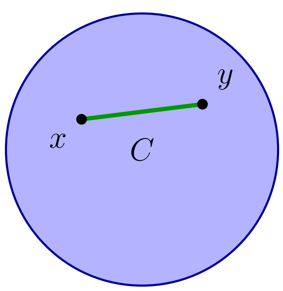
Convex functions are important in optimization because they make optimization problems tractable and computationally efficient.

1 Convex Sets

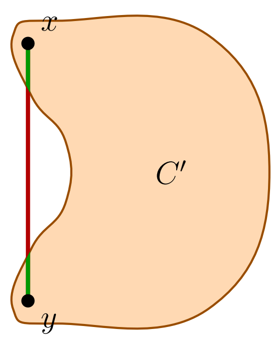
Definition 1 (Convex Set)**.** A set $C \in \mathbb{R}^n$ is convex if and only if for any two elements $x, y \in C$ and $\alpha \in [0, 1]$, we have

$$\alpha x + (1 - \alpha) y \in C$$

That is, in an intuitive sense, we can draw a line between any two points always be inside C; the geometry of C is such that it has no dents. Figure 1 illustrates an example of a convex set and an example of a non-convex set.



(a) Convex set C. The line segment connecting $x, y \in C$ is entirely within C.



(b) Non-convex set C'. Points $x, y \in C'$ are clearly inside the "horns". The segment $x - y$ is not entirely within C'.

Figure 1: Illustrations of convex and non-convex sets.

1.1 Examples

1.1.1 Unit Ball

Theorem 1 (Unit Ball in $\mathbb{R}^n$ is convex)**.** $C = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is convex.

Proof. Let $x, y \in C$ and $\alpha \in [0,1]$. We want that $\alpha x + (1-\alpha)y \in C$.

The condition is true if and only if $\|\alpha x + (1-\alpha)y\| \leq 1$

$$
\begin{aligned}
\|\alpha x + (1-\alpha)y\| &\leq \|\alpha x\| + \|(1-\alpha)y\| && \text{(by Triangle Inequality)} \\
&= \alpha\|x\| + (1-\alpha)\|y\| && \text{(homogeneity)} \\
&\leq \alpha(1) + (1-\alpha)(1) && \text{(since } \|x\| \leq 1, \|y\| \leq 1\text{)} \\
&= \alpha + (1-\alpha) \\
&= 1
\end{aligned}
$$

□