Selected Problems Chapter 2
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem 2.A.11. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \ldots v_m, w$ is linearly independent if and only if $w \notin span(v_1, \ldots, v_m)$.

Proof. For the forward direction, assume for a contradiction that the list $v_1, \ldots v_m, w$ is linearly independent and $w \in span(v_1, \ldots, v_m)$. We can then choose $a_1, \ldots, a_m \in F$ such that

$$w = \sum_{i=1}^{m} a_i v_i,$$

so we have that

$$(\sum_{i=1}^{m} a_i v_i) - w = 0.$$

Since not all of the coefficients are equal to zero, the list $v_1, \ldots, v_m, w$ is not linearly independent, which is a contradiction.

For the backwards direction, assume for a contradiction that $w \notin span(v_1, \ldots, v_m)$ and $v_1, \ldots, v_m, w$ is linearly dependent. We can choose $a_1, \ldots, a_m, a_{m+1} \in F$, where not all of the coefficients are zero, such that

$$(\sum_{i=1}^{m} a_i v_i) + a_{m+1} w = 0.$$

Since $v_1, \ldots, v_m$ are linearly independent, $a_m + 1$ can't be equal to zero, otherwise we'd reach a contradiction. Thus, a_{m+1} is a non-zero coefficient, so we have that

$$w = \sum_{i=1}^{m} \frac{-a_i}{a_{m+1}} v_i$$

, by subtracting $a_{m+1} w$ and dividing out by $-a_{m+1}$. Thus, we conclude that $w \in span(v_1, \ldots, v_m)$, which is a contradiction.

$\square$
