Selected Problems Chapter 2
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Problem 2.A.11. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \ldots v_m, w$ is linearly independent if and only if $w \notin span(v_1, \ldots, v_m)$.

Proof. For the forward direction, assume for a contradiction that the list $v_1, \ldots v_m, w$ is linearly independent and $w \in span(v_1, \ldots, v_m)$. We can then choose $a_1, \ldots, a_m \in F$ such that

$$w = \sum_{i=1}^{m} a_i v_i,$$

so we have that

$$(\sum_{i=1}^{m} a_i v_i) - w = 0.$$

Since not all of the coefficients are equal to zero, the list $v_1, \ldots, v_m, w$ is not linearly independent, which is a contradiction.

For the backwards direction, assume for a contradiction that $w \notin span(v_1, \ldots, v_m)$ and $v_1, \ldots, v_m, w$ is linearly dependent. We can choose $a_1, \ldots, a_m, a_{m+1} \in F$, where not all of the coefficients are zero, such that

$$(\sum_{i=1}^{m} a_i v_i) + a_{m+1} w = 0.$$

Since $v_1, \ldots, v_m$ are linearly independent, $a_m + 1$ can't be equal to zero, otherwise we'd reach a contradiction. Thus, a_{m+1} is a non-zero coefficient, so we have that

$$w = \sum_{i=1}^{m} \frac{-a_i}{a_{m+1}} v_i$$

, by subtracting $a_{m+1} w$ and dividing out by $-a_{m+1}$. Thus, we conclude that $w \in span(v_1, \ldots, v_m)$, which is a contradiction.

$\square$

Problem 2.29 Basis Criterion. A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in F$.

Proof. For the forward direction, assume $v_1, \ldots, v_n$ are vectors in V that form a basis for V. Given $v \in V$, we want v to be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in F$. Since the list forms a basis for V, we can choose $a_1, \ldots, a_n \in F$ such that

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Suppose there exists $b_1, \ldots, b_n \in V$ such that

$$v = b_1 v_1 + \cdots + b_n v_n.$$

. Then,

$$0 = (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n,$$

. so by linear independence, each coefficient must be equal, meaning that v is uniquely determined.

Next, we must show the backwards direction. Assume that every $v \in V$ can be written uniquely as a linear combination of $v_1, \ldots, v_n$. By definition, $v_1, \ldots, v_n$ spans V. We must now show the list in linearly independent. The zero vector is in V, so $0 \in V$ can be written as a linear combination of $v_1, \ldots, v_n$, namely

$$0 = 0v_1 + \cdots + 0v_n,$$

. which is unique by assumption. Thus, the list satisfies the conditions of linear independence. $\square$

Problem 2.B.5. Prove or disprove : there exists a basis p_0, p_1, p_2, p_3 of $P_3(F)$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Proof. This is a true statement. Let $p_0 = 1, p_1 = x, p_2 = x^2 + x^3, p3 = x^3$. We must show that this list of vectors spans $P_3(F)$ and is linearly independent.

Given $g = a + bx + cx^2 + dx^3 \in P_3(F)$, we must show the existence of coefficients in F such that g is in the span of the list of vectors defined earlier. Choosing $c_0 = a, c_1 = b, c_2 = c, c_3 = d - c$, we have that

$$
\begin{aligned}
(\sum_{i=0}^{3} c_i p_i) &= a + bx + c(x^2 + x^3) + (d - c)x^3 \\
&= a + bx + cx^2 + c(x^3 - x^3) + dx^3 \\
&= a + bx + cx^2 + dx^3 \\
&= g.
\end{aligned}
$$

Next, we must show that our list of vectors in $P_3(F)$ is linearly independent. Given $g \in P_3(F)$, suppose there exists $a_0, a_1, a_2, a_3 \in F$ and $b_0, b_1, b_2, b_3 \in F$ such that

$$
g = (\sum_{i=0}^{3} a_i p_i) = (\sum_{i=0}^{3} b_i p_i).
$$

We want a unique representation g. Subtracting the two representations, we have that

$$
\begin{aligned}
0 = (\sum_{i=0}^{3} a_i p_i) - (\sum_{i=0}^{3} b_i p_i) &= (\sum_{i=0}^{3} (a_i - b_i) p_i) \\
&= (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)(x^2 + x^3) + (a_3 - b_3)x^3 \\
&= (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + (a_2 - b_2)(x^3) + (a_3 - b_3)x^3 \\
&.
\end{aligned}
$$

This implies that $a_0 = b_0, a_1 = b_1, a_2 = b_2$. Since $a_2 = b_2$, the fourth term disappears and the final term must also have that $a_3 = b_3$. Thus, all vectors in $P_3(F)$ can be represented uniquely with our list of vectors.

□

Problem 2.B.7. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis for V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis for U.

Proof. This is a false statement. Let $V = \mathbb{R}^4$, with the basis being the standard basis. Consider the span following collection of vectors:

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}
$$

□

Problem 2.C.1. Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Proof. Let $m = \dim U$. We can choose some basis for U to be $u_1, \ldots, u_m \in U$. Since U is a subset of V, we can extend this basis to be a basis of V. However, since all basis for V have the same length, our basis for U can not be extended further, and thus, is already a basis for V.

□

Problem 2.C.6(a). Let $U = \{p \in P_4(F) : p(2) = p(5)\}$. Find a basis for U.

Proof. First, we will propose a basis, and then prove that our list is a basis.

The constraints on the set U lead to the following equation:

$$a + 2b + 4c + 8d + 16e = a + 5b + 25c + 125d + 625e.$$

Solving for b, we get

$$b = -7c - 39d - 203e.$$

Thus, U is spanned by the following vectors:

$$a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ -7 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ -39 \\ 0 \\ 1 \\ 0 \end{pmatrix} + e \begin{pmatrix} 1 \\ -203 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

. The vectors represent the polynomials $1, x^2 - 7x, x^3 - 39x, x^4 - 203x$, which is our proposed basis.

Since the list spans U, it is sufficient to show that the list is linearly independent. Each polynomial is not in the span of the previous polynomials in the list, so by the linear dependence lemma, the list is linearly independent.

□