

# Selected Problems Chapter 6

## Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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**Problem Inner Product Bilinearity.** Let  $V$  be a vector space equipped with an inner product  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ . Show that the inner product is bilinear.

*Proof.* We'll first show additivity in the second slot. We have

$$\begin{aligned}\langle u, v + w \rangle &= \overline{\langle v + w, u \rangle} \\ &= \overline{\langle v, u \rangle + \langle w, u \rangle} \\ &= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle} \\ &= \overline{\langle u, v \rangle} + \overline{\langle u, w \rangle} \\ &= \langle u, v \rangle + \langle u, w \rangle.\end{aligned}$$

For homogeneity in the second slot, we have

$$\begin{aligned}\langle u, \lambda v \rangle &= \overline{\langle \lambda v, u \rangle} \\ &= \overline{\lambda \langle v, u \rangle} \\ &= \lambda \overline{\langle v, u \rangle} \\ &= \lambda \overline{\langle u, v \rangle} \\ &= \lambda \langle u, v \rangle.\end{aligned}$$

□

**Problem Example 6.4(a).** Show that the function  $\langle \cdot, \cdot \rangle : F^n \times F^n \rightarrow \mathbb{C}$  defined by

$$\langle (w_1, \dots, w_n), (z_1, \dots, z_n) \rangle = w_1 \overline{z_1} + \dots + w_n \overline{z_n}$$

is an inner product.

*Proof. Positivity.* Let  $(w_1, \dots, w_n) \in F^n$ . We must show that  $w_1 \overline{w_1} + \dots + w_n \overline{w_n}$  is real and non-negative. It suffices to show that  $w_k \overline{w_k}$  is real and non-negative for each  $k$ . Given  $k \in \{1, \dots, n\}$ , choose  $a, b \in \mathbb{R}$  such that  $w_k = a + bi$ . We have  $w_k \overline{w_k} = (a + bi)(a - bi) = a^2 + b^2 \geq 0$ .

**Definiteness.** For the forward direction, assume that  $\langle w, w \rangle = 0$ . We'll show that each  $w_k = a + bi = 0$ . In the positivity proof, we showed that  $w_k \overline{w_k} \geq 0$ . Since  $\langle w, w \rangle = 0$ , each  $w_k \overline{w_k} = 0$ . We have

$$\begin{aligned} 0 &= w_k \overline{w_k} \\ &= a^2 + b^2 \end{aligned}$$

,so  $a = 0$  and  $b = 0$ . The backward direction is straightforward.

**Additivity in first slot.** Let  $u, v, w \in F^n$ . Then,

$$\begin{aligned} \langle u + v, w \rangle &= (u_1 + v_1) \overline{w_1} + \dots + (u_n + v_n) \overline{w_n} \\ &= (u_1 + \overline{w_1} + \dots + u_n + \overline{w_n}) + (v_1 + \overline{w_1} + \dots + v_n + \overline{w_n}) \\ &= \langle u, w \rangle + \langle v, w \rangle. \end{aligned}$$

**Homogeneity in first slot.** Let  $\lambda \in F$  and let  $u, v \in F^n$ . Then,

$$\begin{aligned} \langle \lambda u, v \rangle &= \lambda u_1 \overline{v_1} + \dots + \lambda u_n \overline{v_n} \\ &= \lambda (u_1 \overline{v_1} + \dots + u_n \overline{v_n}) \\ &= \lambda \langle u, v \rangle. \end{aligned}$$

**Conjugate symmetry.** Let  $u, v \in F^n$ . We have

$$\begin{aligned} \overline{\langle v, u \rangle} &= \overline{v_1 \overline{u_1} + \dots + v_n \overline{u_n}} \\ &= \overline{v_1} u_1 + \dots + \overline{v_n} u_n \\ &= u_1 \overline{v_1} + \dots + u_n \overline{v_n} \\ &= \langle u, v \rangle. \end{aligned}$$

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