Selected Problems Chapter 3
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem Integration. Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ by

$$Tp = \int_0^1 p(x)dx.$$

Show that T is a linear map.

Proof.

Additivity
Given $p, q \in \mathcal{P}(\mathbb{R})$, we want the additivity propriety to hold for T. Applying T to the sum of p and q, we have

$$\begin{aligned} T(p+q) &= \int_0^1 p(x) + q(x)dx \\ &= \int_0^1 p(x)dx + \int_0^1 q(x)dx \\ &= T(p) + T(q), \end{aligned}$$

since integration of a sum is equal to the sum of the integrated parts.
Homogeneity
Given $p \in \mathcal{P}(\mathbb{R})$ and $a \in F$, we want the homogeneity property to hold. Applying T to the scalar multiple of p, we have

$$\begin{aligned} T(a * p) &= \int_0^1 a * p(x)dx \\ &= a * \int_0^1 p(x)dx \\ &= a * T(p), \end{aligned}$$

since constants can be separated in integration.

□

Problem Theorem 3.5. Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m \in W$. Show that there exists a unique linear map $T : V \to W$ such that

$$T(v_j) = w_j$$

for each $j = 1, \dots, n$.

Proof. We must first show the existence of a linear map with the desired properties. Define $T : V \to W$ by

$$T(a_1 v_1 + \dots + a_n v_n) = a_1 w_1 + \dots + a_n w_n$$

, where $a_1, \dots, a_n$ are coefficients in F.

We must show that T is a linear map. Given $a_1 v_1, \dots, a_n v_n \in V$ and $b_1 v_1, \dots, bn_v n \in V$, we have

$$\begin{aligned} T'((a_1 v_1 + \dots + a_n v_n) + ((b_1 v_1 + \dots + b_n v_n))) &= T((a_1 + b_1)v_1 + \dots (a_n + b_n)v_n) \\ &= (a_1 + b_1)w_1 + \dots + (a_n + b_n)w_n \\ &= (a_1 w_1 + \dots + a_n w_n) + (b_1 w_1 + \dots + b_n w_n) \\ &= T(a_1 v_1 + \dots + a_n v_n) + T(b_1 v_1 + \dots + b_n v_n). \end{aligned}$$

Similarly, given $a_1 v_1 + \dots + a_n v_n \in V$ and $\lambda \in F$, we have

$$\begin{aligned} T(\lambda(a_1 v_1 + \dots + a_n v_n)) &= T((\lambda * a_1)v_1 + \dots + (\lambda * a_n)v_n \\ &= (\lambda * a_1)w_1 + \dots + (\lambda * a_n)w_n \\ &= \lambda(a_1 w_1 + \dots + a_n w_n) \\ &= \lambda * T(a_1 v_1 + \dots + a_n v_n) \end{aligned}$$

.

Thus, we have shown T to be a linear map.

Assume the existence of another linear map $T' : V \to W$ with the property

$$T'(v_j) = w_j$$

for each $j = 1, \dots, n$. To show uniqueness, we want that $T(v) = T'(v)$ for all $v \in V$.

Given $v \in V$, we can write $v = a_1 v_1 + \dots + a_n v_n$, since we have a basis for V. Then,

$$\begin{aligned} T(v) &= T(a_1 v_1 + \dots + a_n v_n) \\ &= a_1 T(v_1) + \dots + a_n T(v_n) \\ &= a_1 T'(v_1) + \dots + a_n T'(v_n) \\ &= T'(a_1 v_1 + \dots + a_n v_n) \\ &= T'(v), \end{aligned}$$

since $T(v_i) = w_i = T'(v_i)$ for each $j = 1, \dots, n$. $\square$

Problem 3.A.11. Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Proof. Given a subspace U of v and a linear map $S \in \mathcal{L}(U, W)$, we want to extend S to be a linear map on V. Choose a basis of U to be $u_1, \ldots, u_m$. We can extend our chosen basis for U to be a basis of V as the list $u_1, \ldots, u_m, v_1, \ldots, v_n$. Let

$$w_j = Su_j$$

for $j = 1, \ldots, m$. Thus the linear map S can be explicitly written as

$$\begin{aligned} Su &= S(a_1 u_1 + \cdots + a_m u_m) \\ &= a_1 w_1 + \ldots a_m w_m \end{aligned}$$

, for all $u \in U$.

We are now in the position to define the extension linear map T. We define T by

$$\begin{aligned} Tv &= T(a_1 u_1 + \cdots + a_m u_m + a_{m+1} v_1 + \cdots + a_{m+n} v_n) \\ &= a_1 w_1 + \ldots a_m w_m \end{aligned}$$

, for all $v \in V$.

□