Selected Problems Chapter 5
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

Mustaf Ahmed

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Problem Example 5.8. Suppose $T \in \mathcal{L}(F^2)$ is defined by $T(w, z) = (-z, w)$. Find the eigenvectors and eigenvalues of T if $F = \mathbb{R}$. Find the eigenvectors and eigenvalues of T if $F = \mathbb{C}$

Proof. **Part(a).**

Assume T has eigenvectors and eigenvalues with $F = \mathbb{R}$. The equation $\lambda(w, z) = (-z, w)$ holds and leads to the following system of equations:

$$\lambda w = -z$$
$$\lambda z = w.$$

Solving for λ, we have $\lambda^2 = -1$, which only has solutions in $\mathbb{C}$. This contradiction means T has no eigenvectors and eigenvalues.

Part(b).

In part(a), we showed that the eigenvalues of T must be in the complex numbers. The equation from part(a) $\lambda^2 = -1$ has the solutions $\lambda = i$ and $\lambda = -i$. The eigenvectors corresponding to $\lambda = i$ are of the form $(w, -iw)$ for any $w \in \mathbb{C}$; the eigenvectors corresponding to $\lambda = -i$ are of the form (w, iw).

$\square$

Problem Theorem 5.10 Linearly Independent Eigenvectors. Let $T \in \mathcal{L}(V)$. Suppose $(\lambda_1, \ldots, \lambda_n)$ are distinct eigenvalues of T, and $(v_1, \ldots, v_n)$ are corresponding eigenvectors. Then $(v_1, \ldots, v_n)$ is a linearly independent list.

Proof. For a contradiction, suppose $(v_1, \ldots, v_n)$ is a linearly dependent list. Then choose $a_1, \ldots, a_n \in F$ where not all are zero such that $0 = a_1 v_1 + \ldots a_n v_n$. By the linear dependence lemma, choose the smallest j such that $v_j = \frac{a_1}{a_j} v_1 + \cdots + \frac{a_{j-1}}{a_j} v_{j-1}$. Applying T, we have

$$\lambda_j v_j = \frac{a_1 \lambda_1}{a_j} v_1 + \cdots + \frac{a_{j-1} \lambda_{j-1}}{a_j} v_{j-1}.$$

Subtracting the left side, we have

$$0 = \frac{a_1(\lambda_1 - \lambda_j)}{a_j} v_1 + \cdots + \frac{a_{j-1}(\lambda_{j-1} - \lambda_j)}{a_j} v_{j-1}.$$

Each $(\lambda_k - \lambda_j) \neq 0$ because the eigenvalues are distinct. Since we chose j to be the smallest such that the v_j is in the span of the preceding vectors, $a_k = 0$ for $k = 1, \ldots, j - 1$. Thus, $v_j = 0$, but that is a contradiction because v_j is an eigenvector.

$\square$

Problem Theorem 5.13 Number of Eigenvalues. Suppose V is finite-dimensional. Then each operator on V has at most $dim(V)$ distinct eigenvalues.

Proof. Let $T \in \mathcal{L}(V)$. Let $(\lambda_1, \ldots \lambda_n)$ be a list of distinct eigenvalues in F, and let $(u_1, \ldots, u_m)$ be a corresponding list of eigenvectors. By Theorem 5.10, $(v_1, \ldots, v_n)$ is a linearly independent list. Choose a basis $(v_1, \ldots, v_n)$ of V. By Theorem 2.23, $m \leq n = dim(V)$.
$\square$

Problem 5.A.12. Define $T \in \mathcal{L}(P_4(\mathbb{R}))$ by

$$T(p(x)) = xp'(x)$$

for all $x \in \mathbb{R}$. Find all the eigenvalues and eigenvectors.

Assume T has an eigenvalue λ. Choose a non-zero $p(x) = a_0 + a_1 x + \cdots + a_4 x^4 \in P_4(\mathbb{R})$ such that $T(p(x)) = \lambda p(x)$. This is equivalent to the system of equations

$$\lambda a_0 = 0$$
$$\lambda a_1 = a_1$$
$$\lambda a_2 = 2a_2$$
$$\lambda a_3 = 3a_3$$
$$\lambda a_4 = 4a_4$$

This is equivalent to

$$\lambda a_0 = 0$$
$$a_1(\lambda - 1) = 0$$
$$a_2(\lambda - 2) = 0$$
$$a_3(\lambda - 3) = 0$$
$$a_4(\lambda - 4) = 0$$

.

Let $\lambda \in \{0, 1, 2, 3, 4\}$. Then for each $j \neq \lambda$

$$a_j(\lambda - j) = 0$$
$$a_j = 0$$

by diving by $(\lambda - j)$. The corresponding eigenvector for λ is then $p(x) = a_\lambda x^\lambda$. Let $\lambda \notin \{0, 1, 2, 3, 4\}$. By diving the coefficient $(\lambda - j)$ for $j = 0, 1, 2, 3, 4$, we have $a_j = 0$ for each j. Thus, $p(x) = 0$, which is not an eigenvector.

We've shown that the eigenvalues are $\lambda = 0, 1, 2, 3, 4$ with the corresponding eigenvectors $c, cx, cx^2 cx^3, cx^4$ for $c \in \mathbb{R}$.

Problem 5.A.15. Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.
(a) Prove that T and $S^{-1}TS$ have the same eigenvalues
(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

Proof. **Part(a)**

We must show that λ is an eigenvalue of T if and only if λ is an eigenvalue of $S^{-1}TS$. For the forward direction, assume λ is an eigenvalue of T. Choose $v \in V$ such that $Tv = \lambda v$. Since S is invertible, S is also injective and surjective. We can choose $u \in V$ such that $Su = v$. We have

$$
\begin{aligned}
(S^{-1}TS)u &= (S^{-1}T)(Su) \\
&= (S^{-1}T)(v) \\
&= (S^{-1})(\lambda v) \\
&= \lambda u.
\end{aligned}
$$

For the backward direction, assume λ is an eigenvalue of $S^{-1}TS$. Choose $u \in V$ such that $(S^{-1}TS)u = \lambda u$. Applying S to $(S^{-1}TS)u$, we have

$$
\begin{aligned}
(TS)u &= T(Su) \\
&= \lambda(Su)
\end{aligned}
$$

,

as desired.

Part(b)

If v is an eigenvector of T, then $S^{-1}v$ is an eigenvector of $S^{-1}TS$. if v is an eigenvector of $S^{-1}TS$, Sv is an eigenvector of T.

$\square$

Problem 5.A.18. Show that the operator $T \in \mathcal{L}(C^\infty)$ defined by

$$T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$$

has no eigenvalues.

Proof. Assume that $T(z_1, z_2, \dots) = \lambda(z_1, z_2, \dots)$. This is equivalent to the following system of equations

$$\lambda z_1 = 0$$
$$\lambda z_2 = z_1$$
$$\lambda z_3 = z_2$$
$$\dots$$

The first equation implies $\lambda = 0$ or $z_1 = 0$. If $\lambda = 0$, then $(z_1, z_2, \dots)$ is the zero vector. If $z_1 = 0$ each $z_j = 0$, and thus $(z_1, z_2, \dots)$ is the zero vector. We have that $(z_1, z_2, \dots)$ is the zero vector, which can't be associated with any eigenvalue by definition.

$\square$

Problem 5.A.20. Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(F^\infty)$ defined by

$$T(z_1, z_2, \dots) = (z_2, z_3, \dots)$$

Assume $T(z_1, z_2, \dots) = \lambda(z_1, z_2, \dots)$ with $(z_1, z_2, \dots)$ being non-zero. The previous relation is equivalent to the system of equations

$$\lambda z_1 = z_2$$
$$\lambda z_2 = z_3$$
$$\lambda z_3 = z_4$$
$$\dots$$

By substitution of variables, another equivalent form is

$$\lambda z_1 = z_2$$
$$\lambda^2 z_1 = z_3$$
$$\lambda^3 z_3 = z_4$$
$$\dots$$

Thus, all eigenvectors are of the form $(z_1, z_2, z_3, \dots) = (z_1, \lambda z_1, \lambda^2 z_1, \dots)$ with $z_1 \neq 0$. Each $\lambda \in F$ is an eigen vector.

Problem 5.A.22. Suppose $T \in \mathcal{L}(V)$ and there exists nonzero vectors v and w in V such that $T(v) = 3w$ and $T(w) = 3v$. Prove that 3 and -3 are eigenvalues of T.

Proof. We have $T(v + w) = 3(v + w)$ by linearity. Similarly, $= T(v - w) = 3(w - v) = -3(v - w)$.

We must show that $(v + w)$ and $(v - w)$ are not both the zero vector. For a contradiction suppose $(v + w) = (v - w) = 0$. Adding the two vectors, it is clear that $v = 0$.

$\square$

Problem 5.A.30. Suppose $T \in \mathcal{L}(\mathbb{R}^3)$, and $-4, 5$ and $\sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

Proof. We have $Tx - 9x = (T - 9I)x$. Thus, it is enough to show that $(T - 9I)$ is surjective. We can show that $(T - 9I)$ is surjective by showing that it is injective by Theorem 3.69.

For a contradiction, assume that $(T - 9I)$ is not injective. By Theorem 3.16, choose a non-zero vector x in $\mathbb{R}^3$ such that $(T - 9I)x = 0$. Simplifying the right side , we have

$$Tx - 9x = 0,$$

so

$$Tx = 9x.$$

Thus, 9 is an eigenvalue, so we have 4 eigenvalues in total. By Theorem 5.13, there can be at most 3 eigenvalues, which is a contradiction.

$\square$

Problem 5.A.32. Suppose $(\lambda_1, \ldots, \lambda_n)$ is a list of distinct real numbers. Prove that the list $(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$.

Proof. Let the differentiation operator for real-valued function on $\mathbb{R}$ be $T(f(x)) = f'(x)$. By Theorem 5.10, it suffices to show that $(\lambda_1, \ldots, \lambda_n)$ are eigenvalues with corresponding eigenvectors $(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$.

Applying the differentiation operator for any j, we have

$$T(e^{\lambda_j x}) = \lambda_j e^{\lambda_j x},$$

as desired.

$\square$

Problem 5.B.2. Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Proof. Choose a non-zero vector $v \in V$ such that $Tv = \lambda v$. Applying $(T-2I)(T-3I)(T-4I)$ to v, we have

$$
\begin{aligned}
(T - 2I)(T - 3I)(T - 4I)v &= (T - 2I)(T - 3I)((\lambda - 4)v) \\
&= (T - 2I)((\lambda^2 - 7\lambda + 12)v) \\
&= (\lambda^3 - 9\lambda^2 + 26\lambda - 24)v
\end{aligned}
$$

The polynomial $(\lambda^3 - 9\lambda^2 + 26\lambda - 24)$ has the roots $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$, as desired. $\square$

Problem 5.B.18. Suppose V is finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Define a function $f : \mathbb{C} \to \mathbb{R}$ by

$$f(\lambda) = dim(range(T - \lambda I)).$$

Prove that f is not a continuous function.

Proof. By the rank-nullity theorem, we have

$$dim(range(T - \lambda I)) = dim(V) - dim(null(T - \lambda I)),$$

so we can write $f(\lambda) = dim(V) - dim(null(T - \lambda I))$. Further, If λ is not an eigenvalue of T, $dim(null(T - \lambda I)) = 0$; If λ is an eigenvalue of T, $dim(null(T - \lambda I)) > 0$, because $null(T - \lambda I)$ contains eigenvectors corresponding to λ. Thus, we can write

$$f(\lambda) = \begin{cases} dim(V) - dim(null(T - \lambda I)) & \text{for } \lambda \text{ an eigenvalue of } T \\ dim(V) & \text{for } \lambda \text{ not eigenvalue of } T \end{cases}$$

.

We want a discontinuity at an eigenvalue of T. The vector space V is finite-dimensional, complex and non-zero, so by Theorem 5.27 there is at least one eigenvalue for T. Choose an eigenvalue $\lambda_0 \in \mathbb{C}$.

We want to show a discontinuity at λ_0. Choose $\epsilon = \frac{dim(null(T - \lambda_0 I))}{2}$, which is greater than 0. Let $\delta > 0$. Choose $x \in \mathbb{C}$ such that $|x - \lambda_0| < \delta$ and x is not an eigenvalue; we can do this because there are at most $dim(V)$ eigenvalues. We have

$$\begin{aligned} |f(x) - f(\lambda_0)| &= |dim(V) - dim(V) - dim(null(T - \lambda_0 I)| \\ &= dim(null(T - \lambda_0 I)) \\ &> \frac{dim(null(T - \lambda_0 I))}{2} = \epsilon, \end{aligned}$$

as desired.

$\square$