# Selected Problems Chapter 1
# Real Mathematical Analysis, Pugh, Second Edition

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**Problem 8 Statement.** Suppose that the natural number k is not a perfect $n^{\text{th}}$.

(a) Prove that its $n^{\text{th}}$ root is irrational

(b) Infer that the $n^{\text{th}}$ root of a natural number is either a natural number or it is irrational. It is never a fraction.

**Problem 8 (a).**

*Proof.* We will prove this by contradiction. Suppose the $n^{\text{th}}$ root of $k$ is rational. Choose $p, q \in \mathbb{Z}$, where $q \neq 0$, such that the $n^{\text{th}}$ root $r = \frac{p}{q}$. Then $k = r^n = \frac{p^n}{q^n}$. Since $k$ is an integer, $q$ must divide $p$. This $r$ is an integer, and therefore k is a perfect $n^{\text{th}}$ root, a contradiction. $\square$

**Problem 8 (b).**

A natural number is either a perfect $n^{\text{th}}$ root or it is not. If it is not a perfect $n^{\text{th}}$ root, By (a), we know the $n^{\text{th}}$ root must be irrational. If it is a perfect $n^{\text{th}}$ root, by definition the $n^{\text{th}}$ root must be a an integer.

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**Problem 12 Statement.** Prove that there exists no smallest positive real number. Does there exists a smallest positive rational number? Given a real number $x$, does there exist a smallest real number $y > x$?

*Proof.* We the first part by contradiction. Suppose there did exist a smallest positive real number $x \in \mathbb{R}$. Consider $y = \frac{x}{2}$. Clearly $y < x$ .Similarly, given $x \in \mathbb{Q}$ we can define $y = x - 1$, another rational number, and it is clear that $y = x - 1 < x$.

We will prove the final part by contradiction. Suppose there did exist a smallest number $y > x$. We will construct an even smaller real number that satisfies this property. Let $z = x + \frac{|y-x|}{2}$. We have

$$\begin{aligned} z &= x + \frac{|y-x|}{2} \\ &< x + |y-x| \\ &= y \end{aligned}$$

$\square$

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**Problem 13 Statement.** Let $b =$ l.u.b $S$, where S is a bounded nonempty subset of $\mathbb{R}$.
(a) Given $\epsilon > 0$ show that there exists an $s \in S$ with

$$b - \epsilon \leq s \leq b.$$

(b) Can $s \in S$ always be found so that $b - \epsilon < s < b$.

**Problem 13 (a).**

*Proof.* Since $b =$ Sup $S$, it must be the case that there exist some $s \in S$ satisfying this property, otherwise $b$ would no longer be the Sup for $S$. □

**Problem 13 (b).**

*Proof.* No. Consider $S = \{1, 3\}$ and let $\epsilon = 1$. There does not exist $s \in S$ where

$$3 - 1 < s < 3.$$

□

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**Problem 20 Statement.** Prove that limits are unique, i.e., if $(a_n)$ is a sequence of real numbers that converges to a real number $b$ and also converges to a real number $b^{'}$, then $b = b^{'}$.

**Problem 20.**

*Proof.* Assume that $(a_n)$ converges to $b$ and $b^{'}$ in $\mathbb{R}$. By the epsilon principle, we want :

$$\forall \epsilon > 0 \ , \ |b - b^{'}| < \epsilon.$$

Given $\epsilon > 0$ in $\mathbb{R}$, choose $N, N^{'} \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, \text{ if } n \geq N, \ |a_n - b| < \frac{\epsilon}{2},$$

and

$$\forall n \in \mathbb{N}, \text{ if } n \geq N^{'}, \ |a_n - b^{'}| < \frac{\epsilon}{2}.$$

Let $M = max(N, N^{'})$, and choose $n \in \mathbb{N}$ such that $n \geq M$. Since $n \geq N, N^{'}$, we have

$$|a_n - b^{'}| < \frac{\epsilon}{2},$$

and

$$|a_n - b| < \frac{\epsilon}{2}.$$

Thus, we have

$$-\epsilon < b - a_n + a_n - b^{'} < \epsilon,$$

which means that

$$|b - b^{'}| < \epsilon.$$

□

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**Problem 22 Statement.** A fixed-point of a function $f : A \to A$ is a point $a \in A$ such that $f(a) = a$. The diagonal of $A \times A$ is the set of all pairs $(a, a)$ in $A \times A$.

(a) Show that $f : A \to A$ has a fixed-point if and only if the graph of $f$ intersects the diagonal.

(b) Prove that every continuous function $f : [0, 1] \to [0, 1]$ has at least one fixed-point

(c) Is the same true for continuous functions $f : (0, 1) \to (0, 1)$?

(d) Is the same true for discontinuous functions?

**Problem 22 (a).**

*Proof.* We will first prove the forward direction. Assume that $f : A \to A$ has a fixed-point. Choose $a \in A$ such that $f(a) = a$. Thus, $f : A \to A$ crosses the point $(a, a)$, which is in the diagonal.

We will prove the backward direction. Assume $f : A \to A$ intersects the diagonal. This means that for some $a \in A$ $f(a) = a$, so the function has a fixed-point. □

**Problem 22 (b).**

*Proof.* Given a continuous function $f : [0, 1] \to [0, 1]$ □