Introduction to Modern Algebra I, Spring 2017, Columbia University

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Problem Theorem 1.25. Given an equivalence relation $\sim$ on a set X, the equivalence classes of X form a partition of X. Conversely, if $\mathcal{P} = \{X_i\}$ is a partition of a set X, then there is an equivalence relation on X with equivalence classes X_i.

Proof. For the forward direction, assume that $\sim$ is an equivalence relation on X. Let $x \in X$. The equivalence class $[x]$ is non empty because $x \sim x$. It follows that $\bigcup_{x \in X} [x] = X$. To finish this direction, we need to show that $[x] \cap [y] = \emptyset$ or $[x] = [y]$ for any $x, y \in X$. Assume $[x] \cap [y] \neq \emptyset$. Choose $z \in [x] \cap [y]$. By symmetry and transitivity $x \sim y$, so by transitivity $[y] \subseteq [x]$; a similar argument can be made to show that $[x] \subseteq [y]$. Thus, $[x] = [y]$. Now, assume $[x] \cap [y] = \emptyset$. We are done because trivially follows.

For the backward direction, assume $\mathcal{P} = \{X_i\}$ is a partition of a set X. We'll define the relation $R = \{(x, y) \mid X_i \in \mathcal{P} \text{ and } x, y \in X_i\}$.

Reflexivity. Let $x \in X$. Since $\mathcal{P}$ is a partition, x must be in some X_i. It is clear that x and itself are in the same partition, so R has the reflexive property.

Symmetry. Assume $(x, y) \in R$. Then $x, y \in X_i$ for some i. By the definition of R, $(y, x) \in R$ as well.

Transitivity. Assume $(x, y) \in R$ and $(y, z) \in R$. Then, $x, y \in X_i$ for some i, and $y, z \in X_j$ for some j. We want that $i = j$. Since the partition is formed from mutually disjoint sets, $i = j$. Thus, $x, z \in X_i$, so $x \sim z$ as desired.

□

Problem Corollary 1.26. Two equivalence classes of an equivalence relation are either disjoint or equal.

Proof. Shown in the forward direction of Theorem 1.25.

□

Problem 1. List all subsets of the 3-element set $A = \{1, 2, 3\}$. How many subsets does a set with n elements have? How many of these subsets have at most two elements?

Part (a). The sets are $\emptyset, \{1\}\{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

Part (b). Find how many subsets exist for a set with n elements amounts to summing all possible sizes for combinations of elements in the set:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Part (c). We need to exclude the combinations where $i <= 2$:

$$\begin{aligned}\sum_{i=2}^{n} \binom{n}{i} &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} \\ &= 1 + n + \frac{n(n-1)}{2}\end{aligned}$$

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Problem 3. Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Proof. We want that $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$ and $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$.

$$\begin{aligned} x \in A \setminus (B \cup C) &\iff x \in A \text{ and } x \notin B \cup C \\ &\iff x \in A \text{ and } x \notin B \text{ and } x \notin C \\ &\iff x \in A \text{ and } x \notin B \text{ and } x \in A \text{ and } x \notin C \\ &\iff x \in A \setminus B \text{ and } x \in A \setminus C \\ &\iff x \in (A \setminus B) \cap (A \setminus C) \end{aligned}$$

□

Problem 4. Part (a). Consider sets $A = \{a, b\}$ and $B = \{1, 2, 3\}$. How many injective maps are there from A to B? Give an example of such a map. How many injective maps are there from B to A.

Part (b). Suppose $f : A \to A$ is injective and A is a finite set. Prove that A is bijective. Give an example of an infinite set B and an injective map $f : B \to B$ which is not surjective.

Proof. **Part (a).** We have 3 choices for where to send a in B. We have 2 choices (to avoid hitting the same element from our first choice) left for where to send b in B. Thus, there are $3 * 2 = 6$ possible such functions. An example of an injective map from A to B is $f = \{(a, 1), (b, 2)\}$.

Part (b). We want to show that $A = Im(f)$. Since $f : A \to A$ is injective, every element in $Im(f)$ is unique. Thus, $|Im(f)| = |A|$. Assume for a contradiction that $A \setminus Im(f)$ is non-empty. We have

$$\begin{aligned} |A| &= |Im(f) \cup (A \setminus Im(f))| \\ &= |im(f)| + k \\ &= |A| + k \end{aligned}$$

for some non-zero $k \in \mathbb{N}$. This is a contradiction. Thus, $Im(f) = A$, so f is surjective.

The map $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = 2 * n$ is injective, but it is not surjective because the outputs are always even.

□

Problem 5. Show that for any set A, there is exactly one map f from the empty set $\emptyset$ to A. When f is injective? When f is surjective?

Proof. Let A be a set. Assume that $g : \emptyset \to A$ and $f : \emptyset \to A$. We want that $f = g$. By the definition of a function, $f \subseteq \emptyset \times A$ such that the following holds: for all $x \in \emptyset$, there is unique ordered pair (x, y) with $y \in A$. The set $f \subseteq \emptyset \times A$ is empty because $\emptyset$ contains no elements. Thus, $f = \emptyset$, and the condition for functions holds vacuously. By an identical argument, $g = \emptyset$, and the condition for functions also holds vacuously. We have $f = \emptyset = g$, as desired.

The condition that f is injective is the following: for all $x, y \in \emptyset$ if $f(x) = f(y)$ then $x = y$. This holds vacuously for any such f.

The condition that f is surjective is the following: for all $y \in A$ there exists $x \in \emptyset$ such that $f(x) = y$. The statement is true when $A = \emptyset$.

□

Problem 6. Prove that for sets A, B, C

$$| A \cup B \cup C | = | A | + | B | + | B | - | A \cap B | - | A \cap C | - | B \cap C | + | A \cap B \cap C |$$

Proof. Using a analogous result for 2 sets, we have:

$$\begin{aligned} | A \cup (B \cup C) | &= | A | + | B \cup C | - | A \cap (B \cup C) | \\ &= | A | + | B \cup C | - | (A \cap B) \cup (A \cap C) | \end{aligned}$$

Repeated use of this result yields

$$\begin{aligned} &= | A | + | B \cup C | - | (A \cap B) \cup (A \cap C) | \\ &= | A | + | B | + | C | - | B \cap C | - | (A \cap B) \cup (A \cap C) | \\ &= | A | + | B | + | C | - | B \cap C | - | A \cap C | - | A \cap B | - | A \cap C \cap B | \end{aligned}$$

□

Problem 7. Given maps $f : A \to B$ and $g : B \to C$ such that gf is surjective, prove that g is surjective. Give an example with surjective gf but not surjective f.

Proof. Assume that $c \in C$. We want to find $b \in B$ such that $g(b) = c$. Since gf is surjective, we can choose a $a \in A$ such that $g(f(a)) = c$. All we need is for $f(a) \in B$. Since f is a function from A to B, $f(a) \in B$.

Define $f : \mathbb{N} \to \mathbb{Z}$ by $f(n) = n$. Define $g : \mathbb{Z} \to \mathbb{N}$ by $g(z) = | z |$. The function gf is the identity between the natural number domain and natural number comdomain, so it is surjective. the function f will never hit a negative number in $\mathbb{Z}$, so it is not surjective.

□