Selected Problems Chapter 3
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem Integration. Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ by

$$Tp = \int_0^1 p(x)dx.$$

Show that T is a linear map.

Proof.

Additivity

Given $p, q \in \mathcal{P}(\mathbb{R})$, we want the additivity propriety to hold for T. Applying T to the sum of p and q, we have

$$\begin{aligned} T(p+q) &= \int_0^1 p(x) + q(x)dx \\ &= \int_0^1 p(x)dx + \int_0^1 q(x)dx \\ &= T(p) + T(q), \end{aligned}$$

since integration of a sum is equal to the sum of the integrated parts.

Homogeneity

Given $p \in \mathcal{P}(\mathbb{R})$ and $a \in F$, we want the homogeneity property to hold. Applying T to the scalar multiple of p, we have

$$\begin{aligned} T(a * p) &= \int_0^1 a * p(x)dx \\ &= a * \int_0^1 p(x)dx \\ &= a * T(p), \end{aligned}$$

since constants can be separated in integration.

□

Problem Theorem 3.5. Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m \in W$. Show that there exists a unique linear map $T : V \to W$ such that

$$T(v_j) = w_j$$

for each $j = 1, \dots, n$.

Proof. We must first show the existence of a linear map with the desired properties. Define $T : V \to W$ by

$$T(a_1v_1 + \cdots + a_nv_n) = a_1w_1 + \cdots + a_nw_n$$

, where $a_1, \dots, a_n$ are coefficients in F.

We must show that T is a linear map. Given $a_1v_1, \dots, a_nv_n \in V$ and $b_1v_1, \dots, bn_vn \in V$, we have

$$\begin{aligned} T'((a_1v_1 + \cdots + a_nv_n) + ((b_1v_1 + \cdots + b_nv_n))) &= T((a_1 + b_1)v_1 + \dots (a_n + b_n)v_n) \\ &= (a_1 + b_1)w_1 + \cdots + (a_n + b_n)w_n \\ &= (a_1w_1 + \cdots + a_nw_n) + (b_1w_1 + \cdots + b_nw_n) \\ &= T(a_1v_1 + \cdots + a_nv_n) + T(b_1v_1 + \cdots + b_nv_n). \end{aligned}$$

Similarly, given $a_1v_1 + \cdots + a_nv_n \in V$ and $\lambda \in F$, we have

$$\begin{aligned} T(\lambda(a_1v_1 + \cdots + a_nv_n)) &= T((\lambda * a_1)v_1 + \cdots + (\lambda * a_n)v_n \\ &= (\lambda * a_1)w_1 + \cdots + (\lambda * a_n)w_n \\ &= \lambda(a_1w_1 + \cdots + a_nw_n) \\ &= \lambda * T(a_1v_1 + \cdots + a_nv_n) \end{aligned}$$

.

Thus, we have shown T to be a linear map.

Assume the existence of another linear map $T' : V \to W$ with the property

$$T'(v_j) = w_j$$

for each $j = 1, \dots, n$. To show uniqueness, we want that $T(v) = T'(v)$ for all $v \in V$.

Given $v \in V$, we can write $v = a_1v_1 + \cdots + a_nv_n$, since we have a basis for V. Then,

$$\begin{aligned} T(v) &= T(a_1v_1 + \cdots + a_nv_n) \\ &= a_1T(v_1) + \cdots + a_nT(v_n) \\ &= a_1T'(v_1) + \cdots + a_nT'(v_n) \\ &= T'(a_1v_1 + \cdots + a_nv_n) \\ &= T'(v), \end{aligned}$$

since $T(v_i) = w_i = T'(v_i)$ for each $j = 1, \dots, n$. $\square$

Problem 3.A.11. Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Proof. Given a subspace U of V and a linear map $S \in \mathcal{L}(U, W)$, we want to extend S to be a linear map on V. Choose a basis of U to be $u_1, \ldots, u_m$. We can extend our chosen basis for U to be a basis of V as the list $u_1, \ldots, u_m, v_1, \ldots, v_n$. Let

$$w_j = Su_j$$

for $j = 1, \ldots, m$. Thus the linear map S can be explicitly written as

$$\begin{aligned} Su &= S(a_1u_1 + \cdots + a_mu_m) \\ &= a_1w_1 + \ldots a_mw_m \end{aligned}$$

, for all $u \in U$.

We are now in the position to define the extension linear map T. We define T by

$$\begin{aligned} Tv &= T(a_1u_1 + \cdots + a_mu_m + a_{m+1}v_1 + \cdots + a_{m+n}v_n) \\ &= a_1w_1 + \ldots a_mw_m \end{aligned}$$

, for all $v \in V$. It is clear that $Tu = Su$ for all $u \in U$ by the definition of T; T is also a linear map.

$\square$

Problem 3.A.14. Suppose V is finite-dimensional with $dim\ V \geq 2$. Prove that there exists $S, T \in \mathcal{L}(V, V)$ such that $ST \neq TS$.

Proof. Assume that $n \geq 2$. Choose a basis $v_1, \ldots, v_n \in V$. We will define both linear maps as follows:

$$\begin{aligned} Tv &= T(a_1v_1 + \cdots + a_nv_n) \\ &= a_nv_1 + a_1v_2 + \cdots + a_{n-1}v_n \end{aligned}$$

, which shifts coefficients by 1 in a circle; Also,

$$\begin{aligned} Sv &= S(a_1v_1 + \cdots + a_nv_n) \\ &= a_1v_1. \end{aligned}$$

We will now show that these linear maps do not commute.Let $v = v_1 + 2v_2$. We have

$$\begin{aligned} ST(v) &= ST(v_1 + 2v_2) \\ &= S(2v_1 + v_2) \\ &= 2v_1, \end{aligned}$$

and

$$\begin{aligned} TS(v) &= TS(v_1 + 2v_2) \\ &= T(v_1) \\ &= v_2. \end{aligned}$$

Thus, $ST \neq TS$. $\square$

Problem 3.B.2. Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that $range(S) \subset null(T)$. Prove that $(ST)^2 = 0$.

Proof. Given $u \in V$, we must show that $((ST)(ST))u = 0$. By the definition of the product of linear maps, we have

$$((ST)(ST))u = (ST)(STu)$$

Let $w = (STu)$, which is in $range(S)$. Because $w \in range(S)$, w is also in $null(T)$ by our assumption. We have

$$\begin{aligned}((ST)(ST))u &= (ST)(STu) \\ &= (ST)w \\ &= (STw) \\ &= (S0) \\ &= 0,\end{aligned}$$

as desired.

$\square$

Problem 3.B.12. Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap null(T) = \{0\}$ and $range(T) = \{Tu \mid u \in U\}$.

Proof. Choose a subspace U of V such that $U \oplus null(T) = V$, which is guaranteed by Theorem 2.34. Since U and $null(T)$ form a direct sum, we have

$$U \cap null(T) = \{0\}.$$

The next part is to show that $range(T) = \{Tu \mid u \in U\}$. The right to left inclusion is clear by the definition of range; we must show the left to right inclusion to finish the proof. Given $w \in range(T)$, choose $v \in V$ such that $w = Tv$. Since $U \oplus null(T) = V$, choose $u \in U$ and $n \in null(T)$ such that $v = u + n$. We have

$$\begin{aligned} w &= Tv \\ &= T(u+n) \\ &= T(u) + T(n) \\ &= T(u) + 0 \\ &= T(u), \end{aligned}$$

as desired.

$\square$

Problem Fundamental Theorem of Linear Maps. Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $range(T)$ is finite-dimensional and

$$dim(V) = dim(null(T)) + dim(range(T))$$

Proof. By Theorem 2.34, we can choose a subspace U of V such that $U \oplus null(T) = V$. By Problem 3.B.12, $range(T) = \{Tu \mid u \in U\}$. Let $B_1 = \{u_1, \dots, u_m\}$ of U, and let $B_2 = \{n_1, \dots, n_k\}$ of $null(T)$.

We want $B_1 \cup B_2$ to be a basis for V. Since $U \oplus null(T) = V$, $B_1 \cup B_2$ spans V. We need to show that $B_1 \cup B_2$ is also linearly independent. It suffices to show that the zero-vector can be represented uniquely. Assume that the zero-vector can be written as

$$0 = \sum_{i=1}^{m} \lambda_i u_i + \sum_{i=m+1}^{m+k} \lambda_i n_i$$

and

$$0 = \sum_{i=1}^{m} \alpha_i u_i + \sum_{i=m+1}^{m+k} \alpha_i n_i.$$

Since U and $null(T)$ form a direct sum, any vector in V can be written uniquely as a vector from U and a vector from $null(T)$; Thus, we have

$$\sum_{i=1}^{m} \lambda_i u_i = \sum_{i=1}^{m} \alpha_i u_i$$

and

$$\sum_{i=m+1}^{m+k} \lambda_i n_i = \sum_{i=m+1}^{m+k} \alpha_i n_i.$$

It follows from the fact that B_1 and B_2 are basis lists that $\lambda_i = \alpha_i$ for all i. Thus, $B_1 \cup B_2$ forms a basis for V.

We know that $dim(V) = m + k$ from $B_1 \cup B_2$ being a basis. We also know that $dim(null(T)) = k$ from B_2 being a basis of $null(T)$. It suffices to show that $dim(range(T)) = m$. We must show that $R = \{Tu_1, \dots, Tu_m\}$ is a basis for $range(T)$. By Problem 3.B.12, R spans $range(T)$. For a contradicting, suppose that R is linearly dependent. We can choose Tu_j in $span(Tu_1, \dots, Tu_{j-1})$ by the Linear Dependence Lemma. We have

$$T(u_j) = \lambda_1 T(u_1) + \dots + \lambda_{j_1} T(u_{j-1}),$$

so

$$\begin{aligned} T(u_j) &= T(\lambda_1 u_1 + \cdots + \lambda_{j_1} u_{j-1}) \\ 0 &= T(\lambda_1 u_1 + \cdots + \lambda_{j_1} u_{j-1} - u_j) \end{aligned}$$

Thus, $\lambda_1 u_1 + \cdots + \lambda_{j_1} u_{j-1} - u_j \in null(T)$, with not all the constants being zero, but we know that $U \cap Null(T) = \{0\}$, which is a contradiction.

$\square$