Part III: Continuous Random Variables
Introduction to Probability for Computing

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December 15, 2024

Problem Theorem 9.3. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then $Var(X) = \sigma^2$.

Proof. Assume that $X \sim \mathcal{N}(\mu, \sigma^2)$. Using the definition of $Var(X)$, we have

$$
\begin{aligned}
Var(X) &= E[(X-\mu)^2] \\
&= \int_{\mathbb{R}} (x-\mu)^2 f(x)\, dx \\
&= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\ dx
\end{aligned}
$$

Let $z = (x - \mu)$ for a substitution. Thus,

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}\ dz \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2\sigma^2}}\ dz \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2\sigma^2}}\ dz
\end{aligned}
$$

We can use symmetry of the integrand to change the bounds:

$$
= \frac{2}{\sigma\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-\frac{z^2}{2\sigma^2}}\ dz
$$

Let $y = \frac{z^2}{2\sigma^2}$, so $dz = \frac{\sigma^2}{z} dy$. Hence,

$$
\begin{aligned}
&= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty 2\sigma^2 y e^{-y} \frac{\sigma^2}{z} \ dy \\
&= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty \sqrt{y}\sqrt{2}\sigma^3 e^{-y} \ dy \\
&= \frac{2\sqrt{2}\sigma^3}{\sigma\sqrt{2\pi}} \int_0^\infty \sqrt{y} e^{-y} \ dy \\
&= \frac{2\sqrt{2}\sigma^3}{\sigma\sqrt{2\pi}} \int_0^\infty \sqrt{y} e^{-y} \ dy \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty \sqrt{y} e^{-y} \ dy
\end{aligned}
$$

The integral $\int_0^\infty y^{1/2} e^{-y}\, dy$ is a standard Gamma function, which simplifies to $\Gamma\left(\frac{3}{2}\right)$. Thus,

$$
\begin{aligned}
&= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma(\frac{3}{2}) \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \\
&= \sigma^2,
\end{aligned}
$$

as desired.

$\square$

Problem Theorem 9.3. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then $E(X) = \mu$.

Proof. Assume that $X \sim \mathcal{N}(\mu, \sigma^2)$. Using the definition of $E(X)$, we have

$$\begin{aligned}
E(X) &= \int_{\mathbb{R}} x f(x)\, dx \\
&= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\ dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\ dx
\end{aligned}$$

Let $y = \frac{x-\mu}{\sigma}$, so $dx = \sigma dy$. Substituting,

$$\begin{aligned}
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma y + \mu) e^{-\frac{y^2}{2}} \sigma\ dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma y + \mu) e^{-\frac{y^2}{2}} \sigma\ dy \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2}}\ dy + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\ dy
\end{aligned}$$

The integral $\frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2}}\ dy$ evaluates to 0 since y is an odd function. Hence,

$$\begin{aligned}
&= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\ dy \\
&= \frac{\mu\sqrt{2\pi}}{\sqrt{2\pi}} \\
&= \mu
\end{aligned}$$

$\square$

Lemma 1. *The following lemma will be used in the proof of Theorem 9.5.*

Let $X \sim \mathcal{N}(\mu, \sigma^2)$, and let $Y = aX + b$, where $a > 0$ and $b \in \mathbb{R}$. Let $F_X(x)$ denote the cumulative distribution function of X. Then, the cumulative distribution function of Y, denoted $F_Y(y)$, satisfies:

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right).$$

Proof.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(aX + b \leq y) \\ &= P(X \leq \frac{y-b}{a}) \\ &= F_X\left(\frac{y-b}{a}\right) \end{aligned}$$

□

Problem Theorem 9.5 (Linear Transformation Property). Let $X \sim \mathcal{N}(\mu, \sigma^2)$, and let $Y = aX + b$, where $a > 0$ and $b \in \mathbb{R}$. Then, $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

Proof. Using Lemma 1, $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$. By the fundamental theorem of calculus, differentiating $F_Y(y)$ yields the probability density function for Y. Hence,

$$\begin{aligned} f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) \\ &= \frac{\mathrm{d}}{\mathrm{d}y} F_X\left(\frac{y-b}{a}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y-b}{a}\right) f_X\left(\frac{y-b}{a}\right) \\ &= \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \\ &= \frac{1}{a}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}} \\ &= \frac{1}{a}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-(a\mu+b))^2}{2a^2\sigma^2}} \\ &= \frac{1}{\sqrt{2\pi a^2\sigma^2}} e^{-\frac{(y-(a\mu+b))^2}{2a^2\sigma^2}} \end{aligned}$$

By the definition of the normal distribution, $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$, as desired.

□