

Selected Problems Chapter 3

Linear Algebra Done Wrong, Sergei Treil, 1st Edition

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Problem Uniqueness of Determinant. Let $C \in \mathbb{R}^n$ be a column vector, i.e. $C = (c_i)_{i=1, \dots, n}$.

Show that if $D : (\mathbb{R}^n)^n \rightarrow \mathbb{R}$ satisfies

multi-linearity. linearity in each argument

anti-symmetry. switching arguments induces a sign change

normalization. $D(e_1, \dots, e_n) = 1$

then

$$D(C_1, \dots, C_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{k=1}^n c_{\sigma(k)k}$$

Proof. Let $D : (\mathbb{R}^n)^n \rightarrow \mathbb{R}$ be a function satisfying the three conditions. For each index j , we have $C_j = \sum_i^n c_{ij}e_i$. Repeatedly applying the multi-linear property anwe have

$$\begin{aligned} D(C_1, \dots, C_n) &= D\left(\sum_{i_1}^n c_{i_1 1}e_{i_1}, \dots, \sum_{i_n}^n c_{i_n n}e_{i_n}\right) \\ &= \sum_{i_1}^n c_{i_1 1} D\left(e_{i_1}, \dots, \sum_{i_n}^n c_{i_n n}e_{i_n}\right) \\ &= \dots \\ &= \sum_{i_1}^n \dots \sum_{i_n}^n \prod_{k=1}^n c_{i_k k} D(e_{i_1}, \dots, e_{i_n}) \end{aligned}$$

Simplifying the iterated sum, we have

$$= \sum_{i_1, \dots, i_n} \prod_{k=1}^n c_{i_k k} D(e_{i_1}, \dots, e_{i_n}).$$

By proposition 3.1, $D(e_{i_1}, \dots, e_{i_n}) = 0$ whenever any two of its arguments are the same. Thus, all products in the sum contain a determinant that permutes the standard basis. By anti-symmetry and normalization, we must multiply by the sign of the permutation. We have

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{k=1}^n c_{\sigma(k)k}.$$

□