Selected Problems Chapter 6
Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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Problem Inner Product Bilinearity. Let V be a vector space equipped with an inner product $\langle ., . \rangle : V \times V \to \mathbb{R}$. Show that the inner product is bilnear.

Proof. We'll first show additivity in the second slot. We have

$$\begin{aligned}
\langle u, v + w \rangle &= \overline{\langle v + w, u \rangle} \\
&= \overline{\langle v, u \rangle + \langle w, u \rangle} \\
&= \langle v, u \rangle + \langle w, u \rangle \\
&= \overline{\langle u, v \rangle} + \overline{\langle u, w \rangle} \\
&= \langle u, v \rangle + \langle u, w \rangle.
\end{aligned}$$

For homogenity in the second slot, we have

$$\begin{aligned}
\langle u, \lambda v \rangle &= \overline{\langle \lambda v, u \rangle} \\
&= \overline{\lambda \langle v, u \rangle} \\
&= \lambda \langle v, u \rangle \\
&= \lambda \overline{\langle u, v \rangle} \\
&= \lambda \langle u, v \rangle.
\end{aligned}$$

□

Problem Example 6.4(a). Show that the function $\langle .,. \rangle : F^n \times F^n \to \mathbb{C}$ define by

$$\langle (w_1, \dots, w_n), (z_1, \dots, z_n) \rangle = w_1\overline{z_1} + \cdots + w_n\overline{z_n}$$

is an inner product.

Proof. **Positivity.** Let $(w_1, \dots, w_n) \in F^n$. We must show that $w_1\overline{w_1} + \cdots + w_n\overline{w_n}$ is real and non-negative. It suffices to show that $w_k\overline{w_k}$ is real and non-negative for each k. Given $k \in \{1, \dots, n\}$, choose $a, b \in \mathbb{R}$ such that $w_k = a + bi$. We have $w_k\overline{w_k} = (a+bi)(a-bi) = a^2 + b^2 \geq 0$.

Definiteness. For the forward direction, assume that $\langle w, w \rangle = 0$. We'll show that each $w_k = a + bi = 0$. In the positivity proof, we showed that $w_k\overline{w_k} \geq 0$. Since $\langle w, w \rangle = 0$, each $w_k\overline{w_k} = 0$. We have

$$\begin{aligned} 0 &= w_k\overline{w_k} \\ &= a^2 + b^2 \end{aligned}$$

,so $a = 0$ and $b = 0$. The backward direction is straightforward.

Additivity in first slot. Let $u, v, w \in F^n$. Then,

$$\begin{aligned} \langle u + v, w \rangle &= (u_1 + v_1)\overline{w_n} + \cdots + (u_n + v_n)\overline{w_n} \\ &= (u_1\overline{w_1} + \cdots + u_n\overline{w_n}) + (v_1\overline{w_1} + \cdots + v_n\overline{w_n}) \\ &= \langle u, w \rangle + \langle v, w \rangle. \end{aligned}$$

Homogeneity in first slot. Let $\lambda \in F$ and let $u, v \in F^n$. Then,

$$\begin{aligned} \langle \lambda u, v \rangle &= \lambda u_1\overline{v_1} + \cdots + \lambda u_n\overline{v_n} \\ &= \lambda(u_1\overline{v_1} + \cdots + u_n\overline{v_n}) \\ &= \lambda\langle u, v \rangle. \end{aligned}$$

Conjugate symmetry. Let $u, v \in F^n$. We have

$$\begin{aligned} \overline{\langle v, u \rangle} &= \overline{v_1\overline{u_1} + \cdots + v_n\overline{u_n}} \\ &= \overline{v_1}u_1 + \cdots + \overline{v_n}u_n \\ &= u_1\overline{v_1} + \cdots + u_n\overline{v_n} \\ &= \langle u, v \rangle. \end{aligned}$$

$\square$

Problem Theorem 6.10. Let $v \in V$.

(a). $\|v\| = 0$ if and only if $v = 0$.

(b). $\|\lambda v\| = \lambda \|v\|$ for all $\lambda \in F$.

Proof. **Part (a).** Follows straightforwardly from the fact that $\langle v, v \rangle = 0$ if and only if $v = 0$.

Part (b). Let $\lambda \in F$. We have

$$\begin{aligned}
\|\lambda v\|^2 &= \langle \lambda v, \lambda v \rangle \\
&= \lambda \overline{\lambda} \langle v, v \rangle \\
&= |\lambda|^2 \langle v, v \rangle.
\end{aligned}$$

Taking the square root gives the desired result.

□

Problem Theorem 6.13 Pythagorean Theorem. Suppose u and v are orthogonal vectors in V. Then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof. Taking the norm of $u + v$, we have

$$\begin{aligned}
\|u + v\| &= \sqrt{\langle u + v, u + v \rangle} \\
&= \sqrt{\langle u, u + v \rangle + \langle v, u + v \rangle} \\
&= \sqrt{\langle u, u \rangle + \langle u, v \rangle + \langle v, v \rangle + \langle v, u \rangle} \\
&= \sqrt{\langle u, u \rangle + \langle v, v \rangle} \\
&= \sqrt{\|u\|^2 + \|v\|^2}.
\end{aligned}$$

The result follows from squaring both sides.

□

Problem Theorem 6.15 Cauchy–Schwarz Inequality. Let $u, v \in V$. Then,

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$

Proof. We can write $u = \frac{\langle u,v\rangle}{\|v\|^2}v + w$, where w is orthgonal to v, by Theorem 6.14. Using the Pythagorean Theorem, we have

$$\|u\|^2 = \|\frac{\langle u, v\rangle}{\|v\|^2}v + w\|^2 = \|\frac{\langle u, v\rangle}{\|v\|^2}v\|^2 + \|w\|^2,$$

so

$$\begin{aligned} \|u\|^2 &= \|\frac{\langle u, v\rangle}{\|v\|^2}v\|^2 + \|w\|^2 \\ &= \frac{|\langle u, v\rangle|^2}{\|v\|^2} + \|w\|^2. \end{aligned}$$

Solving for $|\langle u, v\rangle|$, we have

$$\begin{aligned} |\langle u, v\rangle| &= \sqrt{\|u\|^2\|v\|^2 - \|w\|^2\|v\|^2} \\ &\leq \sqrt{\|u\|^2\|v\|^2} \\ &= \|u\|\|v\|, \end{aligned}$$

giving the desired relation.

$\square$

Problem Exercise 6.A.1. Show that the function that takes $((x_1, x_2), (y_1, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$ to $|x_1 y_2| + |x_2 y_2|$ is not an inner product.

Proof. The function does not have additivity in the first slot. For example, choose $u = (1, 1)$, $v = (-1, -1)$ and $w = (1, 1)$. Applying the function to $(u + v, w)$, we have

$$\begin{aligned} f(u + v, w) &= |u_1 w_1 + v_1 w_1| + |u_2 w_2 + v_2 w_2| \\ &= |1 + -1| + |1 + -1| \\ &= 0. \end{aligned}$$

However, $f(u, w) + f(v, w) > 0$. $\square$

Problem Exercise 6.A.5. Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ Prove that $T - \sqrt{2}I$ is invertible.

Proof. The linear operator $T - \sqrt{2}$ is invertible if and only if $\sqrt{2}$ is not an eigenvalue. We will show that $\sqrt{2}$ is not an eigenvalue. For a contradiction, suppose $\sqrt{2}$ is an eigenvalue. Choose $v \in V$ such that $Tv = \sqrt{2}v$. We have

$$\begin{aligned} \|Tv\| &= \|\sqrt{2}v\| \\ &= \sqrt{2}\|v\| \\ &> \|v\|, \end{aligned}$$

a contradiction. $\square$

Problem Exercise 6.A.13. Suppose u and v are non-zero vectors in $\mathbb{R}^2$. Prove that

$$\langle u, v\rangle = \|u\|\|v\|\cos\theta,$$

where θ is the angle between u and v.

Proof. By the law of cosines, we have

$$\|u-v\|^2 = \|u\|^2 + \|v^2\| - 2\|u\|\|v\|cos(\theta)$$

.

Note that

$$\begin{aligned} \|u-v\|^2 &= \langle u, u\rangle + \langle u, u\rangle - 2\langle u, v\rangle \\ &= \|u\|^2 + \|v\|^2 - 2\langle u, v\rangle. \end{aligned}$$

By substituting, we have

$$\begin{aligned} \|u\|^2 + \|v\|^2 - 2\langle u, v\rangle &= \|u\|^2 + \|v^2\| - 2\|u\|\|v\|cos(\theta) \\ -2\langle u, v\rangle &= -2\|u\|\|v\|cos(\theta) \\ \langle u, v\rangle &= \|u\|\|v\|cos(\theta). \end{aligned}$$

□

Problem Exercise 6.A.14. The angle between two nonzero vectors $x, y \in \mathbb{R}^n$ is defined to be

$$arccos \frac{\langle x, y \rangle}{\|x\| \|y\|}.$$

Explain why the Cauchy-Schwarz Inequality is needed to show that this definition makes sense.

Since the domain of *arcos* is $[-1, 1]$, we want $-1 \leq \frac{\langle x,y \rangle}{\|x\|\|y\|} \leq 1$. By the Cauchy-Schwarz inequality, it follows that

$$|\langle x, y \rangle| \leq \|x\| \|y\|,$$

so

$$\frac{|\langle x, y \rangle|}{\|x\| \|y\|} = \mid \frac{\langle x, y \rangle}{\|x\| \|y\|} \mid \leq 1,$$

as desired.

Problem Exercise 6.A.16. Suppose $u, v \in V$ are such that

$$\|u\| = 3, \|u + v\| = 4, \|u - v\| = 6.$$

What number does $\|v\|$ equal?

Proof. By the Parallelogram Equality, we have

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Solving for $\|v\|$ yields $\|v\| = \sqrt{17}$.

□

Problem Theorem 6.30 Vector as linear combination of orthonormal basis. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $v \in V$. Then

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n,$$

and

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2.$$

Proof. Choose $a_1, \ldots, a_n \in \mathbb{F}$ such that $v = a_1 e_1 + \ldots a_n e_n$. Computing the inner product, we have

$$\begin{aligned}
\langle v, e_j \rangle &= \langle \sum_{i=1}^{n} a_i e_i, e_j \rangle \\
&= \sum_{i=1}^{n} \langle a_i e_i, e_j \rangle \\
&= \sum_{i=1}^{n} a_i \langle e_i, e_j \rangle \\
&= a_i \langle e_j, e_j \rangle \\
&= a_i \|e_j\|^2 \\
&= a_i,
\end{aligned}$$

where the fourth equality holds from pair-wise orthogonality. The first part follows, and the second part follows from Theorem 6.25.

□