

# Selected Problems Chapter 6

## Linear Algebra Done Right, Sheldon Axler, 3rd Edition

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**Problem Inner Product Bilinearity.** Let  $V$  be a vector space equipped with an inner product  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ . Show that the inner product is bilinear.

*Proof.* We'll first show additivity in the second slot. We have

$$\begin{aligned}\langle u, v + w \rangle &= \overline{\langle v + w, u \rangle} \\ &= \overline{\langle v, u \rangle + \langle w, u \rangle} \\ &= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle} \\ &= \overline{\langle u, v \rangle} + \overline{\langle u, w \rangle} \\ &= \langle u, v \rangle + \langle u, w \rangle.\end{aligned}$$

For homogeneity in the second slot, we have

$$\begin{aligned}\langle u, \lambda v \rangle &= \overline{\langle \lambda v, u \rangle} \\ &= \overline{\lambda \langle v, u \rangle} \\ &= \lambda \overline{\langle v, u \rangle} \\ &= \lambda \overline{\langle u, v \rangle} \\ &= \lambda \langle u, v \rangle.\end{aligned}$$

□

**Problem Example 6.4(a).** Show that the function  $\langle \cdot, \cdot \rangle : F^n \times F^n \rightarrow \mathbb{C}$  defined by

$$\langle (w_1, \dots, w_n), (z_1, \dots, z_n) \rangle = w_1 \overline{z_1} + \dots + w_n \overline{z_n}$$

is an inner product.

*Proof. Positivity.* Let  $(w_1, \dots, w_n) \in F^n$ . We must show that  $w_1 \overline{w_1} + \dots + w_n \overline{w_n}$  is real and non-negative. It suffices to show that  $w_k \overline{w_k}$  is real and non-negative for each  $k$ . Given  $k \in \{1, \dots, n\}$ , choose  $a, b \in \mathbb{R}$  such that  $w_k = a + bi$ . We have  $w_k \overline{w_k} = (a + bi)(a - bi) = a^2 + b^2 \geq 0$ .

**Definiteness.** For the forward direction, assume that  $\langle w, w \rangle = 0$ . We'll show that each  $w_k = a + bi = 0$ . In the positivity proof, we showed that  $w_k \overline{w_k} \geq 0$ . Since  $\langle w, w \rangle = 0$ , each  $w_k \overline{w_k} = 0$ . We have

$$\begin{aligned} 0 &= w_k \overline{w_k} \\ &= a^2 + b^2 \end{aligned}$$

,so  $a = 0$  and  $b = 0$ . The backward direction is straightforward.

**Additivity in first slot.** Let  $u, v, w \in F^n$ . Then,

$$\begin{aligned} \langle u + v, w \rangle &= (u_1 + v_1) \overline{w_1} + \dots + (u_n + v_n) \overline{w_n} \\ &= (u_1 \overline{w_1} + \dots + u_n \overline{w_n}) + (v_1 \overline{w_1} + \dots + v_n \overline{w_n}) \\ &= \langle u, w \rangle + \langle v, w \rangle. \end{aligned}$$

**Homogeneity in first slot.** Let  $\lambda \in F$  and let  $u, v \in F^n$ . Then,

$$\begin{aligned} \langle \lambda u, v \rangle &= \lambda u_1 \overline{v_1} + \dots + \lambda u_n \overline{v_n} \\ &= \lambda (u_1 \overline{v_1} + \dots + u_n \overline{v_n}) \\ &= \lambda \langle u, v \rangle. \end{aligned}$$

**Conjugate symmetry.** Let  $u, v \in F^n$ . We have

$$\begin{aligned} \overline{\langle v, u \rangle} &= \overline{v_1 \overline{u_1} + \dots + v_n \overline{u_n}} \\ &= \overline{v_1} u_1 + \dots + \overline{v_n} u_n \\ &= u_1 \overline{v_1} + \dots + u_n \overline{v_n} \\ &= \langle u, v \rangle. \end{aligned}$$

□

**Problem Theorem 6.10.** Let  $v \in V$ .

(a).  $\|v\| = 0$  if and only if  $v = 0$ .

(b).  $\|\lambda v\| = |\lambda| \|v\|$  for all  $\lambda \in F$ .

*Proof.* **Part (a).** Follows straightforwardly from the fact that  $\langle v, v \rangle = 0$  if and only if  $v = 0$ .

**Part (b).** Let  $\lambda \in F$ . We have

$$\begin{aligned}\|\lambda v\|^2 &= \langle \lambda v, \lambda v \rangle \\ &= \lambda \bar{\lambda} \langle v, v \rangle \\ &= |\lambda|^2 \langle v, v \rangle.\end{aligned}$$

Taking the square root gives the desired result.

□

**Problem Theorem 6.13 Pythagorean Theorem.** Suppose  $u$  and  $v$  are orthogonal vectors in  $V$ . Then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

*Proof.* Taking the norm of  $u + v$ , we have

$$\begin{aligned}\|u + v\| &= \sqrt{\langle u + v, u + v \rangle} \\ &= \sqrt{\langle u, u + v \rangle + \langle v, u + v \rangle} \\ &= \sqrt{\langle u, u \rangle + \langle u, v \rangle + \langle v, v \rangle + \langle v, u \rangle} \\ &= \sqrt{\langle u, u \rangle + \langle v, v \rangle} \\ &= \sqrt{\|u\|^2 + \|v\|^2}.\end{aligned}$$

The result follows from squaring both sides.

□

**Problem Theorem 6.15 Cauchy–Schwarz Inequality.** Let  $u, v \in V$ . Then,

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

*Proof.* We can write  $u = \frac{\langle u, v \rangle}{\|v\|^2} v + w$ , where  $w$  is orthogonal to  $v$ , by Theorem 6.14. Using the Pythagorean Theorem, we have

$$\|u\|^2 = \left\| \frac{\langle u, v \rangle}{\|v\|^2} v + w \right\|^2 = \left\| \frac{\langle u, v \rangle}{\|v\|^2} v \right\|^2 + \|w\|^2,$$

so

$$\begin{aligned} \|u\|^2 &= \left\| \frac{\langle u, v \rangle}{\|v\|^2} v \right\|^2 + \|w\|^2 \\ &= \frac{|\langle u, v \rangle|^2}{\|v\|^2} + \|w\|^2. \end{aligned}$$

Solving for  $|\langle u, v \rangle|$ , we have

$$\begin{aligned} |\langle u, v \rangle| &= \sqrt{\|u\|^2 \|v\|^2 - \|w\|^2 \|v\|^2} \\ &\leq \sqrt{\|u\|^2 \|v\|^2} \\ &= \|u\| \|v\|, \end{aligned}$$

giving the desired relation. □

**Problem Exercise 6.A.1.** Show that the function that takes  $((x_1, x_2), (y_1, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$  to  $|x_1 y_2| + |x_2 y_2|$  is not an inner product.

*Proof.* The function does not have additivity in the first slot. For example, choose  $u = (1, 1)$ ,  $v = (-1, -1)$  and  $w = (1, 1)$ . Applying the function to  $(u + v, w)$ , we have

$$\begin{aligned} f(u + v, w) &= |u_1 w_1 + v_1 w_1| + |u_2 w_2 + v_2 w_2| \\ &= |1 + -1| + |1 + -1| \\ &= 0. \end{aligned}$$

However,  $f(u, w) + f(v, w) > 0$ .

□

**Problem Exercise 6.A.5.** Suppose  $T \in \mathcal{L}(V)$  is such that  $\|Tv\| \leq \|v\|$ . Prove that  $T - \sqrt{2}I$  is invertible.

*Proof.* The linear operator  $T - \sqrt{2}I$  is invertible if and only if  $\sqrt{2}$  is not an eigenvalue. We will show that  $\sqrt{2}$  is not an eigenvalue. For a contradiction, suppose  $\sqrt{2}$  is an eigenvalue. Choose  $v \in V$  such that  $Tv = \sqrt{2}v$ . We have

$$\begin{aligned} \|Tv\| &= \|\sqrt{2}v\| \\ &= \sqrt{2}\|v\| \\ &> \|v\|, \end{aligned}$$

a contradiction.

□

**Problem Exercise 6.A.13.** Suppose  $u$  and  $v$  are non-zero vectors in  $\mathbb{R}^2$ . Prove that

$$\langle u, v \rangle = \|u\| \|v\| \cos \theta,$$

where  $\theta$  is the angle between  $u$  and  $v$ .

*Proof.* By the law of cosines, we have

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\| \|v\| \cos(\theta)$$

Note that

$$\begin{aligned} \|u - v\|^2 &= \langle u - v, u - v \rangle = \langle u, u \rangle + \langle v, v \rangle - 2\langle u, v \rangle \\ &= \|u\|^2 + \|v\|^2 - 2\langle u, v \rangle. \end{aligned}$$

By substituting, we have

$$\begin{aligned} \|u\|^2 + \|v\|^2 - 2\langle u, v \rangle &= \|u\|^2 + \|v\|^2 - 2\|u\| \|v\| \cos(\theta) \\ -2\langle u, v \rangle &= -2\|u\| \|v\| \cos(\theta) \\ \langle u, v \rangle &= \|u\| \|v\| \cos(\theta). \end{aligned}$$

□

**Problem Exercise 6.A.14.** The angle between two nonzero vectors  $x, y \in \mathbb{R}^n$  is defined to be

$$\arccos \frac{\langle x, y \rangle}{\|x\| \|y\|}.$$

Explain why the Cauchy-Schwarz Inequality is needed to show that this definition makes sense.

Since the domain of  $\arccos$  is  $[-1, 1]$ , we want  $-1 \leq \frac{\langle x, y \rangle}{\|x\| \|y\|} \leq 1$ . By the Cauchy-Schwarz inequality, it follows that

$$|\langle x, y \rangle| \leq \|x\| \|y\|,$$

so

$$\frac{|\langle x, y \rangle|}{\|x\| \|y\|} = \left| \frac{\langle x, y \rangle}{\|x\| \|y\|} \right| \leq 1,$$

as desired.

**Problem Exercise 6.A.16.** Suppose  $u, v \in V$  are such that

$$\|u\| = 3, \|u + v\| = 4, \|u - v\| = 6.$$

What number does  $\|v\|$  equal?

*Proof.* By the Parallelogram Equality, we have

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Solving for  $\|v\|$  yields  $\|v\| = \sqrt{17}$ .

□