NCERT CLASS 12

CHAPTER 10 : EXERCISE 5.13

1. The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one, Find the value of λ.

   Generalized Construction:
   We now that

$$\implies \mathbf{A}^\top = \frac{(\mathbf{B} + \mathbf{C})}{\|\mathbf{B} + \mathbf{C}\|} \tag{1}$$

$$\implies \mathbf{A}^\top (\mathbf{B} + \mathbf{C}) = \|\mathbf{B} + \mathbf{C}\| \tag{2}$$

$$\implies \mathbf{C} = \lambda\mathbf{e}_1 + \mathbf{D} \tag{3}$$

   were,

$$\implies \|\mathbf{B} + \mathbf{C}\| = \sqrt{(\mathbf{B} + \mathbf{C})^\top (\mathbf{B} + \mathbf{C})} \tag{4}$$

   From the Equation(2),We can do

$$\implies \mathbf{A}^\top (\mathbf{B} + \mathbf{C}) = \sqrt{(\mathbf{B} + \mathbf{C})^\top (\mathbf{B} + \mathbf{C})} \tag{5}$$

$$\implies \mathbf{A}^\top (\mathbf{B} + \mathbf{C}) = \sqrt{\|\mathbf{B}\|^2 + 2[\mathbf{B}^\top\mathbf{C}] + \|\mathbf{C}\|^2} \tag{6}$$

$$\implies \mathbf{A}^\top (\mathbf{B} + \mathbf{C}) = \sqrt{\mathbf{B}\mathbf{B}^\top + 2[\mathbf{B}^\top\mathbf{C}] + \mathbf{C}^\top\mathbf{C}} \tag{7}$$

   Substitute the $\mathbf{C}$ Value in the Equation(7),We get

$$\implies \mathbf{A}^\top (\mathbf{B} + \lambda\mathbf{e}_1 + \mathbf{D}) = \sqrt{\mathbf{B}\mathbf{B}^\top + 2\mathbf{B}^\top (\lambda\mathbf{e}_1 + \mathbf{D}) + (\lambda\mathbf{e}_1 + \mathbf{D})^\top (\lambda\mathbf{e}_1 + \mathbf{D})} \tag{8}$$

$$\tag{9}$$

   S.O.B.S,we get

$$\implies (\mathbf{A}^\top (\mathbf{B} + \lambda\mathbf{e}_1 + \mathbf{D}))^2 = \mathbf{B}\mathbf{B}^\top + 2\mathbf{B}^\top (\lambda\mathbf{e}_1 + \mathbf{D}) + (\lambda\mathbf{e}_1 + \mathbf{D})^\top (\lambda\mathbf{e}_1 + \mathbf{D}) \tag{10}$$

$$\implies (\mathbf{A}\lambda\mathbf{e}_1)^2 + (\mathbf{A}^\top\mathbf{B} + \mathbf{D})^2 + 2(\mathbf{A}\lambda\mathbf{e}_1)(\mathbf{A}^\top (\mathbf{B} + \mathbf{D})) = \mathbf{B}\mathbf{B}^\top + 2\mathbf{B}^\top (\lambda\mathbf{e}_1 + \mathbf{D}) + \lambda^2 + 2\lambda\mathbf{e}_1^\top\mathbf{D} + \mathbf{D}\mathbf{D}^\top \tag{11}$$

$$\implies (\lambda^2) + (\mathbf{A}^\top (\mathbf{B} + \mathbf{D}))^2 + 2(\mathbf{A}\lambda\mathbf{e}_1)(\mathbf{A}^\top (\mathbf{B} + \mathbf{D})) = \mathbf{B}\mathbf{B}^\top + 2\lambda(\mathbf{B}^\top\mathbf{e}_1 + \mathbf{e}_1^\top\mathbf{D}) + \mathbf{D}\mathbf{D}^\top + \lambda^2 \tag{12}$$

$$\implies 2\lambda[\mathbf{A}^\top\mathbf{e}_1\mathbf{A}^\top (\mathbf{B} + \mathbf{D}) - (\mathbf{B}^\top\mathbf{e}_1 + \mathbf{e}_1^\top\mathbf{D})] = \mathbf{B}\mathbf{B}^\top + 2\lambda(\mathbf{B}^\top\mathbf{e}_1 + \mathbf{e}_1^\top\mathbf{D}) + \mathbf{D}\mathbf{D}^\top - (\mathbf{A}^\top (\mathbf{B} + \mathbf{D}))^2 \tag{13}$$

$$\implies 2\lambda = \frac{\mathbf{B}\mathbf{B}^\top + 2\mathbf{B}^\top\mathbf{D} + \mathbf{D}\mathbf{D}^\top - (\mathbf{A}^\top (\mathbf{B} + \mathbf{D}))^2}{[\mathbf{A}^\top\mathbf{e}_1\mathbf{A}^\top (\mathbf{B} + \mathbf{D}) - (\mathbf{B}^\top\mathbf{e}_1 + \mathbf{e}_1^\top\mathbf{D})]} \tag{14}$$

$$\implies \lambda = \frac{\mathbf{B}\mathbf{B}^\top + 2\mathbf{B}^\top\mathbf{D} + \mathbf{D}\mathbf{D}^\top - (\mathbf{A}^\top (\mathbf{B} + \mathbf{D}))^2}{2[\mathbf{A}^\top\mathbf{e}_1\mathbf{A}^\top (\mathbf{B} + \mathbf{D}) - (\mathbf{B}^\top\mathbf{e}_1 + \mathbf{e}_1^\top\mathbf{D})]} \tag{15}$$

   Substitute the Given Data in Equation(15),

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}; \mathbf{C} = \begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix}$$

   we get,

$$\implies \lambda = \frac{45 - 14 + 13 - 36}{2(1(6) - 2)} \tag{16}$$

$$\implies \lambda = \frac{44 - 36}{8} \tag{17}$$

$$\impliedby \lambda = \frac{8}{8} \tag{18}$$

$$\implies \lambda = 1 \tag{19}$$