GEOMETRY

Through Algebra

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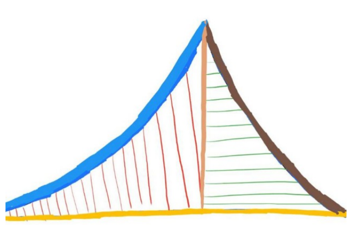

Contents

Introduction

This book shows how to solve problems in geometry using trigonometry and coordinate geometry.

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -2 \\ -5 \end{pmatrix} \tag{1.1}$$

1.1. Matrix

The matrix of vertices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \tag{1.2}$$

$$= \begin{pmatrix} 5 & -5 & -2 \\ -2 & 5 & -5 \end{pmatrix} \tag{1.3}$$

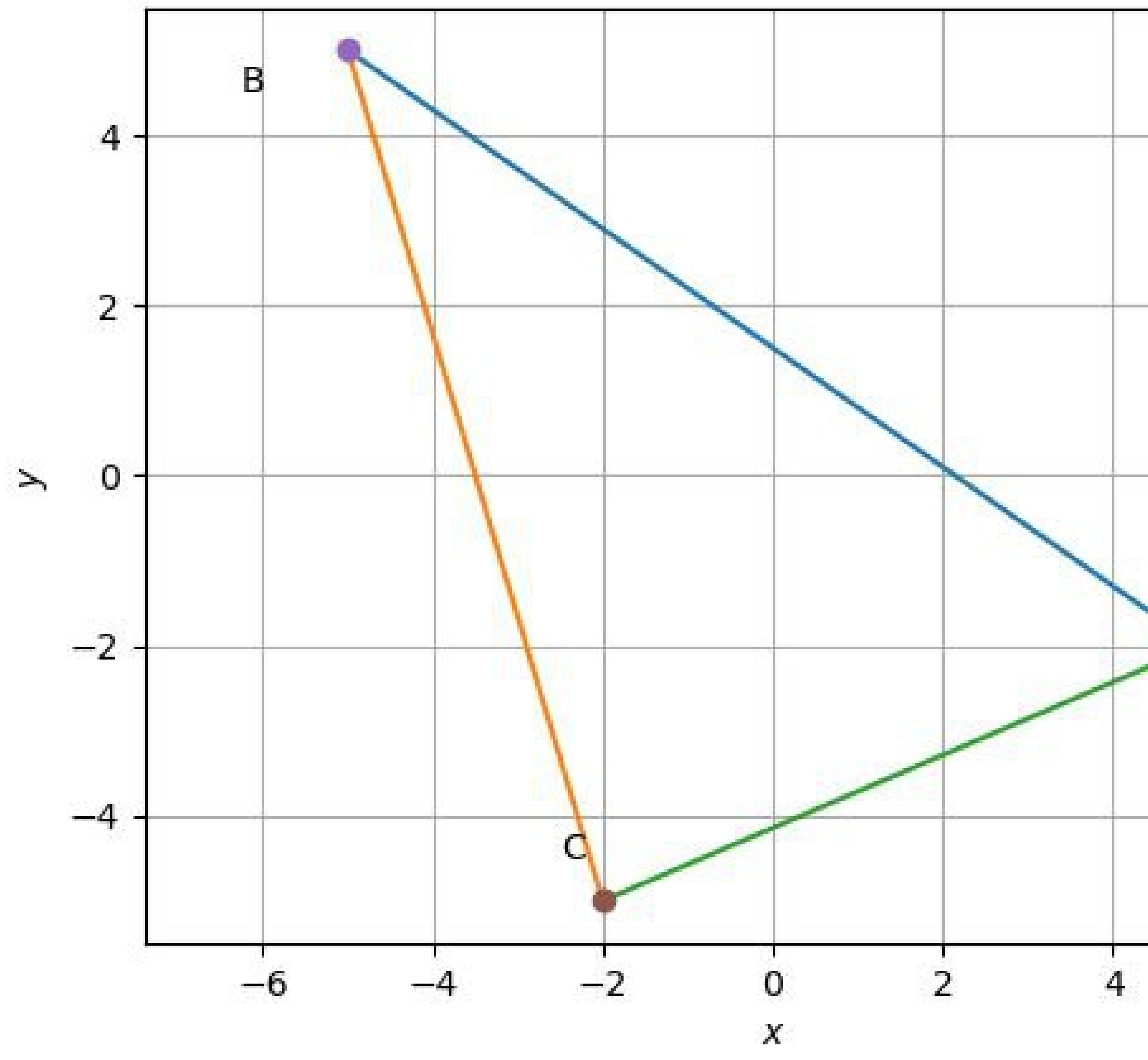

Figure 1.1: △ ABC

1.1.1. Vectors

1.1.1.1. Obtain the direction matrix of the sides of $\triangle \mathbf{ABC}$ defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{1.1.1.1.1}$$

Solution:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{1.1.1.1.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{1.1.1.1.3}$$

$$= \begin{pmatrix} 5 & -5 & -2 \\ -2 & 5 & -5 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{1.1.1.1.4}$$

Using Matrix multiplication

$$\mathbf{M} = \begin{pmatrix} 10 & -3 & -7 \\ -7 & 10 & -3 \end{pmatrix} \tag{1.1.1.1.5}$$

where the second matrix above is known as a circulant matrix. Note that the 2^{nd} and 3^{rd} row of the above matrix are circular shifts of the 1^{st} row.

1.1.1.2. Obtain the normal matrix of the sides of $\triangle \mathbf{ABC}$

Solution:

Considering the rotation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.1.1.2.1}$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R}\mathbf{M} \tag{1.1.1.2.2}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 10 & -3 & -7 \\ -7 & 10 & -7 \end{pmatrix} \tag{1.1.1.2.3}$$

Using matrix multiplication

$$\mathbf{N} = \begin{pmatrix} 7 & -10 & 3 \\ 10 & -3 & -7 \end{pmatrix} \tag{1.1.1.2.4}$$

1.1.1.3. Obtain $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Solution:

The sides vector is obtained as

$$\mathbf{d} = \sqrt{\mathrm{diag}(\mathbf{M}^\top \mathbf{M})} \tag{1.1.1.3.1}$$

$$\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} -7 & 10 \\ 10 & -3 \\ -3 & -7 \end{pmatrix} \begin{pmatrix} 10 & -3 & -7 \\ -7 & 10 & -3 \end{pmatrix} \tag{1.1.1.3.2}$$

Using matrix multiplication

$$\mathbf{M}^\top\mathbf{M} = \begin{pmatrix} 149 & -100 & -49 \\ -100 & 109 & -9 \\ -49 & -9 & 58 \end{pmatrix} \tag{1.1.1.3.3}$$

$$\mathbf{d} = \sqrt{\operatorname{diag}\left(\begin{pmatrix} 149 & -100 & -49 \\ -100 & 109 & -9 \\ -49 & -9 & 58 \end{pmatrix}\right)} \tag{1.1.1.3.4}$$

$$= \begin{pmatrix} \sqrt{149} & \sqrt{109} & \sqrt{58} \end{pmatrix} \tag{1.1.1.3.5}$$

1.1.1.4. Obtain the constant terms in the equations of the sides of the triangle.

Solution:

The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \operatorname{diag}\left\{\left(\mathbf{N}^\top\mathbf{P}\right)\right\} \tag{1.1.1.4.1}$$

$$\mathbf{N}^\top\mathbf{P} = \begin{pmatrix} 7 & 10 \\ -10 & -3 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} 5 & -5 & -2 \\ -2 & 5 & -5 \end{pmatrix} \tag{1.1.1.4.2}$$

$$\tag{1.1.1.4.3}$$

Using matrix multiplication

$$= \begin{pmatrix} 15 & 15 & -64 \\ -44 & 35 & 35 \\ -1 & -20 & 41 \end{pmatrix} \tag{1.1.1.4.4}$$

$$\mathbf{c} = \text{diag}\left(\begin{pmatrix} 15 & 15 & -64 \\ -44 & 35 & 35 \\ -1 & -20 & 41 \end{pmatrix}\right) \tag{1.1.1.4.5}$$

$$= \begin{pmatrix} 15 & 35 & 41 \end{pmatrix} \tag{1.1.1.4.6}$$

1.1.2. Median

1.1.2.1. Obtain the midpoint matrix for the sides of the triangle

Solution:

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.1.2.1.1}$$

$$= \frac{1}{2}\begin{pmatrix} 5 & -5 & -2 \\ -2 & 5 & -5 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.1.2.1.2}$$

Using matrix multiplication

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} & -\frac{3}{2} & 0 \\ 0 & -\frac{7}{2} & \frac{3}{2} \end{pmatrix} \tag{1.1.2.1.3}$$

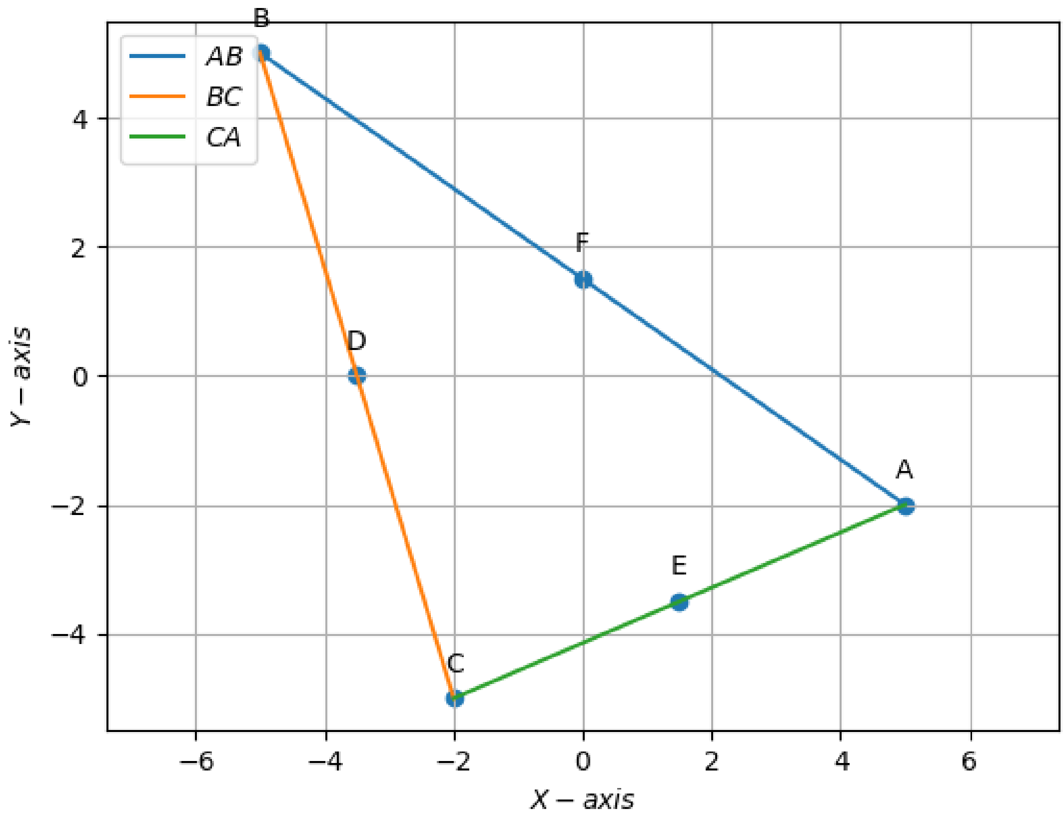

Figure 1.2: mid-points

1.1.2.2. Obtain the median direction matrix.

Solution:

The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{A}-\mathbf{D} & \mathbf{B}-\mathbf{E} & \mathbf{C}-\mathbf{F} \end{pmatrix} \tag{1.1.2.2.1}$$

$$= \begin{pmatrix} \mathbf{A}-\frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B}-\frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C}-\frac{\mathbf{A}+\mathbf{B}}{2} \end{pmatrix} \tag{1.1.2.2.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{1.1.2.2.3}$$

$$= \begin{pmatrix} 5 & -5 & -2 \\ -2 & 5 & -5 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{1.1.2.2.4}$$

Using matrix multiplication

$$\mathbf{M}_1 = \begin{pmatrix} \frac{17}{2} & -\frac{13}{2} & -2 \\ -2 & \frac{17}{2} & -\frac{13}{2} \end{pmatrix} \tag{1.1.2.2.5}$$

1.1.2.3. Obtain the median normal matrix.

Solution:

Considering the rotation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.1.2.3.1}$$

the normal matrix is obtained as

$$\mathbf{N}_1 = \mathbf{R}\mathbf{M}_1 \tag{1.1.2.3.2}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{17}{2} & -\frac{13}{2} & -2 \\ -2 & \frac{17}{2} & -\frac{13}{2} \end{pmatrix} \tag{1.1.2.3.3}$$

$$\mathbf{N}_1 = \begin{pmatrix} 2 & -\frac{17}{2} & \frac{13}{2} \\ \frac{17}{2} & -\frac{13}{2} & -2 \end{pmatrix} \tag{1.1.2.3.4}$$

1.1.2.4. Obtain the median equation constants.

$$\mathbf{c}_1 = \text{diag}\left(\left(\mathbf{N}_1^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix}\right)\right) \tag{1.1.2.4.1}$$

$$\mathbf{N}_1^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} 2 & \frac{17}{2} \\ -\frac{17}{2} & -\frac{13}{2} \\ \frac{13}{2} & -2 \end{pmatrix} \begin{pmatrix} -\frac{7}{2} & \frac{3}{2}0 & 0 \\ 0 & -\frac{7}{2} & \frac{3}{2} \end{pmatrix} \tag{1.1.2.4.2}$$

Using matrix multiplication

$$= \begin{pmatrix} -7 & -\frac{107}{4} & \frac{51}{4} \\ \frac{119}{4} & 10 & -\frac{39}{4} \\ -\frac{91}{4} & \frac{67}{4} & -3 \end{pmatrix} \tag{1.1.2.4.3}$$

$$\mathbf{c}_1 = \text{diag}\left(\left(\begin{pmatrix} -7 & -\frac{107}{4} & \frac{51}{4} \\ \frac{119}{4} & 10 & -\frac{39}{4} \\ -\frac{91}{4} & -\frac{67}{4} & -3 \end{pmatrix}\right)\right) \tag{1.1.2.4.4}$$

$$\mathbf{c}_1 = \begin{pmatrix} -7 & 10 & -3 \end{pmatrix} \tag{1.1.2.4.5}$$

1.1.2.5. Obtain the centroid by finding the intersection of the medians.

Solution:

$$\left(\mathbf{N}_1^\top \;\middle|\; \mathbf{c}^\top\right) = \left(\begin{array}{cc|c} 2 & \frac{17}{2} & -7 \\ -\frac{17}{2} & -\frac{13}{2} & 10 \\ \frac{13}{2} & -2 & -3 \end{array}\right) \tag{1.1.2.5.1}$$

Using Gauss-Elimination method:

$$\left(\begin{array}{cc|c} 2 & \frac{17}{2} & -7 \\ \frac{-17}{2} & -\frac{13}{2} & 10 \\ \frac{13}{2} & -2 & -3 \end{array}\right) \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \left(\begin{array}{cc|c} 1 & \frac{17}{4} & -\frac{7}{2} \\ -\frac{17}{2} & -\frac{13}{2} & 10 \\ \frac{13}{2} & -2 & -3 \end{array}\right) \tag{1.1.2.5.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \frac{17R_1}{2}} \left(\begin{array}{cc|c} 1 & -\frac{17}{4} & -\frac{7}{2} \\ 0 & \frac{237}{8} & -\frac{79}{4} \\ \frac{13}{2} & -2 & -3 \end{array}\right) \tag{1.1.2.5.3}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{13R_1}{2}} \left(\begin{array}{cc|c} 1 & \frac{17}{4} & -\frac{-7}{2} \\ 0 & \frac{237}{8} & -\frac{79}{4} \\ 0 & -\frac{237}{8} & \frac{79}{4} \end{array}\right) \tag{1.1.2.5.4}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{8R_2}{237}} \left(\begin{array}{cc|c} 0 & \frac{17}{4} & -\frac{7}{2} \\ 0 & 1 & \frac{-2}{3} \\ 0 & -\frac{237}{8} & \frac{79}{4} \end{array}\right) \tag{1.1.2.5.5}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{17R_2}{4}} \left(\begin{array}{cc|c} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & -\frac{237}{8} & \frac{79}{4} \end{array}\right) \tag{1.1.2.5.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + \frac{237R_2}{8}} \left(\begin{array}{cc|c} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{array}\right) \tag{1.1.2.5.7}$$

$$\text{Therefore } \mathbf{G} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \tag{1.1.2.5.8}$$

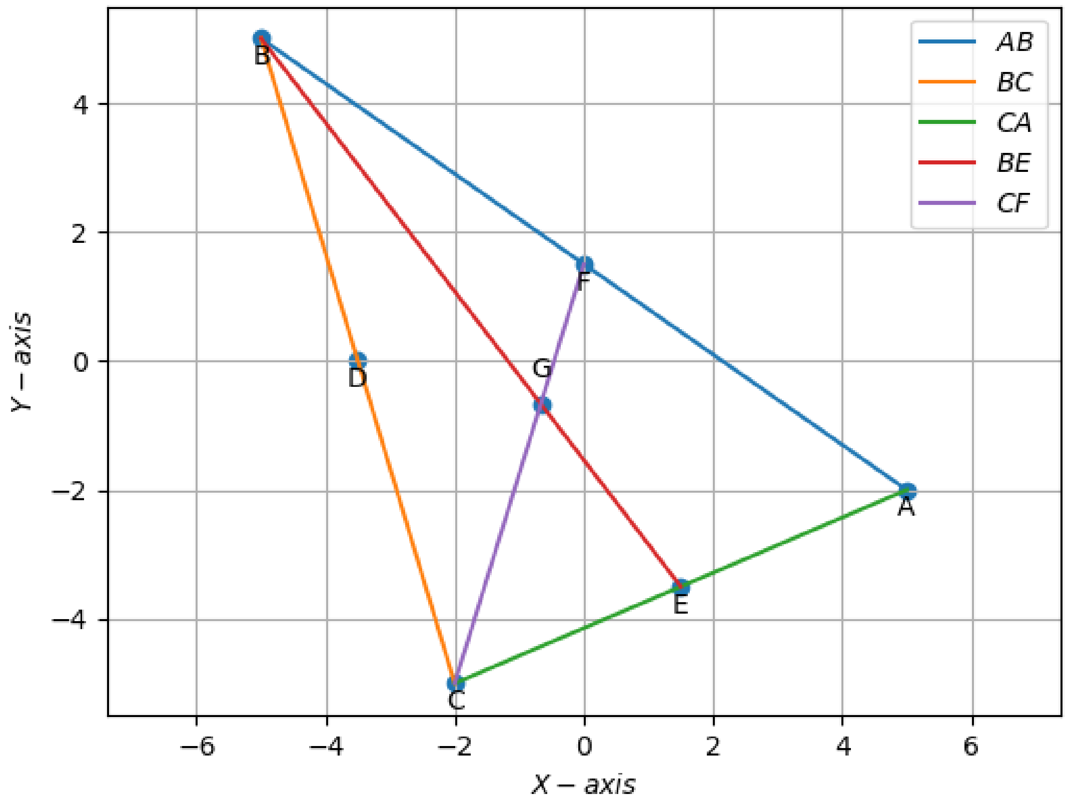

Figure 1.3: centroid of triangle ABC

1.1.3. Altitude

1.1.3.1. Find the normal matrix for the altitudes

Solution: The desired matrix is

$$\mathbf{M}_2 = \begin{pmatrix} \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} & \mathbf{A} - \mathbf{B} \end{pmatrix} \tag{1.1.3.1.1}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tag{1.1.3.1.2}$$

$$= \begin{pmatrix} 5 & -5 & -2 \\ -2 & 5 & -5 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tag{1.1.3.1.3}$$

Using Matrix multiplication

$$\mathbf{M}_2 = \begin{pmatrix} -3 & -7 & 10 \\ 10 & -3 & -7 \end{pmatrix} \tag{1.1.3.1.4}$$

1.1.3.2. Find the constant vector for the altitudes.

Solution:

The desired vector is

$$\mathbf{c}_2 = \text{diag}\left\{\left(\mathbf{M}^\top \mathbf{P}\right)\right\} \tag{1.1.3.2.1}$$

$$\mathbf{M}^\top \mathbf{P} = \begin{pmatrix} -3 & 10 \\ -7 & -3 \\ 10 & -7 \end{pmatrix} \begin{pmatrix} 5 & -5 & -2 \\ -2 & 5 & -5 \end{pmatrix} \tag{1.1.3.2.2}$$

$$\tag{1.1.3.2.3}$$

Using matrix multiplication

$$\mathbf{M}^\top \mathbf{P} = \begin{pmatrix} -35 & 65 & -44 \\ -29 & 20 & 29 \\ 64 & -85 & 15 \end{pmatrix} \tag{1.1.3.2.4}$$

$$\mathbf{c}_2 = \text{diag}\left(\begin{pmatrix} -35 & 65 & -44 \\ -29 & 20 & 29 \\ 64 & -85 & 15 \end{pmatrix}\right) \tag{1.1.3.2.5}$$

$$\mathbf{c}_2 = \begin{pmatrix} -35 & 20 & 15 \end{pmatrix} \tag{1.1.3.2.6}$$

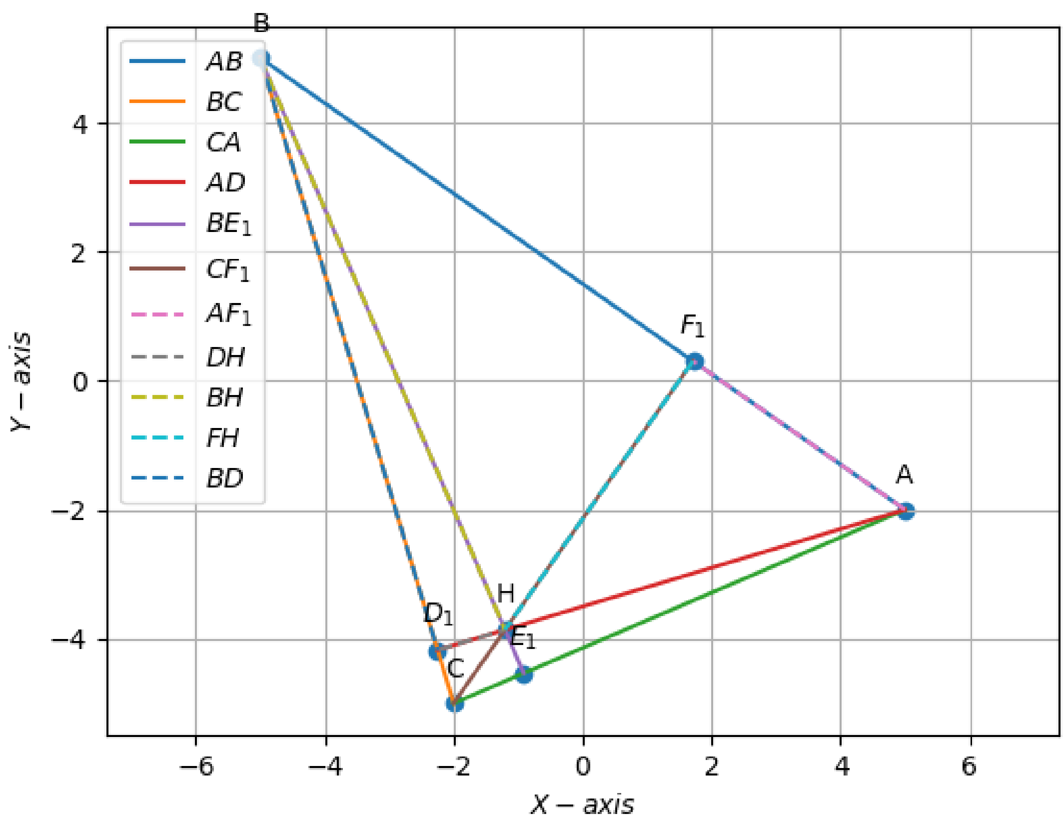

Figure 1.4: Orthocentre of △ ABC

1.1.4. Perpendicular Bisector

1.1.4.1. Find the normal matrix for the perpendicular bisectors

Solution:

The normal matrix is $\mathbf{M}_2$

$$\mathbf{M}_2 = \begin{pmatrix} -3 & -7 & 10 \\ 10 & -3 & -7 \end{pmatrix} \tag{1.1.4.1.1}$$

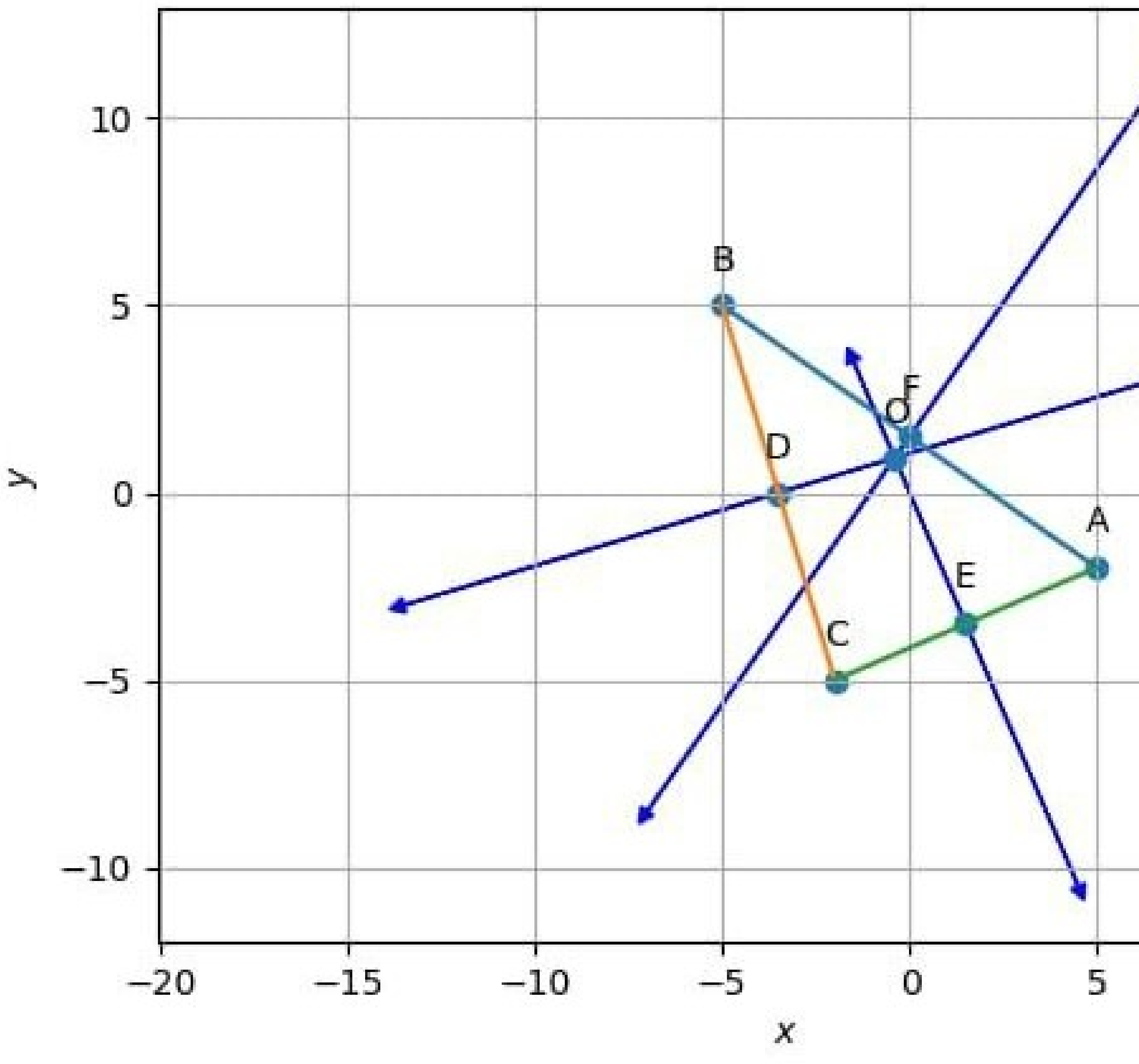

Figure 1.5: plot of perpendicular bisectors

1.1.4.2. Find the constants vector for the perpendicular bisectors.

Solution: The desired vector is

$$\mathbf{c}_3 = \text{diag}\left\{\mathbf{M}_2^\top \begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix}\right\} \tag{1.1.4.2.1}$$

Solution:

$$\mathbf{c}_3 = \text{diag}\left\{\mathbf{M}_2^\top \begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix}\right\} \tag{1.1.4.2.2}$$

$$\mathbf{M}_2^\top \begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix} = \begin{pmatrix}-3 & -10 \\ 7 & -3 \\ 10 & -7\end{pmatrix}\begin{pmatrix}\frac{7}{2} & -\frac{1}{2} & -1 \\ -\frac{3}{2} & -2 & -\frac{7}{2}\end{pmatrix} \tag{1.1.4.2.3}$$

$$\tag{1.1.4.2.4}$$

Using matrix multiplication

$$\mathbf{M}_2^\top \begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix} = \begin{pmatrix}\frac{21}{2} & \frac{-79}{2} & 15 \\ \frac{49}{2} & 0 & -\frac{9}{2} \\ -35 & \frac{79}{2} & -\frac{21}{2}\end{pmatrix} \tag{1.1.4.2.5}$$

$$\mathbf{c}_3 = \text{diag}\left(\begin{pmatrix}\frac{21}{2} & \frac{-79}{2} & 15 \\ \frac{49}{2} & 0 & \frac{-9}{2} \\ -35 & \frac{79}{2} & \frac{-21}{2}\end{pmatrix}\right) \tag{1.1.4.2.6}$$

$$\mathbf{c}_3 = \begin{pmatrix}\frac{21}{2} & 0 & \frac{-21}{2}\end{pmatrix} \tag{1.1.4.2.7}$$

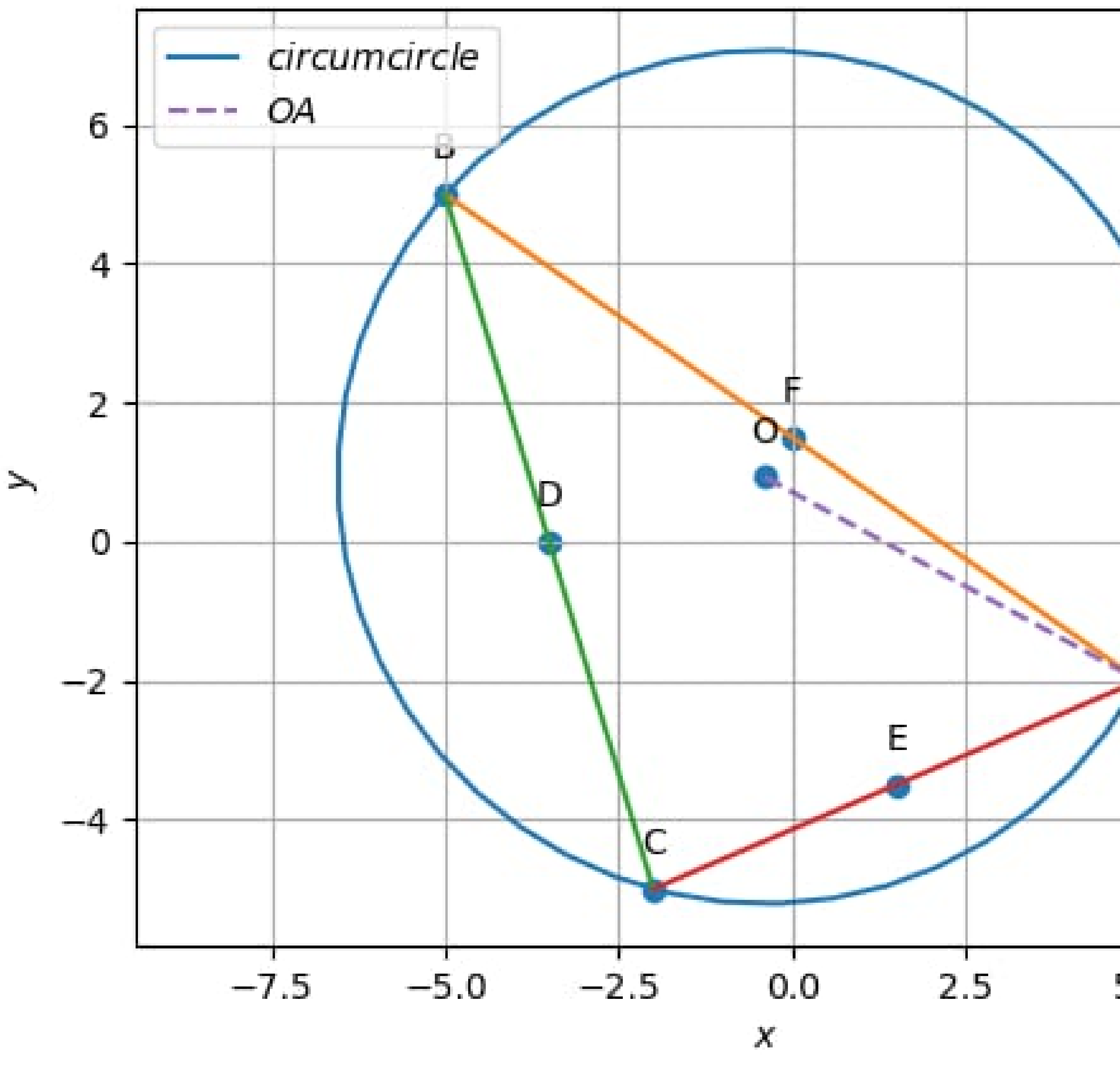

Figure 1.6: circumcentre and circumcircle of △ ABC

1.1.5. Angle Bisector

1.1.5.1. Find the points of contact.

Solution:

The points of contact are given by

$$\begin{pmatrix} \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m} & \frac{m\mathbf{C}+n\mathbf{B}}{m+n} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \frac{n}{b} & \frac{m}{c} & 0 \\ 0 & \frac{p}{c} & \frac{n}{a} \\ \frac{p}{b} & 0 & \frac{m}{a} \end{pmatrix} \tag{1.1.5.1.1}$$

$$\begin{pmatrix} \mathbf{p} & \mathbf{m} & \mathbf{n} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{1.1.5.1.2}$$

$$= \frac{1}{2} \begin{pmatrix} \sqrt{149} & \sqrt{109} & \sqrt{58} \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{1.1.5.1.3}$$

$$= \frac{1}{2} \begin{pmatrix} 12.20655 & 10.4403065 & 7.615773106 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{1.1.5.1.4}$$

Using matrix multiplication

$$\begin{pmatrix}\mathbf{p} & \mathbf{m} & \mathbf{n}\end{pmatrix} = \begin{pmatrix}2.9247 & 4.69101 & 7.51554\end{pmatrix} \tag{1.1.5.1.5}$$

$$\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}\frac{n}{b} & \frac{m}{c} & 0\\ 0 & \frac{p}{c} & \frac{n}{a}\\ \frac{p}{b} & 0 & \frac{m}{a}\end{pmatrix} = \begin{pmatrix}5 & -5 & -2\\ -2 & 5 & -5\end{pmatrix}\begin{pmatrix}\frac{7.51554}{\sqrt{109}} & \frac{4.69101}{\sqrt{58}} & 0\\ 0 & \frac{2.92487}{\sqrt{58}} & \frac{7.51554}{\sqrt{149}}\\ \frac{2.9247}{\sqrt{109}} & 0 & \frac{4.69101}{\sqrt{149}}\end{pmatrix} \tag{1.1.5.1.6}$$

Using matrix multiplication We get the points of contact

$$= \begin{pmatrix}3.03893 & 1.159595 & -3.84709002\\ -2.840375 & 0.688255 & 1.15697245\end{pmatrix} \tag{1.1.5.1.7}$$

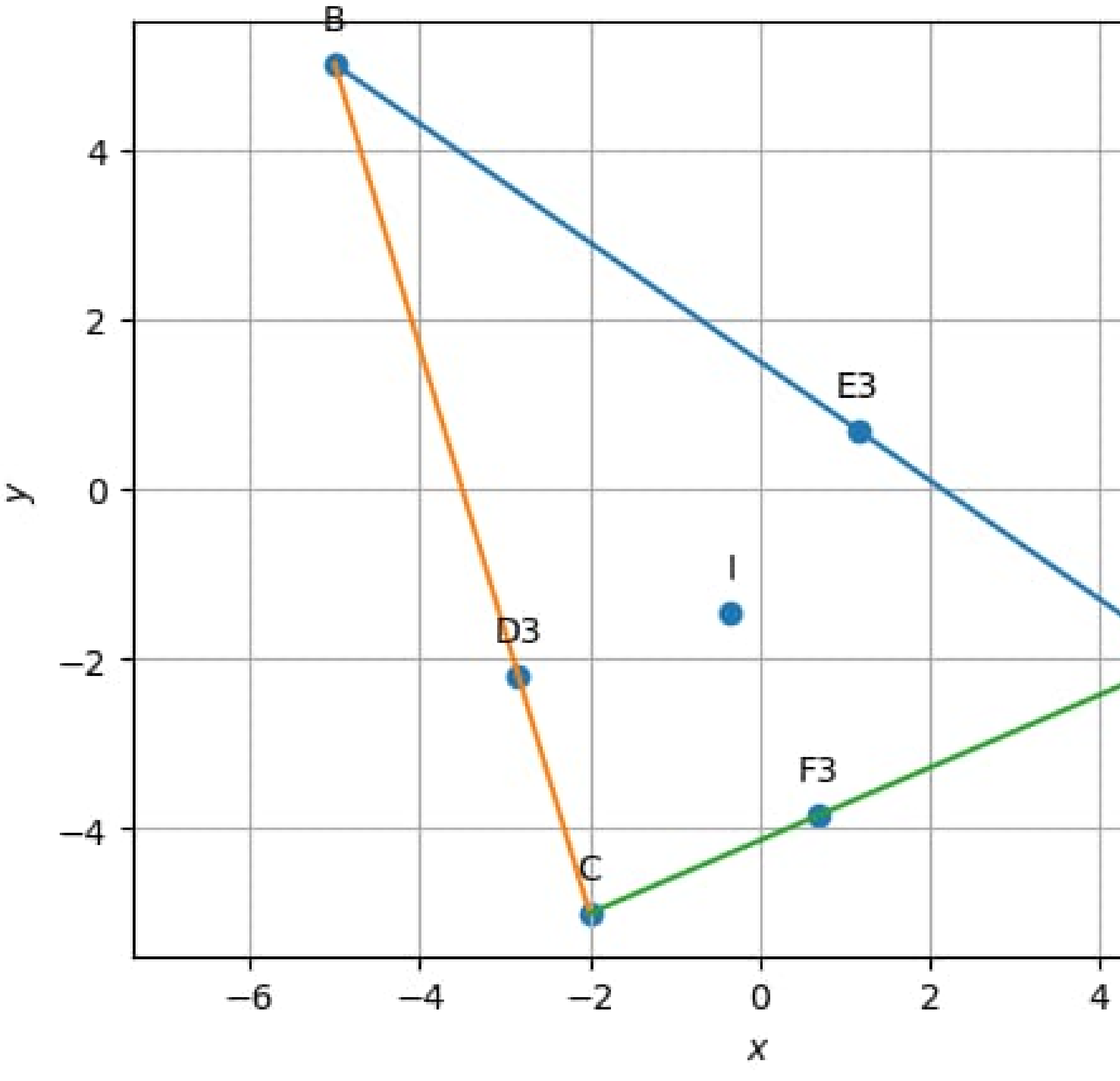

Figure 1.7: Contact points of incircle of *triangle* ABC

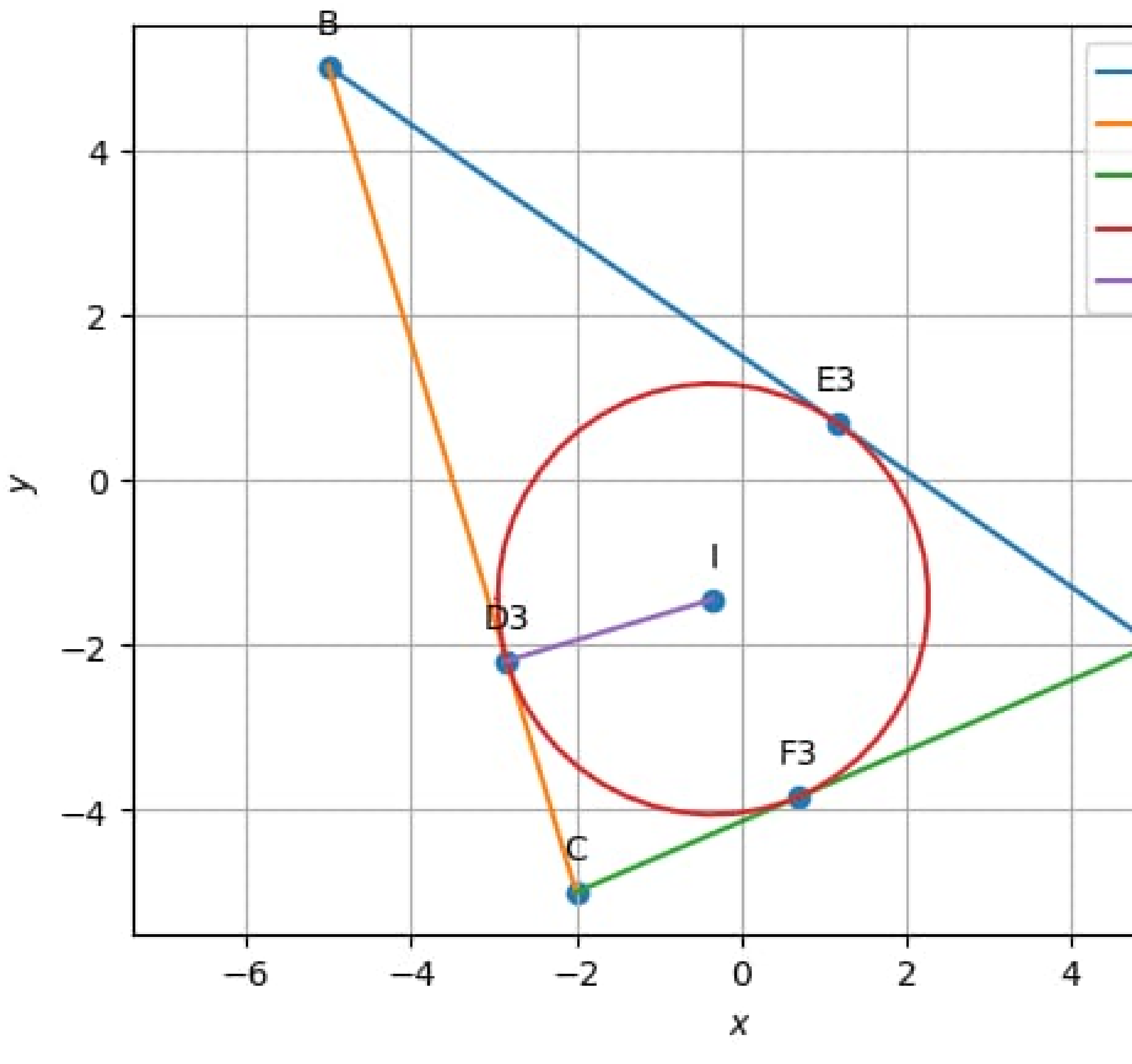

Figure 1.8: Incircle and Incentre of △ ABC