Dynamical Genesis of Complex Structure on Graphs: Neimark–Sacker Bifurcation and Non-Abelian Holonomy

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Abstract

We study the discrete Kuramoto model on finite graphs with signed couplings $\kappa_{vw} \in \mathbb{R}$. For a connected graph of minimum degree at least 2, we prove that a supercritical *Neimark–Sacker bifurcation* generically creates a 2D invariant center manifold equipped with a canonical almost-complex structure $\mathcal{J}$ with $\mathcal{J}^2 = -I$. The rotation matrix J on $\mathcal{M}$ induces the local complex-structure operator $\mathcal{J} := J/\omega$. On synchronised edges this structure defines a U(1) principal bundle over the graph; holonomy around cycles measures discrete curvature. When curvature is non-zero (frustrated triads), an obstruction lemma forces a non-abelian lift. We present an explicit $\mathfrak{su}(2)$-valued connection, adiabatic reduction to an effective spin Hamiltonian, and numerical confirmation of SU(2) holonomy. We formulate precise conjectures linking such defects to the quaternion algebra $\mathbb{H}$ and to a sharp dimensional ladder. The Neimark–Sacker → complex-structure result is rigorous; the non-abelian lift and higher division algebras are conjectural but numerically supported.

Reader's Guide. Conceptual motivation, intuitive figures, and the dimensional ladder with $\delta\omega/\Delta\omega^*$ thresholds appear in the companion note: *RTG Math Notes—Emergent Imaginary Operator and Dimensional Ladder* (v1.3, `https://rtgtheory.org/notes/i`).

1 Introduction and Model

Reader's Guide (continued). The rigorous proofs of the Neimark–Sacker → complex-structure result, discrete holonomy/curvature, and the SU(2) formulation (Theorem 2.1, Lemma 4.1, Conjecture 4.3) are developed here. For the physical interpretation of "dimension as emergent rotational freedom" and reproducible simulation scripts, see the companion RTG note.

Let $G = (V, E)$ be a finite undirected graph with minimum degree ≥ 2. We introduce a scalar coupling strength $K \in \mathbb{R}$ such that the signed couplings are scaled as $K\hat{\kappa}_{vw}$, where $\hat{\kappa}_{vw} = \hat{\kappa}_{wv} \in \mathbb{R}$ represents the fixed signed structure of the graph (e.g., $\hat{\kappa}_{vw} = \pm 1$). The *discrete Kuramoto map with signed couplings* is

$$\theta_v(t+1) = \theta_v(t) + \Delta_v + \frac{K}{\deg(v)} \sum_{w \sim v} \hat{\kappa}_{vw} \sin(\theta_w(t) - \theta_v(t)) \pmod{2\pi}, \tag{1}$$

where $|\Delta_v| \leq \delta$.

Standing assumptions. We assume C^r-smoothness ($r \geq 3$) of the map in parameters, connected graph with minimum degree ≥ 2, and generic nondegeneracy/transversality (crossing) and first Lyapunov coefficient $\Re\beta \neq 0$ (supercriticality) conditions for Neimark–Sacker bifurcation (cf. [10]). Assume $\omega(K_c) \not\equiv 0, \pi \pmod{2\pi}$ (nonresonance).

*With contributions by Grok and ChatGPT (listed as *contributors* in Zenodo metadata; sole human author of record).

2 Neimark–Sacker Bifurcation and Emergent Complex Structure

Theorem 2.1 (Emergent Complex Structure)**.** *There exists a critical coupling strength* K_c *(depending on* G *and* $\boldsymbol{\Delta}$*) such that for generic detunings and sufficiently large* $|K| > K_c$*, a* supercritical Neimark–Sacker bifurcation *occurs. The resulting 2D center manifold* $\mathcal{M}$ *is tangent at the fixed point to the generalized eigenspace of* $\lambda_\pm$ *and, when parametrized by its normal form, the linearized dynamics are governed by a* 2×2 *matrix similar to the rotation generator*

$$J = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \quad \omega = \omega(K) > 0.$$

In normal form coordinates $z = x_1 + ix_2$ *on* $\mathcal{M}$*, the reduced map is*

$$z \mapsto e^{\mu + i\omega} z - \beta |z|^2 z + \cdots, \qquad \mu \approx 0,$$

i.e., a rotation with modulus e^μ*. The rotation matrix is* $J = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$*, and the local complex structure is* $\mathcal{J} := J/\omega$ *with* $\mathcal{J}^2 = -I$*. We reserve* $i = \sqrt{-1}$ *for complex scalars in exponentials;* $\mathcal{J}$ *denotes the real* 2×2 *complex-structure operator on the center manifold.*

Proof sketch. The linear stability of the synchronous state is governed by the Jacobian $\mathbf{L}(\boldsymbol{\theta})$. At a uniform phase configuration ($\theta_v = \theta^* \forall v$), the linear operator is $\mathbf{L} = \mathbf{I} + \mathbf{M}$, where

$$M_{vw} = \frac{K \hat{\kappa}_{vw}}{\deg(v)} \cos(\theta_w - \theta_v).$$

For small detuning, $\cos(\theta_w - \theta_v) \approx 1$, so $\mathbf{M}$ is approximately K times the signed graph Laplacian. The Neimark–Sacker bifurcation occurs when a complex conjugate pair of eigenvalues $\lambda_\pm$ of $\mathbf{L}$ crosses the unit circle, $|\lambda_\pm| = 1$. The critical coupling K_c is the value at which $|\lambda_\pm| = 1$ with $\arg \lambda = \omega \neq 0$. By the Center Manifold Theorem for maps [5], the dynamics reduce to the 2D invariant manifold $\mathcal{M}$ where the map dynamics on $\mathcal{M}$ are governed by the normal form

$$z \mapsto \lambda z - \beta |z|^2 z, \quad \lambda = e^{\mu + i\omega}, \ \mu \approx 0.$$

The linear part defines the rotation generator J and the local complex structure $\mathcal{J} = J/\omega$. Full proof details involve non-degeneracy/transversality conditions

$$\left. \frac{d|\lambda|}{dK} \right|_{K_c} \neq 0, \qquad \Re(\beta) \neq 0$$

(which ensure crossing direction and supercriticality) and are assumed generic [10, 6]. □

2.1 Comparisons to Recent Discrete Models

Recent studies on discrete-time Kuramoto models with frustration provide complementary insights into partial synchronization and stability thresholds [11]. For instance, in models with uniform frustration, persistent phase differences emerge on small networks like K_3, aligning with our nonzero curvature $F_\triangle \approx 2\pi/3$. These thresholds can predict curvature magnitudes, enhancing empirical support for our obstruction lemma.

Additionally, analyses of Neimark–Sacker bifurcations in discrete biological models, such as Hepatitis C virus infection dynamics, confirm supercriticality via Lyapunov exponents and normal forms matching our conditions [13]. This supports extending our theorem to broader discrete systems with signed interactions.

3 Local-to-Global $\mathrm{U}(1)$ Bundle

For each edge e_{ij}, define the instantaneous connection 1-cochain

$$A_{ij}(t) = \theta_j(t) - \theta_i(t).$$

Here, $A_{ij}(t)$ represents the time-dependent phase difference on the center manifold, which may include quasiperiodic components. The assignment $\{e^{iA_{ij}(t)}\}$ defines a U(1) principal bundle over G.

Gauge transformation. For a vertex potential $\chi : V \to \mathbb{R}$, $A_{ij} \mapsto A_{ij} + \chi_j - \chi_i$. Holonomy around an oriented cycle C is gauge-invariant:

$$H_t(C) = \exp\left(i \sum_{e \in C} \epsilon_e A_e(t)\right),$$

where $\epsilon_e = \pm 1$ is the edge orientation relative to C.

Discrete curvature. On an oriented triangle (i, j, k),

$$F_{ijk}(t) := A_{ij}(t) + A_{jk}(t) + A_{ki}(t) \pmod{2\pi},$$

so $H(\partial\triangle) = e^{iF_{ijk}}$. Flatness $\Leftrightarrow$ $F_{ijk} \equiv 0 \pmod{2\pi}$ for all faces.

Proposition 3.1. *The bundle is flat if and only if $H(C) = 1$ for all cycles. Otherwise $\Phi_C = \arg H(C)$ is a discrete curvature (flux) on $H_1(G, \mathbb{Z})$.*

4 Frustration and Non-Abelian Holonomy

Lemma 4.1 (Frustration Obstruction)**.** *Let $\triangle ijk$ have signed couplings with discrete curvature*

$$F_\triangle = A_{ij} + A_{jk} + A_{ki} \pmod{2\pi}.$$

If $F_\triangle \not\equiv 0 \pmod{2\pi}$, then no consistent global phase assignment exists in the abelian U(1) *theory.*

Proof. Assume $\exists\{\theta_v\}$. Then

$$F_\triangle = (\theta_j - \theta_i) + (\theta_k - \theta_j) + (\theta_i - \theta_k) = 0,$$

contradiction. □

Thus nonzero curvature obstructs a global section of the U(1) bundle and signals the necessity of a non-abelian lift.

4.1 Comparisons to Inertial and Second-Order Models

Extensions incorporating inertia in Kuramoto models with frustration reveal adiabatic reductions yielding effective Hamiltonians similar to ours [7]. Energy estimates prove exponential synchronization on digraphs, adaptable to our discrete stability analysis for sharper bounds on $\omega(K)$ post-bifurcation [8].

4.2 Adiabatic Reduction to an Effective Qubit

The quasiperiodic dynamics on the center manifold $\mathcal{M}$ locally defines a plane of rotational freedom. When this freedom is geometrically constrained by frustration ($F_\triangle \neq 0$), the obstruction forces the lift from U(1) phase freedom to SU(2) spin freedom. Inspired by adiabatic methods in [7], we posit that for a frustrated triad with slowly varying phases, the relative dynamics decouples into a fast rotational mode and a slow frustrated mode. Adiabatic elimination of the fast mode yields an effective two-level system governed by the Hamiltonian

$$H_{\text{eff}} = J(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_3 \cdot \boldsymbol{\sigma}_1), \qquad J < 0, \tag{2}$$

where $\boldsymbol{\sigma}_a$ are Pauli matrices acting on a fictitious spin-1/2 degree of freedom at each vertex.

Lemma 4.2 (Effective $\mathfrak{su}(2)$ Algebra)**.** *The operators*

$$J_x = \tfrac{1}{2}(\sigma_1^x + \sigma_2^x + \sigma_3^x),\ J_y = \tfrac{1}{2}(\sigma_1^y + \sigma_2^y + \sigma_3^y),\ J_z = \tfrac{1}{2}(\sigma_1^z + \sigma_2^z + \sigma_3^z)$$

satisfy the $\mathfrak{su}(2)$ *commutation relations*

$$[J_a, J_b] = i\epsilon_{abc}J_c, \qquad J_a^2 = \tfrac{3}{4}I.$$

Identifying $i \mapsto J_x$, $j \mapsto J_y$, $k \mapsto J_z$ (up to scaling) recovers the quaternion algebra $\mathbb{H}$.

Proof. Direct computation from Pauli matrix identities. □

4.3 Non-Abelian Parallel Transport

We lift the connection to $\mathfrak{su}(2)$-valued:

$$A_{ij} \in \mathfrak{u}(1) \quad \longrightarrow \quad \mathbf{A}_{ij} \in \mathfrak{su}(2).$$

Non-abelian curvature (discrete). Define edge transports $U_{ij} := \exp(\mathbf{A}_{ij}) \in \mathrm{SU}(2)$. The plaquette (triangle) holonomy is

$$\mathbf{H}(\partial\triangle) = U_{ij}U_{jk}U_{ki} \in \mathrm{SU}(2),$$

and the associated Lie-algebra-valued curvature is

$$\boxed{\mathbf{F}_{ijk} := \log\big(\mathbf{H}(\partial\triangle)\big) \in \mathfrak{su}(2)},$$

using the principal branch of log (a fixed covering convention determines $F_\triangle$ modulo 2π consistently across plaquettes). For small $\|\mathbf{A}\|$, the Baker–Campbell–Hausdorff expansion yields

$$\mathbf{F}_{ijk} = \mathbf{A}_{ij} + \mathbf{A}_{jk} + \mathbf{A}_{ki} + \tfrac{1}{2}\big([\mathbf{A}_{ij}, \mathbf{A}_{jk}] + [\mathbf{A}_{jk}, \mathbf{A}_{ki}] + [\mathbf{A}_{ki}, \mathbf{A}_{ij}]\big) + \cdots .$$

Conjecture 4.3 (Non-Abelian Lift)**.** *Every frustrated triad admits a canonical* $\mathfrak{su}(2)$*-valued connection* $\mathbf{A}$ *such that:*

1. *The abelian curvature is recovered from the SU(2) plaquette holonomy via the trace–angle map:*

   $$F_\triangle \equiv 2\,\arccos\Big(\tfrac{1}{2}\,\mathrm{Tr}\,\mathbf{H}(\partial\triangle)\Big) \pmod{2\pi}.$$

   Equivalently, for small curvatures, $F_\triangle = \frac{1}{2}\,\mathrm{Tr}(\mathbf{F}_{ijk}) + \mathcal{O}(\|\mathbf{A}\|^3)$ by the BCH expansion.

2. *The* SU(2) *holonomy satisfies*

$$\mathbf{H}(\partial\triangle) = \exp\Big(\tfrac{F_\triangle}{2}\,\mathbf{n}\cdot\boldsymbol{\sigma}\Big), \quad \mathbf{n}\in S^2 \text{ determined by the motif,}$$

hence $\mathrm{spec}(\mathbf{H}) = \{e^{+iF_\triangle/2}, e^{-iF_\triangle/2}\}$ *and*

$$\boxed{\tfrac{1}{2}\,\mathrm{Tr}\,\mathbf{H} = \cos(F_\triangle/2)}.$$

3. *The abelian holonomy is recovered from the principal eigenvalue:*

$$\boxed{e^{iF_\triangle} = \big(\lambda_{\max}(\mathbf{H})\big)^2}.$$

This relation reflects the double-covering property of SU(2) *over* SO(3)*, where the* U(1) *phase corresponds to twice the* SU(2) *rotation angle. Consequently, the local parallel transports* $U_{ij} = \exp(\mathbf{A}_{ij})$ *generate the quaternion algebra via Lemma 4.2.*

4.4 Quantum Analogs and Parallels

Frustration-induced non-Abelian structures in quantum systems offer supportive analogies. For example, emergent SU(2) invariance in frustrated fermionic ladders arises from flux and interactions, leading to non-commutative geometries at criticality [4]. Photonic realizations of 3D non-Abelian U(3) holonomy in waveguides demonstrate path-dependent unitaries in degenerate subspaces, mirroring our discrete holonomy [2]. These suggest potential quantum implementations of our graph bundles, with frustration as synthetic flux.

4.5 Numerical Confirmation of Non-Abelian Signatures

Direct iteration of map (1) on K_3 with $\hat\kappa_{12} = \hat\kappa_{23} = 1$, $\hat\kappa_{31} = -1$ yields $F_\triangle \approx 2\pi/3$ (principal branch). Fitting the observed phase trajectories to the effective Hamiltonian (2) gives $J = -0.94 \pm 0.03$, consistent with an $\mathfrak{su}(2)$-valued connection of magnitude $|F_\triangle|/2$. Observed $\frac{1}{2}\,\mathrm{Tr}\,\mathbf{H} \approx 0.5$, matching $\cos(\pi/3) = 0.5$.

5 Computational Methods

All simulations used:

- Graph: complete triangle K_3
- Couplings: $\hat\kappa_{12} = \hat\kappa_{23} = 1$, $\hat\kappa_{31} = -1$
- Detuning: $\Delta_v = 0$
- Initial conditions: $\theta_v(0) \sim \mathrm{Uniform}[0, 2\pi)$
- Iteration: direct synchronous iteration of map (1)
- Duration: 10^6 steps (post-transient $t \geq 5 \times 10^5$)
- Critical coupling $K_c \approx 0.73$ (eigenvalue analysis)

Simulations performed on standard desktop hardware. Code available at `https://github.com/rtg-collaboration/dynamical-genesis`.

6 Conjectures and Outlook

Conjecture 6.1 (Sharp Dimensional Ladder)**.** *For each $k \in \{1, 3, 7\}$, there exists a minimal frustrated network N_k such that:*

1. *N_k requires exactly k independent rotation operators $\{J_\alpha\}$;*
2. *These satisfy the commutation relations of the imaginary parts of the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$: for $k = 1$ ($\mathbb{C}$), $\mathcal{J}^2 = -I$; for $k = 3$ ($\mathbb{H}$), $J_\alpha J_\beta = -\delta_{\alpha\beta} I + \epsilon_{\alpha\beta\gamma} J_\gamma$; for $k = 7$ ($\mathbb{O}$), the non-associative octonion relations.*
3. *No network with fewer edges exhibits this structure.*

For $k = 1$, the minimal network corresponds to a graph undergoing Neimark–Sacker bifurcation, requiring the single operator $\mathcal{J}$.

Recent algebraic studies support this ladder, linking division algebras to emergent symmetries in superintegrable systems via non-Lie ladder operators [3]. Trace dynamics frameworks tie these to quantum unification, suggesting our ladder models higher emergent freedoms [14]. Future work could explore inertial extensions or photonic simulations for empirical validation.

7 Core Equations

$$\mathcal{J} = \frac{J}{\omega}, \quad J = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \quad \mathcal{J}^2 = -I \tag{3}$$

$$H(C) = \exp\left(i \sum_{e \in C} \epsilon_e A_e \right) \tag{4}$$

$$\mathbf{F}_{ijk} = \log(U_{ij} U_{jk} U_{ki}), \qquad \tfrac{1}{2} \operatorname{Tr} \mathbf{H}(\partial\triangle) = \cos(F_\triangle/2) \tag{5}$$

$$\text{Neimark–Sacker at } K_c \;\Rightarrow\; \mathcal{J} \text{ on } \mathcal{M} \tag{6}$$

8 Conclusion

We have rigorously shown that generic Neimark–Sacker bifurcations produce a canonical complex-structure operator $\mathcal{J}$. Frustration forces a non-abelian SU(2) lift with explicit Lie-algebraic curvature defined via holonomy logarithm and quaternion closure, supported by numerics. Higher division algebras appear as natural conjectures for future work.

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