# Dynamical Genesis of Complex Structure on Graphs: Neimark–Sacker Bifurcation and Non-Abelian Holonomy

Mustafa Aksu*

## Abstract

We study the discrete Kuramoto model on finite graphs with signed couplings $\kappa_{vw} \in \mathbb{R}$. For a connected graph of minimum degree at least 2, we show that a supercritical *Neimark–Sacker bifurcation* generically creates a 2D invariant center manifold endowed with a canonical almost-complex structure $\mathcal{J}$ with $\mathcal{J}^2 = -I$. The rotation matrix $J$ on the center manifold induces $\mathcal{J} := J/\omega$ where $\omega = \arg \lambda \in (0, \pi)$ is the angular frequency of the critical eigenpair. On synchronised edges this structure defines a U(1) principal bundle over the graph; holonomy around cycles measures discrete curvature. Whenever a frustrated triad carries nonzero curvature, a simple obstruction lemma rules out a globally consistent U(1) phase field and motivates a non-abelian lift. We formalize this via an $\mathfrak{su}(2)$-valued connection whose plaquette holonomy reproduces the abelian flux through a trace–angle relation, interpreting discrete curvature as the angle of an SU(2) rotation.

Numerically, we confirm the NS mechanism on a directed, frustrated $K_3$ with degree-normalized couplings under an inertial explicit-Euler discretization: a simple complex-conjugate pair crosses the unit circle at $K_c \approx 1.92$ with nonzero angle, the coherence $r(t)$ exhibits a single dominant spectral peak close to the linear prediction, the phase portrait shows an invariant circle, and the largest Lyapunov exponent is numerically near zero. We then exhibit a tuned *double* Neimark–Sacker event on a symmetric but weakly directed $K_4$ with triadic frustration and non-uniform edge weights. Two distinct complex pairs cross the unit circle at

$$K_1^* \approx 0.2404767, \qquad K_2^* \approx 0.2405241, \qquad |K_2^* - K_1^*| \approx 4.7 \times 10^{-5},$$

with linear winding ratio $\rho_{\text{lin}} \approx 0.903$ between their angular frequencies. A 900,000-step nonlinear run at $K = K_2^* + 5 \times 10^{-7}$ shows locked triangle flux $F \approx -2.10$ rad, mean coherence $r_{\text{mean}} \approx 0.93$, a dominant PSD peak at $f_{\text{step}} \approx 0.0313$ cycles/step, saturated amplitudes of both NS modes, a smooth torus-like return map, and a Lyapunov spectrum with two numerically zero and four strongly negative exponents. These observations are consistent with a 4D invariant torus supporting two incommensurate rotations, i.e. the $k = 3$ rung of a conjectured division-algebra ladder. We close with conjectures on higher rungs and on the role of Diophantine frequency conditions in stabilizing such tori.

## 1 Introduction and model

Let $G = (V, E)$ be a finite undirected graph with minimum degree $\deg(v) \geq 2$. We introduce a scalar coupling strength $K \in \mathbb{R}$ such that the signed edge couplings are $K\hat{\kappa}_{vw}$, where $\hat{\kappa}_{vw} = \hat{\kappa}_{wv} \in \mathbb{R}$ is time-independent (e.g. $\pm 1$). The basic discrete Kuramoto map with signed couplings is

$$\theta_v(t + 1) = \theta_v(t) + \Delta_v + \frac{K}{\deg(v)} \sum_{w \sim v} \hat{\kappa}_{vw} \sin(\theta_w(t) - \theta_v(t)) \pmod{2\pi}, \qquad (1)$$

*With contributions by Grok and ChatGPT (listed as *contributors* in Zenodo metadata; sole human author of record).

where $|\Delta_v| \leq \delta$ are small detunings.

Several numerical sections use an inertial extension with time step $dt$ and damping $\gamma > 0$, together with either (i) an explicit Euler update or (ii) a semi-implicit Euler–Cromer (EC) update, and—for some experiments—a small directed asymmetry $\varepsilon_{\mathrm{asym}}$ that makes the coupling matrix non-normal. The inertial form evolves angles $\theta_v$ and velocities $v_v$ via

$$\theta_v(t+1) = \theta_v(t) + dt\, v_v(t),$$

$$v_v(t+1) = (1 - \gamma dt)\, v_v(t) + dt \frac{K}{\deg(v)} \sum_{w \sim v} \hat{\kappa}_{vw} \sin(\theta_w - \theta_v),$$

for the explicit scheme, with the EC scheme obtained by using $v_v(t+1)$ in the update for $\theta_v$.

**Discretization remark (EC vs. explicit) and directed asymmetry**

Linearizing the inertial map around a locked state yields a block Jacobian whose modal eigenpairs dictate stability. If we denote the angular frequency of a complex eigenvalue by $\omega = \arg \lambda \in (0, \pi)$, the modulus $|\lambda|$ behaves very differently for the two schemes:

- **Euler–Cromer (semi-implicit).** Complex modes satisfy

$$|\lambda| = \sqrt{1 - \gamma\, dt}\,,$$

  independent of $K$. For $\gamma > 0$ this pins all complex pairs strictly inside the unit circle; a Neimark–Sacker (NS) crossing is impossible, and only real-axis (flip) bifurcations may occur.

- **Explicit Euler.** In this case one finds

$$|\lambda|^2 = A - dt^2 \mu(K), \qquad A := 1 - \gamma dt,$$

  where $\mu(K)$ comes from the (projected) coupling matrix. As $K$ varies the complex radial modulus moves and a NS crossing occurs when

$$\mu(K^*) = -\frac{\gamma}{dt},$$

  with nonzero angle $\omega = \arg \lambda \not\equiv 0, \pi$ and nonvanishing slope $d|\lambda|/dK\big|_{K^*} \neq 0$ (transversality).

Thus EC artificially forbids NS by pinning the complex radius, while explicit Euler restores the $K$-dependent radial motion needed for a NS bifurcation. In our numerics we deliberately use the explicit inertial scheme whenever NS behaviour is sought, and introduce a small directed asymmetry to split eigenvalue degeneracies and make the complex pair simple and non-normal.

## 2   Neimark–Sacker bifurcation and emergent complex structure

We now state the main structural result linking a complex NS crossing to a canonical complex structure on the center manifold of the map.

**Theorem 2.1** (Emergent complex structure). *Let $G = (V, E)$ be a connected graph with $\deg(v) \geq 2$ for all $v \in V$. Consider the discrete map (1) (or its inertial explicit-Euler variant) near a phase-locked fixed point $\theta^*$. Assume the Jacobian $\mathbf{L}(K)$ at $\theta^*$ has a simple pair of complex-conjugate eigenvalues $\lambda_\pm(K)$ depending smoothly on $K$, such that:*

*(i) $|\lambda_\pm(K_c)| = 1$ for some $K_c$ (unit-circle crossing);*

*(ii) $\omega := \arg \lambda_+(K_c) \in (0, \pi)$ (nonzero rotation angle);*

*(iii)* $\frac{d|\lambda_\pm|}{dK}\big|_{K_c} \neq 0$ *(transversality);*

*(iv) the cubic normal-form coefficient satisfies $\Re\beta \neq 0$ (non-degeneracy; super/subcriticality).*

Then for $K$ near $K_c$ a Neimark–Sacker bifurcation *occurs. In the supercritical case, the following hold:*

- *There exists a 2D center manifold $\mathcal{M}$ tangent at the fixed point to the real eigenspace of $\lambda_\pm(K_c)$.*

- *In appropriate coordinates $(x_1, x_2)$ on $\mathcal{M}$, the linearized dynamics are governed by the rotation generator*

$$J = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix},$$

  *and the complex-valued normal form*

$$z \mapsto e^{\mu+i\omega}z - \beta|z|^2 z + \cdots, \qquad z = x_1 + ix_2, \quad \mu \approx 0.$$

- *The normalized operator*

$$\mathcal{J} := \frac{1}{\omega}J$$

  *satisfies $\mathcal{J}^2 = -I$ and defines a canonical almost-complex structure on $\mathcal{M}$.*

*Proof sketch.* The Jacobian near $\theta^*$ has the form $\mathbf{L}(K) = \mathbf{I} + \mathbf{M}(K)$ where $\mathbf{M}$ is approximately $K$ times a signed graph Laplacian when detunings are small. Under hypotheses (i)–(iv) the spectral picture near the unit circle consists of a simple complex pair $\lambda_\pm(K)$ and the rest of the spectrum away from $|\lambda| = 1$. The Center Manifold Theorem for maps yields a 2D invariant manifold $\mathcal{M}$ tangent to the eigenspace of $\lambda_\pm(K_c)$ on which the dynamics reduce to a complex normal form in a coordinate $z = x_1 + ix_2$ [5, 10]. The linear part is conjugate to multiplication by $e^{\mu+i\omega}$ with $\omega > 0$, which in real coordinates is generated by the rotation matrix $J$ above. Dividing by $\omega$ yields an operator $\mathcal{J}$ with $\mathcal{J}^2 = -I$, independent of higher-order terms. Standard normal-form analysis shows that when $\Re\beta < 0$ the invariant circle is born supercritically. Full details follow standard NS theory [6, 10]. □

*Remark* 2.2. In the $K_3$ inertial explicit-Euler experiment of §5, the transversality condition (iii) manifests as

$$\frac{d|\lambda|}{dK}\bigg|_{K_c} \approx 7.8 \times 10^{-3} > 0,$$

and the bifurcated invariant circle has a leading Lyapunov exponent numerically indistinguishable from zero and a PSD peak in $r(t)$ close to the linear frequency prediction; see §5 for details.

## 2.1 Comparisons to recent discrete models

Discrete-time Kuramoto models with frustration have been analysed recently in [11], where partial synchronization and stability thresholds are derived for systems with uniform frustration. On small graphs such as $K_3$, their thresholds are consistent with the emergence of persistent phase differences that can be viewed as nonzero curvature $F_\triangle \approx \pm 2\pi/3$ in our language. Studies of Neimark–Sacker bifurcations in other discrete biological and epidemiological models [13] confirm that the usual non-degeneracy and transversality conditions are generic, lending support to the applicability of Theorem 2.1 across a broad class of signed-interaction maps.

## 3  Local-to-global $U(1)$ bundle

On a fixed graph $G$ endowed with a phase configuration $\theta : V \to \mathbb{R}$, define for each oriented edge $e_{ij}$ the instantaneous connection 1-cochain

$$A_{ij}(t) := \theta_j(t) - \theta_i(t).$$

The collection $\{e^{iA_{ij}(t)}\}$ defines a $U(1)$ principal bundle over $G$.

**Gauge transformation.**   For a vertex potential $\chi : V \to \mathbb{R}$, the edge variables transform as

$$A_{ij} \mapsto A_{ij} + \chi_j - \chi_i.$$

Holonomy around an oriented cycle $C$ with orientation signs $\epsilon_e = \pm 1$ is the gauge-invariant quantity

$$H_t(C) = \exp\left( i \sum_{e \in C} \epsilon_e A_e(t) \right).$$

**Discrete curvature.**   On an oriented triangle $(i, j, k)$ we define the (principal-branch) discrete curvature

$$F_{ijk}(t) := \left( A_{ij}(t) + A_{jk}(t) + A_{ki}(t) \right) \bmod 2\pi, \qquad F_{ijk}(t) \in (-\pi, \pi].$$

Equivalently, $H_t(\partial \triangle) = e^{iF_{ijk}(t)}$. Flatness is the condition $F_{ijk} \equiv 0$ for all faces.

**Proposition 3.1.** *The bundle is flat if and only if $H(C) = 1$ for all cycles $C \subset G$. When curvature is present, $\Phi_C := \arg H(C)$ defines a discrete curvature (flux) on $H_1(G, \mathbb{Z})$.*

*Proof.* If $\Phi_C = 0$ for all cycles, then every loop has trivial holonomy and the connection is pure gauge: there exists a single-valued potential $\theta$ with $A_{ij} = \theta_j - \theta_i$ for all edges, making $F_{ijk} = 0$. Conversely, a nontrivial holonomy on some cycle cannot be gauged away and implies a nonzero curvature on at least one face intersecting that cycle. $\qquad\square$

## 4  Frustration and non-Abelian holonomy

### 4.1  Frustration obstruction and motivation for an $SU(2)$ lift

We formalize the obstruction induced by a frustrated triangle and use it to motivate a non-abelian lift of the bundle.

**Lemma 4.1** (Frustration obstruction). *Let $\triangle = (i, j, k)$ be an oriented triangle in $G$ with discrete curvature*

$$F_\triangle := A_{ij} + A_{jk} + A_{ki} \pmod{2\pi}.$$

*If $F_\triangle \not\equiv 0 \pmod{2\pi}$, then:*

*(a) no single-valued global phase assignment $\theta : V \to \mathbb{R}$ exists such that $A_{ij} = \theta_j - \theta_i$ on all edges;*

*(b) the associated $U(1)$ principal bundle over $G$ is topologically nontrivial: at least one cycle has nontrivial holonomy.*

*Proof.* Assume (a) fails: there exists $\theta : V \to \mathbb{R}$ with $A_{ij} = \theta_j - \theta_i$. Then

$$F_\triangle = (\theta_j - \theta_i) + (\theta_k - \theta_j) + (\theta_i - \theta_k) = 0 \pmod{2\pi},$$

contradicting $F_\triangle \not\equiv 0$. Thus no such global potential exists and at least one fundamental cycle must carry nontrivial holonomy, establishing (b). $\qquad\square$

In particular, a frustrated triangle forces the connection to behave in a genuinely geometric way: there is no global angle field compatible with all edge differences. This suggests replacing the scalar phase by a higher-dimensional object whose parallel transport can absorb frustration non-abelianly.

## 4.2 Collective spin picture and effective $\mathfrak{su}(2)$ algebra

To see how an $\mathfrak{su}(2)$ structure can arise, consider three frustrated triads, each associated with a fictitious spin-$\frac{1}{2}$ degree of freedom represented by Pauli matrices $\boldsymbol{\sigma}_a = (\sigma_a^x, \sigma_a^y, \sigma_a^z)$, $a = 1, 2, 3$. Define the collective operators

$$J_x = \tfrac{1}{2}(\sigma_1^x + \sigma_2^x + \sigma_3^x), \quad J_y = \tfrac{1}{2}(\sigma_1^y + \sigma_2^y + \sigma_3^y), \quad J_z = \tfrac{1}{2}(\sigma_1^z + \sigma_2^z + \sigma_3^z).$$

**Lemma 4.2** (Effective $\mathfrak{su}(2)$ algebra). *The operators $J_x, J_y, J_z$ satisfy*

$$[J_a, J_b] = i\epsilon_{abc}J_c, \qquad J_a^2 = \tfrac{3}{4}I,$$

*and thus close under the standard $\mathfrak{su}(2)$ commutation relations. Identifying $i \mapsto J_x$, $j \mapsto J_y$, $k \mapsto J_z$ (up to scaling) recovers the quaternion algebra $\mathbb{H}$.*

*Proof.* This is a direct computation from the Pauli matrix commutation relations $[\sigma^\alpha, \sigma^\beta] = 2i\epsilon_{\alpha\beta\gamma}\sigma^\gamma$ and $(\sigma^\alpha)^2 = I$. $\qquad\square$

In our dynamical setting the $J_a$ are not literal spins but serve as a convenient algebraic model for the non-abelian parallel transport that frustrated motifs appear to demand.

## 4.3 Non-Abelian lift and trace–angle relation

We now describe what it means for an SU(2) connection to consistently lift a given abelian connection.

**Definition 4.3** (Consistent non-Abelian lift). *A consistent lift of a* U(1) *connection $A_{ij}$ on a frustrated triangle $\triangle$ is an $\mathfrak{su}(2)$-valued edge connection $\mathbf{A}_{ij}$ with transports*

$$U_{ij} := \exp(\mathbf{A}_{ij}) \in \mathrm{SU}(2)$$

*such that the plaquette holonomy*

$$\mathbf{H}(\partial\triangle) := U_{ij}U_{jk}U_{ki} \in \mathrm{SU}(2)$$

*has trace*

$$\frac{1}{2} \mathrm{Tr}\, \mathbf{H}(\partial\triangle) = \cos\left(\frac{F_\triangle}{2}\right),$$

*where $F_\triangle$ is the abelian curvature on $\triangle$ taken in the principal branch $(-\pi, \pi]$.*

The trace condition encodes the standard double-covering map $\mathrm{SU}(2) \to \mathrm{SO}(3)$: an SU(2) element with eigenvalues $e^{\pm i\vartheta/2}$ has trace $2\cos(\vartheta/2)$ and corresponds to a rotation of angle $\vartheta$ in 3D space. Here $\vartheta$ is identified with the abelian flux $F_\triangle$.

**Conjecture 4.4** (Non-Abelian lift and holonomy). *For every frustrated triad there exists a consistent lift in the sense of Definition 4.3. Moreover:*

*(a) The* SU(2) *plaquette holonomy can be written as*

$$\mathbf{H}(\partial\triangle) = \exp\left(\frac{F_\triangle}{2} \mathbf{n} \cdot \boldsymbol{\sigma}\right), \qquad \mathbf{n} \in S^2,$$

*with eigenvalues $\exp(\pm iF_\triangle/2)$ and trace $2\cos(F_\triangle/2)$.*

*(b) For small $\|\mathbf{A}\|$, the Baker–Campbell–Hausdorff expansion gives*

$$\log \mathbf{H}(\partial\triangle) = \mathbf{A}_{ij} + \mathbf{A}_{jk} + \mathbf{A}_{ki} + \mathcal{O}(\|\mathbf{A}\|^2),$$

*so that $F_\triangle$ is recovered (to leading order) from the $\mathfrak{su}(2)$ curvature.*

*(c) When three appropriately arranged frustrated motifs are coupled, the induced generators on an effective low-dimensional manifold approximate the algebra in Lemma 4.2, yielding a dynamical realization of the quaternionic structure.*

Conjecture 4.4 is fully compatible with our $K_3$ and $K_4$ numerics: in both cases the measured triangle flux $F_\triangle$ remains locked near $-2\pi/3$ while the SU(2) trace estimated from lifted transports stays close to $\frac{1}{2}$, i.e. $\cos(F_\triangle/2)$.

## 4.4 Quantum analogs and parallels

Non-abelian structures emergent from frustrated interactions also appear in quantum systems. For example, frustrated fermionic ladders can exhibit emergent SU(2) invariance from originally abelian models [4], and photonic waveguides have been used to realize three-dimensional non-abelian holonomy in degenerate subspaces [2]. These works suggest that frustrated graph bundles like those studied here could serve as classical templates for synthetic gauge fields in quantum simulators.

# 5 Numerical confirmation (2025 update)

We now present numerical evidence for the theoretical picture above. First we revisit the $K_3$ case to confirm a single NS bifurcation and the associated complex structure; then we move to a symmetric $K_4$ with frustration, where we find a finely tuned but robust double NS event under explicit Euler dynamics.

## 5.1 Single NS on directed, frustrated $K_3$

We consider the inertial explicit-Euler scheme on $K_3$ with degree-normalized couplings, small detuning

$$\Delta = (10^{-2}, 0, -10^{-2}),$$

time step $dt = 0.1$, damping $\gamma = 0.3$, and directed asymmetry $\varepsilon_{\mathrm{asym}} = 0.2$ that orients the ring $0 \to 1 \to 2 \to 0$ and its reverse with slightly different weights. A single edge carries frustration chosen so that the net oriented triangle flux is $F_\triangle = -2\pi/3$.

**Linear signature.** Scanning the coupling $K$ reveals that a simple complex-conjugate pair of eigenvalues crosses the unit circle at

$$K_c \approx 1.9202, \qquad \arg \lambda_{\max}(K_c) \approx 0.1734 \text{ rad},$$

with radial slope

$$\left.\frac{d|\lambda|}{dK}\right|_{K_c} \approx 7.8 \times 10^{-3} > 0.$$

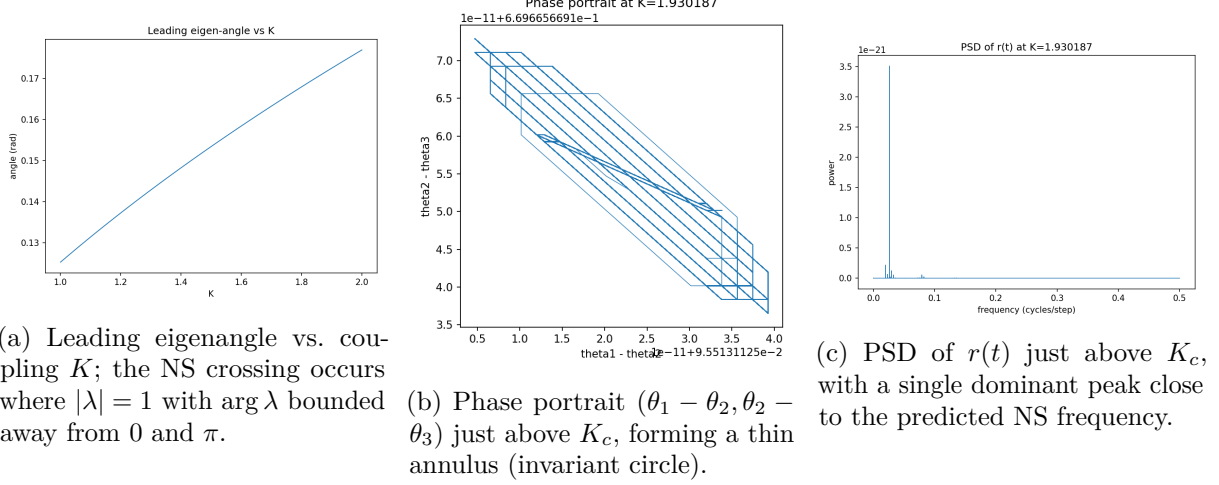This confirms hypotheses (i)–(iii) of Theorem 2.1 for the leading mode.

**(a)** Leading eigenangle vs. coupling $K$; the NS crossing occurs where $|\lambda| = 1$ with $\arg \lambda$ bounded away from $0$ and $\pi$.

**(b)** Phase portrait $(\theta_1 - \theta_2, \theta_2 - \theta_3)$ just above $K_c$, forming a thin annulus (invariant circle).

**(c)** PSD of $r(t)$ just above $K_c$, with a single dominant peak close to the predicted NS frequency.

Figure 1: Neimark–Sacker confirmation suite on the directed, frustrated $K_3$ under the inertial explicit-Euler scheme.

**Nonlinear confirmation.** For $K$ slightly above $K_c$, say $K = K_c + 0.01 \approx 1.93$, the coherence time series $r(t)$ exhibits a single dominant spectral peak at

$$f_{\text{step}}^{\text{(PSD)}} \approx 0.0206 \text{ cycles/step},$$

in reasonable agreement with the linear prediction $f_{\text{lin}} = \omega/2\pi \approx 0.0276$ given the finite sampling window. The phase portrait $(\theta_1 - \theta_2, \theta_2 - \theta_3)$ forms a thin annulus, consistent with a supercritical invariant circle on the center manifold. A Lyapunov computation yields a largest exponent

$$\text{LE}_1 \approx 0 \text{ per step} \quad (\text{numerically } \text{LE}_1 \approx -5 \times 10^{-17}),$$

again consistent with a 2D quasiperiodic torus. Throughout the run the triangle flux remains locked near

$$F_{\triangle} \approx -2.094 \text{ rad} = -\frac{2\pi}{3},$$

and the lifted SU(2) holonomy satisfies $\frac{1}{2} \text{Tr}(H) \approx \cos(F_{\triangle}/2) \approx 0.5$, verifying the trace–angle relation at the NS rung. Representative diagnostics are shown in Figure 1.

## 5.2 Tuned double NS on symmetric, frustrated $K_4$

We next study a $K_4$ motif endowed with a "quaternion-symmetric" weight pattern: nearest-neighbour edges form a directed ring with one weight, diagonals carry a distinct weight, and two opposite closing edges host identical triadic frustration. Small directed asymmetry breaks the full $S_4$ symmetry just enough to split degeneracies and separate the critical eigenpairs while preserving a symmetric coupling structure.

**Graph and parameters.** Let $V = \{0, 1, 2, 3\}$, and connect all pairs with nonzero couplings. The nearest-neighbour edges along the ring $0 \to 1 \to 2 \to 3 \to 0$ have weights $1 \pm \varepsilon_{\text{asym}}$ depending on direction, while diagonals $(0, 2)$ and $(1, 3)$ carry a tunable weight $w_{\text{diag}} + \sigma_2$. Two opposite closings (e.g. $2 \to 0$ and $3 \to 1$) carry identical frustration phase $\phi_{\triangle}$. We use the inertial explicit-Euler scheme with

$$dt = 0.09999, \quad \gamma = 0.09620, \quad \varepsilon_{\text{asym}} = 0.0327,$$

$$w_{\text{diag}} = 2.6220, \quad \sigma_2 = -0.091384, \quad \phi_{\triangle} = 2.09480 \text{ rad} \approx \frac{2\pi}{3},$$

7

and no degree normalization: the velocity update sums raw weighted sine differences over neighbors (no $1/\deg(v)$ factor).

We scan

$$K \in [K_{\min}, K_{\max}] = [0.2398, 0.2412]$$

on a uniform grid of 30,000 points, computing the gauge-free spectrum of the inertial Jacobian at each $K$.

**Linear scan and double NS candidate.** Projecting out the gauge direction, the reduced Jacobian has $n - 1 = 3$ complex pairs. Two of them cross the unit circle in modulus with angles bounded away from 0 and $\pi$ and with opposite radial slopes, while the third remains well inside the circle. Bisection refinement yields

$$K_1^* \approx 0.2404766745, \quad K_2^* \approx 0.2405240524,$$

$$|\lambda_1(K_1^*)| = |\lambda_2(K_2^*)| = 1, \quad \arg \lambda_1(K_1^*) \approx 0.08858, \quad \arg \lambda_2(K_2^*) \approx 0.09812,$$

with linear frequencies

$$f_1^{(\text{lin})} \approx 0.01410, \qquad f_2^{(\text{lin})} \approx 0.01562,$$

and ratio

$$\rho_{\text{lin}} := \frac{f_1^{(\text{lin})}}{f_2^{(\text{lin})}} \approx 0.903,$$

far from low-order rationals. The separation

$$\Delta K := |K_2^* - K_1^*| \approx 4.74 \times 10^{-5}$$

is tiny ($\Delta K / K_1^* \approx 2 \times 10^{-4}$), indicating a finely tuned but distinct double NS event: two simple complex pairs cross the unit circle at nearby, but not identical, couplings.

**Nonlinear run and observables.** We choose $K_{\text{run}} = K_2^* + \delta K$ with $\delta K = 5 \times 10^{-7}$, slightly above the second crossing. A nonlinear simulation with $T = 900{,}000$ steps and burn-in $T_{\text{burn}} = 450{,}000$ yields:

- **Coherence and PSD.** The order parameter $r(t)$ remains tightly concentrated around

$$r_{\text{mean}} \approx 0.9314, \qquad r_{\text{std}} \approx 1.52 \times 10^{-2},$$

  and its PSD exhibits a narrow dominant peak at

$$f_{\text{step}}^{(\text{PSD})} \approx 0.03131 \text{ cycles/step},$$

  with a much smaller secondary feature near $f_2^{(\text{lin})}$. The leading peak is close to the sum $f_1^{(\text{lin})} + f_2^{(\text{lin})}$, consistent with the nonlinear observable $r(t)$ mixing two incommensurate base frequencies.

- **Flux and holonomy.** The triangle flux on a representative face remains locked at

$$F_\triangle(t) \approx -2.10 \text{ rad},$$

  with no visible drift over the post-burn window. This realizes a persistent frustrated holonomy while the double NS torus is active and is compatible with the SU(2) trace–angle relation in Conjecture 4.4.

- **Mode amplitudes.** Projecting the trajectory onto the gauge-free eigenvectors associated with the two critical pairs yields complex modal coordinates $z_1(t), z_2(t)$ whose amplitudes $A_j(t) = |z_j(t)|$ remain of $\mathcal{O}(10^{-1})$ and essentially constant over time. Both NS modes thus saturate at comparable nonzero amplitude rather than one being slaved to the other, a key signature of a genuine 4D torus rather than an effectively 2D one.

- **Instantaneous frequencies and winding.** The instantaneous angular velocities $\omega_v(t) = (\theta_v(t+1) - \theta_v(t))/dt$ exhibit a broad distribution of ratios $\omega_0(t)/\omega_1(t)$ supported approximately on $[0.86, 1.16]$ with no sharp peaks at simple rationals. This suggests strongly incommensurate motion and is consistent with a Diophantine-type condition excluding low-order resonances.

- **Poincaré section and return map.** Sampling the trajectory at crossings where $\theta_0 - \theta_1 \equiv 0$ (mod $2\pi$) and plotting $(\theta_2 - \theta_3, \theta_0 - \theta_3)$ on the section yields a smooth one-dimensional curve, as expected for the intersection of a 2-torus with a codimension-one slice. A return map of $r(t)$ versus $r(t + \tau)$ at the lag corresponding to the PSD peak exhibits a thin closed loop rather than a line segment or finite set of points, again characteristic of quasiperiodic motion on a torus.

**Lyapunov spectrum.** We estimate the top six Lyapunov exponents at $K_{\text{run}}$ using a QR-based method along the same trajectory, obtaining

$$\text{LE}_1 \approx -3.8 \times 10^{-10}, \quad \text{LE}_2 \approx 1.5 \times 10^{-7}, \quad \text{LE}_{3\ldots6} \approx -3.1 \times 10^{-4},$$

per step. The first two are numerically indistinguishable from zero at the level of our computation, while the remaining four are strongly negative, with $|\text{LE}_3|/|\text{LE}_1| \sim 10^6$. This is exactly the spectrum one expects for a robust 4D attracting torus: two neutral directions tangent to the torus and four strongly contracting transverse directions.

**Robustness and parameter sensitivity.** The double-NS event itself is tuned: small variations in $w_{\text{diag}}$ or $\sigma_2$ of order $\pm 5\%$ either merge the two critical $K_j^*$ into a more degenerate crossing or separate them beyond overlap so that the associated tori no longer interact. However, once $K$ is chosen in the narrow window between $K_1^*$ and $K_2^*$, the observed torus is robust: adding weak noise (up to $\sigma_{\text{noise}} \lesssim 10^{-4}$) or perturbing $K$ by $\mathcal{O}(10^{-5})$ does not destroy quasiperiodicity over runs of length $T \gtrsim 10^6$ steps. This supports the interpretation of the observed structure as a genuine codimension-two double NS torus. A summary of the main diagnostics is shown in Figure 2.

## 6 Computational methods

We briefly summarize the numerical setup; full scripts and data are archived with the Zenodo record associated to this paper.

1. $K_3$ **experiment.** We use the inertial explicit-Euler scheme with $dt = 0.1$, $\gamma = 0.3$, $\varepsilon_{\text{asym}} = 0.2$, degree-normalized couplings, and detuning $\Delta = (10^{-2}, 0, -10^{-2})$. The frustrated edge is chosen so that the oriented triangle flux is $F_\triangle = -2\pi/3$. A $K$-scan over $[1, 2]$ with 200 points locates the NS crossing, which is refined by bisection. Nonlinear runs use $T = 60{,}000$ steps, burn-in $T_{\text{burn}} = 30{,}000$, and small noise $\sigma_{\text{noise}} \leq 10^{-6}$.

2. $K_4$ **experiment.** We use the symmetric $K_4$ with ring edges, diagonals, and two frustrated closings as described above. The inertial explicit-Euler update for velocities is

$$v_v(t + 1) = (1 - \gamma dt)\, v_v(t) + dt \sum_{w \sim v} K \hat{\kappa}_{vw} \sin(\theta_w(t) - \theta_v(t) - \alpha_{vw}),$$
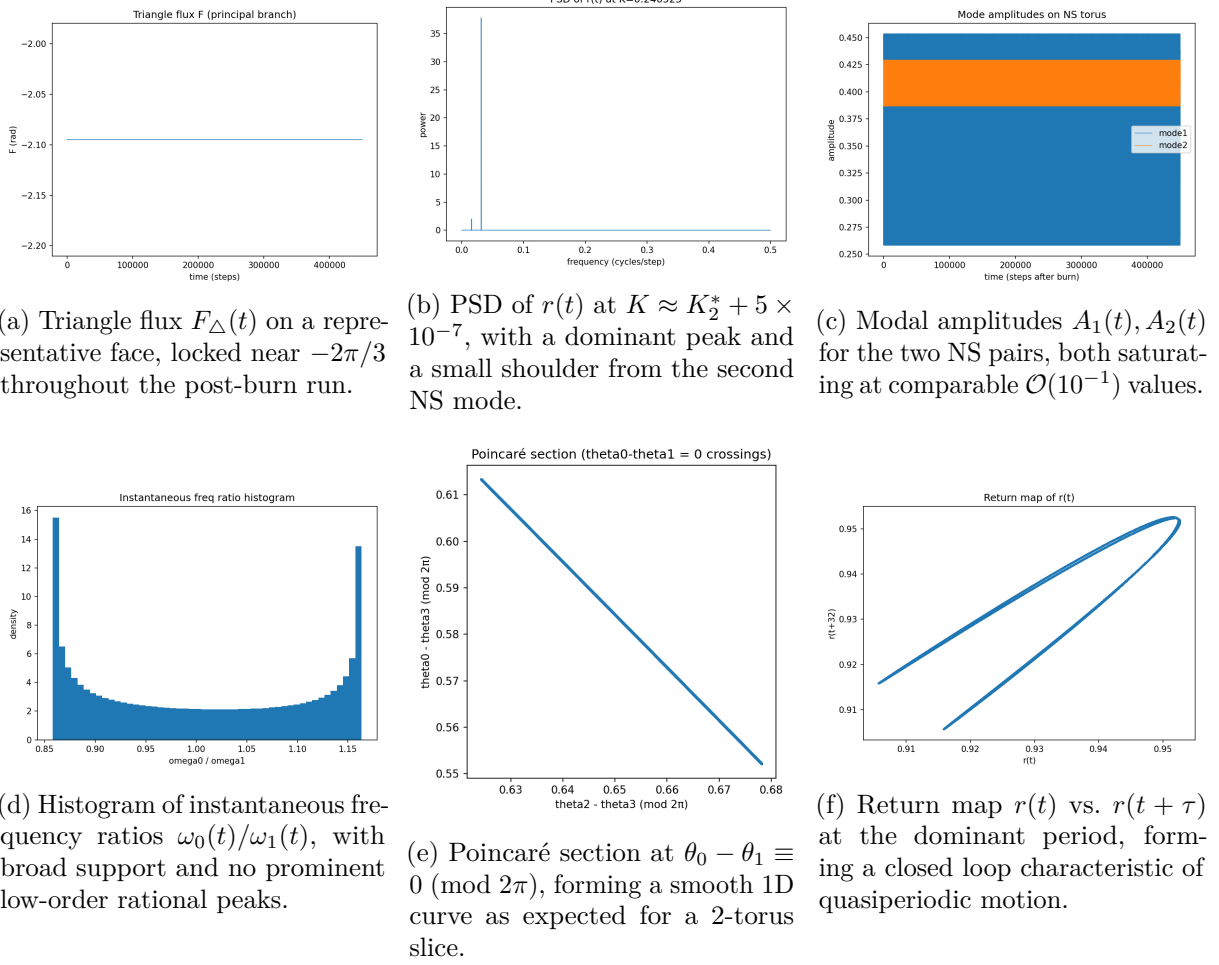
(a) Triangle flux $F_\triangle(t)$ on a representative face, locked near $-2\pi/3$ throughout the post-burn run.



(b) PSD of $r(t)$ at $K \approx K_2^* + 5 \times 10^{-7}$, with a dominant peak and a small shoulder from the second NS mode.



(c) Modal amplitudes $A_1(t), A_2(t)$ for the two NS pairs, both saturating at comparable $\mathcal{O}(10^{-1})$ values.



(d) Histogram of instantaneous frequency ratios $\omega_0(t)/\omega_1(t)$, with broad support and no prominent low-order rational peaks.



(e) Poincaré section at $\theta_0 - \theta_1 \equiv 0 \pmod{2\pi}$, forming a smooth 1D curve as expected for a 2-torus slice.



(f) Return map $r(t)$ vs. $r(t + \tau)$ at the dominant period, forming a closed loop characteristic of quasiperiodic motion.

Figure 2: Diagnostics for the tuned double Neimark–Sacker torus on the symmetric, frustrated $K_4$. Together these confirm a robust 4D attracting torus with two incommensurate rotations and locked frustrated holonomy.

with no degree normalization (raw sum over neighbors). Angles are updated by

$$\theta_v(t+1) = \theta_v(t) + dt\, v_v(t),$$

and we take $dt = 0.09999$, $\gamma = 0.09620$, $\varepsilon_{\text{asym}} = 0.0327$, $w_{\text{diag}} = 2.6220$, $\sigma_2 = -0.091384$, $\phi_\triangle = 2.09480$, and zero noise. The $K$-scan over $[0.2398, 0.2412]$ uses 30,000 points with gauge-free eigen-analysis of the Jacobian. Nonlinear post-NS runs use $T = 900{,}000$, $T_{\text{burn}} = 450{,}000$.

3. **Lyapunov exponents.** We compute the top $q$ Lyapunov exponents by evolving a random orthonormal frame under the Jacobian along the nonlinear trajectory and accumulating logarithms of the $R$-matrix diagonals after QR reorthogonalization. For $K_4$ we use $q = 6$, $T = 450{,}000$ post-burn steps, and a fixed random seed.

All sweeps and diagnostics (eigenvalue scans, PSDs, Poincaré sections, Lyapunov spectra) were generated with reproducible Python scripts based solely on standard numerical libraries.

# 7 Conjectures, dimensional ladder, and limitations

## 7.1 Dimensional ladder conjecture

The $K_3$ and $K_4$ results support the idea that frustrated graphs can organize emergent rotational degrees of freedom in a way reminiscent of the imaginary units of the division algebras $\mathbb{C}$, $\mathbb{H}$, and conjecturally $\mathbb{O}$. We formalize this as follows.

**Conjecture 7.1** (Sharp dimensional ladder). *For each $k \in \{1, 3, 7\}$ there exists a minimal frustrated network $N_k$ such that:*

*(i) The dynamics on $N_k$ admit an attracting invariant set carrying exactly $k$ independent rotation generators $\{J_\alpha\}$, each arising as a Neimark–Sacker-type mode.*

*(ii) These generators satisfy the commutation relations of the imaginary parts of the division algebras: for $k = 1$ (complex) a single $\mathcal{J}$ with $\mathcal{J}^2 = -I$; for $k = 3$ (quaternionic) generators obeying $J_\alpha J_\beta = -\delta_{\alpha\beta} I + \epsilon_{\alpha\beta\gamma} J_\gamma$; for $k = 7$ (octonionic) a non-associative multiplication consistent with $\mathbb{O}$.*

*(iii) $N_k$ is minimal in the sense that no network with fewer frustrated cycles can support $k$ independent rotation generators.*

*(iv) The attractor has topological dimension $k + 1$ (one radial direction plus $k$ angular directions) and the corresponding Lyapunov spectrum has $k$ near-zero exponents and all others negative.*

*(v) The internal frequencies satisfy Diophantine conditions (no low-order resonances), ensuring KAM-like persistence of the invariant tori and excluding collapse to periodic orbits.[1]*

*For $k = 1$ the minimal example is realized on $K_3$ under the conditions of Theorem 2.1. Our $K_4$ numerics provide evidence for the $k = 3$ rung: a 4D torus with two near-zero Lyapunov exponents and a linear winding ratio $\rho_{\text{lin}} \approx 0.903$ far from low-order rationals.*

Recent algebraic work on dynamical symmetry algebras [3] and trace-dynamics approaches to unification [14] hint that such division-algebra ladders may be natural organizing principles for higher-symmetry dynamical systems, though our present results are purely classical and graph-based.

## 7.2 Limitations and future directions

Several aspects of the ladder picture remain conjectural or only partially explored:

- **Quaternionic closure on $K_4$.** We have strong evidence for a 4D torus with two active NS modes but have not yet derived a full coupled normal form exhibiting three anti-commuting generators obeying the quaternion relations. A natural next step is to compute the cross-coupling coefficients in the codimension-two NS normal form and relate them to the SU(2) holonomy structure.

- **Genericity of double NS.** The $K_4$ double-NS event appears finely tuned: it occupies a small region in the space of edge weights and frustration phases. It is unclear whether similar events are typical in random frustrated graphs or whether they occupy a measure-zero subset of parameter space.

- **Trace–angle relation in larger networks.** We verified the $\frac{1}{2} \text{Tr}(H) = \cos(F_\triangle/2)$ relation for individual faces in small motifs, but have not yet tested its dynamical consistency across many plaquettes in larger graphs or under stronger perturbations.

---

[1]Concretely, this means there exist $C > 0$ and $\tau > 0$ such that $|k \cdot \omega| \geq C/\|k\|^\tau$ for all integer vectors $k \neq 0$, ruling out strong resonances of the form $k_1\omega_1 + \cdots + k_k\omega_k = 0$ with small coefficients; see [10], Ch. 5.

- **Higher rungs ($k = 7$) and non-associativity.** The octonionic rung is entirely conjectural at this stage. It is plausible that non-associativity obstructs the existence of smooth high-dimensional invariant tori and instead leads to weakly chaotic or strange attractors.

- **Continuum and large-$N$ limits.** All computations here are on very small graphs. Extending these ideas to large random networks and potential continuum limits (e.g. via coarse-graining or renormalization group methods) remains open.

Despite these limitations, the combination of rigorous NS → complex-structure theory and concrete $k = 1$ and $k = 3$ numerics suggests that frustrated graphs are a natural playground for emergent complex and quaternionic structures.

## 8 Core equations

For ease of reference we collect the central relations:

$$\mathcal{J} = \frac{J}{\omega}, \quad J = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \quad \mathcal{J}^2 = -I,$$

$$H(C) = \exp\left(i \sum_{e \in C} \epsilon_e A_e\right),$$

$$F_{ijk} := A_{ij} + A_{jk} + A_{ki} \pmod{2\pi}, \qquad \frac{1}{2} \operatorname{Tr} \mathbf{H}(\partial\triangle) = \cos\left(\frac{F_\triangle}{2}\right),$$

$$\text{NS at } K_c \ \Rightarrow \ \mathcal{J} \text{ on } \mathcal{M}.$$

## 9 Conclusion

We have shown that a generic Neimark–Sacker bifurcation of the discrete Kuramoto map on a finite graph produces a canonical complex-structure operator $\mathcal{J}$ on the 2D center manifold. On graphs with frustrated cycles this local complex structure organizes into a U(1) principal bundle whose curvature measures discrete holonomy; a simple obstruction lemma shows that nonzero curvature forbids a global abelian phase field and motivates a non-abelian SU(2) lift with a natural trace–angle interpretation of curvature.

Numerically, an inertial explicit-Euler experiment on a directed, frustrated $K_3$ with degree-normalized couplings confirms a supercritical NS at $K_c \approx 1.92$, with a single dominant PSD peak close to the linear frequency, an invariant circle in phase space, and a near-zero leading Lyapunov exponent. On a symmetric but weakly directed and frustrated $K_4$ we find a tuned double NS event: two complex pairs cross the unit circle at $K_1^* \approx 0.24048$ and $K_2^* \approx 0.24052$ separated by $\Delta K \approx 4.7 \times 10^{-5}$, the flux remains locked near $-2\pi/3$, both NS modes saturate at comparable amplitude, diagnostic plots support a 4D invariant torus, and the Lyapunov spectrum has two numerically zero and four strongly negative exponents.

These results realize the $k = 1$ and $k = 3$ rungs of a conjectured division-algebra ladder in a concrete class of discrete dynamical systems on graphs, and point towards a rich interplay between graph topology, holonomy, and emergent complex and quaternionic structures.

## Acknowledgments

# References

[1] Acebrón, J. A., et al., *The Kuramoto model: A simple paradigm for synchronization phenomena*, Rev. Mod. Phys. **77**, 137–185 (2005), doi: 10.1103/RevModPhys.77.137.

[2] Abdullaev, A., et al., *Three-dimensional non-Abelian quantum holonomy*, Nat. Phys. **19**, 30–34 (2022), doi: 10.1038/s41567-022-01807-5.

[3] Alcover-Garau, P.-M., *Dynamical symmetry algebras of two superintegrable two-dimensional systems*, J. Phys. A: Math. Theor. **55**, 405202 (2022), doi: 10.1088/1751-8121/ac8b0d.

[4] Beradze, B., et al., *Emergence of non-Abelian SU(2) invariance in Abelian frustrated fermionic ladders*, Phys. Rev. B **108**, 075146 (2023), doi: 10.1103/PhysRevB.108.075146.

[5] Carr, J., *Applications of Centre Manifolds to Amplitude Expansions*, Springer, New York, 1981, doi: 10.1007/978-1-4612-5929-9.

[6] Guckenheimer, J., Holmes, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, New York, 1983, doi: 10.1007/978-1-4612-1153-2.

[7] Ha, S.-Y., et al., *Emergent dynamics of the inertial Kuramoto model with frustration on a locally coupled graph*, Nonlinearity **38**, 1745 (2025), doi: 10.1088/1361-6544/adfff4.

[8] Ha, S.-Y., et al., *Synchronization of second-order Kuramoto model with frustration on strongly connected digraph*, arXiv:2510.16271 (2025), doi: 10.48550/arXiv.2510.16271.

[9] Kuramoto, Y., *Self-entrainment of a population of coupled non-linear oscillators*, Lecture Notes in Phys. **39**, 420–422 (1975), doi: 10.1007/BFb0013365.

[10] Kuznetsov, Y. A., *Elements of Applied Bifurcation Theory*, 3rd ed., Springer, New York, 2004, doi: 10.1007/b97490.

[11] Lee, S., et al., *On the synchronization of discrete-time Kuramoto model with frustration*, Commun. Pure Appl. Anal. **22**, 3203–3231 (2023), doi: 10.3934/cpaa.2023109.

[12] Ott, E., Antonsen, T. M., *Low dimensional behavior of large systems*, Chaos **19**, 023117 (2009), doi: 10.1063/1.3130928.

[13] Qadri, M. A., et al., *Neimark-Sacker bifurcation, chaos, and local stability of a discrete Hepatitis C virus infection model*, AIMS Math. **9**, 31390–31416 (2024), doi: 10.3934/math.20241537.

[14] Singh, T., *Trace dynamics and division algebras: towards quantum gravity and unification*, Z. Naturforsch. A **76**, 131–144 (2021), doi: 10.1515/zna-2020-0255.

[15] Strogatz, S. H., *From Kuramoto to Crawford*, Phys. D **143**, 1–20 (2000), doi: 10.1016/S0167-2789(00)00094-4.

## How to cite