Assignment 4

COMP232 SECTION (PP)

Mustafa Alawadi (40217764)

Presented to Professor Hassan Hajjdiab

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I certify that this submission is my original work and meets the Faculty's Expectations of Originality.

Mustafa Alawadi

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1.

1. The addition table for Z6:

Z6 = {0, 1, 2, 3, 4, 5}

+6	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

The cells in the table represent the result of adding the rows and columns elements modulo 6.

The multiplication table for Z6:

x6	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

The cells in the table represent the result of multiplying the rows and colums elements of 6.

2.

2. $f : Z^2 \to Z^2$ is given by $f(m,n) = (m-n, n)$ for
$f_k(m,n) = f(f_k(m,n))$ for $k \in Z^+$

Assume that $f(k)$ is stating the following:
$f_k(m,n) = (m - kn, n)$ where $k \in Z^2$

Basis: $n=1$ $f^{1}(m,n) = f(m,n) = (m-n, n) = (m-1n, n)$

Inductive Hypothesis: $n=k$. Assume $f^{k}(m,n) = (m-kn, n)$

Inductive step: $n=k+1$

$$f^{k+1}(m,n)$$
$$f(f^{k}(m,n))$$ inductive step hypothesis
$$f(m-kn, n)$$
$$(m-kn-n, n)$$
$$(m-(k+1)n, n)$$

3.

A) $1 \times 1! + 2 \times 2! + \ldots + n \times n! = (n+1)! - 1$

Checking for the base case: $P(n)$

$$1 \times 1! = (1+1)! - 1$$
$$1 = 1 \checkmark$$

Inductive hypothesis:

$P(k)$

$$1 \times 1! + 2 \times 2! + \ldots + k \times k! = (k+1)! - 1$$

Thus for $P(k+1)$

Left side: $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + k \times k! + (k+1) \times (k+1)!$

Right side: $(k+1)! - 1 + (k+1)! \times (k+1) = (k+1)! \times (k+1+1) - 1$

$$= (k+1)! \times (k+2) - 1 = (k+2)! - 1$$

Therefore, $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + k \times k! + (k+1) \times (k+1)! = (k+2)! - 1$

In conclusion, this is true for all $n \in Z^+$.

B) If $n>6$, then $3^n < n!$

Since n is bigger than 6, then for basis step : $n=7$, at $P(7)$

$$3^7 < 7!$$

$$2187 < 5040$$

The basis step is true

Inductive hypothesis: replace n by $k+1$, such as $P(k+1)$

$$3^k < k!$$

now for $P(k+1)$:

$$3^{k+1} < (k+1)!$$

$$3^k \cdot 3 < (k)! \quad \text{multiply by } 3$$

$$3^k \cdot 3 < 3 \cdot k!$$

$$3 \cdot k! < (k+1) \cdot k!$$

This inequality stands such as $3 < k+1$ for $k \geq 2$.

Thus: $3^{k+1} = 3^k \cdot 3 < 3 \cdot k! < (k+1) \cdot k! = (k+1)!$

4.

$f_0 = 0$, $f_1 = 1$ $3 \mid f_{4n}$, for all $n \geq 0$

4. $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$

$f_0 = 0$
$f_1 = 1$
$f_2 = f_1 + f_0 = 1 + 0 = 1$
$f_3 = f_2 + f_1 = 1 + 1 = 2$
$f_4 = f_3 + f_2 = 2 + 1 = 3$
$\vdots$

Our base case for $n = 0$:

$3 \mid F(0)$ is 0 because $\frac{0}{3} = 0$, thus the base case holds

Inductive:

Assume $3 \mid F(4k)$ for $k \geq 0$:

Let's show that $3 \mid F(4(k+1))$

$F(4(k+1)) = F(4k+4)$

recursive definition of fibonacci sequence:

$F(4k+4) = F(4k+3) + F(4k+2)$

we can now do $F(4k+2)$ and $F(4k+3)$ as being divisible by 3

$F(4k+2) = 3m$ m, n are integers
$F(4k+3) = 3n$

If $F(4k+4) = F(4k+3) + F(4k+2)$ then $F(4k+4) = 3m + 3n$

When we factor 3 out we get $F(4k+4) = 3(m+n)$

And this proves by mathematical induction that 3 divides $F(4n)$ for all $n \geq 0$

5.

5. Basic step: $n = 18$ since $n \geq 18$

$P(18)$ is true since two 7-cents and one 4-cents stamps can be used to get an 18 cents. Ex: $(2 \times 7) + 4 = 18$

Strong Inductive Hypothesis:

The statement holds for $18 \leq n \leq K$. Therefore, $18 \leq n \leq K$ can be produced using 4 cents and 7 cents stamps. Also, if we let $K \geq 21$

$$P(18), P(19), P(20), P(21) \ldots P(K+1)$$

If we know that $K-3$ can be produced ~~cents~~ using 4 and 7 cents stamps, thus, to form $K+1$ cents, we can add a 4 cents stamp to the $K-3$ cents.

We conclude that $P(K+1)$ is true since $K-3$ can be represented by 4 cents and 7 cents, thus $K+1$ can also be represented.

6.

6.A) $S = \{(a,b) : a \in Z^+, b \in Z^+, \text{ and } a|b\}$

This relation can be described for positive integers a and b, a divided by b gives an integer.

Base case: $(n,n) \in S$ for any positive integer n. This states that any positive integer is divisible by 1.

Recursive case: Using a and b: $(a,b) \in S$ then: $(a,bk) \in S$ for all $K \in Z^+$.

b) This relation can be described for positive integers a and b, the sum of a and b divided by 3 will give an integer, in other words:

$3m = (a+b)$

Ex: $a=1$ and $b=2$ or $a=2$ and $b=1$; will end such as $= 3$, divisible by 3

Ex: $a=2$ and $b=4$ or $a=4$ and $b=2$; will end such as $=6$, divisible by 3

Ex: $(a=4$ and $b=5)$ or $(a=5$ and $b=4)$; will end such as $a+b=9$, divisible by 3

Recursive case: Using a and b

$$(a,b) \in S \Rightarrow (a+3,b) \in S$$

$$(a,b) \in S \Rightarrow (a,b+3) \in S$$

$$(a,b) \in S \Rightarrow (a+1,b+2) \in S$$

$$(a,b) \in S \Rightarrow (a+2,b+1) \in S$$

7.

7. A) $x+2y=0$

Reflexive: It is not reflexive because if $x=1$, then $1+2(1)$ is not equal to 0.

Symmetric: It is not symmetric because if $x=2$ and $y=-1$, then $2+2(-1)=0$, however, $-1+2(2)$ is not equal to 0.

Anti-Symmetric: It is indeed anti-symmetric because when we swap x and y and negate the whole expression, it becomes $-(y+2x)$ which is not equal to the original expression.

Transitive: It is not transitive, take the case: $(x+2y)+(y+2z)$

B) $x=2y$

Reflexive: It is not reflexive since $x=1$, $1\neq 2(1)$

Symmetric: It is not symmetric $x=6$ and $y=2$. $6=2(3)$, however $2\neq 2(3)$

Anti-Symmetric: It is anti-symmetric $x=2y$, $y=2x$ as long as $x=y$

Transitive: It is not transitive, take case: $(x+2y)+(y+2z)$

c) $x - y$ is a rational number

Reflexive: It is reflexive $x-y$ for $x = y$, $x - y = 0$

Symmetric: It is symmetric (a,b) or (b,a) as if $x-y$ is rational, then $-(x-y) = y-x$

Anti-Symmetric: It is not anti-symmetric (a,b) since a is $\neq b$, this also stands for (b,a)

Transitive: It is transitive, take case when $x-y = \frac{a}{b}$ and $y-z = \frac{c}{d}$ and $x-z = \frac{c}{d} - \frac{a}{b}$

d) $xy = 0$

Reflexive: It is not reflexive because $(0,0) = 0$

Symmetric: It is symmetric because (x,y) and (y,x) have $xy = 0$ and $yx = 0$

Anti-Symmetric: It is not anti-symmetric since $x \neq y$ $xy = 0$, and $yx = 0$, even if $x \neq y$

Transitive: It is not transitive because $x = 0$ and $y = 1$, (x,y) and (y,z) are in R but (x,z) is not.

e) $xy \geq 0$

Reflexive: It is reflexive, take x, y for $y = x$ then $x^2 \geq 0$ for all x.

Symmetric: It is symmetric, take x, y for (x,y) and (y,x). This is true for $xy \geq 0$ and $yx \geq 0$.

Anti-Symmetric: It is not anti-symmetric because when $x \neq y$ then $xy \geq 0$ and $yx \geq 0$ is not always equal.

Transitive: It is not transitive, (x,y) and (y,z), if $xy \geq 0$ and $yz \geq 0$ and $xz \geq 0$

f) $X=1$ or $y=1$

Reflexive: It is not reflexive, $3 \in R$, but not in $3R3$.

Symmetric: It is symmetric take $x=1$ or $y=1$; $y=1$ or $x=1$; $y R x$

Anti-Symmetric: It is not anti-symmetric take $x=1$ and $y=5 \not\Rightarrow 1=5$

Transitive: It is not transitive; take $2R1$ and $1R5 \not\Rightarrow 2R5$.

g) x is a multiple of y

Reflexive: It is reflexive as every element of x is a multiple of itself.

Symmetric: It is not symmetric, counterexample; $x=4$, $y=2$. $x=2y$, note that y is not a multiple of x because 2 is not equal to any integer multiple of 4.

Anti-Symmetric: It is anti-symmetric because $x \neq y$ then xy and yx cannot both be true.

Transitive: It is transitive, take $x=ky$, $k \in Z$ also take $y=nz$, $n \in Z$

h) $xy = 1$

Reflexive: It is not reflexive, $x = y = 1$, $|x| = 1$ Satisfies the condition, but $x = 2$ $y = 1/2$ does not satisfies.

Symmetric: It is symmetric for (x, y) $xy = yx = 1$

Anti-Symmetric: It is not anti-symmetric $xy = yx = 1$ this does not satisfy, it shouldn't be equal to 1.

Transitive: It is not transitive, take $x = 2$, $y = 1/2$, $z = 2$; $xy = 1$, $yz = 1$, but $xz \neq 1$.

8.

8. A) If we let A be the set of even integers
$A = \{x \in Z \mid x = 2n \text{ for } n \in Z\}$
If we let B be the set of odd integers
$B = \{x \in Z \mid x = 2n+1 \text{ for } n \in Z\}$
The condition for partitions stand for both A and B because every integers are either odd or even, but not both.

B) The set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remainder of 2 when divided by 3.

Let A be the set of integers divisible by 3: $A \in Z$: $A: \{\ldots -9, -6, -3, 3, 6, 9 \ldots\}$
Let B be the set of integers leaving a remainder of 1 when divided by 3: $B \in Z$: $B: \{\ldots -8, -5, -2, 1, 4, 7, 10 \ldots\}$
Let C be the set of integers leaving a remainder of 2 when divided by 3: $C \in Z$: $C: \{\ldots -10, -7, -4, -1, 2, 5, 8, 11, \ldots\}$
Because all sets A, B and C are the disjaint sets and the union of 3 sets makes the complete sets of integers. Then we can conclude that the set of integers leaving a remainder of 1 when divided by 3 and the set of integers leaving a remainder of 2 when divided by 3 are all the partitions of the set of integers Z.

C) Let A be the set of integers less than -100: $A \in \mathbb{Z}$;
$A: \{\ldots, -104, -103, -102, \ldots\}$; $A = \{x \in \mathbb{Z} \mid x \leq -100\}$
Let B be the set of integers with absolute value not exceeding 100:
$B = \{x \in \mathbb{Z} \mid |x| \leq 100\}$
Let C be the set of integers greater than 100: $C = \{x \in \mathbb{Z} \mid x \geq 100\}$
This is a partition of the set of integers, the initial condition in the question stands for this; because every integer is either less than -100, between (less than -100) and 100 (inclusive) or greater than 100. Thus, we conclude this will cover all $\mathbb{Z}$.

D) Let A be the set of integers not divisible by 3, $A \in \mathbb{Z}$:
$A = \{\ldots -4, -2, -1, 1, 2, 4, 5, 7, \ldots\}$
Let B be the set of even integers, $B\{x \in \mathbb{Z} \mid x = 2n \; n \in \mathbb{Z}\}$
Let C be the set of integers that leave a remainder of 3 when divided by 6, $C = \{x \in \mathbb{Z} \mid x \bmod 6 = 3\}$; $C = \{\ldots, -15, -3, 3, 9, 15 \ldots\}$

This isn't a partition of the set of integers, there is at least one case that does not stand; for example 6 is an integer both divisible by 3 and is an even number. Because 6 is an integer that belongs to more than one subset.