Laplace Transform Lab: Solving ODEs using Laplace Transform in MATLAB

This lab will teach you to solve ODEs using a built in MATLAB Laplace transform function `laplace`.

There are five (5) exercises in this lab that are to be handed in. Write your solutions in a separate file, including appropriate descriptions in each step.

Include your name and student number in the submitted file.

Student Information

```
Student Name: Mustafa Khan

Student Number: 1006717037
```

Using symbolic variables to define functions

In this exercise we will use symbolic variables and functions.

```
syms t s x y

f = cos(t)
```

f = $\cos(t)$

```
h = exp(2*x)
```

h = e^{2x}

Laplace transform and its inverse

```
% The routine |laplace| computes the Laplace transform of a function

F=laplace(f)
```

F =

$$\frac{s}{s^2+1}$$

By default it uses the variable `s` for the Laplace transform But we can specify which variable we want:

```
H=laplace(h)
```

H =

$$\frac{1}{s-2}$$

```
laplace(h,y)
```

ans =

$$\frac{1}{y-2}$$

```
% Observe that the results are identical: one in the variable |s| and the
% other in the variable |y|
```

We can also specify which variable to use to compute the Laplace transform:

```
j = exp(x*t)
```

j = $e^{t\,x}$

```
laplace(j)
```

ans =

$$\frac{1}{s-x}$$

```
laplace(j,x,s)
```

ans =

$$\frac{1}{s-t}$$

```
% By default, MATLAB assumes that the Laplace transform is to be computed
% using the variable |t|, unless we specify that we should use the variable
% |x|
```

We can also use inline functions with `laplace`. When using inline functions, we always have to specify the variable of the function.

```
l = @(t) t^2+t+1
```

```
l = function_handle with value:
    @(t)t^2+t+1
```

```
laplace(l(t))
```

ans =

$$\frac{s+1}{s^2}+\frac{2}{s^3}$$

MATLAB also has the routine `ilaplace` to compute the inverse Laplace transform

```
ilaplace(F)
```

ans = $\cos(t)$

```
ilaplace(H)
```

ans = e^{2t}

```
ilaplace(laplace(f))
```

ans = $\cos(t)$

If `laplace` cannot compute the Laplace transform, it returns an unevaluated call.

```
g = 1/sqrt(t^2+1)
```

g =

$$\frac{1}{\sqrt{t^2+1}}$$

```
G = laplace(g)
```

G =

$$\mathrm{laplace}\left(\frac{1}{\sqrt{t^2+1}}, t, s\right)$$

But MATLAB "knows" that it is supposed to be a Laplace transform of a function. So if we compute the inverse Laplace transform, we obtain the original function

```
ilaplace(G)
```

ans =

$$\frac{1}{\sqrt{t^2+1}}$$

The Laplace transform of a function is related to the Laplace transform of its derivative:

```
syms g(t)
laplace(diff(g,t),t,s)
```

ans = $s\,\mathrm{laplace}(g(t), t, s) - g(0)$

Exercise 1

Objective: Compute the Laplace transform and use it to show that MATLAB 'knows' some of its properties.

Details:

(a) Define the function `f(t)=exp(2t)*t^3`, and compute its Laplace transform `F(s)`. (b) Find a function `f(t)` such that its Laplace transform is `(s - 1)*(s - 2))/(s*(s + 2)*(s - 3)` (c) Show that MATLAB 'knows' that if `F(s)` is the Laplace transform of `f(t)`, then the Laplace transform of `exp(at)f(t)` is `F(s-a)`

(in your answer, explain part (c) using comments).

Observe that MATLAB splits the rational function automatically when solving the inverse Laplace transform.

```
%Part a)
f(t) = exp(2*t)*t^3
```

f(t) = $t^3\, e^{2\,t}$

```
F = laplace(f)
```

F =

$$\frac{6}{(s-2)^4}$$

```
%Part b)
g = ilaplace(((s - 1)*(s - 2))/(s*(s + 2)*(s - 3)))
```

g =

$$\frac{6\,e^{-2\,t}}{5}+\frac{2\,e^{3\,t}}{15}-\frac{1}{3}$$

```
syms f(t) t s a

%Part c)
F = laplace(f(t))
```

F = $\mathrm{laplace}(f(t), t, s)$

```
laplace(exp(a*t) * f(t))
```

ans = $\mathrm{laplace}(f(t), t, s-a)$

```
% By looking at the outputs of the previous two lines
% we see that if F = laplace(f(t),t,s), then the output
% of the laplace transform of exp(a*t) * f(t) is
% laplace(f(t), t, s-a).

% Thus, MATLAB 'knows' that if F is the laplace
% transform of f(t), then the laplace transform of
% exp(at)f(t) is F(s-a).
```

Heaviside and Dirac functions

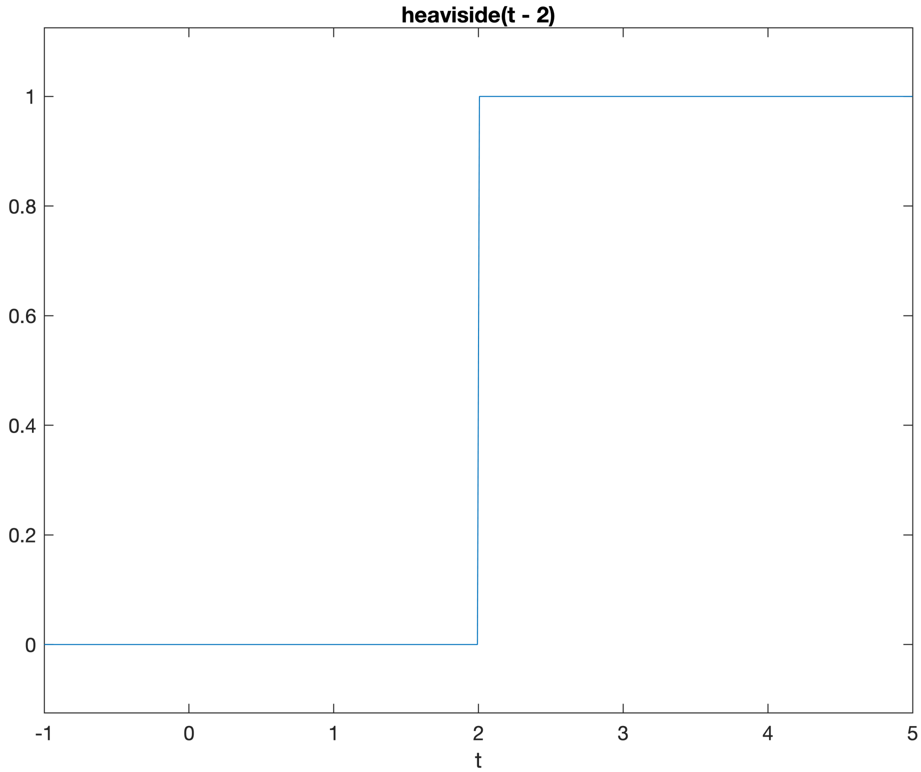
These two functions are builtin to MATLAB: `heaviside` is the Heaviside function `u_0(t)` at `0`

To define `u_2(t)`, we need to write

```
f=heaviside(t-2)
```

f = $\mathrm{heaviside}(t-2)$

```
ezplot(f,[-1,5])
```




```
% The Dirac delta function (at |0|) is also defined with the routine |dirac|
g = dirac(t-3)
```

g = $\delta(t-3)$

```
% MATLAB "knows" how to compute the Laplace transform of these functions
laplace(f)
```

ans =

$\frac{e^{-2s}}{s}$

```
laplace(g)
```

ans = e^{-3s}

Exercise 2

Objective: Find a formula comparing the Laplace transform of a translation of `f(t)` by `t-a` with the Laplace transform of `f(t)`

Details:

- Give a value to a
- Let `G(s)` be the Laplace transform of `g(t)=u_a(t)f(t-a)` and `F(s)` is the Laplace transform of `f(t)`, then find a formula relating `G(s)` and `F(s)`

In your answer, explain the 'proof' using comments.

```
syms f(t) t s

%Let a be 5
a=5
```

```
a = 5
```

```
%the laplace of f(t)
F = laplace(f(t))
```

F = $\mathrm{laplace}(f(t), t, s)$

```
%this is the laplace transform of a shifted f function
G = laplace(f(t-a)*heaviside(t-a))
```

G = $e^{-5s}\,\mathrm{laplace}(f(t), t, s)$

```
% From the code above, F(s) is laplace(f(t),t,s) and
% G(s) is e^(-5s)laplace(f(t),t,s). Therefore, the
% formula that relates G(s) to F(s) is G(s) = e^(-as)F(s)
% which in the case of a=5 is G(s) = e^(-5s)F(s).
```

Solving IVPs using Laplace transforms

Consider the following IVP, `y''-3y = 5t` with the initial conditions `y(0)=1` and `y'(0)=2`. We can use MATLAB to solve this problem using Laplace transforms:

```
% First we define the unknown function and its variable and the Laplace
% tranform of the unknown
syms y(t) t Y s

% Then we define the ODE
ODE=diff(y(t),t,2)-3*y(t)-5*t == 0
```

ODE =

$$\frac{\partial^2}{\partial t^2}\, y(t) - 5\,t - 3\,y(t) = 0$$

```
% Now we compute the Laplace transform of the ODE.
L_ODE = laplace(ODE)
```

L_ODE =

$$s^2\,\mathrm{laplace}(y(t),t,s) - s\,y(0) - \left(\left(\frac{\partial}{\partial t}\;y(t)\right)\Big|_{t=0}\right) - \frac{5}{s^2} - 3\,\mathrm{laplace}(y(t),t,s) = 0$$

```
% Use the initial conditions
L_ODE=subs(L_ODE,y(0),1)
```

L_ODE =

$$s^2\,\mathrm{laplace}(y(t),t,s) - s - \left(\left(\frac{\partial}{\partial t}\;y(t)\right)\Big|_{t=0}\right) - \frac{5}{s^2} - 3\,\mathrm{laplace}(y(t),t,s) = 0$$

```
L_ODE=subs(L_ODE,subs(diff(y(t), t), t, 0),2)
```

L_ODE =

$$s^2\,\mathrm{laplace}(y(t),t,s) - s - \frac{5}{s^2} - 3\,\mathrm{laplace}(y(t),t,s) - 2 = 0$$

```
% We then need to factor out the Laplace transform of |y(t)|
L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
```

L_ODE =

$$Y\,s^2 - s - 3\,Y - \frac{5}{s^2} - 2 = 0$$

```
Y=solve(L_ODE,Y)
```

Y =

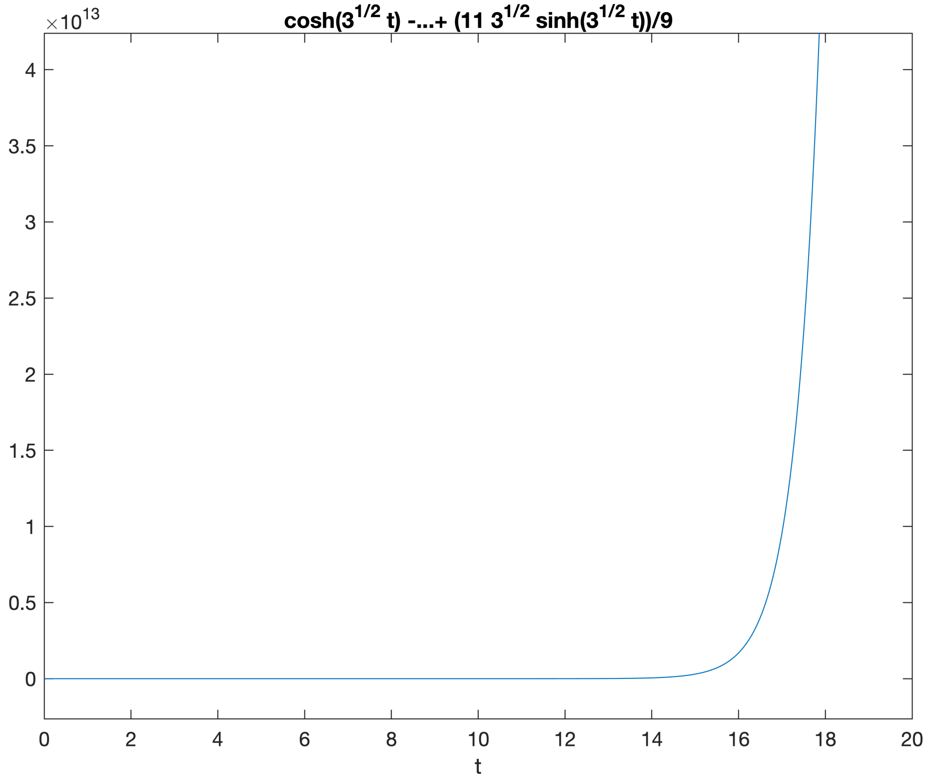
$$\frac{s + \frac{5}{s^2} + 2}{s^2 - 3}$$

```
% We now need to use the inverse Laplace transform to obtain the solution
% to the original IVP
y = ilaplace(Y)
```

y =

$$\cosh(\sqrt{3}\;t) - \frac{5\,t}{3} + \frac{11\;\sqrt{3}\;\sinh(\sqrt{3}\;t)}{9}$$

```
% We can plot the solution
ezplot(y,[0,20])
```




```
% We can check that this is indeed the solution
diff(y,t,2)-3*y
```

ans = $5\,t$

Exercise 3

Objective: Solve an IVP using the Laplace transform

Details: Explain your steps using comments

- Solve the IVP
- `y'''+2y''+y'+2*y=-cos(t)`
- `y(0)=0`, `y'(0)=0`, and `y''(0)=0`
- for `t` in `[0,10*pi]`
- Is there an initial condition for which `y` remains bounded as `t` goes to infinity? If so, find it.

```
% First we define the unknown function and its variable and the Laplace
% tranform of the unknown
syms y(t) t Y s

% Then we define the ODE
ODE=diff(y(t),t,3)+2*diff(y(t),t,2)+diff(y(t),t,1)+2*y(t)+cos(t) == 0
```

ODE =

$$\frac{\partial^3}{\partial t^3}\, y(t) + 2\,\frac{\partial^2}{\partial t^2}\, y(t) + \frac{\partial}{\partial t}\, y(t) + \cos(t) + 2\, y(t) = 0$$

```
% Now we compute the Laplace transform of the ODE.
L_ODE = laplace(ODE)
```

L_ODE =

$$s\,\sigma_1 - y(0) - 2\,s\,y(0) - s\,\left(\left(\frac{\partial}{\partial t}\, y(t)\right)\Big|_{t=0}\right) + \frac{s}{s^2+1} + 2\,s^2\,\sigma_1 + s^3\,\sigma_1 - 2\,\left(\left(\frac{\partial}{\partial t}\, y(t)\right)\Big|_{t=0}\right) - s^2\,y(0) - \left(\left(\right.\right.$$

where

$$\sigma_1 = \mathrm{laplace}(y(t), t, s)$$

```
% Use the initial conditions
L_ODE=subs(L_ODE,y(0),0)
```

L_ODE =

$$s\,\sigma_1 - s\,\left(\left(\frac{\partial}{\partial t}\, y(t)\right)\Big|_{t=0}\right) + \frac{s}{s^2+1} + 2\,s^2\,\sigma_1 + s^3\,\sigma_1 - 2\,\left(\left(\frac{\partial}{\partial t}\, y(t)\right)\Big|_{t=0}\right) - \left(\left(\frac{\partial^2}{\partial t^2}\, y(t)\right)\Big|_{t=0}\right) + 2\,\sigma_1 = 0$$

where

$$\sigma_1 = \mathrm{laplace}(y(t), t, s)$$

```
L_ODE=subs(L_ODE,subs(diff(y(t), t), t, 0),0)
```

L_ODE =

$$s\,\mathrm{laplace}(y(t), t, s) + \frac{s}{s^2+1} + 2\,s^2\,\mathrm{laplace}(y(t), t, s) + s^3\,\mathrm{laplace}(y(t), t, s) - \left(\left(\frac{\partial^2}{\partial t^2}\, y(t)\right)\Big|_{t=0}\right) + 2\,\mathrm{laplace}($$

```
L_ODE=subs(L_ODE, subs(diff(y(t),t,2), t, 0),0)
```

L_ODE =

$$s\,\mathrm{laplace}(y(t), t, s) + \frac{s}{s^2+1} + 2\,s^2\,\mathrm{laplace}(y(t), t, s) + s^3\,\mathrm{laplace}(y(t), t, s) + 2\,\mathrm{laplace}(y(t), t, s) = 0$$

```
% We then need to factor out the Laplace transform of |y(t)|
L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
```

L_ODE =

$$2\,Y + Y\,s + \frac{s}{s^2+1} + 2\,Y\,s^2 + Y\,s^3 = 0$$

```
Y=solve(L_ODE,Y)
```

Y =

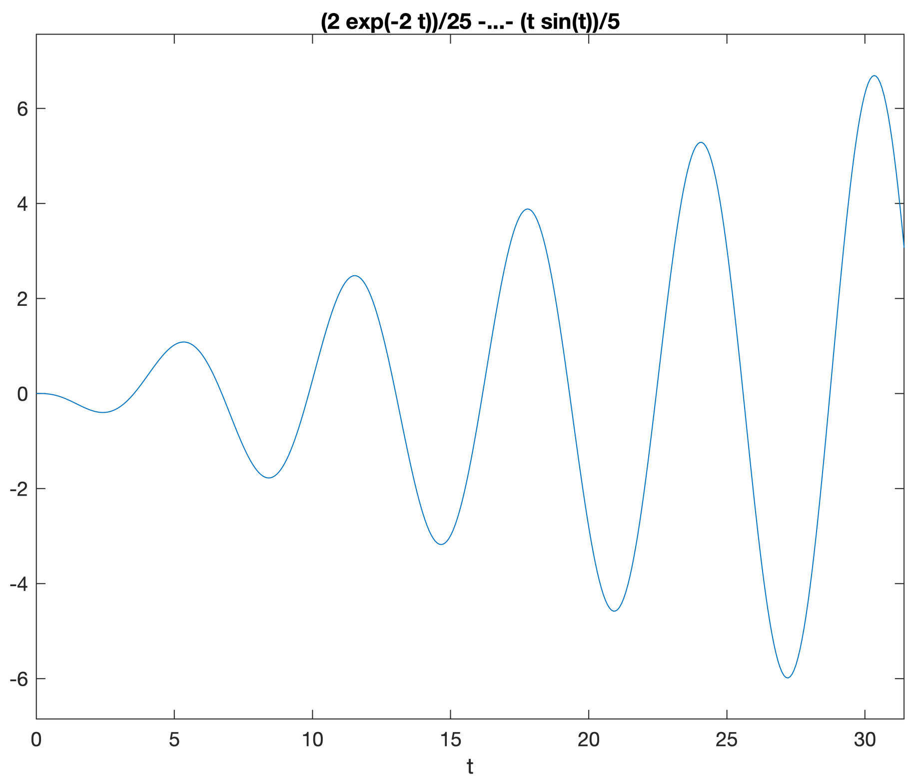
$$-\frac{s}{(s^2+1)\ (s^3+2\,s^2+s+2)}$$

```
% We now need to use the inverse Laplace transform to obtain the solution
% to the original IVP
y = ilaplace(Y)
```

y =

$$\frac{2\,e^{-2\,t}}{25}-\frac{2\cos(t)}{25}+\frac{3\sin(t)}{50}+\frac{t\cos(t)}{10}-\frac{t\sin(t)}{5}$$

```
% We can plot the solution
ezplot(y,[0,10*pi])
```




```
% We can check that this is indeed the solution
diff(y,t,3)+2*diff(y,t,2)+diff(y,t,1)+2*y
```

ans = $-\cos(t)$

```
%Question: Is there an initial condition for which y
% remains bounded as t goes to infinity? If so, find it.

%Answer: No there is no initial condition that can keep
```

```
% y bounded as t goes to infinity.
% By running the above code and commenting out the three
% lines where the initial conditions are plugged in (i.e.:
% L_ODE=subs(L_ODE,y(0),0)
% L_ODE=subs(L_ODE,subs(diff(y(t), t), t, 0),0)
% L_ODE=subs(L_ODE, subs(diff(y(t),t,2), t, 0),0) )
% We see that the general equation of the solution to the
% differential equation is as follows:
% y=c1e^(-2t) + c2cos(t) + c3sin(t) -tsin(t)/5 + tcos(t)/10
% where c1, c2 and c3 are coefficients that can be impacted
% by the initial conditions. However, the growth of the
% y values in the solution as t increases is not affected
% by the terms with the coefficients. They are impacted
% by the last two terms which have a factor of t.
% No initial condition can cause these terms to go to 0
% which would be the only way to stop the growth of the
% y values and bound them.

% Additional observation:
% If you were to look at the laplace transform of the ODE
% you get an equation of Y(s). This has terms which
% are impacted by the initial conditions and terms
% which are not. The former are from the LHS of the ODE
% while the latter are from the RHS, aka the forcing
% function g(t). Taking the inverse laplace of Y(s)
% gives us y(t) and we notice once again that there
% are terms with coefficients dependent on initial
% conditions and other terms such as:
% -tsin(t)/5 + tcos(t)/10 as mentioned above which don't
% depend on coefficients. These happen to be the
% terms that arise due to the forcing function as well.
% Thus, the observation we make (or an alternative
% explanation for why no initial conditions would bound
% the solution) is that the terms causing the growth of
% the solution all arise from the forcing function.
```

Exercise 4

Objective: Solve an IVP using the Laplace transform

Details:

- Define
- `g(t) = 3 if 0 < t < 2`
- `g(t) = t+1 if 2 < t < 5`
- `g(t) = 5 if t > 5`
- Solve the IVP
- `y''+2y'+5y=g(t)`
- `y(0)=2 and y'(0)=1`
- Plot the solution for `t` in `[0,12]` and y in `[0,2.25]`.

In your answer, explain your steps using comments.

```
% First we define the unknown function and its variable and the Laplace
% tranform of the unknown
syms y(t) g(t) t Y s

%Define g(t) using the heaviside() function
%g(t) = 3*heaviside(t+1) - 3*heaviside(t-2) + (t+1)*heaviside(t-2) - (t+1)*heaviside(t
%which we simplify into:
g(t) = 3*heaviside(t+1) + (t-2)*heaviside(t-2) + (4 - t)*heaviside(t-5)
```

g(t) = $3\,\mathrm{heaviside}(t+1) + \mathrm{heaviside}(t-2)\;(t-2) - \mathrm{heaviside}(t-5)\;(t-4)$

```
% Then we define the ODE
ODE=diff(y(t),t,2)+2*diff(y(t),t,1)+5*y(t)-g(t) == 0
```

ODE =

$$\frac{\partial^2}{\partial t^2}\;y(t) + 2\,\frac{\partial}{\partial t}\;y(t) - 3\,\mathrm{heaviside}(t+1) + 5\;y(t) - \mathrm{heaviside}(t-2)\;(t-2) + \mathrm{heaviside}(t-5)\;(t-4) = 0$$

```
% Now we compute the Laplace transform of the ODE.
L_ODE = laplace(ODE)
```

L_ODE =

$$2\,s\,\sigma_1 - 2\;y(0) - s\;y(0) - \frac{e^{-2\,s}}{s^2} + s^2\,\sigma_1 - \left(\left(\frac{\partial}{\partial t}\;y(t)\right)\Big|_{t=0}\right) - \frac{3}{s} + \frac{e^{-5\,s}\;(s+1)}{s^2} + 5\,\sigma_1 = 0$$

where

$$\sigma_1 = \mathrm{laplace}(y(t), t, s)$$

```
% Use the initial conditions
L_ODE=subs(L_ODE,y(0),2)
```

L_ODE =

$$2\,s\,\mathrm{laplace}(y(t), t, s) - 2\,s - \frac{e^{-2\,s}}{s^2} + s^2\,\mathrm{laplace}(y(t), t, s) - \left(\left(\frac{\partial}{\partial t}\;y(t)\right)\Big|_{t=0}\right) - \frac{3}{s} + \frac{e^{-5\,s}\;(s+1)}{s^2} + 5\,\mathrm{laplace}(y$$

```
L_ODE=subs(L_ODE,subs(diff(y(t), t), t, 0),1)
```

L_ODE =

$$2\,s\,\mathrm{laplace}(y(t), t, s) - 2\,s - \frac{e^{-2\,s}}{s^2} + s^2\,\mathrm{laplace}(y(t), t, s) - \frac{3}{s} + \frac{e^{-5\,s}\;(s+1)}{s^2} + 5\,\mathrm{laplace}(y(t), t, s) - 5 = 0$$

```
% We then need to factor out the Laplace transform of |y(t)|
L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
```

L_ODE =

$$5\,Y-2\,s+2\,Y\,s-\frac{e^{-2\,s}}{s^2}+Y\,s^2-\frac{3}{s}+\frac{e^{-5\,s}\,(s+1)}{s^2}-5=0$$

```
Y=solve(L_ODE,Y)
```

Y =

$$\frac{2\,s+\frac{e^{-2\,s}}{s^2}+\frac{3}{s}-\frac{e^{-5\,s}\,(s+1)}{s^2}+5}{s^2+2\,s+5}$$

```
% We now need to use the inverse Laplace transform to obtain the solution
% to the original IVP
y = ilaplace(Y)
```

y =

$$\mathrm{heaviside}(t-2)\,\left(\frac{t}{5}+\frac{2\,e^{2-t}\,\left(\cos(2\,t-4)-\frac{3\,\sin(2\,t-4)}{4}\right)}{25}-\frac{12}{25}\right)-\mathrm{heaviside}(t-5)\,\left(\frac{t}{5}+\frac{2\,e^{5-t}\,\left(\sigma_3-\right.}{25}\right.$$

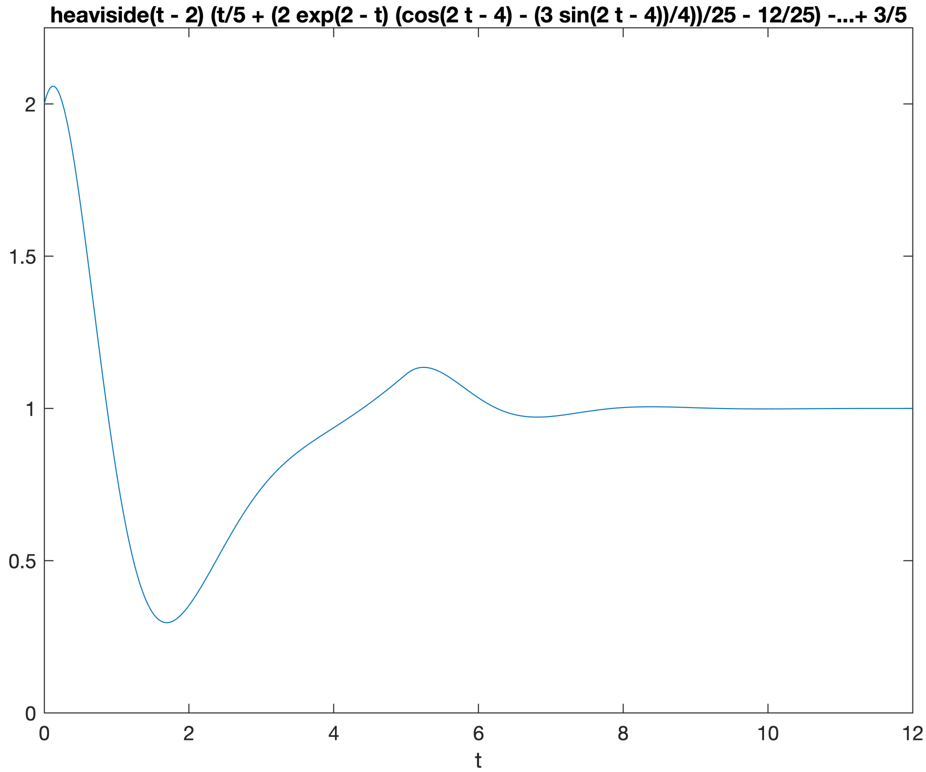
where

$$\sigma_1=\frac{\sin(2\,t)}{2}$$

$$\sigma_2=\sin(2\,t-10)$$

$$\sigma_3=\cos(2\,t-10)$$

```
% We can plot the solution between the specified bounds
ezplot(y,[0,12,0,2.25])
```




```
% We can check that this is indeed the solution
diff(y,t,2)+2*diff(y,t,1)+5*y
```

```
ans =
```

$$\frac{3\,\mathrm{e}^{-t}\,(\sigma_2+\sigma_1)}{5} - 2\,\mathrm{e}^{-t}\,(\sigma_2-\sigma_1) - \mathrm{heaviside}(t-5)\,\left(\frac{\mathrm{e}^{5-t}\,(4\,\sigma_{16}+2\,\sigma_{17})}{5} + \frac{2\,\mathrm{e}^{5-t}\,(\sigma_{16}-2\,\sigma_{17})}{5} - \sigma_{13}\right) - 2\,\dot{\imath}$$

where

$$\sigma_1 = 2\sin(2\,t)$$

$$\sigma_2 = 4\cos(2\,t)$$

$$\sigma_3 = \frac{\sin(2\,t)}{2}$$

$$\sigma_4 = \frac{t}{5} + \sigma_9 - \frac{27}{25}$$

$$\sigma_5 = \frac{t}{5} + \sigma_{11} - \frac{12}{25}$$

$$\sigma_6 = \frac{2\,\mathrm{e}^{5-t}\,\sigma_{10}}{25} + \sigma_9 - \frac{1}{5}$$

$$\sigma_7 = \frac{2\,\mathrm{e}^{2-t}\,\sigma_{12}}{25} + \sigma_{11} - \frac{1}{5}$$

$$\sigma_8 = \frac{\mathrm{e}^{5-t}\,(\sigma_{16}-2\,\sigma_{17})}{5} - \sigma_{13}$$

$$\sigma_9 = \frac{2\,\mathrm{e}^{5-t}\,\left(\sigma_{16}-\frac{3\,\sigma_{17}}{4}\right)}{25}$$

$$\sigma_{10} = \frac{3\,\sigma_{16}}{2} + 2\,\sigma_{17}$$

$$\sigma_{11} = \frac{2\,\mathrm{e}^{2-t}\,\left(\sigma_{14}-\frac{3\,\sigma_{15}}{4}\right)}{25}$$

$$\sigma_{12} = \frac{3\,\sigma_{14}}{2} + 2\,\sigma_{15}$$

$$\sigma_{13} = \frac{\mathrm{e}^{5-t}\,\left(\sigma_{16}+\frac{\sigma_{17}}{2}\right)}{5}$$

$$\sigma_{14} = \cos(2\,t-4)$$

$$\sigma_{15} = \sin(2\,t-4)$$

Exercise 5

Objective: Use the Laplace transform to solve an integral equation

Verify that MATLAB knowns about the convolution theorem by explaining why the following transform is computed correctly.

```
syms t tau y(tau) s
I=int(exp(-2*(t-tau))*y(tau),tau,0,t)
```

I =

$$\int_0^t e^{2\tau-2t}\, y(\tau)\mathrm{d}\tau$$

```
laplace(I,t,s)
```

ans =

$$\frac{\mathrm{laplace}(y(t),t,s)}{s+2}$$

```

% I is the convolution of two functions, f and g.
% where f=e^(-2t) and g=y(t).

% According to convolution theorem, the laplace transform
% of two convolved functions is equal to L(f*g) = L(f)L(g),
% or in other words, it is equal to the multiplication of
% the laplace transforms of each individual functions.

% The laplace transform of f=e^(-2t) is given by:
% L(e^-2t) = -1/(-s-2) = 1/(s+2). We can define
% the laplace of g=y(t) as L(y(t)) = laplace(y(t),t,s).

% Clearly, to compute the laplace of the integral,
% MATLAB is invoking the convolution theorem
% as it produces an answer of the form L(f)L(g)
% where the multiplication of the laplace transforms
% of both functions results in laplace(y(t),t,s)/(s+2)
```