Second-Order Lab: Second-Order Linear DEs in MATLAB

In this lab, you will learn how to use `iode` to plot solutions of second-order ODEs. You will also learn to classify the behaviour of different types of solutions.

Moreover, you will write your own Second-Order ODE system solver, and compare its results to those of `iode`.

Opening the m-file lab5.m in the MATLAB editor, step through each part using cell mode to see the results.

There are seven (7) exercises in this lab that are to be handed in on the due date of the lab.

Student Information

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Iode for Second-Order Linear DEs with constant coefficients

In the `iode` menu, select the `Second order linear ODEs` module. It opens with a default DE and a default forcing function `f(t) = cos(2t)`. The forcing function can be plotted along with the solution by choosing `Show forcing function` from the `Options` menu.

Use this module to easily plot solutions to these kind of equations.

There are three methods to input the initial conditions:

Method 1. Enter the values for `t0`, `x(t0)`, and `x'(t0)` into the `Initial conditions` boxes, and then click `Plot solution`.

Method 2. Enter the desired slope `x'(t0)` into the appropriate into the `Initial conditions` box, and then click on the graph at the point `(t0,x(t0))` where you want the solution to start.

Method 3. Press down the left mouse button at the desired point `(t0,x(t0))` and drag the mouse a short distance at the desired slope `x'(t0)`. When you release the mouse button, `iode` will plot the solution.

Growth and Decay Concepts

We want to classify different kinds of behaviour of the solutions. We say that a solution:

`grows` if its magnitude tends to `infinity` for large values of `t`, that is, if either the solution tends to `+∞` or `−∞`,

`decays` if its magnitude converges to `0` for large values of `t`,

`decays while oscillating` if it keeps changing sign for large values of `t` and the amplitude of the oscillation tends to zero,

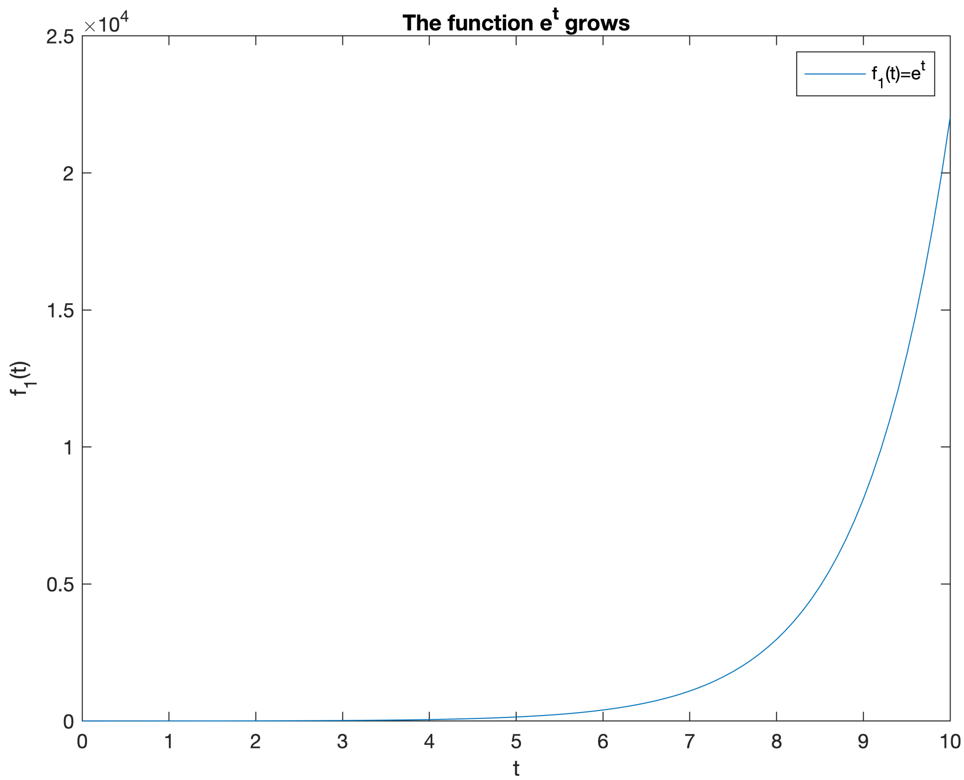
`grows while oscillating` if it keeps changing sign for large values of `t` and the amplitude of the oscillation tends to infinity.

Example

```
t = 0:0.1:10;
```

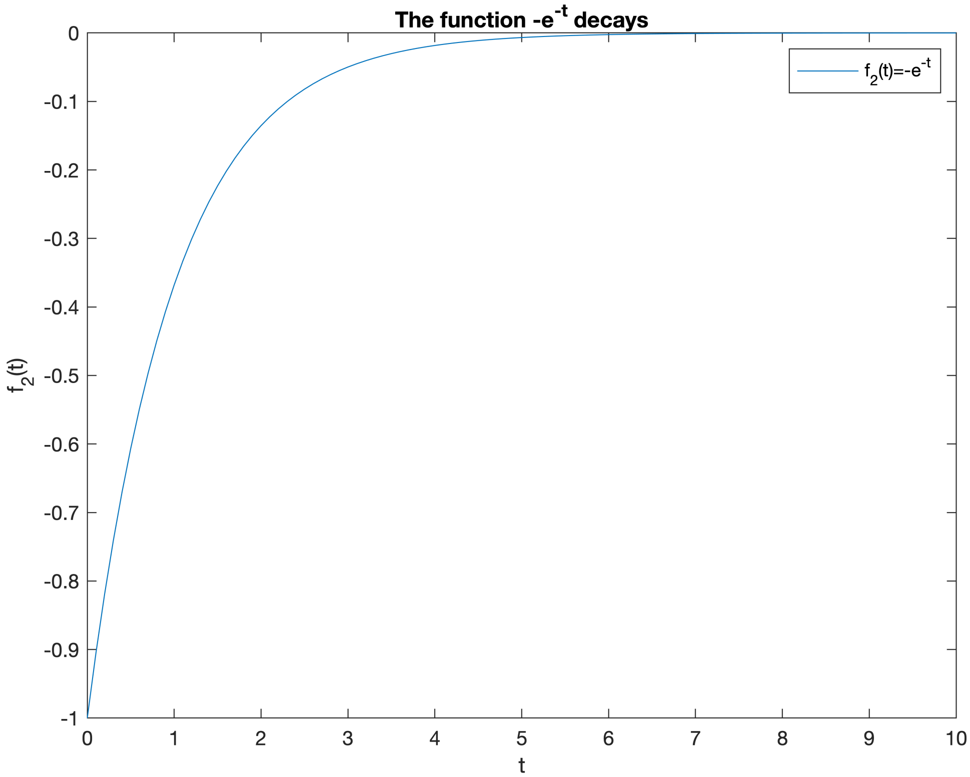
```
% Example 1
figure();
y1 = exp(t);
plot(t,y1)

% Annotate the figure
xlabel('t');
ylabel('f_1(t)');
title('The function e^t grows');
legend('f_1(t)=e^t');
```



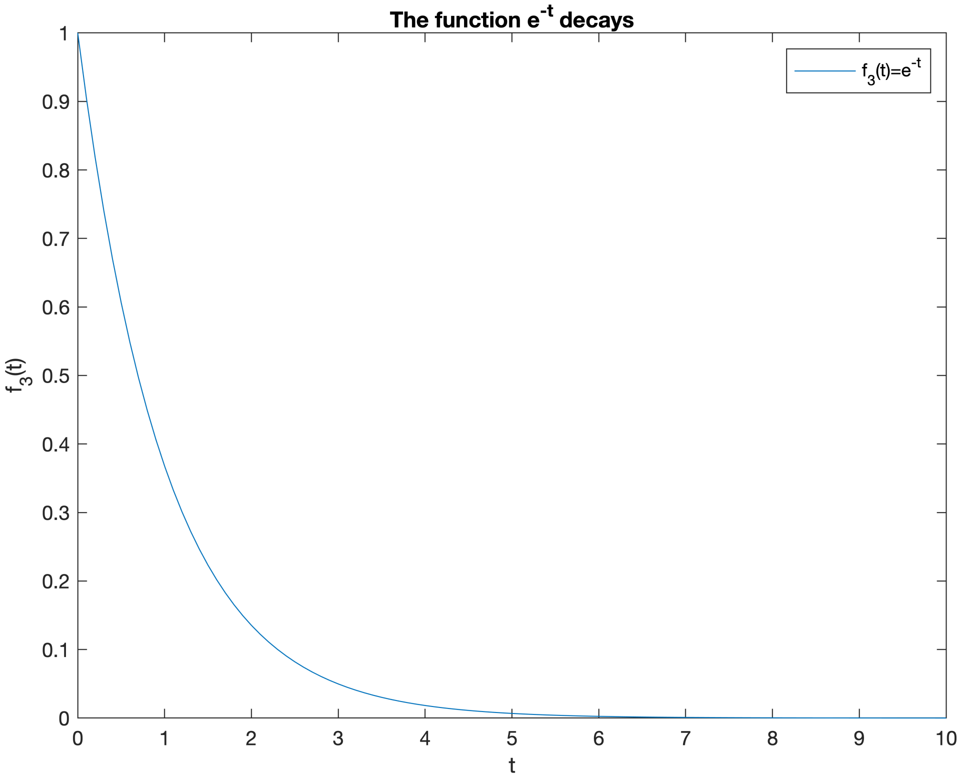

```
% Example 2
figure();
y2 = -exp(-t);
plot(t,y2)

% Annotate the figure
xlabel('t');
ylabel('f_2(t)');
title('The function -e^{-t} decays');
legend('f_2(t)=-e^{-t}');
```



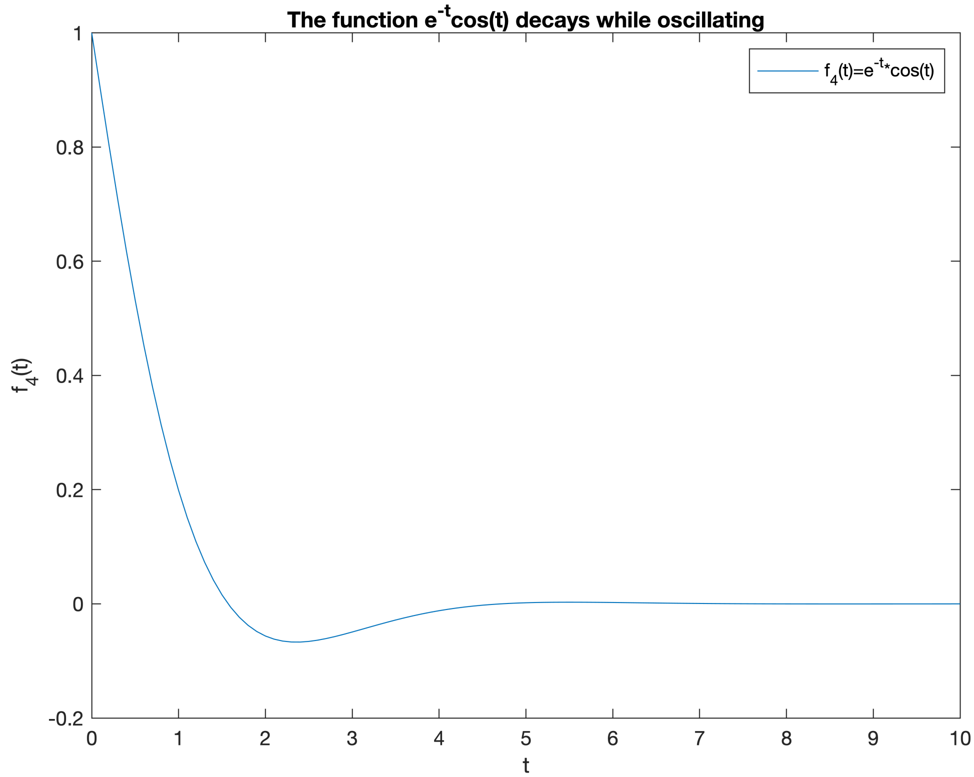

```
% Example 3
figure();
y3 = exp(-t);
plot(t,y3)

% Annotate the figure
xlabel('t');
ylabel('f_3(t)');
title('The function e^{-t} decays');
legend('f_3(t)=e^{-t}');
```



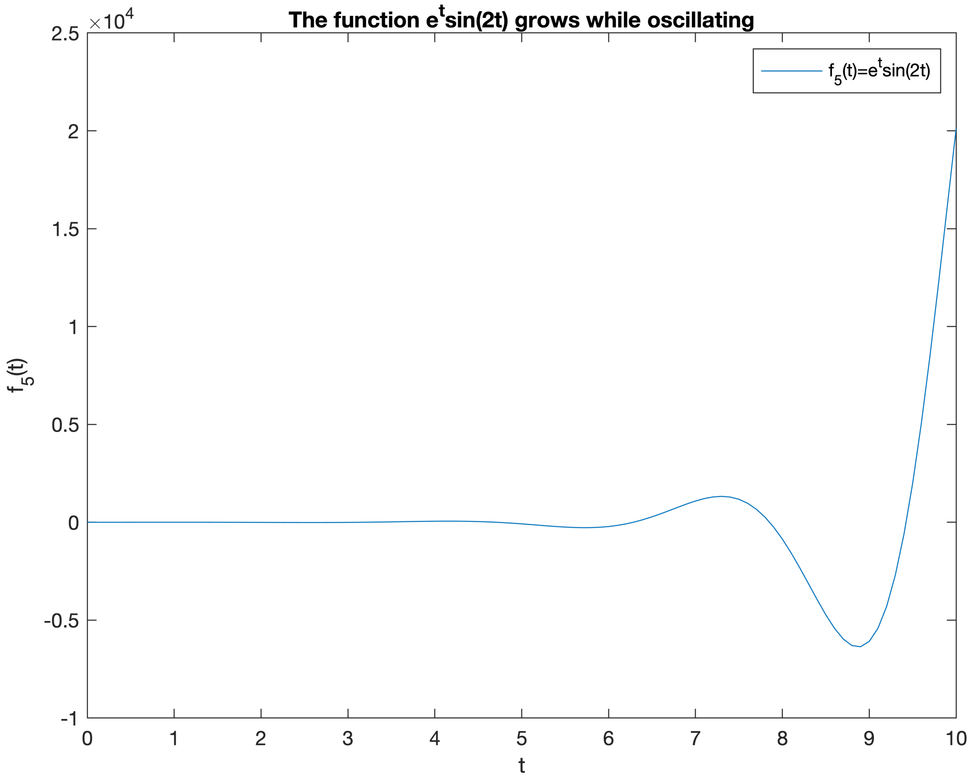

```
% Example 4
figure();
y4 = exp(-t).*cos(t);
plot(t,y4)

% Annotate the figure
xlabel('t');
ylabel('f_4(t)');
title('The function e^{-t}cos(t) decays while oscillating');
legend('f_4(t)=e^{-t}*cos(t)');
```



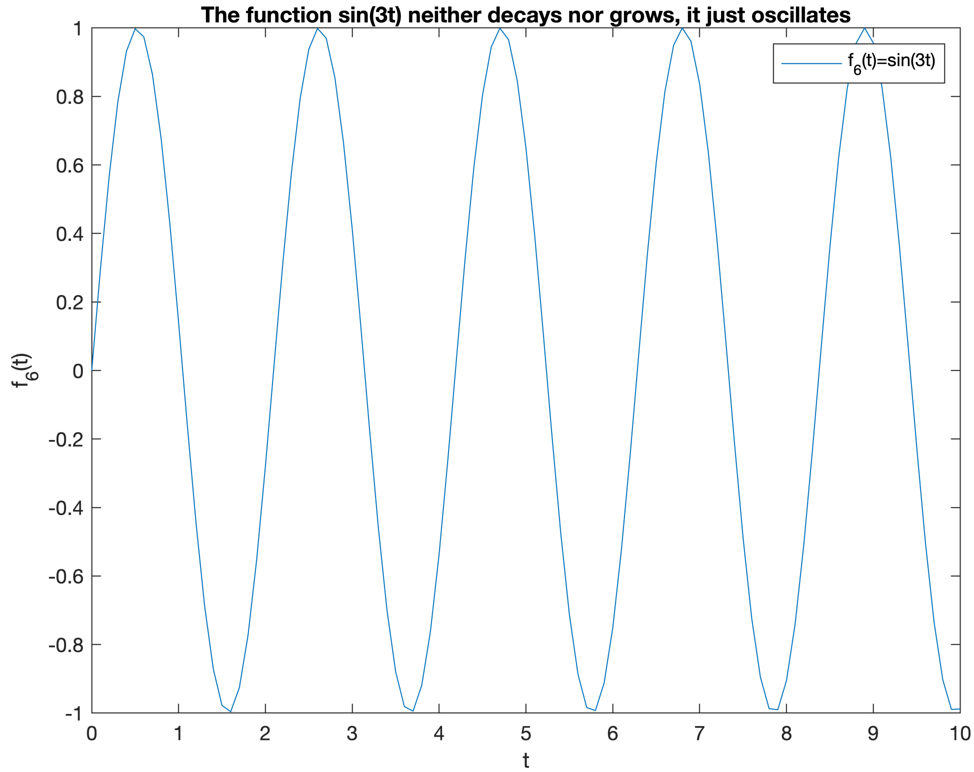

```
% Example 5
figure();
y5 = exp(t).*sin(2*t);
plot(t,y5)

% Annotate the figure
xlabel('t');
ylabel('f_5(t)');
title('The function e^{t}sin(2t) grows while oscillating');
legend('f_5(t)=e^{t}sin(2t)');
```




```
% Example 6
figure();
y6 = sin(3*t);
plot(t,y6)

% Annotate the figure
xlabel('t');
ylabel('f_6(t)');
title('The function sin(3t) neither decays nor grows, it just oscillates');
legend('f_6(t)=sin(3t)');
```




```
% |Remark.| A function which |grows while oscillating| doesn't |grow|,
% because it keeps changing sign, so it neither tends to $+\infty$ nor to
% $-\infty$.
```

Exercise 1

Objective: Use `iode` to solve second-order linear DEs. And classify them.

Details: Consider the ODE:

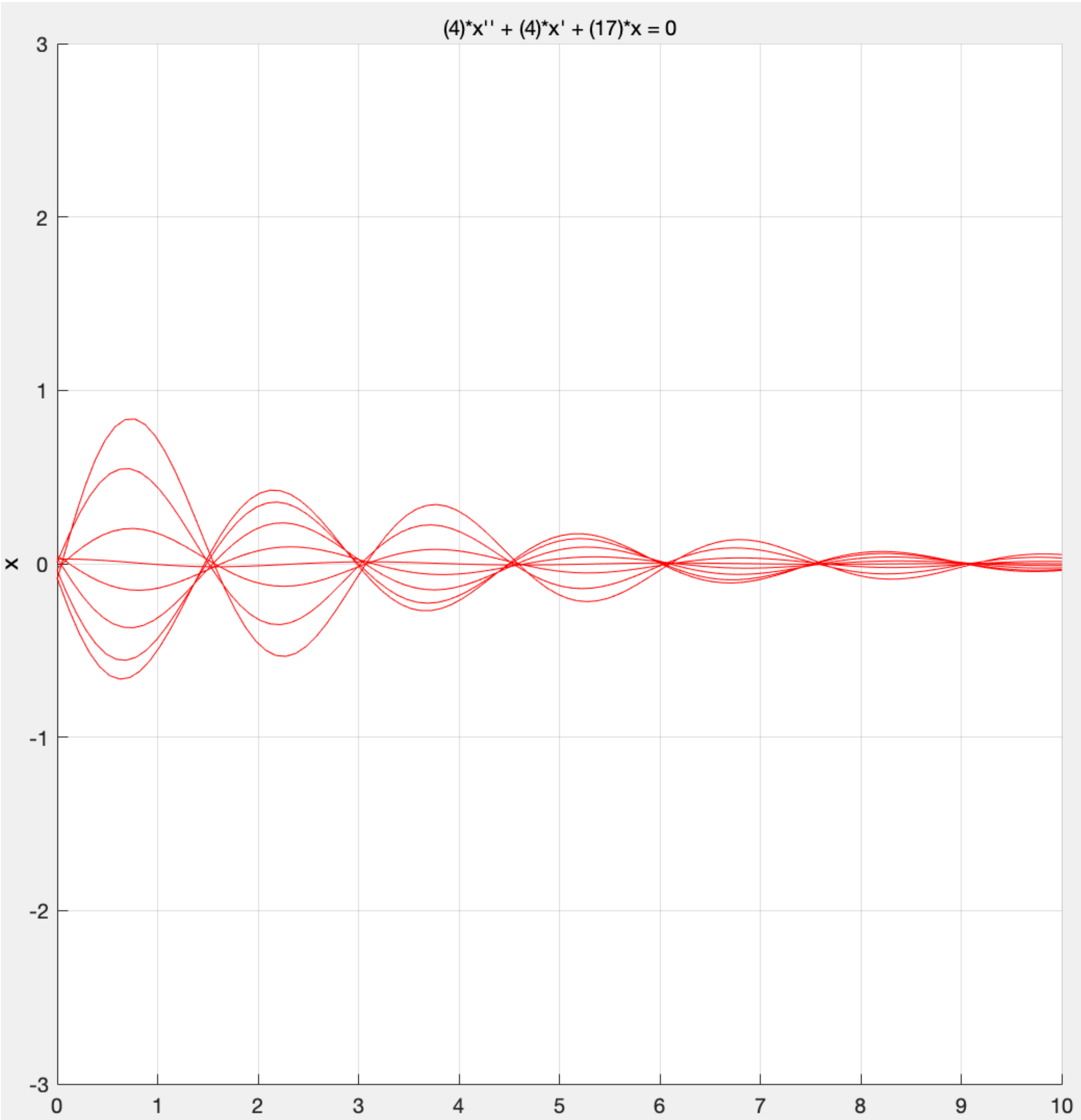
```
4y'' + 4 y' + 17 y = 0
```

(a) Use `iode` to plot six (6) numerical solutions of this equation with "random" initial data (use Method 3 above) and press-and-drag at various initial points, with some of the slopes being positive and some negative)

Use only initial points in the part of the window where `0<t<1` and `-1<x<1` and take all initial slopes between `-3` and +3.

Change the window to `[0,10]` x `[-3,3]`. Attach a cropped screenshot to your answers file.

```
%iode
```



(b) Based on the results of (a), state what percentage of solutions decay, grow, grow while oscillating, or decay while oscillating.

```
%Answer: All (100% of) solutions decay while oscillating out of
% the sample of 6 shown above within the region specified by
% the question.
```

(c) Solve the DE and write the exact solution. Explain why this justifies your answer in (b).

```
%DE: 4y'' + 4y' + 17y = 0

%Characteristic equation: 4λ^2 + 4λ + 17 = 0
%Using the quadratic formula, we get:
% λ = −1/2 + 2i and λ = −1/2 − 2i as our complex eigenvalues.
%Thus, plugging this into the general equation of the solution
%to a second order differential equation, we get:
%y = e^(−t/2)(c1 cos(2t) + c2 sin(2t)) as the exact solution.
```

```
%Observe that as the limit of t approaches infinity,
%the e^(-t/2) term approaches 0. Thus, y approaches 0 as well
%for large values of t. This justifies the answer in (b) because
%regardless of what c1 and c2 are (i.e.: regardless of the initial
%conditions), the solution, y, always tends to 0.
```

Exercise 2

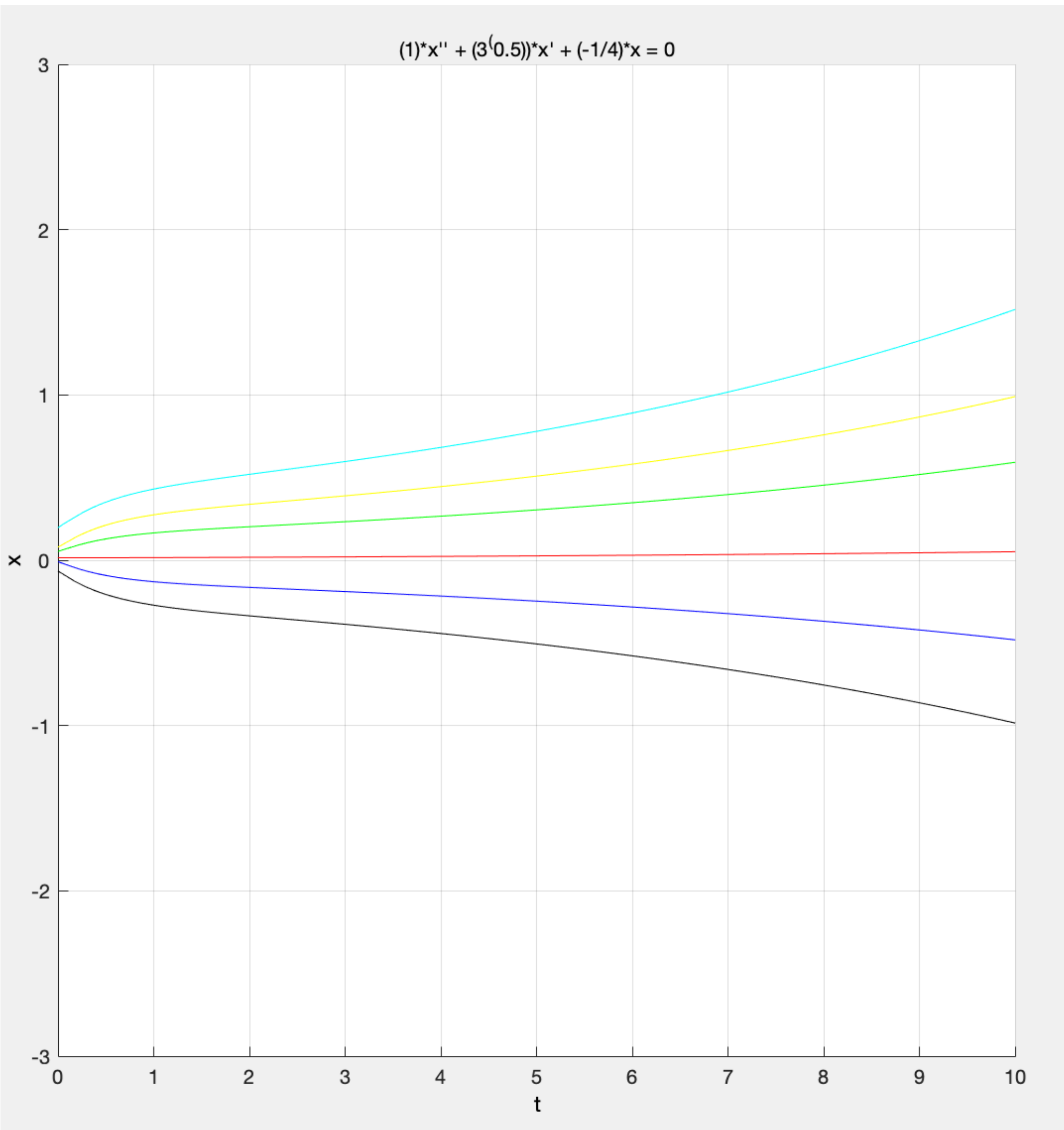
Consider the ODE:

```
y'' + sqrt(3) y' - y/4 = 0
```

Repeat (a), (b), (c) from Exercise 1 with this DE.

(a) Plot using iode.

```
%iode
```



(b) Percentage of solutions that decay, grow, etc.

```
%Answer: All (100% of) solutions grow out of
% the sample of 6 shown above within the region specified by
% the question.
```

(c) Solve the DE and write the exact solution.

```
% DE: y'' + sqrt(3)y' -y/4 = 0

% Rewrite the equation with y = exp(γt) and then form the characteristic
% equation:
% exp(γt)(γ^2 + sqrt(3)γ - 1/4) = 0

% Solving for γ we get γ_1 = (-sqrt(3)+2)/2 and γ_2 = (-sqrt(3)-2)/2

% Since we have two real, distinct roots the general equation of the ODE
% is:
% y = c1exp(γ_1 t) + c2exp(γ_2 t) and so the exact solution is:
% y = c1exp(((-sqrt(3)+2)/2)t) + c2exp(((-sqrt(3)-2)/2)t)

%Observe that as t->∞, the c2exp(((-sqrt(3)-2)/2)t) term in the exact
%solution will tend towards 0 while c1exp(((-sqrt(3)+2)/2)t) will grow
%exponentially. Of course, the intiial conditions will determine which
%effect is greater but for the 6 values selected in iode and for the vast
%majority of cases, the solutions grow.
```

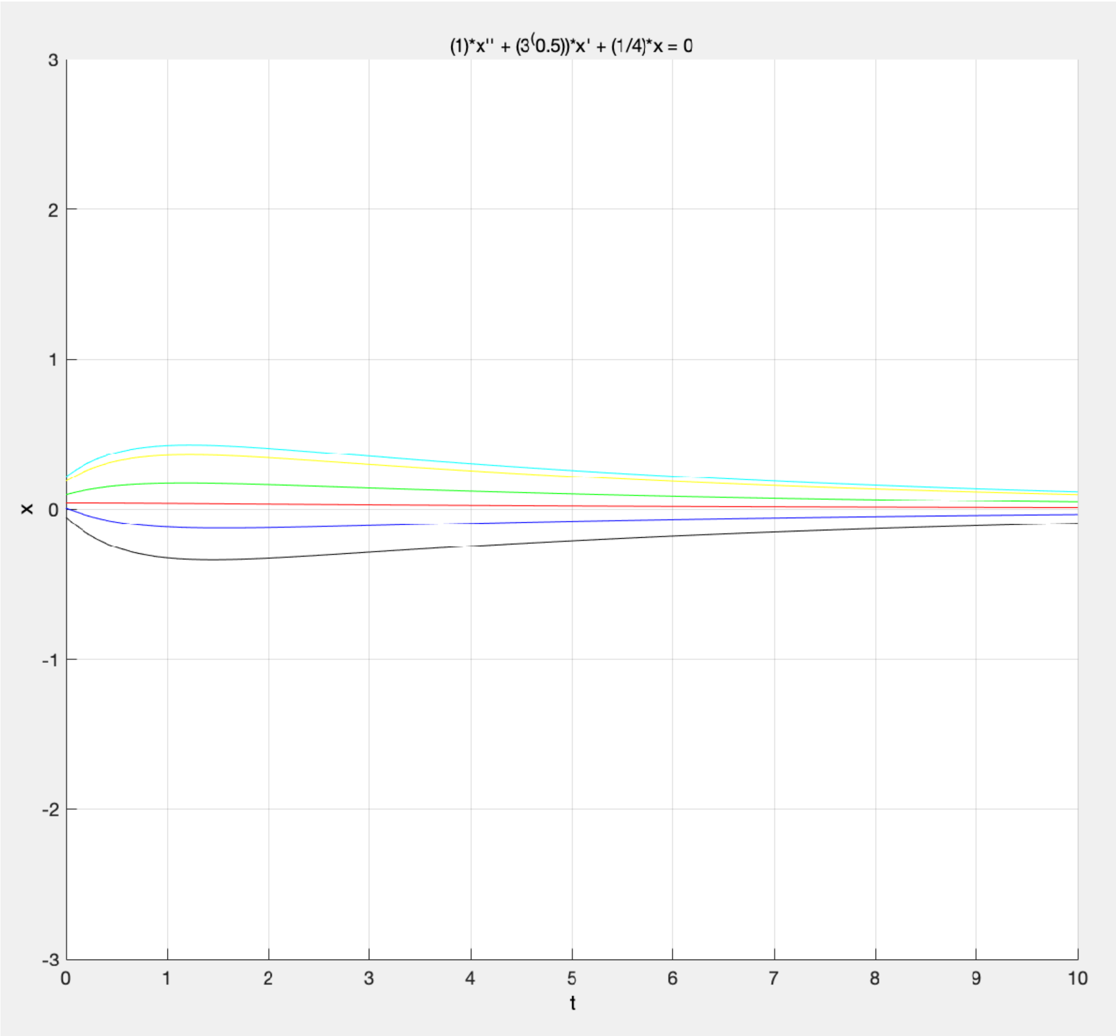
Exercise 3

Consider the ODE:

```
y'' + sqrt(3) y' + y/4 = 0
```

Repeat (a), (b), (c) from Exercise 1 with this DE.

(a) Plot using iode.



(b) Percentage of solutions that decay, grow, etc.

```
%Answer: All (100% of) solutions decay out of
% the sample of 6 shown above within the region specified by
% the question.
```

(c) Solve the DE and write the exact solution.

```
% DE: y'' + sqrt(3)y' + y/4 = 0

% Rewrite the equation with y = exp(γt) and then form the characteristic
% equation:
% exp(γt)(γ^2 + sqrt(3)γ + 1/4) = 0

% Solving for γ we get γ_1 = (−sqrt(3)+sqrt(2))/2 and γ_2 = (−sqrt(3)−sqrt(2))/2

% Since we have two real, distinct roots the general equation of the ODE
% is:
% y = c1exp(γ_1 t) + c2exp(γ_2 t) and so the exact solution is:
% y = c1exp(((−sqrt(3)+sqrt(2))/2)t) + c2exp(((−sqrt(3)−sqrt(2))/2)t)

%Observe that as t->∞, the c1exp(((−sqrt(3)+sqrt(2))/2)t) term and the
```

```
% c2exp(((-sqrt(3)-2)/2)t) term in the exact solution will tend towards 0.
% This is why the solutions shown in (a) all decay.
```

Example

Consider the ODE:

```
y'' + 2y' + 10y = 0
```

The solution is

```
y(t) = e^(-t) ( c1 cos(3t) + c2 sin(3t) )
```

From this, it is easy to see that all solutions decay while oscillating.

Similarly, for the equation

```
y'' - 2y' + 10y = 0
```

The solution is

```
y(t) = e^t ( c3 cos(3t) + c4 sin(3t) )
```

which grows while oscillating.

Exercise 4

Consider the fourth-order ODE:

```
y'''' + 2 y''' + 6 y'' + 2 y' + 5 y = 0
```

(a) Find the general solution for this problem. You can use MATLAB to find the roots of the characteristic equation numerically with `roots`.

```
p = [1 2 6 2 5]; %coefficients of characteristic equation/polynomial
r = roots(p) %solving for roots of polynomaials to get eigenvalues
```

```
r = 4×1 complex
  -1.0000 + 2.0000i
  -1.0000 - 2.0000i
   0.0000 + 1.0000i
   0.0000 - 1.0000i
```

```

%There are 4 roots:
% -> -1 + 2i
% -> -1 - 2i
% -> +i
% -> -i

%Thus, the general solution of this differential equation is:
% y = C1e^(-t)cos(2t) + C2e^(-t)sin(2t) + C3cos(t) + C4sin(t)
```

(b) Predict what percentage of solutions with random initial data will grow, decay, grow while oscillating, and decay while oscillating. Explain.

```
%Answer: All (100% of) solutions are neither growing nor decaying but are oscillating.
% This is because as t approaches infinity, the first two terms,
```

```
% C1e^(-t)cos(2t) and C2e^(-t)sin(2t), will go to 0, however, the last two
% terms in the general solution, C3cos(t) and C4sin(t), will continue
% oscillating and will continue doing so.
```

Exercise 5

Objective: Classify equations given the roots of the characteristic equation.

Details: Your answer can consist of just a short sentence, as `grows` or `decays while oscillating`.

Consider a second-order linear constant coefficient homogeneous DE with `r1` and `r2` as roots of the characteristic equation.

Summarize your conclusions about the behaviour of solutions for randomly chosen initial data when.

(a) `0 < r1 < r2`

(b) `r1 < 0 < r2`

(c) `r1 < r2 < 0`

(d) `r1 = alpha + beta i` and `r2 = alpha - beta i` and `alpha < 0`

(e) `r1 = alpha + beta i` and `r2 = alpha - beta i` and `alpha = 0`

(f) `r1 = alpha + beta i` and `r2 = alpha - beta i` and `alpha > 0`

```
% Assuming non-zero coefficients:
% (a) Grows and not oscillating
% (b) Grows and not oscillating
% (c) Decays and not oscillating
% (d) Decays while oscillating
% (e) Constant but oscillating
% (f) Grows while oscillating
```

Numerical Methods for Second-Order ODEs

One way to create a numerical method for second-order ODEs is to approximate derivatives with finite differences in the same way of the Euler method.

This means that we approximate the first derivative by:

```
y'(t[n]) ~ ( y[n] - y[n-1] ) / h
```

and

```
y''(t[n])  ~ ( y'(t[n+1]) - y'(t[n]) ) / h ~ ( y[n+1] - 2 y[n] + y[n-1]
) / ( h^2 )
```

By writing these approximations into the ODE, we obtain a method to get `y[n+1]` from the previous two steps `y[n]` and `y[n-1]`.

The method for approximating solutions is:

1. Start with `y[0]=y0`

2. Then we need to get `y[1]`, but we can't use the method, because we don't have two iterations `y[0]` and `y[-1]`(!!). So we use Euler to get

```
y[1] = y0 + y1 h
```

y1 is the slope given by the initial condition

3. Use the method described above to get `y[n]` for n=`2,3,...`.

Exercise 6

Objective: Write your own second-order ODE solver.

Details: Consider the second-order ODE

```
y'' + p(t) y' + q(t) y = g(t)
```

Write a second-order ODE solver using the method described above.

This m-file should be a function which accepts as variables (t0,tN,y0,y1,h), where t0 and tN are the start and end points of the interval on which to solve the ODE, y0, y1 are the initial conditions of the ODE, and h is the stepsize. You may also want to pass the functions into the ODE the way `ode45` does (check MATLAB lab 2). Name the function DE2_<UTORid>.m.

Note: you will need to use a loop to do this exercise.

Exercise 7

Objective: Compare your method with `iode`

Details: Use `iode` to plot the solution of the ODE `y'' + exp(-t/5) y' + (1-exp(-t/5)) y = sin(2*t)` with the initial conditions `y(0) = 1, y'(0) = 0`

Use the window to `[0,20]x[-2,2]` Without removing the figure window, plot your solution (in a different colour), which will be plotted in the same graph.

Comment on any major differences, or the lack thereof.

```
p = @(t) exp(-t/5);
q = @(t) 1 - exp(-t/5);
g = @(t) sin(2.*t);
t0 = 0;
tN = 20;
y0 = 1;
y1 = 0;
h = 0.001;
t = t0:h:tN;
y = DE2_khanm382(p,q,g,t0,tN,y0,y1,h);

%%Comments: The plot from iode and the plot from my numerical solver are almost
%%identical. Of course, there are likely minor deviation in the exact
%%values at different points but as can be seen below the general shape of
%%the solution is the same.
```

```
%iode
%To overlay the plot, simply run plot(t,y, '--') in the command window.
```

Note: Iode doesn't have the option to add a legend so I'll explain through words what each curve is below. The blue curve is iode's plot of the solution with the given initial conditions. The dashed red line is the solution from my numerical solver (dashed so that it is easier to differentiate since both solutions almost perfectly overlap).