CTA200H homework assignment for Yuanyuan Wang
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Faraday rotation

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This homework assignment aims at developing code to perform simulated observations of Faraday-rotated polarization from point-like radio sources, as well as some first analysis steps.

In radio astronomy, observations are often described in terms of Stokes parameters. Stokes I describes the total intensity of radiation, the two parameters Q and U the linearly polarized contribution, and the parameter V the circularly polarized contribution. In the following we will assume that $V = 0$. The two Stokes parameters describing linear polarization are often combined into one complex number $P \in \mathbb{C}$ via

$$P = Q + i\,U = |P|\,\mathrm{e}^{2i\,\chi}, \tag{1}$$

so that $Q \in \mathbb{R}$ and $U \in \mathbb{R}$ are the real and imaginary part, respectively, and $\chi \in (-\pi/2, \pi/2]$ is an angle that describes the orientation of the polarization plane. Finally, the polarization fraction is defined as

$$p = \frac{|P|}{I}. \tag{2}$$

1 Noiseless observations of a single source

Suppose we are observing a single source with the intrinsic properties

$$\begin{aligned} Q_0 &= 0.5\,\mathrm{Jy}, \\ U_0 &= 0.0\,\mathrm{Jy}, \\ \phi_0 &= 10.0\,\mathrm{rad\,m^{-2}}, \end{aligned}$$

where ϕ_0 is the Faraday depth of the source. The complex polarization that we see is then

$$P = P_0\,\mathrm{e}^{2i\,\phi_0\,\lambda^2} = (Q_0 + i\,U_0)\,\mathrm{e}^{2i\,\phi_0\,\lambda^2} \tag{3}$$

and depends on the wavelength λ of the observation.

Write a python script that accomplishes the following tasks.

- Create a complex variable that stores the intrinsic polarization, $P_0 = Q_0 + i\,U_0$.
- Set up a numpy array with the frequencies ν of our observations. For this, assume that we have eight observations spaced equidistantly in frequency between 520 MHz and 800 MHz. Set up arrays for the values of λ and λ^2 as well.
- Calculate the observed complex polarization at each of the eight frequencies. The result should be a complex numpy array.
- Calculate the angle χ for the observed polarization. Make sure it is defined in the correct interval. Hint: Do some research online to see if there is a function in numpy that does this for you.
- Create plots of the observed values of Q, U, and χ as a function of ν, λ, and λ^2. Make sure the axis labels make sense and give the correct units. It may be useful to use a denser frequency sampling to get a clearer picture of what is going on. Describe the shape of the curves you find when plotting the three quantities versus λ^2. Can you use any of these plots to verify that the source you have simulated is indeed at a Faraday depth of $\phi_0 = 10.0\,\mathrm{rad\,m^{-2}}$?
- Plot the eight data points in the Q-U-plane, i.e., plot the observed value of U against the observed value of Q for all frequencies. What trajectory is described by the curve as you step through the frequencies? Again, play around with denser frequency sampling to get a clearer picture of what is happening.

2 Noisy observations of a single source

Now suppose that our observations contain some measurement errors. We will assume that the observations are such that they lead to independent Gaussian errors on the observed polarization angle.[1]

- Use your script from the previous problem to simulate an angle χ for each frequency.
- Add noise to these angles, i.e., calculate new angles $\chi^{(\mathrm{obs})} = \chi + \delta\chi$, where $\delta\chi$ is drawn from a zero-mean Gaussian distribution, independently for each frequency, with some fixed variance σ^2.

[1] In reality, this assumption is at best approximately valid.

We will try to use our simulated data to constrain the Faraday depth ϕ_0 and intrinsic polarization angle χ_0 of the source (let's assume that we know the value of $|P_0|$, since it can be read off directly from the observations in this idealized scenario).

- Set up a two-dimensional array of possible tuples of values for ϕ_0 and χ_0.
- For each point in the grid, calculate the predicted values for χ.
- Using these values, calculate the log-likelihood for each of the tuples, i.e.,

$$\frac{1}{2\sigma^2}\sum_{\nu}\left(\chi^{(\mathrm{obs})}-\chi\right)^2+\mathrm{const}, \tag{4}$$

  ignoring the constant.
- Make a two-dimensional color-plot of the likelihood and/or the log-likelihood. Make sure your grid covers the complete area of interest (i.e., the area where the likelihood is significantly different from zero). How does this plot change when you change the value of σ?
- Assuming flat priors on ϕ_0 and χ_0, this likelihood is the same as the joint posterior for these two parameters. Calculate the one-dimensional posterior for ϕ_0 by numerically integrating over the χ_0-dimension. Make a plot of this posterior as well.
- If you make your grid large enough, can you find multiple maxima of the likelihood?

3 Multiple sources

Now suppose that along the same line of sight there is a second source, with the intrinsic properties

$$\begin{aligned}Q_1 &= 0.0\,\mathrm{Jy},\\ U_1 &= 0.25\,\mathrm{Jy}\\ \phi_1 &= -20\,\mathrm{rad\,m^{-2}}.\end{aligned}$$

The polarization we see will be the sum of the Faraday-rotated polarization of the first source and the Faraday-rotated polarization of the second source,

$$P(\lambda) = P_0\,\mathrm{e}^{2i\,\phi_0\,\lambda^2} + P_1\,\mathrm{e}^{2i\,\phi_1\,\lambda^2}. \tag{5}$$

- Add the second source to your simulation code. Create the same plots as in Sect. 1, this time including the results for the first source, for the second source, and for their superposition. Is the superposition qualitatively distinct from the two single-source cases in any of the plots?

With two sources, we can try to constrain a lot more parameters, namely $\chi_0, \chi_1, |P_0|, |P_1|, \phi_0, \phi_1$, or equivalently $Q_0, Q_1, U_0, U_1, \phi_0, \phi_1$.

- Using the simulated superposition of two sources, add Gaussian noise to the observed polarization angles again.
- See if you can find a way to sample the six-dimensional likelihood (or the posterior, assuming flat priors for all parameters). For plotting purposes, six dimensions are too much. Marginalize over four of the six dimensions and create two-dimensional color plots as before. Try a few combinations, e.g., ϕ_0 and ϕ_1, ϕ_0 and χ_0, χ_0 and χ_1.