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In [1]:
```
from __future__ import division

import numpy as np
np.random.seed(0)

import mltools as ml
import sys
sys.path.append('code')

import matplotlib.pyplot as plt   # use matplotlib for plotting with inline plots
plt.set_cmap('jet');
%matplotlib inline
import warnings
warnings.filterwarnings('ignore'); # for deprecated matplotlib functions
```

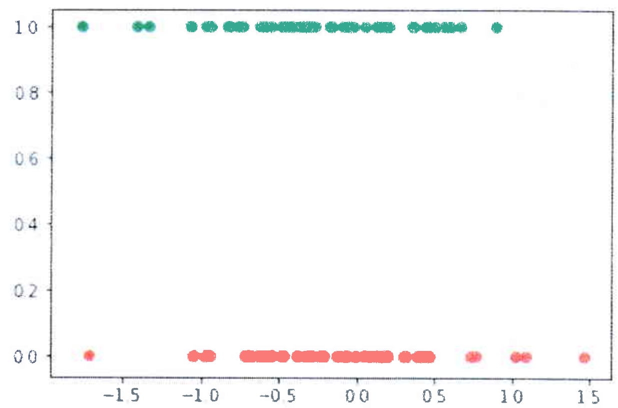
Problem 1: Logistic Regression

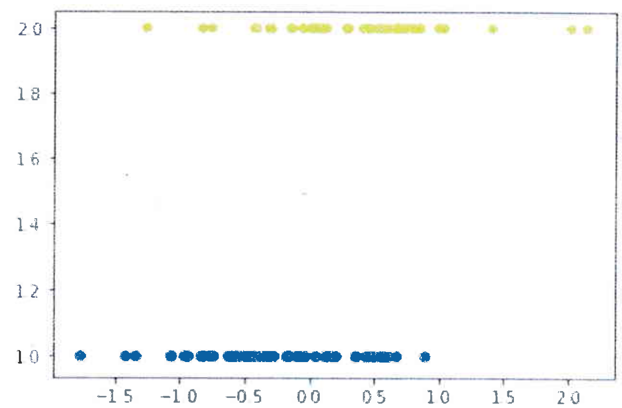
In [2]:
```
iris = np.genfromtxt("data/iris.txt",delimiter=None)
X, Y = iris[:,0:2], iris[:,-1]   # get first two features & target
X,Y  = ml.shuffleData(X,Y)       # reorder randomly rather than by class label
X,_  = ml.transforms.rescale(X)  # rescale to improve numerical stability, speed co
nvergence

XA, YA = X[Y<2,:], Y[Y<2]        # Dataset A: class 0 vs class 1
XB, YB = X[Y>0,:], Y[Y>0]        # Dataset B: class 1 vs class 2
```

P1.1

For each of the two datasets, create a separate scatter plot in which the training data from the two classes is plotted in different colors. Which of the two datasets is linearly separable?

In [3]:
```
XA0, YA0 = X[Y==0,:], Y[Y==0]
XA1, YA1 = X[Y==1,:], Y[Y==1]
#print("shapes of XA0 and YA0:",XA0.shape,YA0.shape)
#print("XA0\n", XA0)
#print("YA0\n", YA0)
XA0_avg = []
for pairing in XA0:
    XA0_avg.append(.5 * (pairing[0]+pairing[1]))
XA1_avg = []
for pairing in XA1:
    XA1_avg.append(.5 * (pairing[0]+pairing[1]))
#print(XA0_avg)
plt.scatter(x=XA0_avg, y=YA0, c="r")
plt.scatter(x=XA1_avg, y=YA1, c="g")
plt.show()


XB0, YB0 = X[Y==1,:], Y[Y==1]
XB1, YB1 = X[Y==2,:], Y[Y==2]
XB0_avg = []
for pairing in XB0:
    XB0_avg.append(.5 * (pairing[0]+pairing[1]))
XB1_avg = []
for pairing in XB1:
    XB1_avg.append(.5 * (pairing[0]+pairing[1]))
plt.scatter(x=XB0_avg, y=YB0, c="b")
plt.scatter(x=XB1_avg, y=YB1, c="y")
plt.show()
```

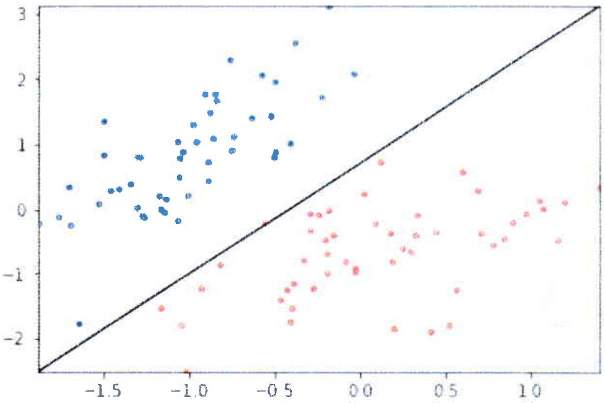




We can clearly see that both datasets are linearly separable. The first plot can be separated with the line y=0.5 and the second with y=1.5.

P1.2

In [11]:
```
def myPlotBoundary(self, X,Y):
    """ Plot the (linear) decision boundary of the classifier, along with data """
    if len(self.theta) != 3: raise ValueError('Data & model must be 2D');
    ax = X.min(0),X.max(0); ax = (ax[0][0],ax[1][0],ax[0][1],ax[1][1]);
    ## TODO: find points on decision boundary defined by theta0 + theta1 X1 + theta
2 X2 == 0
    x1b = np.array([ax[0],ax[1]]);  # at X1 = points in x1b
    x2b = np.array([ax[2],ax[3]]);      # TODO find x2 values as a function of x1's
values
    ## Now plot the data and the resulting boundary:
    A = Y==self.classes[0]; # and plot it:
    plt.plot(X[A,0],X[A,1],'b.',X[~A,0],X[~A,1],'r.',x1b,x2b,'k-'); plt.axis(ax); p
lt.draw();


# Create a shell classifier
class logisticClassify2(ml.classifier):
    classes = []
    theta = np.array( [-1, 0, 0] )    # initialize theta to something
    plotBoundary = myPlotBoundary     #
    predict = None                    # these functions will be implemented later
    train = None

learnerA = logisticClassify2()
learnerA.classes = np.unique(YA)       # store the class values for this problem
learnerA.theta = np.array([2, 6, -1])#NotImplementedError;  # TODO: insert hard-cod
ed values
learnerA.plotBoundary(XA,YA)

plt.show()
```



P1.3

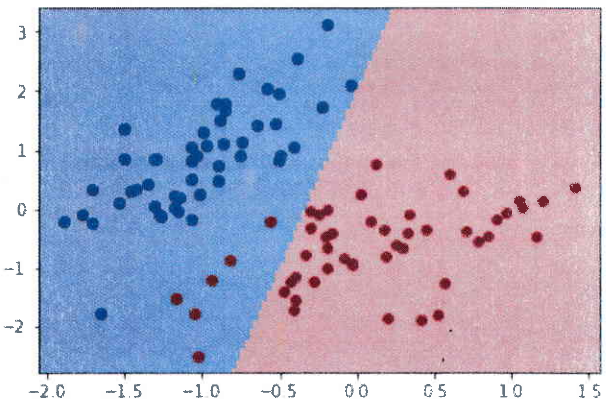
In [36]:
```
# Should go in your logistic2 class:
def myPredict(self,X):
    """ Return the predictied class of each data point in X"""
    #raise NotImplementedError
    Yhat = []
    for i in X:
        val  = self.theta[0] + (self.theta[1] * i[0]) + (self.theta[2] * i[1])
        if val > 0:
            Yhat.append(self.classes[1])
        else:
            Yhat.append(self.classes[0])
    ## TODO: compute linear response r[i] = theta0 + theta1 X[i,1] + theta2 X[i,2]
for each i
    ## TODO: if r[i] > 0, predict class 1:  Yhat[i] = self.classes[1]
    ##       else predict class 0:  Yhat[i] = self.classes[0]
    return np.array(Yhat)


# Update our shell classifier definition
class logisticClassify2(ml.classifier):
    classes = []
    theta = np.array( [-1, 0, 0] )    # initialize theta to something
    plotBoundary = myPlotBoundary     #
    predict = myPredict
    train = None

learnerA = logisticClassify2()
learnerA.classes = np.unique(YA)       # store the class values for this problem
learnerA.theta = np.array([2, 6, -1])  #NotImplementedError;  # TODO: insert hard-c
oded values

print("Error A: ", learnerA.err(XA,YA))

learnerB = logisticClassify2()
learnerB.classes = np.unique(YB)       # store the class values for this problem
learnerB.theta = np.array([2, 6, -1])  #NotImplementedError;  # TODO: insert hard-c
oded values

print("Error B: ", learnerB.err(XB,YB))
```

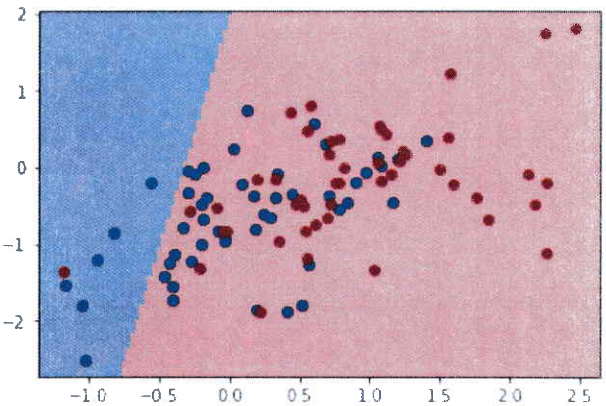
```
Error A:  0.0606060606061
Error B:  0.454545454545
```

P1.4

If predict is implemented, then the inherited 2D visualization function should work; you can verify your decision boundary from P1.2:

In [38]:
```
ml.plotClassify2D(learnerA,XA,YA)
plt.show()
ml.plotClassify2D(learnerB,XB,YB)
plt.show()
```





Here is an example of latex equations that may be useful for expressing the gradient:

P1.5 Gradient of NLL

Our negative log-likelihood loss is:

$$J_j(\theta) = -\begin{cases} \log(\sigma(x^{(i)} \cdot \theta)) & \text{if } y^{(i)} = 1 \\ \log(1 - \sigma(x^{(i)} \cdot \theta)) & \text{if } y^{(i)} = 0 \end{cases}$$

Thus, its gradient is:

$$\nabla J_j(\theta) = (something)$$

For my solution to 1.5, see the handwritten solutions at the end of the document. Problems 2 and 3 are found there as well. Thank you.

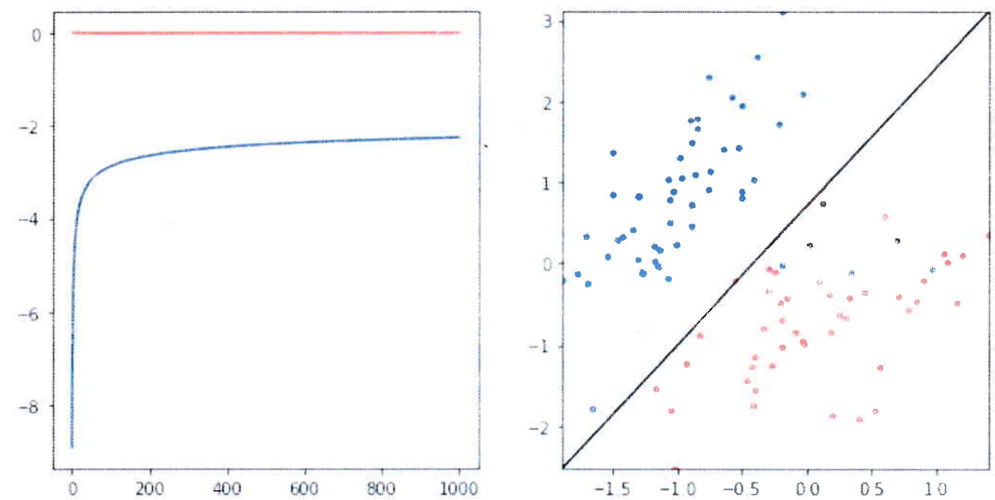
P1.6 and 1.7

Now define the train function and complete its missing code.

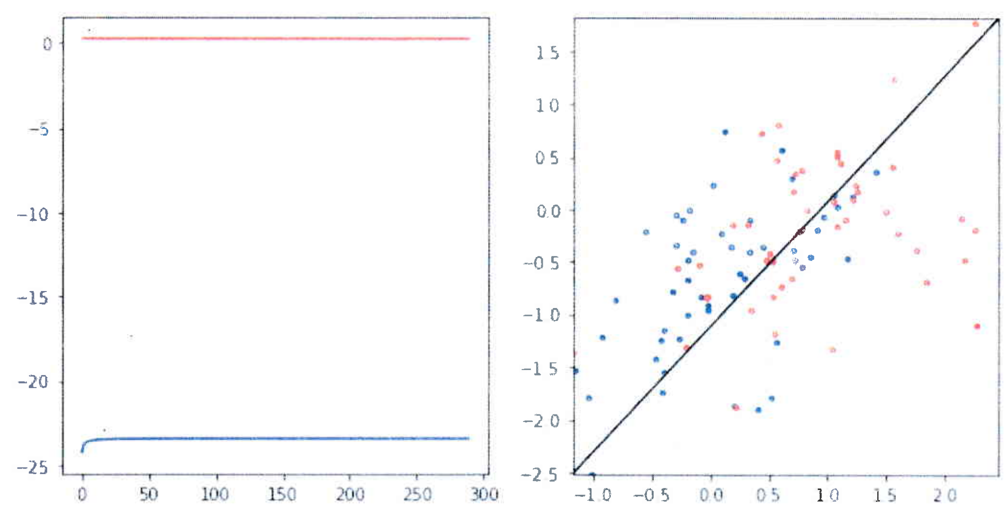
In [61]:
```
def sigmoid(z):
    """Sigmoid function. """
    return 1. / (1 + np.exp(-z))
```

In [84]:
```
def myTrain(self, X, Y, initStep=1.0, stopTol=1e-4, stopEpochs=5000, plot=None):
    """ Train the logistic regression using stochastic gradient descent """
    from IPython import display
    M,N = X.shape;                     # initialize the model if necessary:
    self.classes = np.unique(Y);       # Y may have two classes, any values
    XX = np.hstack((np.ones((M,1)),X)) # XX is X, but with an extra column of ones
    YY = ml.toIndex(Y,self.classes);   # YY is Y, but with canonical values 0 or 1
    if len(self.theta)!=N+1: self.theta=np.random.rand(N+1);
    # init loop variables:
    epoch=0; done=False; Jnll=[]; J01=[];
    while not done:
        stepsize, epoch = initStep*2.0/(2.0+epoch), epoch+1; # update stepsize
        # Do an SGD pass through the entire data set:
        for i in np.random.permutation(M):
            #print("[",self.theta[0], self.theta[1], self.theta[2],"]")
            ri    = np.dot(XX[i], self.theta);     # TODO: compute linear response
r(x)
            gradi = (sigmoid(ri) - YY[i])*XX[i];     # TODO: compute gradient of NL
L loss
            self.theta -= stepsize * gradi;  # take a gradient step

        J01.append( self.err(X,Y) )  # evaluate the current error rate

        ## TODO: compute surrogate loss (logistic negative log-likelihood)
        #Jsur = sum_i [ (log si) if yi==1 else (log(1-si)) ]
        Jsur = 0
        for ind in range(XX.shape[0]):
            if YY[ind] == 0:
                Jsur += np.log10([1-sigmoid(np.dot(XX[ind], self.theta))])[0]
            else:
                Jsur += np.log10([sigmoid(np.dot(XX[ind], self.theta))])[0]
        Jnll.append(Jsur) # TODO evaluate the current NLL loss
        display.clear_output(wait=True);
        plt.subplot(1,2,1); plt.cla(); plt.plot(Jnll,'b-',J01,'r-'); # plot losses
        if N==2: plt.subplot(1,2,2); plt.cla(); self.plotBoundary(X,Y); # & predict
or if 2D
        plt.pause(.01);                    # let OS draw the plot

        ## For debugging: you may want to print current parameters & losses
        # print self.theta, ' => ', Jnll[-1], ' / ', J01[-1]
        # raw_input()   # pause for keystroke

        # TODO check stopping criteria: exit if exceeded # of epochs ( > stopEpoch
s)
        if epoch > stopEpochs or (epoch > 1 and(Jnll[len(Jnll)-1]-Jnll[len(Jnll)-
2]) < stopTol):
            done = True;   # or if Jnll not changing between epochs ( < stopTol )
```

In [86]:
```
# Update our shell classifier definition
class logisticClassify2(ml.classifier):
    classes = []
    theta = np.array( [-1, 0, 0] )    # initialize theta to something
    plotBoundary = myPlotBoundary    #
    predict = myPredict              # Now all parts are implemented
    train = myTrain

plt.rcParams['figure.figsize'] = (10,5)    # make a wide figure, for two subplots

learnerA = logisticClassify2()
learnerA.theta = np.array([0.,0.,0.]);
learnerA.train(XA,YA,initStep=1e-1,stopEpochs=1000,stopTol=1e-5);
```

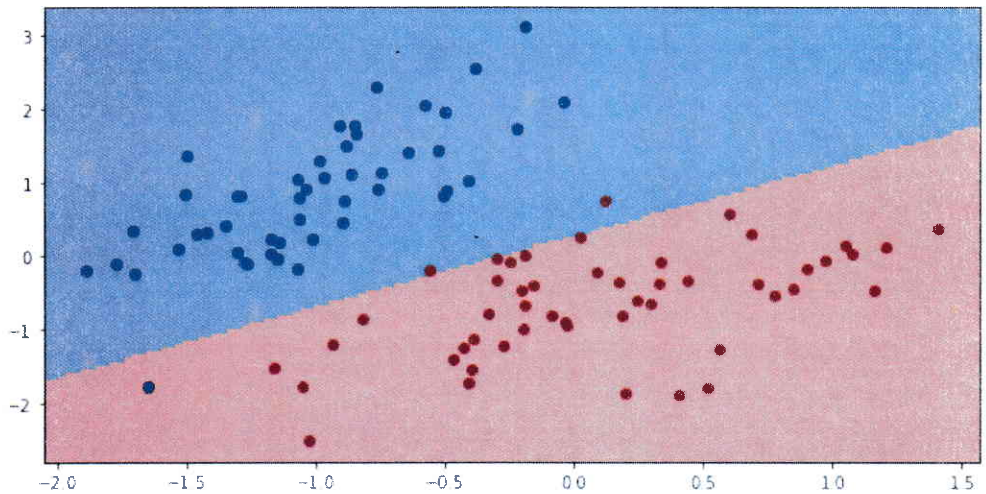


In [87]:
```
plt.rcParams['figure.figsize'] = (10,5)    # make a wide figure, for two subplots

learnerB = logisticClassify2()
learnerB.theta = np.array([0.,0.,0.]);
learnerB.train(XB,YB,initStep=1e-1,stopEpochs=1000,stopTol=1e-5);
```

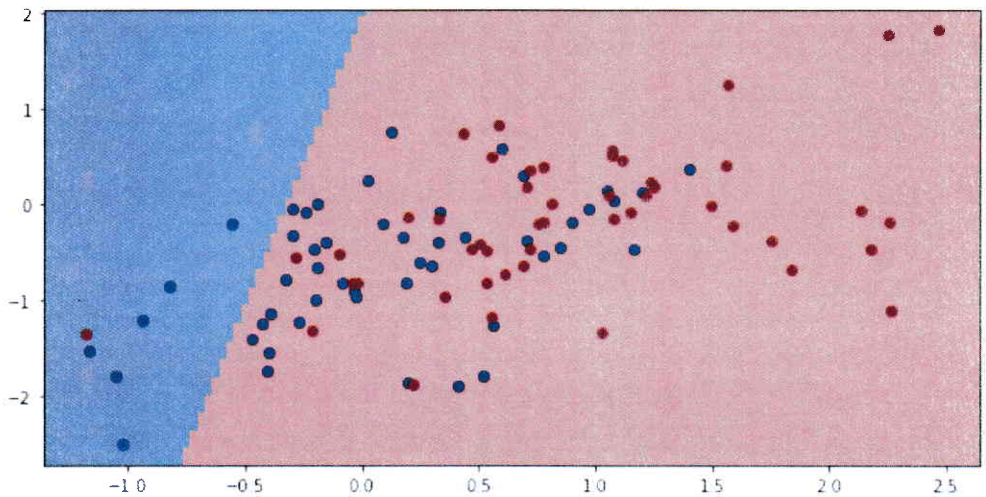


In [65]:
```
ml.plotClassify2D(learnerA,XA,YA)
print("Training error rate A: ",learnerA.err(XA,YA))

plt.show()
ml.plotClassify2D(learnerB,XB,YB)
print("Training error rate B: ",learnerB.err(XB,YB))

plt.show()
```

```
Training error rate A:  0.030303030303
```




```
Training error rate B:  0.454545454545
```



1.5) Show that, for $J_i(\theta) = -y_i \log \sigma(x_i \cdot \theta) - (1-y_i)\log(1-\sigma(x_i \cdot \theta))$,

and $\sigma(r) = (1+\exp(-r))^{-1}$ (and, subsequently, $\sigma'(r) = \sigma(r)(1-\sigma(r))$),

$\frac{dJ(\theta)}{d\theta} = \nabla J_i = (\sigma(x_i \cdot \theta) - y_i)x_i$. Assume log and ln are synonymous.

$$\frac{dJ}{d\theta} = \frac{d}{d\theta}\left(-y_i \log \sigma(x_i \cdot \theta)\right) - \frac{d}{d\theta}\left((1-y_i)\log(1-\sigma(x_i \cdot \theta))\right)$$

$$= -y_i \frac{\frac{d}{d\theta}(\sigma(x_i \cdot \theta))}{\sigma(x_i \cdot \theta)} - (1-y_i)\frac{d}{d\theta}\left(\log(1-\sigma(x_i \cdot \theta))\right)$$

$$= -x_i y_i (1-\sigma(x_i \cdot \theta)) - (1-y_i)\frac{\frac{d}{d\theta}(1-\sigma(x_i \cdot \theta))}{1-\sigma(x_i \cdot \theta)}$$

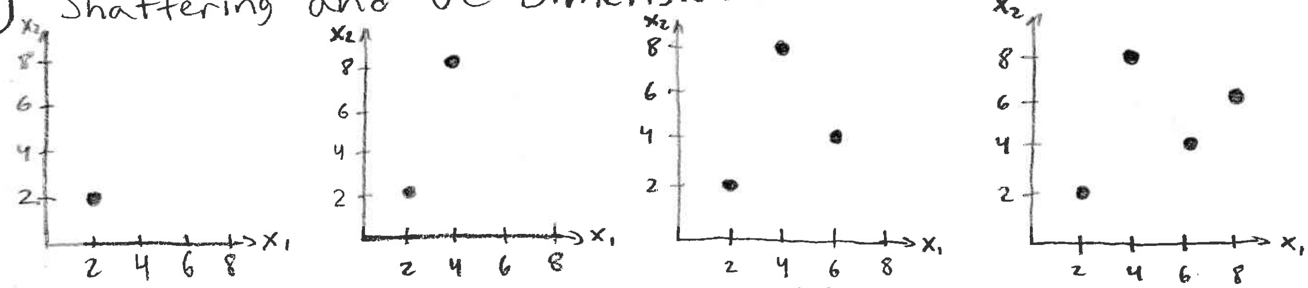
$$= -x_i y_i (1-\sigma(x_i \cdot \theta)) - (1-y_i)(\cancel{1} + \sigma(x_i \cdot \theta))(-x_i)$$

$$= -x_i y_i + x_i y_i \sigma(x_i \cdot \theta) - (y_i - 1)(x_i \sigma(x_i \cdot \theta))$$

$$= -x_i y_i + \cancel{x_i y_i \sigma(x_i \cdot \theta)} - \cancel{x_i y_i \sigma(x_i \cdot \theta)} + x_i \sigma(x_i \cdot \theta)$$

$$\nabla J_i = x_i \sigma(x_i \cdot \theta) - x_i y_i = \boxed{x_i(\sigma(x_i \cdot \theta) - y_i)}$$

2) Shattering and VC Dimension:



Datasets: (a) (b) (c) (d)

$T[z] = +1$ for $z \geq 0$, $T[z] = -1$ for $z < 0$

2.1) $z = a + bx_1$. There is no way this learner could shatter (c) or (d); say that, in (c), $x_1 = 4$ was classified as positive while the others were negative. The learner cannot generate a negative class, then a positive one, and then negative again because it can only make, at most, one decision boundary. Understanding this, we need only test if it can shatter set (b), because it is obvious that the boundary can be on either side of 1 datapoint, thus shattering (a).

$x_1 = 2$,	$x_1 = 4$	
+	+	✓ let a and b both be ≥ 0
+	−	✓ let $2 \leq a \leq 3$ and $b = -1$
−	+	✓ let $a < -2$, $b = 1$
−	−	✓ let a and b both be < 0

$\Rightarrow$ $\boxed{\text{sets (a) and (b) can both be shattered.}}$

$\Rightarrow$ $\boxed{\text{VC dim} = 2}$

2.2) $z = abx_1 + \frac{c}{a}x_2$

(2,2),	(4,8),	(6,4)	
+	+	+	✓ $a, b, c > 0$
+	+	−	X $\Rightarrow$ cannot shatter!

$6ab + 4\frac{c}{a}$, let $a = 1$

$6b + 4c$ $2b + 2c$, $4b + 8c$

$|-4c| > |6b|$ $|-2c| > |2b|$ $|-8c| > |4b|$

$c < -\frac{3}{2}b$ $c < -b$ $c < -\frac{1}{2}b$

(2,2),	(4,8)	
+	+	✓ $a, b, c = 1$
+	−	✓ $a, b = 1$; $c = -1$
−	+	✓ $a, c = 1$; $b = -2$
−	−	✓ $a = 1$; $b, c = -1$

$\boxed{\text{VC Dim} = 2}$

$\Rightarrow$ $\boxed{\text{sets (a) and (b) can be shattered}}$

$2ab + 2\frac{c}{a}$, $4ab + 8\frac{c}{a}$

$ab + \frac{c}{a} < 0$, $ab + 2\frac{c}{a} \geq 0$

$\frac{c}{a} > 0$

2.3) $z = (x_1 - a)^2 + (x_2 - b)^2 + c$

(2,2),	(4,8),	(6,4)	
+	+	+	✓ $a, b, c = 0$
+	+	−	✓ $a = 7, b = 8, c = -6$
+	−	+	✓ $a = 4, b = 8, c = -1$
+	−	−	✓ $a = 7, b = 8, c = -10$
−	+	+	✓ $a = 2, b = 2, c = -1$
−	+	−	✓ $a = 4, b = 4, c = -9$
−	−	+	✓ $a = 2, b = 6, c = -17$
−	−	−	✓ $a = 2, b = 8, c = -37$

sets (a), (b), (c) shattered!
(next page, test for (d))

$(2-a)^2 + (2-b)^2 + c$, $(4-a)^2 + (8-b)^2 + c$, $(6-a)^2 + (4-b)^2 + c$

$a = 2$, $b = 8$

$36 + c$, $4 + c$, $16 + 16 + c$

$a = 2$, $b = 2$

c, $4 + 36 + c$, $16 + 4$

$a = 3$, $b = 6$

$1 + 16 + c$, $1 + 4 + c$, $9 + 4 + c$

$a = 2$, $b = 6$

$16 + c$, $4 + 4 + c$, $16 + 4 + c$

$c = -17$

2.3 cont'd)

$(2,2),\ (4,8),\ (6,4),\ (8,6)$ $\qquad (2-a)^2+(2-b)^2+c,\ (4-a)^2+(8-b)^2+c,\ (6-a)^2+(4-b)^2+c,\ (8-a)^2+(6-b)^2+c$

+	+	+	+	✓	$a=0, b=0, c=0$
+	+	+	−	✓	$a=8, b=6, c=-1$
+	+	−	+	✓	$a=6, b=4, c=-1$
+	+	−	−	✓	$a=7, b=5, c=-3$
+	−	+	+	✓	$a=4, b=8, c=-1$
+	−	+	−	✓	$a=6, b=7, c=-6$
+	−	−	+	✓	$a=5, b=6, c=-6$
+	−	−	−	✓	$a=7, b=5, c=-10$
−	+	+	+	✓	$a=2, b=2, c=-1$
−	+	+	−	✗	cannot make only $(2-a)^2+(2-b^2)+c$ and $(8-a)^2+(6-b)^2+c$ negative. (coordinates are too far apart)
−	+	−	+	✓	$a=2, b=2, c=-21$
−	+	−	−		
−	−	+	+		
−	−	+	−		
−	−	−	+	✓	$a=2, b=2, c=-41$
−	−	−	−	✓	$a=7, b=5, c=-35$

$\Rightarrow$ Only sets (a), (b), and (c) are shattered (VC dim = 3)

2.4) $(2,2),\ (4,8),\ (6,4),\ (8,6)$ $\qquad z_1 = a + bx_1 + cx_2,\quad z_2 = d + bx_1 + cx_2$

$a+2b+2c$	$a+4b+8c$	$a+6b+4c$	$a+8b+6c$
$d+2b+2c$	$d+4b+8c$	$d+6b+4c$	$d+8b+6c$

+	+	+	+	✓ $a=b=c=d=0$
+	+	+	−	
+	+	−	+	✓ $a=-5, b=1, c=-\frac{1}{2}, d=-2$
+	+	−	−	✓ $a=-3, b=1, c=-1, d=-2$
+	−	+	+	✓ $a=3, b=\frac{1}{2}, c=-1, d=7$
+	−	+	−	✓ $a=1, b=\frac{1}{2}, c=-1, d=7$
+	−	−	+	✗ Same problem as 2.3
+	−	−	−	✓ $a=d=5, b=c=-1$
−	+	+	+	✓ $a=b=c=1; d=-5$
−	+	+	−	
−	+	−	+	✓ $a=0, b=-1, c=\frac{1}{2}, d=4$
−	+	−	−	✓ $a=0, b=-1, c=\frac{1}{2}, d=5$
−	−	+	+	✓ $a=b=c=1; d=-11$
−	−	+	−	
−	−	−	+	✓ $a=b=c=1; d=-13$
−	−	−	−	✓ $a=b=c=1; d=-15$

sets (a), (b), (c) can be shattered $\Rightarrow$ VC dim = 3

Assume that if $T[z_1] = T[z_2]$, $\hat{y} = +1$. $\hat{y} = -1$ otherwise.

3) Statement of Collaboration.

I only consulted Piazza and the discussion Jupyter Notebooks. No other sources were used.