Big O: A Review

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Big O: Definition

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  - E.g., $n^2 + 42n + 7 = O(n^2)$ means:
    - The function $f(n) = n^2 + 42n + 7$ is in the set $O(n^2)$

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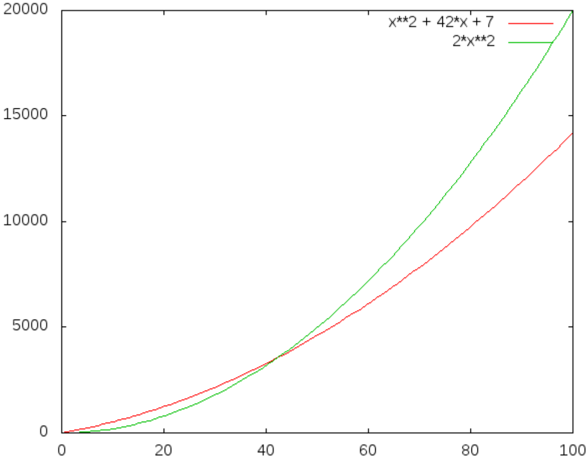

$$n^2 + 42n + 7 \leq 2n^2 \text{ for all } n \geq 50$$

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- So, $n^2 + 42n + 7 \le 50n^2$ for all $n \ge 1$
- $n^2 + 42n^2 + 7n^2 = O(n^2)$ [$c = 50,\ n_0 = 1$]

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- $5n\log_2 n + 8n - 200 \le 13n\log_2 n$ for all $n \ge 2$
- $5n\log_2 n + 8n - 200 = O(n\log_2 n)$ $[\ c = 13,\ n_0 = 2\ ]$

Some common relations

- $O(n^{c_1}) \subset O(n^{c_2})$ for any $c_1 < c_2$
- For any constants $a, b, c > 0$,

$$O(a) \subset O(\log n) \subset O(n^b) \subset O(c^n)$$

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- Examples: $O(n^{1.5}) \subseteq O(n^{1.5} \log n)$

An indulgence

- In this course, we have seen expressions like $O(n - i)$
  - Two argument function $g(n, i) = n - i$
  - *For the purposes of this course*, we will take $O(g(n, i))$ to be

$$O(g(n,i)) = \{f(n,i) : \begin{array}{l}\text{there exists positive constants } c \text{ and } n_0 \\ \text{such that } f(n,i) \leq cg(n,i) \text{ for all } n \geq n_0 \\ \text{and all valid arguments } i\}\end{array}$$

- For example (Lists) valid values of i are $\{0, \ldots, n-1\}$ or (sometimes) $\{0, \ldots, n\}$

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- Easier just to say $O(n)$ (constant-time) operations