Predator and Prey Model Writeup

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1 The Predator-Prey Model

The Predator-Prey model is a mathematical model used to describe the dynamics of two species in an ecosystem. The model consists of two first-order non-linear differential equations that describe the interaction between the two species. The model is also known as the Lotka-Volterra model and looks like the following:

$$\frac{dx}{dt} = ax - bxy = x(a - by) \tag{1}$$
$$\frac{dy}{dt} = -cy + exy = y(-c + ex) \tag{2}$$

where a, b, c, and e are ≥ 0. The parameters have the following meanings:

- x: the population of the prey species (e.g. rabbits)
- y: the population of the predator species (e.g. foxes)
- a: the natural growth rate of prey in the absence of predation
- b: the death rate per encounter of prey due to predation
- c: the natural death rate of the predator in the absence of prey
- e: the efficiency of turning predated prey into predator offspring

2 Dimensionless Equations

To simplify the stability analysis of the model, we can make the equations dimensionless. We achieve this by introducing the following dimensionless variables:

$$u(\tau) = \frac{ex(t)}{c}$$
$$v(\tau) = -\frac{by(t)}{a}$$
$$\tau = at$$

Substituting these variables into the original equations, we get the following dimensionless equations:

$$\frac{du}{d\tau} = u(1 - v) \tag{3}$$
$$\frac{dv}{d\tau} = \alpha v(u - 1) \tag{4}$$

Where $\alpha = \dfrac{c}{a}$.

3 Equilibrium Points

The equilibrium points of the system are found by setting the right hand side of the equations to zero. This gives us the following equilibrium points, $(0,0)$ and $(1,1)$.
The Jacobian matrix of the system is given by:

$$J(u,v) = \begin{bmatrix} 1-v & -u \\ \alpha v & \alpha(u-1) \end{bmatrix}$$

The eigenvalues of the Jacobian matrix at the equilibrium points are

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix}$$

$$J(1,1) = \begin{bmatrix} 0 & -1 \\ \alpha & 0 \end{bmatrix}$$

The eigenvalues of $J(0,0)$ are 1 and $-\alpha$, and the eigenvalues of $J(1,1)$ are $\pm i\sqrt{\alpha}$.

4 Stability Analysis

Since the eigenvalues of $J(0,0)$ are both real and that $\lambda_1\lambda_2 < 0$, the equilibrium point $(0,0)$ is a saddle point. Since both of the eigenvalues of $J(1,1)$ are complex, the equilibrium point $(1,1)$ is a center, or stable/unstable spiral.

- The equilibrium point $(0,0)$ represents when both species are extinct.
- The equilibrium point $(1,1)$ represents when both species coexist in time.
- A center can lead to a regular, cyclic behavior in both populations that oscillate around the equilibrium point in a stable manner.
- A stable spiral indicates that there can be a perturbation in one of the populations, but they will eventually return to the equilibrium state.
- An unstable spiral suggests that the populations can experience population collapses or rapid growth due to a small perturbation.

Consider the following example where we have the following values for the parameters: $a = 0.1$, $b = 0.02$, $c = 0.2$, and $e = 0.01$. Then the dimensionless equations are as follows:

$$\frac{du}{d\tau} = u(1-v)$$
$$\frac{dv}{d\tau} = 2v(u-1)$$

This has the following Jacobian matrices at the equilibrium points:

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad J(1,1) = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

The eigenvalues of $J(0,0)$ are 1 and -2, and has the following phase plane analysis:

- Trace $= 1 - 2 = -1$
- Determinant $= 1 \cdot -2 = -2$
- Discriminant $= (-1)^2 - 4(-2) = 1 + 8 = 9$

Since the discriminant is positive and the determinant is negative, the equilibrium point $(0,0)$ is a saddle point.

The Jacobian matrix at $(1,1)$ has eigenvalues $\pm i\sqrt{2}$, and the phase plane analysis is as follows:

- Trace $= 0$
- Determinant $= 0 - (-2) = 2$
- Discriminant $= 0^2 - 8 = -8$

Since the discriminant is negative and the trace is zero, the equilibrium point $(1,1)$ is a center. This is also justified by the phase-plane diagram below:

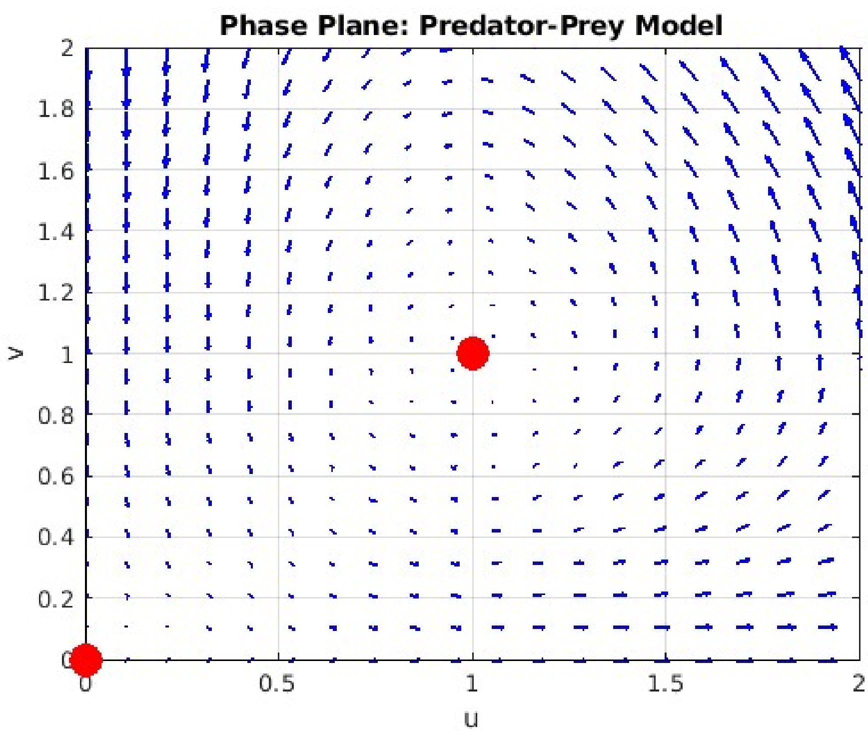

Figure 1: Phase Plane Diagram with $\alpha = 2$