# Heston-Nadi GARCH

$$x_k = \ln(S_k/S_{k-1})$$

$$\mathbb{P}: \begin{cases} x_k = (r+\lambda h_k) + \sqrt{h_k}\,\varepsilon_k \\ h_{k+1} = \omega + \beta h_k + \alpha(\varepsilon_k - \gamma\sqrt{h_k})^2 \end{cases}$$

$\varepsilon_k \overset{\mathbb{P}}{\sim} N(0,1)$

$r$: risk free interest rate

---

$$\mathbb{Q}: \begin{cases} x_k = (r - \frac{h_k^*}{2}) + \sqrt{h_k^*}\,\varepsilon_k^* \\ h_{k+1}^* = \omega^* + \beta h_k^* + \alpha^*(\varepsilon_k^* - \rho^*\sqrt{h_k^*})^2 \end{cases}$$

$\varepsilon_k^* \overset{\mathbb{Q}}{\sim} N(0,1)$

$$\omega^* = \frac{\omega}{1-2\alpha\eta_2} \qquad \alpha^* = \frac{\alpha}{(1-2\alpha\eta_2)^2} \qquad \lambda^* = \lambda(1-2\alpha\eta_2)$$

$$\gamma^* = \gamma(1-2\alpha\eta_2) \qquad \rho^* = \lambda^* + \gamma^* + \frac{1}{2}$$

(Let $\eta_2 = 0$)

---

log-likelihood under $\mathbb{P}$

$\{y_i\}$ — daily log return of stock price
$i = 1, 2, \cdots, N$

Let $h_0$ = variance of $\{y_i\}$

$$\varepsilon_i = [y_i - (r+\lambda h_i)]/\sqrt{h_i}$$

$$h_{i+1} = \omega + \beta h_i + \alpha(\varepsilon_i - \gamma\sqrt{h_i})^2$$

$$= \omega + \beta h_i + \alpha\left[\frac{y_i - (r-\lambda h_i)}{\sqrt{h_i}} - \gamma\sqrt{h_i}\right]^2$$

Thus, we obtain $\{h_i\}$ from $\{y_i\}$.

The log-likelihood can be computed as

$$Y_1 = -\frac{1}{2}\sum_{i=1}^{N}\left\{\ln(h_i) + (y_i - (r+\lambda h_i))^2/h_i\right\}$$

---

log-likelihood under $\mathbb{Q}$ (American option)

Assuming $\underset{\substack{\uparrow \\ \text{implied volatility} \\ \text{from market price}}}{\sigma_{i,t}} = \underset{\substack{\uparrow \\ \text{implied volatility} \\ \text{from HN-GARCH model}}}{\underbrace{\sigma_{i,t}(C_{i,t}(h_t(\xi^*))}} + \varepsilon_{i,t}$

$\varepsilon_{i,t} \sim N(0, \sigma_\varepsilon^2)$.

Given parameters $\omega, \beta, \alpha, \gamma, \lambda$, Compute the Corresponding parameters $\omega^*, \alpha^*, \lambda^*, \gamma^*, \xi^*$ and price American option under $Q$.

Then, Compute the Corresponding implied volatility under BS framework by the willow tree structure.

Consider all strike prices and maturities at each day, the log-likelihood under $Q$ for the implied volatilities is

$$Y_2 = -\frac{1}{2}\sum_{i=1}^{M}\left[2\ln\sigma_\varepsilon + (\sigma_i - \sigma_{imp,i})^2/\sigma_\varepsilon^2\right]$$

implied volatility from models

implied volatility on market price

M is the total number of options

---

Combine the log-likelihood P and Q

$$Y_{joint} = \frac{(N+M)}{2N} Y_1 + \frac{N+M}{2M} Y_2$$

Then use fmincon to solve for $\omega, \beta, \alpha, \gamma, \lambda$ and $\sigma_\varepsilon$ with the objective function $(-Y_{joint})$ with the nonnegativity Constraint on all six parameters.