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# Two-factor Heston model equipped with regime-switching: American option pricing and model calibration by Levenberg–Marquardt optimization algorithm

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## Abstract

In this paper, we consider the pricing of American options under a regime-switching double Heston model, such that the interest rate and mean-reversion level parameters in both stochastic volatility models shift in various states. We develop a semi-analytical formula for double Heston partial differential equation by using the equivalent European put option price and standard portfolio-consumption model. Then through the moment-generating function of this particular model, the American put option price is evaluated. We employ the Levenberg–Marquardt optimization method to calibrate the regime-switching double Heston model. Numerical experiments have also been performed to demonstrate the accuracy of the proposed formula and the performance of regime change mechanism on option pricing. Ultimately, through an experimental application, we indicate the proposed model is premier to the double Heston model, which illustrates the importance of considering regime-switching factor.


*Keywords:* American option; Double Heston model; Regime-switching; Calibration; Levenberg–Marquardt method

## 1. Introduction

American option is one of the most important derivatives that are traded by financial market participants in large volumes on trading days, so it is significant to obtain a closed form solution of this type securities. Since the holder of American options has the right to exercise it at any time before the expiration date, its pricing remains one of the most challenging problems in financial mathematics. It has long been known that the Black–Scholes (BS) model leads to the evaluation of a systematically biased option. This has led to several generalizations of the BS model based on stochastic volatilities [58], classic [31] and 3/2 model of Heston [32].

Due to the evidence in [59] and [26], when asset dynamics are described by the stochastic volatility model based on the one-factor, the constructions of implied volatility are not maintained over time horizons. To dominate

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this problem, more advanced models were offered by academics and researchers. Gatheral et al. [27] considered fractional Brownian motion instead to the Brownian motion in the variance process and Mehrdoust et al. [44] investigated that the Brownian motions of volatility and stock processes can be replaced by mixed fractional Brownian motions. Da Fonseca et al. [15] showed that the option price obtained by the Wishart multi-factor random oscillation model is more consistent than the one-factor models and better fits market data. Christoffersen et al. [11] introduced the double-Heston (DH) model by adding another stochastic volatility factor based on the Cox–Ingersoll–Ross (CIR) process to the Heston dynamics. Their main motivation for introducing this model was more flexibility to modeling the structure of fluctuations and better empirical fitting with European option prices. This has recently been confirmed by Mehrdoust et al. [47] and Zhang and Feng [63] for American options. Li and Zhang [39] by examining a set of option index data confirmed the presented results from [11]. More specifically, the DH model includes of two independent CIR mean-reverting processes that its computational tools are similar to the standard Heston model, but the obtained results in the option pricing under this model can be very desirable. Recently, Fallah and Mehrdoust [23] analyzed the strong convergence of the DH model. Existence of the solution of this model and its uniqueness were investigated by Fallah and Mehrdoust [24]. In this paper, we present the DH model based on the Markov regime-switching (MRS) process for American put option pricing.

In the financial literature, there are wide applications of MRS models. The use of regime-switching (RS) models in financial literature was first introduced by Hamilton [30]. After that, interest in MRS model programs has increased, including option valuation, stock and energy markets modeling, risk management and etc. Local risk minimizing option in the DH model equipped with RS factor can be found in [16]. Valuation of European option under this model along with stochastic interest rate and jump factor was recently studied by Lyu et al. [42]. Furthermore, among the valuable works that have been done recently in the American option pricing under RS models, we can mention [7], [22] and [62]. As one of the applications of RS model in energy markets, Mehrdoust and Noorani [46] analyzed the behavior of Nord Pool electricity spot prices and forward contracts by MRS dynamics. Bo and Capponi [6] investigated the hidden regime impact on credit risk portfolios.

The purpose of MRS models is to show the stochastic treatment of a phenomenon by distinct states, which is done with separate random processes. In fact, the mechanism adopted between switching model regimes follows a hidden Markov chain. One of the primary features of the MRS process is that the model dynamics under this process changes suddenly. From [10], the MRS models can be investigated as extensions of hidden Markov models. For the applications of hidden Markov models to finance and insurance, see [20,37,50,60] and [57]. As is well known, the mean-reverting models are used in many stochastic volatility models. Depending on the parameters of this type of models on the hidden Markov chain allows the oscillation process to operate differently under different regimes. This change of regimes can be an alteration in government policy, inflation and recession, war, and so on. In the financial and economic literature, other models based on the RS processes called threshold-RS (TRS) models are used to analyze the financial markets, that have been studied by Gonzalo and Pitarakis [28]. The major discrepancy between the TRS models and the MRS models is that in the first case, the switching mechanism between the regimes is visible, while in the second case, it is hidden. Here, we consider MRS models and for simplicity, we call these models RS.

To assess the stock price dynamics, we consider the DH model such that the interest rate and the mean-reversion levels of the volatilities depend on hidden Markov chain and we call it the regime-switching double Heston (RSDH) model. As stated by Tankov [59], finite difference method can be used to price derivatives. In this method, issues related to the stability and accuracy of the solution should be examined, which can complicate the process. We note that in the RSDH model, in addition to adding a more stochastic factor to the asset dynamics, adding a switching mode factor also complicates the numerical problem and we encounter a higher dimensional problem. Thus, when the dynamics of assets are described by the two-factor model dependent on the Markov chain, the valuation of securities such as the American put option via numerical methods like the finite differences complicates the problem. Since the 1970s, when Black and Scholes presented the popular BS formula for option pricing, it has been a challenge for researchers, finding a clear and unambiguous solution to the American option. Newly, by using the standard portfolio-consumption equation, this gap was filled by Alghalith [1] and Mehrdoust et al. [47], under the geometric Brownian motion (GBM) model and the DH model. In this paper, we consider this approach under the RSDH model to price the American option. We also by calibrating this model with the American put option data of SPDR S&P 500 ETF Trust (SPY) market, show that the proposed RSDH model fits the considered option market data better than the basic models such as the DH and BS models. Most recently, based on the stochastic volatility models, several numerical approaches for pricing American option have been studied: the finite difference methods of Ikonen and Toivanen [33], Ballestra and Sgarra [3] and Zhu and Chen [64] and some generalizations, such as Kunoth et al. [38], Salmi et al. [54] and Burkovska et al. [8]. Using binomial tree methods Monte Carlo simulation to price American option are another approaches that has been studied by many researchers over the past two decades: the Monte-Carlo simulation method of Rogers [52], least square Monte-Carlo simulation method of Longstaff and Schwartz [40], Mehrdoust et al. [47], Mehrdoust et al. [44], Samimi et al. [55] and binomial tree methods of Beliaeva [4], Ruckdeschel et al. [53] and Mehrdoust and Noorani [45]. In addition, American option pricing problem based on Fourier methods was introduced as a significant approach by Lord et al. [41] and Fang and Oosterlee [25].

All calibration algorithms by minimizing the error criterion (loss function) search a region of parameters space in a intelligent way (see [13,43,49] and [9]). Furthermore, the loss function can be considered as the inverse BS vega on the option market prices (see [35,36,61] and [12]). A common method to calibration of parametric models, such as the Heston model and its generalizations, is to use the least squares method for the option price. In order to calibrate the RSDH model, we minimize the loss function which defined as mean square error (MSE) of option market prices and the semi-analytical solution of option price by applying the Levenberg–Marquardt (LM) method. In this framework, we follow the strategy developed by Cui et al. [13], which was applied for the calibration of Heston model.

This paper is structured as follows. Section 2 introduces the RSDH model. In Section 3, the American put option price by using the standard portfolio-consumption equation and the European put option price is studied. In Section 4, according to the proposed semi-analytic formula for the American put option price, we apply the LM optimization algorithm to calibrate the option prices obtained from the RSDH model into a dataset of option price observations in the SPY market. In Section 5, we present the numerical results. Then, through an experimental application, the advantage of the RSDH model compared to the DH and BS models is shown. Section 6, concludes the article.

## 2. The RSDH model

This section introduces a model in which dynamics of the asset have two different volatility processes that follow CIR process. In this model, interest rate parameter and mean-reversion level parameters in both volatility dynamics depend on hidden (latent) Markov chain. The available model is called the RS double Heston model. Due to this feature, the states of different jumps follow a Markov chain and the RS effect on contingent claim prices is considered. Suppose that the underlying price and two volatilities represented by $S_t$, $V_t^{(1)}$, and $V_t^{(2)}$, respectively. According to the provided descriptions, the RSDH model under the risk-neutral probability measure[1] is implemented as follows:

$$\begin{aligned} dS_t &= r_{Z_t} S_t dt + \sqrt{V_t^{(1)}} S_t dW_t^{(1)} + \sqrt{V_t^{(2)}} S_t dW_t^{(2)}, \quad S_0 = s, \\ dV_t^{(1)} &= \kappa(\theta_{Z_t} - V_t^{(1)})dt + \xi \sqrt{V_t^{(1)}} dW_t^{(3)}, \quad V_0^{(1)} = \upsilon^{(1)}, \\ dV_t^{(2)} &= \lambda(\beta_{Z_t} - V_t^{(2)})dt + \sigma \sqrt{V_t^{(2)}} dW_t^{(4)}, \quad V_0^{(2)} = \upsilon^{(2)}, \end{aligned} \tag{2.1}$$

where, $dW_t^{(1)} dW_t^{(3)} = \rho_1 dt$ and $dW_t^{(2)} dW_t^{(4)} = \rho_2 dt$, $\rho_1, \rho_2 \in (-1, 1)$, whereas the Brownian motions $W_t^{(1)}$ and $W_t^{(2)}$ are independent. Further, $Z_t$ is a continuous-time hidden Markov Chain process which is independent of the four Wiener processes. In particular, $Z_t$ is defined as[2]

$$Z_t = \begin{cases} 1, & \textit{when the economy is in regime } 1, \\ 2, & \textit{when the economy is in regime } 2, \end{cases}$$

such that the transition between the states of chain follows a Poisson process, that is,

$$\mathbb{P}(t_{ij} > t) = e^{-\pi_{ij} t}, \quad \textit{for } j, i = 1, 2, \ j \neq i.$$

[1] Under risk-neutral measure which is equivalent to objective measure, the discounted stock price is martingale and the market does not has arbitrage.

[2] We focus on 2-state Markov chain, but it is straightforward to extend all the results of this work to a model with more than two regimes.

Notice that $\pi_{ij}$ is the transition rate of the hidden Markov chain from regime $i$ to regime $j$, and $t_{ij}$ is the elapsed time in regime $i$ before transitioning to regime $j$. It is clear that when the two transition rates $\pi_{12}$ and $\pi_{21}$ are equal to zero, the proposed model would certainly convert to the DH model.

From [19], the state space of $Z_t$ can be considered as a set of unit vectors $\{e_1, e_2\}$, $e_1 = (1, 0)^\top$ and $e_2 = (0, 1)^\top$. Here, $u^\top$ denotes the transpose of the vector $u$. In this case, the interest rate and the mean-reversion levels of the volatilities can be derived through

$$
\begin{aligned}
r_{Z_t} &= \langle \bar{r}, Z_t \rangle = \sum_{i=1}^{2} r_i \langle Z_t, e_i \rangle, \\
\theta_{Z_t} &= \langle \bar{\theta}, Z_t \rangle = \sum_{i=1}^{2} \theta_i \langle Z_t, e_i \rangle, \\
\beta_{Z_t} &= \langle \bar{\beta}, Z_t \rangle = \sum_{i=1}^{2} \beta_i \langle Z_t, e_i \rangle,
\end{aligned}
$$

where $\bar{r} = (r_1, r_2)^\top$, $\bar{\theta} = (\theta_1, \theta_2)^\top$, $\bar{\beta} = (\beta_1, \beta_2)^\top$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{2\times 1}$.

Elliott et al. [18] showed that the semi-martingale representation of the hidden Markov chain process $Z$ can be expressed as follows:

$$
Z_t = Z_0 + \int_0^t \Pi Z_s ds + M_t, \tag{2.2}
$$

where $\{M_t\}$ is an $\mathbb{R}^{2\times 1}$-valued martingale process with respect to the filtration generated by $Z$ under $\mathbb{P}$, and $\Pi = (\pi_{ij})_{i,j=1,2}$ is the rate matrix, such that the dynamics of the chain are generated.

To simplify the method of extracting the American option pricing formula, which will be discussed in the next section, we express the dynamics of the model with the following matrix form

$$
dU_t = M(U_t)dt + \Sigma(U_t)dB_t, \tag{2.3}
$$

where $U_t$ and $B_t$ are defined as

$$
U_t = \begin{bmatrix} X_t \\ V_t^{(1)} \\ V_t^{(2)} \end{bmatrix}, \quad B_t = \begin{bmatrix} B_t^{(1)} \\ B_t^{(2)} \\ B_t^{(3)} \\ B_t^{(4)} \end{bmatrix}, \tag{2.4}
$$

with $X_t = \ln(S_t)$, and the four Brownian motions, $B_t^{(1)}$, $B_t^{(2)}$, $B_t^{(3)}$ and $B_t^{(4)}$ are independent of each other. The drift term $M(U_t)$ can be specified as

$$
M(U_t) = \begin{bmatrix} r_{Z_t} - \frac{1}{2}V_t^{(1)} - \frac{1}{2}V_t^{(2)} \\ \kappa(\theta_{Z_t} - V_t^{(1)}) \\ \lambda(\beta_{Z_t} - V_t^{(2)}) \end{bmatrix} = K_0 + K_1 U_t, \tag{2.5}
$$

where

$$
K_0 = \begin{bmatrix} r_{Z_t} \\ \kappa\theta_{Z_t} \\ \lambda\beta_{Z_t} \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\kappa & 0 \\ 0 & 0 & -\lambda \end{bmatrix}.
$$

Moreover, the diffusion part $\Sigma(U_t)$ in Eq. (2.3) is as follows

$$
\Sigma(U_t) = \begin{bmatrix} \sqrt{V_t^{(1)}} & \sqrt{V_t^{(2)}} & 0 & 0 \\ \rho_1\xi\sqrt{V_t^{(1)}} & 0 & \xi\sqrt{(1-\rho_1^2)V_t^{(1)}} & 0 \\ 0 & \rho_2\sigma\sqrt{V_t^{(2)}} & 0 & \sigma\sqrt{(1-\rho_2^2)V_t^{(2)}} \end{bmatrix}. \tag{2.6}
$$

Let $\mathcal{F}_t^U$ and $\mathcal{F}_t^Z$ be the natural filtrations generated by the four Brownian motions, $B_t^{(1)}$, $B_t^{(2)}$, $B_t^{(3)}$ and $B_t^{(4)}$ and the RS Markov chain, $Z_t$, up to time t, respectively, that is

$$\mathcal{F}_t^U = \sigma\Big\{B_s^{(1)}, B_s^{(2)}, B_s^{(3)}, B_s^{(4)} :\ s \le t\Big\}, \quad \mathcal{F}_t^Z = \sigma\Big\{Z_s :\ s \le t\Big\}.$$

In the following section, we present the RSDH model dynamics, and we discuss under this model, the pricing of American options through European option and standard portfolio-consumption.

## 3. Options pricing problem

This section is subdivided into two subsections. In the first subsection a semi-analytical solution for the pricing of European option is derived. For this purpose, we first determine the closed-form solution of the implied moment-generating function of model as unknown part of the pricing formula. In the second subsection, we develop a semi-analytical solution to the partial differential equation of DH by using the European put option price and the standard portfolio-consumption model. Then through the affine form of the model, we provide the American put option price.

### 3.1. Pricing of European options

We start this subsection by demonstrating Eq. (2.3) has affine form. As shown earlier in Eq. (2.5), the drift term $M(U_t)$ can be presented as an affine function. In particular, $\Sigma(U_t)\Sigma^\top(U_t)$ can be formulated as

$$\Sigma(U_t)\Sigma^\top(U_t) = \begin{bmatrix} V_t^{(1)} + V_t^{(2)} & \rho_1 \xi V_t^{(1)} & \rho_2 \sigma V_t^{(2)} \\ \rho_1 \xi V_t^{(1)} & \xi^2 V_t^{(1)} & 0 \\ \rho_2 \sigma V_t^{(2)} & 0 & \sigma^2 V_t^{(2)} \end{bmatrix} = H \otimes U_t, \tag{3.1}$$

where $H := (h_{ij})$ is a $3 \times 3 \times 3$ matrix, with $(h_{ij})_{i,j=1,2,3} \in \mathbb{R}^{3\times 1}$, such that

$$h_{11} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad h_{21} = h_{12} = \begin{bmatrix} 0 \\ \rho_1 \xi \\ 0 \end{bmatrix}, \quad h_{22} = \begin{bmatrix} 0 \\ \xi^2 \\ 0 \end{bmatrix},$$

and

$$h_{31} = h_{13} = \begin{bmatrix} 0 \\ 0 \\ \rho_2 \sigma \end{bmatrix}, \quad h_{32} = h_{23} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad h_{33} = \begin{bmatrix} 0 \\ 0 \\ \sigma^2 \end{bmatrix}.$$

We note that, if $H$ be the $3 \times 3 \times 3$ matrix and $b \in \mathbb{R}^{3\times 1}$, then $H \otimes b$ denotes the inner product of each element of $H$ and vector $b$.
Combining Eqs. (2.5) and (3.1), we can conclude that the SDE (2.3) has affine form.

Assuming that $C^E(U_t, Z_t, t)$ with strike price $K$ and maturity time $T$ presents the European put option price at time $t \le T$, we have

$$\begin{aligned} C^E(U_t, Z_t, t) =& \mathbb{E}\Big[e^{-\int_t^T r_{Z_s} ds}(K - S_T)^+ \Big| \mathcal{F}_t^U, \mathcal{F}_t^Z\Big] \\ =& \mathbb{E}\Big[e^{-\int_t^T r_{Z_s} ds}(K - e^{X_T})^+ \Big| \mathcal{F}_t^U, \mathcal{F}_t^Z\Big] \\ =& \mathbb{E}\Big[e^{-\int_t^T r_{Z_s} ds}(K - e^{\epsilon_1 \cdot U_T})^+ \Big| \mathcal{F}_t^U, \mathcal{F}_t^Z\Big] \\ =& \mathbb{E}\Big[e^{-\int_t^T r_{Z_s} ds}(K - e^{\epsilon_1 . U_T})\mathbf{I}_{\ln(K) \ge \epsilon_1 \cdot U_T} \Big| \mathcal{F}_t^U, \mathcal{F}_t^Z\Big] \\ =& K G_{0,\epsilon_1}\Big(\ln(K); U_t, Z_t, t, T\Big) - G_{\epsilon_1,\epsilon_1}\Big(\ln(K); U_t, Z_t, t, T\Big), \end{aligned} \tag{3.2}$$

where $\mathbf{I}_{\mathcal{A}}$ denotes the indicator function of the set (event) $\mathcal{A}$, and

$$G_{a,b}\Big(c; U_t, Z_t, t, T\Big) = \mathbb{E}\Big[e^{-\int_t^T r_{Z_s} ds} e^{a \cdot U_T} \mathbf{I}_{c \ge b \cdot U_T} \Big| \mathcal{F}_t^U, \mathcal{F}_t^Z\Big],$$

with $\epsilon_1 = (1, 0, 0)^\top$ and $a, b \in \mathbb{R}^{3\times 1}$.

Following [17], we have

$$G_{a,b}\Big(c; U_t, Z_t, t, T\Big) = \frac{\Psi(a, U_t, Z_t, t, T)}{2} - \frac{1}{\pi}\int_0^{+\infty} \frac{\mathrm{Im}\Big[\Psi\Big(a + i\nu b, U_t, Z_t, t, T\Big)e^{-i\nu c}\Big]}{\nu} d\nu, \tag{3.3}$$

where $i = \sqrt{-1}$ and $\mathrm{Im}(\cdot)$ indicates the imaginary term. Moreover, $\Psi(\phi, U_t, Z_t, t, T)$ denotes the moment-generating function as follows

$$\Psi(\phi, U_t, Z_t, t, T) = \mathbb{E}\Big[e^{-\int_t^T r_{Z_s} ds} e^{\phi \cdot U_t}\Big|\mathcal{F}_t^U, \mathcal{F}_t^Z\Big], \tag{3.4}$$

with $\phi = (\phi_1, \phi_2, \phi_3)$.

Without applying the imaginary unit, the Fourier transform applied on the probability density function of $U_t$ is called a moment-generating function (see [59], for more details). What is more important is the analytical expression of the expectation in Eq. (3.4), which forms the core of this article. And leads to finding a closed form for American option pricing. As pointed out by Elliott et al. [21], to obtain the moment-generating function, we first evaluate the conditional expectation of a derivative with respect to the information on the sample path of the Markov chain $Z$ from $t = 0$ to expiry date $t = T$, i.e., $\mathcal{F}_T^Z$. It can be easily handled for a given realized trajectory of $Z_t$, the parameters depend on hidden Markov chain $r_{Z_t}$, $\theta_{Z_t}$ and $\beta_{Z_t}$ behave as deterministic functions of time.

Given the existence of the Markov chain, the direct calculation of the expectation involved in Eq. (3.4) is extremely difficult. To comfort the calculation, we consider the moment-generating function $\Psi(\phi, U_t, Z_t, t, T)$ as follows:

$$\Psi(\phi, U_t, Z_t, t, T) = \mathbb{E}\Big[\mathbb{E}\big[e^{-\int_t^T r_{Z_s} ds} e^{\phi \cdot U_t}\big|\mathcal{F}_t^U, \mathcal{F}_T^Z\big]\Big|\mathcal{F}_t^Z\Big]. \tag{3.5}$$

On the other hand, before we can achieve moment-generating function, we must first implement internal expectation. For this propose, we define a new moment-generating function $\mu\big(\phi, U_t, t, T\big|\mathcal{F}_T^Z\big)$ on all the generated filtration of the process $Z$ until the maturity date $T$. Then, we can write

$$\mu\big(\phi, U_t, t, T\big|\mathcal{F}_T^Z\big) = \mathbb{E}\Big[e^{-\int_t^T r_{Z_s} ds} e^{\phi \cdot U_t}\Big|\mathcal{F}_t^U, \mathcal{F}_T^Z\Big], \tag{3.6}$$

its expression can be explicitly derived and is provided in the following proposition.

**Proposition 3.1.** *Suppose that the stock dynamics follows the system (3.4), then the implied moment-generating function* $\mu := \mu\big(\phi, U_t, t, T\big|\mathcal{F}_T^Z\big)$ *is expressed as follows*

$$\mu = e^{C(\phi;\tau) + D(\phi;\tau)\cdot U_t}, \tag{3.7}$$

*where " $\cdot$" denotes the dot product for vectors,* $\tau = T - t$,

$$D(\phi; \tau) = \begin{bmatrix} D_1(\phi;\tau) \\ D_2(\phi;\tau) \\ D_3(\phi;\tau) \end{bmatrix},$$

*and*

$$\begin{aligned}
D_1(\phi; \tau) &= \phi_1, \\
D_2(\phi; \tau) &= \frac{d_1 + \big(\rho_1 \xi \phi_1 + \xi^2 \phi_2 - \kappa\big)}{\xi^2} \cdot \frac{1 - e^{d_1 \tau}}{1 - g_1 e^{d_1 \tau}} + \phi_2, \\
D_3(\phi; \tau) &= \frac{d_2 + \big(\rho_2 \sigma \phi_1 + \sigma^2 \phi_3 - \lambda\big)}{\sigma^2} \cdot \frac{1 - e^{d_2 \tau}}{1 - g_2 e^{d_2 \tau}} + \phi_3, \\
C(\phi; \tau) &= (\phi_1 - 1)\int_t^T \langle \bar{r}, Z_s\rangle ds + \kappa \int_t^T \langle \bar{\theta}, Z_s\rangle D_2(\phi; T - s) ds + \lambda \int_t^T \langle \bar{\beta}, Z_s\rangle D_3(\phi; T - s) ds, \\
d_1 &= \sqrt{\big(\rho_1 \xi \phi_1 + \xi^2 \phi_2 - \kappa\big)^2 + \xi^2\left(\phi_1 - \phi_1^2 + 2\kappa\phi_2 - 2\rho_1 \xi \phi_1 \phi_2 - \xi^2 \phi_2^2\right)}, \\
d_2 &= \sqrt{\big(\rho_2 \sigma \phi_1 + \sigma^2 \phi_3 - \lambda\big)^2 + \sigma^2\left(\phi_1 - \phi_1^2 + 2\lambda\phi_3 - 2\rho_2 \sigma \phi_1 \phi_3 - \sigma^2 \phi_3^2\right)},
\end{aligned}$$

$$g_1 = \frac{\left(\rho_1 \xi \phi_1 + \xi^2 \phi_2 - \kappa\right) + d_1}{\left(\rho_1 \xi \phi_1 + \xi^2 \phi_2 - \kappa\right) - d_1},$$
$$g_2 = \frac{\left(\rho_2 \sigma \phi_1 + \sigma^2 \phi_3 - \lambda\right) + d_2}{\left(\rho_2 \sigma \phi_1 + \sigma^2 \phi_3 - \lambda\right) - d_2}.$$

**Proof.** The implied expectation $\mu\left(\phi, U_t, t, T \middle| \mathcal{F}_T^Z\right)$ can be performed by solving the PDE that carried out by the Feynman–Kac theorem as follows:

$$-\frac{\partial \mu}{\partial \tau} + \left(K_0 + K_1 U_t\right) \cdot \frac{\partial \mu}{\partial U} + \frac{1}{2}(H \otimes U_t)\frac{\partial^2 \mu}{\partial U^2} = r_{Z_t} \mu. \tag{3.8}$$

Following the solution method used by Duffie et al. [17], the solution of the above PDE can be assumed to be of the form

$$\mu = e^{C(\phi;\tau)+D(\phi;\tau)\cdot U_t}.$$

Substituting this function into the PDE (3.8), gives the following PDEs:

$$\frac{\partial}{\partial \tau} D(\phi; \tau) = K_1^\top D + \frac{1}{2} D^\top H D,$$
$$\frac{\partial}{\partial \tau} C(\phi; \tau) = K_0 \cdot D - r_{Z_t},$$

with the initial conditions

$$D(\phi; 0) = \phi, \quad C(\phi; 0) = 0.$$

Due to the definition of $D(\phi; \tau)$, we have

$$\frac{\partial}{\partial \tau} D_1 = 0, \quad D_1(\phi, 0) = \phi_1,$$
$$\frac{\partial}{\partial \tau} D_2 = \frac{1}{2}\xi^2 D_2^2 + D_2(\rho_1 \xi D_1 - \kappa) + \frac{1}{2} D_1(D_1 - 1), \quad D_2(\phi, 0) = \phi_2,$$
$$\frac{\partial}{\partial \tau} D_3 = \frac{1}{2}\sigma^2 D_3^2 + D_3(\rho_2 \sigma D_1 - \lambda) + \frac{1}{2} D_1(D_1 - 1), \quad D_3(\phi, 0) = \phi_3.$$

Obviously, $D_1(\phi; \tau)$ can be immediately derived as a constant $\phi_1$. In order to solve the ODEs of $D_2(\phi; \tau)$ and $D_3(\phi; \tau)$, due to the fact that the initial conditions are inhomogeneous, we convert it to homogeneous case with $\hat{D}_2(\phi; \tau) = D_2(\phi; \tau) - \phi_2$ and $\hat{D}_3(\phi; \tau) = D_3(\phi; \tau) - \phi_3$. We obtain

$$\frac{\partial}{\partial \tau} \hat{D}_2 = \frac{1}{2}\xi^2 \hat{D}_2^2 + \left(\rho_1 \xi \phi_1 + \xi^2 \phi_2 - \kappa\right) \hat{D}_2 + \left(\phi_2 \phi_1 \rho_1 \xi - \phi_2 \kappa + \frac{1}{2}\phi_1^2 - \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2^2 \xi^2\right),$$
$$\hat{D}_2(\phi; 0) = 0,$$
$$\frac{\partial}{\partial \tau} \hat{D}_3 = \frac{1}{2}\sigma^2 \hat{D}_3^2 + \left(\rho_2 \sigma \phi_1 + \sigma^2 \phi_3 - \lambda\right) \hat{D}_3 + \left(\phi_3 \phi_1 \rho_2 \sigma - \phi_3 \lambda + \frac{1}{2}\phi_1^2 - \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2^2 \sigma^2\right),$$
$$\hat{D}_3(\phi; 0) = 0.$$

which are the Riccati equations with homogeneous initial conditions. Then the expressions of $\hat{D}_2(\phi; \tau)$ and $\hat{D}_3(\phi; \tau)$ can be derived as

$$\hat{D}_2(\phi; \tau) = \frac{d_1 + \left(\rho_1 \xi \phi_1 + \xi^2 \phi_2 - \kappa\right)}{\xi^2} \cdot \frac{1 - e^{d_1 \tau}}{1 - g_1 e^{d_1 \tau}},$$
$$\hat{D}_3(\phi; \tau) = \frac{d_2 + \left(\rho_2 \sigma \phi_1 + \sigma^2 \phi_3 - \lambda\right)}{\sigma^2} \cdot \frac{1 - e^{d_2 \tau}}{1 - g_2 e^{d_2 \tau}}.$$

With available of $D(\phi; \tau)$, the expression $C(\phi; \tau)$ can be derived by integrating on both sides of the ODE with respect to $t$. This completes the proof. □

In order to obtain the function $\Psi(\phi, U_t, Z_t, t, T)$, valuation of the outer expectation in Eq. (3.5) is the remaining work, which can be simplified as follows

$$\begin{aligned}\Psi(\phi, U_t, Z_t, t, T) &= \mathbb{E}\left[\mu\big(\phi, U_t, t, T \big| \mathcal{F}_T^Z\big) \big| \mathcal{F}_t^Z\right] \\ &= e^{D(\phi;\tau)\cdot U_t}\mathbb{E}\left[e^{C(\phi;\tau)} \big| \mathcal{F}_t^Z\right].\end{aligned} \tag{3.9}$$

The following proposition states the implied expectation $\mathbb{E}\left[e^{C(\phi;\tau)} \big| \mathcal{F}_t^Z\right]$:

**Proposition 3.2.** *If the stock dynamics follows the RSDH model (2.1), then the implied expectation* $\mathbb{E}\left[e^{C(\phi;\tau)} \big| \mathcal{F}_t^Z\right]$ *is given by*

$$\mathbb{E}\left[e^{C(\phi;\tau)} \big| \mathcal{F}_t^Z\right] = \left\langle e^{\Pi^\top \tau + G} Z_t, \mathbf{1}\right\rangle, \tag{3.10}$$

*where* $\mathbf{1} = (1, 1)^\top$, *and the rate matrix of the Markov chain* $Z_t$ *defined as*

$$\Pi = \begin{bmatrix} -\pi_{12} & \pi_{12} \\ \pi_{21} & -\pi_{21} \end{bmatrix}.$$

*Moreover,*

$$G = diag\Big[(\phi_1 - 1)\bar{r}\tau + \kappa\bar{\theta}\int_t^T D_2(\phi; s-t)ds + \lambda\bar{\beta}\int_t^T D_3(\phi; s-t)ds\Big], \tag{3.11}$$

*such that* $diag[\cdot]$ *is an operator on* $n \times 1$ *vector and gives the* $n \times n$ *matrix, where the elements on the main diagonal are the same as the vector elements and the rest are zero.*

**Proof.** Let

$$\mathcal{L}_{t,T} = Z_T e^{C(\phi;\tau)}.$$

Then by using Itô's Formula to $\mathcal{L}_{t,T}$, we have

$$\begin{aligned} d\mathcal{L}_{t,T} &= e^{C(\phi;\tau)}\Big((\Pi Z_T)^\top dt + (dM_T)^\top\Big) + e^{C(\phi;\tau)} Z_T^\top \Big((\phi_1 - 1)\langle\bar{r}, Z_T\rangle + \kappa\langle\bar{\theta}, Z_T\rangle D_2(\phi;\tau) \\ &\qquad + \lambda\langle\bar{\beta}, Z_T\rangle D_3(\phi;\tau)\Big)dT \\ &= e^{C(\phi;\tau)}(dM_T)^\top + e^{C(\phi;\tau)} Z_T^\top\Big(\Pi^\top + diag\big[(\phi_1 - 1)\bar{r} + \kappa\bar{\theta}D_2(\phi;\tau) + \lambda\bar{\beta}D_3(\phi;\tau)\big]\Big)dT. \end{aligned}$$

Thus, we get

$$\begin{aligned} \mathcal{L}_{t,T} &= Z_t + \int_t^T e^{C(\phi;\tau)}(dM_s)^\top \\ &\quad + \int_t^T e^{C(\phi;s-t)} Z_s^\top\Big(\Pi^\top + diag\big[(\phi_1 - 1)\bar{r} + \kappa\bar{\theta}D_2(\phi;s-t) + \lambda\bar{\beta}D_3(\phi;s-t)\big]\Big)ds. \end{aligned}$$

Let $\varphi_{t,T}$ denote the expectation of $\mathcal{L}_{t,T}$ conditional on $\mathcal{F}_t^Z$, that is,

$$\varphi_{t,T} = \mathbb{E}\Big[\mathcal{L}_{t,T}\Big|\mathcal{F}_t^Z\Big].$$

Then, from above we obtain

$$\varphi_{t,T} = Z_t + \int_t^T \Big(\Pi^\top + diag\big[(\phi_1 - 1)\bar{r} + \kappa\bar{\theta}D_2(\phi;s-t) + \lambda\bar{\beta}D_3(\phi;s-t)\big]\Big)\varphi_{t,s}ds. \tag{3.12}$$

Suppose that $\Phi_{t,s}$ be the $2 \times 2$ matrix, which is the solution of the linear system of ODEs in Eq. (3.12), in this case we obtain

$$\frac{d\Phi_{t,s}}{ds} = \Big(\Pi^\top + diag\big[(\phi_1 - 1)\bar{r} + \kappa\bar{\theta}D_2(\phi;s-t) + \lambda\bar{\beta}D_3(\phi;s-t)\big]\Big)\Phi_{t,s}, \quad \Phi_{t,t} = diag(\mathbf{1}).$$

It can be easily shown that,

$$\varphi_{t,T} = \Phi_{t,T} Z_t.$$

Due to fact that $\langle Z_T, \mathbf{1}\rangle = 1$, we have

$$\begin{aligned}\mathbb{E}\Big[e^{C(\phi;\tau)}\Big|\mathcal{F}_t^Z\Big] &= \mathbb{E}\Big[\langle Z_T e^{C(\phi;\tau)}, \mathbf{1}\rangle\Big|\mathcal{F}_t^Z\Big]\\ &= \mathbb{E}\Big[\langle \mathcal{L}_{t,T}, \mathbf{1}\rangle\Big|\mathcal{F}_t^Z\Big]\\ &= \langle \varphi_{t,T}, \mathbf{1}\rangle\\ &= \langle \Phi_{t,T} Z_t, \mathbf{1}\rangle = \langle e^{\Pi^\top \tau + G} Z_t, \mathbf{1}\rangle.\end{aligned}$$

Therefore, the result follows. □

Now, we are ready to propose a semi-analytical solution for European put option $C^E(U_t, Z_t, t)$. From Eqs. (3.2) and (3.3) we have

$$\begin{aligned}C^E(U_t, Z_t, t) &= \frac{K\,\Psi\big(0, U_t, Z_t, t, T\big) - \Psi\big(\epsilon_1, U_t, Z_t, t, T\big)}{2}\\ -\ \frac{1}{\pi}&\int_0^{+\infty}\Big(\frac{K\,\mathrm{Im}\big[\Psi\big(i\nu\epsilon_1, U_t, Z_t, t, T\big)e^{-i\nu\ln K}\big] - \mathrm{Im}\big[\Psi\big(\epsilon_1(1+i\nu), U_t, Z_t, t, T\big)e^{-i\nu\ln K}\big]}{\nu}\Big)d\nu, \end{aligned} \tag{3.13}$$

where $\epsilon_1 = (1, 0, 0)^\top$ and $i = \sqrt{-1}$. Moreover, due to Propositions 3.1 and 3.2, $\Psi\big(\phi, U_t, Z_t, t, T\big)$ worked out as follows

$$\Psi\big(\phi, U_t, Z_t, t, T\big) = e^{D(\phi;\tau).U_t}\langle e^{\Pi^\top\tau+G} Z_t, \mathbf{1}\rangle, \tag{3.14}$$

where

$$G = \begin{bmatrix}(\phi_1 - 1)r_1\tau + \kappa\theta_1 p(\phi;\tau) + \lambda\beta_1 q(\phi;\tau) & 0\\ 0 & (\phi_1 - 1)r_2\tau + \kappa\theta_2 p(\phi;\tau) + \lambda\beta_2 q(\phi;\tau)\end{bmatrix},$$

with

$$p(\phi;\tau) = \frac{1}{\xi^2}\Big(\big[d_1 - \big(\rho_1\xi\phi_1 + \xi^2\phi_2 - \kappa\big)\big]\tau - 2\ln\big(\frac{1 - g_1 e^{d_1\tau}}{1 - g_1}\big)\Big) + \phi_2\tau, \tag{3.15}$$

$$q(\phi;\tau) = \frac{1}{\sigma^2}\Big(\big[d_2 - \big(\rho_2\sigma\phi_1 + \sigma^2\phi_3 - \lambda\big)\big]\tau - 2\ln\big(\frac{1 - g_2 e^{d_2\tau}}{1 - g_2}\big)\Big) + \phi_3\tau. \tag{3.16}$$

So far we have developed a semi-analytical expression for pricing European put option with a maturity $T$ and a strike $K$. For this purpose, the closed-form solution of the moment-generating function of $U_t$ conditional upon the filtration generated by the market development was derived. In the next subsection, we deduce a semi-analytical expression for pricing American put options by using the equivalent European put option price and standard portfolio-consumption model.

### 3.2. Pricing of American options

Assume $\Lambda(t)$ is the value of the agent's wealth at time $t$, such that he (she) invests this asset in the money market account expressed by the RSDH model (2.1). Notice that the interest rate of the model depends on hidden Markov chain and its changes among different states.

Suppose the investor has $\Delta_t$ shares of asset. Let the $\Delta_t$ portfolio be stochastic, such that it is adopted with filtration generated by market developments (four Brownian motions and Markov chain). The remainder of the wealth value, $\Lambda_t - \Delta_t S_t$, is funded in the money market account. From [5], $\Lambda_t$ satisfies the following equation

$$\begin{aligned}d\Lambda(t) &= \Delta_t dS_t + r_{Z_t}\big(\Lambda_t - P_t - \Delta_t S_t\big)dt\\ &= \Delta_t\big(\alpha_{Z_t} S_t dt + \sqrt{V_t^{(1)}}dW_t^{(1)} + \sqrt{V_t^{(2)}}dW_t^{(2)}\big) + r_{Z_t}\big(\Lambda_t - P_t - \Delta_t S_t\big)dt\\ &= \big(r_{Z_t}\Lambda_t + \Delta_t(\alpha_{Z_t} - r_{Z_t})S_t - P_t\big)dt + \Delta_t\big(\sqrt{V_t^{(1)}}dW_t^{(1)} + \sqrt{V_t^{(2)}}dW_t^{(2)}\big).\end{aligned} \tag{3.17}$$

where $P \geq 0$ and $\alpha_{Z_t}$ are the consumption rate and the mean rate of return depending on Markov chain $Z_t$, respectively. The consumption rate factor was applied in Eq. (3.17), because if the agent does not exercise the American contingent claim at the optimal time, he (she) may consume a deduction of wealth (see, [14]).

According to studies by Guo [29], we know that a financial market described by an RS model is incomplete. On the other hand, if every bounded contingent claim is attainable then the market model is complete, that is, the wealth process $\Lambda$ and American contingent claim $C^A$ are equivalent (see [51], for more information). Due to the fact that the RSDH model describes the incomplete market, we have $\Lambda_t \geq C_t^A$. In this case, we can write

$$\Lambda_t = C_t^A + P_t,$$

where $P_t \geq 0$ is called the consumption rate.

Since the writer of the American option may not have exerted the fraction of the buyer's wealth at the optimal time and by assuming that $0 \leq \gamma < 1$ is a fraction of the wealth spent, we have

$$P_t = \gamma_t \Lambda_t = \gamma_t \big(C_t^A + P_t\big).$$

Thus

$$P_t = \frac{\gamma_t}{1-\gamma_t} C_t^A.$$

Under the hypothesis that $\gamma$ is martingale, we obtain

$$\eta \equiv \mathbb{E}\gamma_s = \gamma_t.$$

In this case

$$P_t = \frac{\eta}{1-\eta} C_t^A. \tag{3.18}$$

Recently, Alghalith [1] and Mehrdoust et al. [47], have presented an analytical solution for the American put option pricing problem under GBM and DH models by using the generalized BS formula and Fourier transform method, respectively. In this work, according to the equivalent European contingent claim and applying the consumption rate process, this meditation to valuation of the American put option under the RSDH model is generalized. Due to the intended portfolio and by using the discounted Feynman–Kac, we obtain

$$\begin{aligned}
&\frac{\partial C^A}{\partial t} + \frac{1}{2}(V^{(1)} + V^{(2)})\frac{\partial^2 C^A}{\partial S^2} + \frac{1}{2}\xi^2 V^{(1)}\frac{\partial^2 C^A}{\partial V^{(1)2}} + \frac{1}{2}\sigma^2 V^{(2)}\frac{\partial^2 C^A}{\partial V^{(2)2}} + \rho_1 \xi V^{(1)}\frac{\partial^2 C^A}{\partial S \partial V^{(1)}} \\
&+ \rho_2 \sigma V^{(2)}\frac{\partial^2 C^A}{\partial S \partial V^{(2)}} - \frac{1}{2}(V^{(1)} + V^{(2)})\frac{\partial C^A}{\partial S} + \kappa(\theta_Z - V^{(1)})\frac{\partial C^A}{\partial V^{(1)}} + \lambda(\beta_Z - v_2)\frac{\partial C^A}{\partial V^2} = r_Z C^A - P.
\end{aligned}$$

Thus, according to the results obtained by Alghalith [1], the value of American put option with the consumption rate $P_t$ in Eq. (3.18) can be obtained from the following equation

$$C^A(U_t, Z_t, t) = \mathbb{E}\Big[e^{-\int_t^T \Upsilon_{Z_s} ds}(K - S_T)^+ \Big| \mathcal{F}_t^U, \mathcal{F}_t^Z\Big], \tag{3.19}$$

where $\Upsilon_{Z_t} = \frac{r_{Z_t} - \eta(1+r_{Z_t})}{1-\eta}$, such that

$$\Upsilon_{Z_t} = \langle \bar{\Upsilon}, Z_t \rangle = \sum_{i=1}^{2} \Upsilon_i \langle Z_t, e_i \rangle,$$

with $\bar{\Upsilon} := (\Upsilon_1, \Upsilon_2) = \Big(\frac{r_1 - \eta(1+r_1)}{1-\eta}, \frac{r_2 - \eta(1+r_2)}{1-\eta}\Big)^\top \in \mathbb{R}^{2\times 1}$.

So far, the American put option price have been expressed based on the conditional expectation (3.19). A close look at this formula shows that the factor $e^{-\int_t^T \Upsilon_{Z_s} ds}$ is placed instead of the factor $e^{-\int_t^T r_{Z_s} ds}$ in the European put option price formula (first line in Eq. (3.2)). In this case, by replacing $\Upsilon_{Z_t}\mu$ instead of $r_{Z_t}\mu$ on the right hand side of Eq. (3.8), we can obtain the American put option price similar to the European put option. Therefore, according to the expression (3.13), the pricing formula for American put option worked out as follows

$$\begin{aligned}
C^A(U_t, Z_t, t) &= \frac{K\hat{\Psi}\big(0, U_t, Z_t, t, T\big) - \hat{\Psi}\big(\epsilon_1, U_t, Z_t, t, T\big)}{2} \\
&- \frac{1}{\pi}\int_0^{+\infty}\Big(\frac{K\,\mathrm{Im}\big[\hat{\Psi}\big(i\nu\epsilon_1, U_t, Z_t, t, T\big)e^{-i\nu \ln K}\big] - \mathrm{Im}\big[\hat{\Psi}\big(\epsilon_1(1+i\nu), U_t, Z_t, t, T\big)e^{-i\nu \ln K}\big]}{\nu}\Big)d\nu,
\end{aligned} \tag{3.20}$$

where $\epsilon_1 = (1, 0, 0)^\top$ and $i = \sqrt{-1}$. Moreover, due to the Eq. (3.14), $\hat{\Psi}\big(\phi, U_t, Z_t, t, T\big)$ worked out as follows

$$\hat{\Psi}\big(\phi, U_t, Z_t, t, T\big) = e^{D(\phi;\tau).U_t} \langle e^{\Pi^\top \tau + \hat{G}} Z_t, \mathbf{1}\rangle,$$

where

$$\hat{G} = \begin{bmatrix} (\phi_1 r_1 - \Upsilon_1)\tau + \kappa\theta_1 p(\phi;\tau) + \lambda\beta_1 q(\phi;\tau) & 0 \\ 0 & (\phi_1 r_2 - \Upsilon_2)\tau + \kappa\theta_2 p(\phi;\tau) + \lambda\beta_2 q(\phi;\tau) \end{bmatrix},$$

and $p(\phi;\tau)$ and $q(\phi;\tau)$ are Eqs. (3.15) and (3.16), respectively.

This result is quite intuitive, if we do not have a consumption rate, i.e, $\eta$ be equal to zero, the prices resulting from the European and American put options are equivalent. In addition, as expected, the value of American put option is higher than the equivalent European contingent claim. To provide numerical results, the integral involved in Eq. (3.20) is approximated by Gauss–Laguerre quadrature method.

In the next section, the estimated values of $\eta$ and the other model parameters are obtained through the LM method.

## 4. Implementation of LM method

When utilizing the calibrated model to price derivatives, the most important aspect of calibration is its ability to correctly reproduce prices and hedge ratios based on calibrated instruments, such that the stability of the obtained results is maintained over time. In fact, determining model parameters proportionate with option market prices is generally mentioned to as the calibration problem of model. Increasing the complexity of model calibration is the cost to pay for more realistic models such as the DH model (or a generalization such as the DH model with RS). As mentioned by Jacquier and Jarrow [34], "The estimation method is as important as the model itself".

In order to calculate option prices by RSDH model, we need input parameters that are not observable from market data, thus we cannot use empirical estimates. Bakshi et al. [2] stated that the implicit structural parameters vary notably with the estimated parameters from samples in the time series. For instance, the time series correlation coefficient for asset returns and its volatilities from the estimated daily prices is lower than their implied volatility. The set of model parameters should be as well as possible making the model price in accordance with the prices observed in the market. Hence, one of the complexities in solution of the calibration problem is that the market data is not sufficient to accurately identify the parameters. In practice, accurate matching of observed prices is not possible and meaningful. Therefore, the model calibration problem is formulated as a nonlinear optimization problem. Our goal is to minimize the error in the option price obtained by the model and the market price traded for a set of options. A common way to measure this error is to use the mean square error between the market price and the model, which leads to solutions of the nonlinear least square problem.

Assume that $C^*(K_i, T_i)$ be the market price of an American put option with strike $K_i$ and expiry date $T_i$ and $C(\Gamma, K_i, T_i)$ be the American put option price calculated by the RSDH analytical form (3.20), with the parameter vector

$$\Gamma := \big(\eta, r_1, r_2, \theta_1, \theta_2, \beta_1, \beta_2, \rho_1, \rho_2, \pi_{12}, \pi_{21}, v^{(1)}, v^{(2)}, \kappa, \lambda, \xi, \sigma\big)^\top.$$

For model calibration, we modulate the residuals for the $n$ option prices as follows:

$$E_i(\Gamma) := C(\Gamma; K_i, T_i) - C^*(K_i, T_i), \quad i = 1, \dots, n.$$

We set $\mathbf{E}(\Gamma) \in \mathbb{R}^{n\times 1}$ as the residual vector, that is,

$$\mathbf{E}(\Gamma) := \big(E_1(\Gamma), E_2(\Gamma), \dots, E_n(\Gamma)\big)^\top.$$

We consider the calibration of the RSDH model as the nonlinear least-square problem

$$\min_{\Gamma\in\mathbb{R}^{m\times 1}} \Xi(\Gamma), \tag{4.1}$$

where $m = 17$ represents the number of parameters, and

$$\Xi(\Gamma) = \frac{1}{2}\|\mathbf{E}(\Gamma)\|^2 = \frac{1}{2}\mathbf{E}^\top(\Gamma)\mathbf{E}(\Gamma). \tag{4.2}$$

Given that the observations is much higher than the number of model parameters, that is, $n \gg m$, to calibrate the model, we are dealing with overdetermined system.

We denote $\nabla = \partial/\partial\Gamma$ as the gradient operator with respect to the parameter vector $\Gamma$ and use $\nabla\nabla^\top$ as the Hessian operator.

Let $\mathcal{J} := (\mathcal{J}_{ji})_{\substack{i=1,\ldots,n,\\ j=1,\ldots,m}} = \nabla\mathbf{E}^\top \in \mathbb{R}^{m\times n}$ be the Jacobian matrix of the residual vector $\mathbf{E}$, such that

$$\mathcal{J}_{ji} = \left[\frac{\partial \mathbf{E}_i}{\partial \Gamma_j}\right] = \left[\frac{\partial C\left(\Gamma; K_i, T_i\right)}{\partial \Gamma_j}\right].$$

Moreover, assume that $\mathcal{H}(E_i) := \nabla\nabla^\top E_i \in \mathbb{R}^{m\times m}$ be the Hessian matrix of each residual $E_i$ with elements $\mathcal{H}_{jk}\left(E_i\right) = \left[\frac{\partial^2 E_i}{\partial\Gamma_j\partial\Gamma_k}\right]$. According to the nonlinear least-square expressions (4.1) and (4.2), one can easily write the gradient and Hessian of the objective function $\varXi$ as follows

$$\nabla\varXi = \mathcal{J}\mathbf{E},$$
$$\nabla\nabla^\top\varXi = \mathcal{J}\mathcal{J}^\top + \sum_{i=1}^{n} E_i\mathcal{H}(E_i). \tag{4.3}$$

Now, we use the LM method for calibration of the RSDH model. The LM method is a common tool for solving nonlinear least square models such as Eq. (4.1). This algorithm was introduced in 1944 and developed in the early 1960's to optimize nonlinear least squares problems. The LM algorithm is an iterative scheme that locates the minimum of a several variables function expressed under the form of the sum of squares of non-linear real-valued functions (see [48]). It became a standard technique for non-linear least-squares problems, widely used and analyzed in diverse fields related to applied mathematics. LM algorithm can be seen as a combination of two well-known numerical minimization methods: the Steepest descent and the Gauss–Newton method. When the current solution is far from the optimal one, the scheme behaves like a steepest descent method: not fast, but converges. Otherwise, it becomes a Gauss–Newton method. The search step of this algorithm is given by

$$\varDelta\Gamma = (\mathcal{J}\mathcal{J}^\top + \varpi\mathcal{I})^{-1}\nabla\varXi, \tag{4.4}$$

where $\varpi$ and $\mathcal{I}$ are the damping factor and the identity matrix, respectively. Using adjustment parameter $\varpi$, when the objective value in a specified step is far from optimal value, then the Gauss–Newton and the steepest descent methods are applied. To this end, a large amount must be added to $\varpi$, so that the Hessian matrix is dominated by a diagonal matrix such an identity matrix and

$$\nabla\nabla^\top\varXi \approx \varpi\mathcal{I}.$$

If the target and optimal values are close to each other at a given step, with the Gauss–Newton approximation, $\varpi$ is given a miniature quantity to dominate the Hessian matrix. In this case, we have

$$\nabla\nabla^\top\varXi \approx \mathcal{J}\mathcal{J}^\top, \tag{4.5}$$

which vanishes the second part in Eq. (4.3). The approximation expression in Eq. (4.5) is trustworthy when $E_i$ or $\mathcal{H}(E_i)$ is little. The first case occurs when we are faced with a so-called small residual problem, and the second occurs when $\varXi$ is almost linear. Notice that this model should produce a small residue around the optimal range, since otherwise it is an unsuitable model.

Algorithm 1 provides LM method to calibrate the RSDH model. In lines 1–5 of this Algorithm, the American put option pricing function is computed. In lines 1–8, the gradient function is computed. The constant parameters $\omega$ and $\zeta_0$ are given in line 1 as default values. Finally, in line 4, we solve a $17 \times 17$ linear system.

Note that the comparison between RSDH model and the DH model without regime changes (standard DH model) is due to the point that the relevant parameters are considered similarly in both models. However, it has been confirmed by many researchers that any financial model needs to be calibrated to compare with market data. Here we also calibrate the option pricing problem under the RSDH model with the American put option market data. Obviously, the estimated parameters of the DH and RSDH models are not the same after calibration. Therefore, we are yet not confident of the efficiency of the RSDH model in the American put option pricing. After determining the parameters from the actual market data, we must evaluate the efficiency of both models. This issue will be demonstrated in the next section. Estimation of the parameters of the RSDH model and the standard DH model are presented in Table 1, which uses the real data of American put option in SPY market with maturity period of August 2019-December 2019. Moreover, underlying price with no-dividend yield is 287.97, for starting day

**Algorithm 1** The RSDH model calibration by using LM algorithm.

1: **Require:** The initial guess $\Gamma_0$, initial damping factor $\varpi_0 = \omega \max\{\text{diag}[\mathcal{J}_0]\}$, tolerance level Tol and $\zeta_0 = 2$
2: Calculate $\|\mathbf{E}(\Gamma_0)\|$ and $\mathcal{J}_0$
3: **for** $i = 0, 1, 2, \ldots$ **do**
4: Obtain values $\Gamma_i$ by solving system (4.4)
5: Calculate $\Gamma_{i+1} = \Gamma_i + \Delta\Gamma_i$ and $\|\mathbf{E}(\Gamma_{i+1})\|$
6: Calculate $\varrho = \Delta\Gamma_i^\top\big(\varpi_k \Delta\Gamma_i + \mathcal{J}_i \mathbf{E}(\Gamma_k)\big)$ and $\varrho' = \|\mathbf{E}(\Gamma_k)\| - \|\mathbf{E}(\Gamma_{k+1})\|$
7: **if** $\varrho > 0$ and $\varrho' > 0$ **then**
8: Calculate $\mathcal{J}_{i+1}, \varpi_{i+1} = \varpi_i, \zeta_{i+1} = \zeta_i$
9: **else**
10: Set $\varpi_i = \varpi_i \zeta_i, \zeta_i = 2\zeta_i$ and repeat from line 4
11: **end if**
12: **if** $\|\mathbf{E}(\Gamma_i)\| \leq$ Tol **then**
13: Break
14: **end if**
15: **end for**

**Table 1**
Estimated parameters.

| Parameters | RSDH model | Standard DH model |
|---|---|---|
| $\eta$ | 0.1883 | 0.2857 |
| $r_1(r)$ | 0.0665 | 0.0560 |
| $r_2$ | 0.0854 | – |
| $\theta_1(\theta)$ | 0.3282 | 0.3372 |
| $\theta_2$ | 0.4167 | – |
| $\beta_1(\beta)$ | 0.1659 | 0.1537 |
| $\beta_2$ | 0.2555 | – |
| $\rho_1$ | −0.7131 | −0.6003 |
| $\rho_2$ | −0.2040 | −0.4543 |
| $\pi_{12}$ | 2.6268 | – |
| $\pi_{21}$ | 3.2150 | – |
| $\upsilon^{(1)}$ | 0.3945 | 0.5152 |
| $\upsilon^{(2)}$ | 0.3105 | 0.1495 |
| $\kappa$ | 5.0041 | 5.0044 |
| $\lambda$ | 7.6296 | 5.9786 |
| $\xi$ | 0.5816 | 0.4454 |
| $\sigma$ | 0.5311 | 0.6346 |

(August, 07, 2019), strike price is 255, bid price is 32.49 and ask price is 32.97. Here, the data set for the SPY option prices are collected from $ivolatility.com$ on a trading day from August, 07, 2019 with maturity period of August 2019-December 2019.

## 5. Numerical results and empirical application

In this section, we show some numerical results for the American put option valuation under the RSDH model and evaluate the effect of model parameters on the American put option prices. As a comparative method, we also provide the numerical results for option prices under the standard DH model. In the following section, considering the standard DH model as a criterion, we evaluate the performance of the proposed RSDH model through experimental application. To do this, the dataset is provided with several suitable filters. Ultimately, the empirical results are introduced and analyzed.

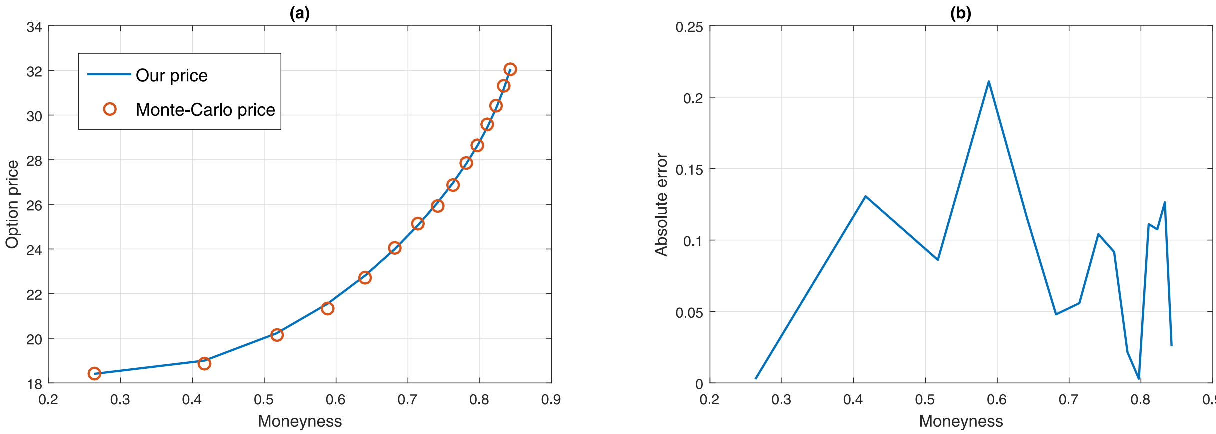


**Fig. 1.** Comparison of the proposed price and Monte-Carlo price (a) with various moneyness ($\frac{\text{underlying price}}{\text{strike price}}$); Absolute error between the proposed price and Monte-Carlo price (b).

### *5.1. Numerical results*

In this subsection, we investigate the impact of the RS mechanism on the American option pricing. Before examining the model features, the proposed formula must be accredited to ensure that no algebraic errors are inducted in the extraction process. In the following, unless otherwise stated, the consumed wealth parameter $\eta$ is equal to 0.27. The mean-reversion speed parameter of both volatilities, that is, $\kappa$ and $\lambda$, are set to 5.2 and 6.1, respectively. The volatility of volatility in both models, that is, $\sigma$ and $\xi$, are set to 0.3, 0.5, respectively. The interest rate levels of the asset process $S$ in state $e_1$ and state $e_2$, that is, $r_1$ and $r_2$, are set to 0.05 and 0.08, respectively. The mean-reversion rate parameter of the first volatility process $V^{(1)}$ in regime 1 and regime 2, that is, $\theta_1$ and $\theta_2$, are set to 0.2 and 0.4, respectively. In regimes 1 and 2, this parameter in the second volatility process $V^{(2)}$, that is, $\beta_1$ and $\beta_2$, are set to 0.1 and 0.2, respectively. The correlation factors $\rho_1$ and $\rho_2$ are assumed to be $-0.6$ and $-0.4$, respectively. Moreover, the time to maturity $\tau$ is 1 year. Both strike price $K$ and underlying price $S_0$ are given by values 55 and 15, respectively. The spot values of the volatilities, that is, $\upsilon^{(1)}$ and $\upsilon^{(2)}$, are set to 0.06 and 0.05, respectively. In order to evaluate the effect of the transition rate on the American put option price, we consider the transition rate matrix as $\Pi = \begin{pmatrix} -\pi & \pi \\ \pi & -\pi \end{pmatrix}$ and set $\pi = 0.05$ (a similar idea for European option can be found in [56]).

Fig. 1 shows a comparison between the option price obtained by the proposed semi-analytical formula and the naive Monte-Carlo simulation method with 1000 trajectories and 50 time steps. As can be seen from the figure, the American put prices obtained by these two methods are very close to each other, so that the obtained maximum absolute error is 0.217 (see Fig. 1(b)). This means that the extracted semi-analytical formula in practice is accurate and can be applied in pricing problems. It should be noted that the numerical results presented in this figure are based on the fact that the current state of Markov chain is in the regime 1. We notice that when the state of Markov chain is in regimes 1 and 2, we have $Z_t = (1, 0)^\top$ and $Z_t = (0, 1)^\top$, respectively.

Acknowledging the proposed pricing formula, the impact of RS on the American put option prices can now be examined. Given that transition rates in both regimes are the same, the option prices with various transition rates are plotted in Fig. 2. As shown in Fig. 2, for different $\pi$, the option price in state $e_2$ is higher than state $e_1$ and with increasing $\pi$, which the probability of the chain $Z$ transiting between two states will increase, the prices in both states converge. It can be inferred from this figure when the transition rates is small and the translation probability of $Z$ from one state to another state is low, the American put option in state $e_2$ is more expensive than in state $e_1$. It is clear that when the transition probability $Z$ from state $e_1$ to state $e_2$ and vice versa is high, the option price in both states should be close to each other. Therefore, for large transition rate values, the option price behaves similarly in both states.

To further study the efficacy of the parameters $\theta$ and $\beta$ under the first regimes in both volatility processes, we fixed the parameters $\theta_2$, $\beta_2$ and changed the parameters $\theta := \theta_1$, $\beta := \beta_1$. Also to evaluate the efficacy of the

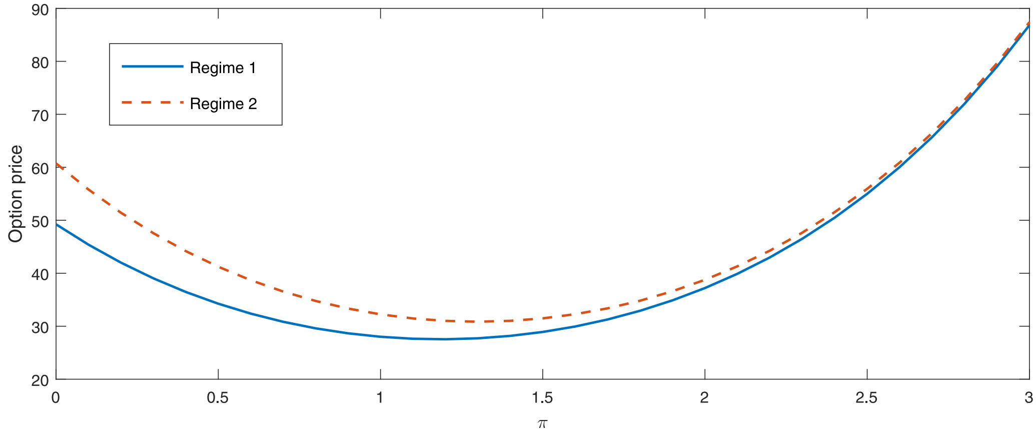


**Fig. 2.** American put option prices corresponding to different $\pi$.

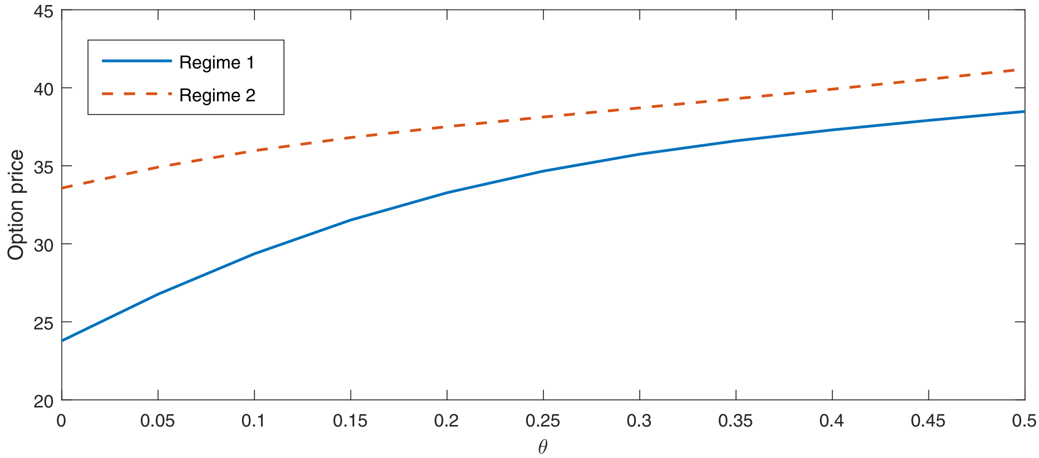


**Fig. 3.** American put option prices corresponding to different $\theta$.

parameter $\theta$ and $\beta$ under second regime in both volatility processes, we fixed the parameters $\theta_1$, $\beta_1$ and changed the parameters $\theta := \theta_2$, $\beta := \beta_2$. The American put option prices with various mean-reversion level parameters $\theta_1(\theta_2)$ and $\beta_1(\beta_2)$ in regime 1 (regime 2) are provided in Figs. 3 and 4, respectively. From these figures, when we compare the effect of the mean-reversion level parameters of the both volatilities in both regimes, it is observed that the value of the option price in state $e_2$ is higher than state $e_1$. According to this figures, we find that when the mean-reversion parameters $\theta$ and $\beta$ in the both volatilities are in state $e_2$, the option price are greater than the corresponding values in state $e_1$. Moreover, the impact of volatility of volatility parameters $\sigma$ and $\xi$ on the American put option price is provided in Fig. 5.

Table 2 shows the effect of the consumed wealth parameter $\eta$ and strike price $K$ on the value of American put option. According to the results obtained in this table, with increasing the value of parameters $\eta$ and $K$, the American put option price increases and its value becomes more expensive than European put options. Note that, when the consumed wealth value is zero, the American and European put options are exactly equal.

### 5.2. Empirical studies

In this subsection, to illustrate the practical usefulness of the proposed RS model, we provide an empirical application of the RSDH model. More precisely, we discuss the accuracy of fitting the SPY American put option data using the semi-analytical formula provided by the RSDH model compared to other conventional models. Particularly,

**Table 2**
$\dfrac{\text{American put option prices}}{\text{(American put option price} - \text{European put option price)\%}}$ with various $\eta$ and $K$.

| $K$ \ $\eta$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|
| 60 | 35.47 | 35.48 | 35.53 | 35.65 | 36.86 | 36.23 | 36.87 | 37.94 | 39.57 |
| | 0.0% | 1.53% | 6.79% | 18.03% | 39.03% | 76.15% | 140.16% | 247.59% | 410.08% |
| 65 | 37.67 | 37.71 | 37.80 | 37.95 | 38.20 | 38.64 | 39.38 | 40.59 | 42.41 |
| | 0.0% | 3.92% | 12.13% | 27.14% | 53.06% | 96.86% | 170.31% | 291.60% | 473.95% |
| 70 | 39.84 | 39.90 | 40.02 | 40.20 | 40.51 | 41.02 | 41.85 | 43.20 | 45.23 |
| | 0.0% | 6.34% | 17.57% | 36.40% | 67.33% | 117.92% | 200.96% | 336.40% | 539.14% |
| 75 | 41.97 | 42.05 | 42.20 | 42.42 | 42.78 | 43.36 | 44.29 | 45.78 | 48.02 |
| | 0.0% | 8.80% | 23.08% | 45.79% | 81.80% | 139.28% | 232.08% | 381.91% | 605.51% |
| 80 | 44.05 | 44.17 | 44.34 | 44.61 | 45.02 | 45.66 | 46.69 | 48.33 | 50.78 |
| | 0.0% | 11.30% | 28.67% | 55.31% | 96.45% | 160.92% | 263.61% | 428.08% | 672.92% |
| 85 | 46.11 | 46.24 | 46.45 | 46.76 | 47.22 | 47.93 | 49.06 | 50.85 | 53.52 |
| | 0.0% | 13.82% | 34.31% | 64.92% | 111.27% | 182.80% | 295.52% | 474.83% | 741.48% |
| 90 | 48.13 | 48.29 | 48.53 | 48.87 | 49.39 | 50.18 | 51.40 | 53.35 | 56.24 |
| | 0.0% | 16.36% | 40.01% | 74.64% | 126.24% | 204.91% | 327.77% | 522.13% | 810.91% |
| 95 | 50.12 | 50.31 | 50.57 | 50.96 | 51.53 | 52.39 | 53.72 | 55.82 | 58.93 |
| | 0.0% | 18.93% | 45.76% | 84.44% | 141.35% | 227.23% | 360.33% | 569.94% | 881.23% |
| 100 | 52.08 | 52.29 | 52.59 | 53.02 | 53.64 | 54.57 | 56.01 | 58.26 | 61.60 |
| | 0.0% | 21.51% | 51.56% | 94.32% | 156.58% | 249.73% | 393.18% | 618.20% | 952.36% |

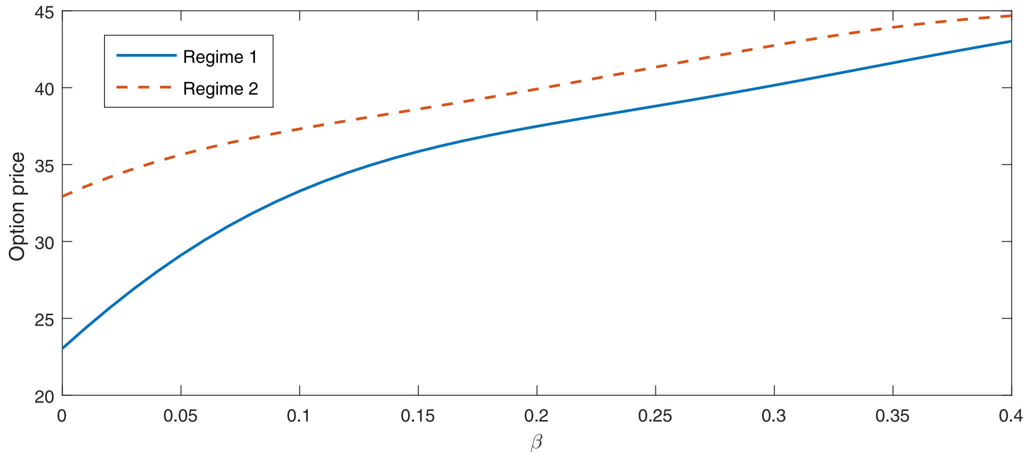


**Fig. 4.** American put option prices corresponding to different $\beta$.

we compare the BS, DH and RSDH models by evaluating out-of-sample prediction error (OSPE) and in-sample fitting error (ISFE), such that the parameters of each model are calibrated according to the SPY American put option data. As in numerical examples, we only consider a two regime case for illustration. Here, we consider the American put prices as the arithmetic average of the option prices in states $e_1$ and $e_2$, which were calculated from Eq. (3.20). It is necessary to mention that the chosen data to measure the ISFE and the OSPE are propounded from the SPY American put option prices with maturity period of August 2019-December 2019 and January 2020, respectively.

The calibrated parameters in the evaluation of OSPE are based on the estimated parameters with the corresponding in-sample data. We consider the root mean square error (RMSE) as a proxy for errors. The RMSE between the SPY American put option prices in market and the prices obtained by BS, DH and RSDH models in each of the in-sample and out-of-sample situations are reported in Table 3. According to this table, the RSDH model has the best performance, that is, the proposed pricing formula in this study has a lowest RMSE in both ISFE and OSPE. We note that the parameters used for evaluating ISFE and OSPE are estimated based on the SPY option

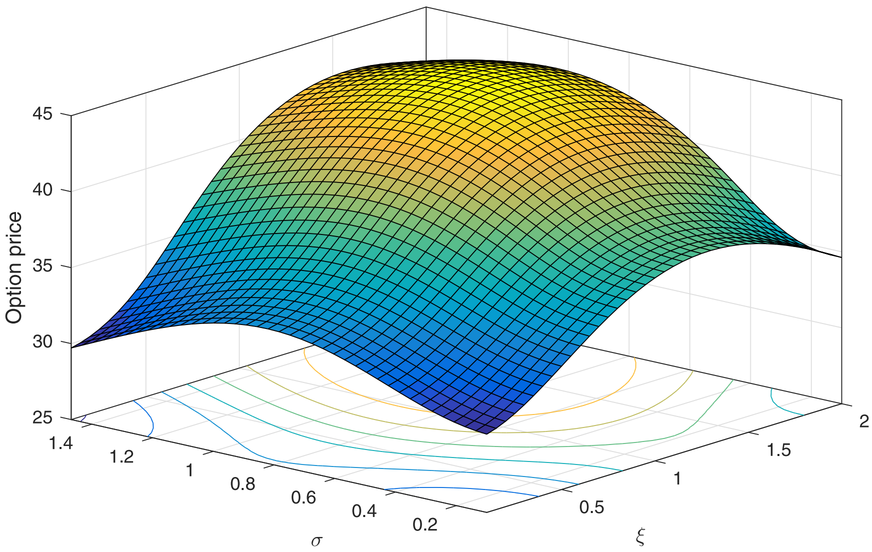


**Fig. 5.** American put option prices with changing $\sigma$ and $\xi$.

**Table 3**
Comparison of ISFE and OFPE for the various models.

| Errors | RSDH model | DH model | BS model |
|---|---|---|---|
| In-sample | 0.1710 | 0.4357 | 0.7811 |
| Out-of-sample | 0.2676 | 0.5767 | 0.8297 |

data with maturity period of August 2019-December 2019. The estimated parameters of the RSDH and DH models are reported in Table 1. Moreover, the interest rate and volatility parameters in the BS model are obtained as 0.0041 and 0.3102, respectively.

Although the ISFE acts almost identically on the BS and DH models, but the OSPE of the DH model is almost twice that of the RSDH model. The OSPE of the BS model are more than seven times of the RSDH model. As a result, the RSDH model presents considerable experimental betterments to existing models, containing the DH and BS models.

## 6. Conclusion

This paper suggests an American put options pricing under the DH model with the Markov-switching mechanism. After solving the moment-generating function of the underlying price dynamics and its two volatilities based on the standard portfolio-consumption, we derive a semi-analytical formula for American put options under the RSDH model. We also minimized the loss function using the LM method, which is an efficient and fast algorithm in optimization, then estimated all available parameters. Numerical experiments were performed to show the performance of the proposed semi-analytical formula and the different features of the American put option prices under the RSDH model. The results show that adding the RS factor to the model has a notable effect on the American put option price. Eventually, through an experimental study for a test data set, we showed that the proposed RS model performs better than the DH and BS models.

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