

# Joint Calibration Thesis

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# 1. Estimation with GARCH Models

In our paper we will be taking a look at two different GARCH models and how they are represented under both  $\mathbb{P}$  and  $\mathbb{Q}$  measures. The two models we will take a closer look is the Heston-Nandi GARCH(1, 1) model and the Duan (1995) model.

## 1.1. GARCH Processes

Before we are able to look, we need the general picture of the GARCH process where under the physical measure  $\mathbb{P}$ , we assume the underlying stock price process follows a conditional distribution  $D$  where we can express the log returns  $R_t$  at time period  $t$  as:

$$\begin{aligned} R_t \equiv \ln\left(\frac{S_t}{S_{t-1}}\right) &= \mu_t - \gamma_t + \varepsilon_t & \varepsilon_t \mid F_{t-1} &\sim D(0, h_t) \\ &= \mu_t - \gamma_t + \sqrt{h_t} z_t & z_t \mid F_{t-1} &\sim D(0, 1) \end{aligned} \quad (1)$$

$$\varepsilon_t = \sqrt{h_t} z_t$$

- $S_t$ : Stock price at time  $t$
- $h_t$ : Conditional variance of the log return in period  $t$

## 1.2. Parameter Definitions

- $\omega$ : Long term average variance constant
- $\alpha$ : Coefficient for lagged innovation
- $\beta$ : Coefficient for lagged variance
- $\gamma$ : The Asymmetry coefficient
- $\lambda$ : Price of risk or risk premium
- $r$ : Risk-free rate or discount rate

Relationships:

- $\alpha$  and  $\beta$  ensures model's stationarity ( $\alpha + \beta < 1$ )
- $\gamma > 0$  indicates negative shocks have a larger impact on future volatility.

## 1.3. Risk Neutralization

- $\mathbb{P}$ : Physical Measure
- $\mathbb{Q}$ : Risk-Neutral Measure

Using the Radon-Nikodym derivative, we can convert the physical measure to the risk-neutral measure. Let  $z_t$  i.i.d  $N(0, 1)$ , then  $\gamma_t = \frac{1}{2}h_t$  since  $\exp(\gamma_t) = E_{t-1}[\exp(\varepsilon_t)]$

The Radon-Nikodym derivative is defined as:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \mid F_t = \exp\left(-\sum_{i=1}^t \left(\frac{\mu_i - r_i}{h_i} \varepsilon_i + \frac{1}{2} \left(\frac{\mu_i - r_i}{h_i}\right)^2 h_i\right)\right) \quad (2)$$

We also get under the RN-Derivative:

Defined by  $\varepsilon_t \mid F_{t-1} \sim N(-(\mu_t - r_t), h_t)$

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = r_t - \frac{1}{2}h_t + \varepsilon_t^* \quad (3)$$

with  $\varepsilon_t^* \mid F_{t-1} \sim N(0, h_t)$  and  $E^Q\left[\frac{S_t}{S_{t-1}} \mid F_{t-1}\right] = \exp(r_t)$

### 1.3.1. Heston-Nandi GARCH

$\mathbb{P}$ :

$$\begin{aligned} R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r + \lambda h_t + \varepsilon_t \\ h_t &= \omega + \beta h_{t-1} + \alpha(\varepsilon_{t-1} - c\sqrt{h_{t-1}})^2 \end{aligned} \quad (4)$$

Assume:

- $r_t = r$
- $\mu_t = r + \left(\lambda + \frac{1}{2}\right)h_t$

To risk neutralize Equation 4 we substitute it along with the assumptions stated into Equation 2 to get the corresponding RN-Derivative for the Heston-Nandi Model:

$$\frac{dQ}{dP} \mid F_t = \exp\left(-\sum_{i=1}^t \left(\left(\lambda + \frac{1}{2}\right)\varepsilon_i + \frac{1}{2}\left(\lambda + \frac{1}{2}\right)^2 h_i\right)\right) \quad (5)$$

Risk-neutral innovations of the form:

$$\varepsilon_t^* = \varepsilon_t + \lambda h_t + \frac{1}{2}h_t$$

$\mathbb{Q}$ :

$$\begin{aligned} R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r - \frac{1}{2}h_t + \varepsilon_t^* \\ h_t &= \omega + \beta h_{t-1} + \alpha\left(z_{t-1}^* - \left(\gamma + \lambda + \frac{1}{2}\right)\sqrt{h_{t-1}}\right)^2 \end{aligned} \quad (6)$$

Using  $z_t^* \stackrel{Q}{\sim} N(0, 1)$ ,  $\varepsilon_t^* = \sqrt{h_t}z_t^*$  and  $\rho^* = \gamma + \lambda + \frac{1}{2}$  into Equation 6 we get Equation 7:

$$\begin{aligned} R_t &= \left(r - \frac{1}{2}h_t\right) + \sqrt{h_t}z_t^* \\ h_t &= \omega + \beta h_{t-1} + \alpha\left(z_{t-1}^* - \rho^*\sqrt{h_{t-1}}\right)^2 \end{aligned} \quad (7)$$

### 1.3.2. Duan (1995)

$\mathbb{P}$ :

$$\begin{aligned} R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r_t + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \varepsilon_t \\ h_t &= \omega + \beta h_{t-1} + \alpha \varepsilon_{t-1}^2 \end{aligned} \quad (8)$$

Assume:

- Price of risk  $\lambda$  is assumed to be constant
- $r_t = r$
- $\mu_t = r + \lambda\sqrt{h_t}$  or  $\lambda = \frac{\mu_t - r}{\sqrt{h_t}}$

To risk neutralize Equation 8 we substitute it along with the assumptions stated into Equation 2 to get the corresponding RN-Derivative for the Duan (1995) Model:

$$\frac{dQ}{dP} \mid F_t = \exp\left(-\sum_{i=1}^t \left(\frac{\varepsilon_i}{\sqrt{h_i}}\lambda + \frac{1}{2}\lambda^2\right)\right) \quad (9)$$

Risk-neutral innovations of the form:

$$\varepsilon_t^* = \varepsilon_t + \mu_t - r_t = \varepsilon_t + \lambda\sqrt{h_t}$$

$\mathbb{Q}$ :

$$\begin{aligned} R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r - \frac{1}{2}h_t + \varepsilon_t^* \\ h_t &= \omega + \beta h_{t-1} + \alpha(\varepsilon_{t-1}^* - \lambda\sqrt{h_{t-1}})^2 \end{aligned} \quad (10)$$

Let  $\varepsilon_t^* = z_t^* \sqrt{h_t}$  with  $z_t^* \stackrel{Q}{\sim} N(0, 1)$ :

$$\begin{aligned} R_t &= r - \frac{1}{2}h_t + z_t^* \sqrt{h_t} \\ h_t &= \omega + \beta h_{t-1} + \alpha(\sqrt{h_{t-1}}(z_{t-1}^* - \lambda))^2 \end{aligned} \quad (11)$$

## **Bibliography**