Joint Calibration Thesis

Mustafif Khan

September 17, 2024

Contents

1. Estimation with GARCH Models

In our paper we will be taking a look at two different GARCH models and how they are represented under both $\mathbb{P}$ and $\mathbb{Q}$ measures. The two models we will take a closer look is the Heston-Nandi GARCH(1, 1) model and the Duan (1995) model.

1.1. GARCH Processes

Before we are able to look, we need the general picture of the GARCH process where under the physical measure $\mathbb{P}$, we assume the underlying stock price process follows a conditional distribution D where we can express the log returns R_t at time period t as:

$$
\begin{aligned}
R_t \equiv \ln\left(\frac{S_t}{S_{t-1}}\right) &= \mu_t - \gamma_t + \varepsilon_t & \varepsilon_t \mid F_{t-1} \sim D(0, h_t) \\
&= \mu_t - \gamma_t + \sqrt{h_t} z_t & z_t \mid F_{t-1} \sim D(0, 1)
\end{aligned}
\tag{1}
$$

$\varepsilon_t = \sqrt{h_t} z_t$

- S_t: Stock price at time t
- h_t: Conditional variance of the log return in period t

1.2. Parameter Definitions

- ω: Long term average variance constant
- α: Coefficient for lagged innovation
- β: Coefficient for lagged variance
- γ: The Asymmetry coefficient
- λ: Price of risk or risk premium
- r: Risk-free rate or discount rate

Relationships:

- α and β ensures model's stationarity ($\alpha + \beta < 1$)
- $\gamma > 0$ indicates negative shocks have a larger impact on future volatility.

1.3. Risk Neutralization

- $\mathbb{P}$: Physical Measure
- $\mathbb{Q}$: Risk-Neutral Measure

Using the Radon-Nikodym derivative, we can convert the physical measure to the risk-neutral measure. Let z_t i.i.d $N(0, 1)$, then $\gamma_t = \frac{1}{2} h_t$ since $\exp(\gamma_t) = E_{t-1}[\exp(\varepsilon_t)]$

The Radon-Nikodym derivative is defined as:

$$
\frac{\mathrm{dQ}}{\mathrm{dP}} \mid F_t = \exp\left(-\sum_{i=1}^{t}\left(\frac{\mu_i - r_i}{h_i}\varepsilon_i + \frac{1}{2}\left(\frac{\mu_i - r_i}{h_i}\right)^2 h_i\right)\right) \tag{2}
$$

We also get under the RN-Derivative:

Defined by $\varepsilon_t \mid F_{t-1} \sim N(-(\mu_t - r_t), h_t)$

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = r_t - \frac{1}{2}h_t + \varepsilon_t^* \tag{3}$$

with $\varepsilon_t^* \mid F_{t-1} \sim N(0, h_t)$ and $E^Q\left[\frac{S_t}{S_{t-1}} \mid F_{t-1}\right] = \exp(r_t)$

1.3.1. Heston-Nandi GARCH

$\mathbb{P}$:

$$\begin{aligned} R_t \equiv \ln\left(\frac{S_t}{S_{t-1}}\right) &= r + \lambda h_t + \varepsilon_t \\ h_t &= \omega + \beta h_{t-1} + \alpha\left(\varepsilon_{t-1} - c\sqrt{h_{t-1}}\right)^2 \end{aligned} \tag{4}$$

Assume:

- $r_t = r$
- $\mu_t = r + \left(\lambda + \frac{1}{2}\right)h_t$

To risk neutralize Equation 4 we subsitute it along with the assumptions stated into Equation 2 to get the corresponding RN-Derivative for the Heston-Nandi Model:

$$\frac{\mathrm{dQ}}{\mathrm{dP}} \mid F_t = \exp\left(-\sum_{i=1}^{t}\left(\left(\lambda + \frac{1}{2}\right)\varepsilon_i + \frac{1}{2}\left(\lambda + \frac{1}{2}\right)^2 h_i\right)\right) \tag{5}$$

Risk-neutral innovations of the form:

$\varepsilon_t^* = \varepsilon_t + \lambda h_t + \frac{1}{2}h_t$

$\mathbb{Q}$:

$$\begin{aligned} R_t \equiv \ln\left(\frac{S_t}{S_{t-1}}\right) &= r - \frac{1}{2}h_t + \varepsilon_t^* \\ h_t &= \omega + \beta h_{t-1} + \alpha\left(z_{t-1}^* - \left(\gamma + \lambda + \frac{1}{2}\right)\sqrt{h_{t-1}}\right)^2 \end{aligned} \tag{6}$$

Using $z_t^* \overset{Q}{\sim} N(0,1)$, $\varepsilon_t^* = \sqrt{h_t}z_t^*$ and $\rho^* = \gamma + \lambda + \frac{1}{2}$ into Equation 6 we get Equation 7:

$$\begin{aligned} R_t &= \left(r - \frac{1}{2}h_t\right) + \sqrt{h_t}z_t^* \\ h_t &= \omega + \beta h_{t-1} + \alpha\left(z_{t-1}^* - \rho^*\sqrt{h_{t-1}}\right)^2 \end{aligned} \tag{7}$$

1.3.2. Duan (1995)

$\mathbb{P}$:

$$\begin{gathered} R_t \equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r_t + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \varepsilon_t \\ h_t = \omega + \beta h_{t-1} + \alpha\varepsilon_{t-1}^2 \end{gathered} \tag{8}$$

Assume:

- Price of risk λ is assumed to be constant
- $r_t = r$
- $\mu_t = r + \lambda\sqrt{h_t}$ or $\lambda = \frac{\mu_t - r}{\sqrt{h_t}}$

To risk neutralize Equation 8 we subsitute it along with the assumptions stated into Equation 2 to get the corresponding RN-Derivative for the Duan (1995) Model:

$$\frac{\mathrm{dQ}}{\mathrm{dP}} \mid F_t = \exp\left(-\sum_{i=1}^{t}\left(\frac{\varepsilon_i}{\sqrt{h_i}}\lambda + \frac{1}{2}\lambda^2\right)\right) \tag{9}$$

Risk-neutral innovations of the form:

$\varepsilon_t^* = \varepsilon_t + \mu_t - r_t = \varepsilon_t + \lambda\sqrt{h_t}$

$\mathbb{Q}$:

$$\begin{gathered} R_t \equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r - \frac{1}{2}h_t + \varepsilon_t^* \\ h_t = \omega + \beta h_{t-1} + \alpha\left(\varepsilon_{t-1}^* - \lambda\sqrt{h_{t-1}}\right)^2 \end{gathered} \tag{10}$$

Let $\varepsilon_t^* = z_t^*\sqrt{h_t}$ with $z_t^* \overset{Q}{\sim} N(0,1)$:

$$\begin{gathered} R_t = r - \frac{1}{2}h_t + z_t^*\sqrt{h_t} \\ h_t = \omega + \beta h_{t-1} + \alpha\left(\sqrt{h_{t-1}}(z_{t-1}^* - \lambda)\right) \end{gathered} \tag{11}$$

1.4. Log Likelihoods

1.4.1. Return Log Likelihood

The log likelihood of the return process is calculated under the $\mathbb{P}$ measure.

Let Y_1 represent the returns log likelihood and N be the number of days in the returns sample, we can then compute it as:

$$Y_1 = -\frac{1}{2} \sum_{i=1}^{N} \left\{ \ln(h_i) + \frac{(R_i - \mu_i + \gamma)^2}{h_i} \right\} \tag{12}$$

From the GARCH Process we can deduce the following:

$$\begin{aligned} R_t &= \mu_t - \gamma + \sqrt{h_t} z_t \\ \sqrt{h_t} z_t &= R_t - \mu_t + \gamma \end{aligned} \tag{13}$$

Substituting the above into Equation 12 we get the following:

$$Y_1 = -\frac{1}{2} \sum_{i=1}^{N} \{ \ln(h_i) + z_i^2 \} \qquad\qquad z_i \sim D(0,1) \tag{14}$$

1.4.2. Options Log Likelihood

The log likelihood of the options process is calculated under the $\mathbb{Q}$ measure.

Assume

$$\underbrace{\sigma_{i,t}}_{\text{Imp vol. from market price}} = \underbrace{\sigma_{i,t}\left(C_{i,t}\left(h_t(\xi^*)\right)\right)}_{\text{Imp vol. from GARCH model}} + \varepsilon_{i,t} \tag{15}$$

where $\varepsilon_{i,t} \sim N\left(0, \sigma_\varepsilon^2\right)$

Let Y_2 represent the options log likelihood and M be the number of options, we can then compute it as:

$$Y_2 = -\frac{1}{2} \sum_{i=1}^{M} \left\{ 2\ln(\sigma_\varepsilon) + \frac{\left(\overbrace{\sigma_i}^{\text{Imp vol. on market price}} - \overbrace{\sigma_{\text{imp},i}}^{\text{Impl vol. from models}} \right)^2}{\sigma_\varepsilon^2} \right\} \tag{16}$$

1.4.3. Joint Log Likelihood

$$Y_{\text{joint}} = \frac{N+M}{2N} Y_1 + \frac{N+M}{2M} Y_2 \tag{17}$$

In our Calibration Artificial Neural Network, we will use $-Y_{\text{joint}}$ as our objective function to to solve for the parameters $\theta = (\omega, \alpha, \beta, \gamma, \lambda, \sigma_\varepsilon)$, with the non-negativity constraint on the parameters in θ.