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Estimating and using GARCH models with VIX data for option valuation

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ABSTRACT

This paper uses information on VIX to improve the empirical performance of GARCH models for pricing options on the S&P 500. In pricing multiple cross-sections of options, the models' performance can clearly be improved by extracting daily spot volatilities from the series of VIX rather than by linking spot volatility with different dates by using the series of the underlying's returns. Moreover, in contrast to traditional returns-based Maximum Likelihood Estimation (MLE), a joint MLE with returns and VIX improves option pricing performance, and for NGARCH, joint MLE can yield empirically almost the same out-of-sample option pricing performance as direct calibration does to in-sample options, but without costly computations. Finally, consistently with the existing research, this paper finds that non-affine models clearly outperform affine models.

1. Introduction

The empirical option pricing performance of the GARCH family models has been well studied in the recent literature (see Christoffersen et al., 2012, and references therein). For option valuation, GARCH model parameters are often estimated by the Maximum Likelihood Estimation (MLE) method using return series, Non-linear Least-Squares (NLS) on (multiple) cross-sections of option data, or both returns and option data. However, the MLE approach with returns only does not necessarily yield good estimates for option pricing; therefore, one might prefer to estimate structural parameters directly using information on option price observations (see, e.g., Christoffersen and Jacobs, 2004; Christoffersen et al., 2012). On the other hand, as pointed out, for example, in (Broadie and Detemple, 2004 and Duan and Yeh, 2010), model calibration on option prices over a long period can result in challenging and costly computations, especially if a closed-form analytical solution is not available. Importantly, in most GARCH models, no (semi-closed) solutions are available for option valuation, and especially with non-affine models, option prices can be computed only through Monte Carlo simulation or by using approximations (Duan et al., 2006).

In this paper, we investigate alternative approaches to estimating parameters and to valuating index options using information on the VIX index, aiming in general (i) to improve the option pricing performance of GARCH models and (ii) to reduce the computational burden. Many VIX-related papers consider VIX derivatives (see, for example, Lin and Chang, 2010, and references therein) or volatility-forecasting (see, for example, Poon and Granger, 2003, and references therein), but here the goal is different: *to use information on the VIX index to estimate GARCH models and to improve their performance for pricing multiple cross-sections of options on the S&P 500.*

A few recent papers on continuous time volatility models have focused on estimation procedures using volatility proxies constructed from volatility indices such as VXO and VIX (see, for example, Jones, 2003; Bakshi et al., 2006; Aït-Sahalia and Kimmel, 2007; Duan and Yeh, 2010; Kanniainen, 2011). Aït-Sahalia and Kimmel (2007) provide a maximum likelihood estimator for three continuous time models using VIX as a volatility proxy, yet their closed-form results can be easily applied also to other popular continuous time-stochastic volatility models. Duan and Yeh (2010) also used information from the VIX index jointly with returns on the S&P 500 and introduced a maximum likelihood estimation method for a class of continuous-time stochastic volatility models in a jump diffusion framework. Moreover, Duan and Yeh (2012) developed an estimation method to capture the VIX term-structure jointly with returns. However, these papers did not investigate empirically, based on observations of option prices on the S&P 500, how inclusion of the VIX index in parameter estimation improved the models' option pricing performance. One welcome exception in continuous-time stochastic volatility is Kaeck and Alexander (2012), who estimated several continuous-time models with the Markov chain Monte Carlo using information

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on the VIX term-structure and testing parameter estimates with extensive option data samples.

Most importantly, to our knowledge, the VIX index has not been widely used to estimate or calibrate discrete-time GARCH models, perhaps because the spot volatility state is already inherently available when historical asset returns are used for given GARCH structural parameters. A welcome exception are Hao and Zhang, 2013, who recently proposed a joint likelihood estimation with returns and VIX. However, they used no option data to compare and verify the option pricing performance of different estimation methodologies, and in addition and unlike in this paper, assumed no autocorrelation in disturbances.

As we show in this paper, with GARCH models, we must implement MLE with autoregressive disturbances, because VIX errors (differences between observations and model values) are highly autocorrelated. We show that proper inclusion of information on VIX in MLE can substantially decrease the option pricing error over traditional MLE with returns data only. Even more interestingly, so far in the literature models have been calibrated to option prices over multiple days by linking spot volatility with different dates by using return series. However, the empirical evidence in this paper shows that, in fact, spot volatilities should be extracted from VIX rather than from returns. According to our extensive empirical analysis, using VIX (rather than return series) with multiple cross-sections of options can substantially improve the models' option pricing performance.

Our approaches are based on the fact that VIX approximates the 30-day variance swap rate on the S&P 500 index (see, e.g., Carr and Wu, 2006; Bollerslev et al., 2011).[1] The first approach is to improve calibrations and option valuation to multiple cross-sections of options. When information on option contracts is used over multiple days, spot volatility on different dates has traditionally been 'updated' using the time series of the underlying's returns for given structural parameter values. Instead of calculating daily spot volatilities from returns series, we suggest extracting spot volatilities from lagged values of VIX. For at least two reasons, this VIX-based volatility extraction may work better than the traditional returns-based approach.

First, both VIX and options prices contain forward-looking information, whereas asset returns do not. On the one hand, as VIX approximates the 30-day variance swap rate, which can be interpreted to measure the risk-neutral expectation of integrated variance within the month (see, for example, Carr and Wu, 2006), this approach provides forward-looking parameter estimates under the risk-neutral measure. VIX, on the other hand, can be regarded as the value of a portfolio of options (while at the same it approximates the variance swap rate), representing aggregated information on option contracts Carr and Wu (2006). Second, the returns-based volatility extraction approach is applied under the physical measure, which may pose a problem, if the price of the return risk cannot be identified from the option data.

In the second approach, we aim to improve MLE estimations by incorporating information on VIX into the likelihood function of the bivariate system with autoregressive disturbances. In order to calculate VIX disturbances, we must solve VIX for given structural parameters and return-based filtered conditional spot volatilities. Parameter estimates are then found by maximizing the joint likelihood of returns and VIX. Because VIX represents a portfolio of options, VIX-based parameter estimates can potentially yield better option pricing performance than pure return-based maximum likelihood estimates, and one can reasonable expect that VIX-implied option pricing errors are not far from minima. Compared to the traditional approach of calibrating GARCH models on option price observations, this joint VIX-Returns-MLE definitely saves computation time, especially with non-affine models, which have no analytical formulae for option prices. Instead of using Monte-Carlo methods to repeatedly value a large set of option prices to minimize the option pricing error, VIX-based parameter estimates can be obtained without computationally expensive option valuations.

This paper is organized as follows. In Section 2, we discuss the GARCH models used in the paper and in Section 3 the calibration approaches using information on option data. In addition, we show how conditional spot volatilities can be alternatively extracted from VIX. Section 4 introduces the joint MLE approach with autoregressive disturbances. In Section 5, we describe our data sets and estimate various GARCH models by four methods and examine the option pricing performance of the different models and estimation methods. The final section discusses the results and draws conclusions.

2. Models

In this section, we introduce three widely recognized specifications we employed in this study. We chose a set of models for comparing different estimation and volatility extraction approaches. In particular, for diversified analysis, we chose the GARCH specifications that incorporate volatility asymmetry differently and do not nest each other. The models are the non-affine model by Glosten et al. (1993); the non-affine NGARCH-specification of Engle and Ng (1993); and the affine model originally proposed by Heston and Nandi (2000). Hereafter, the models are referred to as GJR, NGARCH, and HN, respectively.

The HN specification gives rise to a quasi-closed-form solution to European options, which expedites option valuation, whereas GJR and NGARCH are applied with Monte Carlo methods.[2] On the other hand, in some previous papers, non-affine models outperformed affine models (see, e.g.,Hsieh and Ritchken, 2005; Christoffersen et al., 2010a); therefore, GJR and NGARCH serve as interesting benchmarks for HN. The potential advantage of GJR is that under the risk-neutral measure, the price of the return risk and the physical leverage parameter can be identified from option data, whereas with HN and NGARCH only their combination can be estimated. This is important especially in extracting (updating) spot volatilities over a multi-day data set with returns series.

GJR

With GJR, total return dynamics are

$$\begin{aligned} R_{t+1} &\equiv \ln\left(\frac{S_{t+1}}{S_t}\right) = r + \lambda\sqrt{h_{t+1}} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1}, \\ h_{t+1} &= \beta_0 + h_t\left[\beta_1 + \beta_2 z_t^2 + \beta_3 \max(0, -z_t)^2\right], \end{aligned} \tag{1}$$

where $\beta_0 > 0,\ \beta_1,\ \beta_2,\ \beta_3 \geqslant 0$ for the positive conditional variance and $\lambda > 0$ for the positive equity risk-premium. Moreover, z_t is the iid standard normal random variable. According to Duan's (Duan, 1995) locally risk-neutral pricing framework, total return dynamics can be expressed under the risk-neutral measure as

[1] On September 22, 2003, the CBOE reformulated its VIX index to use the *model-free* implied volatility approach on the S&P 500 and created a historical record for a changed S&P 500 VIX dating back to 1990. The old index, based on the Black–Scholes model (and hence not model-free), was renamed VXO. Specifically, VXO is an average of Black–Scholes implied volatility quotes on eight near-the-money options at two nearby maturities on the S&P 100. For further information, see CBOE Documentation 2003. Note that this study uses VIX, not VXO.

[2] Heston and Nandi (2000) provide a closed-form solution to the characteristic function for future asset prices, but for European option pricing, we must rely on the inversion of this characteristic function, which involves numerical integration.

$$R_{t+1} = r - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\tilde{z}_{t+1},$$
$$h_{t+1} = \beta_0 + h_t\left[\beta_1 + \beta_2(\tilde{z}_t - \lambda)^2 + \beta_3 \max(0, -\tilde{z}_t + \lambda)^2\right]. \quad (2)$$

The variance is weak stationary under the physical and risk-neutral measures if $\Psi := \beta_1 + \beta_2 + \beta_3/2 < 1$ and if $\tilde{\Psi} := \beta_1 + [\beta_2 + \beta_3 N(\lambda)](1+\lambda^2) + \beta_3\lambda n(\lambda)$, where $N(\cdot)$ and $n(\cdot)$ denote the standard normal cumulative and density distribution functions, respectively (see also Duan et al., 2006). Here Ψ and $\tilde{\Psi}$ denote the volatility persistence under the physical and risk-neutral measures, respectively.

NGARCH

The second model we employed was NGARCH, introduced by Engle and Ng (1993), under which the dynamics are as follows:

$$R_{t+1} = r + \lambda\sqrt{h_{t+1}} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1},$$
$$h_{t+1} = \beta_0 + h_t\left[\beta_1 + \beta_2(z_t - \beta_3)^2\right], \quad (3)$$

where $\beta_0 > 0,\ \beta_1,\ \beta_2 \geqslant 0$ for the positive conditional variance and $\lambda > 0$ for the positive equity risk-premium. The leverage effect is modeled with β_3, which shifts the innovation function. Note that NGARCH assumes the same risk-return relation as GJR, but its leverage effect is built differently. Under the risk-neutral measure, the dynamics are

$$R_{t+1} = r - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\tilde{z}_{t+1},$$
$$h_{t+1} = \beta_0 + h_t\left[\beta_1 + \beta_2(\tilde{z}_t - \tilde{\beta}_3)^2\right], \quad (4)$$

where $\tilde{\beta}_3 \equiv \beta_3 + \lambda$. In contrast to GJR, β_3 and λ do not appear separately here, which implies that only their combination can be identified under the risk-neutral measure. The variance is weak stationary under the physical and risk-neutral measures if $\Psi := \beta_1 + \beta_2(1+\beta_3^2) < 1$ and if $\tilde{\Psi} := \beta_1 + \beta_2\left(1+\tilde{\beta}_3{}^2\right) < 1$, respectively.

Heston–Nandi

Under HN, the dynamics of total returns and volatility on the underlying stock are

$$R_{t+1} = r + \gamma h_{t+1} + \sqrt{h_{t+1}}z_{t+1},$$
$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2\left(z_t - \beta_3\sqrt{h_t}\right)^2, \quad (5)$$

where h_t is time-varying squared volatility, $\beta_0 > 0,\ \beta_1,\ \beta_2 \geqslant 0$ to ensure that the conditional variance remains positive and that $\gamma > -\frac{1}{2}$ for the positive risk-return trade-off. Notice that HN assumes a linear relation between returns and *squared* volatility. The total return dynamics can be expressed under the risk-neutral measure as

$$R_{t+1} = r - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\tilde{z}_{t+1},$$
$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2\left(\tilde{z}_t - \tilde{\beta}_3\sqrt{h_t}\right)^2, \quad (6)$$

where $\tilde{z} \sim N(0,1)$ under the risk-neutral measure and $\tilde{\beta}_3 \equiv \beta_3 + \gamma + \frac{1}{2}$. The variance is weak stationary under the physical and risk-neutral measures if $\Psi := \beta_1 + \beta_2\beta_3^2 < 1$, and if $\tilde{\Psi} := \beta_1 + \beta_2\tilde{\beta}_3^2 < 1$, respectively. Again, the leverage effect is built differently than in the two other models. The main advantage of this characterization is that it permits a quasi-closed-form solution for pricing vanilla European options (see Heston and Nandi, 2000), whereas the two other models require Monte-Carlo methods or analytical approximations such as in Duan et al. (2006). Similarly to NGARCH, the price of the return risk and the leverage parameter appear only combined under the risk-neutral measure and cannot thus be identified separately from option prices.

Apart from the GARCH models used in our paper, several alternative models for describing volatility dynamics and pricing volatility derivatives have been proposed recently. For example, in (Zhu and Zhang, 2007), risk-neutral dynamics for instantaneous variance from the forward variance curve is derived. Sepp (2008) proposes a square root model for variance dynamics is proposed assuming compound Poisson jumps in variance, and provides closed-form pricing formulas for volatility derivatives on implied and realized variance of S&P 500 index. Aït-Sahalia and Mancini (2008) analyze the forecasts of quadratic variation based on several stochastic volatility models, including Heston, jump-diffusion models with jumps in variance, log-volatility models, etc. Moreover, Egloff et al. (2010) suggest that models with two risk factors are necessary in capturing the empirical features of variance swap rates. Specifically, they use another square root process to model the mean of the instantaneous variance. Gatheral (2008) discusses the performance of several two factor models concluding that the double lognormal model and double CEV model perform well according to empirical studies about pricing derivatives on index returns, risk-neutral and realized variance. Recently, Bakshi et al. (forthcoming) developed a new model framework to describe the volatility tail behavior and incorporates the investors' heterogeneity in beliefs about volatility. Based on the empirical tests using data of VIX options, this model has better performance compared with a model with opposite assumptions in the left tail of the volatility distribution. In our future research, we will be interested in investigating these alternatives in pricing index and volatility derivatives.

3. Calibration methods on option prices

This section deals with approaches to calibrate GARCH models on option prices. Calibration is based on nonlinear least-squares estimation (NLS) using information on exchange-traded option prices to find parameters that minimize the error between prices given by the model and those observed in the market. Bakshi et al. (1997), among others, employ NLS using loss functions to minimize the pricing error of the daily cross-section of options. Since their work, a wealth of literature has emerged on the evaluation of stochastic volatility models using empirical information on option prices (see, e.g., Chernov and Ghysels, 2000; Christoffersen and Jacobs, 2004; Huang and Wu, 2004; Barone-Adesi et al., 2008; Bams et al., 2009; Christoffersen et al., 2012). The question we specifically consider in this paper is how to filter conditional spot variance to price cross-sections of options on different days.

3.1. Nonlinear least-squares with option prices and returns (Options-Returns-NLS)

The existing papers on option pricing with GARCH specifications seem to use *returns* to filter conditional spot volatility. To demonstrate this method, we follow Heston and Nandi (2000) and Christoffersen and Jacobs (2004) and set initial spot volatility at time zero to equal unconditional volatility 250 days before the first date included in the sample. If (and because) the data set consists of data spanning several periods, spot volatility at the beginning of each day, i.e., h_t, cannot be assumed to be some predetermined constant but can be extracted as "observable" using the volatility updating rule (see also Christoffersen and Jacobs, 2004, On-Line Appendix). For example, for GJR, the volatility updating rule can be expressed as

$$h_{t+1} = \beta_0 + h_t\left[\beta_1 + \beta_2 z_t^2 + \beta_3 \max(0, -z_t)^2\right],$$
$$z_t = \left[R_t - \left(r + \lambda\sqrt{h_t} - \frac{1}{2}h_t\right)\right] \Big/ \sqrt{h_t}. \quad (7)$$

By iterating the above expression for $t = 1, 2, \ldots, n$, we obtain a returns-based proxy for spot variances $\{h_t^R;\ t = 1, 2, \ldots, n\}$ for the given structural parameters θ. With NGARCH or HN, the leverage parameter, β_3, and the price of the return risk, λ or γ, cannot, however, be separately identified under the risk-neutral probability measure, because only their combination can be estimated. Therefore, with NGARCH and HN, using the volatility updating rule practically requires that we assume a zero market price for the return risk, i.e., that we calculate z_t by ignoring the terms of $\lambda\sqrt{h_t}$ and γh_t, respectively (see Christoffersen and Jacobs, 2004, On-Line Appendix). Overall, even though this approach primarily uses option data, also the time series of stock returns is required to link volatility on different dates. Consequently, this approach is called "Option-Returns-NLS".

As Broadie et al. (2007) argue, minimizing the error between model and market option dollar-prices places a greater weight on expensive in-the-money and long-maturity options, whereas the implied volatility metric provides an intuitive weighting of options across strikes and maturities. Because our option samples include also in-the-money options, it is not reasonable to use the pure dollar price error as a loss function. Implied volatility errors, on the other hand, can be approximated by scaling dollar prices by Black (1976) vegas without extensive computations (see Carr and Wu, 2007; Trolle and Schwart, 2009; Christoffersen et al., 2012). This linear vega-approximation of implied volatility is applied also in this paper:

$$\mathcal{V}\mathrm{RMSE}(\theta) = 100 \times \sqrt{\frac{1}{N_M}\sum\nolimits_{t,i}\left(\frac{c_t(h_t^R;\theta) - \hat{c}_{i,t}}{\widehat{\mathcal{V}}_t}\right)^2}, \tag{8}$$

which is minimized with respect to the structural parameters θ. Here $c_{i,t}$ is the price of the ith option at time t given by the model, $\hat{c}_{t,i}$ is the corresponding price of the option observed in the market, h_t^R is a return-implied proxy for spot variance at time t, and θ denotes the structural parameters. $\widehat{\mathcal{V}}_{t,i}$ denotes (Black, 1976) vega computed at the true market prices of options. Moreover, $N_M = \sum_{t=1}^M N_t$, where N_t is the number of option prices in the sample at time t, and M is the total number of days in the sample.

The Options-Returns-NLS and Returns-MLE approaches differ mainly in that the former provides option-implied forward-looking estimates under the risk-neutral measure, whereas the latter is performed under the physical measure with return data. To calibrate a model with the nonlinear least-squares approach using option data, a large set of option contracts must be valued repeatedly to minimize the pricing error. If (quasi-) closed-form solutions are available for option prices, computation is not necessarily a major problem. However, if a GARCH specification requires Monte Carlo methods, the Option-Returns-NLS approach becomes computationally very demanding compared to Returns-MLE.

3.2. Nonlinear least-squares with option prices and VIX (Options-VIX-NLS)

As described in the previous section, the Options-Returns-NLS approach minimizes the option pricing error over multiple days of data linking the spot volatility on different dates by using the time series of the underlying's returns with the volatility updating rule. We suggest here another procedure to make spot volatility observable for a time-series of cross-sections of options: instead of calculating spot volatilities from return series with the volatility updating rule, daily spot volatilities can be extracted from the series of VIX.

Since VIX approximates the 30-day variance swap rate on the S&P 500 index, it can be interpreted to measure the risk-neutral expectation of integrated variance within a month (see, e.g., Carr and Wu, 2006; Bollerslev et al., 2011). In discrete time and in the absence of jumps, this can be written as

$$\frac{1}{\tau}\left(\frac{\mathrm{VIX}_t}{100}\right)^2 \cong \frac{1}{T}\tilde{\mathbb{E}}_t\sum_{j=1}^{T} h_{t+j},$$

where $\tilde{\mathbb{E}}(\cdot)$ is an expectation under the risk-neutral measure. The annualizing parameter $\tau = 365$ or $\tau = 252$ depends on whether the calendar or trading day count convention, respectively, is applied. Similarly, for a 30-day VIX, $T = 30$ or $T = 22$ also depends on the choice of day count convention.[3] In this paper, we apply the trading day count convention, because the return likelihood must be estimated with trading day returns.[4] In fact, Hao and Zhang (2013) used a mixed approach: $T = 30$ but $\tau = 252$.

Notice that under GARCH, h_{t+1} is known at time t. It is straightforward to show that the conditional n step prediction under the risk-neutral measure can be expressed as

$$\tilde{\mathbb{E}}_t h_{t+n} = h_{t+1}\tilde{\Psi}^{n-1} + (\beta_0 + \mathbf{1}_{\mathrm{HN}}\beta_2)\sum_{i=1}^{n-1}\tilde{\Psi}^{i-1} = h_{t+1}\tilde{\Psi}^{n-1} + \tilde{h}(1 - \tilde{\Psi}^{n-1}),$$

with

$$\tilde{h} = \frac{\beta_0 + \mathbf{1}_{\mathrm{HN}}\beta_2}{1 - \tilde{\Psi}},$$

where $\mathbf{1}_{\mathrm{HN}}$ stands for the indicator function of HN, i.e., $\mathbf{1}_{\mathrm{HN}} = 1$ when HN is used and zero otherwise. Notice that $\tilde{h}$ is, in fact, the unconditional variance under the risk-neutral measure; therefore, the expected variance $\tilde{\mathbb{E}}_t h_{t+n}$ can be expressed as a linear combination of the conditional spot variance h_{t+1} and the unconditional variance $\tilde{h}$, weighted by $\tilde{\Psi}^{n-1}$.

Consequently, for a given spot volatility and structural parameters, VIX_t can be computed from the return series using the relation

$$\begin{aligned}\frac{1}{\tau}\left(\frac{\mathrm{VIX}_t}{100}\right)^2 &= \frac{1}{T}\sum_{j=1}^{T}\left(h_{t+1}\tilde{\Psi}^{j-1} + \tilde{h}\left(1 - \tilde{\Psi}^{j-1}\right)\right)\\ &= h_{t+1}\frac{1 - \tilde{\Psi}^T}{(1 - \tilde{\Psi})T} + \tilde{h}\left(1 - \frac{1 - \tilde{\Psi}^T}{(1 - \tilde{\Psi})T}\right)\end{aligned} \tag{9}$$

It can be shown that for $0 < \Psi < 1,\ (1 - \tilde{\Psi}^T)/((1 - \tilde{\Psi})T)$ is between 1 and 0 and is decreasing in T. Therefore, VIX converges on long term variance as T increases and on spot variance when T decreases to 1 day. Intuitively, the higher T, the more VIX is explained by the unconditional long-term variance $\tilde{h}$ and the less by the conditional spot variance h_{t+1}. Moreover, the larger the $\tilde{\Psi}$, the more VIX is explained by spot variance.

Relation (9) directly implies that a proxy for the conditional variance can be extracted from the series of VIX using the following expression:

$$h_t = \frac{(1 - \tilde{\Psi})T}{1 - \tilde{\Psi}^T}\left[\frac{1}{\tau}\left(\frac{\mathrm{VIX}_{t-1}}{100}\right)^2 - \tilde{h}\left(1 - \frac{1 - \tilde{\Psi}^T}{(1 - \tilde{\Psi})T}\right)\right] \tag{10}$$

Note that expression (10) sets a new condition for the parameter values: because spot variances must be positive for any $t > 0$, the right side of (10) must be positive as well. This means that there is a "critical value" for VIX: if an empirical observation of VIX is less than that of the critical value, then the spot variance turns negative, suggesting that a model specification and/or parameter estimates are invalid.

[3] In addition, because CBOE uses calendar days to calculate VIX (see CBOE's documentation), squared observations on VIX must be multiplied by $(30/365) \times (252/22)$ when the trading day count convention is applied.

[4] We thank the reviewer for pointing this out.

To calibrate the models on option prices, we may use expression (10) for spot volatilities to calculate option values in the loss function (8). Overall, this approach aims to minimize the following expression with respect to θ:

$$\mathcal{V}\mathrm{RMSE}(\theta) = 100 \times \sqrt{\frac{1}{N_M}\sum_{t,i}\left(\frac{c_{t,i}\left(h_t^{\mathrm{VIX}};\theta\right) - \hat{c}_{i,t}}{\widehat{\mathcal{V}}_{t,i}}\right)^2}, \tag{11}$$

where h_t^{VIX} is obtained from VIX_{t-1} with (10). Since the above approach uses joint information on specific option contracts and VIX, it is called "Options-VIX-NLS".

As mentioned in the introduction, Options-VIX-NLS can be expected to work better than Options-Returns-NLS: most importantly, both VIX and option prices contain forward-looking information, whereas asset returns do not. In addition, the returns-based volatility updating procedure requires an estimate of the market price of risk but, at the same time, when using option data, such an estimate and the leverage parameter are not separately available for GJR and HN, a condition that can reduce the reliability of the traditional Option-Returns-NLS approach. We can avoid this problem with Options-VIX-NLS, because there spot volatility is determined under the risk-neutral measure using information on VIX without having separately to identify the price of the return risk.

4. Joint maximum likelihood estimation using information on the VIX index and returns

This section introduces joint maximum likelihood estimation using observations on returns and the VIX index with autoregressive disturbances. Recently, Hao and Zhang (2013) estimated GARCH models with both returns and VIX. However, in our experience, the disturbances (the differences between the market observations and model values of the VIX index) are highly autocorrelated; therefore, in contrast to Hao and Zhang (2013), we implemented MLE with autoregressive disurbances. In this paper, this approach is called "VIX-Returns-MLE".

The joint likelihood on returns and VIX is obtained by maximizing

$$\ln L = \ln L^{\mathrm{T}}(\mathbf{R};\theta) + \ln L^{\mathrm{VIX}}(\mathbf{VIX};\theta,\rho),$$

where $\mathbf{R}$ is the $n \times 1$ vector of returns and $\mathbf{VIX}$ is the $n \times 1$ vector of VIX index, and L^{R} and L^{VIX} are the log-likelihood functions for returns and VIX, respectively. Moreover, ρ is an autoregressive parameter introduced below.

To obtain a returns likelihood, we maximize the log-likelihood function based on given observations of returns $\{R_t;\ t = 1,2,\ldots,n\}$ and the initial spot volatility, h_1. For GJR and NGARCH, we maximize the following log-likelihood function

$$\ln L^{\mathrm{T}}(\mathbf{R};\theta) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\left\{\ln h_t - \frac{\left(R_t - r - \lambda\sqrt{h_t} + \frac{1}{2}h_t\right)^2}{h_t}\right\} \tag{12}$$

and for HN

$$\ln L^{\mathrm{T}}(\mathbf{R};\theta) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\left\{\ln h_t - \frac{(R_t - r - \gamma h_t)^2}{h_t}\right\}. \tag{13}$$

For the given parameters $\theta = \{\beta_0,\beta_1,\beta_2,\beta_3,\lambda\}$ or $\theta = \{\beta_0,\beta_1,\beta_2,\beta_3,\gamma\}$, the spot variance $\{h_t; t = 2,3,\ldots,n\}$ is calculated from $\{R_t; t = 1,2,\ldots,n\}$ according to return dynamics. To make the parameter estimates consistent with the observed long-term variance in MLE under the physical measure, we used the relation $\beta_0 = h(1-\Psi) - \mathbf{1}_{\mathrm{HN}}$, where h is the unconditional variance under the physical measure, calculated directly from preceding daily return series (see Christoffersen et al., 2010a, for a discussion about the relation between long-term variance and persistence).

With the likelihood function on VIX, we use the following model to describe autoregressive disturbances,

$$u_t = \rho u_{t-1} + e_t,$$

where $e_t \sim NID(0,\sigma^2)$ and

$$u_t = \mathrm{VIX}_t^{\mathrm{Market}} - \mathrm{VIX}_t^{\mathrm{Model}}(h_{t-1};\theta),$$

where $\mathrm{VIX}_t^{\mathrm{Model}}(h_{t-1};\theta)$ is obtained with Eq. (9). In fact, calculating u_t requires data on both returns and VIX, because spot volatilities are filtered from returns using the volatility updating rule (see Eq. (7)); therefore, strictly speaking, the likelihood function on VIX should be denoted as $\ln L^{\mathrm{VIX}}(\mathbf{VIX},\mathbf{R};\theta,\rho)$.

In the existing literature, Hao and Zhang (2013) used an additive error model without autoregressive disturbances. On the other hand, a multiplicative error model data has recently been applied by Kaeck and Alexander (2012) for the VIX term structure with maturities of 1 and 12 months, and by Duan and Yeh (2012) with maturities of 1, 3, and 6 months. Moreover, Kaeck and Alexander (2012) used autoregressive disturbances for continuous time models and the MCMC method to estimate parameters. After comparing option pricing performance between additive and multiplicative error models (both with autoregressive disturbances), we ended up using an additive model (the difference was not large, but we preferred an additive error model).

We follow Beach and MacKinnon (1979) in applying MLE with autoregressive disturbances with the VIX index. Let us assume that u_t is a normally distributed random variable with a mean zero and a contemporaneous variance Σ. We may then obtain that (see Beach and MacKinnon, 1979)

$$\begin{aligned}\ln L^{\mathrm{VIX}}(\mathbf{VIX};\theta,\rho) = &-\frac{n}{2}\left(\ln(2\pi) + \ln(\Sigma(1-\rho^2))\right)\\ &+\frac{1}{2}\left(\ln(\Sigma(1-\rho^2)) - \ln(\Sigma)\right)\\ &-\frac{1}{2\Sigma}\left(u_1^2 + \sum_{t=2}^{n}\frac{(u_t-\rho u_{t-1})^2}{1-\rho^2}\right).\end{aligned}$$

This could be easily extended to a multivariate case (Beach and MacKinnon, 1979). We return to the information on VIX term structure in further research.

5. Empirical analysis

5.1. Data and setup

This study used returns data, reformulated model-free daily VIX data, and option data from January 3, 1996 through December 31, 2010, covering thus 15 years. According to Kaeck and Alexander (2012), calculating VIX indices from volatility surface can lessen systematic biases from CBOE's methodology; therefore, instead of using CBOE's data, we calculated VIX from the volatility surface provided by OptionMetrics, a widely used high-quality source. Similarly, we used daily return data on the S&P 500 from the above period. In particular, the total return version of the index was used to take dividends into account. Both the return and VIX series have 3,759 daily observations available for study.

Moreover, to empirically study the models' option pricing performance, we used S&P 500 call and put option data from 1996 through 2010, obtained from OptionMetrics. OptionMetrics' data files comprise date, expiration date, call or put identifier, strike price, best bid and best offer, implied volatility, interest rates,[5]

[5] Zero curves were derived from the BBA LIBOR rates and settlement prices of CME Eurodollar futures (for more, see OptionMetrics File and Data Reference Manual).

and future dividend estimates based on put-call parity. Because calibrating GJR and NGARCH with option data is expensive and time-consuming, to lessen this computational burden and yet to use a time series of cross-sections of option prices, we used Wednesday's and Thursday's call and put option data. In particular and like Christoffersen et al. (2010b), we included Wednesdays as in-sample data and Thursdays as out-sample data. This decision was justified by these days being less likely than others such as Monday or Friday to be affected by day-of-the-week effects (see Christoffersen et al., 2010b). In fact, in the last 15 years, several studies have used only Wednesday's option data (see, e.g.,Dumas et al., 1998; Heston and Nandi, 2000; Christoffersen and Jacobs, 2004; Christoffersen et al., 2010a). Overall, our data samples are the following:

- *Sample A*: January 1996–December 2009, Wednesdays, Calls and Puts.
- *Sample B*: January 1996–December 2009, Thursdays, Calls and Puts.
- *Sample C*: January 2010–December 2010, Wednesdays and Thursdays, Calls and Puts.

Sample A is in-Sample Data (used for calibration), and samples B and C served as out-of-sample data sets to verify option pricing performance with data on a different weekday (B) and a following period (C). Moreover, option pricing performance was evaluated by using calls or puts solely in the samples.

Following Bakshi et al. (1997), quotes less than \$3/8 or not satisfying standard no-arbitrage conditions were filtered out. Furthermore, options with maturities longer than 365 calendar days were not included. Finally, to use data on liquid options only, options with a daily volume of less than 100 were filtered out. Even after these criteria for selecting option data, 28,096 call option and 40,432 put option contracts remained in Sample A. The corresponding numbers for Sample B were 28,118 and 40,210 and for Sample C 7463 and 11,885, respectively. Computationally, they were thus considerably large samples. To further lighten the computational burden on data, distributed computing and variance reduction techniques, including empirical martingale methods by Duan and Simonato (1998) and antithetic variables, were used. Tables 1–3 show the properties of option data divided into different maturity and moneyness bins with moneyness defined as S/K. The average bid-ask spread was about 5% of the average mid-price in both samples. In most contracts, times to expiration were less than 60 days. Importantly, there were more deep-out-of-the-money[6] puts than calls available in the market, which was reflected in our samples. The difference is, in fact, substantial as there are 7877 calls and 18,961 puts in this category in Sample A. This clearly affected the results between call and put options but also represented the actual markets.

To obtain parameter optima with option data, all three models on option Sample A were calibrated multiple times using Options-VIX-NLS approaches for a 30-day VIX and Options-Returns-NLS. Consequently, 6 sets of parameters were calibrated from option data (Table 4). Second, we applied joint MLE by using time-series data on returns and VIX from 1996 to 2009. This procedure was somewhat sensitive to initial parameter values; therefore, to avoid local minima, all models were estimated 500 times with randomized initial values when Returns-MLE or VIX-Returns-MLE approaches were used, and results with the highest loglikelihood were reported. In fact, these results were achieved repeatedly from different initial values, which partly verified their validity. Third, to study the option pricing performance of the different estimation approaches and different models, option $\mathcal{V}$RMSEs were calculated and compared using all three option samples for each set of parameter estimates.

Some restrictions were used in calibration: $\beta_0,\ \beta_1,\ \beta_2,\ \beta_3$, and λ were restricted to be non-negative and for HN, $\gamma \geqslant -\frac{1}{2}$. In addition, persistence, $\tilde{\Psi}$, was restricted to be less than one. Loss functions were minimized using the Nelder–Mead simplex algorithm, a derivative-free method for unconstrained multivariable function minimization, as implemented in the MATLAB `fminsearch` code MathWorks (2010). The same optimization algorithm was used with Monte Carlo simulations at least by Barone-Adesi et al. (2008). Options were priced by simulating 10,000 paths with the empirical martingale methods of Duan and Simonato (1998) by also adding antithetic variables.

5.2. Parameter estimates and empirical results

Table 4 reports NLS estimates on in-sample option prices using returns- and VIX-based volatility extraction procedures (i.e., Options-Returns-NLS and Options-VIX-NLS approaches) and 5 shows Maximum Likelihood estimate returns and the VIX index from 1990 through 2009. In addition to parameter estimates and standard errors, the tables show volatility persistences and annualized standard deviations implied by the models. Notice that with the Options-NLS and VIX-Returns-MLE approaches, persistences and standard deviations cannot be determined under the physical measure for NGARCH and HN, because estimates of the market price of risk and the leverage parameter are not separately available.

Parameter estimates were roughly of the same order of magnitude across different approaches, and especially those of Options-Returns-NLS and Options-VIX-NLS differed only little. The estimates were precise in general, but an important exception was GJR's β_2, which was poorly estimated and close to zero, indicating that volatility responds only to negative return shocks.[7] All models and estimation approaches clearly indicate the leverage effect.

Importantly, the estimates of ρ indicate that VIX errors (u_t) are highly autocorrelated in MLE. This is in contrast to Kaeck and Alexander (2012), who found that for continuous-time stochastic volatility models (with or without jumps and the second stochastic volatility factor) the autocorrelation coefficient is close to zero for 30-day VIX and shows only weak autocorrelation in the error term.[8] This difference between their and our findings stems from different models, different estimation methods, and the filtering of spot volatility. In particular, the properties of model-based spot volatility obviously affect autocorrelation. Under GARCH models, the conditional variance can be made directly observable from returns, whereas in Kaeck and Alexander (2012), Markov chain Monte Carlo (MCMC) is needed, because in diffusion models volatility is a latent variable. Moreover, in our other research with MCMC, we noticed that methods to filter volatility dynamics significantly affect the model-implied VIX. If a method tends to filter a more volatile volatility curve, the model-implied VIX will more be volatile; consequently, daily errors between the model and market VIX can vary widely from day to day, implying a lower autocorrelation coefficient.

In accordance with the existing literature, the implied persistence under the risk-neutral measure is high, from 0.987 to

[6] $S/K < 0.94$ for calls or $S/K \geqslant 1.06$ for puts.

[7] This was confirmed using several statistical packages for the maximum likelihood estimation of the GJR model. They all indicated that β_2 was insignificant for the given data. However, β_2 becomes significant when using long-term data from 1950 and therefore it seems that this property (insignificant β_2) can be identified from the data of the past decades (1990–2010). The size of the data in MLE does not explain this, because β_2 was found to be insignificant with NLS calibrations to a large option data sample, too. Somewhat similar results have been reported for example in Barone-Adesi et al. (2008) with FHS method.

[8] In Kaeck and Alexander (2012), the estimated autocorrelation of the 1-year VIX error for all the 1-factor models is close to 1.

Table 1
Properties of option Sample A. The table shows the number of contracts, the average price (in parentheses), and the average bid-ask spread {in braces} across the moneyness and maturity of call and put option data, reported by dividing the data into three groups based on maturity (calendar days) and six groups based on moneyness (S/K). Sample A covers Wednesdays from January 1996 through December 2009.

Sample A: Calls					Sample A: Puts				
Moneyness	Maturity (days to expiration)				Moneyness	Maturity (days to expiration)			
S/K	<60	60–180	$\geqslant 180$	Total	S/K	<60	60–180	$\geqslant 180$	Total
<0.94	3430	2662	1785	7877	$\geqslant 1.06$	10,146	5827	2988	18,961
	(3.63)	(11.09)	(25.70)	(11.15)		(4.17)	(12.71)	(23.09)	(9.78)
	{0.86}	{1.36}	{1.86}	{1.26}		{0.75}	{1.27}	{1.69}	{1.06}
0.94–0.97	3799	1395	474	5668	1.03–1.06	3959	1435	585	5979
	(6.61)	(22.57)	(57.05)	(14.75)		(8.84)	(27.30)	(47.59)	(17.07)
	{0.88}	{1.71}	{2.31}	{1.20}		{0.98}	{1.81}	{2.13}	{1.29}
0.97–1.00	5177	1827	672	7676	1.00–1.03	5161	1905	778	7844
	(15.29)	(40.38)	(76.41)	(26.61)		(16.25)	(36.50)	(58.41)	(25.35)
	{1.25}	{2.05}	{2.39}	{1.54}		{1.31}	{1.93}	{2.22}	{1.55}
1.00–1.03	3027	979	450	4456	0.97–1.00	3401	1459	604	5464
	(31.33)	(56.11)	(91.90)	(42.89)		(30.31)	(50.12)	(74.51)	(40.49)
	{1.76}	{2.20}	{2.40}	{1.92}		{1.82}	{2.16}	{2.29}	{1.96}
1.03–1.06	846	237	111	1194	0.94–0.97	734	335	212	1281
	(57.07)	(80.46)	(114.32)	(67.03)		(58.65)	(74.61)	(95.29)	(68.89)
	{2.05}	{2.34}	{2.46}	{2.14}		{2.43}	{2.53}	{2.44}	{2.46}
$\geqslant 1.06$	742	322	161	1225	<0.94	414	272	217	903
	(145.10)	(199.28)	(197.06)	(166.17)		(159.23)	(156.40)	(156.87)	(157.81)
	{2.17}	{2.12}	{2.35}	{2.18}		{3.43}	{3.55}	{2.95}	{3.35}
Total	17,021	7422	3653	28,096	Total	23,815	11,233	5384	40,432
	(21.59)	(36.78)	(57.49)	(30.27)		(15.67)	(28.79)	(44.86)	(23.20)
	{1.26}	{1.77}	{2.12}	{1.51}		{1.16}	{1.66}	{1.96}	{1.41}

Table 2
Properties of option Sample B. It covers Thursdays from January 1996 through December 2009.

Sample B: Calls					Sample B: Puts				
Moneyness	Maturity (days to expiration)				Moneyness	Maturity (days to expiration)			
S/K	<60	60–180	$\geqslant 180$	Total	S/K	<60	60–180	$\geqslant 180$	Total
<0.94	3132	2709	1700	7541	$\geqslant 1.06$	9526	6039	2891	18,456
	(3.83)	(11.51)	(25.62)	(11.50)		(4.31)	(12.77)	(23.92)	(10.15)
	{0.89}	{1.43}	{1.89}	{1.31}		{0.77}	{1.32}	{1.77}	{1.11}
0.94–0.97	3764	1471	479	5714	1.03–1.06	3909	1499	570	5978
	(6.31)	(23.24)	(56.11)	(14.84)		(8.58)	(27.10)	(48.13)	(17.00)
	{0.85}	{1.72}	{2.38}	{1.20}		{0.97}	{1.80}	{2.22}	{1.30}
0.97–1.00	5279	1868	607	7754	1.00–1.03	5138	1993	745	7876
	(15.06)	(39.44)	(72.53)	(25.43)		(15.87)	(36.77)	(60.49)	(25.38)
	{1.26}	{2.02}	{2.32}	{1.53}		{1.30}	{1.93}	{2.29}	{1.55}
1.00–1.03	3111	1090	411	4612	0.97–1.00	3502	1533	584	5619
	(31.11)	(55.62)	(90.98)	(42.24)		(29.91)	(49.10)	(73.23)	(39.65)
	{1.83}	{2.21}	{2.58}	{1.98}		{1.84}	{2.07}	{2.27}	{1.94}
1.03–1.06	856	268	139	1263	0.94–0.97	791	350	218	1359
	(57.14)	(80.72)	(112.86)	(68.28)		(57.88)	(76.15)	(98.72)	(69.14)
	{2.12}	{2.53}	{2.62}	{2.26}		{2.47}	{2.52}	{2.48}	{2.48}
$\geqslant 1.06$	746	326	162	1234	<0.94	431	316	175	922
	(139.31)	(198.07)	(174.94)	(159.51)		(149.02)	(154.47)	(160.35)	(153.04)
	{2.28}	{2.52}	{2.57}	{2.38}		{3.70}	{3.55}	{2.99}	{3.52}
Total	16,888	7732	3498	28,118	Total	23,297	11,730	5183	40,210
	(21.60)	(36.98)	(55.99)	(30.11)		(15.92)	(29.14)	(45.15)	(23.54)
	{1.29}	{1.82}	{2.18}	{1.55}		{1.19}	{1.68}	{2.02}	{1.44}

0.999, whereas persistence under the physical measure would show lower values. Notice that if $\tilde{\Psi} = 0.999$, then VIX is largely contributed by spot variance. The implied annualized standard deviations from the Options-NSL estimates were comparable across the models, whereas the Returns-MLE and VIX-Returns-MLE estimates yielded substantially lower standard deviations. Based on loglikelihood values with Returns-MLE estimates, the models ranked from best to worst are GJR, NGARCH, and HN (GJR's loglikelihood value is only slightly higher than that of NGARCH). More importantly, based on VIX-Returns-MLE loglikelihood values, the order is NGARCH, GJR, and HN. NGARCH clearly outperformed GJR, indicating that it is superior in capturing the joint dynamics of VIX and returns. Similarly, the in-sample $\mathcal{V}$RMSEs with Options-Returns-NLS and Options-VIX-NLS estimates favored NGARCH over GJR and GJR over the affine HN model. Overall, in the light of in-sample statistics, especially NGARCH seemed well matched, whereas the in-sample performance of HN was somewhat poor.

Let us now take a closer look at Tables 6–8, which compare option pricing errors ($\mathcal{V}$RMSE) across data samples, option type, estimation approaches, and models. In particular, the tables can be used to answers three questions: (i) which volatility extraction approach, (ii) which estimation approach, and (iii) which GARCH model to use to valuate multiple cross-sections of options. Notice that in the tables, ‘‘Option-NLS'' refers to either ‘‘Options-

Table 3
Properties of option Sample C. It covers Wednesdays and Thursdays from January 2010 through December 2010.

Sample C: Calls					Sample C: Puts				
Moneyness	Maturity (days to expiration)				Moneyness	Maturity (days to expiration)			
S/K	< 60	60–180	⩾ 180	Total	S/K	< 60	60–180	⩾ 180	Total
< 0.94	950	911	477	2338	⩾ 1.06	3826	2575	893	7294
	(2.04)	(7.68)	(21.89)	(8.29)		(3.75)	(11.79)	(31.91)	(10.04)
	{0.72}	{1.51}	{2.66}	{1.42}		{0.88}	{1.62}	{2.70}	{1.37}
0.94–0.97	1055	381	130	1566	1.03–1.06	922	371	112	1405
	(5.12)	(20.82)	(53.58)	(12.97)		(10.37)	(33.94)	(72.20)	(21.52)
	{0.98}	{2.27}	{3.55}	{1.51}		{1.47}	{2.70}	{3.46}	{1.95}
0.97–1.00	1145	444	162	1751	1.00–1.03	1079	551	181	1811
	(14.34)	(38.74)	(72.12)	(25.87)		(18.54)	(44.79)	(86.33)	(33.30)
	{1.71}	{2.83}	{3.69}	{2.17}		{1.93}	{2.90}	{3.89}	{2.42}
1.00–1.03	698	378	165	1241	0.97–1.00	542	274	117	933
	(30.31)	(52.50)	(87.99)	(44.74)		(32.24)	(60.36)	(101.38)	(49.17)
	{2.38}	{2.94}	{3.77}	{2.74}		{2.69}	{3.23}	{3.94}	{3.00}
1.03–1.06	187	66	34	287	0.94–0.97	145	49	30	224
	(54.97)	(75.29)	(104.73)	(65.54)		(58.63)	(84.36)	(119.31)	(72.38)
	{3.03}	{2.84}	{4.07}	{3.11}		{3.86}	{4.02}	{5.55}	{4.12}
⩾ 1.06	159	81	40	280	< 0.94	98	77	43	218
	(117.97)	(122.23)	(169.18)	(126.52)		(160.54)	(193.72)	(202.52)	(180.54)
	{6.13}	{3.45}	{6.69}	{5.43}		{3.76}	{3.62}	{4.59}	{3.87}
Total	4194	2261	1008	7463	Total	6612	3897	1376	11,885
	(17.63)	(29.56)	(53.51)	(26.09)		(12.95)	(26.49)	(55.49)	(22.32)
	{1.64}	{2.25}	{3.33}	{2.05}		{1.39}	{2.08}	{3.15}	{1.82}

Table 4
Parameter estimates from Options-Returns-NLS and Options-VIX-NLS. NLS estimates from option prices using two different volatility extraction approaches (Options-Returns-NLS and Options-VIX-NLS) on Sample A (January 1996–December 2009, Wednesdays). The models were estimated directly by fitting observed option prices in Sample A and using nonlinear least-squares to minimize the squared option-pricing error ($\mathcal{V}$RMSE). Standard errors are reported below each parameter estimate in parentheses. At bottom, the table shows the option-pricing fit ($\mathcal{V}$RMSE, Eq. (8)) at parameter optima. In the table, Options-Returns-NLS refers to the returns-based extraction approach and Options-VIX-NLS to the VIX-based extraction approach.

Model	*GJR*		*NGARCH*		*Heston–Nandi*	
Volatility proxy	h_t^R	h_t^{VIX}	h_t^R	h_t^{VIX}	h_t^R	h_t^{VIX}
β_0	1.349E−06	1.084E−06	1.869E−06	1.131E−06	3.155E−20	1.22E−13
	(4.436E−09)	(6.350E−09)	(5.483E−09)	(6.849E−09)	(2.292E−08)	(4.999E−09)
β_1	0.8920	0.6490	0.8556	0.8229	0.6699	0.2294
	(3.290E−05)	(6.547E−04)	(4.048E−05)	(4.415E−04)	(5.395E−04)	(5.500E−05)
β_2	7.562E−16	5.990E−13	3.290E−02	0.0841	1.528E−06	2.258E−06
	(5.982E−06)	(4.384E−05)	(1.055E−05)	(2.377E−04)	(1.330E−09)	(3.656E−10)
β_3	0.0209	0.0199				
	(9.046E−06)	(3.850E−05)				
λ/γ	1.9665	4.053,539,827				
	(5.565E−04)	(5.540E−04)				
$\tilde{\beta}_3$			1.7757	1.0349	459.9564	579.0795
			(0.0112)	(4.281E−04)	(0.8500)	(0.1877)
$\tilde{\Psi}$	0.994	0.995	0.992	0.997	0.993	0.987
$\sqrt{\bar{h} \times 252}$	0.2339	0.2557	0.2464	0.3134	0.2375	0.2068
Option $\mathcal{V}$RMSE	*4.5461*	*3.3340*	*4.4747*	*3.2815*	*4.9591*	*3.8748*

Returns-NLS'' or ''Options-VIX-NLS'', depending on which volatility extraction approach was used. The italicized numbers represent the best volatility extraction approach for a given model and option sample and the bold number represent the best model for a given volatility extraction approach and option sample.

Which volatility extraction approach to use to valuate multiple cross-sections of options?

First, by looking at the row marked ''Option-NLS'' in Tables 6–8, we can see that option $\mathcal{V}$RMSEs can be optimized substantially lower by using the VIX-based volatility extraction approach rather than the return-based approach, regardless of the underlying model or sample. For example, Table 6, third row, shows that GJR's $\mathcal{V}$RMSE is 4.9012 with the returns-based volatility updating rule (representing the Options-Returns-NLS parameter optima), but if we apply the VIX-based extraction approach instead of the return-based updating rule, $\mathcal{V}$RMSE drops to 3.7986 (representing the Options-VIX-NLS parameter optima). Similarly, when parameter estimates from Returns-MLE or VIX-Returns-MLE were used in Sample A, B, or C with GJR and NGARCH, VIX-based volatility extraction generally yielded substantially lower option $\mathcal{V}$RMSEs than the return-based volatility updating rule.

Overall, in terms of call and put options, option $\mathcal{V}$RMSEs are always lower when the VIX-based volatility extraction approach rather than the return-based approach is used. When

Table 5
Parameter estimates from MLE-Returns and VIX-Return-MLE. Parameter estimates from MLE-Returns and VIX-Returns-MLE using daily data from January 1990 to December 2009 on total returns of the S&P 500 and VIX index. Standard errors are reported below each parameter estimate in parentheses. At bottom, the table shows the loglikelihood at parameter optima.

Model	*GJR*		*NGARCH*		*Heston–Nandi*	
Estimation data	R	R, VIX	R	R, VIX	R	R, VIX
β_0^*	1.512E−07	4.76E−07	2.143E−07	7.383E−07	2.401E−07	8.12E−07
	(2.048E−08)	(2.845E−08)	(2.221E−08)	(2.366E−08)	(6.771E−08)	(2.875E−08)
β_1	0.9498	0.9371	0.9117	0.7819	0.9252	0.7331
	(3.252E−03)	(1.282E−03)	(5.048E−03)	(3.440E−03)	(6.496E−03)	(8.349E−03)
β_2	1.026E−03	1.138E−09	0.0453	0.0264	2.597E−06	1.765E−06
	(4.460E−03)	(8.560E−04)	(3.283E−03)	(5.560E−04)	(1.112E−07)	(5.151E−08)
β_3	0.0861	0.0871	0.8691	2.4728	158.1884	364.0355
	(5.881E−03)	(1.061E−03)	(0.0949)	(0.0526)	(10.3945)	(10.5526)
λ/γ	0.071732	0.22963148	0.0931	0.2130	5.4917	19.5630
	(1.189E−02)	(1.243E−02)	(0.0109)	(0.0084)	(1.1407)	(0.8204)
ρ_{VIX}		0.9340		0.9683		0.9521
		(1.088E−03)		(9.475E−04)		(5.397E−04)
$\tilde{\Psi}$	0.999	0.999	0.999	0.999	0.995	0.993
$\sqrt{h} \times 252$	0.1952	0.3464	0.2324	0.4313	0.3875	0.3141
Log-likelihood	10,985	20,743	10,981	21,422	10,904	20,652

Table 6
Option $\mathcal{V}$RMSEs for different volatility extraction approaches, different parameter estimates and different volatility models. Sample A: Wednesdays from January 1996 through December 2009.

Volatility proxy	(a) Calls and puts		(b) Calls		(c) Puts	
	h_t^R	h_t^{VIX}	h_t^R	h_t^{VIX}	h_t^R	h_t^{VIX}
GJR						
Options-NLS	4.5461	*3.3340*	**4.5092**	*3.6565*	4.5715	*3.0901*
Returns-MLE	6.0687	*4.2011*	6.8224	*5.0664*	5.4844	*3.4751*
VIX-Returns-MLE	5.7121	*3.9408*	6.4845	*4.7013*	5.1071	*3.3110*
NGARCH						
Options-NLS	**4.4747**	***3.2815***	4.5221	***3.6041***	**4.4415**	***3.0371***
Returns-MLE	**5.8102**	***3.9571***	**6.2993**	***4.6891***	**5.4445**	***3.3557***
VIX-Returns-MLE	**5.2185**	***3.4087***	**5.4063**	***3.7516***	**5.0840**	***3.1484***
Heston–Nandi						
Options-NLS	4.9591	*3.8748*	4.8163	*4.4592*	5.0559	*3.4103*
Returns-MLE	10.0161	*6.1150*	12.6391	*7.6178*	7.6831	*4.8012*
VIX-Returns-MLE	7.5848	*5.0132*	8.9868	*6.0038*	6.4331	*4.1891*

calls and puts were analyzed separately, only 3 cases out of 108 had a lower $\mathcal{V}$RMSE than when the return-based approach was used.[9] Therefore, *in-sample and out-of-sample analyses placed VIX-based volatility extraction above the traditional return-based volatility updating rule, regardless of the underlying model or estimation method: Consequently, VIX-based volatility extraction is certainly favored in option pricing*. The greatest difference between the volatility extraction approaches was achieved with the GJR model and Returns-MLE -parameter estimates, in which case the VIX-based extraction approach yielded a 36% lower $\mathcal{V}$RMSE compared to the traditional returns-based volatility updating rule.

Which estimation approach to use to valuate options?

According to Table 6–8, *VIX-Returns-MLE clearly outperformed traditional Returns-MLE in option pricing, regardless of the model, option sample, or volatility extraction approach*. This is not surprising as VIX values are aggregated from option prices; therefore, it is natural that joint likelihood can improve option pricing performance, especially if return data samples do not extend over several decades.

Importantly, in terms of the option pricing performance of the NGARCH model, the VIX-Returns-MLE approach is competitive with direct calibrations to option prices, especially when VIX-based volatility extraction is used (see Tables 6–8, columns "h_t^{VIX}" and rows under NGARCH). In fact, the differences in out-of-sample $\mathcal{V}$RMSEs were negligible. In particular, in Samples A, B, and C (calls and puts), the $\mathcal{V}$RMSE ratios between VIX-Returns-MLE and Options-VIX-NLS for NGARCH were 1.0388, 1.0419, and 1.0796, respectively. The corresponding ratios for GJR and HN were higher, varying from 1.1820 to 1.3600. Therefore, *for NGARCH, VIX-Returns-MLE can have almost the same out-of-sample option pricing performance as Options-NLS but without recourse to computationally expensive calibrations to several cross-sections of option prices.*

Overall, VIX-Returns-MLE can yield better results that the traditional Returns-MLE approach. On the other hand, with NGARCH, VIX-Returns-MLE performed well against direct calibrations to options prices. However, with GJR and HN, neither VIX-Returns-MLE nor Returns-MLE were competitive. Therefore, unless one prefers to run computationally expensive calibrations, one could just as well apply the VIX-Returns-MLE approach with NGARCH.

[9] In particular, the three exceptions appeared in Sample C with MLE estimates, where calls' $\mathcal{V}$RMSE was sometimes lower when the return-based approach was used (see Table 6, third and fourth column). However, VIX-based volatility extraction is always favored when we look at $\mathcal{V}$RMSE values for both calls and puts with these parameter estimates).

Table 7
Option $\mathcal{V}$RMSEs for different volatility extraction approaches, different parameter estimates and different volatility models. Sample B: Thursdays from January 1996 through December 2009.

Volatility proxy	(a) Calls and puts		(b) Calls		(c) Puts	
	h_t^R	h_t^{VIX}	h_t^R	h_t^{VIX}	h_t^R	h_t^{VIX}
GJR						
Options-NLS	**4.9012**	*3.7986*	**4.7041**	*3.9514*	5.0344	*3.6880*
Returns-MLE	6.355	*4.7929*	6.9007	*5.5099*	**5.9438**	*4.2198*
VIX-Returns-MLE	6.0676	*4.5142*	6.6786	*5.1171*	5.6009	*4.0394*
NGARCH						
Options-NLS	4.9092	***3.7510***	4.8821	***3.9104***	**4.9281**	***3.6354***
Returns-MLE	**6.2126**	***4.5374***	**6.5529**	***5.1124***	5.9631	***4.0876***
VIX-Returns-MLE	**5.5531**	***3.9080***	**5.6007**	***4.0819***	**5.5195**	***3.7817***
Heston–Nandi						
Options-NLS	5.4238	*4.4498*	5.2788	*4.947*	5.5229	*4.0662*
Returns-MLE	9.9206	*6.5871*	12.2566	*8.0286*	7.8862	*5.3533*
VIX-Returns-MLE	7.6207	*5.5565*	8.7466	*6.4939*	6.7223	*4.7933*

Table 8
Option $\mathcal{V}$RMSEs for different volatility extraction approaches, different parameter estimates and different volatility models. Sample C: Wednesdays and Thursdays from January 2010 through December 2010.

Volatility proxy	(a) Calls and puts		(b) Calls		(c) Puts	
	h_t^R	h_t^{VIX}	h_t^R	h_t^{VIX}	h_t^R	h_t^{VIX}
GJR						
Options-NLS	5.0032	*3.4598*	4.2239	***3.7629***	5.4357	*3.2551*
Returns-MLE	**5.7885**	*5.1287*	***5.1203***	6.4128	6.1713	*4.1227*
VIX-Returns-MLE	5.4193	*4.7055*	*4.8244*	5.859	5.7615	*3.8065*
NGARCH						
Options-NLS	**4.6869**	***3.3639***	**3.8655**	*3.7678*	5.1360	***3.0834***
Returns-MLE	5.8128	***4.7801***	*5.1469*	***5.8737***	6.1944	***3.9413***
VIX-Returns-MLE	**5.2041**	***3.6315***	**4.8041**	***4.1007***	5.4402	***3.3029***
Heston–Nandi						
Options-NLS	5.0710	*3.9326*	5.2381	*4.4116*	**4.9631**	*3.5994*
Returns-MLE	9.0740	*6.4956*	12.3729	*8.6330*	**6.1572**	*6.6016*
VIX-Returns-MLE	6.5505	*5.2384*	8.3674	*6.6016*	**5.0881**	*4.1600*

Which GARCH model to use to valuate options?

Generally, NGARCH yielded the best results, whereas HN performed poorly. With Samples A and B, NGARCH was clearly superior, especially when spot volatilities were extracted from VIX, regardless of the estimation methodology (see the bold numbers in the tables). With Samples A and B, HN was not successful and performed poorly when call options were priced with Returns-MLE. Tables 1–3 show that the samples contained fewer short-term, out-of-the-money call options than short-term, out-of-the-money put options (and in the markets), which can make $\mathcal{V}$RMSE relatively higher for call options. This feature stands out with a poor model with poor parameter estimates. With Sample C, the results are mixed. There NGARCH outperformed the others in general, but, interestingly, HN priced put options reasonably well with the returns-based volatility extraction approach.

Let us take a closer look at columns "h_t^{VIX}" in Tables 6–8, where the VIX-based spot volatility extraction approach is applied in option valuations for different parameter estimates. In particular, we find that non-affine NGARCH with parameters estimates from VIX-Returns-MLE (that uses jointly VIX and returns for estimation rather than direct option prices) yielded a *better* option pricing performance than the HN model with direct – and computationally expensive – calibrations to option prices.[10]

This result suggests that it is better to use a non-affine GARCH model, and in particular NGARCH, even without direct use of option prices for estimation, rather than the option-calibrated affine HN, whose analytical tractability is no longer a clear advantage in estimating a model for option valuation. Moreover, as Christoffersen et al. (2012) argue, Monte-Carlo pricing is very precise and reasonably fast and in our experience very useful with modern acceleration technologies. Therefore, this paper's empirical results clearly argue for using non-affine characterizations for pricing options.

Overall, based on our analysis, *the best model is the non-affine model NGARCH and the worst the affine HN*. Moreover, as to comparing GJR and NGARCH, when data on VIX and/or options is used, GJR has one extra degree of freedom, which, in fact, clearly favors NGARCH over GJR.[11] The result on the superiority of NGARCH and GJR over the Heston–Nandi is in line with the existing literature that favors non-affine stochastic volatility models (see, e.g. Hsieh and Ritchken, 2005; Christoffersen et al., 2010a,b, 2012; Duan and Yeh, 2010; Kaeck and Alexander, 2012). Recently, Kaeck and Alexander (2012) reported very clear evidence in the favor of non-affine models. In particular, according to their paper, the out-of-sample root mean square error of continuous time affine specifications was approximately 20% higher than with (continuous-time) non-affine models. They also found that inclusion of jumps is less important than allowing for non-affine dynamics and, in fact, jump-augmented non-affine models showed no improvement over simple diffusion models in their out-of-sample tests.

[10] In the second, fourth, and sixth columns in Tables 6–8, the $\mathcal{V}$RMSEs of NGARCH with parameter estimates from VIX-Returns-MLE are lower than the $\mathcal{V}$RMSEs of HN with estimates from Options-NLS.

[11] We thank the reviewer for pointing this out.

Finally, a few words about computational times. On a single laptop PC (Intel Core i5, M540, 4×2.53 GHz, 6 GB RAM), the overall running-time (the wall clock time used) to valuate once the 32,729 options in Sample A using the HN model was 25 s with the HN quasi-closed-form solution and 249 s with the Monte-Carlo methods. Moreover, with cloud-computing with the Techila middleware[12] on 173 Azure extra small virtual machines (173×1 GHz CPU, 768 MB RAM) and the task divided into 775 jobs according to 775 sample dates, the overall wall clock time was 55 s and the total CPU time 44 min and 33 s. The Monte-Carlo running-times were approximately the same for GJR and NGARCH. Substantially shorter wall clock times can be recorded if more workers (virtual machines) are available on the cloud or if the workers are larger (more efficient). Then the wall clock time differs very little between the HN model with the quasi-closed-form solution on a local computer and the HN model or some other GARCH model (such as GJR or NGARCH) with the Monte-Carlo methods on a cloud computing platform. Consequently, with modern acceleration technologies closed-form solutions are no longer a critical requirement for option valuation.

However, even though valuating such an options sample is not very time intensive, calibrating GARCH models on the option prices of such a sample (i.e., 32,729 options) can take a long time, even with a closed-form solution or Monte-Carlo methods on a cloud. The optimization procedure required as many as 200–700 iterations, depending on the initial parameter values, implying that the overall wall clock time was as much as 4 h with a closed-form solution and 8 h with the Monte-Carlo methods on our cloud. On our local laptop, calibrating GJR and NGARCH may take up to 35 h. Compared to direct calibrations on multiple cross-sections of option prices, VIX-Returns-MLE and Returns-MLE are dramatically faster. For example, on our local laptop it took less than a minute to run an estimation procedure using VIX-Returns-MLE with over 700 iterations! Therefore, instead of spending several hours in a modern cloud environment to calibrate a non-affine model on multiple cross-sections of option prices, one can obtain competitive parameter estimates in a few seconds on a local computer by using information on VIX rather than option prices directly.

6. Conclusions

When several cross-sections of options are priced, spot volatilities must be obtained on different dates. According to our empirical analysis, spot volatilities should be extracted from the lagged values of VIX rather than relying on the traditional approach to link volatility to different dates using time-series of stock returns. For samples including call and put options, the VIX-based spot volatility extraction approach yielded 11–38% lower option pricing errors (Vega-RMSE) than the traditional returns-based volatility updating rule.

This paper tested maximum likelihood estimation by using joint information on the VIX index and S&P500 returns for option valuation. In terms of option pricing, this joint likelihood estimation outperforms the traditional approach, which relies only on the underlying's returns. In fact, for the NGARCH model, the VIX-based method can yield competitive parameter estimates with respect to direct calibrations to multiple cross-sections of options over a long period but without costly computations.

In terms of option pricing, both in-sample and out-of-sample analyses favor the non-affine NGARCH models over GJR and clearly over the affine Heston–Nandi characterization, regardless of the spot volatility extraction approach or estimation method. Moreover, the Heston–Nandi yielded also lower likelihood values than the non-affine models, indicating that its performance was poorer also in terms of modeling the dynamics of returns and VIX. This paper included also some additional research on the trade-off between the affine Heston–Nandi model with a quasi-closed-form solution and the non-affine models (especially NGARCH) with Monte-Carlo methods.[13] Our results suggest that compared to the option-calibrated Heston–Nandi model, the non-affine NGARCH model can price options surprisingly accurately even without directly using option prices for estimation. One can well employ an MLE procedure on joint information on VIX and returns without costly computations and yet achieve good option pricing performance with NGARCH. A need for Monte-Carlo methods constitutes a real computational challenge in calibrating the non-affine models to option prices, but because computationally expensive calibration can be bypassed in estimation by using observations of VIX rather than option prices, a closed-form solution is no longer a critical requirement.

If only option data is used to estimate parameters, overfitting is a potential problem, as argued by Christoffersen et al. (2012). In future research, we will extend estimation approaches towards constructing an objective function to combine different data sources, not only returns and VIX but also realized volatility from high-frequency returns data. Moreover, in future research, we will study the estimation and option pricing performance of GARCH models with VIX term-structure data.

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[12] For more information on a platform we used, see www.techila.fi. Moreover, a case study on the use of different computational platforms in volatility simulation can be found at www.hpcfinance.eu.

[13] Christoffersen et al. (2012) contains a good discussion on the trade-off between affine and non-affine models.

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