Making Mathematical Statements

CMEE Maths Week

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1 Functions

A few notes on functions

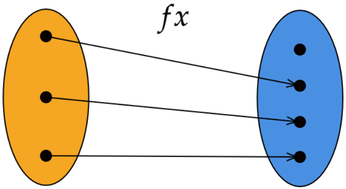
An important property of a function is whether it can be classified within any of the following:

- Injective
- Surjective
- Bijective

This determines whether information is lost in the process of mapping the domain to the codomain (or vice versa), or if that information can be recovered through the function's inverse.

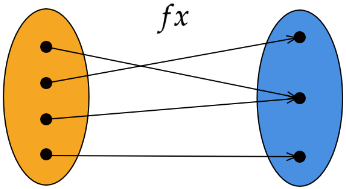
Injectivity

An function is injective if every element of the domain can be uniquely mapped to one element of the codomain.



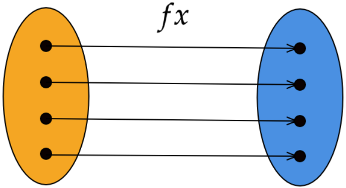
Surjectivity

A function is surjective if each element of the codomain can be mapped from a least one element of the domain.



Bijectivity

A function is bijective if it is both injective and surjective.



Invertibility

- A function is invertible if and only if it is bijective.
- Injective functions can be made invertible (replace codomain with image).
- Injective and surjective functions are reversible, in other words, they have a left and right inverse, respectively.

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Polynomials

The General expression for polynomials is:

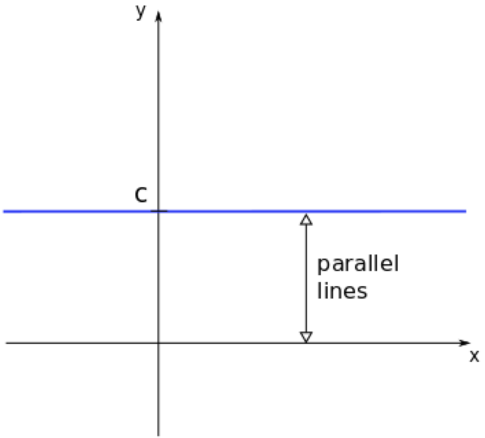
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$$

- n is a non-negative integer.
- The coefficients $\{a_n, a_{n-1}, \ldots, a_0\}$ are in $\mathbb{R}$, can be extended to $\mathbb{C}$.
- n defines the degree of the polynomial, given $a_n \neq 0$.
- The roots of a polynomial are all of the $x \in \mathbb{C}$ such that $f(x) = 0$.

Degree 0

Degree 0 polynomials are simply constant functions.

$$f(x) = a_0$$
$$c = a_0$$



Degree 1

Degree 1 polynomials are linear functions and are given by:

$$f(x) = mx + b$$

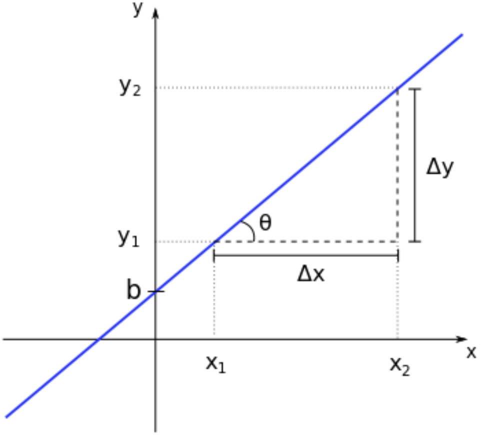
- $m \neq 0$
- m is the slope of the function
- b is the y-intercept

Building a line

Given two points on a line, we can find its slope and intercept.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$b = y_1 - x_1 \frac{\Delta y}{\Delta x}$$



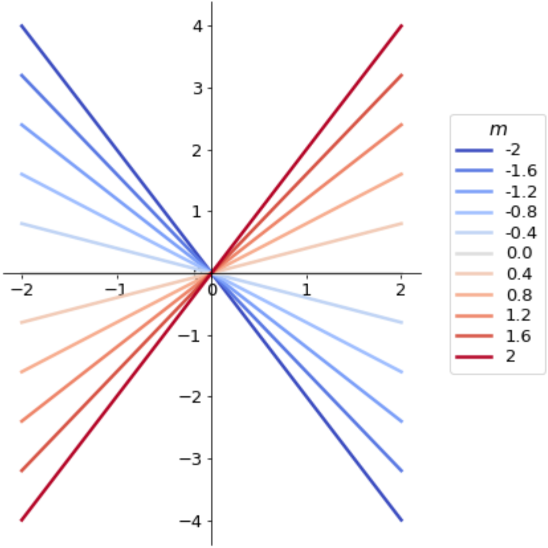
A Line as an Increasing/Decreasing Function

With lines handy, it is useful to define increasing and decreasing functions. Given any x_2 and x_1 such that $x_2 > x_1$:

- If $f(x_2) > f(x_1) \Rightarrow f(x)$ is an increasing function.
- If $f(x_2) < f(x_1) \Rightarrow f(x)$ is a decreasing function.

From here we can see that a line's slope defines whether it is increasing or decreasing.

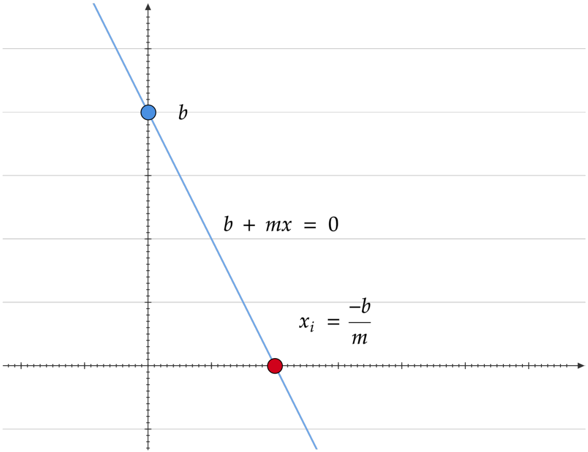
- Values of $m > 0$ offer us **increasing** functions.
- Values of $m < 0$ render **decreasing** functions.



The x-intercept

Assuming both m and b are real numbers,there exists an $x_i \in \mathbb{R}$ for which $f(x_i) = 0$.

$$x_i = \frac{-b}{m}$$



Degree 2

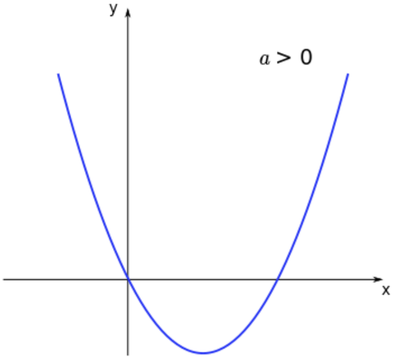
The general equation for degree 2 polynomials is:

$$f(x) = ax^2 + bx + c$$

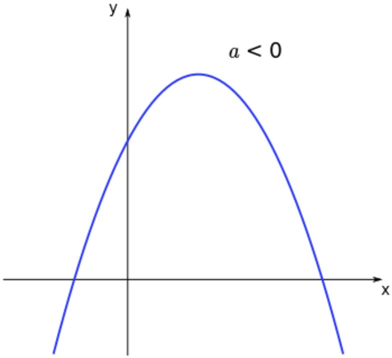
- $a \neq 0$
- Generates a parabola upon graphing.
- Properties are well described.

Concavity

Concave up (positive concavity)



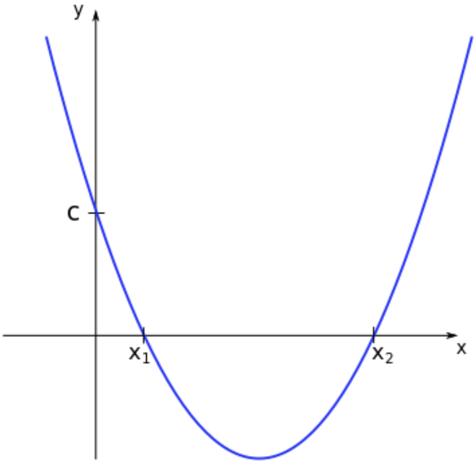
Concave down (negative concavity)



Parabola Intercepts

Evidently, the y-intercept is c. Setting $f(x) = 0$, completing the square, and solving for x gives us the x-intercepts:

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
$$x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$



Symmetry and the Vertex

We can note that the quadratic function is symmetric. This convenient property allows us to determine the parabola's vertex, and therefore, its maxmium/minimum. In general, it is worth noting that:

- Even degree polynomials are symmetric about their vertex.
- Odd degree polynomials are anti-symmetric about their infelction point.

Finding the Vertex

Given the parabola's symmetry, it is straightforward to see that the vertex lies at the midpoint between both x-intercepts.

$$x_v = \frac{x_2 + x_1}{2} = -\frac{b}{2a}$$

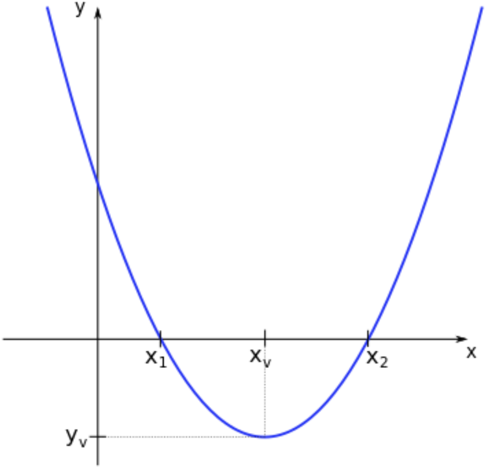
Finding the Max/Min

Having found x_v, we can plug it in to $f(x)$ to find the max/min.

$$y_v = f(x_v) = -\frac{(b^2 - 4ac)}{4a}$$

Additionally:

- If $a > 0$ then (x_v, y_v) is a minimum.
- If $a < 0$ then (x_v, y_v) is a maximum.



The Discriminant

Looking closely at the solutions for the x-intercepts and the vertex, we can note the appearance of $(b^2 - 4ac)$ in both, this value is denominated as the discrimintant:

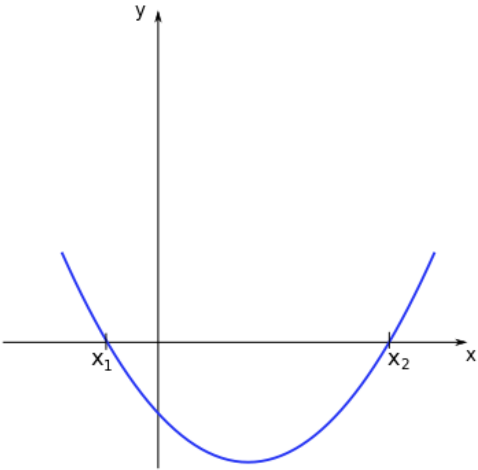
$$\Delta = (b^2 - 4ac)$$

The value of Δ has a few important implications.

Positive Discriminant

If $\Delta > 0$ there exist two real roots.

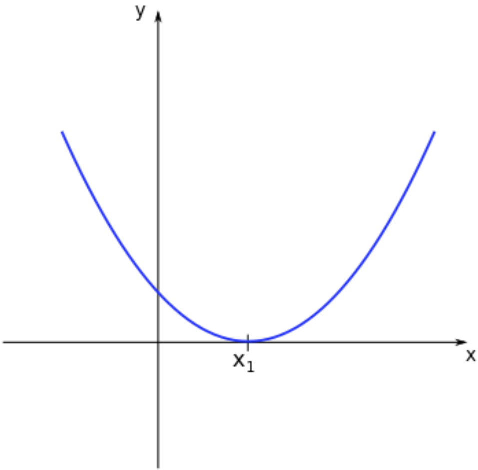
- $\sqrt{\Delta}$ is a non-zero real.
- $y_v < 0$ for $a > 0$.
- $y_v > 0$ for $a < 0$.



Null Discriminant

If $\Delta = 0$ there exists one real root, $x_1 = x_2$.

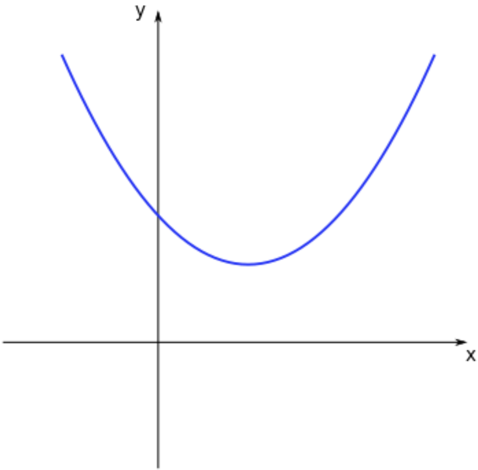
- $\sqrt{\Delta} = 0$.
- $y_v = 0$.



Negative Discriminant

If $\Delta < 0$ both roots are complex, meaning, $f(x)$ never crosses the x-intercept.

- $\sqrt{\Delta}$ is imaginary.
- $y_v > 0$ for $a > 0$.
- $y_v < 0$ for $a < 0$.



Fundamental Theorem of Algebra

Looking at polynomials in general, it can be proven that for $n \geq 1$ the function:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$$

Has at least one, and at most n complex roots. Note that complex roots can arise regardless of whether the coefficients are all real.

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3 Rational Functions

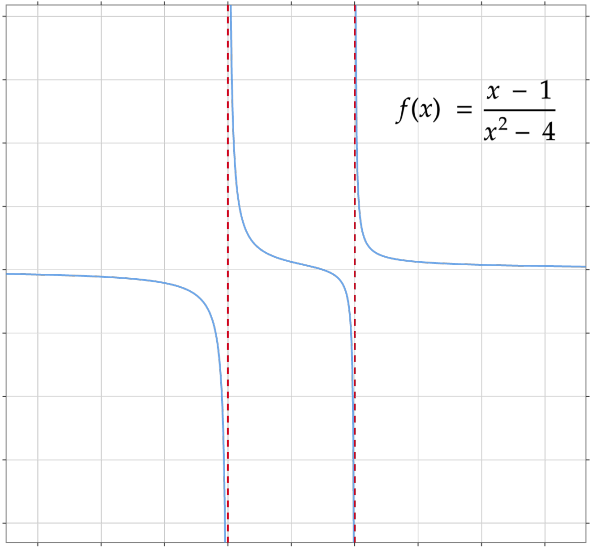
Rational functions are simply quotients of polynomial functions:

$$f(x) = \frac{p(x)}{q(x)}$$

With $p(x)$ and $q(x)$ both being polynomials. Additionally, we need to make sure that $q(x) \neq 0$ for every point in the domain.

Vertical Asymptotes

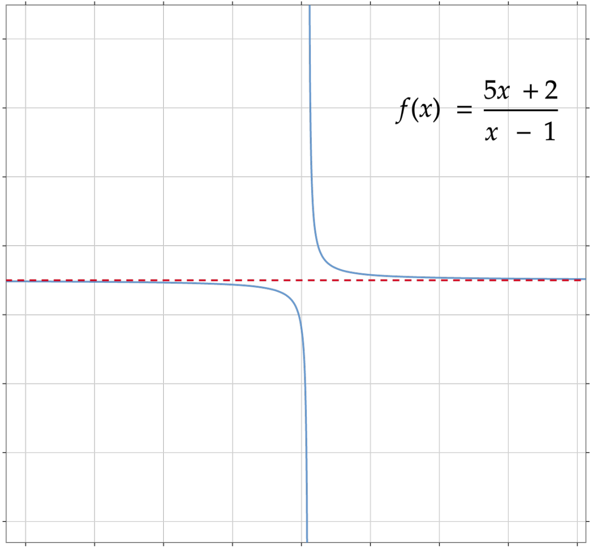
Assuming $p(x)$ shares no common roots with $q(x)$, $f(x)$ approaches a vertical line on both sides of any of $q(x)$'s roots. These are the **vertical asymptotes**.



Horizontal Asymptotes

If $f(x)$ approaches some finite value c as $x \to \pm\infty$, then we say c is a **horizontal asymptote** of $f(x)$.

This means that as $|x|$ becomes very large, $f(x) \approx c$.



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4 Power Functions

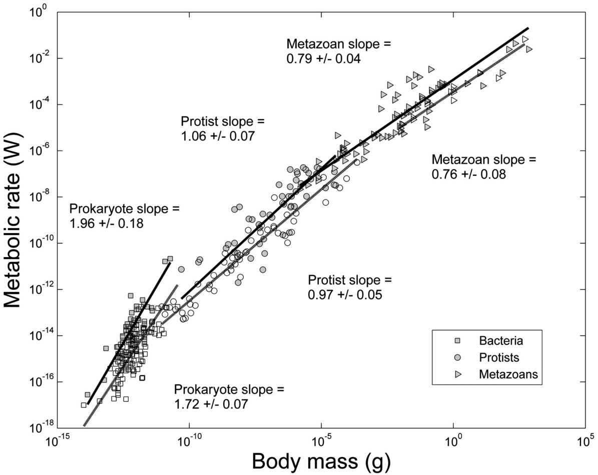
Another series of functions that come up often in biology are power functions or power laws, given by:

$$f(x) = Cx^{\alpha}$$

Where C and α are both real constants and $C \neq 0$.

Power Laws in Biology

Power laws have many applications in biology (e.g. allometric scaling laws, allele fixation, biogeography, etc.), most of which consider $x \geq 0$. In this regime we can make a few observations.

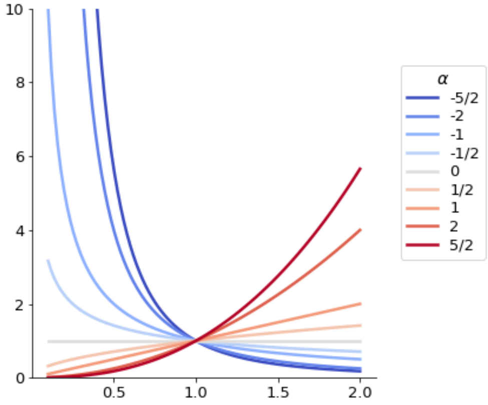


DeLong *et al.*, 2010, PNAS

Power Laws in the Positive Regime

For a given $C > 0$ we can note that:

- If $\alpha > 0$ then $f(x)$ is an increasing function.
- If $\alpha < 0$ then $f(x)$ is a decreasing function.



Scale Invariance

In general, all power laws exhibit **scale invariance**, meaning that the behaviour of the law is functionally identical for every observable scale. For any change of scale $\lambda > 0$:

$$f(\lambda x) = C(\lambda x)^{\alpha} = C\lambda^{\alpha} x^{\alpha} = \lambda^{\alpha} f(x).$$

So modifying your scale by λx is equivalent to changing the response by a factor of λ^{α}.

Universality, as the name suggests, allows us to draw similarities between systems which share functional responses. Namely, two systems with the same functional response usually have analogous organizing principles, despite being governed by a different set of relationships.

While the mathematics behind universality are well beyond the scope of this course, the implications are far-reaching and well worth a read.

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5 Exponential Functions

Exponential Functions

Another function nearly ubiquitous within biology is the exponential function, defined by:

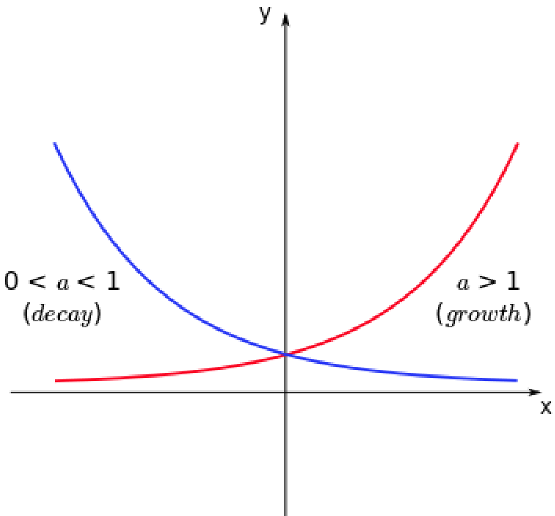
$$f(x) = a^x$$

Where a is a positive real called the **base** and x is the **exponent**. While power functions are somewhat similar to exponentials, their properties are wholly different.

Growth and Decay

Exponential functions are always positive and can display two basic behaviours:

- Growth for $a > 1$.
- Decay for $a < 1$.



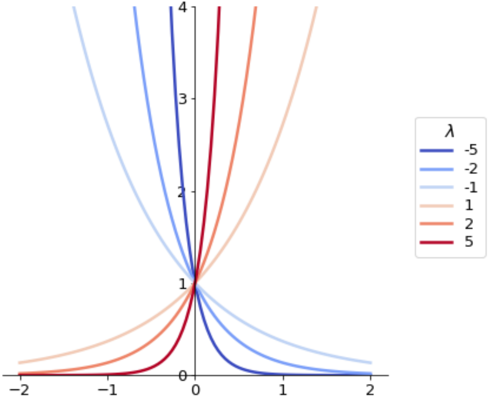
A Common Base

The base, a, can be refactored to display a more general form:

$$f(x) = a^x = (e^\lambda)^x = e^{\lambda x}$$

Here e represents Euler's number and $e^\lambda = a$. We can note that under this new form we have:

- Growth for $\lambda > 0$.
- Decay for $\lambda < 0$.



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6 Logarithmic Functions

Logarithmic Functions

Working backwards from exponentials, we encounter the logarithm, whereby the expression:

$$\log_a b = x \text{ is equivalent to } a^x = b$$

This defines the logarithm as the inverse operation of the exponential, denoted by the function:

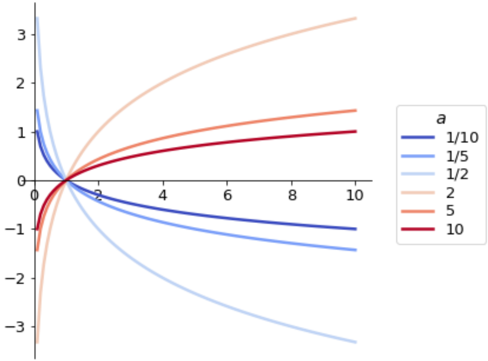
$$f(x) = \log_a x$$

This is the **logarithm of x in base a**, where $a > 0$. We can note that $\log_a x$ is only defined for $x > 0$.

Shape of the Logarithm

Much like exponential functions, the logarithm's behaviour is largely determined by its base.

- $\log_a x$ is an increasing function for $a > 1$.
- $\log_a x$ is an decreasing function for $a < 1$.



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The Exponential as an Injection

The exponential function is a good example of an injective function.

$$\exp(x) : \mathbb{R} \to \mathbb{R}$$

For every $x \in \mathbb{R}$, $\exp(x)$ maps each value of x to a unique element of the codomain. Note that $\exp(x)$ is not surjective in $\mathbb{R}$ because it is never negative valued. If we want $exp(x)$ to be surjective, we need to redefine the codomain:

$$exp(x) : \mathbb{R} \to \mathbb{R}^+$$

Under this region, $exp(x)$ is a bijection and has a well defined inverse function.

Making the Quadratic Invertible

Notice that $f(x) = x^2$ defined as $f : \mathbb{R} \to \mathbb{R}$ is neither injective nor surjective.

- $f(x) = f(-x)$ - Two elements of the domain get mapped to a single element of the codomain.
- $f(x) \geq 0$ - $f(x)$ is not negative valued.

We can make $f(x)$ injective and surjective by redefining its domain and codomain.

Making the Quadratic Invertible

7 Injectivity and Surjectivity Revisited

To make $f(x) = x^2$ injective we simply replace the codomain with $\mathbb{R}^+$, and analogously, to make it surjective we specify that the domain is $\mathbb{R}^+$:

$$f(x) : \mathbb{R}^+ \to \mathbb{R}^+$$

Having redefined both the domain and the codomain, this version of $f(x) = x^2$ is bijective and has a well defined inverse function

$$f^{-1}(x) = \sqrt{x} \quad \text{and} \quad f^{-1}(x) : \mathbb{R}^+ \to \mathbb{R}^+$$