Describing Shapes and Patterns

CMEE Maths Week

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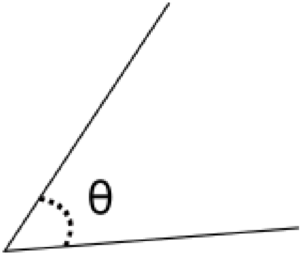
Angles constitute one of the primary elements of Euclidean geometry. We can define them by the **arc** separating two intersecting line segments



Measuring Angles

Angles are usually measured in either **radians** or **degrees**, both units of measurement are equivalent via the following relationship:

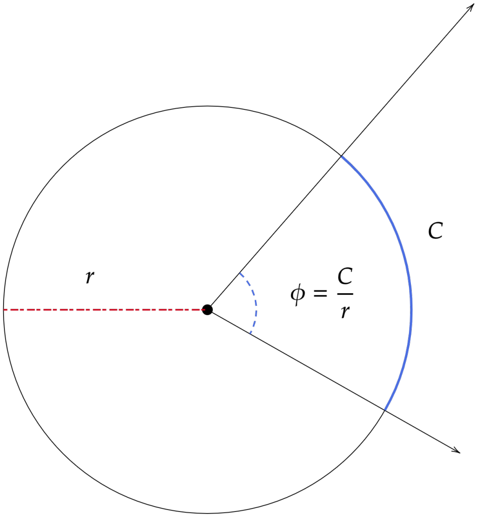
$$\frac{360^\circ}{\theta} = \frac{2\pi}{\phi}$$

Here θ is the angle measured in degrees and ϕ represents the angle in radians.

The Radian as a Fundamental Unit

Draw an arc of arbitrary radius r, between two intersecting lines. The angle in radians is the ratio of that arc's length C relative to the radius of the circumscribing circle.

$$\phi = \frac{C}{r}$$



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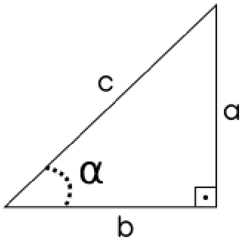
Trigonometric Basics

We define two basic trigonometric relationships. Given a right triangle with **catheti** a and b, and **hypothenuse** c:

$$\sin(\alpha) = \frac{a}{c}$$

$$\cos(\alpha) = \frac{b}{c}$$

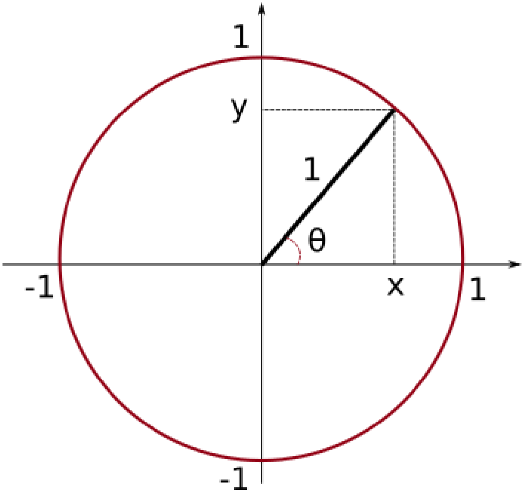
Where α is the angle adjacent to cathetus b.



Trigonometry on the Unit Circle

Considering a circle of radius **1**, we find a relationship between the angle θ, separating the line segment in the **radial** direction and the horizontal axis:

$$y = \sin(\theta)$$
$$x = \cos(\theta)$$



Trigonometric Identities

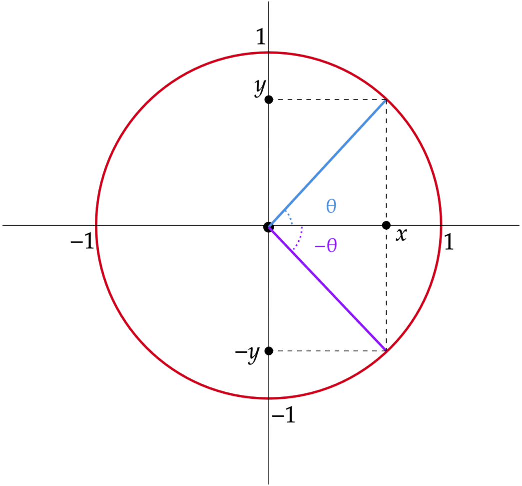
With the aid of the unit circle and **Pythagora's Theorem**, a few properties of *sines* and *cosines* become clear:

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\cos(\theta) = \cos(\theta + 2\pi)$
- $\sin(\theta) = \sin(\theta + 2\pi)$

Negative Angles and Trigonometric Symmetries

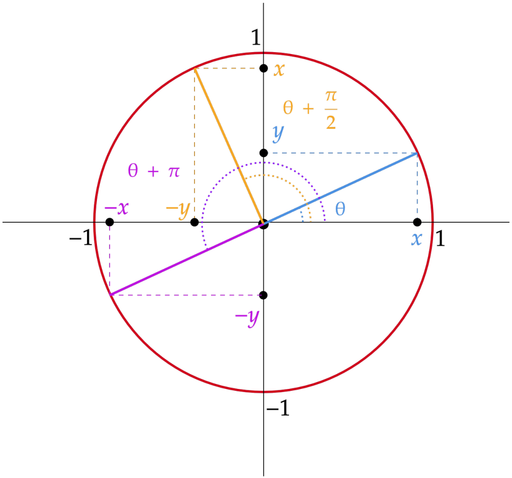
Defining negative angles as clockwise translations along the unit circle, we find the following:

- $\cos(\theta) = \cos(-\theta)$.
- $\sin(\theta) = -\sin(-\theta)$.



Another key observation is how *Sines* and *Cosines* respond to $\frac{\pi}{2}$ rotations.

$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin(\theta)$$
$$\sin\left(\theta + \frac{\pi}{2}\right) = +\cos(\theta)$$



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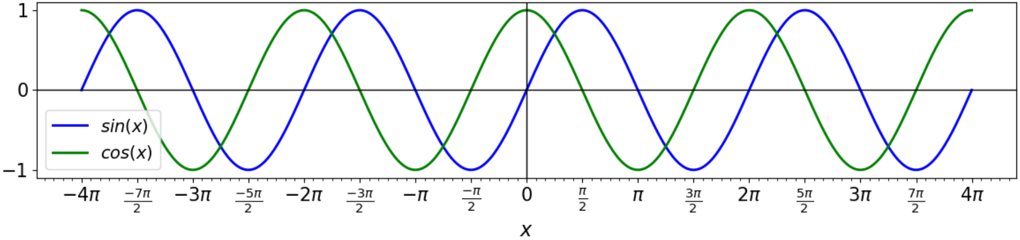
Trigonometric Functions

Having gone through the basic properties of *Sines* and *Cosines*, we can express them as functions defined for $x \in \mathbb{R}$.

$$f(x) = \cos(x)$$
$$g(x) = \sin(x)$$

Note, however, that both $f(x)$ and $g(x)$ are bounded by the closed interval $[-1, 1]$.
(i. e. $\left|f(x)\right| \leq 1$ for any $x \in \mathbb{R}$).

$\sin(x)$ and $\cos(x)$ on the Graph



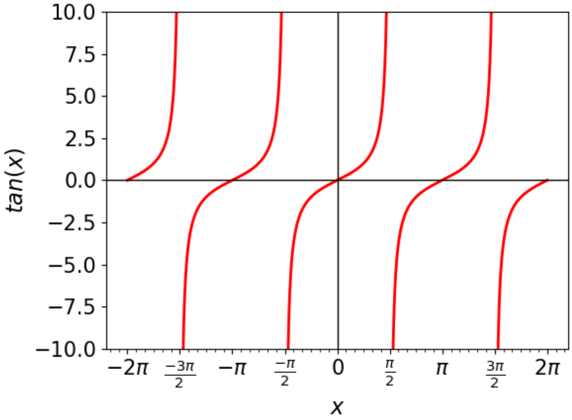
Observe how the graphs of $\sin(x)$ and $\cos(x)$ are identical except for a **phase shift** of $\frac{\pi}{2}$ to the right.

The Tangent

We can now go ahead and construct the *tangent* of x:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Note that $\tan(x)$ is not defined for $\cos(x) = 0$.



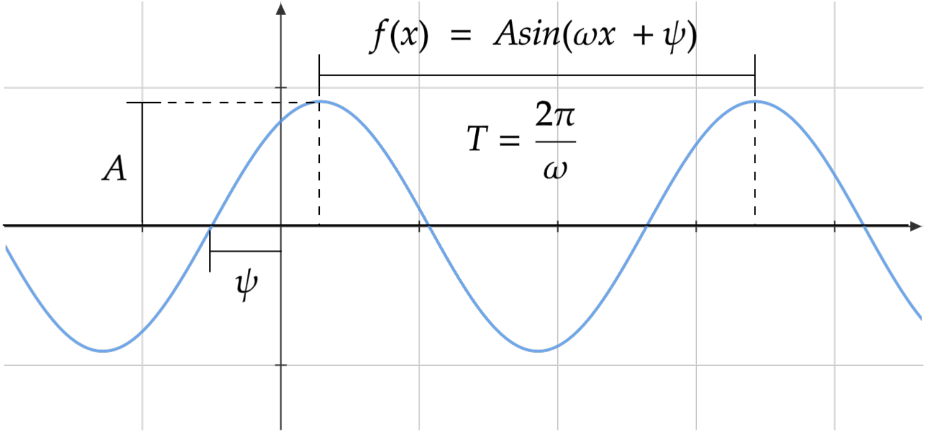
Trigonometric Functions as Periodic Relationships

These three trigonometric functions all satisfy $f(x) = f(x + 2\pi)$,in other words, they are **periodic**. In general, a function is periodic whenever there exists an a such that for any x in the domain:

$$f(x + a) = f(x)$$

If a is the smallest number which satisfies this property we call it the **period** of $f(x)$.

Making Waves



The Wave Function

3 Trigonometric Functions

For an arbitrary wave function - $f(x) = A\sin(\omega x + \psi)$:

- A is the **Amplitude**.
- $\omega = \frac{2\pi}{T}$ is the **angular frequency**.
- ψ is the **phase shift**

Note how it doesn't matter whether you choose $\sin$ or $\cos$ for your wave function.

Modelling Complex Periodic Phenomena

It is possible to model increasingly complex periodic behaviours through the composition of wave functions. The resulting process is called a **Fourier Series**.

$$F_T(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(n\omega_0 x\right) + b_n \sin\left(n\omega_0 x\right) \right]$$

Fourier series are the backbone of several modern computational tools and a discipline all of its own.

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Transformations on Graphs

Having defined a few elementary functions we can apply some basic transformations to create new functions. For now we'll look at what happens when you add or multiply constants to the argument or the function as a whole.

- Modifying the argument - $g(x) = f(ax + c)$
- Modifying the function - $g(x) = af(x) + c$

This process is called **function composition** and the result is a **composite function**.

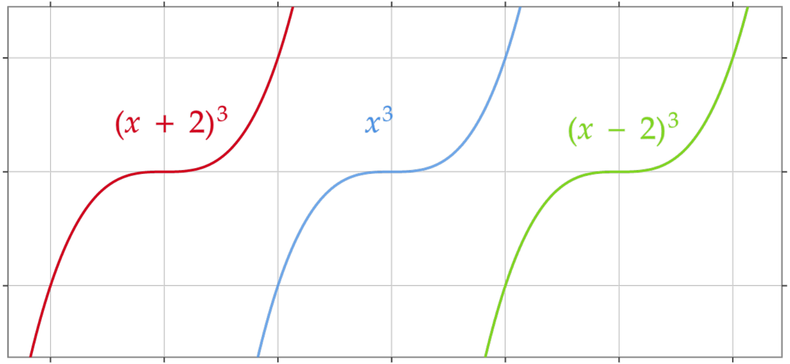
Horizontal Translations

Adding a constant to the argument shifts the whole function along the x-axis.

- $f(x - c)$ shifts $f(x)$, c units in the positive direction.
- $f(x + c)$ shifts $f(x)$, c units in the negative direction.

Worth noting how the directions are reversed.

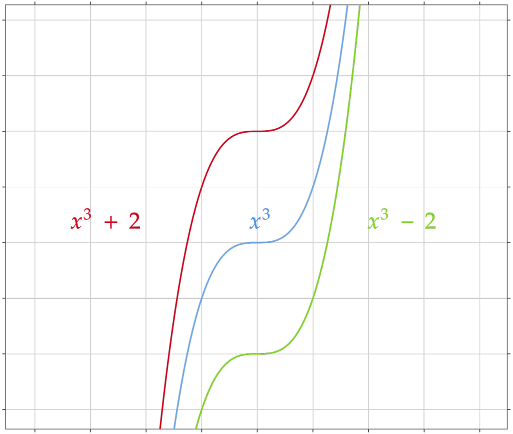
Horizontal Translations



Vertical Translations

Adding a constant to the function, results in a shift along the y-axis.

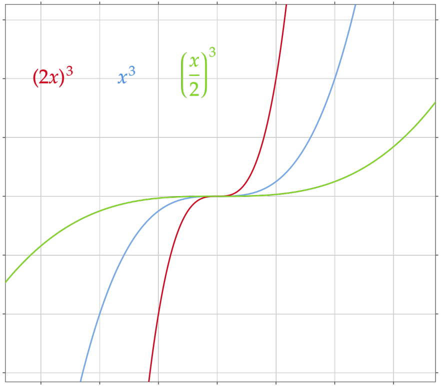
- $f(x) + c$ shifts $f(x)$, c units upwards.
- $f(x) - c$ shifts $f(x)$, c units downwards.



Horizontal Stretching/Compression

Whenever the argument of a function is multiplied by a positive constant c, it either compresses or stretches the graph horizontally.

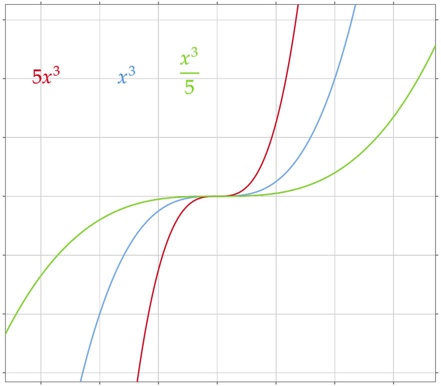
- If $c > 1$ then $f(x)$ gets compressed by a factor of $\frac{1}{c}$.
- If $0 < c < 1$ it gets stretched by a factor of $\frac{1}{c}$.



Vertical Stretching/Compression

Directly multiplying a function by a positive constant c, produces analogous results vertically.

- If $c > 1$ then $f(x)$ gets stretched by a factor of c.
- If $0 < c < 1$ it gets compressed by a factor of c.



Reflection

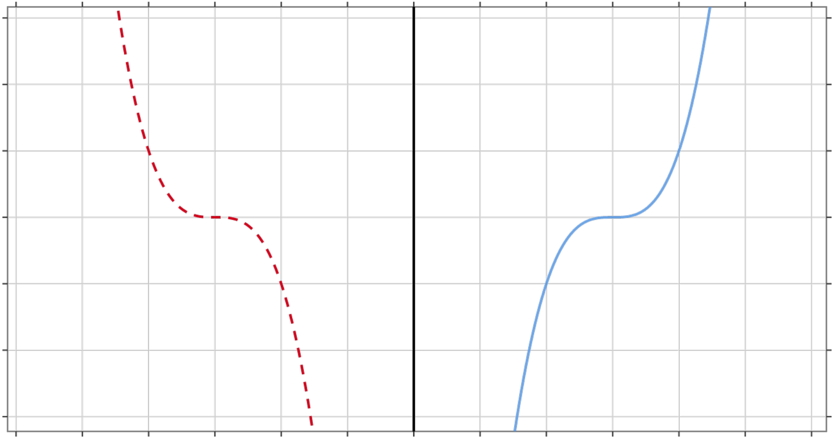
Multiplying the argument or the function by -1 will produce a reflection on either of the axes:

- $f(-x)$ becomes reflected on the y-axis.
- $-f(x)$ becomes reflected on the x-axis

Note that multiplying by a negative constant $-c$, generates both a reflection and the corresponding compression/stretch.

Reflection on the y-axis

4 Transformations on Graphs



Reflection on the x-axis