Lecture 03

The Solow model

Where our journey starts?

We are about to study the mechanics of capital accumulation as a driver of the **very** long-run trend dynamics of real GDP

Demand side			Supply side	
	Goods Market		**OFFICE HOURS**	**Mondays**
Money Market (CB Policy)	IS – MP			**Room S1024**
Forex Market	IS – LM – BP			4:40 – 6:40 pm
	AD	AS	LRAS dynamics	
		Labour market	Capital accumulation	
			Economic growth	

ILOs

By the end of this block you should be able to:

- explain modern growth trends across countries
- discuss growth in potential output
- formulate the Solow model of economic growth (analytically)
- illustrate the ‘growth effects’ and ‘level effects’ of exogenous shocks (graphically)
- explain the convergence hypothesis
- analyse the growth performance of rich and poor countries
- describe Malthus’ forecast of eventual starvation and how technical progress and capital accumulation made this forecast wrong

Subject Guide / Block 18

Macroeconomics

Block	Title	BVFD Chapter
11	Introduction to macroeconomics	17
12	Supply-side economics and economic growth	18
13	Output and aggregate demand	19, 20
14	Money and banking; interest rates and monetary transmission	21 (except Maths 21.2, 22 (except Maths 22.1)
15	Monetary and fiscal policy	23 (except 23.6 and the appendix)
16	Aggregate demand and aggregate supply	24
17	Inflation	25 (except 25.1)
18	Unemployment	26 (except Maths 26.1)
19	Exchange rates and the balance of payments	27(except Maths A27.1)
20	Open economy macroeconomics	28 (except Maths 28.1)

Technology
in the Solow model

The neoclassical growth model with exogenous technical progress and non-optimising households

Inputs to production

Capital (endogenous), K

- output per worker may increase with capital per worker

Labour (exogenous), L

- population growth
- participation rates
- human capital

Technology (public), A

- Inventions
- R&D
- Education

Land (fixed), T

- fixed supply, but quality may be improved

Raw materials, R

- important distinction between **depletable** resources (coal, oil)
- **renewable** resources (timber, fish)

Inputs to production

Production function

shows the maximum output that can be produced using specified quantities of inputs, given existing technical knowledge

Output is a function of capital (K), labour (L), technology (A), land (T), raw materials, (R), human capital, (H), etc.:

$$Y = F(K, L, \color{red}{A}, T, R, H, \dots)$$

In a basic model we abstract from any production factors except capital, labour and technology

Technical knowledge

The state of technical knowledge changes due to:

- **Inventions** – scientific discoveries of new knowledge
- **Innovations** – the embodiment of knowledge into actual production techniques
- **Learning by doing** – increases in productivity

Technology is free and is publicly available as

- a **non-rival** good - its consumption or use by others does not preclude my own
- a n**on-excludable** good - it is impossible to prevent the person from using it or from consuming it

R&D can make it costly and private: patent systems address a market failure and secure ‘fair’ returns on R&D

The Solow residual

Growth accounting

Our results are largely shaped by the production function:

$$Y(t) = F(K(t), L(t), A(t))$$

Let's decompose output growth rates by applying

Growth accounting:

$$\frac{\dot{Y}}{Y} = \frac{K}{Y}\frac{\partial Y}{\partial K}\frac{\dot{K}}{K} + \frac{L}{Y}\frac{\partial Y}{\partial L}\frac{\dot{L}}{L} + \frac{A}{Y}\frac{\partial Y}{\partial A}\frac{\dot{A}}{A} \equiv \alpha_K\frac{\dot{K}}{K} + \alpha_L\frac{\dot{L}}{L} + R$$

where $\alpha_K \equiv \frac{\partial lnY}{\partial lnK} = \frac{\partial Y}{\partial K}\frac{K}{Y}$ is the capital income share

The Solow residual $R \equiv \frac{A}{Y}\frac{\partial Y}{\partial A}\frac{\dot{A}}{A}$

accounted for **more than a half** of GDP growth on average since the end of the WWII

The Solow residual

The Solow residual (the rate of growth of total factor productivity - TFP) reflects all sources of growth other than the contribution of capital and labour

With CRS production function $\alpha_K + \alpha_L = 1$. If $\alpha_K \equiv \alpha$ then

$$R = \frac{\dot{Y}}{Y} - \left[\alpha \frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L}\right]$$

The Solow residual may be interpreted as a measure of the contribution to growth of technology

Labour-augmenting neoclassical aggregate production function

Neoclassical growth theory

The theory we are studying is neoclassical because it does not ask how actual output gets to potential output

- Over a **long enough** period, actual and potential output are equal

The neoclassical growth theory rests on the **neoclassical aggregate** production function, which explains the evolution of output by the dynamics of 3 inputs:

$$Y(t) = F(K(t), L(t), A(t))$$

- $K(t)$ − evolution of capital stock
- $L(t)$ − evolution of total employment
- $A(t)$ − evolution of technology

Hereinafter, we will skip time t in our notations

Neoclassical production function and technology

Labour-augmenting (Harrod-neutral)

$$Y = F(K, A \cdot L)$$

- a technological innovation raises output in the same way as an increase in the stock of labor

Capital-augmenting (Solow-neutral)

$$Y = F(A \cdot K, L)$$

- a technological improvement increases production in the same way as an increase in the stock of capital

Hicks-neutral technology

$$Y = A \cdot F(K, L)$$

- the ratio of marginal products remains unchanged for a given capital-labor ratio

Neoclassical production function

If the production function is Cobb–Douglas

$$Y = A \cdot K^{\alpha} L^{1-\alpha}$$

augmenting A, K, or L will not matter for the results

- A is then interpreted as a total factor productivity (TFP)

Labour-augmenting production function

$$Y = F(K, AL)$$

In the Cobb–Douglas case, we will be safe in assuming that technological progress is labor augmenting

- because in a competitive setting, the factor-income shares are constant
- A is then interpreted as ‘effectiveness of labour’

Labour-augmenting technical change

Review of Economic Studies (2002) **69**, 781–809
0034-6527/02/00310781$02.00

Directed Technical Change

DARON ACEMOGLU
MIT

First version received May 2001; *final version accepted February* 2002 (*Eds.*)

For many problems in macroeconomics, development economics, labour economics, and international trade, whether technical change is biased towards particular factors is of central importance. This paper develops a simple framework to analyse the forces that shape these biases. There are two major forces affecting equilibrium bias: the price effect and the market size effect. While the former encourages innovations directed at scarce factors, the latter leads to technical change favouring abundant factors. The elasticity of substitution between different factors regulates how powerful these effects are, determining how technical change and factor prices respond to changes in relative supplies. If the elasticity of substitution is sufficiently large, the long run relative demand for a factor can slope up.

I apply this framework to develop possible explanations to the following questions: why technical change over the past 60 years was skill biased, and why the skill bias may have accelerated over the past 25 years? Why new technologies introduced during the late eighteenth and early nineteenth centuries were unskill biased? What is the effect of biased technical change on the income gap between rich and poor countries? Does international trade affect the skill bias of technical change? What are the implications of wage push for technical change? Why is technical change generally labour augmenting rather than capital augmenting?

Labour-augmenting technical change

LABOR- AND CAPITAL-AUGMENTING TECHNICAL CHANGE

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http://www.jstor.org/stable/40005140

Abstract
I analyze an economy in which firms can undertake both labor- and capital-augmenting technological improvements. In the long run, the economy resembles the standard growth model with purely labor-augmenting technical change, and the share of labor in GDP is constant. Along the transition path, however, there is capital-augmenting technical change and factor shares change. Tax policy and changes in labor supply or savings typically change factor shares in the short run, but have no or little effect on the long-run factor distribution of income. (JEL: O33, O14, O31, E25)

Neoclassical production function

Constant returns to scale (CRS)

Production function exhibits constant returns to scale:

$$F(\lambda K, \lambda AL) = \lambda F(K, AL)$$

Mathematically, F is a homogeneous function of degree 1

Euler's theorem

if the factor prices equal the respective marginal products, the factor payments just exhaust the total output

$$F(K, AL) = Y = \frac{\partial Y}{\partial K} K + \frac{\partial Y}{\partial (AL)} AL = MPK \cdot K + MPL \cdot L$$

Neoclassical production function

Essentiality

$$F(0, AL) = F(K, 0) = 0$$

- At very low levels of income, savings may be zero as all resources are needed for consumption
- So capital cannot be created through investment, and output may not be able to grow through time
- Positives levels of A and L are also essential for production

Efficient worker and state variable

Technology (labour efficiency) is growing

Can output per worker be a state variable?

The Solow model

Labour-augmenting neoclassical production function

$$Y = F(K, A \cdot L)$$

The evolution of the technical change is exogenous:

$$\dot{A} = g_A \cdot A$$

- In most textbooks on economic growth it is denoted as g

The UoL SG uses $Y = F(K, EL)$ where labour 'efficiency' is denoted as E so EL – units of effective labour ('worker equivalents' in BVFD). The growth rate of labour efficiency is denoted as $\dot{E} = t \cdot E$

Let's use the notation $g \equiv g_A$ for the technical growth rate:

$$\dot{A} = g \cdot A$$

In discrete time: $A_{t+1} = (1 + g)A_t$

The Solow model

Capital accumulation

With **no government** spending and no international linkages a constant depreciation rate, δ, implies that the change in the capital stock is equal to:

$$\dot{K} = Y - C - \delta K = I - \delta K$$

where C is consumption and I is investment

Since saving is equal to investment, $S = I$

The **saving rate**, s, is assumed to be **constant**:

$$S = sY$$

Thus, capital accumulation equation becomes:

$$\dot{K} = sY - \delta K$$

The Solow model

Capital consumption

1. Capital stock wear and tear $\Rightarrow$ linear depreciation δ
2. Capital per worker decreases due to population growth n
   - Recall the difference between capital widening and deepening
3. Capital becomes obsolete due to technical progress g
   - If new technologies are introduced, workers become more productive
   - Less labour is required to produce the same amount of output
   - Some workers become available for other uses
   - Technical progress is therefore equivalent to an increase in the number of workers, in other words to population growth

The overall rate of capital consumption becomes $\delta + g + n$

BGP

Balanced growth path (BGP)

Recall, that net capital investment is equal to

$$\dot{K} = sY - \delta K$$

This implies $\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta = const$

Hence the output to capital ratio is constant $\frac{Y}{K} = const$

The balanced growth path (or steady state) requires

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = const$$

How to calculate the BGP growth rate
for a particular technology?

Practice

SECTION C: Calculation questions

Derive the BGP growth rate of GDP and APL for: $Y = K^{\alpha}(AL)^{1-\alpha}$

Solution.
$$\frac{\dot{Y}}{Y} = \alpha\frac{\dot{K}}{K} + (1-\alpha)\left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L}\right)$$
$$\frac{\dot{Y}}{Y} - \alpha\frac{\dot{K}}{K} = \frac{\dot{Y}}{Y}(1-\alpha) = (1-\alpha)\left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L}\right)$$

The balanced path growth rates of output and capital are:
$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = g + n$$

On the balanced growth path GDP per capita (or average product of labour) grow at:
$$g_{\left(\frac{Y}{L}\right)} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = g + n - n = g$$

Steady state
in the Solow model

GEM and BGP vs steady state

Language level	Problem ❓		Solution ☑		
General	**Real-life situation**	Critical ⇨ thinking	**Practical answer**	**G**	GDP per capita growth rate tends to be constant in the very long run
Conceptualisation	⇩		⇧	Intuition	
Economic	**Question**	Economic → reasoning	**Forecast**	**E**	On the BGP output per efficient worker is constant due to CRS
Formalisation	⇩		⇧	Proposition	
Mathematic	**Model**	Analytical ⇒ solution	**Theorem**	**M**	In the steady state $\dot{k} = 0$, where the state variable is $k \equiv K/AL$

The Solow model

Consider labour-augmenting production function

$$Y = F(K, AL)$$

- Technology is interpreted as efficiency per worker, thus the level of technology improves labour productivity

Representative ‘effective’ consumer/firm produces:

$y \equiv Y/AL$ – output per unit of effective labour (or efficient worker)

$k \equiv K/AL$ – capital per unit of effective labour

Recall the CRS property $F(\lambda K, \lambda AL) = \lambda F(K, AL)$

Use $\lambda = 1/AL$ to obtain

Intensive form production function

$$y = f(k) = F(K/AL, 1) = F(K, AL)/AL$$

The Solow model

Capital accumulation per unit of effective labour

The growth rate of $k \equiv K/AL$ is $g_k = g_K - g - n$ or

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - (g + n)$$

From $\dot{K} = sY - \delta K$ we have $\frac{\dot{K}}{K} = \frac{sY}{K} - \delta = \frac{\frac{sY}{AL}}{\frac{K}{AL}} - \delta = \frac{sy}{k} - \delta$

Thus

$$\frac{\dot{k}}{k} = \frac{sf(k)}{k} - (n + g + \delta)$$

$$\boxed{\dot{k} = sf(k) - (n + g + \delta)k}$$

The Solow model

The steady state

The ratio of capital to efficiency units of labour (the state variable $k \equiv K/AL$) is constant, so the growth rate of it is zero

$$\frac{\dot{k}}{k} = 0$$

In the steady state, there is a certain level of k^*, such that

$$sf(k^*) = (n + g + \delta)k^*$$

Ignoring the trivial solution $k^* = 0$ we consider only $k^* > 0$

The Solow model

Balanced growth path and steady state

In the **steady state**, $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - (g + n) = 0$

$$\frac{\dot{K}}{K} = g + n$$

Since $y = f(k)$ it follows that $\dot{y} = 0$. Therefore output grows at the same rate (implying **balanced growth path**)

$$\frac{\dot{Y}}{Y} = \frac{\dot{y}}{y} + (g + n) = g + n$$

Output per-worker (or GDP per capita or APL) grows at TFP growth rate:

$$g_{\left(\frac{Y}{L}\right)} \equiv \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = (n + g) - n = g$$

The Solow diagram

The Solow model: graphical solution

Capital formation equation

$$\dot{k} = sf(k) - (n + g + \delta)k$$

Can be solved for k analytically as a first order differential equation for a given $f(k)$ and initial conditions

- We limit our analysis to a closed form solution for a steady-state level of k by solving

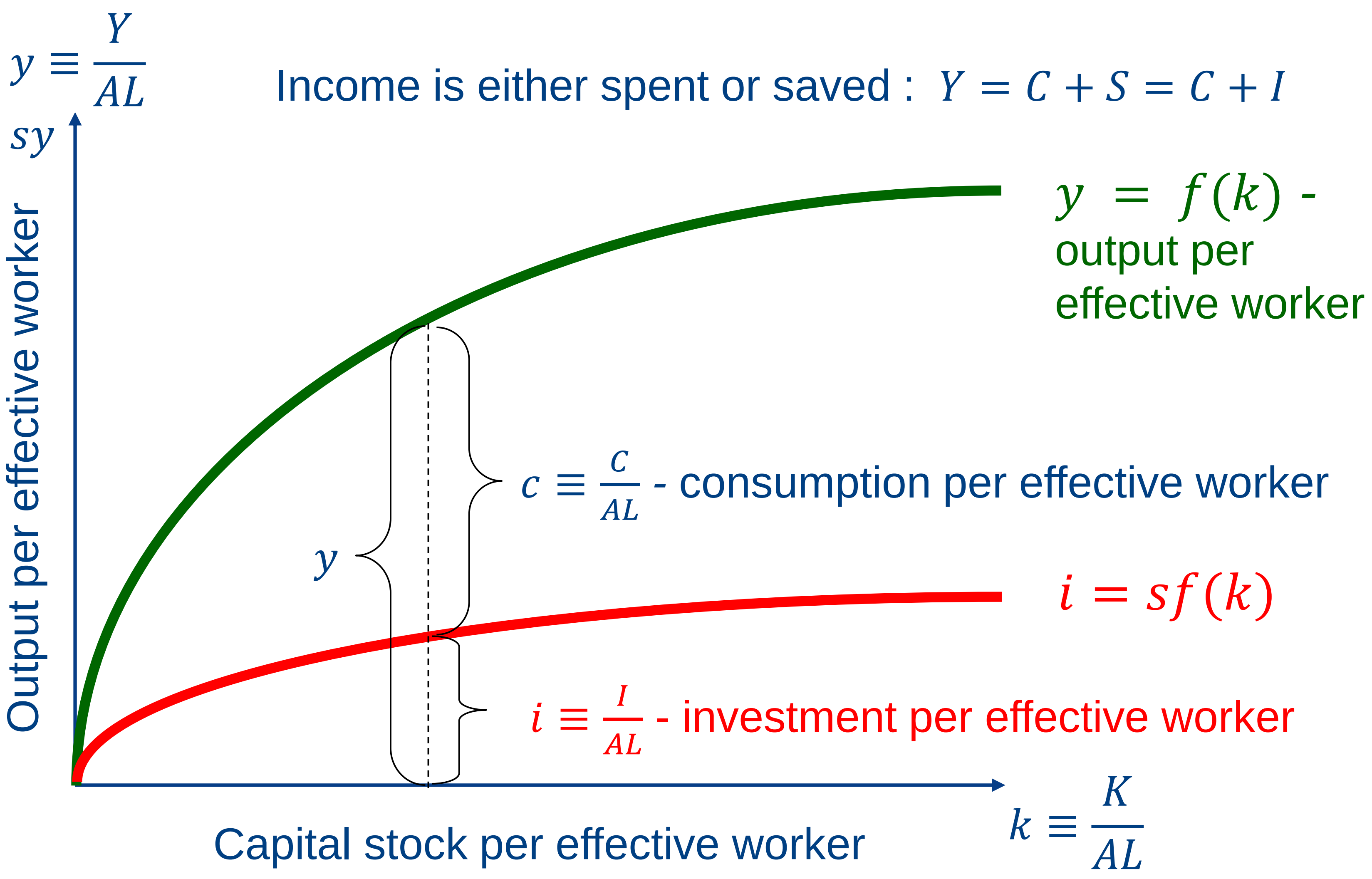
$$sf(k^*) = (n + g + \delta)k^*$$

for **CRS** production function

- An elegant and easy way to find graphically the steady-state k^* is by using the famous Solow diagram
- When capital accumulation (actual investment) is equal to capital consumption (break-even investment) the curve $sf(k)$ crosses the line $(n + g + \delta)k$

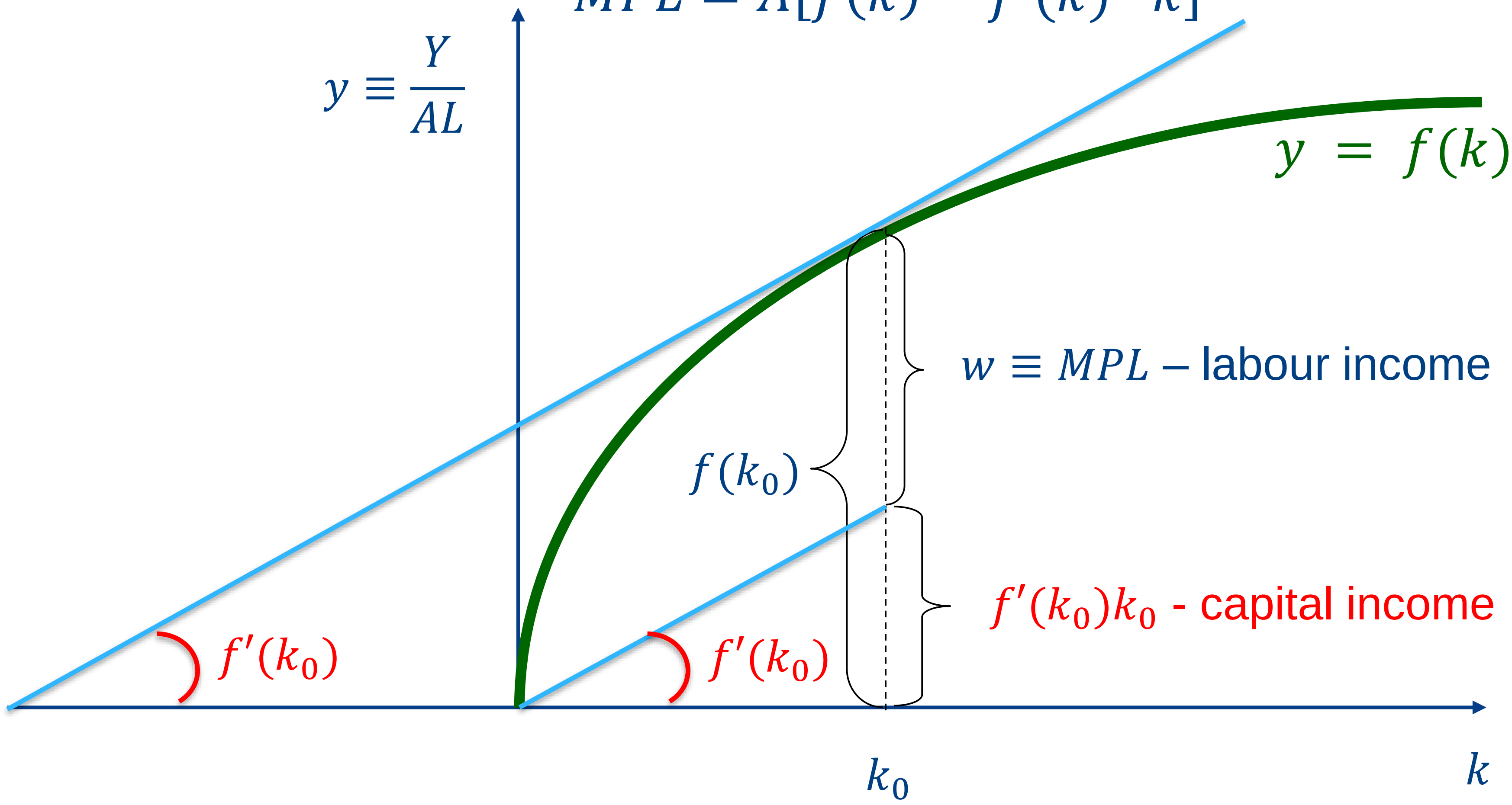
The Solow diagram: capital accumulation

Income is either spent or saved : $Y = C + S = C + I$

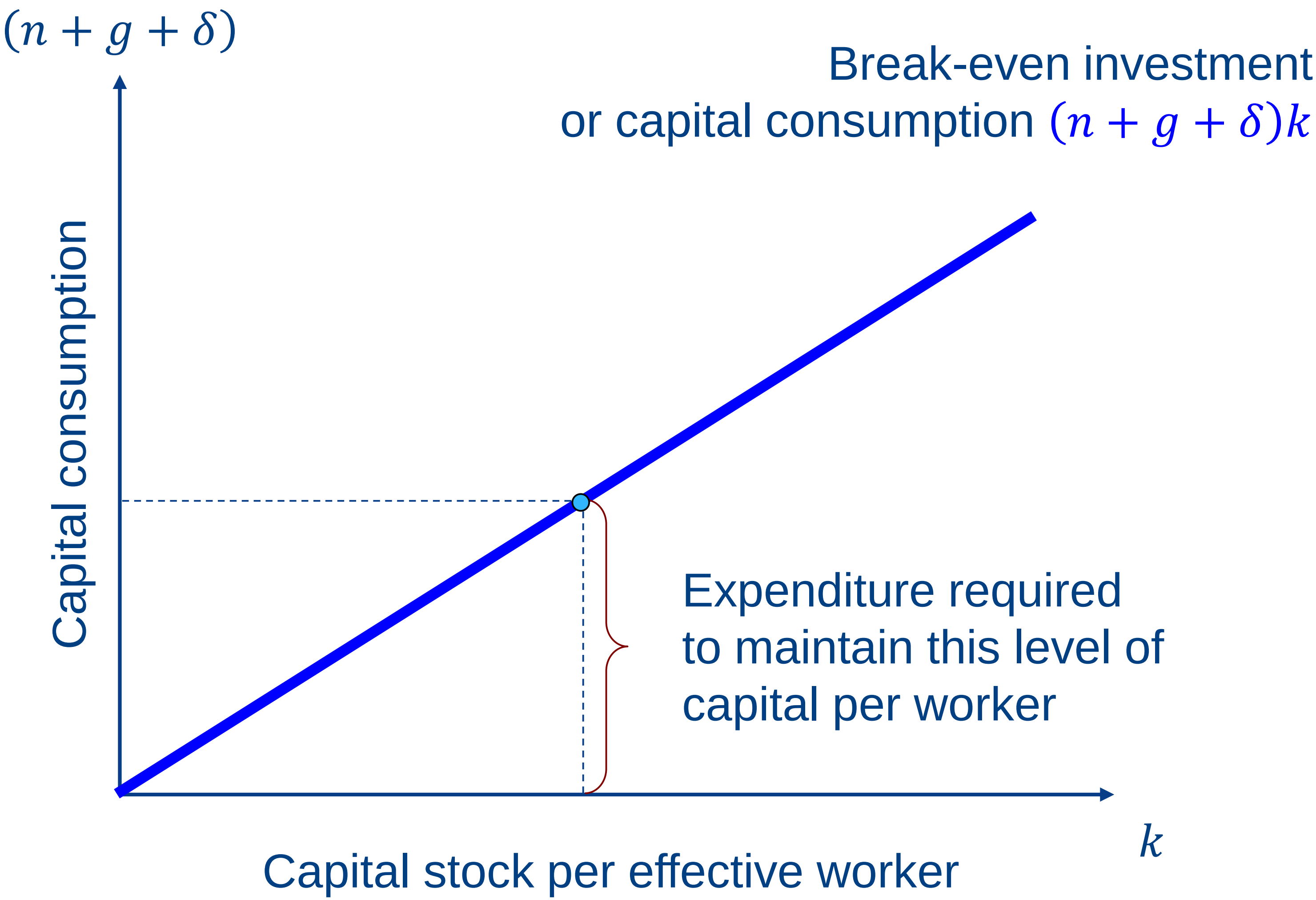


The Solow diagram and marginal products

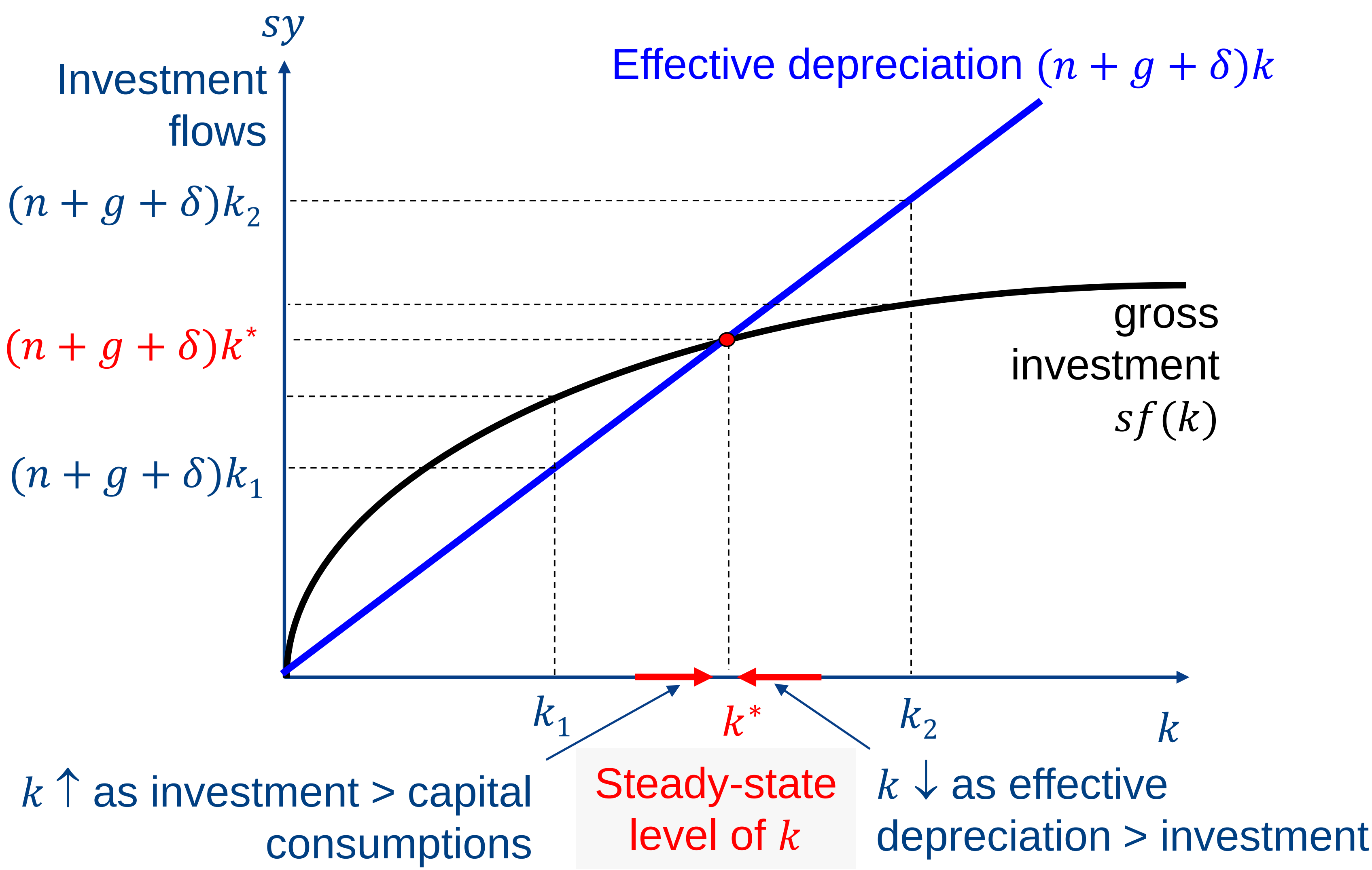
$$MPK = f'(k)$$
$$MPL = A[f(k) - f'(k) \cdot k]$$



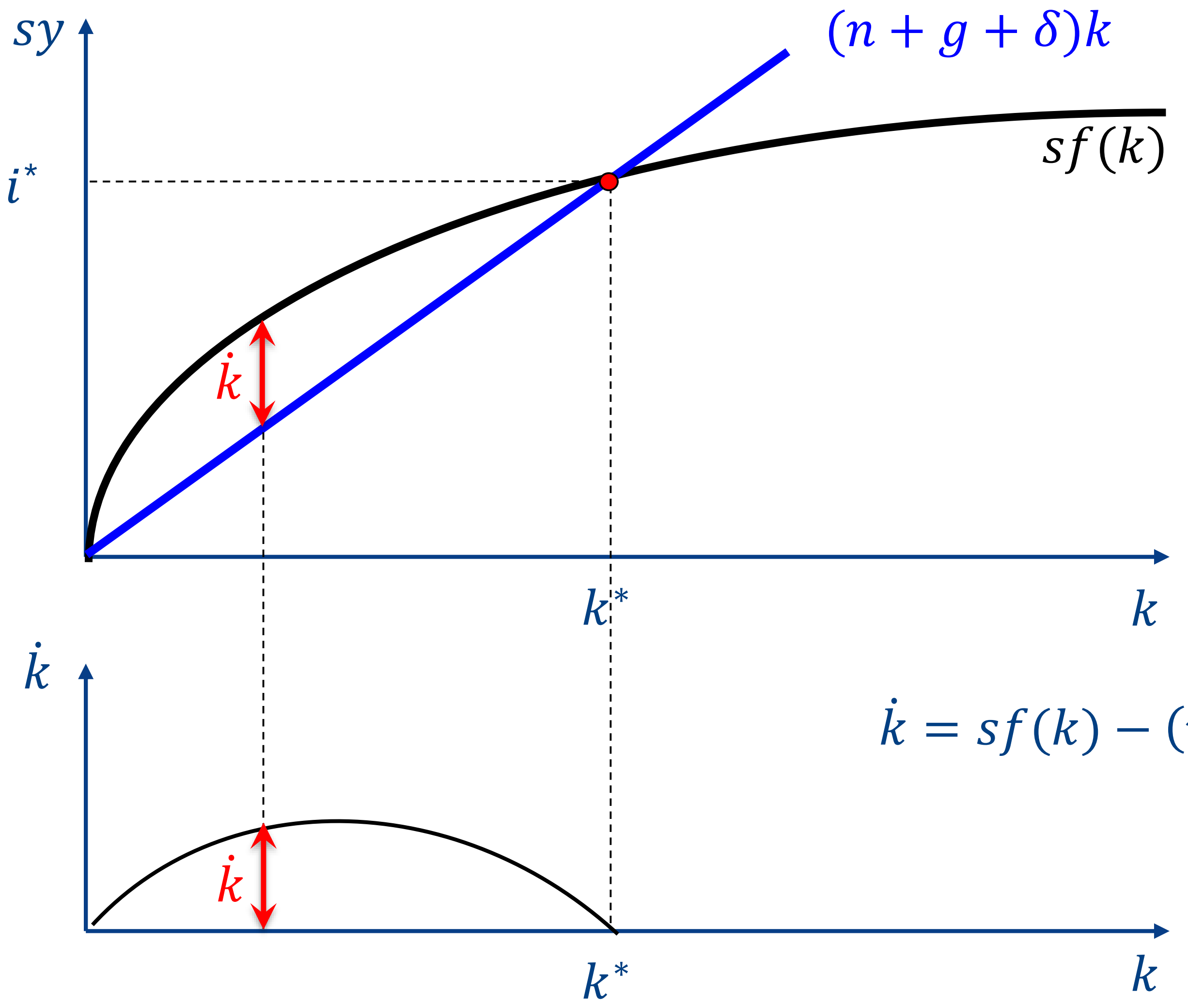
The Solow diagram: capital consumption



Convergence to the steady state



Convergence and phase diagram

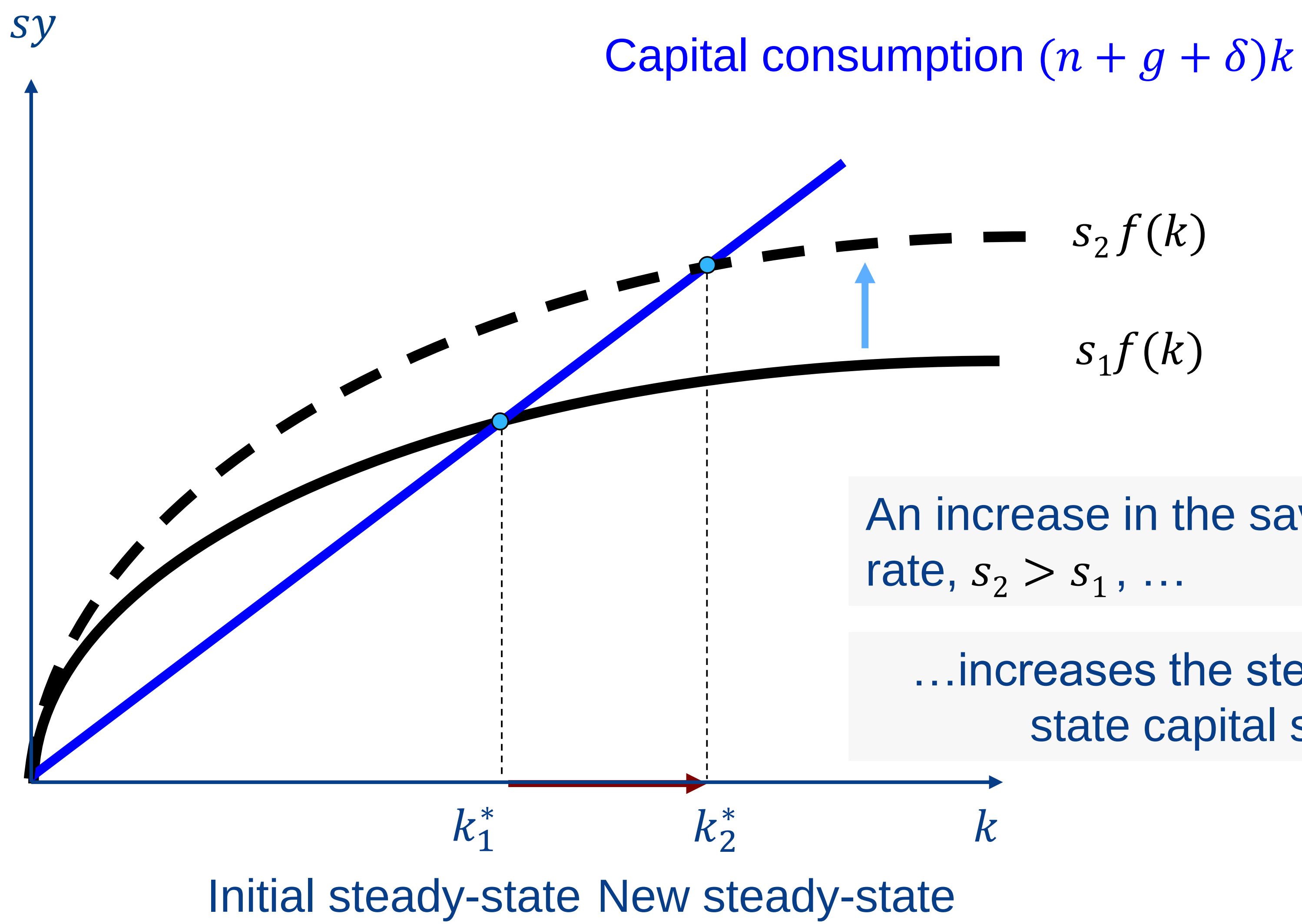


$$\dot{k} = sf(k) - (n + g + \delta)k$$

Wither growth polices?

Will greater saving rate
lead to greater investment
and thus higher output
per effective unit of labour?

Saving rate and steady state



An increase in the saving rate, $s_2 > s_1$, …

…increases the steady-state capital stock

The role of saving rate

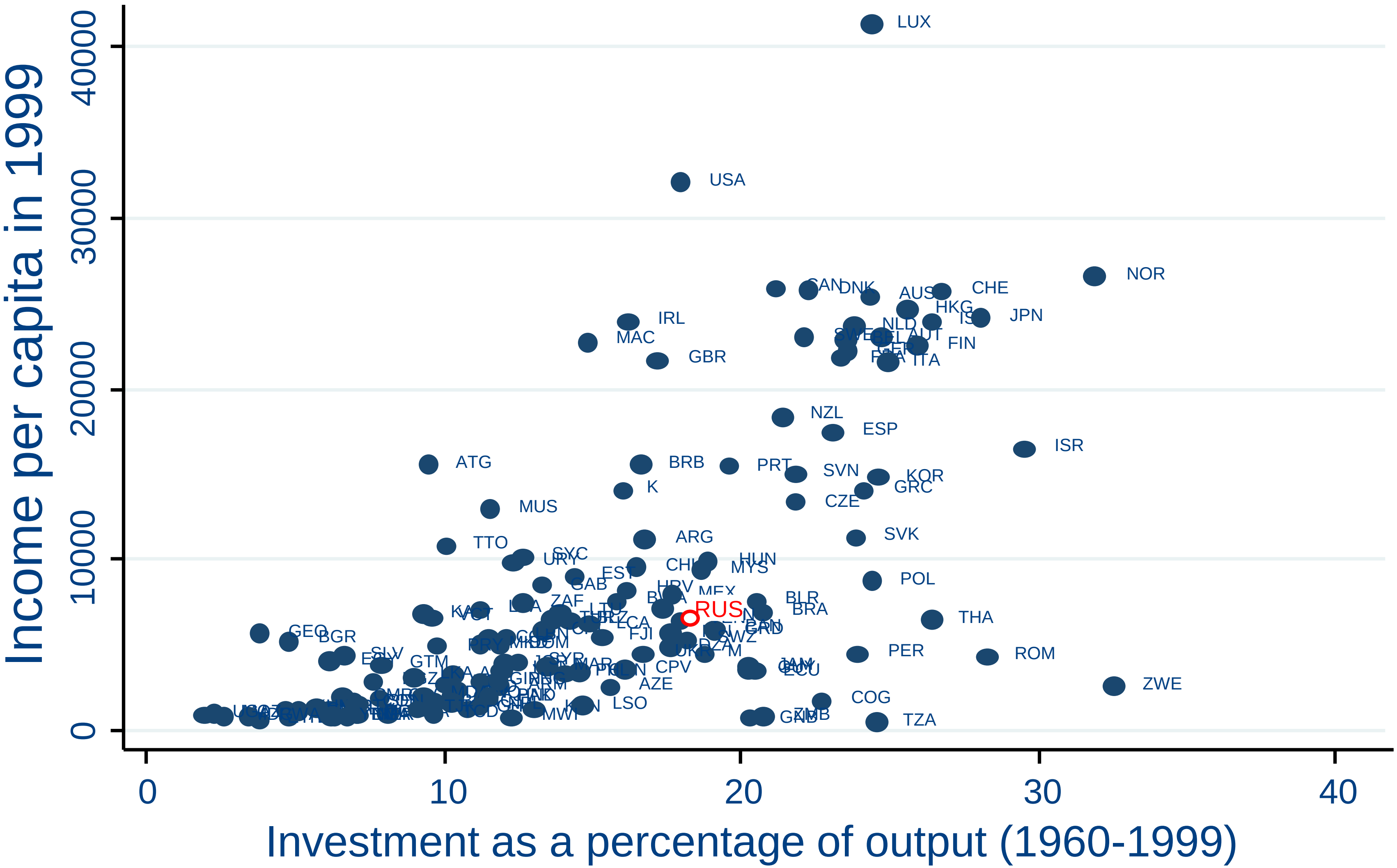
"Can *income per capita* grow forever by simply saving and investing in physical capital?"

NO!

If the production function is neoclassical
and there is no technical growth

Before reaching $s = 1$, the economy would reach an 'optimal' saving rate s_{GR}, so that further increases in saving rates would put not be 'beneficial' on the balanced growth path

Saving rates vary across countries



Capital formation vs capital consumption

Recall, that capital formation $\dot{k} = sf(k) - (n + g + \delta)k$ can be expressed in terms of growth rate:

$$\frac{\dot{k}}{k} = \frac{sf(k)}{k} - (n + g + \delta)$$

where

- $\frac{sf(k)}{k} = \frac{i}{k} = \frac{I}{K}$ is the investment to capital ratio
- $(n + g + \delta)$ is the rate of capital consumption

Mind, that $\frac{sf(k)}{k}$ is a convex function that decreases in k

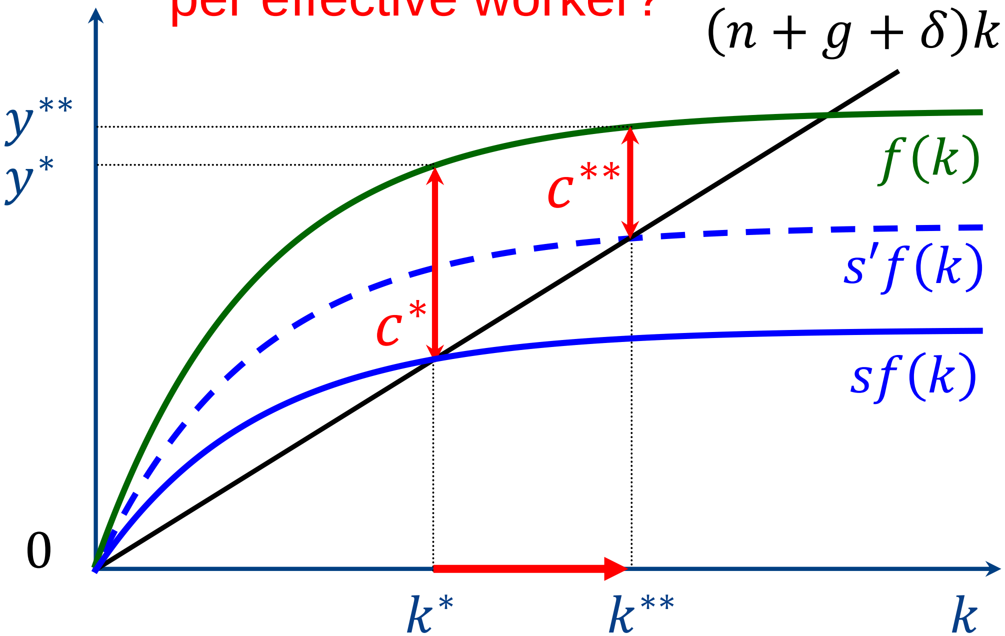
$\frac{\dot{k}}{k} > 0$, if $\frac{sf(k)}{k} > n + g + \delta$

Higher saving rate

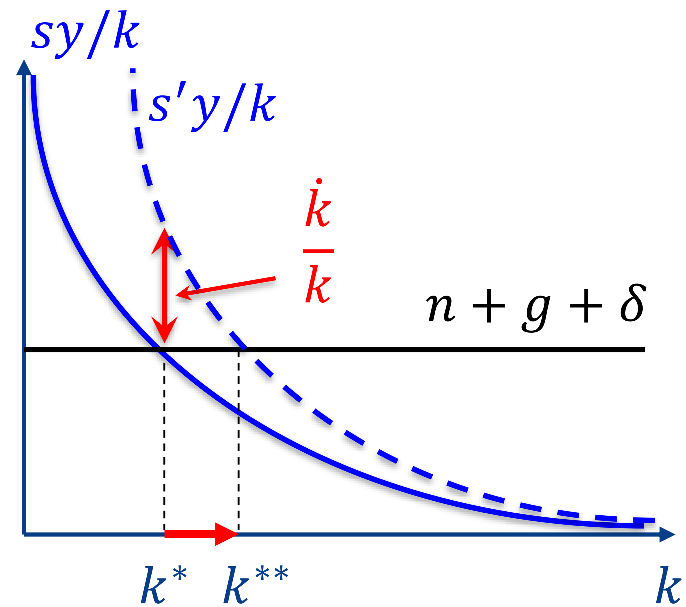
Higher s leads to higher **level** of GDP per capita both in the SR (but without initial hike) and in the LR

The **growth rate** of GDP per capita rises in the SR but converges to g in the LR

What about consumption per effective worker?



What about consumption per worker growth rate?



Golden rule*

Golden rule (Phelps)

The golden rule (GR) level of capital per effective worker k_{GR} is defined as the steady-state level of capital per effective worker that maximises consumption per effective worker:

$$c^* = (1-s)f(k^*) = f(k^*) - sf(k^*)$$

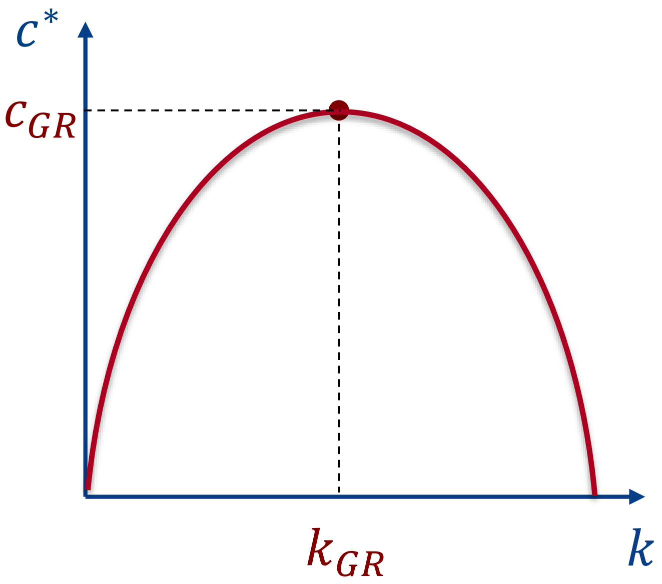
For each steady-state $sf(k^*) = (n+g+\delta)k^*$

$$c^* = f(k^*) - (n+g+\delta)k^*$$

FOC:

$$\boxed{f'(k_{GR}) = n + g + \delta}$$

Golden rule (Phelps)



$$f'(k_{GR}) = n + g + \delta$$

The marginal product of capital (per effective worker) equals the rate of effective depreciation of capital (the sum of the physical capital depreciation rate δ, the growth rates of technological progress, g and the labour force growth rate, n

$$r \equiv MPK - \delta = n + g$$

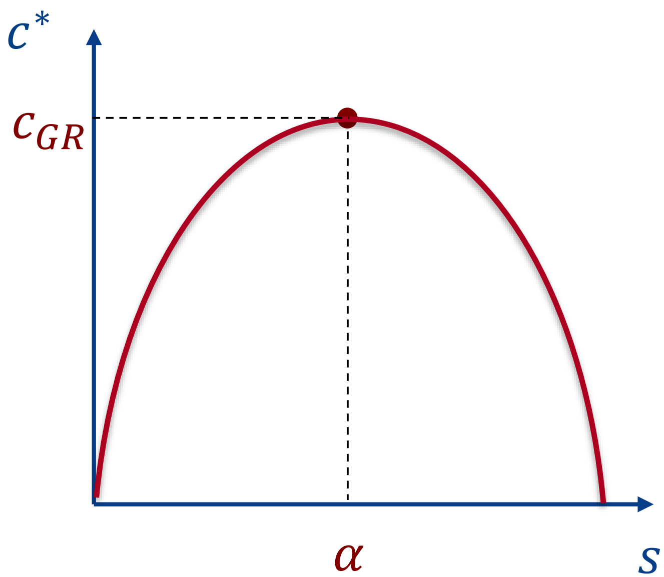
‘Optimal’ saving rate

How to find an ‘optimal’ saving rate s_{GR}?

The saving rate that maximises consumption per effective worker on the balanced growth path should obey:

$$f'(k_{GR})\frac{k}{y} = (n+g+\delta)\frac{k}{y}$$

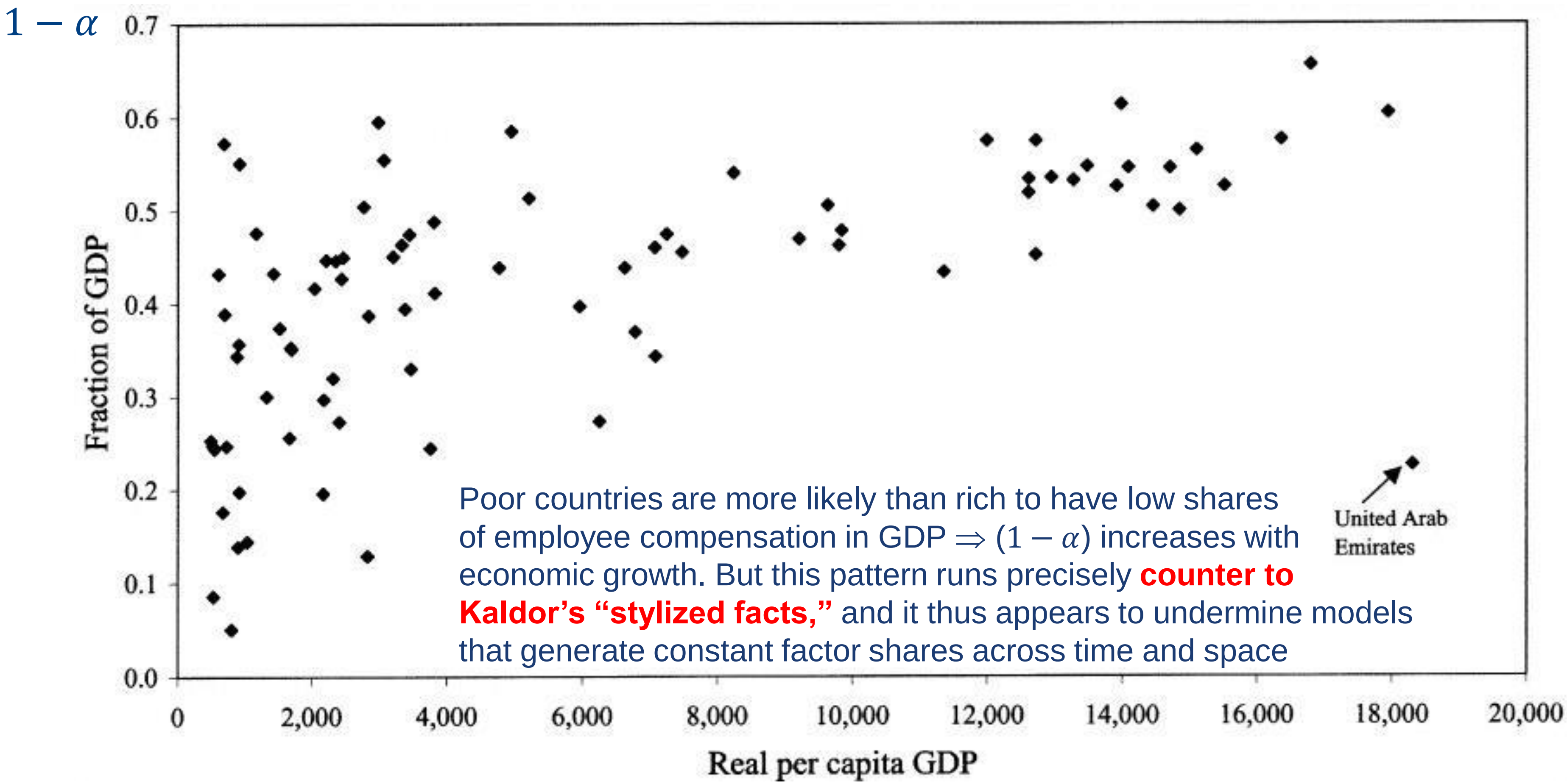
$$\varepsilon_k^y \equiv \alpha = \frac{sy}{y} = s_{GR}$$



If all capital income is saved and reinvested, all labour income is consumed

This condition maximises consumption in the steady-state

Labour income shares vary across countries



Poor countries are more likely than rich to have low shares of employee compensation in GDP $\Rightarrow (1-\alpha)$ increases with economic growth. But this pattern runs precisely **counter to Kaldor's "stylized facts,"** and it thus appears to undermine models that generate constant factor shares across time and space

Employee compensation share of GDP, 81 countries, (1987–92). Sources: Employee compensation shares are taken from United Nations (1994); data on real per capita GDP are taken from the Penn World Tables, version 5.6.

Douglas Gollin, *Getting Income Shares Right*. *Journal of Political Economy* 2002 110:458-474. DOI: 10.1086/338747 Copyright © 2002 The University of Chicago

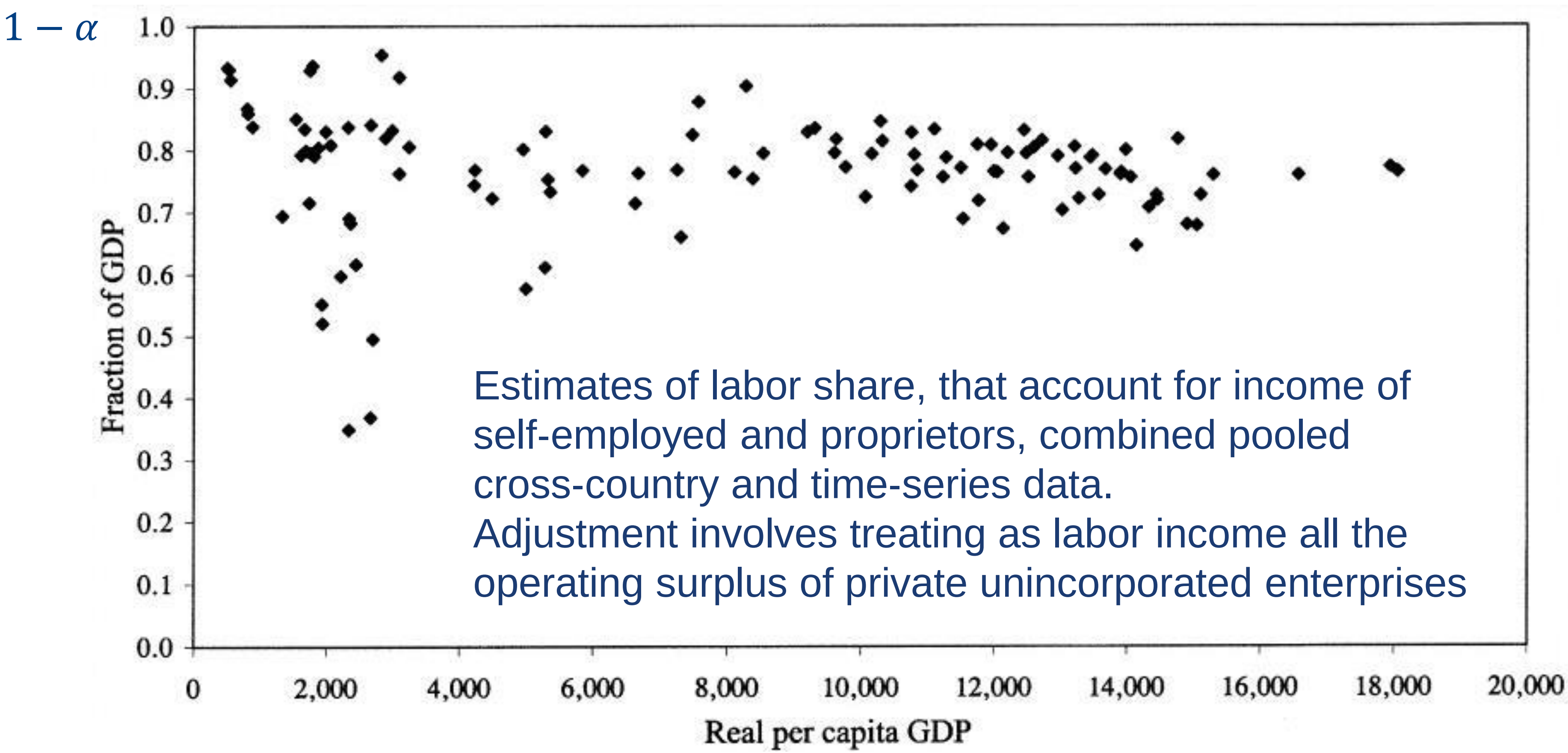
Possible explanations of different income shares across countries

1. Factor shares differ across countries because each country in fact faces a different aggregate technology
   - However, it is not clear why the relationship between inputs and outputs should suddenly shift at a national frontier
   - it is not clear why production technologies should vary with per capita GDP rather than with geography
2. The aggregate technology displays a nonunitary elasticity of substitution between capital and labor
   - However, employee compensation shares remained stable over time, even as accumulation has changed relative factor prices
3. Factor markets are noncompetitive in some countries and that factors are not paid their marginal products
   - Indeed, capital owners may have greater market power in poor countries, however, this argument fades in a world of increasingly mobile captial

Possible explanations of different income shares across countries

4. Finally, the measurement is poor or, more precisely, that employee compensation shares are a poor measure of labor shares
   - the disparities in employee compensation shares may reflect changes in the sectoral composition of output
   - these disparities may reflect changes in the structure of employment – especially in the importance of self-employment

Labour income shares vary across countries



Estimates of labor share, that account for income of self-employed and proprietors, combined pooled cross-country and time-series data.
Adjustment involves treating as labor income all the operating surplus of private unincorporated enterprises

Sources: Raw data on factor shares are taken from United Nations (1994). Data on real per capita GDP are taken from the Penn World Tables, version 5.6.

Douglas Gollin, *Getting Income Shares Right*. *Journal of Political Economy* 2002 110:458-474. DOI: 10.1086/338747

Is the Solow model applicable?

Shall we follow
the golden rule of capital accumulation
if growth rates are exogenously
determined?

Practice: golden rule saving rate

Example: Cobb-Douglas production function

$$Y = K^{\alpha}(AL)^{1-\alpha}$$

In the intensive form: $y = f(k) = k^{\alpha}$

The golden rule implies: $f'(k_{GR}) = \alpha k_{GR}{}^{\alpha-1} = n + g + \delta$

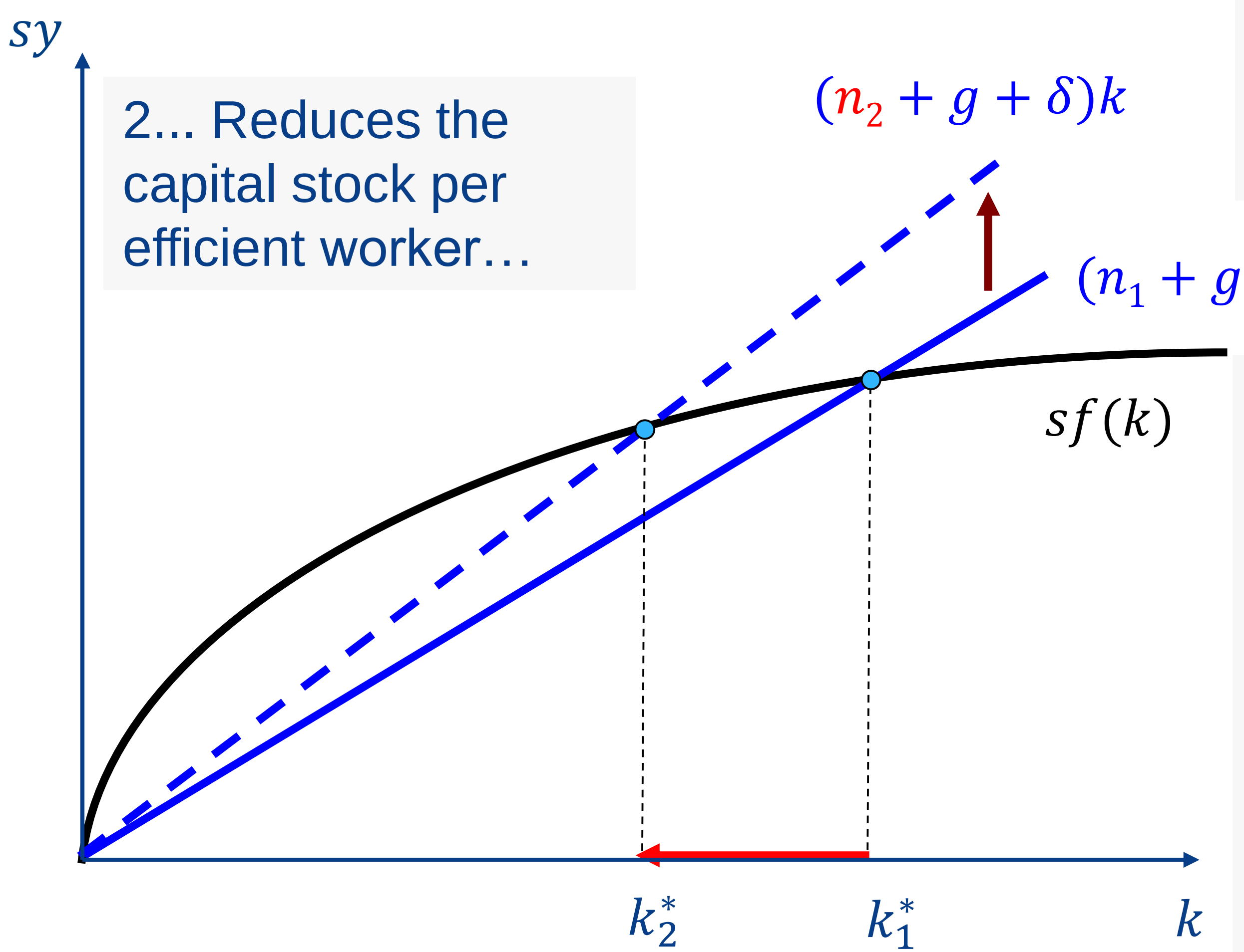
The golden rule steady-state level of k is then:

$$k_{GR} = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

This is the special case ($s_{GR} = \alpha$) in the general steady-state formula:

$$k^{*} = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

Exogenous parameter shock: a higher demographic growth rate n



1. A higher growth rate of the population

2... Reduces the capital stock per efficient worker…

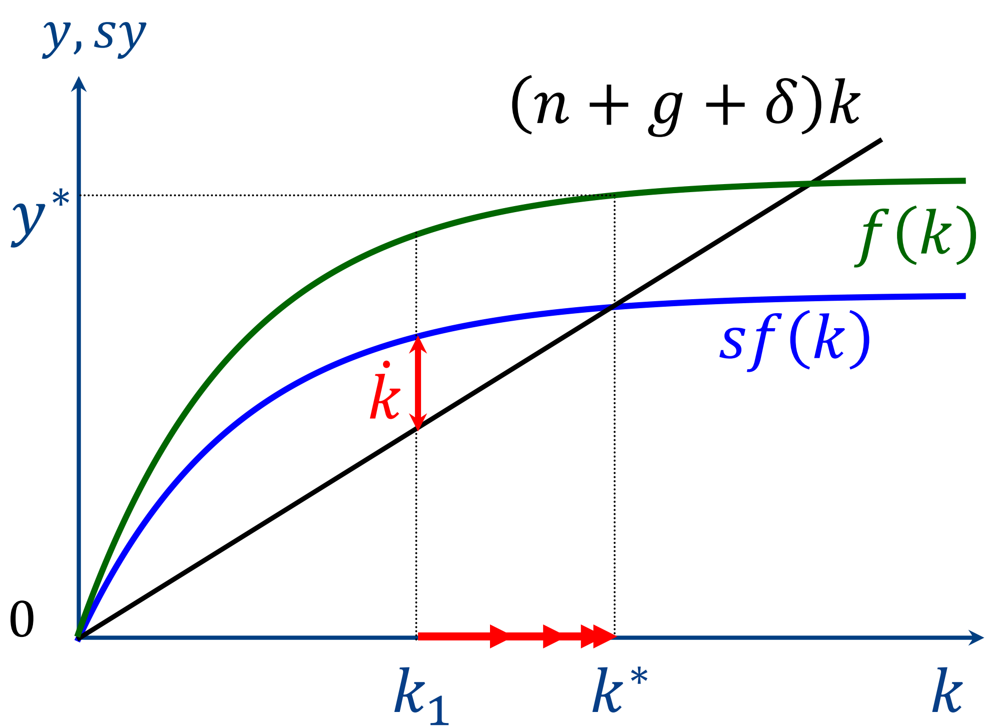
3. …And therefore reduces the steady-state capital stock

Countries with high demographic growth rates should have a **lower** level of per-capita income, *ceteris paribus*

Endogenous variable shock: a decrease in the stock of capital K

When K ⬇, the state variable $k \equiv \frac{K}{AL}$ ⬇ $(k_1 < k^*)$

No curves will shift!



The **growth rate** of capital first jumps up: $\left.\frac{\dot{k}}{k}\right|_{k=k_1} > 0$ then gradually diminishes to zero at k^*