ICEF MACROECONOMICS - 1

Lecture 02

The Solow model

Where our journey starts?

We are about to study the mechanics of capital accumulation as a driver of the **very** long-run trend dynamics of real GDP

Demand side		Supply side	
	Goods Market	**OFFICE HOURS**	**Mondays**
Money Market (CB Policy)	IS – MP		**Room S1043**
Forex Market	IS – LM – BP		4:40 – 6:00 pm
	AD	AS	LRAS dynamics
		Labour market	Capital accumulation
			Economic growth

ILOs

By the end of this block you should be able to:

- explain modern growth trends across countries
- discuss growth in potential output
- formulate the Solow model of economic growth (analytically)
- illustrate the ‘growth effects’ and ‘level effects’ of exogenous shocks (graphically)
- explain the convergence hypothesis
- analyse the growth performance of rich and poor countries
- describe Malthus’ forecast of eventual starvation and how technical progress and capital accumulation made this forecast wrong

Subject Guide / Block 18

Macroeconomics

Block	Title	BVFD Chapter
11	Introduction to macroeconomics	17
12	Supply-side economics and economic growth	18
13	Output and aggregate demand	19, 20
14	Money and banking; interest rates and monetary transmission	21 (except Maths 21.2, 22 (except Maths 22.1)
15	Monetary and fiscal policy	23 (except 23.6 and the appendix)
16	Aggregate demand and aggregate supply	24
17	Inflation	25 (except 25.1)
18	Unemployment	26 (except Maths 26.1)
19	Exchange rates and the balance of payments	27(except Maths A27.1)
20	Open economy macroeconomics	28 (except Maths 28.1)

Stylized facts about economic growth

The hockey stick

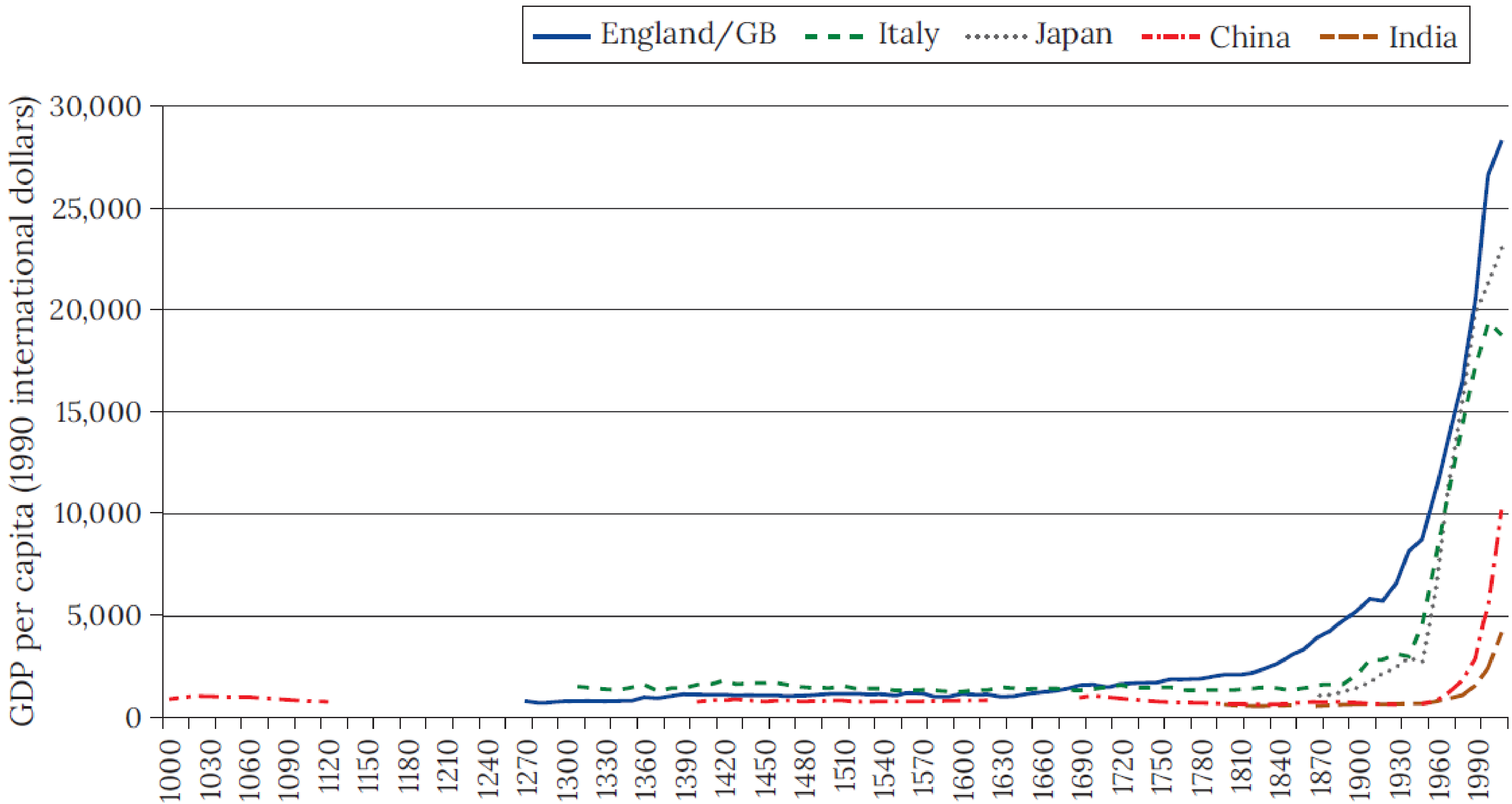


Figure 16.1 GDP per capita (1990 international dollars): 1000–2010.

Source: Broadberry (2022).

The East, the West and the Rest

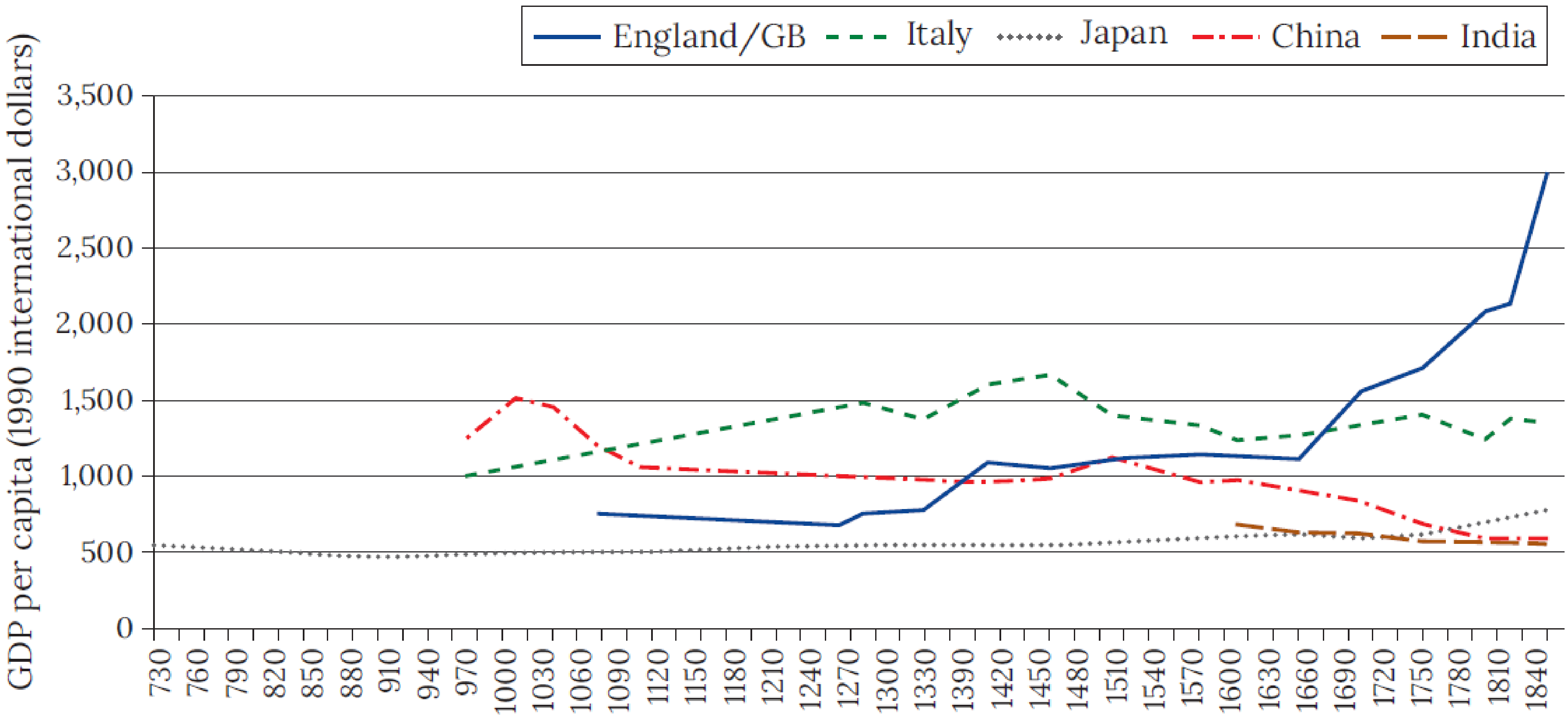


Figure 16.2 GDP per capita (1990 international dollars): 730–1850.

Note: All series have been linearly interpolated to create annual series.

Source: Broadberry et al. (2015).

The East, the West and the Rest

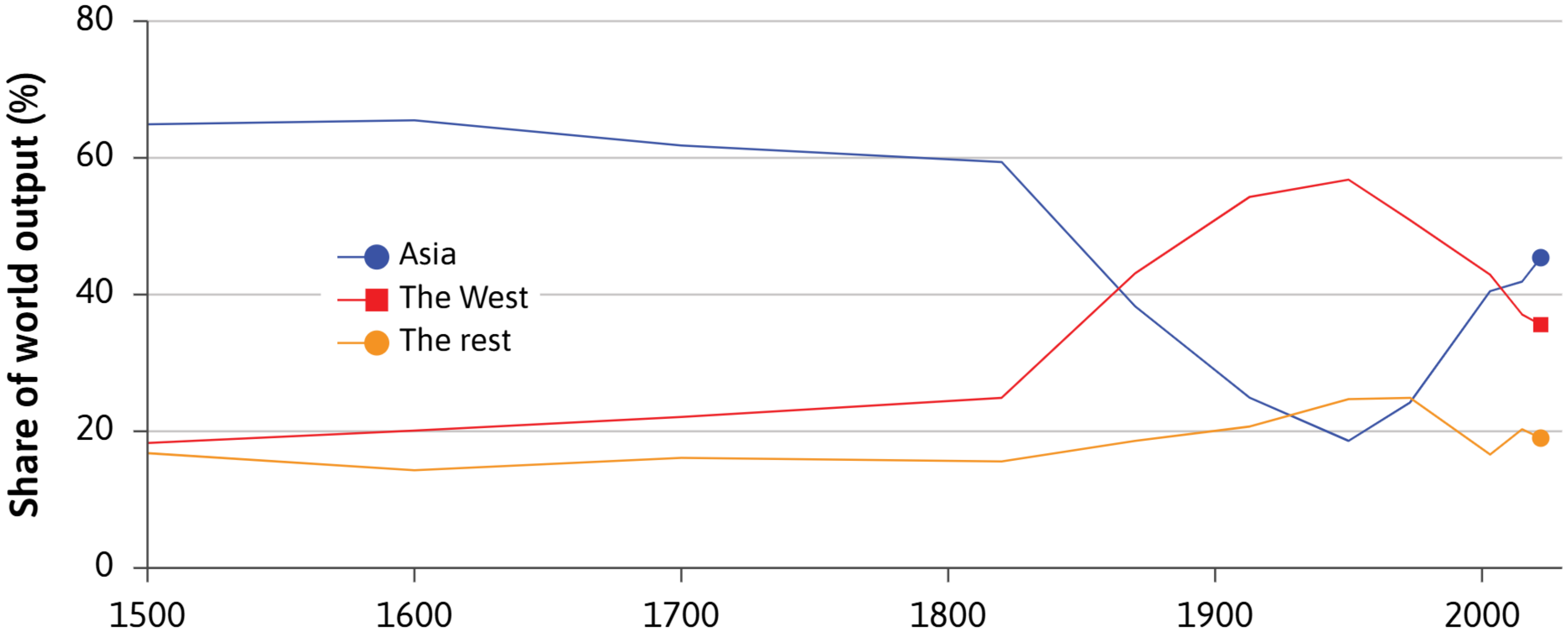


Figure 1.18 Shares of world output produced in Asia, the West, and the Rest. The West refers to western Europe, the US, Canada, Australia, and New Zealand. The Rest includes Latin America, Japan, Africa, and eastern Europe (the latter including Russia and other countries ruled by the Communist Party in the second half of the twentieth century).

Angus Maddison. 2007. Contours of the world economy 1–2030 AD: Essays in macro-economic history. OUP Oxford.; IMF World Economic Outlook. 2022.

Economic growth across continents

- In 1150, **Angkor Wat** was the most highly populated city in the world with about 1mln inhabitants (compared to 50-80k in Paris and 30-50k in **Novgorod**)
- In 1600, India, China, and the rest of Asia produced 3/5 of the world's output, three times as much as western Europe
- In 1750, China produced a third of the world's manufacturing output and India's textile and other manufacturing output was greater than in the whole of Europe
- And only in 1850 did **London** displace Beijing as the world's most populous city

Great divergence

In 2023, Russia's Gross National Income per capita reached 14250 USD, and, consequently, Russia was classified as a high-income country for the first time since 2015.

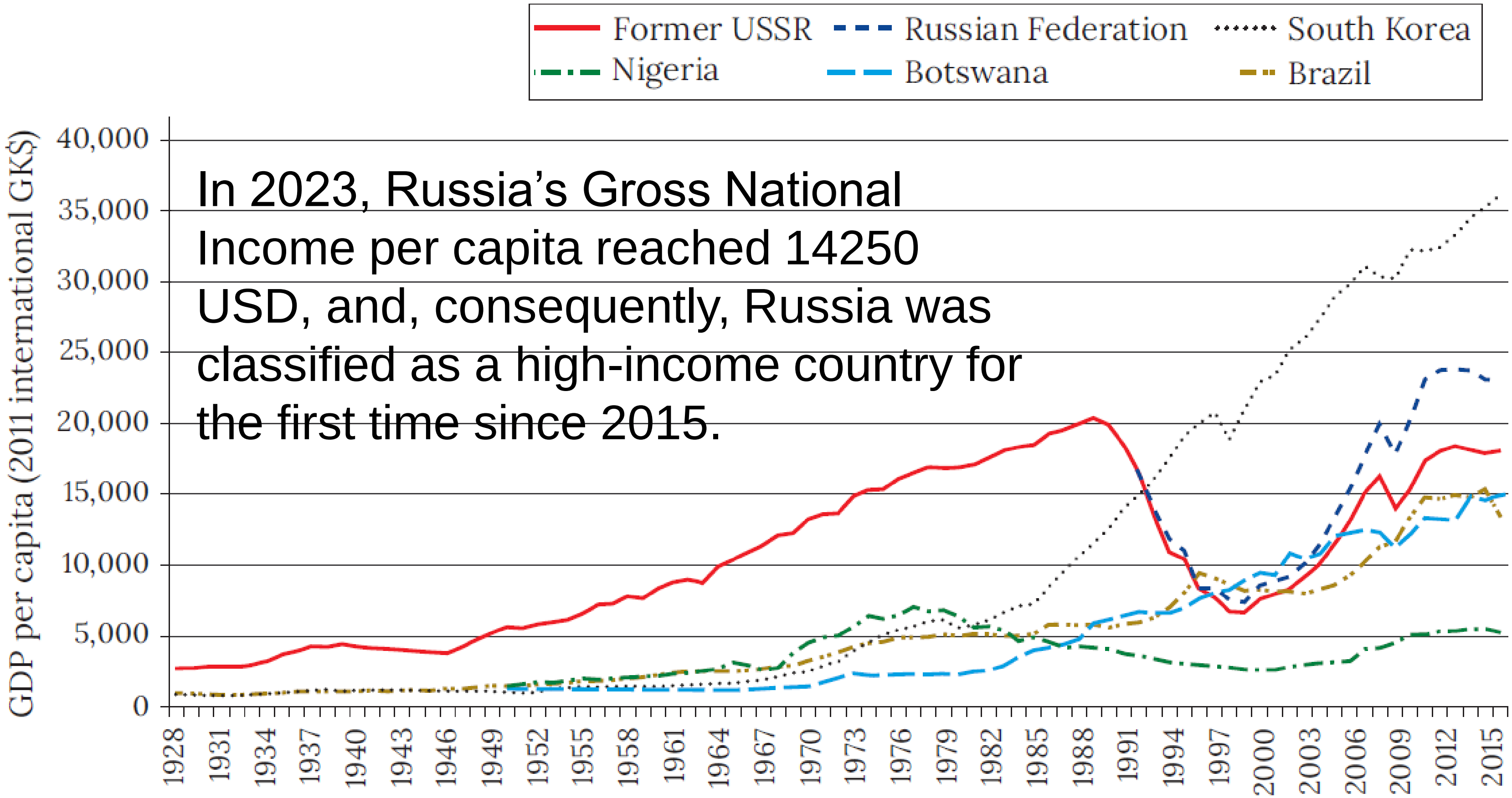
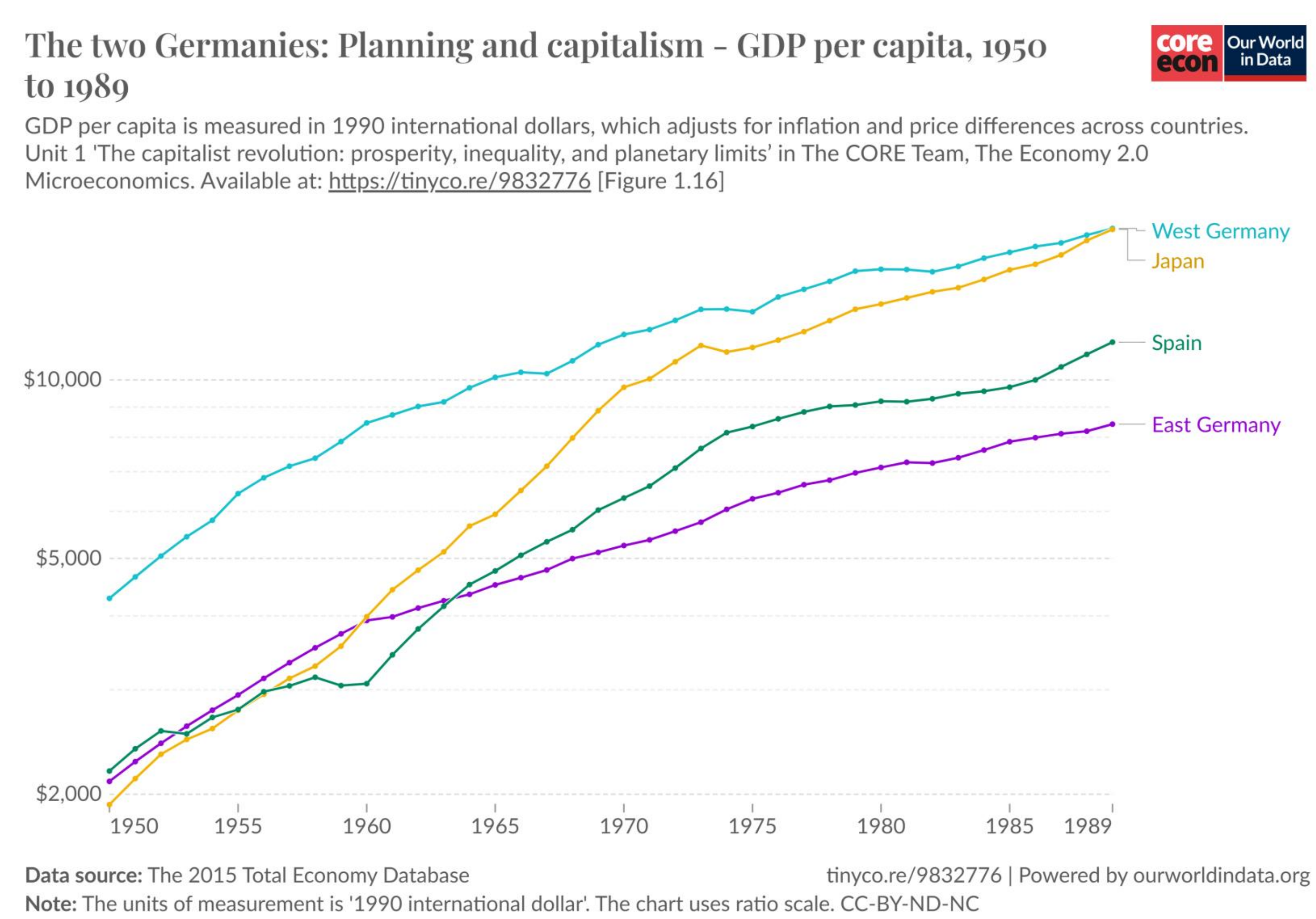


Figure 16.3 GDP per capita (2011 international Geary-Khamis dollars): 1928–2016.

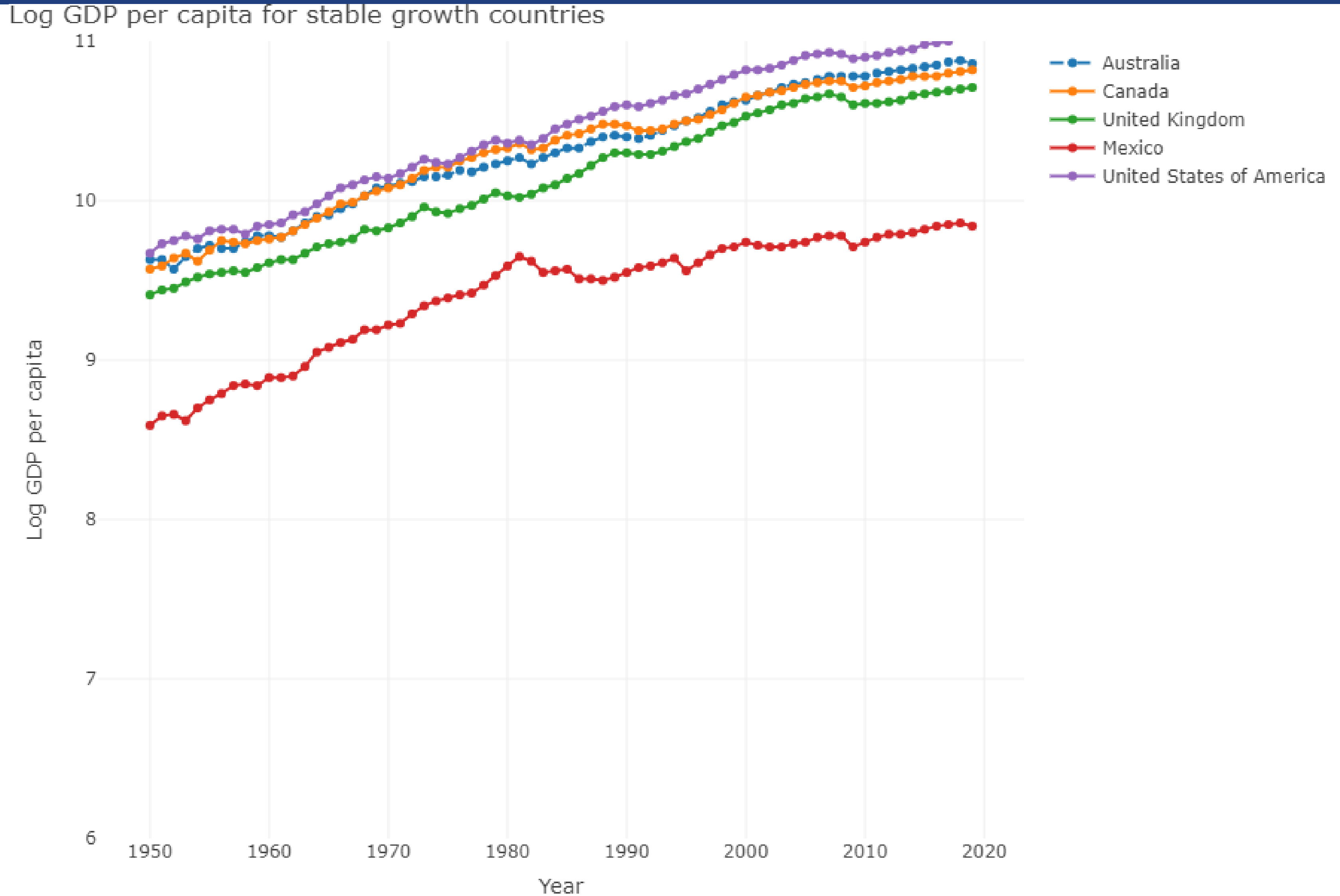
Note: Where data are missing, series have been linearly interpolated to create annual series.

Source: New Maddison Project Database (2018 version).

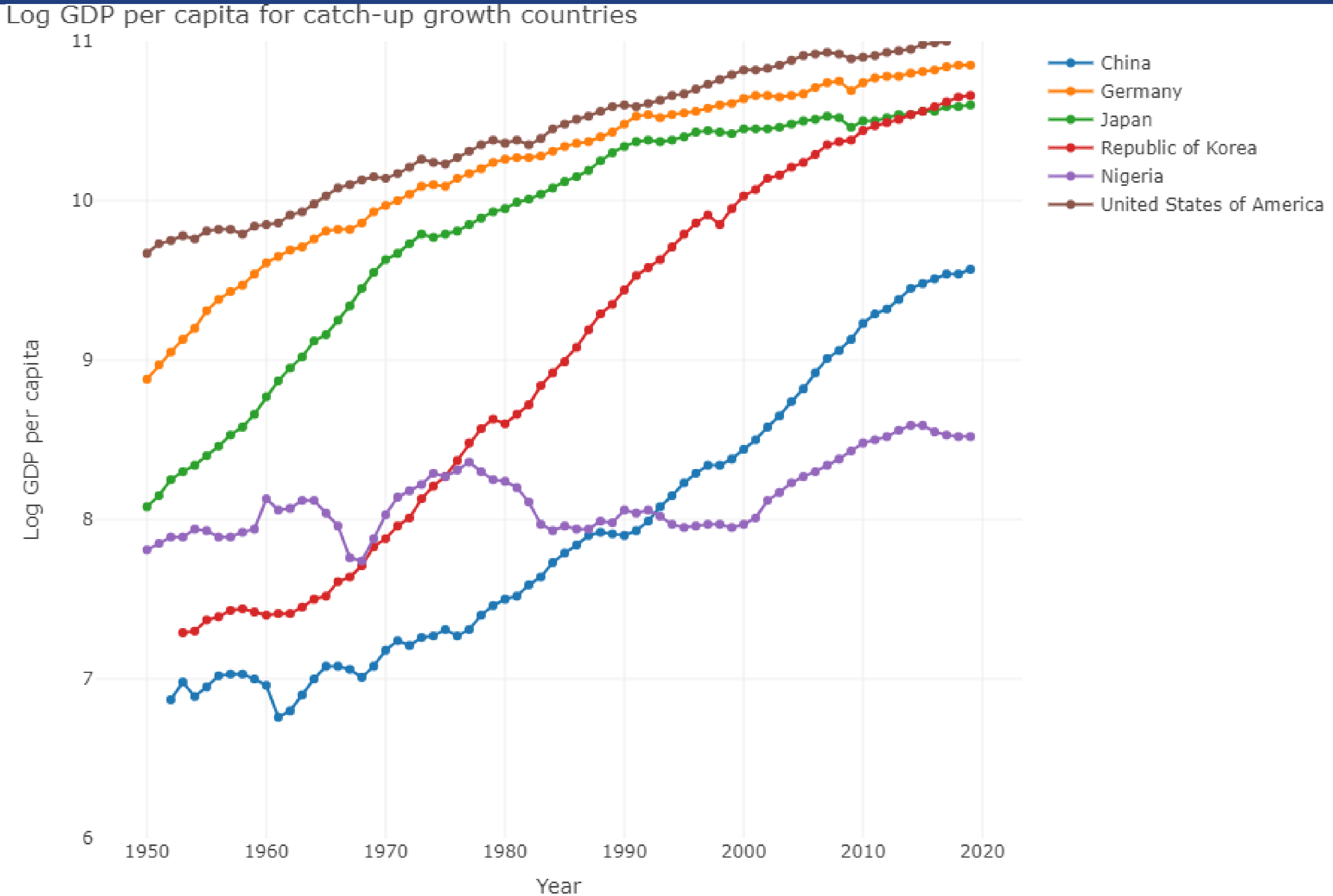
The East and the West Germany after WWII



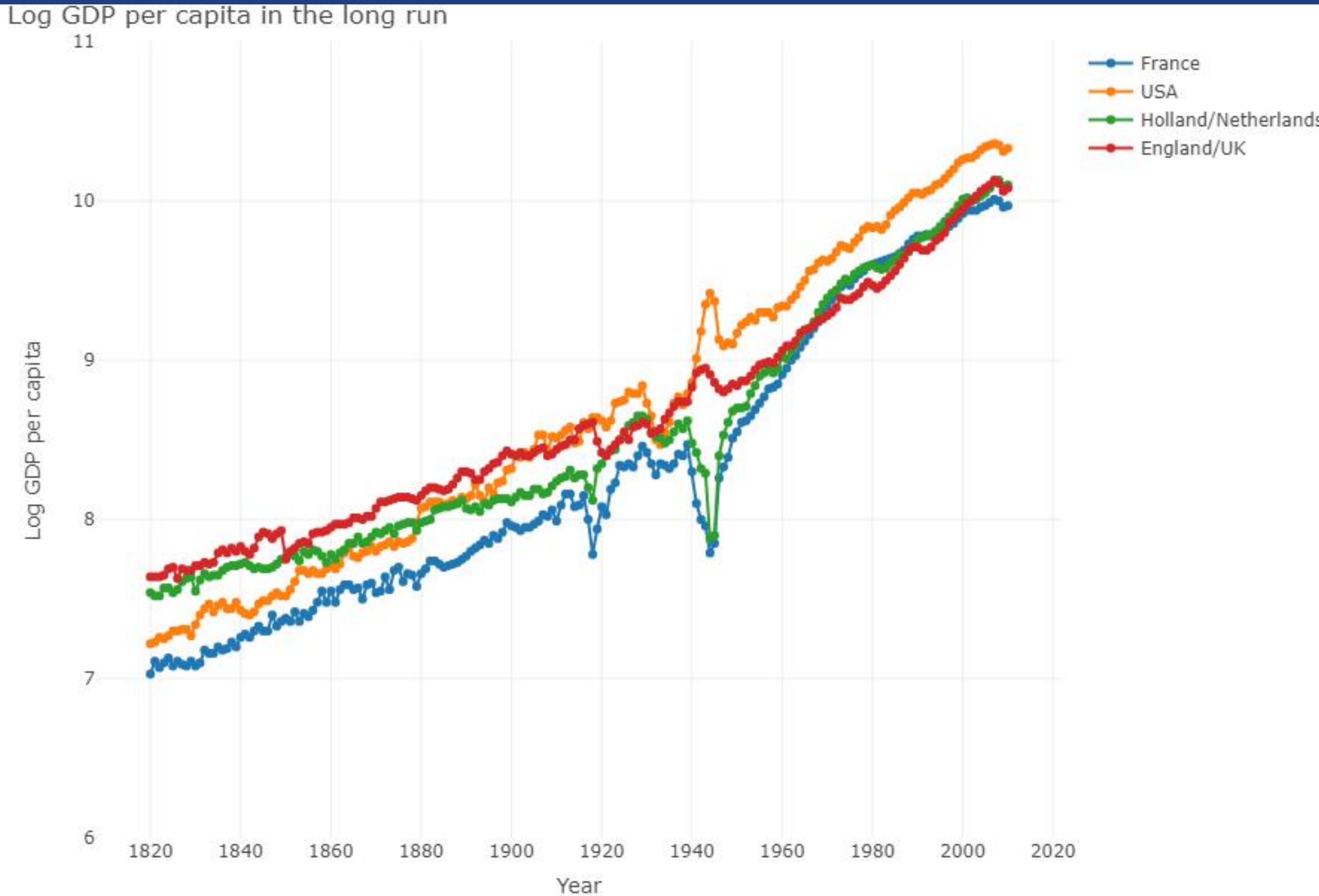
Stable growth countries



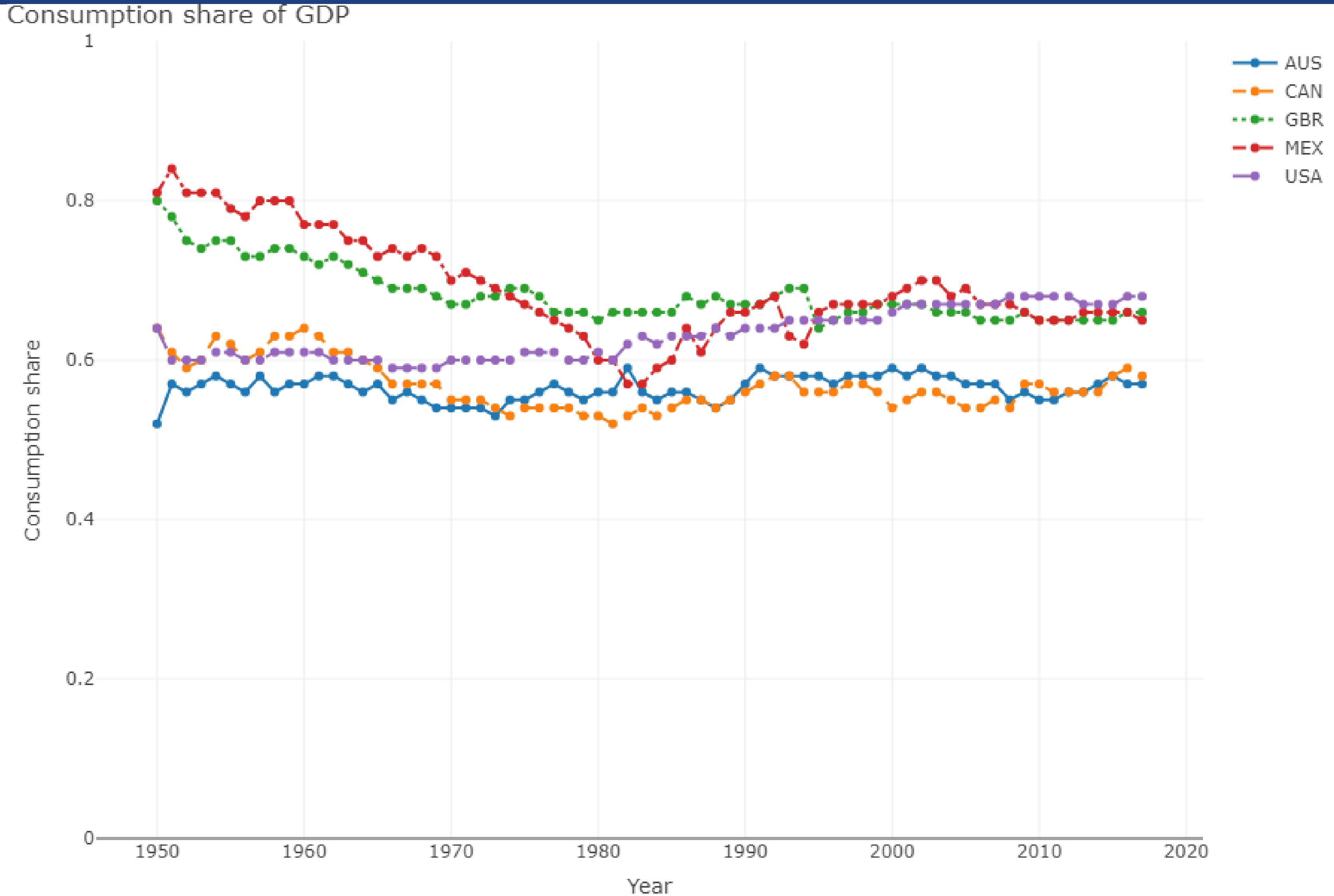
Catch-up growth countries



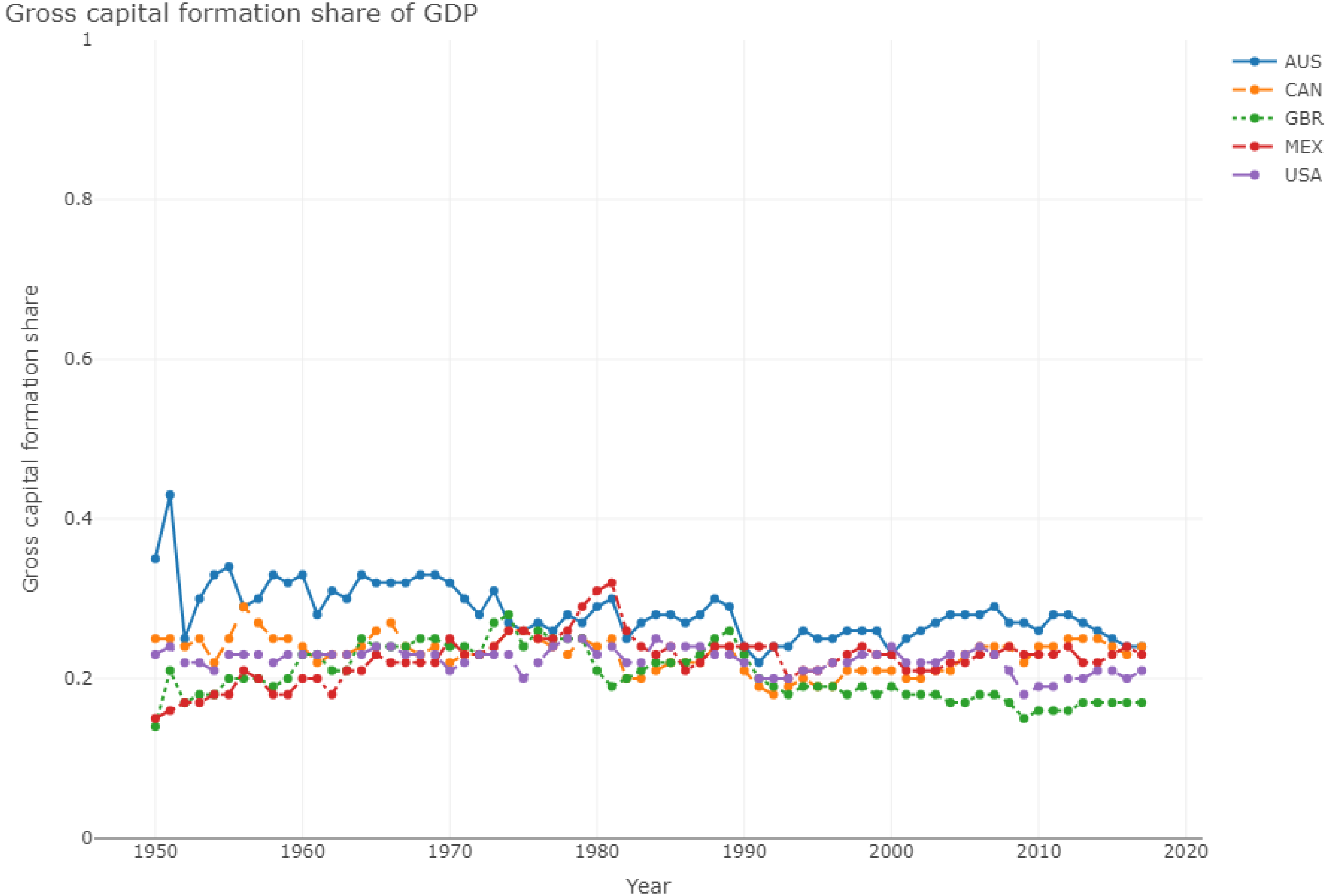
Economic growth since XIX century



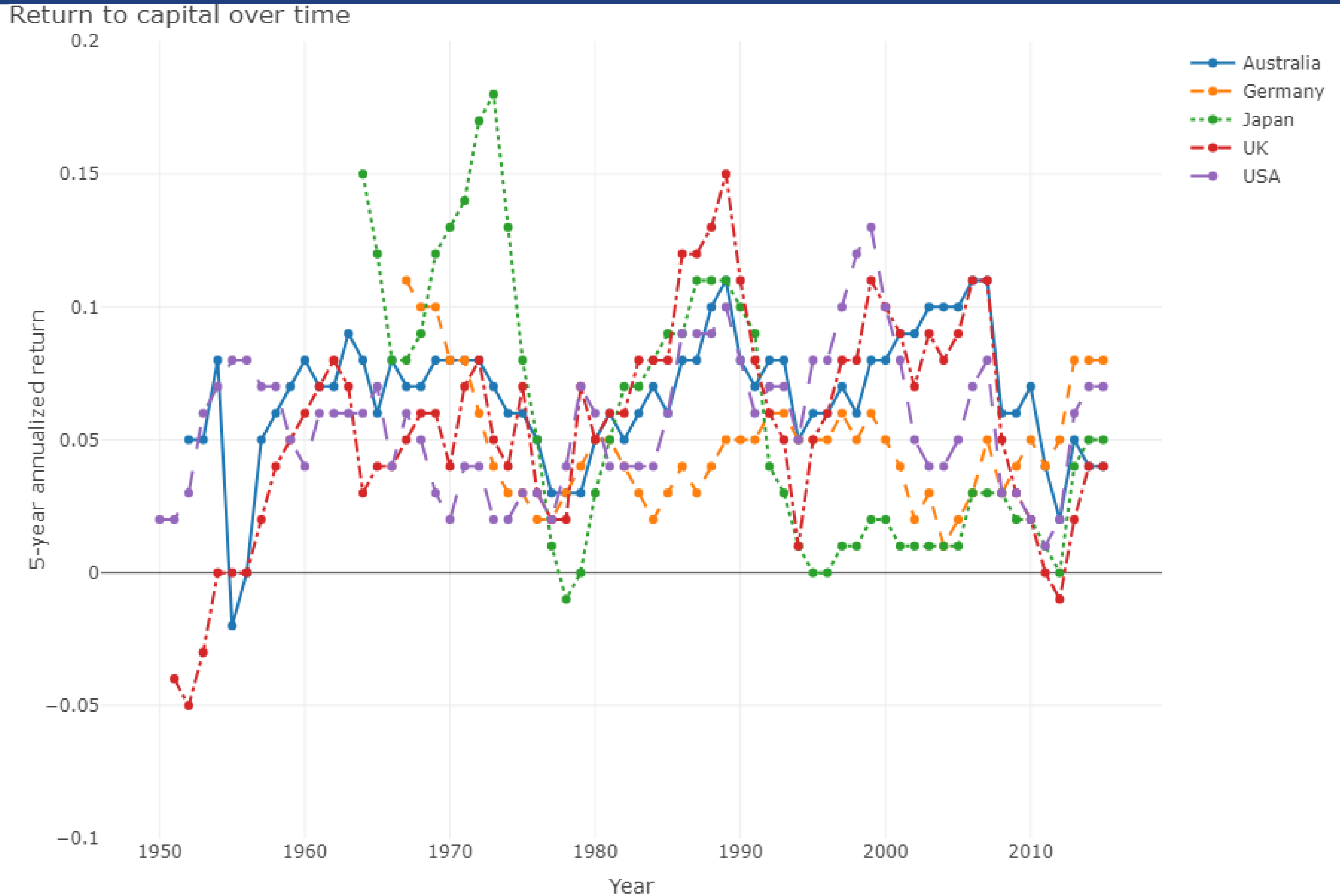
Consumption share of GDP



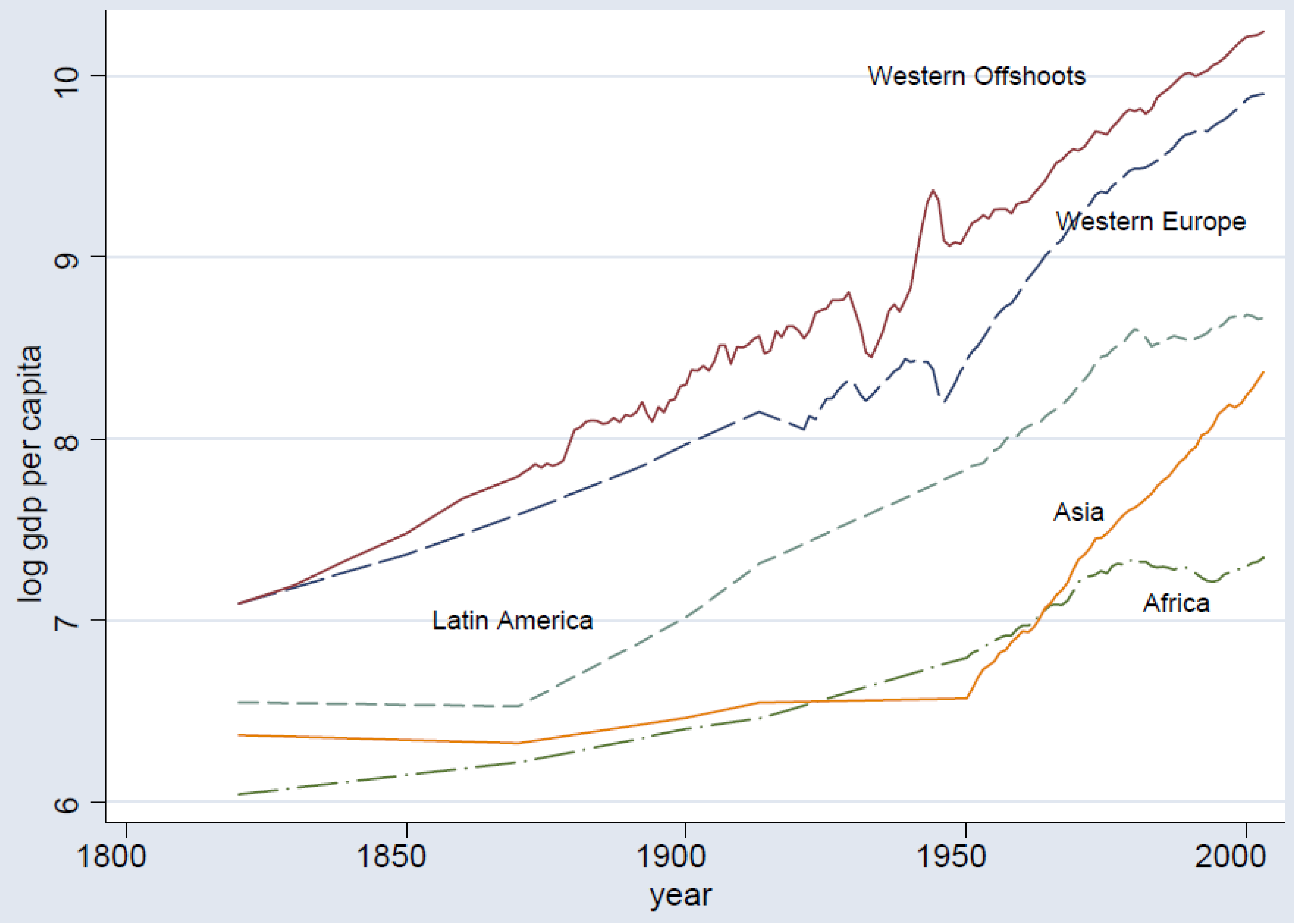
Gross capital formation share of GDP



Return to capital over time



Economic growth across continents



Stylized facts about economic growth

Kaldor (1963)

1. Per **capita** output grows over time: $g_{\left(\frac{Y}{L}\right)} > 0$
   - and its growth does not tend to diminish
2. Physical capital per **worker** grows over time: $g_{\left(\frac{K}{N}\right)} > 0$
3. The rate of return to capital $MPK \approx const$
4. Output to capital ratio $Y/K \approx const$
5. The shares of labour and physical capital in national income are nearly constant:
   - $wL/Y = const, (r + \delta)K/Y = const$
6. The growth rate of output per worker (labour productivity) differs substantially across countries

A Robinson Crusoe economy

Robinson Crusoe economy

- No markets and firms!
- A single person lives on a remote island (a metaphorical closed economy without a government)
- A single final good (eg. corn) can be consumed, saved (stored)
- Robinson owns the input and manages the technology that transforms input (corn seeds) to output (corncob)
- All the saved corn is invested to accumulate ‘capital’ and used as input in production together with labour
- Labour supply is totally inelastic, so there is NO leisure-consumption choice and population equals employment, $L = N = 1$

Robinson Crusoe economy

Production function

Two inputs, physical capital stock $K(t)$, and labour, $L(t)$, are used at time t to produce the flow of output $Y(t)$ according to the following production function

$$Y(t) = F(K(t), L(t), t)$$

Output can be consumed $C(t)$, or invested, $I(t)$, to create new units of physical capital, $K(t)$.

In a closed economy output equals income, and the amount invested equals the amount saved

$$S(t) = I(t)$$

Robinson Crusoe economy

Capital accumulation

Net increase in the stock of physical capital at a point in time equals gross investment less depreciation (assumed to be linear function of existing capital stock,δK_t)

In discrete time, $\Delta K \equiv K_{t+1} - K_t = I_t - \delta K_t$

In continuous time, $\Delta t \to 0$

$$\frac{\partial K}{\partial t} \equiv \dot{K} = I - \delta K$$

Saving rate

Let $0 \leq s \leq 1$ be the fraction of output Y that is saved and thus invested. The dynamics of capital is determined by

$$\dot{K} = sY - \delta K$$

A representative agent model

Households

A representative agent model

- A closed economy is inhabited by a large number of identical households who own all the assets and inputs
- They all work, i.e. serve as producers, and rent capital to themselves
- The economy admits a **representative consumer** – the demand and labor supply side of the economy can be represented as if it resulted from the behavior of a single household
- Households save a **constant** exogenous fraction s of their disposable income – irrespective of what else is happening in the economy – not optimising!

Firms

- All firms have access to the same production function for the final good, Y, or output
- All input markets – labour and capital – are perfectly competitive, i.e. all factors are paid their marginal products
- The final good market is perfectly competitive so profits are zero
- The economy admits a **representative firm**, with a representative (or aggregate) production function
- The number of firms and households is the same so all households are (self)-employed and there is no difference between per capita and per worker terms, $L = N$

Inputs to production

Capital (endogenous), K

- output per worker may increase with capital per worker

Labour (exogenous), L

- population growth
- participation rates
- human capital

Technology (public), A

- Inventions
- R&D
- Education

Land (fixed), T

- fixed supply, but quality may be improved

Raw materials, R

- important distinction between **depletable** resources (coal, oil)
- **renewable** resources (timber, fish)

Inputs to production

Production function

shows the maximum output that can be produced using specified quantities of inputs, given existing technical knowledge

Output is a function of capital (K), labour (L), technology (A), land (T), raw materials, (R), human capital, (H), etc.:

$$Y = F(K, L, A, T, R, H, \dots)$$

In a basic model we abstract from any production factors except capital, labour and technology

Neoclassical aggregate production function without technical progress

Neoclassical growth theory

The theory we are studying is neoclassical because it does not ask how actual output gets to potential output

- Over a **long enough** period, actual and potential output are equal

The neoclassical growth theory rests on the **neoclassical aggregate** production function, which explains the evolution of output by the dynamics of 2 inputs:

$$Y(t) = F(K(t), L(t))$$

- $K(t)$ − evolution of capital stock
- $L(t)$ − evolution of total employment

Hereinafter, we will skip time t in our notations

Neoclassical production function

Constant returns to scale (CRS)

Production function exhibits constant returns to scale:

$$F(\lambda K, \lambda L) = \lambda F(K, L)$$

Mathematically, F is a homogeneous function of degree 1

Euler's theorem

if the factor prices equal the respective marginal products, the factor payments just exhaust the total output

$$F(K, AL) = Y = \frac{\partial Y}{\partial K} K + \frac{\partial Y}{\partial L} L$$

GEM and global profit tax reform

Language level	Problem ?		Solution ☑		
General	**Real-life situation**	Critical ⇨ thinking	**Practical answer**	**G**	all markets are perfectly competitive ⇨ profits are zero
Conceptualisation	⇩		⇧	Intuition	
Economic	**Question**	Economic → reasoning	**Forecast**	**E**	all factors of production are paid their marginal products ⇨ CRS
Formalisation	⇩		⇧	Proposition	
Mathematic	**Model**	Analytical ⇒ solution	**Theorem**	**M**	Aggregate production function $F(K, AL)$ is homogenous of degree 1

Neoclassical production function

Positive and diminishing returns to inputs

- Decreasing returns to capital refers to the property that increases in capital lead to smaller and smaller increases in output as the level of capital increases

$$MPK = \frac{\partial Y}{\partial K} = \frac{\partial F}{\partial K} > 0, \qquad \frac{\partial^2 F}{\partial K^2} < 0$$

- Decreasing returns to labor refers to the property that increases in labor, given capital, lead to smaller and smaller increases in output as the level of labor increases

$$MPL = \frac{\partial Y}{\partial L} = \frac{\partial F}{\partial L} > 0, \qquad \frac{\partial^2 F}{\partial L^2} < 0$$

Neoclassical production function

Essentiality

$$F(0, L) = F(K, 0) = 0$$

- At very low levels of income, savings may be zero as all resources are needed for consumption
- So capital cannot be created through investment, and output may not be able to grow through time
- Positives level of L is also essential for production

The Solow model

Saving

(since technology and labour growth rates are exogenous) is driven by capital accumulation in transition to the balanced growth path

Robert Solow (1924-2023)

Published his seminal paper in 1956

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1987

Nobel Prize motivation:

"for his contributions to the theory of economic growth."

*‘All theory depends on **assumptions** which **are not quite true**. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive. A **"crucial" assumption** is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be **reasonably realistic**. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect.’*

The Solow model

Assumptions

Crucial and simplifying

A1 Factors of production are **substitutes** and not complements

A2 Saving rate s is exogenous. Saving $S = sY$ generates investment, which is consistent with the **neoclassical** interpretation of the saving/investment balance $S = I$

A3 The interest rate is perfectly flexible and instantaneously adjusts investment and saving

A4 Wages adjust so that the supply of labour (set exogenously by the growth rate of the population n) and the demand for labour adjusts perfectly

The Solow model

The evolution of the labour

In continuous time: $\frac{dL(t)}{dt} \equiv \dot{L}(t) = n \cdot L(t)$

The growth rate of labour n is exogenous

$$n = \frac{\dot{L}(t)}{L(t)} = \frac{d\ln L(t)}{dt} = \frac{d\ln L(t)}{dL(t)} \frac{dL(t)}{dt} = \frac{1}{L(t)} \dot{L}(t)$$

The rate of change of the log of L is constant and = n

$$\int d\ln L(t) = \int n dt \qquad \Longrightarrow \quad \ln L(t) = \ln L(0) + nt$$

Exponentiating both sides of this equation gives us:

$$L(t) = L(0) e^{nt}$$

In discrete time: $L_{t+1} - L_t = n \cdot L_t$

The Solow model

Capital accumulation

With **no government** spending and no international linkages a constant depreciation rate, δ, implies that the change in the capital stock is equal to:

$$\dot{K} = Y - C - \delta K = I - \delta K$$

where C is consumption and I is investment

Since saving is equal to investment, $S = I$

The **saving rate**, s, is assumed to be **constant**:

$$S = sY$$

Thus, capital accumulation equation becomes:

$$\dot{K} = sY - \delta K$$

The Solow model

Capital consumption (1) - depreciation

- Capital stock is reduced by physical wear and tear and moral **depreciation**
- As the capital stock grows older, its value is discounted
- The amount of discounting is given by the **discount rate** δ

The Solow model

Capital consumption (2) – population growth

Capital-widening

- extends the existing capital **per worker** to new extra workers, so $\frac{K}{L}$ is constant

$$\frac{\dot{K}}{K} = \frac{\dot{L}}{L} \equiv n \quad \Rightarrow \quad g_{\left(\frac{K}{L}\right)} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = 0$$

Capital deepening

- raises capital **per worker** for all workers, $\frac{K}{L}$ is growing

$$g_{\left(\frac{K}{L}\right)} > 0$$

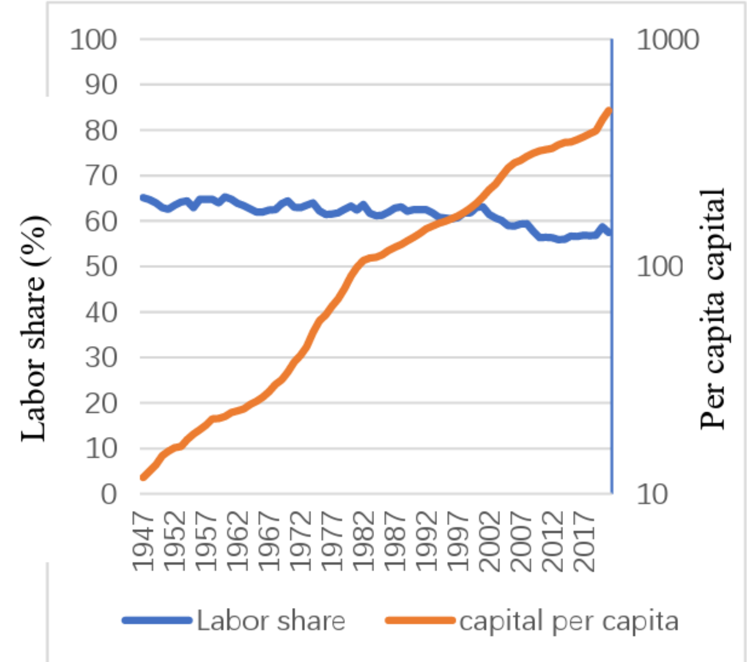


Figure 1: Capital Deepening and Labor Share Changes in the USA

The Solow model

Capital consumption (3) – technical progress

- If new technologies are introduced, workers become more productive
- Less labour is required to produce the same amount of output
- Some workers become available for other uses
- Technical progress is therefore equivalent to an increase in the number of workers, in other words to population growth (we shall call this growth t)

$$\frac{\dot{A}}{A} \equiv g$$

BGP

A country is said to have a **balanced growth path** (BGP) if the following four ratios are stable:

1. The growth rate of GDP per capita Y/L (g_Y)
2. The ratio of gross capital formation to GDP (I/Y)
3. Labor's income share in GDP (wL/Y)
4. The capital/output ratio (K/Y)

Balanced growth path

Balanced growth path (BPG)

a condition of the economy in which all **endogenous** variables grow with constant rates and remain stable (balanced) over time relative to one another

- In the production function $Y = F(K, L)$
  - The rate of population growth is exogenous $\frac{\dot{L}}{L} \equiv n$

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = const$$

⇨ the output to capital ratio is constant $\frac{Y}{K} = const$

Balanced growth path (BGP)

Recall, that net capital investment is equal to

$$\dot{K} = sY - \delta K$$

This implies $\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta = const$

Hence the output to capital ratio is constant $\frac{Y}{K} = const$

The balanced growth path requires

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = const$$

How to calculate the BGP growth rate for a particular technology?

Mathematical methods

Recall, that the growth rate of a variable $Y(t)$ equals the time derivative of its log:

$$\frac{\partial lnY(t)}{\partial t} = \frac{\frac{\partial Y(t)}{\partial t}}{Y(t)} = \frac{\dot{Y}(t)}{Y(t)}$$

We can skip time factor t to simplify our notations

Consider the following function: $Z(t) = X(t)Y(t)$

- Taking logs: $lnZ(t) = lnX(t) + lnY(t)$
- Then differentiating wrt time t we obtain the formula:

$$\dot{Z}(t)/Z(t) = [\dot{X}(t)/X(t)] + [\dot{Y}(t)/Y(t)]$$

$$\frac{\dot{Z}}{Z} = \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y}$$

Mathematical methods

Applying the same technique for:

$$Z = \frac{X}{Y}$$

we obtain

$$\frac{\dot{Z}}{Z} = \frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}$$

Applying again for:

$$Z = X^{\alpha}$$

we obtain

$$\frac{\dot{Z}}{Z} = \alpha \frac{\dot{X}}{X}$$

Steady state in the Solow model without technical progress

$$A = 1$$

The Solow model with $A = 1$

Consider a neoclassical production function

$$Y = F(K, AL)$$

Assumption (to be relaxed later):

- labour productivity A (≈ labour-augmented technical progress) is NOT growing and normalized to $A = 1$

Representative consumer/firm produces:

$y \equiv Y/L$ – output per unit of labour = out per (efficient?) worker

$k \equiv K/L$ – capital per unit of labour = capital per worker

Recall the CRS property: $F(\lambda K, \lambda L) = \lambda F(K, L)$

Use $\lambda = 1/L$ to obtain production function in per worker terms

Intensive form production function

$$y = f(k) = F(K/L, 1) = F(K, L)/L$$

The Solow model with $A = 1$

Capital accumulation per worker

The growth rate of $k \equiv K/L$ is

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - n$$

From $\dot{K} = sY - \delta K$ we have $\frac{\dot{K}}{K} = \frac{sY}{K} - \delta$

Thus

$$\frac{\dot{k}}{k} = \frac{sY/L}{K/L} - (n+\delta) = \frac{sf(k)}{k} - (n+\delta)$$

$$\boxed{\dot{k} = sf(k) - (n+\delta)k}$$

The Solow model with $A = 1$

The steady state

The ratio of capital to labour (the state variable $k \equiv K/L$) is constant, so the growth rate of it is zero

$$\frac{\dot{k}}{k} = 0$$

In the steady state, there is a certain level of k^*, such that

$$sf(k^*) = (n + \delta)k^*$$

Ignoring trivial solution $k^* = 0$ we consider only $k^* > 0$

The Solow model with $A = 1$

Balanced growth path and steady state

In the **steady state**, $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - n = 0$

$$\frac{\dot{K}}{K} = n$$

Since $y = f(k)$ it follows that $\dot{y} = 0$. Therefore output grows at the same rate (implying **balanced growth path**)

$$\frac{\dot{Y}}{Y} = \frac{\dot{y}}{y} + n = n$$

Output per-worker (or GDP per capita) grows at rate

$$g_{\left(\frac{Y}{L}\right)} \equiv \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = n - n = 0$$